The Temporal Network Autocorrelation Model*

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Abstract

We introduce the temporal network autocorrelation model (TNAM), a statistical technique designed to model the behavior or attributes of actors embedded in a network. This highly flexible model is ideally suited for incorporating not only network and spatial dependencies into the model specification, but also doing those things over time and can accommodate virtually any model structure from generalized linear models to mixed effects models. The ability to account for network effects while modeling actor behavior is essential because the inclusion of such effects may affect other estimates produced by the model, thus risking faulty inference if the data generating process is not adequately modeled. Depending on the application, such effects may also be of primary interest to the researcher. The TNAM, and its companion software, offers a powerful and flexible tool for modeling the interplay between actors and the networks in which they are embedded while also being intuitive and easy-to-use.

1 Introduction

Scholars with network data tend to be interested either in how connections in the network form, or how being imbedded in the network affects the behavior or attributes

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of the actors. Much work has been dedicated to the prior in recent years (see e.g. large literatures on the exponential random graph model (Cranmer and Desmarais 2011; Cranmer et al. 2012a,b) and latent space network model (Ward and Hoff 2007; Ward et al. 2011; Dorff and Ward 2013)), while comparatively little has been dedicated to the latter. The current work aims to provide researchers with an expanded toolkit for modeling the behaviors and attributes of actors imbedded in a network.

The nodes embedded in a network, typically some sort of actor in a social science context, possess any variety of attributes and/or exhibit any number of behaviors. For example, legislators embedded in a collaboration network possess a variety of attributes, such as their race, gender, party, education, home district, and so forth. They also exhibit behaviors that political scientists might wish to model, such as their voting behavior and their tendencies to introduce new legislation. As a second example, and one we will examine in detail in the application below, consider states embedded in international networks. These states possess attributes, such as their GDPs, their populations, their military strength and so forth. They also exhibit behaviors analysts might wish to model, such as their tendencies to engage in conflict, their internal stability, and their regime types. Throughout the discussion below, we will use the terms actor attributes and behaviors interchangeably, while, from a statistical perspective, they are both simply variables measured at the unit level and can serve as either the outcome variable or predictors depending on the substance of the problem.

Modeling the actor behavior in the context of a network is complicated by the fact that the statistical techniques generally used to model actor-level phenomena are designed with an intrinsic assumption of independence. That is, a typical regression analysis assumes that actors do not influence, interact, or interfere with one another. This assumption is generally reasonable in the context for which it was developed and largely employed: survey research. However, this assumption is highly problematic in the network context where actor-to-actor influence and the diffusion of traits across the network is often what is primarily interesting to the researcher.

Most applications in political science that have faced this challenge have a either assumed the network away, or modeled it in an overly simplistic fashion. For example, a study that employs a simple regression to model political party choice based on several actor-level attributes – such as income, education, and race – assumes that one's family and friends have no influence on party choice, or at least that their influence is completely orthogonal to any of the actor attributes used to predict the outcome. Some studies have improved upon this rather unsatisfactory state of affairs by computing simple actor-level network measures (e.g. centrality) and including those as predictors in the regression. While this is a step in the right direction, it leaves much to be desired as such an approach does not provide a systematic or holistic model of the network processes at play. Recently, political science has seen the introduction of much more sophisticated models for actors in networks, most notably the M-STAR model (Hays et al. 2010; Franzese et al. 2012) that a makes use of a spatial lags approach to modeling network dependencies that dates back to the early 1980's in other fields.

Here, we introduce a technique to flexibly model how the interdependencies between actors embedded in a network affect their attributes and behaviors. Our model builds upon existing network autocorrelation models but adds distinct value to the progression of these models by incorporating a wide range of endogenous network features and fine-grained temporal dependencies into the modeling process, both essential for modeling diffusion processes on networks. As such, we call our model the temporal network autocorrelation model (TNAM). Below, we trace the evolution of network autocorrelation models and their respective shortcomings leading up to the TNAM. We then introduce the TNAM itself and consider its specification in detail. Finally, we illustrate the TNAM's abilities the application to the study of the spread of democracy in international politics.

2 The Temporal Network Autocorrelation Model (TNAM)

2.1 Background: Space, in Geography and Networks

The network autocorrelation model finds its roots in spatial statistics. Spatial statistical models aim to account for the fact that two actors who are physically proximate to one another are likely to effect one another. The effect of one proximate actor on the other is clearly a violation of the independence assumptions made by most statistical models and thus require a correction in the structure of the model. This spatial dependence between actors is an old problem in statistics and dates back at least to Galton. In 1889, Sir Francis Galton famously criticized a paper presented by Sir Edward Tyler investigating cross-national differences in the social practices of kinship and marriage. Galton argued that the conclusions of the paper were potentially misleading, as the author had failed to account for the possibility of diffusion of social practices across national borders. Particularly within anthropology, the statistical issues arising from the interdependence of observations became known as "Galton's problem." Galton thus was not only pivotal in introducing concepts such as correlation and regression to the mean into social research, he also identified social interaction as a significant barrier to using such statistical methods to draw inferences about social behavior. Galton's original critique was based on a verbal argument, but statisticians later provided more rigorous mathematical treatments and proposed solutions for his conjecture (e.g., Student (Gosset); Moran (1950)).

