Multivariate Diferrentiation and Integration

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Stats I

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Dealing with Multiple Dimensions

- In science it is common to perform experiments where the effect of a single treatment is studied (e.g. a drug dose).
- This design is uncommon in political science, possibly to our detriment, so we need to build a toolkit that can manage multi-dimensional processes.
- Even a hypothesis stated, "I believe y is an increasing function of x", cannot be studied in that form.
- The empirical test of this hypothesis takes the form, "I believe y is a function of x_1, x_2, $..x_k$, and that $\frac{\partial y}{\partial x_j} > 0$", which uses the primary differentiation concept from multivariate calculus, the partial derivative.

The Partial Derivative I

Differentiation and integration in multiple dimensions (the variants we will explore, find additional reading if you are interested in total derivatives) represent iterative application of the integration and differentiation concepts you've already learned.

Whereas in one dimension $\frac{d}{dx}f(x)$ was used to denote the first derivative of $f(x)$ with respect to x, $\frac{\partial}{\partial x_j}f(x_1, x_2..x_k)$ is used to denote the partial derivative of $f(x_1, x_2..x_k)$ with respect to x_j.

To find $\frac{\partial}{\partial x_j}f(x_1, x_2..x_k)$, you simply find $\frac{d}{dx_j}f(x_1, x_2..x_k)$, creating a problem of univariate differentiation by treating all $x_i \forall\ i \neq j$ as numeric constants.

The Partial Derivative II

What is $\frac{\partial}{\partial x_j} f(x_1, x_2)$ where

$$f(x_1, x_2) = x_2 e^{x_1^2}$$

The gradient of a function of k variables is the column vector of first partial derivatives of that function.

What is the gradient of the above function?

Multiple Integration I

Where the univariate definite integral was denoted $\int_a^b f(x)dx$, the multiple integral is denoted

$$\int_{a_k}^{b_k} .. \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2..x_k)dx_1 dx_2..dx_k$$

Note the order of the subscripts on (a, b) and the order of the j in the dx_j. The order of integration and the respective bounds is set by this order, and is always read from inside out. Multiple integration is performed by applying univariate integration to the inner-most integral, treating other variables as constants, until no integral is left.

Multiple Integration II

$$\int_{a_k}^{b_k} .. \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x_1, x_2..x_k) dx_1 dx_2..dx_k$$

$$= \int_{a_k}^{b_k} .. \int_{a_2}^{b_2} f_2(a_1, b_1, x_2..x_k) dx_2..dx_k$$

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$$= \int_{a_k}^{b_k} f_k(a_1, b_1, ..x_k) dx_k$$

Multiple Integration III

Suppose you are a policy-maker presented with the opportunity to choose two parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^{2+}$ of a policy (say benefit levels) that result in an outcome pair $(x_1, x_2) \in [0, 1]^2$, say economic growth and inequality. The happiness you derive from the policy outcome pair (your utility function) is given by

$$U(x_1, x_2) = 2x_1 + x_2$$

Suppose that given a choice (α_1, α_2), your, beliefs in the Bayesian sense, about the policy follow this distribution

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\frac{1}{1+\alpha_1} + \frac{1}{1+\alpha_2}}(x_1^{\alpha_1} + x_2^{\alpha_2})$$

What is your expected (mean) utility given a policy choice (α_1, α_2)?