# Time Series Cheatsheet

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*January 24, 2017*

## Mean

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f(x) dx \text{ for } \int_{-\infty}^{\infty} |x| f(x) dx < \infty, \text{ undefined otherwise} \\
\mu_t &= E[Y_t] \text{ for } t = 0, \pm 1, \pm 2, \dots \\
E[aX + bY + c] &= aE[X] + bE[Y] + c \\
E[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y) dx dy
\end{aligned}
$$

## Variance

$$
\begin{aligned}
Var(X) &\geq 0 \text{ non-negative} \\
Var(a + bX) &= b^2 Var(X) \\
Var(X + Y) &= Var(X) + Var(Y) \text{ for independent } X, Y \\
Var(X) &= E[X^2] - \big(E[X]\big)^2
\end{aligned}
$$

## Covariance

$$
\begin{aligned}
Cov(a + bX, c + dY) &= bdCov(X, Y) \\
Var(X + Y) &= Var(X) + Var(Y) + 2Cov(X, Y) \\
Cov(X + Y, Z) &= Cov(X, Z) + Cov(Y, Z) \\
Cov(X, X) &= Var(X) \\
Cov(X, Y) &= Cov(Y, X) \\
Cov(X, Y) &= 0 \text{ for independent } X, Y \\
Cov(X, Y) &= E\big[(X - E[X])(Y - E[Y])\big] \\
&= E\big[(X - \mu_X)(Y - \mu_Y)\big] \\
&= E\big[XY - XE[Y] - YE[X] + E[X]E[Y]\big] \\
&= E[XY] - E\big[XE[Y]\big] - E\big[YE[X]\big] + E\big[E[X]E[Y]\big] \\
&= E[XY] - E[X]E[Y]
\end{aligned}
$$

## Correlation

If $X^*$ is a standardized $X$ and $Y^*$ is a standardized $Y$, then

$$\rho = [X^* Y^*]$$

Else,

$$\rho = Corr(X, Y) = \frac{Corr(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

$$
\begin{aligned}
-1 \leq Corr(X, Y) \leq 1 \\
Corr(a + bX, c + dY) = sign(bd)Corr(X, Y) \\
Corr(X, Y) = \pm 1 iff \exists a, b \text{ s.t. } P(Y = a + bX) = 1
\end{aligned}
$$

## Events

$$
\begin{aligned}
e_i &= \text{event} i \\
Y_0 &= e_0 \\
Y_t &= e_0 + e_1 + e_2 + \ldots + e_t
\end{aligned}
$$

## Autocovariance

$$
\begin{aligned}
\gamma_{t,t} &= Var(Y_t) \\
\gamma_{t,s} &= \gamma_{s,t} \\
\left|\gamma_{t,s}\right| &\leq \sqrt{\gamma_{t,t}\gamma_{s,s}}
\end{aligned}
$$

$$
\begin{aligned}
\gamma_{t,s} &= Cov(Y_t, Y_s) \text{ for } t, s = 0, \pm 1, \pm 2, \ldots \\
Cov(X, Y) &= E\big[(X - \mu_X)(Y - \mu_Y)\big] \\
&= E[XY] - E[X]E[Y]
\end{aligned}
$$

$$
\begin{aligned}
Cov(Y_t, Y_s) &= E\big[(Y_t - \mu_t)(Y_s - \mu_s)\big] \\
&= E[Y_t Y_s] - \mu_t \mu_s
\end{aligned}
$$

For stochastic process $e_1, e_2, \ldots$ with mean 0, variance $\sigma^2$,

$$
\begin{aligned}
\gamma_{t,t} &= Var(Y_t) \\
&= Var(\sum_{i=1}^{t} e_i) \\
&= \sum_{i=1}^{t} Var(e_i) \\
&= t\sigma^2 \\
\gamma_{t,s} &= Cov(Y_t, Y_s) \\
&= Cov(e_1 + e_2 + \ldots + e_t, e_1, e_2, \ldots + e_s) \\
&= Cov\big([\sum_{i=1}^{n} c_i Y_{t_i}, \sum_{j=1}^{n} d_j Y_{s_j}]\big) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{n} c_i d_j Cov(Y_{t_i}, Y_{s_j}) \\
&= \sum_{i=1}^{t} \sum_{j=1}^{s} Cov(e_i, e_j) \text{ (when } i = j \text{, you get variance, when } i \neq j \text{ you get 0)} \\
&\text{for } 1 \leq t \leq s \\
&= t\sigma^2
\end{aligned}
$$

If $c_1, c_2, \ldots, c_m$ and $d_1, d_2, \ldots, d_n$ are constants and $t_1, t_2, \ldots, t_m$ and $s_1, s_2, \ldots, s_n$ are time points, then

$$Cov\left[\sum_{i=1}^{m} c_i Y_{t_i}, \sum_{j=1}^{n} d_j Y_{s_j}\right] = \sum_{i=1}^{m} c_i d_j Cov(Y_{t_i}, Y_{s_j})$$

**Special case**

$$Var\left[\sum_{i=1}^{n} c_i Y_{t_i}\right] = \sum_{i=1}^{n} c_i^2 Var(Y_{t_i}) + 2\sum_{i=2}^{n}\sum_{j=1}^{i-1} c_i c_j Cov(Y_{t_i}, Y_{t_j})$$

## Autocorrelation

$$\begin{aligned}\rho_{t,t} &= 1\\ \rho_{t,s} &= \rho_{s,t}\\ \left|\rho_{t,s}\right| &\leq 1\end{aligned}$$

$$\rho_{t,s} = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}, \gamma_{s,s}}}$$

## Strictly Stationary

The joint distribution of $Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}$ is the same as $Y_{t_1-k}, Y_{t_2-k}, \ldots, Y_{t_n-k} \forall t_i, i = 1, 2, \ldots, n$ and lag $k$. Also, $Var(Y_t) = Var(Y_{t-k})$ (constant variance over time).

$Cov(Y_{t-k}, Y_{s-k}) \forall t, s$ and $k$. Putting $k = s$ and $k = t$, we get

$$\begin{aligned}\gamma_{t,s} &= Cov(Y_{t-s}, Y_0)\\ &= Cov(Y_0, Y_{s-t})\\ &= Cov(Y_0, Y_{|t-s|})\\ &= \gamma_{0,|t-s|}\\ \gamma_k &= Cov(Y_t, Y_{t-k})\\ \rho_k &= Corr(Y_t, Y_{t-k})\\ \rho_k &= \frac{\gamma_k}{\gamma_0}\\ \gamma_0 &= Var(Y_t)\\ \gamma_k &= \gamma_{-k}\\ |\gamma_k| &\leq \gamma_0\\ \rho_0 &= 1\\ \rho_k &= \rho_{-k}\\ |\rho_k| &\leq 1\end{aligned}$$

## Weakly Stationary

1. Mean function is **constant over time**
2. $\gamma_{t,t-k} = \gamma_{0,k}$ for all time t and lag k