One of the most powerful solutions to this problem of spatial dependencies is the spatial lag model (often called the spatial autoregressive model or SAR) first introduced by Cliff and Ord (1969). To understand the spatial lag model, consider a matrix of spatial weights, W. W is a square matrix with as many rows and columns as actors in the dataset. The value of w_{ij} captures actor i's proximity to actor j. A regression model using these weights can be specified by including the matrix in the linear regression function and is called an autocorrelation model:

$$Y = \alpha + \rho W Y + X\beta + \epsilon, \tag{1}$$

where Y is the $N \times 1$ outcome vector, X is an $N \times K$ matrix of exogenous covariates, W is a square $N \times N$ spatial weight matrix, ϵ is the error term, and ρ and β are estimates. In other words, the dependencies are captured by matrix multiplication of the spatial weights matrix and the outcome vector. Actor i's outcome is determined by his or her covariate values and the outcome values of his or her connections, where connection is defined by outgoing relations (e.g. i names j as a friend) and where connections can be binary (they exist or do not) or weighted (i.e., closer/stronger friends' outcome values are valued higher in i's consideration than weaker friends' outcome values). For individual actors in Y, equation (1) can be rewritten as

$$y_i = \alpha + \rho \sum_{j=1}^{n} w_{ij} y_j + \sum_{k=1}^{K} \beta_k x_{ik} + \epsilon, \tag{2}$$

providing an actor-level interpretation of the model.

One of the major advantages of the spatial lags approach is that the form of the spatial dependency is highly flexible and somewhat generic. That is to say, the specific mechanism by which proximate actors affect one another need not be known and spatial dependencies that are not of interest to the researcher need not be modeled elaborately in order to provide unbiased hypothesis tests for other effects of interest. However, if the specific mechanism is hypothesized, the weights matrix W can often be operationalized so as to capture that mechanism and thus test specific hypotheses.

Early in its development, scholars realized that the spatial lag model could be used to model dependencies not only in geographic space, but in networks as well. The spatial autocorrelation model was thus applied to culture and anthropology and framed as a *network* autocorrelation problem by White et al. (1981), Dow et al. (1982), and Dow et al. (1984).

Such spatial models can be used rather easily to model network-based dependencies as well. Consider a network, which can be weighted (ties have numeric values) or unweighted (ties are binary on or off) as well as directed (a tie from i to j does not imply a tie from j to i) or undirected (all ties are symmetric by definition), to be represented by the adjacency matrix A.[1] In such an adjacency matrix, the connection of vertex i to vertex j is represented by a_{ij}. One can turn the simple spatial model into a model that accounts for some network dependencies simply by setting $W = A$. Then, "proximity" in the spatial sense refers directly to "neighbors" in whatever network structure is under study. That is, the spatial estimate would capture the effect of the outcomes for the nodes to which focal node i is directly connected on i's outcome.

These spatial models are also quite flexible and can be generalized to accommodate other distributional types of the outcome vector by plugging the linear predictor in equation (1) into a GLM with link function $g(\cdot)$. That is, $g(\mu_i) = \eta_i$ with commonly used probability densities being Gaussian, binomial, and Poisson among others and $g()$ being any link function appropriate to transform μ_i to the space of all real numbers. This allows us to model vertex attributes of any distributional type while still controlling for dependencies. Generalized linear network autocorrelation models are common in the literature on spatial autocorrelation and are most commonly estimated using the maximum likelihood principle (Doreian 1980, 1981).

A major restriction to the network autocorrelation models just described is that they only accommodate one type of network dependency: the effect of direct connections (i.e. "neighbors") in the network on a given focal vertex. While this setup may be useful in a number of cases, it is dissatisfying as a general model of vertex behavior conditioned on networks. Some substantial progress on relaxing this restriction was made by Leenders

[1] Strictly speaking, the term "Adjacency matrix" refers to networks with binary ties, but we use the term more generally here.

(2002), who proposed alternative specifications for the weight matrix W as a function of the adjacency matrix A observed on the actors, $W = f(A)$. While the original weight or distance matrix captures the "total", or accumulated, behavior of direct outward connections ($W := A$), alternative specifications include

- The row-standardized weight matrix with entries $w_{ij} := \frac{a_{ij}}{a_{i.}}$, which captures the average, rather than total, influence of connections,
- The column-standardized weight matrix $w_{ij} := \frac{a_{ij}}{a_{.j}}$, where the weight is proportional to the in-degree of each vertex rather than the out-degree (thus capturing exerted rather than received influence),
- The influence of second- or third-order indirect rather than direct contacts, $W := A^2$ or $W = A^3$, respectively, and
- Comparison with other actors who have similar tie profiles and thus structural similarity, as measured by a Euclidean proximity matrix (defined over both incoming and outgoing relations of each node pair by Leenders 2002).

The spatial approach to modeling the behavior of actors embedded in networks was introduced to political science by Franzese and Hays (2007), who also introduced a spatio-temporal autocorrelation element to the model (hence its namesake as the STAR model). The technique was further honed by Hays et al. (2010) and Franzese et al. (2012), who combined the spatio-temporal dependencies of the STAR model with endogenous network evolution (network–behavior co-evolution) using a Markov transition approach. The establishment of this latter model, the M-STAR model, was significant because it was the first time the spatial lag approach had been used to model co-evolutionary processes between the behavior of actors and the structure of the network. An alternative coevolutionary model exists – called the stochastic actor oriented model – but it's specification of network dependencies on the behavior of actors is that of a simple regression while it's model of the network structure itself is rather more elaborate (Snijders et al. 2010; Steglich et al. 2010, 2006; Veenstra et al. 2013).

