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WHY THE BASS MODEL FITS WITHOUT DECISION VARIABLES

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Over a large number of new products and technological innovations, the Bass diffusion model (Bass 1969) describes the empirical adoption curve quite well. In this study, we generalize the Bass model to include decision variables such as price and advertising. The generalized model reduces to the Bass model as a special case and explains why the Bass model works so well without including decision variables. We compare our generalized Bass model to other approaches from the literature for including decision variables into diffusion models, and our results provide both theoretical and empirical support for the generalized Bass model. We also show how our generalized Bass model can be used for product planning purposes
(Diffusion; Marketing Mix; New Product Research; Pricing Research)

1. Introduction

It has been found that over a large number of new products and technological innovations the Bass Model (Bass 1969) describes the empirical adoption curve quite well. Underlying each of these curves is a sequence of prices and other influences. Russell (1980) and other economists have suggested that the Bass Model is incomplete because it does not combine contagion effects with traditional economic variables such as price. A major purpose of this study is to develop an understanding of why the Bass Model works even though it does not include decision variables. In order to do this it is necessary to develop a model (or models) that, under certain circumstances (regularity of change behavior in time involving the decision variables), will reduce to (be observationally equivalent to) the Bass Model. We have developed a generalized version of the Bass Model that has this property. We shall refer to this model as the Generalized Bass Model (GBM). In this study we examine empirically GBM with price and advertising and suggest ways to use the model for new product planning purposes.

2. The Bass Model

Because the Bass Model has been the basis for extensions to include decision variables it will be useful to review it briefly here. The basic underlying premise of the model is that the probability of adoption of a new product or technology at time T given that it has not yet been adopted would depend linearly on two forces, one of which is independent of the number of previous adopters and one of which would be positively influenced by

previous adopters. Bass described the first force, represented by p, as the coefficient of innovation and the second force by q as the coefficient of imitation. Lekvall and Wahlbin (1973) and others have referred to these as "external influence" and "internal influence".

Thus, if $f(T)$ is the density function describing the time of adoption of a population and $F(T)$ is the cumulative function, the hazard function describing the conditional probability of adoption at time T (or proportion of those not yet adopting) was proposed as:

$$f(T)/(1 - F(T)) = p + qF(T) \quad \text{or} \tag{1}$$

$$dF/[p + (q - p)F - qF^2] = dT. \tag{2}$$

Assuming $F(0) = 0$, the solution to the differential equation in (2) is given by:

$$F(T) = (1 - e^{-(p+q)T})/(1 + (q/p)e^{-(p+q)T}). \tag{3}$$

The density function then becomes:

$$f(T) = ((p + q)^2/p)e^{-(p+q)T}/(1 + (q/p)e^{-(p+q)T})^2. \tag{4}$$

If the total number of ultimate adopters (the market potential) is given by m and, assuming each adopter buys only one unit, the sales rate, $S(T) = mf(T)$ where $f(T)$ is given by (4). This is the Bass Model in the time domain. In the cumulative adoption domain, it is possible to write:

$$S(T) = pm + (q - p)Y(T) - (q/m)[Y(T)]^2, \quad \text{where } Y(T) = mF(T). \tag{5}$$

In the original paper (Bass 1969) the parameters were estimated by regressing sales on cumulative sales and cumulative sales squared, from which the parameters p, q, and m are uniquely identified. Later, Srinivasan and Mason (1986) showed how to estimate the parameters in the time domain of $mf(T)$ by non-linear least squares. In empirical studies (with and without decision variables) parameters have been estimated, for the most part, in the cumulative adoption domain but sometimes in the time domain. Models in the two domains, of course, are equivalent if there is no disturbance term, but they are not exactly equivalent if there is a random disturbance. An advantage of the time domain version of the model is that it helps one understand the behavior of the adoption curve in time as related to parameter variation and, when decision variables are involved, as related to changes in these variables. The time domain version, of course, requires a closed-form solution that connects the time domain to the cumulative adoption domain.

3. The Generalized Bass Model

We seek here to develop a generalization of the Bass Model that: (1) includes decision variables, (2) has a closed-form solution in the time domain and, (3) reduces to the Bass Model as a special case under plausible regularity conditions for the decision variables. Generalization of the hazard function for the Bass Model will necessarily involve the right hand side of equation (1). The most obvious generalization is one in which p and q are permitted to vary with time. We seek to preserve the fundamental character of equation (1) and thus we suggest:

$$f(T)/[1 - F(T)] = [p + qF(T)]x(T). \tag{6}$$

We shall call $x(T)$ "current marketing effort" to reflect the current effect of dynamic marketing variables on the conditional probability of adoption at time T. Although we call this term "current marketing effort," it is important to note that it can reflect the effects of lags in the decision variables. Thus, we can have "carryover effects" of advertising and other lagged effects being mapped to $x(T)$. See §6 for further analysis of carryover effects of advertising and for level effects. At this stage we leave $x(T)$ as a completely

general non-negative function. Later we suggest a particular functional form for $x(T)$ and we suggest a way for including lags in decision variables.

In the generalization, $x(T)$ can serve to shift the hazard function upward or downward. Subsequently, we shall test a more general model in which p and q are different and distinct functions of "current marketing effort," but, for now, we shall explore the implications of (6). The more general model does not have a closed-form solution, but (6) does.

We want to know not just the effects of marketing variables at time T on the diffusion process, but the effects of these variables over time. To study these effects we must solve the following differential equation:

$$dF/[p + (q - p)F - qF^2] = x(T)dT, \tag{7}$$

or,

$$\int dF/[p + (q - p)F - qF^2] = \int x(T)dT. \tag{8}$$

Then,

$$F = (1 - pe^{-(X(T)+C)(p+q)})/(1 + qe^{-(X(T)+C)(p+q)}). \tag{9}$$

Assuming $F(0) = 0$,

$$pe^{-(X(0)+C)(p+q)} = 1 \quad \text{and} \quad C = [1/(p + q)] \operatorname{Ln}(p) - X(0).$$

Substituting this result into (9), we have:

$$F(T) = (1 - e^{-(X(T)-X(0))(p+q)})/((q/p)e^{-(X(T)-X(0))(p+q)} + 1). \tag{10}$$

$X(T)$ may be thought of as cumulative marketing effort and thus cumulative adoption is a function of cumulative marketing effort. Differentiating (10) with respect to T yields:

$$f(T) = x(T)[(p + q)^2/p]e^{-(X(T)-X(0))(p+q)}/((q/p)e^{-(X(T)-X(0))(p+q)} + 1)^2. \tag{11}$$

The density function indicated in equation (11) shows that the time to adoption depends on both current marketing effort and cumulative marketing effort from previous periods. If $x(T)$ is known for all T, then the explicit functional connection between f and T is defined by equation (11). Clearly, if $x = 1$ for all T, then $f(T)$ in (11) is identical to the solution for $f(T)$ in equation (4). If X is unknown but highly correlated with time, then the original Bass Model, although possibly misspecified, will produce a good description of the diffusion pattern. Thus, if the data are generated by equation (11) and equation (4) is presumed to be the true model, we have a classic example of the general principle that if omitted variables (X) are highly correlated with included variables (T) the misspecified model will provide a good fit to the data. Only when the omitted variables are loosely connected to time will the misspecified model fail to provide a good fit to the data. From a forecasting viewpoint, even in the presence of omitted variables, equation (4) may provide the basis for good forecasts. However, from the perspective of an understanding of the effects of decision variables, correct specification of the model is essential.

