

Time Series Cheatsheet

Brian Detweiler

January 24, 2017

Mean

$$\begin{aligned}E[X] &= \int_{-\infty}^{\infty} xf(x)dx \text{ for } \int_{-\infty}^{\infty} |x|f(x)dx < \infty, \text{ undefined otherwise} \\ \mu_t &= E[Y_t] \text{ for } t = 0, \pm 1, \pm 2, \dots \\ E[aX + bY + c] &= aE[X] + bE[Y] + c \\ E[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y)dx dy\end{aligned}$$

Variance

$$\begin{aligned}\text{Var}(X) &\geq 0 \text{ non-negative} \\ \text{Var}(a + bX) &= b^2\text{Var}(X) \\ \text{Var}(X + Y) &= \text{Var}(X) + \text{Var}(Y) \text{ for independent } X, Y \\ \text{Var}(X) &= E[X^2] - (E[X])^2\end{aligned}$$

Covariance

$$\begin{aligned}\text{Cov}(a + bX, c + dY) &= bd\text{Cov}(X, Y) \\ \text{Var}(X + Y) &= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) \\ \text{Cov}(X + Y, Z) &= \text{Cov}(X, Z) + \text{Cov}(Y, Z) \\ \text{Cov}(X, X) &= \text{Var}(X) \\ \text{Cov}(X, Y) &= \text{Cov}(Y, X) \\ \text{Cov}(X, Y) &= 0 \text{ for independent } X, Y \\ \text{Cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E[(X - \mu_X)(Y - \mu_Y)] \\ &= E[XY - XE[Y] - YE[X] + E[X]E[Y]] \\ &= E[XY] - E[XE[Y]] - E[YE[X]] + E[E[X]E[Y]] \\ &= E[XY] - E[X]E[Y]\end{aligned}$$

Correlation

If X^* is a standardized X and Y^* is a standardized Y , then

$$\rho = E[X^*Y^*]$$

Else,

$$\rho = \text{Corr}(X, Y) = \frac{\text{Corr}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

$$\begin{aligned}
-1 &\leq \text{Corr}(X, Y) \leq 1 \\
\text{Corr}(a + bX, c + dY) &= \text{sign}(bd) \text{Corr}(X, Y) \\
\text{Corr}(X, Y) &= \pm 1 \text{ if } \exists a, b \text{ s.t. } P(Y = a + bX) = 1
\end{aligned}$$

Events

$$\begin{aligned}
e_i &= \text{event } i \\
Y_0 &= e_0 \\
Y_t &= e_0 + e_1 + e_2 + \dots + e_t
\end{aligned}$$

Autocovariance

$$\begin{aligned}
\gamma_{t,t} &= \text{Var}(Y_t) \\
\gamma_{t,s} &= \gamma_{s,t} \\
|\gamma_{t,s}| &\leq \sqrt{\gamma_{t,t} \gamma_{s,s}} \\
\gamma_{t,s} &= \text{Cov}(Y_t, Y_s) \text{ for } t, s = 0, \pm 1, \pm 2, \dots \\
\text{Cov}(X, Y) &= E[(X - \mu_X)(Y - \mu_Y)] \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

$$\begin{aligned}
\text{Cov}(Y_t, Y_s) &= E[(Y_t - \mu_t)(Y_s - \mu_s)] \\
&= E[Y_t Y_s] - \mu_t \mu_s
\end{aligned}$$

For stochastic process e_1, e_2, \dots with mean 0, variance σ^2 ,

$$\begin{aligned}
\gamma_{t,t} &= \text{Var}(Y_t) \\
&= \text{Var}\left(\sum_{i=1}^t e_i\right) \\
&= \sum_{i=1}^t \text{Var}(e_i) \\
&= t\sigma^2 \\
\gamma_{t,s} &= \text{Cov}(Y_t, Y_s) \\
&= \text{Cov}(e_1 + e_2 + \dots + e_t, e_1 + e_2 + \dots + e_s) \\
&= \text{Cov}\left(\left[\sum_{i=1}^n c_i Y_{t_i}, \sum_{j=1}^n d_j Y_{s_j}\right]\right) \\
&= \sum_{i=1}^m \sum_{j=1}^n c_i d_j \text{Cov}(Y_{t_i}, Y_{s_j}) \\
&= \sum_{i=1}^t \sum_{j=1}^s \text{Cov}(e_i, e_j) \text{ (when } i = j, \text{ you get variance, when } i \neq j \text{ you get 0)} \\
&\text{for } 1 \leq t \leq s \\
&= t\sigma^2
\end{aligned}$$

If c_1, c_2, \dots, c_m and d_1, d_2, \dots, d_n are constants and t_1, t_2, \dots, t_m and s_1, s_2, \dots, s_n are time points, then

$$Cov \left[\sum_{i=1}^m c_i Y_{t_i}, \sum_{j=1}^n d_j Y_{s_j} \right] = \sum_{i=1}^m c_i d_j Cov(Y_{t_i}, Y_{s_j})$$

Special case

$$Var \left[\sum_{i=1}^n c_i Y_{t_i} \right] = \sum_{i=1}^n c_i^2 Var(Y_{t_i}) + 2 \sum_{i=2}^n \sum_{j=1}^{i-1} c_i c_j Cov(Y_{t_i}, Y_{t_j})$$

Autocorrelation

$$\rho_{t,t} = 1$$

$$\rho_{t,s} = \rho_{s,t}$$

$$|\rho_{t,s}| \leq 1$$

$$\rho_{t,s} = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}, \gamma_{s,s}}}$$

Strictly Stationary

The joint distribution of $Y_{t_1}, Y_{t_2}, \dots, Y_{t_n}$ is the same as $Y_{t_1-k}, Y_{t_2-k}, \dots, Y_{t_n-k} \forall t_i, i = 1, 2, \dots, n$ and lag k . Also, $Var(Y_t) = Var(Y_{t-k})$ (constant variance over time).

$Cov(Y_{t-k}, Y_{s-k}) \forall t, s$ and k . Putting $k = s$ and $k = t$, we get

$$\begin{aligned} \gamma_{t,s} &= Cov(Y_{t-s}, Y_0) \\ &= Cov(Y_0, Y_{s-t}) \\ &= Cov(Y_0, Y_{|t-s|}) \\ &= \gamma_{0,|t-s|} \\ \gamma_k &= Cov(Y_t, Y_{t-k}) \\ \rho_k &= Corr(Y_t, Y_{t-k}) \\ \rho_k &= \frac{\gamma_k}{\gamma_0} \\ \gamma_0 &= Var(Y_t) \\ \gamma_k &= \gamma_{-k} \\ |\gamma_k| &\leq \gamma_0 \\ \rho_0 &= 1 \\ \rho_k &= \rho_{-k} \\ |\rho_k| &\leq 1 \end{aligned}$$

Weakly Stationary

1. Mean function is **constant over time**
2. $\gamma_{t,t-k} = \gamma_{0,k}$ for all time t and lag k