We end this discussion with a cautionary note. Some have suggested that one can use co-evolutionary models of actor behavior and network structure to differentiate between two processes that are often difficult to parse out: homophily (the tendency of actors with similar attributes to tie) and influence (Steglich et al. 2010). However, resent research, most notably by (Shalizi and Thomas 2011), has pointed out that this is not generally possible and heroic assumptions are necessary to make this differentiation in even the most trivial examples. We therefore to not attempt to parse influence from homophily in a co-evolutionary process, but rather to further the specification of models focusing on actor behavior conditioning on the network in which they are embedded.

2.2 A More General Model: The Temporal Network Autocorrelation Model (TNAM)

The previous discussion made clear how the tools of spatial analysis can be adapted to model network interdependencies between actors. Here, we introduce an extended framework for the spatial autoregression model such that it can be adjusted to model nearly any sort of dependencies that can exist between actors embedded in a network. While this extension to the spatial framework is rather straightforward, it is significant because it allows for the flexible specification of the weights matrix. This extension allows us to easily operationalize and include an extensive set of dependence specifications, much like the specification of an exponential random graph model (a technique for modeling dyadic ties famed for its ability to model nearly any sort of network dependency). As such, our extension to the network autocorrelation model's framework and the accompanying suite of network dependancies it allows us to include in the model specification, opens the door to much more detailed modeling capabilities than was previously possible. In other words, existing approaches, while they make use of spatial dynamics, are inflexible with respect to the types of dependencies allowable and thus restrict the set of hypotheses that the researcher may subject to rigorous testing. Our technique, which we call the temporal network autocorrelation model (TNAM) relaxes these restrictions.

We generalize the spatial approach described above not just via the weights matrix, but with arbitrary functions of (subsets of) Y, A, X, or any combination thereof. Most critically, dependencies are often not spatial, but temporal. That is to say, Y, W, and X may be observed longitudinally and the temporal dynamics of the model may be critical to its fit. Consider a system containing N vertices (actors) for each of which there are T consecutive observations of their behavior and covariates. A behavioral observation of vertex i at time point t is denoted as y_i^t, and all observations are saved in an $N \times T$ matrix Y. The vertices are embedded in a network, which can change over time and is thus recorded as a series of adjacency matrices and stored in an $N \times N \times T$ array called A. A single adjacency matrix at time t is denoted A^t, and an edge between nodes i and

j at time t is denoted by a_{ij}^t. The adjacency matrices that populate A can be binary or contain weighted edges with arbitrary distributional forms. For example, edges following Gaussian, Poisson, binomial, Bernoulli, ordered categorical, and unordered categorical distributions are all easily accommodated. Besides the (possibly changing or possibly constant) network, there are K attributes associated with the vertices and/or time steps. We use them as predictors and store them in a three-dimensional $N \times K \times T$ array denoted X. An individual attribute at a particular time step is denoted x_i^t.

In our temporal network autocorrelation model (TNAM), the behavior of vertex i at time t is conditional not only on predictors and the behavior of other vertices j at the same time step, but also on the vertex's own previous behavior and the previous behavior of the other vertices j. As such, the TNAM is the most fully featured model for the behavior of actors embedded in a network yet achieved. The order of the temporal dependence, D, has an upper bound of $D \in \{0, 1, \ldots, T-1\}$ previous time steps that are relevant for explaining y_i^t. Dependencies may be functions of any subset of current and/or recent networks $A^{t-D,\ldots,t}$, predictors $X^{t-D,\ldots,t}$, and behavior $Y^{t-D,\ldots,t-1}$ while $Y_{j\neq i}^t$. The predicted vertex behavior is a function of all these dependencies. As such, the probability of a specific outcome behavior for vertex i is

$$P(y_i^t | A^{t-D,\ldots,t}, X^{t-D,\ldots,t}, Y^{t-D,\ldots,t-1}, Y_{j\neq i}^t, \boldsymbol{\theta}) = g^{-1}([A^{t-D,\ldots,t}, X^{t-D,\ldots,t}, Y^{t-D,\ldots,t-1}, Y_{j\neq i}^t]\boldsymbol{\beta}), \tag{3}$$

where g^{-1} is a mean function appropriate for the PDF assumed for the distribution of Y.

The analyst may also add further structure to the model in just that same way that one would with a generalized linear model. One useful addition to the model may be to model time more elaborately. The TNAM can accept most any temporal structure one would want to impose on it. For instance, the model can accommodate a pth order polynomial temporal trend for the form

$$\phi(t) = \phi_0 + \phi_1 t + \phi_2 t^2 + \cdots + \phi_p t^p, \tag{4}$$

with the time index t as the argument. Often, the first and second order polynomials (e.g. the linear and quadratic terms) will be sufficient to model the time trend, but the TNAM can incorporate an arbitrary polynomial. But the TNAM's ability to mode time is not restricted to polynomials: one could include an exponential or logistic time trend just as easily.

Furthermore, if the vertices or time periods exhibit different baseline probabilities or slopes, one may add fixed or random effects without needing to adjust the specification of the TNAM. Additionally, one could use the same structure of the linear predictor to estimate models for censored data (e.g. Tobit), mixture models such as those with zero inflation, and many more. In sum, the TNAM has robust and flexible abilities to accommodate many network-based effects into the specification all while maintaining the same basic GLM framework that lends itself so well to the imposition of additional structure.