A Special Case

It is possible to differentiate equation (4) with respect to T to find that the time of peak adoption is:

$$T^* = [1/(p + q)] \operatorname{Ln}(q/p). \tag{12}$$

The original Bass Model is symmetric about the peak time, T^*. As discussed in Mahajan, Muller and Bass (1990), various "flexible diffusion models" have been developed around

modifications of the Bass Model. These models allow for asymmetric distributions about the time of the peak and, in this sense, are more flexible than the original Bass Model. Suitable choice of the function $X(T)$ in equation (11) can produce any desired time of peak and can accommodate a great variety of shapes for the density function f.

Let us examine the case where current marketing effort, $x(T)$, is constant. Then $X(T) = cT$. Substitution in equation (11) and treating $X(0) = 0$ yields:

$$f(T) = [(p' + q')^2/p']e^{-(p'+q')T}/((q/p)e^{-(p'+q')T} + 1)^2. \tag{13}$$

In equation (13), $p' = cp$ and $q' = cq$ and hence, with constant current marketing effort, equation (13) is observationally equivalent to equation (4). When current marketing effort is constant but not observed, the original Bass Model will describe the data well and will provide a basis for good forecasts. However, if marketing effort had been at a different level, the shape of the curve would have been influenced by this different level and the time of the peak would have been shifted inversely proportional to the different level. Differentiating equation (13) with respect to T and solving for T^*, we find:

$$T^*(c) = [1/(p' + q')] \text{ Ln } (q'/p') = 1/c[1/(p + q)] \text{ Ln } (q/p). \tag{14}$$

Substitution of (14) into (13) and solving for $S(T^*(c))$ we find that $S(T^*(c))$ $= [m(p + q)^2/4q]c$ where $S(T) = mf(T)$ and m is the ultimate market potential for the product. Therefore, sales at the peak will vary in direct proportion to the level of current marketing effort.

It is noteworthy that if "current marketing effort" is constant and $X(0) = 0$, equation (11) becomes:

$$f(T) = c[(p + q)^2/p]e^{-c(p+q)T}/((q/p)e^{-c(p+q)T} + 1)^2. \tag{15}$$

Equation (15) is observationally equivalent to (4), and we can use this result to aid in understanding why the Bass Model works without including decision variables. If the decision variables are changing in a more or less regular pattern such that the mapping function that connects the decision variables to $x(T)$ is approximately constant, the Bass Model will fit the diffusion process quite well. We can use this fact in searching for an appropriate mapping function that connects the decision variables to $x(T)$.

The Mapping Function

In developing the mapping function that maps decision variables to current market effort, we want to take into account the basic character of the diffusion model and apply first principles. Therefore, in deriving the function we use the following sequence of logical deductions:

(1) We know that GBM without decision variables reduces to BM, i.e., GBM with $x(T) = 1$ or GBM with $x(T) = c$. In mapping decision variables to $x(T)$, then, we must consider the period-to-period changes in the decision variables as relevant.

(2) We know that if $x(T) = c$, then $X(T) = cT$. We have studied the behavior of $X(T)$ empirically by solving equation (10) for $X(T)$ by replacing $F(T)$ with $\Sigma S_T/m$ and we let $X(0) = 0$. The solution for $X(T)$ will then depend on $p + q$ and q/p. We have fixed these at several different values and plotted the empirical values of $X(T)$ for each pair of values of $p + q$ and q/p for 12 different products. In each instance we find that $X(T)$ is approximately linear in T. That is, $X(T)$ is empirically approximately equal to cT. For details, see Bass and Krishnan (1992). We seek a mapping function such that a plausible regular behavior of decision variables would map to approximately a constant.

(3) Let us consider two decision variables: price, denoted as $\text{Pr}(T)$, and advertising, denoted as $\text{ADV}(T)$. Taking first differences: $\Delta \text{Pr}(T) = \text{Pr}(T) - \text{Pr}(T - 1)$ and $\Delta \text{ADV}(T) = \text{ADV}(T) - \text{ADV}(T - 1)$. Using the diminishing returns argument, we

expect the effect of a given Δ to be less as the base level from which the change takes place increases. We, therefore, take as variables to be weighted in the mapping function: $\Delta \Pr(T)/\Pr(T-1)$ and $\Delta \text{ADV}(T)/\text{ADV}(T-1)$. Our mapping function then becomes:

$$x(T) = 1 + [\Delta \Pr(T)/\Pr(T-1)]\beta_1 + [\Delta \text{ADV}(T)/\text{ADV}(T-1)]\beta_2. \quad (16)$$

The expected sign of β_1 is negative and the expected sign of β_2 is positive, and our mapping function may be easily extended to include additional variables.

(4) In the mapping function for $x(T)$, the first term on r.h.s., 1 translates into $X(T) = T$ thus seeking to preserve the fundamental character of the Bass Model, while the coefficients β_1 and β_2 reflect the effectiveness of price and advertising *over* the simple time-based diffusion.

(5) In the discussion above we have treated $\Pr(T)$ and $\text{ADV}(T)$ as the observed (historical) values of price and advertising at time T and we may use (16) for estimation purposes with historical data on adoption rates and prices and advertising. Suppose, however, that there is a new product to be introduced and no data are yet available. For policy purposes we require a mapping function that will map different levels or patterns of decision variables to $x(T)$. We evaluate policies relative to some base level values. Let $\Pr_b(T)$ and $\text{ADV}_b(T)$ be baseline values for price and advertising for all T and let $\Pr(T)$ and $\text{ADV}(T)$ be contemplated price and advertising policies. Then, using the principles developed above, we have the mapping function:

$$x(T) = 1 + [(\Pr(T) - \Pr_b(T))/\Pr_b(T)]\beta_1$$
$$+ [(\text{ADV}(T) - \text{ADV}_b(T))/\text{ADV}_b(T)]\beta_2$$
$$+ [\Delta \Pr(T)/\Pr(T-1)]\beta_1 + [\Delta\text{ADV}(T)/\text{ADV}(T-1)]\beta_2. \quad (17)$$

Note that if $\Pr(T) = \Pr_b(T)$ and $\text{ADV}(T) = \text{ADV}_b(T)$, for all T, (17) is the same as (16). The second and third terms in (17) may be thought of as reflecting the level effect of the decision variables and the fourth and fifth terms may be thought of as reflecting the period effect. It is possible to write (16) as:

$$x(T) = 1 + [(\Delta \Pr(T)/(T - (T-1)))/\Pr(T-1)]\beta_1$$
$$+ [(\Delta\text{ADV}(T)/(T - (T-1)))/\text{ADV}(T-1)]\beta_2$$

or,

$$x(T) = 1 + [(\Delta \Pr(T)/(\Delta T))/\Pr(T - \Delta T)]\beta_1$$
$$+ [\Delta\text{ADV}(T)/(\Delta T))/\text{ADV}(T - \Delta T)]\beta_2. \quad (18)$$

Treating time as continuous and taking limits as $\Delta T \to 0$, we have:

$$x(T) = dX(T)/dT$$
$$= 1 + [(d \Pr(T)/dT)/\Pr(T)]\beta_1 + [(d\text{ADV}(T)/dT)/\text{ADV}(T)]\beta_2. \quad (19)$$

Equation (19) is, then, the continuous time version of (16). Integration of (19) between 0 and T gives the continuous time version of cumulative marketing effort $X(T)$:

$$X(T) = T + \text{Ln}(\Pr(T)/\Pr(0))\beta_1 + \text{Ln}(\text{ADV}(T)/\text{ADV}(0))\beta_2. \quad (20)$$

4. Properties of GBM

A Special Case

Note that if decision variables are constant for all T, GBM reduces to BM. Furthermore, if the decision variables are changing from period to period, but the changes are such that the value of each variable in each period is a constant proportion of the value in the preceding period so that $\Pr(T) = \alpha_1 \Pr(T - 1)$ and $ADV(T) = \alpha_2 ADV(T - 1)$, (20) becomes:

$$X(T) = cT, \quad \text{where } c = 1 + \beta_1 \text{ Ln } (\alpha_1) + \beta_2 \text{ Ln } (\alpha_2). \tag{21}$$

It seems plausible that the percentage change in decision variables from period-to-period would be approximately constant and thus the mapping function is consistent with the empirical observation of $X(T)$ that is approximately linear in T. A constant percentage decline in price is consistent with the often-observed exponential shape of price patterns for new technologies.

