Homework 4

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1. Calculate and sketch the autocorrelation functions ρ_k for the following stationary processes.

(a) $Y_t = -0.9Y_{t-1} + e_t$

Answer: For this AR(1) model, we let $\phi = -0.9$, such that

$$
\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \\
&= \phi(\phi Y_{t-2+e_{t-1}}) + e_t \\
&= \phi(\phi(Y_{t-3} + e_{t-2}) + e_{t-1}) + e_t \\
&= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3}
\end{aligned}
$$

Continuing this expansion indefinitely we get

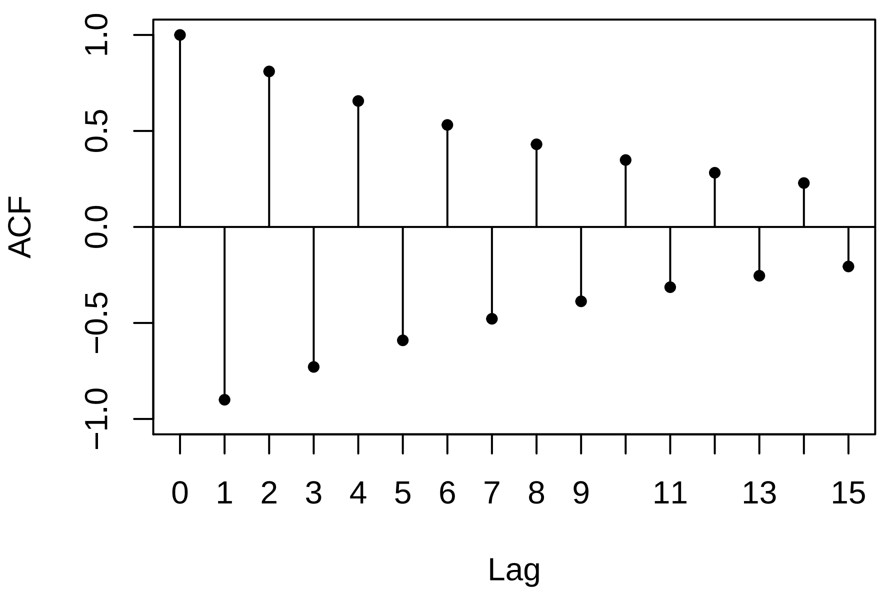
$$
\begin{aligned}
&Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \dots \\
&\rho_k = \phi^k, s.t.|\rho_k| \leq 1
\end{aligned}
$$