2.3 New Specifications of Spatial Network Dependencies

Four general types of effects may be included in the specification of a TNAM: terms involving exogenous effects X, terms involving the network (as an adjacency matrix) A and the outcome Y, terms including the outcome Y and an exogenous variable X, and terms including only the network A. Additionally, the model can accommodate intricate temporal dependences, fixed/random effects, and most other structural modifications that may be used with generalized linear models.

Exogenous Effects (X). First is the simple case of exogenous effects. By exogenous we mean effects that are not related to the outcome or the network in any way. These are simple a node-level attribute measure that could be included in any regression. For example, exogenous effects might be education and income when the outcome is individual party choice. This is the simplest case because it requires no structural modification to the model, involves only the sort of variables researchers are familiar with working with as independent variables, and - were the specification populated with only such effects - would not require anything but a basic generalized linear model. One can also use node-level covariates to account for autoregression in the model; that is, to use the effect of previous behavior as a predictor of current behavior.

Spatial Lags (A and Y). Second, spatial lags may be included. A spatial lag refers to the case where a spatial weights matrix is multiplied by the outcome vector. More specifically an $N \times N$ relational matrix (A) is created that reflects some relationship between any given actor i and any other actor j through A_{ij}. The relationships between i and j may be binary or weighted and directed or undirected. Such relational matrices may be populated in a variety of ways. For example, A could simply be the network of interest, it could reflect connections out to multiple (arbitrary) degrees of separation, it could consist of the geodesic distances between vertices, and a wide variety of alternative specifications involving the outcome network. Once the relational matrix is populated, it is multiplied by the outcome vector and included as a predictor in the model. Moreover, temporal dynamics may come into play. The spatial lag may be constructed using either current or recent adjacency matrices and/or current or recent actor behavior. This process is sufficiently general that it may serve as the means by which most network dependencies can be taken into account in the model specification. Because this process is quite general, it is useful to consider several individual specifications that might prove useful (though hardly a comprehensive list of dependencies one could adjust for via spatial lag).

One can also use spatial lags to capture the network autocorrelation inherent in networks. For example, when political actors are members of a policy network, their success of achieving policy outcomes is not independent from each other. Most likely, being connected to policy winners increases the success rate. In many settings, indirect effects may be important as well: how does the behavior of my friends' friends affect my own behavior? In some contexts, spatio-temporal lags are useful: how does the past behavior of my friends affect my current behavior?

A spatial lag can also be used to capture the structural similarity of a focal node's connections compared to other nodes in the network. For such a term, the matrix A would be populated with structural similarity measures and then multiplied by the outcome. The intuition is that behavior is sometimes affected by comparison with structurally similar nodes. For example, a worker may be impressed by the performance of other workers who are embedded in the same team or who report to the same bosses. As with the other model terms, temporal lags are possible.

Co-membership in a clique, be it a formal group or not, may allow nodes in that clique to exert higher than normal influence on the focal node. Such co-membership is operationalized as a spatial lag with the analyst deciding how to define the groups; be they formally coded or inductively discovered based on the link pattern in a network. Returning to our democracy example for intuition, one might code co-membership in a clique as membership in the G-20 or any number of regional trade groups. Alternatively, one could use the very structure of the international trade network to inductively infer the groups present in the network. As additional examples, one could include binary indications of co-membership in some group or matching on some attribute, a difference the absolute values of some node attribute between the two nodes or nearly any other measure of a relationship at the dyadic level (e.g. the original measure from which the spatial weights matrix drew its name: distance in geographic space).

Attribute Similarity (Y and X). Third, many theories, most notably in sociology, draw heavily on the notion of homophily: that ties should be more likely to form between actors with the same or similar attributes. For example, people with similar religious beliefs may be more likely to become friends. Homophiliy is often (and sometimes inextricably) confounded with influence; also a process of great interest in many areas of the social sciences. While it is not always possible – and indeed is usually a matter of research design – to disentangle homophily and influence, the TNAM easily incorporates a similarity effect on any given attribute via a spatial lag. The appropriate predictor variable may be computed by multiplying the outcome vector by a $n \times n$ similarity matrix, where the similarity matrix is computed on the predictive attributed and can be either an indicator of attribute matching or an absolute difference in attribute values (depending on how the attributed is coded). For example, the application below examines the spread of democracy through the international system. The outcome variable is thus an indicator for whether the state under examination is a democracy or not. Much theory on the spread of democracy relates to the wealth of states, so, if one hypothesizes an influential effect, one might naturally examine the absolute difference in per-capita GDPs between the focal state and others as a predictor of democratic spread.

Network Effects (A) Lastly, the TNAM can accommodate effects based on the structure of the network directly. Typically, these will be computed as node-level measures of network phenomena. As an example, one could include an actor's degree, centrality (using any conventional definition of centrality), local clustering coefficient, and the like as predictors of the behavior under study.

To better illustrate the utility of these network effects, consider a actor's prominence in the network of interest. The behavior of an actor is often thought to be affected by its position in a social network. Nodes that are more prominent in such an network may display markedly different behavior than nodes that are less prominent. To continue with the democracy example, states that have many allies may both exert and receive more pressure on their regime types than states with few allies. This assumes a degree centrality understanding of network prominence. Many forms of centrality are possible including closeness, and betweenness, where the interpretation of the effect would be different.