Adoption Rate and Price Decline Rate

The mapping function has the property that the rate of adoption increases as the relative rate of price decline increases. If this relative rate is constant, consumers will adapt their adoption timing to this assumed rate of price decline. If, however, the rate of decline should increase then, using equation (17) to adjust from the regular pattern, consumers will accelerate their adoption plans (reservation prices will be reached sooner) and buy earlier than they otherwise would have. Therefore, the model is consistent with a "rational expectations" hypothesis concerning the timing of adoption (see Stokey 1981). For further discussion of rational expectations and GBM see §6.

GBM and the Hazard Rate

Unlike other models that modify the Bass Model to include decision variables, GBM retains the properties of BM. The essential property of equation (1) is that the conditional probability of adoption at time T depends on (increases with) cumulative adoption at T. Thus, an increase (or impulse) in adoption at T will have an effect at T but also have a "carrythrough" effect that will increase adoption in the future. The hazard rate, or proportion adopting at time T given that adoption has not yet occurred, of GBM is given by equation (6). Assuming $X(0) = 0$ and substituting the solution for $F(T)$ in equation (10) into equation (6) gives the hazard rate for GBM:

$$f(T)/[1 - F(T)] = \lambda(X(T)) = x(T)(p + q)/[1 + (q/p)e^{-(p+q)X(T)}]. \tag{22}$$

It is possible to write the adoption rate at $T + \tau$ as:

$$f(T + \tau) = \lambda(X(T + \tau))(1 - F(T + \tau)). \tag{23}$$

To assess the impact on adoption at $T + \tau$ of an impulse in current marketing effort at T, $x(T)$, we differentiate equation (23) with respect to $x(T)$:

$$df(X(T + \tau))/dx(T) = [d\lambda(X(T + \tau))/dx(T)](1 - F(T + \tau))$$
$$- \lambda(X(T + \tau))dF(T + \tau)/dx(T). \tag{24}$$

Equation (24) may be expressed as:

$$df(X(T + \tau))/dx(T) = f(X(T + \tau))dx(T)[(q - p) - 2qF(X(T + \tau))]. \tag{25}$$

If the impulse in current marketing effort, $dx(T)$, is positive, the sign of the impact on adoption at $T + \tau$ will be determined by the sign of the third term in equation (25). Noting that equation (23) may be written as:

$$f(T + \tau) = x(T + \tau)[p + (q - p)F - qF^2], \qquad (26)$$

and differentiating with respect to F in order to find peak sales:

$$f' = x[(q - p) - 2qF] = 0.$$

Peak sales will then occur when $[(q - p) - 2qF] = 0$ and sales will be increasing as long as this expression is positive. Hence, the effect of an impulse in current marketing effort at T, $x(T)$, will be positive at $T + \tau$ as long as sales have not peaked at that time. Equation (25) shows the impulse response at $T + \tau$ of an impulse in current marketing effort at T.

Market Potential

The Generalized Bass Model is a model of the adoption rate. It may be used in conjunction with market potential that is constant and not influenced directly by decision variables (m = constant) as in the Bass Model, or it may be treated as a function of decision variables ($m = m(T)$) and estimated jointly with the parameters of GBM. We shall estimate m in both ways as well as in conjunction with other models of the diffusion rate.

Comparison of GBM with Desirable Properties

We have developed 10 properties that we think are desirable, and we compare GBM with each of these.

(1) Although the Bass Model has been shown to provide good descriptions of the adoption data for a large number of new products and technologies, there are instances where, for a variety of reasons, this will not be the case. However, when the model does fit the adoption data well, a desirable property of extensions of it to include other variables is that these extensions reduce to the Bass Model as a special case. Only in this way is it possible to understand the conditions that explain why the model works without additional variables. GBM does have this desirable property.

(2) It is desirable for the model to be as general as possible. It would be nice to be able to express p, q, and m as separate functions of decision variables. GBM is not the most general possible model.

(3) In order to more easily understand the behavior of the model in the time domain, a closed-form solution is desirable. GBM does have a closed-form.

(4) The model should lead to plausible and statistically significant estimates of the parameters of the diffusion process, the market potential, and the decision variables. We shall show that in the instances where we have the required data, GBM does have this property.

(5) The model should be demonstrated to work with several data sets. We have data for only three product categories, but GBM works for all of these.

(6) We think that it is desirable for the model to maintain the essential property of the Bass Model. We have shown that GBM has this property.

(7) The model should explain deviations of the empirical observations from the predicted values of the smooth curve of the Bass Model. We shall provide comparisons of the predictive power of GBM vis-a-vis BM.

(8) It is desirable for the model to demonstrate how the adoption curve would be shifted when different sequences of the decision variables are involved. Use of equation (17) with GBM permits comparison of the effects of widely varying policies.

(9) The model should be flexible and encompass a great variety of shapes. GBM has this property.

(10) The model should permit the incorporation of lags in the decision variables in order to understand the carryover effects of previous decisions on current adoption. We

TABLE 1

Diffusion Model Studies that Include Price and/or Advertising

Ref #	Author Reference	Res* Goal	Model Proposed	Reduces to Bass Model only if	Closed-form Solution	Empirical Support	Optimization Results
1	Robinson & Lakhan, 1975	N	$Sales = (M - Y)(p + qY)e^{-k*Pr(t)}$; Y = Cumulative Sales; *Price affects adoption rate*	price is constant over time	Nil	Nil	Optimal price mimics diffusion (increases until peak sales, and decreases later)
2	Bass, 1980	T	$Sales(t) = f_{Bass}(t)*K* Pr^{-\eta}$; *Price affects market potential*	price is constant over time	Yes	6 products (poor fit on 2 of them)	Optimal price decreases monotonically
3	Bass & Bultez, 1982	N					
4	Dolan & Jeuland, 1981	N	Same as [1]	price is constant over time	Nil	Nil	Optimal price mimics diffusion; drops monotonically for large discount rate
5	Jeuland & Dolan, 1982	N					
6	Kalish, 1983	N	$Sales(t) = f_1(Y)f_2(Pr)$; Generalizes [1], [4]; *Price affects market potential*	price is constant over time	Nil	Nil	Optimal price mimics diffusion with any cost function & no discounting OR constant cost & discounting
7	Kalish, 1985	Both N, T	Adv. influences awareness; price is a trade-off for uncertainty; previous adopters create awareness and reduce uncertainty. *Adv., price and even previous adopters affect market potential*	Not clear, because it is a different model altogether	Nil	1 product is tested on the reduced version of the model	With no uncertainty, optimal price mimics diffusion. With zero discounting, adv. drops monotonically; with positive discounting adv. is U shaped
8	Horsky, 1990	Both N, T	Individual level utility model is built based on product benefits, income and price, and then aggregated. Contagion process is modelled at aggregate level; *Price affects market potential*	price is constant over time	Nil	4 products (data included repeat purchases possibly)	Optimal price mimics diffusion
9	Jeuland, 1981	Both N, T	Individual level model is built using uncertainty, information flow, and price effect. *Price affects both market potential and adoption rate (indirect)*	price is constant over time	Nil	Nil	As per the author, optimal price is supportive of [1]
10	Jain & Rao, 1990	1	$Sales(t) = (M - Y)(p + qF_{Bass}) Pr^{-\eta}$ where, F_{Bass} is the solution to the Bass Model. *As per authors, price can be taken as affecting either market potential or adoption rate*	price is constant over time	Nil	4 products	—**