Substituting in our value of -0.9 for ϕ, we get

$$\rho_k = -0.9^k$$

Such an autocorrelation function might look like this:

```
n <- 15
ACF <- ARMAacf(ar = c(-0.9), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



(b) $Y_t = 8 + e_t - 0.75e_{t-1} + 0.5e_{t-2} - 0.25e_{t-3}$

Answer:

Looking at this MA(3) model, we have

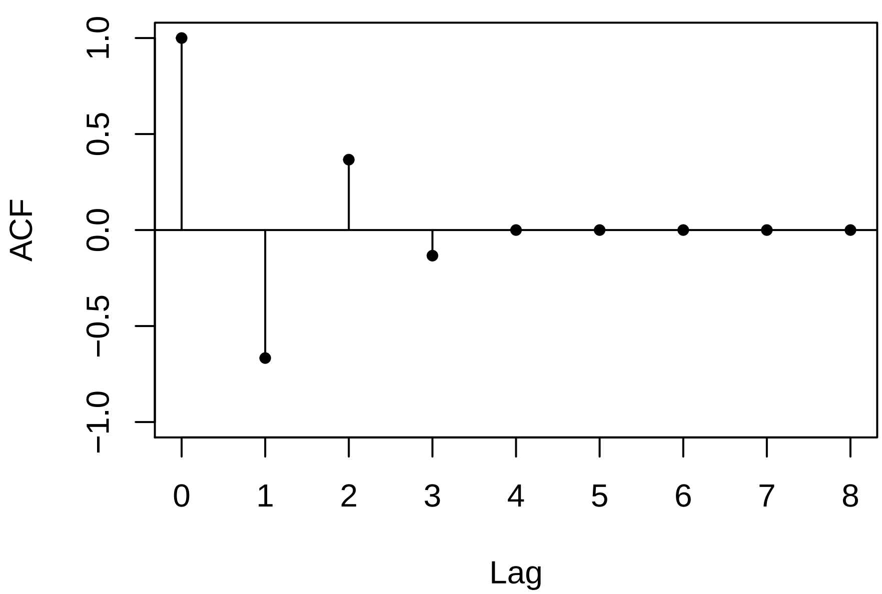
$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \theta_3 e_{t-3}$$
$$\rho_k = \begin{cases} 1 & \text{if } k = 0 \\ \frac{-\theta_1+\theta_1\theta_2+\theta_2\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} & \text{for } k \pm 1 \\ \frac{-\theta_2+\theta_2\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} & \text{for } k \pm 2 \\ \frac{-\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} & \text{for } k \pm 3 \end{cases}$$

Letting $\theta_1 = 0.75, \theta_2 = -0.5$ and $\theta_3 = 0.25$, we get

$$Y_t = e_t - 0.75e_{t-1} - (-0.5)e_{t-2} - 0.25e_{t-3}$$
$$\rho_k = \begin{cases} 1 & \text{if } k = 0 \\ \frac{-0.75+(0.75)(-0.5)+(-0.5)(0.25)}{1+(0.75)^2+(-0.5)^2+(0.25)^2} = \frac{-1.25}{1.875} = \frac{-2}{3} & \text{for } k \pm 1 \\ \frac{-(-0.5)+(-0.5)(0.25)}{1+(0.75)^2+(-0.5)^2+(0.25)^2} = \frac{0.375}{1.875} = \frac{1}{5} & \text{for } k \pm 2 \\ \frac{-(0.25)}{1+(0.75)^2+(-0.5)^2+(0.25)^2} = \frac{-0.25}{1.875} = \frac{-2}{15} & \text{for } k \pm 3 \end{cases}$$

Such an autocorrelation function might look like this:

```r
n <- 8
ACF <- ARMAacf(ma = c(-0.75, 0.5, -0.25), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



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2. Verify that for an MA(1) process

$$\max_{-\infty<\theta<\infty} \rho_1 = 0.5 \text{ and } \min_{-\infty<\theta<\infty} \rho_1 = -0.5$$

Answer:

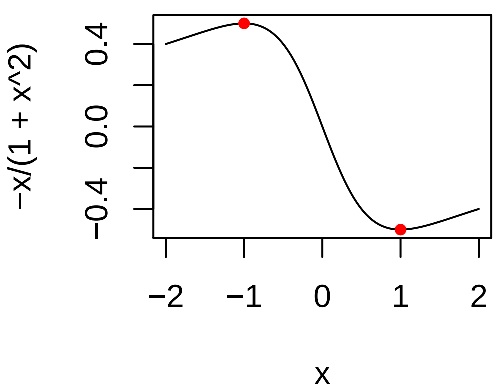
For MA(1), using $k = 1$, we know $\rho_1 = \frac{-\theta}{1+\theta^2}$

We can find the global maxima and minima at the inflection points by taking the derivative and setting it equal to zero.

$$\begin{aligned}
\frac{-\theta}{1+\theta^2}\frac{d}{d\theta} &= \frac{t^2-1}{(t^2+1)^2} \\
\frac{t^2-1}{(t^2+1)^2} &= 0 \\
\frac{t^2}{(t^2+1)^2} - \frac{1}{(t^2+1)^2} &= 0 \\
\frac{t^2}{(t^2+1)^2} &= \frac{1}{(t^2+1)^2} \\
t^2 &= 1 \\
t &= \pm 1
\end{aligned}$$