In the appendix, we provide mathematical details for some of the effects that are likely to be most frequently used. These effects are pre-implemented in the `xergm` package. While we hope the those details prove useful and convenient, the appendix hardly provides an exhaustive taxonomy.

3 Case Study: The Diffusion of Democracy

We illustrate the Temporal Network Autocorrelation Model using an empirical real-data example. The case study we choose is both well-studied and suitable for spatio-temporal network modeling. To show the added value of the TNAM, we first replicate findings of Gleditsch and Ward (2006) on the diffusion of democracy and then add several TNAM-specific model terms.

Gleditsch and Ward (2006) observe a binary `autocracy` variable, where 1 indicates that a country is an autocracy in a given year t, and 0 indicates that the country is a democracy. The variable is a dichotomized version of the democracy variable from the Polity IV dataset (Marshall and Jaggers 2002). To model the diffusion of democracy, Gleditsch and Ward (2006) estimate a transition model with two separate equations and processes. One equation models whether countries that are democracies at $t-1$ switch to autocracies at t or not ("autocratic transition"). The other equation captures whether countries that are autocracies at $t-1$ remain autocracies until t ("autocratic regime survival"). Both equations are estimated using generalized linear models for binary outcome variables (logit or probit link).

The authors include two sets of explanatory variables: domestic factors, which may lead to an autocratic transition or autocratic survival inside a country, and diffusion mechanisms, which relate the events inside a country to the international system. These spatio-temporal diffusion variables are interesting from a TNAM perspective and can be extended using the model terms presented in this paper.

Table 1 presents four models: the autocratic transition model by Gleditsch and Ward (2006) in column 1, their autocratic survival model in column 2, and the same models with additional TNAM specifications in columns 3 and 4. Note that while the original study is based on years 1951–1998, we restrict our analysis to 1965–1998 because some of the covariates we employ are only available for these years. Except for one diffusion variables, the findings from 1965 to 1998 are qualitatively consistent with the longer time period covered in the original study. Model 1 includes the following domestic factors:

	(1) Transition	(1) Survival	(2) Transition	(2) Survival
Original Model				
(Intercept)	$7.74\ (2.56)^{**}$	$8.03\ (1.47)^{***}$	$4.90\ (2.79)$	$9.65\ (1.62)^{***}$
Logged GDP per capita	$-1.26\ (0.29)^{***}$	$-0.17\ (0.16)$	$-1.18\ (0.32)^{***}$	$-0.26\ (0.19)$
Proportion of Neighboring Democracies	$-1.35\ (0.80)$	$-1.63\ (0.52)^{**}$	$-0.57\ (0.82)$	$-0.42\ (0.58)$
Civil War	$0.53\ (0.54)$	$0.83\ (0.53)$	$0.25\ (0.60)$	$0.63\ (0.54)$
Years of Peace at Territory	$0.01\ (0.01)$	$-0.01\ (0.00)^{***}$	$0.01\ (0.01)$	$0.00\ (0.01)$
Economic Growth	$-0.04\ (0.03)$	$0.01\ (0.02)$	$-0.04\ (0.03)$	$0.01\ (0.02)$
Global Proportion of Democracies	$-3.42\ (2.81)$	$-5.66\ (1.71)^{***}$	$-2.59\ (2.80)$	$-10.28\ (2.05)^{***}$
Neighboring Transition to Democracy		$-0.28\ (0.25)$		$-0.25\ (0.26)$
Additional TNAM Terms				
CliqueLag (Autocracy, Alliances)			$-0.24\ (0.11)^{*}$	$0.01\ (0.08)$
Clustering (Alliances)			$0.65\ (0.62)$	$-1.85\ (0.43)^{***}$
Interaction (Cliquelag, Clustering)			$0.29\ (0.10)^{**}$	$0.22\ (0.09)^{*}$
WeightLag (Autocracy, Trade)			$3.98\ (1.10)^{***}$	$0.37\ (0.70)$
AIC	252.70	539.23	241.46	511.41
BIC	290.15	587.21	300.30	583.37
Log Likelihood	-119.35	-261.62	-109.73	-243.70
Deviance	238.70	523.23	219.46	487.41
Num. obs.	1555	2972	1555	2972

$^{***}p < 0.001$, $^{**}p < 0.01$, $^{*}p < 0.05$

Table 1: A Temporal Network Autocorrelation Model applied to the Diffusion of Democracy

Logged GDP per Capita The higher the living standard in the previous year (as measured by the log of the gross domestic product), the more likely people will claim self-determination and thus democracy; therefore, negative coefficients of this variable can be found in both sub-models, meaning that more wealth leads to lower odds of autocratic transition and survival (though only significant in the transition model).

Economic Growth A measure of economic growth is used to test whether economic decline is associated with regime change. The negative coefficient in the first column indicates that lower values on the economic growth variable lead to higher odds of autocratic transition (though it is not significant at the 95 percent level).

Civil War Controls whether a state was involved in a civil war as domestic conflict may cause a regime change. However, this is not supported by the data.

Years of Peace at Territory This variable is a "count of the number of years that a country has remained at peace on its territory as a proxy for the stability of peace" (Gleditsch and Ward 2006: 923) and is based on the Correlates of War dataset. Enduring peace slightly promotes autocratic regime survival, as indicated by the positive coefficient in column 2.