	Author & year	Goal	Model	Assumption		Products	Comments
11	Kamakura & Balasubramanian, 1988	1	Sales = $(Mpr^{-\eta} - Y)\,Pr^{-\eta 2}(p + qY)$. A nested modelling approach to compare various nested models. *Results show price affects only adoption rate*	price is constant over time	Nil	6 products (Repeat purchases possibly included)	—
12	Horsky & Simon, 1983	Both N, T	$f/(1 - F) = p_1 + qF + p_2 \log$ (adv.) *Adv. affects adoption rate*	adv. is constant or exponential in F	Nil	1 product	Optimal adv. drops monotonically. Note price is held constant.
13	Teng & Thompson, 1983	N	$f/(1 - F) = p_1 + q_1F + (p_2 + q_2F)$adv. *Adv. affects adoption rate*	adv. is constant over time	Nil	Nil	If A = maximum adv. and a = minimum adv., then optimal adv. is a for all time, or A upto peak sales and a later, or a-A-a cycle.
14	Thompson & Teng, 1984	N	$f/(1 - F) = [(p_1 + q_1F) + (p_2 + q_2F)$adv.$]e^{-APt}$ *Both price and adv. affect adoption rate*	price and adv. are constant	Nil	Nil	Optimal price not evaluated; if price is constant, optimal adv. is same as [12]
15	Simon & Sebastian, 1987	1	Sales = $K + [p_1 + q_1F + q_2f/(adv)F] \times [M - Y]$ where f(adv.) at t follows Nerlove-Arrow-Model *Adv. affects adoption rate*	adv. is constant over time	Nil	1 product	—
16	Dockner & Jorgenson, 1988	N	General model: Sales = $f(Y, adv.)$ Specific model: $f/(1 - F) = (p_1 + q_1F') + (p_2 + q_2F)f_2(adv)$ Generalizes [11] *Adv affects adoption rate*	adv. is constant over time	Nil	Nil	For general model, with very low discounting, optimal adv. is U shaped. With specific model, $q_2 = 0$ ($p_2 = 0$) implies a monotonically decreasing (increasing) optimal adv. No result with +ve discounting and dropping cost.
17	Chatterjee & Eliashberg, 1990	N	Aggregated from micro-model developed by them in 1989. Sales = $f_1(Y)\,f_2(Y, adv.)$ *Adv. affects adoption rate*	adv. is constant over time	Nil	Nil	With low discounting, optimal adv. drops monotonically. With high discounting, it is inverted U shaped.
18	Jeuland, 1993	T	Sales = $e^{-(k1+k2Pr)F}$ $[p_1 + p_2 \text{ Log (adv.)} + qF]$ *Adv. affects adoption rate*	price and adv are constant	Nil	Nil	—
19	Jain, 1992	T	$f/(1 - F) = (p + qF_{Bass})e^{-k1*Pr+k2*adv}$ This is a current-effects model *Price and adv. affect adoption rate*	price and adv. are constant	In discrete time domain	3 products	—

* Res. Goal: Research goal of the paper
** : The purpose of the research is not optimization; hence no optimal policies
T: Main goal is to develop a *theory* to incorporate price and advertising in the diffusion framework.
N: Main goal is to derive optimal price and advertising, *assuming* a theoretical model for the purpose
I: Main goal is to *investigate* the role of price or advertising by empirically comparing many possible extensions to the Bass Model.

have shown that the use of equation (16) with GBM permits lags and maintains a closed-form solution.

Previous Studies

In Table 1 we summarize previous studies in which price or advertising has been incorporated in diffusion models. We have attempted to be exhaustive in our listing with respect to models that include price or advertising, although there are other studies that include other factors. Lilien, Rao, and Kalish (1981), for example, have included detailing (of pharmaceuticals) in a diffusion model and Jones and Ritz (1991) have included distribution. However, we feel for our purpose that Table 1 characterizes the appropriate set of models that include decision variables. Although there have been several studies involving decision variables, when the studies are broken out into categories, the set of models with desirable characteristics becomes quite small. Many of the studies have had normative purposes and there is no empirical analysis of data in connection with the model employed. In Table 2 we provide a taxonomy of the studies listed in Table 1.

As indicated in Tables 1 and 2 there is only one previous study in which a closed-form solution is involved for a continuous model in the time domain and there is no study in which a model has been used that has the property that the Bass Model is a special case if the decision variables are not constant.

5. Parameter Estimation

Initially, we shall estimate GBM with the assumption of fixed m. We have estimated the parameters of GBM using the method initially proposed by Srinivasan and Mason (1986) and used by Jain and Rao (1990) and by Jain (1992). SYSNLIN (SAS (1984)) has been employed in the estimation. We have data for price and advertising for three product categories as well as adoption data. These are: room air conditioners, color television and clothes dryers.

Variable Measures

Because we have measured price in $x(T)$ for GBM as a percentage change and because we have found that a deflator tends to induce irregular noise in our measure, we have chosen to use actual prices in estimating GBM. Also, behaviorally, actual percentage changes in price, as opposed to deflated percentage changes, may be a more realistic representation of consumer perception.

Simon (1982) has suggested that change in advertising be measured as $\max \{0, \Delta A(T)\}$. The argument is that changes in advertising should have impact only if they are positive and thus the effects of advertising change are asymmetric. For GBM we have taken Simon's operationalization as the measure of the change in calculating percentage change. Thus, we have $[\max \{0, \Delta A(T)\}]/A(T-1)$ as the measure of percentage change and for the continuous case we have for cumulative marketing effort,

$$X(T) = T + \text{Ln} \left[\Pr(T)/\Pr(0)\right]\beta_1 + \text{Ln} \left[\hat{A}(T)/A(0)\right]\beta_2,$$

where $\hat{A}(T)$ is the imputed value from Simon's operationalization. That is, it is the actual value if there has been a positive change in advertising and otherwise it is the last value for which there was a positive change. Because, with our data, advertising is generally increasing, $\hat{A}$ is ordinarily the actual value.

Because our measure of $X(T)$ depends on the initial values of price and advertising, $A(0)$ and $\Pr(0)$, we have been required to impute these values from $A(1)$ and $\Pr(1)$. Although we have experimented with different initial values and have found that the estimates are not very sensitive to them, we present in Table 3 estimates generally derived from imputed values: $A(0) = A(1)$ and $\Pr(0) = \Pr(1)$.

TABLE 2

A Summary of Marketing Mix Models Incorporating Price and Advertising

	Purpose of the Model	Theoretical				
	Model Characteristics	With Empirical Support	With No Empirical Support	Normative	Investigative	Total 19 Models
Models with closed-form solution		[2] [19]	Nil	Nil	Nil	2 out of 19
Decision	Models with price alone	[2] [8]	[9]	[1] [3] [4] [5] [6] [8] [9]	[10] [11]	10 out of 19
Variables	Models with adv. alone	~[12]	Nil	[12] [13] [16] [17]	~[15]	5 out of 19
	Models with both price & adv.	Nil	~[7]* [18]	[14]**	Nil	3 out of 19
Models that reduce to Bass Model when the decision variables vary over time		Nil	Nil	Nil	Nil	0 out of 19***

[·] Refers to the Reference # in Table 1.
[·]* Only price is considered in the empirical study.
[·]** Optimal price trajectory is not evaluated.
~[·] Only one product category is tested.
[·]*** Horsky & Simon model [12] may reduce to Bass Model under the unlikely condition that advertising is exponential in F

Optimization Results: A Summary
   Optimal Price mimics diffusion: [1] [4] [5] [6] [7] [8] [9]
   Optimal Price decreases monotonically: [3]
   Optimal Advertising decreases monotonically: [7] [12] [16] [17]
   Optimal Advertising increases monotonically: [16]
   Optimal Advertising is U- or inverted U-shaped: [7] [13] [14] [16] [17]

Parameter Estimates and Model Comparisons

The parameter estimates and asymptotic standard errors are shown in Table 3 for the Bass Model and the Generalized Bass Model.