Now we just need to evaluate at $t = \pm 1$ and we can see the global maximum and minimum:

$$\begin{aligned}
\frac{-1}{1+1^2} &= \frac{-1}{2} \\
\frac{-(-1)}{1+(-1)^2} &= \frac{1}{2}
\end{aligned}$$



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3. Consider the ARMA(1, 2) model

$$Y_t = 0.7Y_{t-1} + e_t + 0.8e_{t-1} - 0.6e_{t-2}$$

Assume that $\{e_t\}$ is a white noise process with zero mean and unit variance ($\sigma_e^2 = 1$). Find the numerical values of ρ_0, ρ_1 and ρ_2 by hand. Also find a recursive relationship between ρ_k and ρ_{k-1} for $k > 2$.

Answer:

As always, $\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$.

To find ρ_1 and ρ_2, we'll first need $Var(Y_t)$.

$$\begin{aligned}
Var(Y_t) &= \phi_1^2 Var(Y_{t-1}) + Var(e_t) + \theta_1^2 Var(e_{t-1}) - \theta_2^2 Var(e_{t-2}) \\
&= \phi_1^2\gamma_0 + \sigma_e^2 + \theta_1^2\sigma_2^2 - \theta_2^2\sigma_e^2 \\
\gamma_0 &= \phi_1^2\gamma_0 + \sigma_e^2(1 + \theta_1^2 - \theta_2^2) \\
\gamma_0 - \phi_1^2\gamma_0 &= \sigma_e^2(1 + \theta_1^2 - \theta_2^2) \\
\gamma_0(1 - \phi_1^2) &= \sigma_2^2(1 + \theta_1^2 - \theta_2^2) \\
\gamma_0 &= \frac{\sigma_e^2(1 + \theta_1^2 - \theta_2^2)}{1 - \phi_1^2}
\end{aligned}$$

Substituting in values for $\phi_1, \theta_1, \theta_2$, and $\sigma_e^2 = 1$ we have

$$\begin{aligned}
\gamma_0 &= \frac{1 + (0.8)^2 - (0.6)^2}{1 - (0.7)^2} \\
&= \frac{1 + (0.8)^2 - (0.6)^2}{1 - (0.7)^2} \\
&= \frac{1 + 0.64 - 0.36}{1 - 0.49} \\
&= \frac{1 + 0.64 - 0.36}{1 - 0.49} \\
&= \frac{1.28}{0.51} \\
&\approx 2.5098
\end{aligned}$$

Now we need γ_k for $k = 1, 2$.

$$\begin{aligned}
\gamma_1 &= Cov(Y_t, Y_t - k) \\
&= \phi_1 Cov(Y_{t-1}, Y_{t-k}) + \theta_1\sigma_e^2 - \theta_2\sigma_e^2 \\
&= \phi_1\gamma_0 + \theta_1\sigma_e^2 - \theta_2\sigma_e^2 \\
\gamma_2 &= \phi_1\gamma_1 - \theta_2\sigma_e^2 \\
\gamma_k &= \phi_1\gamma_{k-1}
\end{aligned}$$

With substituted values, we have

$$\begin{aligned}
\gamma_1 &= 0.7(2.5098) + 0.8(1) - 0.6(1) \\
&\approx 1.9569 \\
\gamma_2 &= 0.7(1.9569) - 0.6(1) \\
&\approx 0.7698
\end{aligned}$$

Thus, ρ_k is

$$\begin{aligned}
\rho_k &= \frac{\gamma_k}{\gamma_0} \\
&= \frac{\phi_1 \gamma_{k-1}}{\frac{\sigma_e^2(1+\theta_1^2-\theta_2^2)}{1-\phi_1^2}} \\
&= \frac{0.7\gamma_{k-1}}{2.5098} \\
&\approx 0.2789\gamma_{k-1}
\end{aligned}$$

