Homework 4

Brian Detweiler

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1. Calculate and sketch the autocorrelation functions ρ_k for the following stationary processes.

(a) $Y_t = -0.9Y_{t-1} + e_t$

Answer: For this AR(1) model, we let $\phi = -0.9$, such that

$$
\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \\
&= \phi(\phi Y_{t-2+e_{t-1}}) + e_t \\
&= \phi(\phi(Y_{t-3} + e_{t-2}) + e_{t-1}) + e_t \\
&= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3}
\end{aligned}
$$

Continuing this expansion indefinitely we get

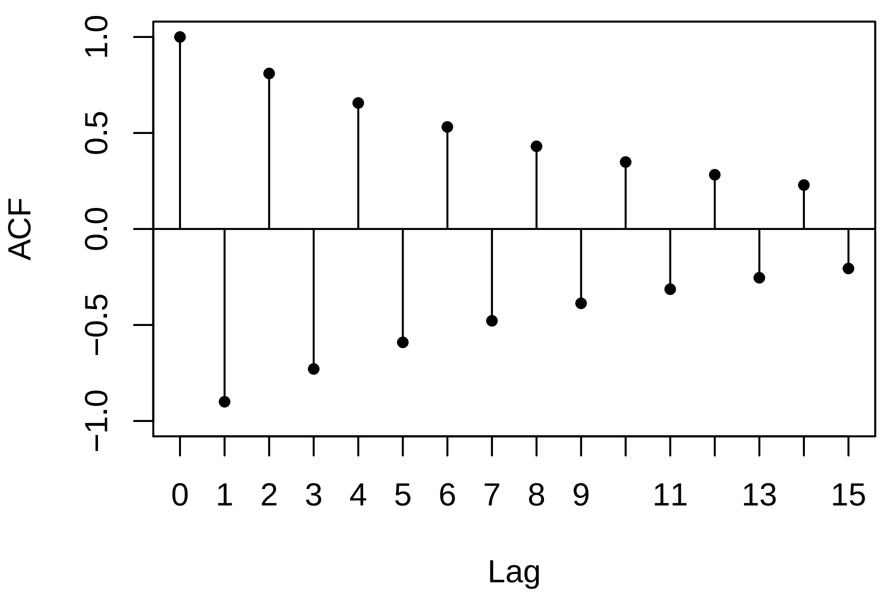
$$
\begin{aligned}
&Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \dots \\
&\rho_k = \phi^k, s.t. |\rho_k| \le 1
\end{aligned}
$$