It is clear that the fits of each of the models to the data for each product are quite good. The parameter estimates of β_1 and β_2 for GBM are significant (by traditional standards) and have the correct signs. It is worth noting that the estimates of p and q and m for BM are reasonably close to the estimates of these parameters for GBM. As we have seen, BM is a special case of GBM when the variables change by a constant percentage in each period and thus when the percentage change varies only slightly, the parameter estimates for BM would be expected to be close to those of GBM. In this, we have an explanation of why BM fits the data well without including decision variables in that it produces a smooth curve that deviates only slightly from the results produced by the true model.

Predictive Comparisons

We have seen in Table 3 that the fit of GBM, as expected, is superior to the fit of BM. However, there remains the issue of the predictive qualities of the two models. Following Lattin and Roberts (1989) in comparing the predictive qualities of the two models, we use step-ahead forecasting. We first fit each model for n periods and then forecast adoption for the $n + 1$th period. We then re-estimate the model for $n + 1$ periods and forecast

TABLE 3

Parameter Estimates and Asymptotic Standard Errors for BM and GBM

	R^2	SSE $*10^6$	p	q	$m *10^3$	β_1	β_2
			Room Air Conditioners				
BM	.9394	309613.63	.0093	.3798	18320.78	—	—
			(.0021)	(.0417)	(1122)		
GBM	.9746	129735.02	.00516	.3309	19502.49	−1.3691	.61859
		[58.10%]	(.0015)	(.0248)	(1129)	(.6468)	(.2583)
			Color TV				
BM	.9846	830593.13	.0049	.6440	39524.27	—	—
			(.0011)	(.0447)	(1404)		
GBM	.9935	351188.22	.00416	.5965	39753.69	−4.81295	n.s.
		[57.72%]	(.0007)	(.0329)	(996)	(1.866)	
			Clothes Dryer				
BM	.9278	187783.21	.0134	.3317	16239.1	—	—
			(.0023)	(.0373)	(1011.1)		
GBM	.9676	84309.71	.0118	.2959	16755.92	−.8117	.6583
		[55.10%]	(.003)	(.0275)	(910)	(.567)	(.267)

(...): Asymptotic Standard Errors
SSE: Sum of the Squared Errors
[...]: Percentage SSE reduction in using GBM over BM

adoption for the n + 2th period and so on. We have taken n as the last period before (or around) the peak in the adoption rate. In Table 4 we report the mean squared error of the forecasts of the two models. In each instance the percentage reduction in forecast error for GBM is substantial. The reason for this, we think, as shown in Figures 1 and 2, is that the decision variables will forecast the irregular deviations of sales from the smooth curve of the BM forecasts.

Empirical Estimation of Some Previous Models with Decision Variables

The first study in which the Bass Model was modified to include decision variables was conducted by Robinson and Lakhani (1975). We shall refer to this as R-L. The purpose of their study was to study the normative implications of the effect of price on adoption. Other studies have also utilized the R-L model for normative purposes, notably

TABLE 4

One-step Ahead Forecasting Results

|	Mean Squared Error $*10^6$			
Product	With Bass Model (BM)	With Generalized Bass Model (GBM)	Percentage Reduction in Error by Using GBM Over BM	Average Over
Room Air Conditioners	149312.82	107991.12	27.68%	4 forecast-periods
Color TV	1745836.50	927357.14	46.88%	2 forecast-periods
	1675176.90	1065349.60	36.40%	3 forecast-periods
Clothes Dryers	122536.88	50554.673	58.74%	5 forecast-periods
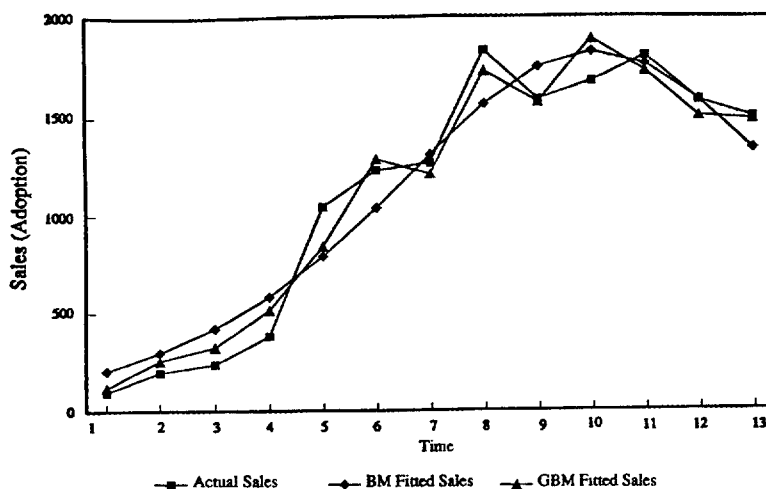
FIGURE 1. Comparison of Actual Sales and Fitted Sales for Room Air Conditioners.

Dolan and Jeuland (1982), Thompson and Teng (1984), and Kalish (1983). Perhaps surprisingly, the R-L model has never been estimated with empirical data. We shall do so here.

The model employed in R-L and other studies is:

$$S(T) = [pm + (q - p)Y - (q/m)Y^2]e^{-k\Pr(T)}, \qquad (27)$$

where $\Pr(T)$ is the price at time T. The expression in the brackets is, of course, the cumulative adoption domain version of the Bass Model indicated in equation (5). The model has the property that a reduced price today will increase sales today and will also enhance tomorrow's adoption through its effect on cumulative adoption. It is not possible to derive a closed-form expression for $S(T)$ unless a specific function of time is specified for $\Pr(T)$ and it is important to note that although the R-L model employs the Bass Model, it does not reduce to the Bass Model unless $\Pr(T)$ is constant. We have tried, without success, to obtain plausible and significant estimates of the R-L model for the three products for which we have data. We have estimated equation (27) using the nonlinear least squares SAS (1984) program entitled SYSNLIN.
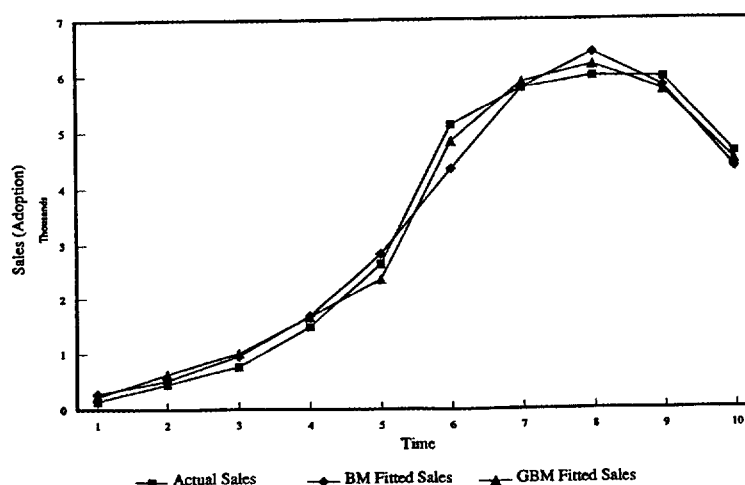

FIGURE 2. Comparison of Actual Sales and Fitted Sales for Color TV.