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4. Consider a "AR(1)" process satisfying $Y_t = \phi Y_{t-1} + e_t$, where $t > 0$, ϕ can be any number and $\{e_t\}$ is a white noise process with zero mean and variance σ_e^2. Let Y_0 be a random variable with mean μ and variance σ_0^2. Show that for $t > 0$ we have

(a) $Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \ldots + \phi^{t-1} e_1 + \phi^t Y_0$

Answer:

Using recursion for a simple case, we have

$$\begin{aligned} Y_t &= \phi Y_{t-1} + e_t \\ &= \phi(\phi Y_{t-2} + e_{t-1}) + e_t \\ &= \phi(\phi(\phi Y_{t-3} + e_{t-2}) + e_{t-1}) + e_t \\ &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 Y_{t-3} \end{aligned}$$

Extending this down to Y_0, we get

$$\begin{aligned} Y_t &= \phi Y_{t-1} + e_t \\ &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \ldots + \phi^{t-1} e_1 + \phi^t Y_0 \end{aligned}$$

(b) $E[Y_t] = \phi^t \mu$**.**

Using the result from (a), we get

$$\begin{aligned} E[Y_t] &= E[\phi Y_{t-1} + e_t] \\ &= E[e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \ldots + \phi^{t-1} e_1 + \phi^t Y_0] \\ &= E[e_t] + E[\phi e_{t-1}] + E[\phi^2 e_{t-2}] + E[\phi^3 e_{t-3}] + \ldots + E[\phi^{t-1} e_1] + E[\phi^t Y_0] \\ &= 0 + \phi \cdot 0 + \phi^2 \cdot 0 + \phi^3 \cdot 0 + \ldots + \phi^{t-1} \cdot 0 + \phi^t E[Y_0] \\ &= \phi^t E[Y_0] \\ &= \phi^t \mu \end{aligned}$$

(c)

$$Var(Y_t) = \begin{cases} \frac{1-\phi^{2t}}{1-\phi^2}\sigma_e^2 + \phi^{2t}\sigma_0^2 & \text{for } \phi \neq 1 \\ t\sigma_e^2 + \sigma_0^2 & \text{for } \phi = 1 \end{cases}$$

Similarly,

$$\begin{aligned} Var(Y_t) &= Var(\phi Y_{t-1} + e_t) \\ &= Var(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \ldots + \phi^{t-1} e_1 + \phi^t Y_0) \\ &= Var(e_t) + Var(\phi e_{t-1}) + Var(\phi^2 e_{t-2}) + Var(\phi^3 e_{t-3}) + \ldots + Var(\phi^{t-1} e_1) + Var(\phi^t Y_0) \\ &= Var(e_t) + \phi^2 Var(e_{t-1}) + \phi^4 Var(e_{t-2}) + \phi^6 Var(e_{t-3}) + \ldots + \phi^{2(t-1)} Var(e_1) + \phi^{2t} Var(Y_0) \\ &= \sigma_e^2 + \phi^2\sigma_e^2 + \phi^4\sigma_e^2 + \phi^6\sigma_e^2 + \ldots + \phi^{2(t-1)}\sigma_e^2 + \phi^{2t}\sigma_0^2 \end{aligned}$$