Besides these domestic factors, model 1 contains three diffusion variables:

Global Proportion of Democracies The more prevalent democracy is in the world, the more a country may be influenced towards democracy. Indeed, this variable is associated with low odds of autocratic regime survival, as indicated by the strong negative coefficient in the second column.

Proportion of Neighboring Democracies The number of democracies in a 500 km radius divided by the number of countries in the same radius. In TNAM parlance, this is a row-normalized NetLag term where the outcome variable y is a democracy dummy and the adjacency matrix is a binary network matrix where values of 1 indicate that countries i and j are less than 500 km apart. In spatial regression parlance, this is a row-normalized spatial lag term. In SAOM parlance, this is an average alter effect. The rationale is that exposure to democracies within a local or regional context might inhibit autocratic transition or survival.

Neighboring Transition to Democracy A binary variable indicating whether a transition to democracy takes place in a neighboring state within a 500 km radius. The parameter for this term is constrained to be 0 in the transition model and estimated along with the other parameters in the autocratic survival model because neighboring democratic transitions are expected to affect autocracies but not other democracies.

Model 2 presents the same specification with additional TNAM-related model terms. The model fit as measured by AIC is improved by including the additional variables. If additional model terms are penalized as in the BIC measure, the transition model fares slightly worse while the new autocratic survival model still fits better than the original model. Four new model terms are introduced. All of them make extensive use of additional network specifications. As in the original model, all model terms are lagged by one year. That is, if the transition between $t-1$ and t is modeled, the variables are measured in year t. This means that network dependencies are usually modeled as spatio-temporal lags.

The first three additional TNAM terms are related to bilateral defensive alliances as measured in the ATOP dataset (Leeds et al. 2002). `CliqueLag (Autocracy, Alliances)` is a spatial/network lag where the network matrix is a clique co-membership matrix. Cliques are maximal complete subgraphs of the alliance network, i. e., a clique is a subset of nodes where all nodes are connected by an alliance to all other nodes. In this model term, cliques are of minimum size $n = 3$. For example, if countries i and j are both members of the same five international alliance cliques, they have an edge weight of 5 in the clique co-membership matrix. A dummy variable for autocracy at $t-1$ is multiplied by the clique co-membership matrix (i. e., a spatio-temporal lag is computed). Substantively, this captures the extent to which autocratic cliques in which a country is a member influence the country's decision to transform into an autocracy or sustain an autocratic status. The transition sub-model indicates that embeddedness in autocratic clubs leads to stability; democratic countries that have strong alliance ties with clubs of dictators are in fact less likely to become an autocracy. Social stability in terms of alliances is important for democratic stability, even if the peers are autocracies.

`Clustering (Alliances)` is a network-type dependency term, which means that the covariate is only a function of the network, not the outcome variable. This model term computes the local clustering coefficient for each country, which is defined as the extent to which the allies of a country have alliance ties among each other. In other words, is country i more (or less) likely to transition to an autocracy or remain an autocracy if its allies are allied, that is, if the country is embedded in a stable social structure? The negatively significant coefficient in the autocratic regime survival model suggests that such a socially stable defense arrangement increases the chances that an autocracy becomes a democracy. This effect supports the previous finding on stability and is a complementary specification.

`Interaction (CliqueLag, Clustering)` is a centered interaction term of the previous two model terms. As both variables are related to alliances and social closure, they are correlated. An interaction term allows us to evaluate their separate and joint effects despite the correlation. As a reminder, the `CliqueLag` term captures if embeddedness in autocratic clubs affects autocratic transition or survival, and the `Clustering` term captures whether embeddedness in alliance clubs per se affects autocratic transition or survival, irrespective of the type of ally. The model term yields positive and significant coefficients in both sub-models. Substantively, this means that the more polyadic or clique closure exists between a country and its neighbors, the more likely their joint property of being a club of autocracies plays out and incentivizes the country to become or remain an autocracy.

The final model term is `WeightLag (Autocracy, Trade)`. This is a classic spatial lag with row normalization (except for the fact that the outcome variable and the variable used to create the spatial lag are not identical). The autocracy dummy variable is multiplied by a row-normalized matrix of international trade flows. The model term captures in how far trade partners exert an influence on the regime type of a country. If a large proportion of a country's strong trade partners are autocracies, this increases the chances that the country becomes an autocracy as well. On the other hand, trade ties with autocracies do not have a significant effect on autocratic regime survival.

In addition to this simple logistic model, we have estimated the same model with an additional linear and quadratic time trend variable or, alternatively, a linear mixed-effects model with random effects for countries or years. All of these models yield approximately the same results; therefore the simpler model presented here is sufficient.

The additional TNAM variables not only add valuable information about the data-generating process. Controlling for these network effects also renders two of the original model terms insignificant. Most importantly, the proportion of neighboring democracies—one of the main findings of the original paper—may be a spurious effect that is better captured by alliances and trade network effects.

This underlines the importance of modeling nodal attributes as functions of latent network dependencies. The theory section suggested several ways to do this, and the accompanying software package `xergm` makes it easy to implement such specifications, as shown in this case study. International relations is a prime example where seemingly independent units are connected by latent multiplex network relations. Social mechanisms between countries operate via various channels, such as direct contagion, comparison with structurally similar nodes, or polyadic closure, and they may have spatio-temporal effects on local outcomes. However, similar processes can be expected in other group settings, for example legislators nested in parliaments or interest groups and government agencies nested in policy domains, where mutual relevance plays a role.