Table 5 shows the estimation results for the three products. For color television we could not achieve convergence, and for the other two products only the parameter m was significant. The R^2's for these two products are less than those reported for the Bass Model in Table 3 because the estimate of k, although not significant, is not 0. The problem, we think, lies in the inclusion of price exponentially. The model is redundant in the sense that it has a very loosely connected parameter structure. For a discussion of redundancy in the estimation of nonlinear models see Reich (1981) and Seber and Wild (1989). The problem is that in the form specified by the R-L model, the estimation becomes "confused" and it is impossible to distinguish between the parameters unless price is a constant or the coefficient near 0, in which case the model reduces to approximately the Bass Model. To examine this, we set k equal to various values and found that we could get plausible and significant estimates of p, q, and m only when k was close to 0. When $k = 0$, of course, we have the Bass Model.

For our data sets, then, it is not possible to obtain satisfactory estimates of the parameters of the R-L model. Moreover, we think that, in general, the R-L model will present severe problems of estimation that possibly have nothing to do with variable collinearity. In combining a model that is in the cumulative adoption domain with exponential effects there is a confounding of time domain influences through $\Pr(T)$ with influences that operate through the cumulative adoption domain.[1] The result is a model that is either dominated by time domain effects or by cumulative adoption effects. If the diffusion data have been generated by the R-L model, except under very restrictive conditions, the parameters will be unrecoverable empirically. This conclusion clearly has implications for the value of normative results derived from study of the R-L model.

In addition to the R-L model for pricing, we estimated models with advertising. We have examined the model initially estimated for a banking product by Horsky and Simon (1983) and generalized by Dockner and Jorgensen (1988) as well as studied normatively by Teng and Thompson (1983). The model is:

$$S(T) = (m - Y)[p_1 + p_2 f(\mathrm{ADV})] + Y(m - Y)[q_1 + q_2 f(\mathrm{ADV})]$$

$$= (m - Y)[p_1 + q_1 Y + p_2 f(\mathrm{ADV}) + q_2 Y f(\mathrm{ADV})]. \tag{28}$$

In the model indicated by equation (28) ADV represents advertising expenditures. Dockner and Jorgensen, and Horsky and Simon took $q_2 = 0$ and $f(\mathrm{ADV}) = \log(\mathrm{ADV})$, while Teng and Thompson took $f(\mathrm{ADV}) = \mathrm{ADV}$.

We estimated the model with various specifications and found that the estimates were either inconsistent (the value of one parameter estimate implied $p_2 < 0$) or the nonsignificance of some parameter estimates left other parameters under-identified. Hence we have concluded that this model, like the R-L model, is not generally viable for empirical estimation of the effects of decision variables. We have also estimated a hybrid model that combines elements of the Horsky-Simon model with the Robinson-Lakhani model (Horsky-Simon advertising operationalization and R-L price operationalization) and find that the estimates are, in general, not significant.

Estimates of More General Models

The Generalized Bass Model may be used in conjunction with a fixed m or an m that is tied to decision variables. Our earlier estimates were based on an assumption of fixed m, but here we shall examine the possibility that m is dynamic and tied to prices. In

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[1] Jain (1992) has developed a "current effects" model that combines time effects with price and advertising effects that enter exponentially and that does yield empirical parameter estimates. However, unlike R-L, the components of this model are entirely in the time domain.

TABLE 5

Statistical Estimates of Parameters of the R-L Model

Parameters	Standard Errors in Parentheses		
	Room Air Conditioners	Color TV	Clothes Dryers
R^2	0.8953	No Convergence	0.8962
p	0.0117 (0.0219)		0.0289 (0.0796)
q	0.3275 (0.4152)		0.4948 (1.231)
m	17,275.5 (1861.9)		14,963.2 (2474)
k	-0.071 (0.3738)		0.1491 (1.05)

addition, we shall estimate models in which there is a differential effect of marketing effort on p and q. We utilize the following functions:

Dynamic m: $m(T) = m[\Pr(T)]^{-\eta}$.
Differential Effects on p and q:

$$x_1(T) = 1 + \beta_{11}[(d\Pr(T)/dT)/\Pr(T)] + \beta_{12}[(d\mathrm{ADV}(T)/dT)/\mathrm{ADV}(T)],$$

$$x_2(T) = 1 + \beta_{21}[(d\Pr(T)/dT)/\Pr(T)] + \beta_{22}[(d\mathrm{ADV}(T)/dT)/\mathrm{ADV}(T)],$$

where $x_1(T)$ represents the influence of decision variables on p and where $x_2(T)$ represents the influence of decision variables on q. We have estimated seven different models:
Differential Effects with Fixed m:

$$S = m[1 - Y/m][px_1(T) + qx_2(T)Y/m] \qquad \text{Model 1}$$

Differential Effects with Dynamic m (One Way):

$$S = m[\Pr(T)]^{-\eta}[1 - Y/m][px_1(T) + qx_2(T)Y/m] \qquad \text{Model 2}$$

Differential Effects with Dynamic m (Second Way):

$$S = [m[\Pr(T)]^{-\eta} - Y][px_1(T) + qx_2(T)Y/(m[\Pr T)]^{-\eta})] \qquad \text{Model 3}$$

GBM with Dynamic m (One Way):

$$S = m[\Pr(T)]^{-\eta}[1 - Y/m][p + qY/m]x_1(T) \qquad \text{Model 4}$$

GBM with Dynamic m (Second Way):

$$S = [m[\Pr(T)]^{-\eta} - Y][p + qY/(m[\Pr(T)]^{-\eta})]x_1(T) \qquad \text{Model 5}$$

BM with Dynamic m (One Way):

$$S = m[\Pr(T)]^{-\eta}[1 - Y/m][p + qY/m] \qquad \text{Model 6}$$

BM with Dynamic m (Second Way):

$$S = [m[\Pr(T)]^{-\eta} - Y][p + qY/(m[\Pr(T)]^{-\eta})]. \qquad \text{Model 7}$$

Estimates of the parameters of the seven models for the three products are shown in Table 6. In every instance with the differential effects models the price and advertising coefficients are not significant. We conclude, then, that to the extent that differential effects exist, it is not possible to distinguish these effects in estimation while, as previously shown, the GBM model yields plausible and significant estimates of the effects of decision variables.

In examining the possibility of dynamic market potential we find that in almost every instance the estimate of η is not significant or has the wrong sign. GBM, as previously indicated, may be used in conjunction with either a fixed market potential or one that

is made to be dynamic and depend on decision variables. In practice, when using the model to study the effects of policies on adoption for a new product when no data are available for estimation, we think that it is probably best to treat m as fixed because guesses about a fixed m are probably intuitively more feasible than guesses about the influence of decision variables on m.

6. Policy Changes and Guessing Parameters, Price Elasticities, Rational Expectations, Level Effects, and Carryover Effects

Policy Changes and Guessing Parameters

From a policy perspective, a key issue in the use of diffusion models is the effect of different policies on the location of the diffusion curve and timing of the peak. We observe the diffusion pattern associated with observed historical values of the decision variables, but how would the curve have been shifted by very different policies? GBM provides a way to evaluate widely different policies. The mapping function given by equation (17) may be used to evaluate policy changes relative to baseline policies.