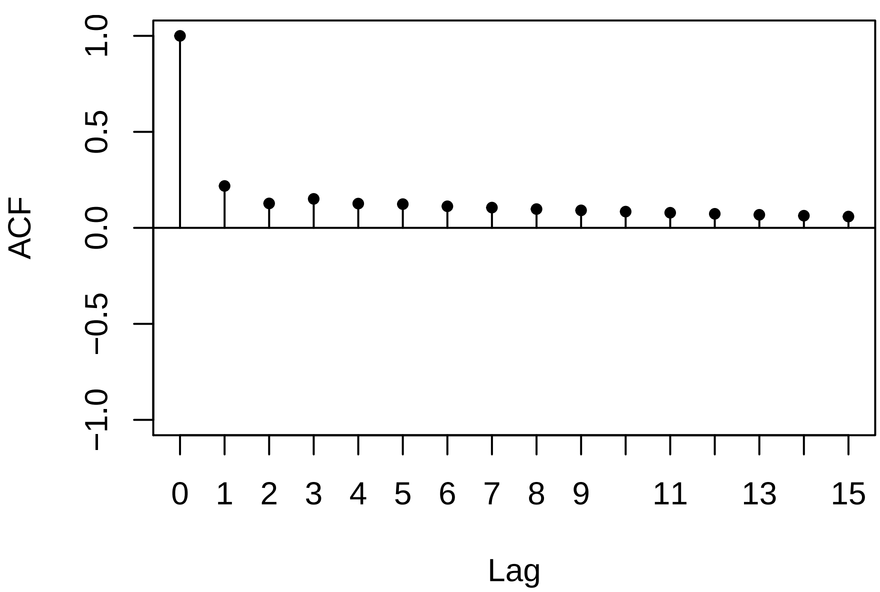
Letting $\phi = 1$, it is clear that $Var(Y_t) = t\sigma_e^2 + \sigma_0^2$.

If $\phi \neq 1$, we can see that $Var(Y_t) = \sigma_e^2(1 + \phi^2 + \phi^4 + \ldots + \phi^{2(t-1)}) + \phi^{2t}\sigma_0^2$. The expanded series identity for $(1 + \phi^2 + \phi^4 + \ldots + \phi^{2(t-1)})$ is $\frac{1-\phi^{2t}}{1-\phi^2}$, and thus, for $\phi \neq 1$, we have $Var(Y_t) = \frac{1-\phi^{2t}}{1-\phi^2}\sigma_e^2 + \phi^{2t}\sigma_0^2$.

(d) Suppose $\mu = 0$. Show that if $\{Y_t\}$ is stationary, then $Var(Y_t) = \frac{\sigma_e^2}{1-\phi^2}$.

5. The following command in R will plot the theoretical autocorrelation function of an ARMA(2, 2) model $Y_t = 0.5Y_{t-1} + 0.4Y_{t-2} + e_t - 0.7e_{t-1} - 0.6e_{t-2}$ for the first 15 lags:

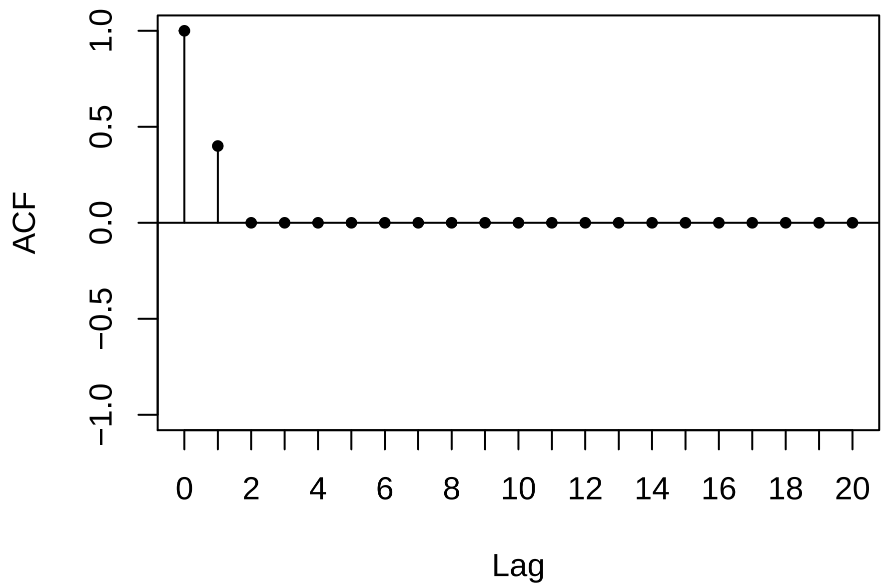
```r
n <- 15
ACF <- ARMAacf(ar = c(0.5, 0.4), ma = c(-0.7, -0.6), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



Modify the code to generate the theoretical autocorrelation functions up to 20 lags of the following ARMA processes:

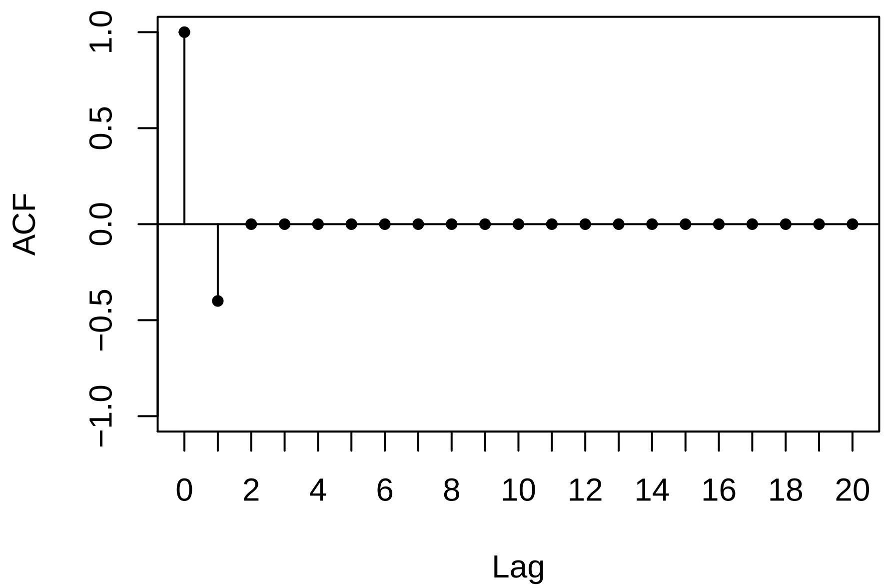
(a) MA(1) with $\theta = 0.5$

```r
n <- 20
ACF <- ARMAacf(ma = c(0.5), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