Substituting in our value of -0.9 for ϕ, we get

$$\rho_k = -0.9^k$$

Such an autocorrelation function might look like this:

```
n <- 15
ACF <- ARMAacf(ar = c(-0.9), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



(b) $Y_t = 8 + e_t - 0.75e_{t-1} + 0.5e_{t-2} - 0.25e_{t-3}$

Answer:

Looking at this MA(3) model, we have

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \theta_3 e_{t-3}$$
$$\rho_k = \begin{cases} 1 & \text{if } k = 0 \\ \frac{-\theta_1 + \theta_1\theta_2 + \theta_2\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} & \text{for } k \pm 1 \\ \frac{-\theta_2 + \theta_2\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} & \text{for } k \pm 2 \\ \frac{-\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} & \text{for } k \pm 3 \end{cases}$$

Letting $\theta_1 = 0.75, \theta_2 = -0.5$ and $\theta_3 = 0.25$, we get

$$Y_t = e_t - 0.75e_{t-1} - (-0.5)e_{t-2} - 0.25e_{t-3}$$
$$\rho_k = \begin{cases} 1 & \text{if } k = 0 \\ \frac{-0.75+(0.75)(-0.5)+(-0.5)(0.25)}{1+(0.75)^2+(-0.5)^2+(0.25)^2} & \text{for } k \pm 1 \\ \frac{-(-0.5)+(-0.5)(0.25)}{1+(0.75)^2+(-0.5)^2+(0.25)^2} & \text{for } k \pm 2 \\ \frac{-(0.25)}{1+(0.75)^2+(-0.5)^2+(0.25)^2} & \text{for } k \pm 3 \end{cases}$$

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2. Verify that for an MA(1) process

$$max_\theta \rho_1 = 0.5 \text{ and } min_\theta \rho_1 = -0.5$$

3. Consider the ARMA(1, 2) model

$$Y_t = 0.7Y_{t-1} + e_t + 0.8e_{t-1} - 0.6e_{t-2}$$

Assume that $\{e_t\}$ is a white noise process with zero mean and unit variance ($\sigma_e^2 = 1$). Find the numerical values of ρ_0, ρ_1 and ρ_2 by hand. Also find a recursive relationship between ρ_k and ρ_{k-1} for $k > 2$.

Answer:

4. Consider a "AR(1)" process satisfying $Y_t = \phi Y_{t-1} + e_t$, where $t > 0$, ϕ can be any number and $\{e_t\}$ is a white noise process with zero mean and variance σ_e^2. Let Y_0 be a random variable with mean μ and variance σ_0^2. Show that for $t > 0$ we have

(a) $Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \ldots + \phi^{t-1} e_1 + \phi^t Y_0$

(b) $E[Y_t] = \phi^t \mu$.

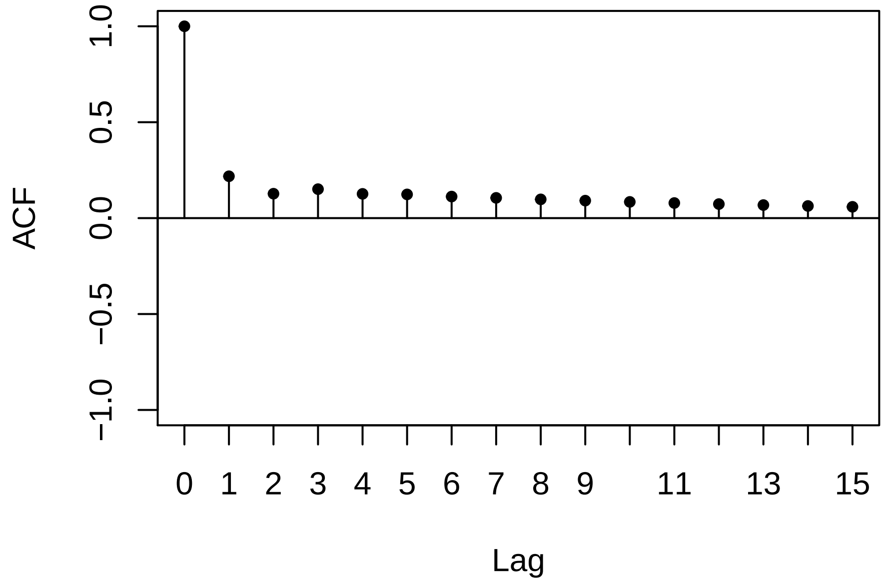
(c)

$$Var(Y_t) = \begin{cases} \frac{1-\phi^{2t}}{1-\phi^2}\sigma_e^2 + \phi^{2t}\sigma_0^2 & \text{for } \phi \neq 1 \\ t\sigma_e^2 + \sigma_0^2 & \text{for } \phi = 1 \end{cases}$$

(d) Suppose $\mu = 0$. Show that if $\{Y_t\}$ is stationary, then $Var(Y_t) = \frac{\sigma_e^2}{1-\phi^2}$.

5. The following command in `R` will plot the theoretical autocorrelation function of an ARMA(2, 2) model $Y_t = 0.5Y_{t-1} + 0.4Y_{t-2} + e_t - 0.7e_{t-1} - 0.6e_{t-2}$ for the first 15 lags:

```r
n <- 15
ACF <- ARMAacf(ar = c(0.5, 0.4), ma = c(-0.7, -0.6), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



Modify the code to generate the theoretical autocorrelation functions up to 20 lags of the following ARMA processes:

(a) MA(1) with $\theta = 0.5$

(b) MA(1) with $\theta = -0.5$

(c) MA(2) with $\theta_1 = \theta_2 = 0.1$

(d) AR(1) with $\phi = 0.4$

(e) AR(1) with $\phi = -0.4$

(f) AR(2) with $\phi_1 = 0.5$ **and** $\phi_2 = -0.9$

(g) ARMA(1, 1) with $\phi = 0.7$ **and** $\theta = 0.4$

(h) ARMA(1, 2) given in Question 3