4 Conclusion

It is well known that failure to properly model network dependencies, when they are at play, results in biased estimates of the effects that are included in the model (Cranmer and Desmarais 2011). Failure to model such dependencies is essentially a form of model misspecification. In the TNAM, we have introduced a flexible and stable model that can adjust for a wide variety of network dependencies while modeling actor-level outcomes. Not only does this ability allow the analyst to avoid biases that would compromise the validity of statistical inferences, but it opens the door to the empirical testing of a wide variety of theoretical mechanisms that either defied testing or rendered it much more difficult before the advent of the TNAM.

To describe the TNAM as flexible is something of an understatement. Network dependencies can be added in a fashion very similar to the temporal exponential random graph model (famous for its flexibility). That is to say, the spatial-lag framework can model nearly any type and any order of network dependencies for outcomes measured at the actor (vertex) level. Custom dependencies are easy to build and include as model terms can always be computed separately prior to inclusion in the final specification. What is more, the analyst may account for time via lags, polynomials, other functions, fixed effects, or random effects. Estimation and model structure are flexible: the GLM family of models is fully accessible to the TNAM, as are other models such as mixed effects, Bayesian, and even non-parametric models. Further still, many of these abilities, the model terms and families likely to be most widely used, are implemented in our easy to use companion software: the `tnam` functionality in the **xergm** package (Leifeld et al. 2016).

Looking forward, it is possible to use the TNAM in concert with the TERGM to create something similar to a co-evolutionary model of actor behavior and network structure. To accomplish this, one would, at each time step, fit a TERGM with a specification including TNAM terms and a TNAM whose specification includes TERGM terms.

We want to offer a similar approach to spatio-temporal dependencies, possibly with TERGMs as a network evolution component.

A Appendix: Details on TNAM Terms

This appendix provides mathematical details on the most common terms one can include in a TNAM based on the `xergm` package.

A.1 Network lag

The *network lag* term is defined as

$$\text{netlag}_i^t(y, N, D, r, k, o, f) = \sum_k^o \sum_{j \neq i} y_j^{t-D}[g_{ji}^{t-D} = k](r[g_{ij}^{t-D} \leq k] + (1-r))f(k)s \tag{5}$$

where the user can choose between a version without standardization,

$$s = 1, \tag{6}$$

with row standardization,

$$s = \left[\sum_k^o \sum_{j \neq i} [g_{ji}^{t-D} = k](r[g_{ij}^{t-D} \leq k] + (1-r))f(k)\right]^{-1}, \tag{7}$$

or with column standardization,

$$s = \left[\sum_k^o \sum_{j \neq i, j \neq l} N_{jl}^{t-D}[g_{ji}^{t-D} = k](r[g_{ij}^{t-D} = k] + (1-r))f(k)\right]^{-1}. \tag{8}$$

Details:

- The network lag is computed for node i by looking at the attribute values of nodes j who are connected to i (directly or indirectly).
- y denotes the focal attribute vector. In many cases, y is in fact the outcome vector of the regression model (therefore the "lag" component in "netlag"), but could also choose some other vector, the values of which in other nodes have an influence on node i.
- t denotes the current time step at which the model term is computed as a covariate.
- D is the temporal lag. By default, $D = 0$ means there is no lag, but arbitrary non-negative integer values up to $D = T - 1$ can be used.
- N is the network matrix. N_{ji} indicates the corresponding value in the network matrix.
- r is a binary argument that equals 1 if only reciprocal edges should be considered and 0 if all incoming edges should be considered. $(r[g_{ij}^{t-D} \leq k] + (1-r))$ therefore checks if a) only reciprocal edges should be considered *and* the reciprocal tie is indeed there (at k steps or less), or b) all incoming edges should be considered (in which case the term becomes 1 under any circumstance).
- k is the minimum path length to be considered, by default $k = 1$. o is the maximum path length to be considered, by default $o = 1$, with a maximum of $o = m - 1$. The model term can include only direct network connections if $k = o = 1$, or it can include indirect connections, e. g., third-order connections $k = o = 3$, or combinations of multiple orders, e. g., $k = 1$ and $o = 3$.
- $[g_{ji}^{t-D} = k]$ is an indicator function that equals 1 if the geodesic distance between j and i is exactly k steps and 0 otherwise. In the simplest case where only direct connections in a binary network are counted, this reduces to $[g_{ji}^{t-D} = k] = N_{ji}$.
- f is a decay function that downweights higher numbers of k. By default, a geometric decay function $f(k) = k^{-1}$ is used, but any other function can be used alternatively. For example, decay can be switched off by setting $f(k) = 1$.
- By default, standardization is switched of, i. e., $s = 1$. If row standardization is used (Equation 7), this divides the total exposure of i by the indegree of i (over k paths). This is called the "average alter" effect in SAOM parlance. Column

standardization (Equation 8) divides total exposure of i by the number of outgoing connections of nodes j, that is, it standardizes by capacity of j to distribute influence.