(b) MA(1) with $\theta = -0.5$

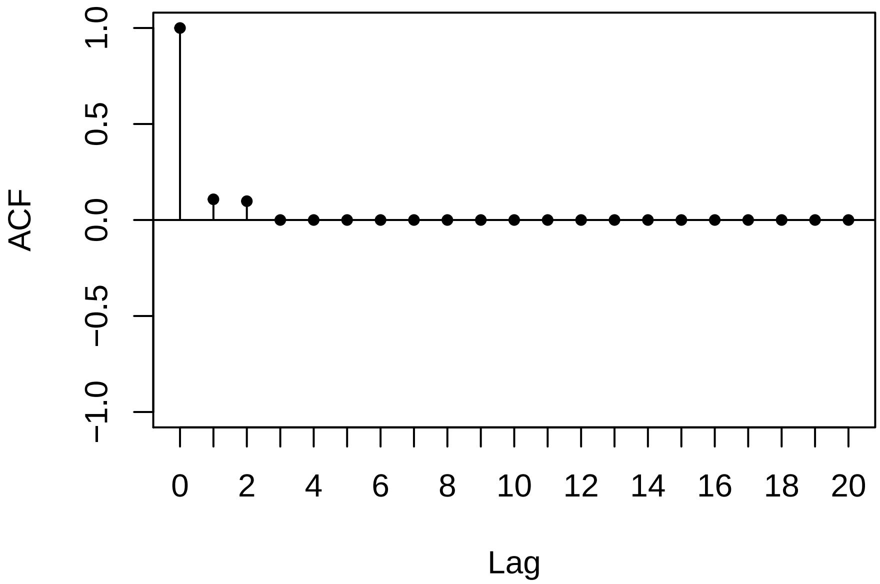
```
n <- 20
ACF <- ARMAacf(ma = c(-0.5), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



(c) MA(2) with $\theta_1 = \theta_2 = 0.1$

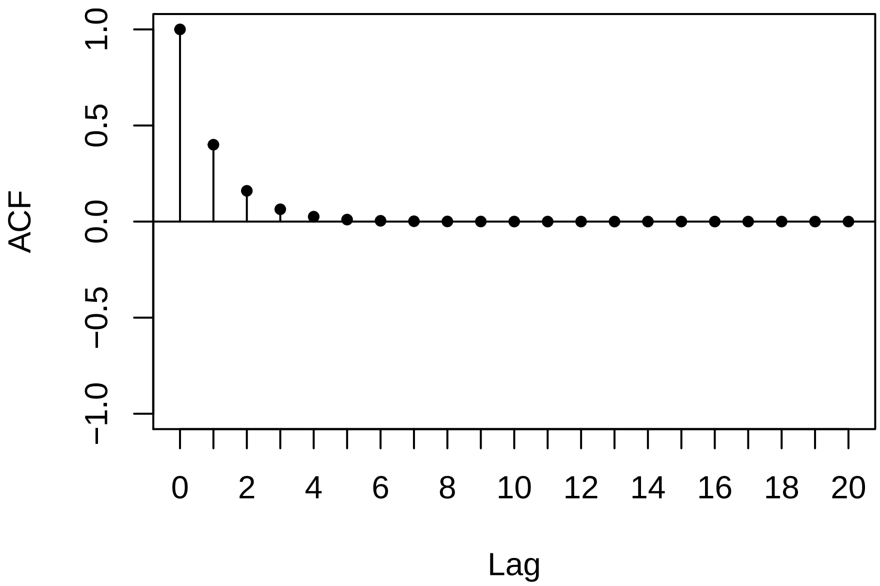
```
n <- 20
ACF <- ARMAacf(ma = c(0.1, 0.1), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
```

```r
points(0:n, ACF, pch = 20)
abline(h = 0)
```



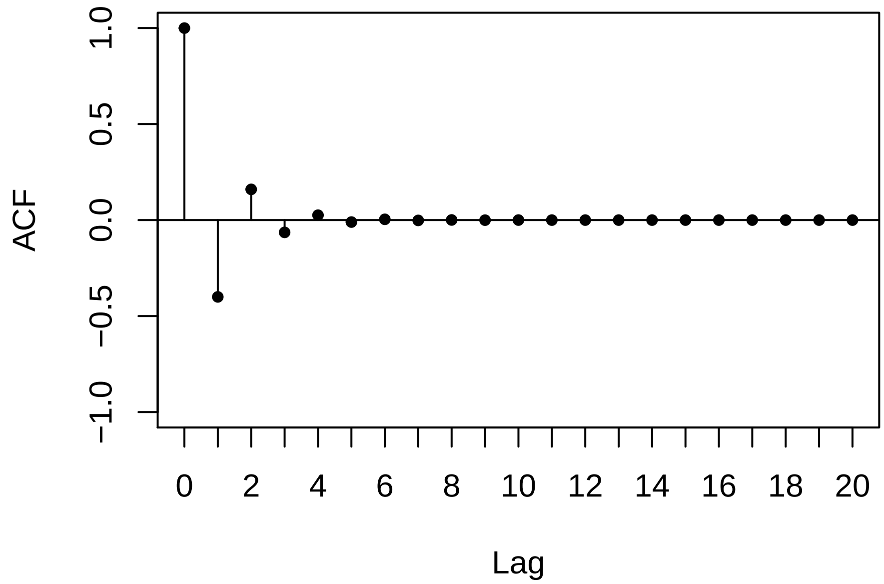
(d) AR(1) with $\phi = 0.4$

```r
n <- 20
ACF <- ARMAacf(ar = c(0.4), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



