From Equation 3.2.41 we see that the solution to Equation 13 is the smallest value of n such that

$$\alpha \geq \sum_{i=0}^{r-1} \binom{n}{i} (1-q)^i q^{n-i} \tag{14}$$

just as in Equation 9.

In fact, it can be shown, with the aid of calculus (see Noether, 1967a), that for the two-sided tolerance limits and for both types of one-sided tolerance limits, the sample size n depends on the solution to

$$\alpha \geq \sum_{i=0}^{r+m-1} \binom{n}{i} (1-q)^i q^{n-i} \tag{15}$$

which is somewhat surprising in that Equation 15 depends on the sum $r + m$ but does not depend on whether we wish to choose as our interval all values to the right of $X^{(r+m)}$, or all values of the left of $X^{(n+1-r-m)}$, or all values between $X^{(r)}$ and $X^{(n+1-m)}$, or any combination of two order statistics whose ranks have a difference of $n + 1 - m - r$.

The use of Table A3 to solve Equation 15 is, at best, frustrating. Therefore Tables A5 and A6 are given for the most popular values $r + m = 1$ and $r + m = 2$. The approximation in Equation 1 is furnished without proof by Scheffé and Tukey (1944). Equation 2 is obtained by solving Equation 1 for q. Graphs to aid in finding n are given by Murphy (1948) and Birnbaum and Zuckerman (1949). More extensive tables are given by Owen (1962). □

Tolerance limits may also be used with two samples (Danziger and Davis, 1964), with a single censored sample (Bohrer, 1968), or for deciding from which of two possibly multivariate populations a sample was obtained (Quesenberry and Gessaman, 1968). Usage of tolerance limits on discrete random variables is examined by Hanson and Owen (1963). An application of tolerance intervals to the regression problem is discussed by Bowden (1968). Other articles dealing with tolerance limits are given by Mack (1969) and Goodman and Madansky (1962).

3.3 EXERCISES

1. What must the sample size be to be 90% sure that at least 95% of the population lies within the sample range?
   - **(a)** Use the exact table.
   - **(b)** Use the approximation.

2. What must the sample size be to be 95% certain that at least 90% of the population equals or exceeds $X^{(1)}$?

(a) Use the exact table.

(b) Use the approximation.

3. What must the sample size be in order for there to be a probability 0.90 that at least 85% of the population is $\leq X^{(n)}$?

(a) Use the exact table.

(b) Use the approximation.

4. What must the sample size be in order for there to be a 95% chance that 99% of the population is $\geq X^{(2)}$?

(a) Use the exact table.

(b) Use the approximation.

5. What must the sample size be so there is probability 0.90 that at least 50% of the population is between $X^{(5)}$ and $X^{(n-4)}$ inclusive?

6. What must the sample size be in Exercise 5 if the probability is 0.95 instead of 0.90? How about 0.99?

7. A fitness gym has measured the percentage of fat on 86 of its members.

(a) At least what percent of its members have fat percentages between the smallest and largest of the percentages measured on the 86 members in the sample, with 95% certainty? With 90% certainty?

(b) At least what percent of its members have fat percentage between $X^{(2)}$ and $X^{(85)}$ with 95% certainty? With 90% certainty?

8. A mail order catalog company surveyed 146 of its customers to find out the shipping time (from order date to the date of delivery) for their recent orders using regular U.S. mail.

(a) At least what percentage of its customers can expect delivery between $X^{(1)}$ and $X^{(142)}$ as given by its sample observations, with 95% certainty? With 90% certainty.

(b) Notice that the endpoints of the tolerance interval are not symmetric in this case. What is an advantage of using these asymmetric endpoints in this problem?

9. An engineer writing the acceptance specs for a load of steel reinforcement rods would like to specify that at least 90% of the rods are between the sixth longest and the sixth shortest rods in a random sample she selects. In order to have 99% confidence in this statement, what should the sample size be?

10. A computer model is developed to simulate the conditions within a combat unit (e.g., a communications center) in battle conditions. One of the items of interest, determined by the computer model, is the minimum number of people required to maintain a satisfactory level of operation of the combat unit. We want to staff the combat unit with enough people so it will operate satisfactorily during 90% of the battles.

(a) How many computer runs are necessary so we can be 99.9% sure that the number of people required is no more than $X^{(n)}$, the largest observed number in the computer runs?

(b) How many computer runs are necessary so we can be 99.9% sure that the number of people required is between $X^{(1)}$ and $X^{(n)}$?