A.2 Weight lag

In the network lag statistic, the default setting is $k = o = 1$ for first order network lag, $f(k) = 1$ for switching off the decay, and $r = 0$ for switching off the reciprocity argument. This reduces to a simple spatial or spatio-temporal lag of the form

$$\text{weightlag}_i(y, N, D) = \sum_{j \neq i} y_j^{t-D} N_{ji}^{t-D}, \tag{9}$$

which we call a *weight lag* term and which can be conveniently computed for the y observations of all nodes i by computing

$$\text{weightlag}(y, N, D) = y^{t-D} N^{t-D}, \tag{10}$$

i. e., via matrix multiplication. Note that the more complicated network lag specifications can only be computed via multiple nested loops, therefore the weight lag term is computationally more efficient. Moreover, the weight lag term is a noteworthy sub-specification of a network lag because it works with weighted network specifications of N (unlike more complex network lag specifications). If N is a weighted matrix, this corresponds to the classic spatial autoregressive model suggested by Cliff and Ord (1969).

A.3 Attribute similarity

The analyst may wish to include effects to capture homophily between a node of interest i and its alters j based on a predictor x. For such purposes, the *attribute similarity* term is defined as

$$attsim_{ij}^t(y, X, D) = \sum_{j \neq i} (|x_i^{t-D} - x_j^{t-D}|) y_i^{t-D}, \tag{11}$$

if x is numeric, and as

$$attsim^t_{ij}(y, X, D) = \sum_{j \neq i} \delta_{ij}(x_i^{t-D}, x_j^{t-D}) y_i^{t-D}, \tag{12}$$

If x is qualitative. In the latter equation, Kronecker's delta is 1 if both of its arguments are the same. In words, this term computes a similarity matrix based on the node attribute x and uses the matrix to construct a spatial lag by multiplying it by the outcome vector y. If the attribute x is numeric, the absolute difference between x_i and x_j is computed. Whereas, if x is qualitative, the similarity matrix is populated with one's wherever x_i and x_j have the same value and zero otherwise.

The intuition behind this model term is that node i's behavior may be influenced by node j's behavior if nodes i and j are similar on another dimension. For example, if i and j both smoke while k does not smoke, j's alcohol consumption may affect i's alcohol consumption to a larger extent than node k's alcohol consumption. In this example, the outcome variable y is alcohol consumption and the attribute x is smoking.

A.4 Structural similarity

The analyst may also be interested in homophily effects based on the local structure of the network surrounding nodes i and j. The *structural similarity* effect captures such phenomena and is coded

$$structsim^t_{ij}(y, D, s, N) = \sum_{j \neq i} (|s_i(N^{t-D}) - s_j(N^{t-D})|) y_i^{t-D}, \tag{13}$$

where $s()$ is some structural statistic computed at the vertex level. In words, this term computes a similarity matrix populated by the differences between structural statistics computed for the each vertex pair and then multiplies this similarity matrix by the outcome vector. The definition of the $s()$ statistic is flexible such that any type of structural statistic computed on the vertex level may be used.

The intuition is that behavior is sometimes affected by comparison with structurally similar nodes. For example, a worker may be impressed by the performance of other workers who are embedded in the same team or who report to the same bosses. As with the other model terms, temporal lags are possible.

A.5 Centrality

The *centrality* model term computes a centrality index for the nodes in a network or matrix:

$$cent_i^t(c, N) = c(N_i) y_i^{t-D}, \tag{14}$$

where $c()$ is any centrality statistic. In words, the centrality score for a particular node is multiplied by the corresponding value of the outcome vector

This can capture important structural effects because being central often implies certain constraints or opportunities more peripheral nodes do not have. For example, central nodes in a network of employees might be able to perform better.

Most forms of centrality can be accommodated, including the common forms of degree, betweenness, and eigenvalue.

A.6 Clique lag

The *cliquelag* model term computes a clique co-membership matrix and multiplies this matrix with the outcome variable:

$$cliquelag_i^t(y, N, D, s) = \sum_{j \neq i} \delta_{ij}(s(N_i^{t-D}), s(N_j^{t-D})) y_i^{t-D}, \tag{15}$$

where $s()$ is a statistic on the network that produces a clique membership value for vertex i and Kronecker's delta is again 1 if both of its arguments are the same.

The intuition behind this is that in some settings individuals may be influenced to a particularly strong extent by peers in the same cliques. A clique is defined as a maximal connected subgraph of size k. For example, a deviant behavior of a person may be conditioned by the deviant behavior of the person's friends ? but only if these friends are tied to each other as well so that a clique among these persons exists. A minimal and a maximal k may be defined, where k is the size of the cliques. In the clique co-membership matrix, all cliques with $min(k) \leq k \leq max(k)$ are included.

A.7 Clustering

The *clustering* model term computes the local clustering coefficient, which is also known as transitivity.

This index has high values if the direct neighborhood of a node is densely interconnected. For example, if one's friends are friends with each other, this may have repercussions on ego's behavior.

To build the statistic, consider the local clustering coefficient

$$C_i = \frac{|\{e_{jk} : v_j, v_k \in R_i, e_{jk} \in E\}|}{k_i(k_i.1)} \tag{16}$$

for an undirected network and

$$C_i = \frac{2|\{e_{jk} : v_j, v_k \in R_i, e_{jk} \in E\}|}{k_i(k_i.1)} \tag{17}$$

for a directed network. In both equations, E is the set of all edges, e_{ij} is a particular edge, v_k is a particular vertex, R indicates the region or neighborhood on which the statistic is computed, and k_i is the number of vertices $|R_i|$ in the region, R_i of a vertex. The denominator of these equations is a normalizer because $k_i(k_i - 1)$ is the number of edges that could exist within the region of i. As such, the clustering coefficient reflects the proportion of edges between the vertices within a neighborhood over the number of edges that could exist in that same neighborhood.

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