From an application viewpoint probably the most important use of diffusion models for planning purposes is before the product has been introduced and there are no data available with which to estimate parameters. Lawrence and Lawton (1981) have developed a method for guessing the parameters of the Bass Model when the technology is still in the planning stage and no data are available. This method has apparently been used successfully in a number of instances (see Lawrence and Lawton in Wind, Mahajan, and Cardozo, 1981, pp. 532–535 for examples of applications). We shall outline here a simple method for guessing parameters without data.

In real applications of any model with decision variables three pieces of information will be required: (1) values of the diffusion parameters, (2) an estimate of the market potential, and (3) estimates of the response coefficients for the decision variables. As discussed previously, and as confirmed by the results in Table 3, for GBM the estimates of the diffusion parameters, p and q, are close to those obtained from BM. This result increases the feasibility of applying GBM when no data are available. Hence, the Lawrence and Lawton method for BM may be applied as well to GBM. That method is based on the empirical observation that estimates of $p + q$ usually lie between 0.3 and 0.7, with a mode of perhaps 0.4. Managers can guess the "contagion rate" of the new product to get estimates of $p + q$. If $p + q$ is known then $p + q$ and "First Year Sales" will determine q/p. Thus managers can apply intuition to guess "First Year Sales." The market potential, m, is a number about which managers ordinarily have strong intuitive feel. Only β_1 and β_2 remain to be guessed. These may be determined by having managers guess price and advertising elasticities along with specified and planned relative rates of change in decision variables. If $k(\mathrm{Pr})$ is the relative rate of change in price (i.e., $\Delta \mathrm{Pr}/\mathrm{Pr}$) and $k(\mathrm{ADV})$ is the relative rate of change in advertising then it can be shown that:

$$\beta_1 = \eta(\mathrm{Pr})/(1 - k(\mathrm{Pr})\eta(\mathrm{Pr}) - k(\mathrm{ADV})\eta(\mathrm{ADV})), \qquad (29)$$

$$\beta_2 = \eta(\mathrm{ADV})/(1 - k(\mathrm{Pr})\eta(\mathrm{Pr}) - k(\mathrm{ADV})\eta(\mathrm{ADV})), \qquad (30)$$

where $\eta(\mathrm{Pr})$ and $\eta(\mathrm{ADV})$ are price and advertising elasticities and $k(\mathrm{Pr}) < 0$ and $k(\mathrm{ADV}) > 0$.

Parameter guesses for GBM are both simple and feasible, and we think that the model will be useful for planning purposes for new products.

Price Elasticity

In developing an analytical expression for price elasticity, it is convenient to use the approximation: $f(T) = F(T) - F(T - 1)$. Srinivasan and Mason (1986) showed that for the Bass Model this approximation provides a good fit for observed data. Hence,

TABLE 6

Parameter Estimates of Seven Generalized Diffusion Models

Model	p	q	m	B11	B12	B21	B22	η	R^2
				Room Air Conditioners					
1	.0083*	.3515	18475	−16*	~−.56*	−.24*	.811	—	.9693
2	.0083#	.3515#	18475#	−16*	~−.56*	−.24*	.811	.00003*	.9693
3	.0083#	.3515#	18475#	−16*	~−.56*	−.24*	.811	.000002*	.9693
4	.0052+	.3309+	19502+	−2 15	.8896	—	—	.003*	.9238
5	.0052+	.3309+	19502+	−1.72	.810	—	—	~−.005*	.9244
6	.0093+	.3798+	16697	—	—	—	—	~−.025	.8847
7	.0093+	.3798+	756*	—	—	—	—	~−.57*	.8639
				Color Television					
1	.0191	.5502	34905	~88	2.08*	−36.2*	~−.45	—	.9901
2	.0191#	.5502#	34905#	~88	2.08*	−36.2*	~−.45	.000002	.9901
3	.0191#	.5502#	34905#	~88	2.08*	−36.2*	~−.45	.000018	.9901
4	.0042+	.5965+	39754+	−13.9	.12*	—	—	.0174	.9607
5	.0042+	.5965+	39754+	−11.1	.04*	—	—	~−.01*	.9209
6	.0049+	.6440+	34906	—	—	—	—	~−.03	.9302
7	.0049+	.6440+		No Convergence					
				Clothes Dryers					
1	.0169	.2959	15716	~2.84*	~−.26*	−2*	1.07*	—	.9534
2	.0169#	.2959#	15716#	~2.84*	~−.26*	−2*	1.07*	.00014*	.9534
3	.0169#	.2959#	15716#	~2.84*	~−.26*	−2*	1.07*	.000007*	.9534
4	.0118+	.2959+	16756+	−1.4*	.873	—	—	.0053*	.9337
5	.0118+	.2959+	16756+	−.825*	.76	—	—	~−.0023*	.9313
6	.0134+	.3317+	14953	—	—	—	—	~−.022	.8877
7	.0134+	.3317+	13084*	—	—	—	—	~−.04*	.8659

*: Estimates with very high standard errors.
~: Estimates with wrong sign.
#: Exogenously fixed values. Refer to Model 1 estimates.
+: Exogenously fixed values. Refer to Table 3.
—: Parameter not in the model.

$$\frac{df(T)}{d\,\Pr(T)} = \frac{dF(T)}{d\,\Pr(T)}, \quad \text{since } \frac{dF(T-1)}{d\,\Pr(T)} = 0,$$

$$\eta = \frac{df(T)}{d\,\Pr(T)} \frac{\Pr(T)}{f(T)} = \frac{dF(T)}{d\,\Pr(T)} \frac{\Pr(T)}{f(T)}. \tag{31}$$

For GBM,

$$F_T = \frac{1 - e^{-aX(T)}}{1 + be^{-aX(T)}} \quad \text{and} \quad f_T = \frac{a^2}{p} \frac{e^{-aX(T)}}{(1 + be^{-aX(T)})^2} x(T) \tag{32}$$

where $a = (p + q)$ and $b = q/p$. Therefore,

$$\frac{dF(T)}{d\,\Pr(T)} = \frac{dF(T)}{dX(T)} \frac{dX(T)}{d\,\Pr(T)} = \frac{dF(T)}{dX(T)} \beta_1/\Pr(T) = \frac{a^2}{p} \frac{e^{-aX(T)}}{(1 + be^{-aX(T)})^2} \frac{\beta_1}{\Pr(T)}. \tag{33}$$

Substituting (32) and (33) for $f(T)$ and $dF(T)/d\,\Pr(T)$ in (31), we get $\eta_T = \beta_1/x(T)$, where

$$x(T) = 1 + \beta_1 \left(\frac{\Pr(T) - \Pr(T-1)}{\Pr(T-1)} \right) + \left(\beta_2 \max \left\{ 0, \frac{A(T) - A(T-1)}{A(T-1)} \right\} \right).$$

Under conditions, then, when percentage changes in the variables are constant, price elasticity will also be constant. We have averaged the price elasticities over all time periods for the three products we have studied. The results are: Room Air Conditioners: -1.129, Color TV: -4.501, Clothes Dryers: -0.668. Elasticity values, as we have just shown, may be used for policy purposes in conjunction with the model.

Rational Expectations

We previously referred to the rational expectations interpretation of the GBM model. We elaborate briefly here on this interpretation. Shown in Figure 3 are price patterns with different (constant) rates of decline in price. For each of these patterns there is a corresponding GBM diffusion curve that is observationally equivalent to the BM diffusion curve. With regularity in the behavior of the decision variables, consumers will learn this and adjust their adoption timing accordingly. Figure 4 shows how the diffusion curves are shifted to the left as the rate of price decline increases.