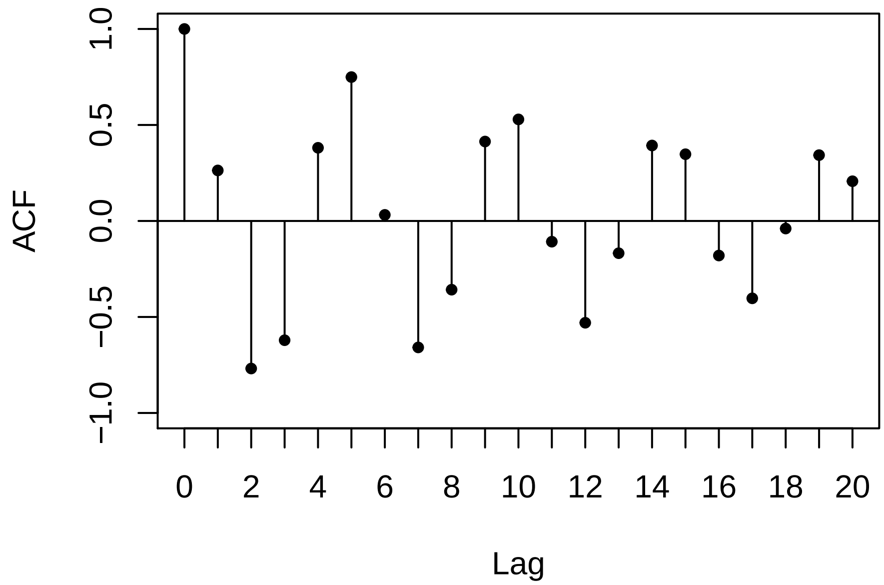
(e) AR(1) with $\phi = -0.4$

```
n <- 20
ACF <- ARMAacf(ar = c(-0.4), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



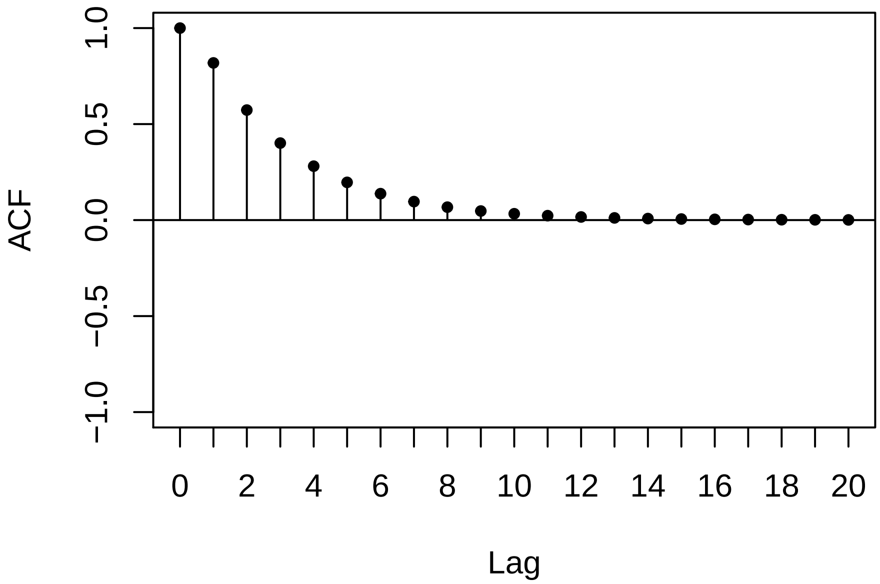
(f) AR(2) with $\phi_1 = 0.5$ and $\phi_2 = -0.9$

```
n <- 20
ACF <- ARMAacf(ar = c(0.5, -0.9), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



(g) ARMA(1, 1) with $\phi = 0.7$ and $\theta = 0.4$

```r
n <- 20
ACF <- ARMAacf(ar = c(0.7), ma = c(0.4), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



(h) ARMA(1, 2) given in Question 3

```r
n <- 20
ACF <- ARMAacf(ar = c(0.7), ma = c(0.8, -0.6), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```