Level Effects

Because our formulation of $x(T)$ is in terms of rate of change in the decision variables, it is not obvious what the effects of different levels of the decision variables will be. To analyze the effects of level differences equation (17) must be used. Differentiation of (17) with respect to Pr_b indicates that:

$$dx/d\,\mathrm{Pr}_b(T) = -\mathrm{Pr}(T)\beta_1/[\mathrm{Pr}_b(T)]^2. \tag{34}$$

Because β_1 is less than 0, equation (34) will be positive and the higher the baseline price pattern, the higher x and the more the curve will be shifted to the left by the new (lower) price pattern. Also,

$$dx/d\,\mathrm{Pr}(T) = \beta_1/\mathrm{Pr}_b + \beta_1/\mathrm{Pr}(T-1). \tag{35}$$

The higher the new price pattern, $\mathrm{Pr}(T)$, the less the curve will be shifted to the left.

Carryover Effects of Advertising

Because of saturation and nonlinearity, analysis of the carryover effects of advertising is not straightforward. An increase in advertising at $T - 1$ will have three effects: (1) it will increase adoption at $T - 1$ and, because of the "carrythrough" effects of GBM, this will tend to increase adoption at T; (2) it will increase the base level from which changes in advertising at T will have an influence and thus accelerate diminishing returns at T; and (3) it will increase market saturation at T, thereby making increased adoption more difficult. Whether the net effect of an increase in advertising at $T - 1$ will be positive or negative at T is a complex issue but clearly will depend upon the net impact of the three forces. Differentiation of the conditional probability of adoption at time T indicated in equation (22) with respect to $\mathrm{ADV}(T - 1)$ provides a basis for evaluating the increase (or decrease) in the conditional adoption probability at time T as $\mathrm{ADV}(T-1)$ increases. We find that the necessary condition for there to be positive carryover effects of advertising is:

$$xq(1 - F)dX/d\mathrm{ADV}(T - 1) > -dx/d\mathrm{ADV}(T - 1). \tag{36}$$

$dX/d\mathrm{ADV}(T - 1)$ will be positive and $dx/d\mathrm{ADV}(T - 1)$ will be negative. The LHS of (36) will decrease with F as saturation effects tend to offset the increases in advertising. The result is that carryover effects of advertising tend to be positive early in the adoption process, but can turn negative as saturation increases. Of course, the exact carryover effect depends, as indicated in (36), on the rate of increase in advertising.
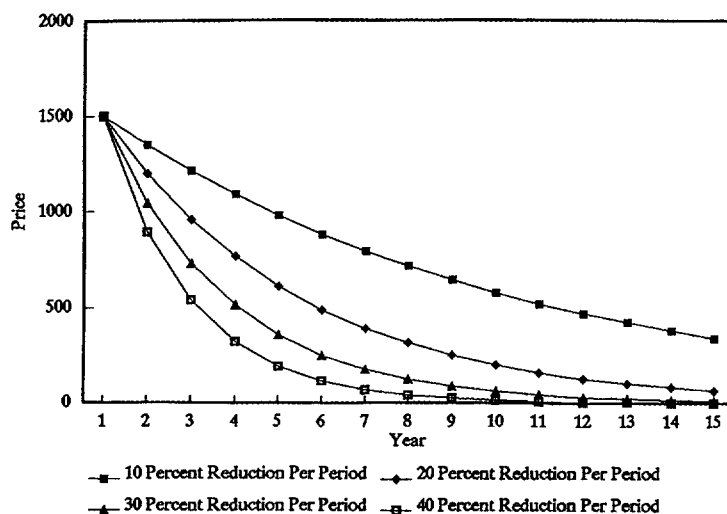
FIGURE 3. Price Paths for Different Constant Percentage Price Reductions: Price Changes by Constant Percent.

7. Conclusions

When $x(T)$ is constant the Generalized Bass Model will be observationally equivalent to the Bass Model and thus when the Bass Model provides a good fit to adoption data, GBM will provide the same fit if $x(T)$ is constant. This statement provides an explanation of why the Bass Model fits without decision variables. If percentage changes in period-to-period values of the decision variables are approximately, but not exactly, constant, GBM will provide approximately the same fit as BM and when the coefficients on the decision variables are statistically significant GBM will provide explanations for the deviations of actual data from the smooth curve of BM and an improved fit. GBM is a continuous model with the desirable properties of having a closed-form solution in the time domain and maintaining the "carry-through" character of BM.

The Generalized Bass Model is both flexible and simple. We think that it easily can be used for planning purposes. Although we think that the functional form we have used
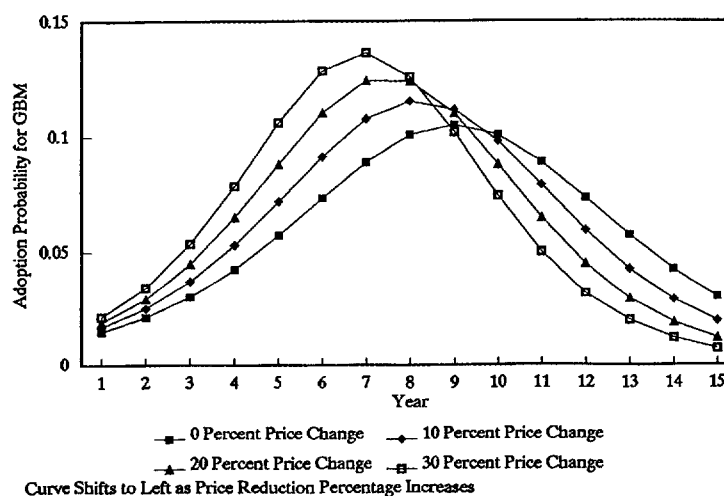
FIGURE 4. Adoption Rates for Different Price Paths: Constant Percent Reductions.

for $x(T)$ is a good one, further study of appropriate functional forms to be used in conjunction with the model is warranted. Normative exploration of GBM, we think, will lead to interesting and useful results. In order to permit and encourage further examination of the influence of decision variables on adoption we are publishing the data for the three products we have studied in Table 7.[2]

[2] This paper was received June 29, 1992, and has been with the authors 7 months for 2 revisions. Processed by James M. Lattin, Area Editor.

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TABLE 7
Sales, Price, and Advertising Data

Yr	1	2	3	4	5	6	7	8	9	10	11	12	13
					Room Air Conditioners								
Ss	96	195	238	380	1045	1230	1267	1828	1586	1673	1800	1580	1500
Pr	410	370	365	388	335	341	320	293	310	279	269	275	259
Ad	0.0	0.615	1.198	3.196	5.340	14.372	9.391	13.610	16.785	9.238	5.863	3.923	1.493
Sd	1.0	1.615	2.198	4.196	6.340	15.372	15.372	19.591	22.766	22.766	22.766	22.766	22.766
					Color TV								
Ss	147	438	747	1463	2646	5118	5777	5982	5962	4631			
Pr	610	566	555	551	560	535	525	521	515	505			
Ad	1.0	3.7767	5.2115	6.0171	8.9443	8.4297	14.4217	14.7369	19.7050	25.2183			
Sd	1.0	3.7767	5.2115	6.0171	8.9443	8.9443	14.9363	15.2515	20.2196	25.7329			
					Clothes Dryers								
Ss	106	319	492	635	737	890	1397	1523	1294	1240	1425	1260	1236
Pr	213	232	245	236	219	211	210	217	215	215	195	189	185
Ad	0.0	0.844	3.385	2.772	3.852	2.137	4.092	6.752	7.483	6.065	8.657	5.358	1.361
Sd	1.0	1.844	4.385	4.385	5.465	5.465	7.420	10.080	10.811	10.811	13.403	13.403	13.403

Ss: Unit sales in thousands
Pr: Average price in $
Ad: Advertising expenditures in million $
Sd. Modified Advertising expenditures following Simon's operationalization

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