Homework Assignment 2

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Note: $\{e_t\}$ always denotes a sequence of independent, identically distributed random variables with mean zero and constant variance σ^2

1. If $Y_t = e_t - e_{t-7}$, show that $\{Y_t\}$ is (weakly) stationary and that, for $k > 0$, its autocorrelation function is nonzero only for lag $k = 7$.

Answer:

For stochastic process $\{Y_t\}$ to be weakly stationary, two properties must hold: The mean function must be zero, and the variance must be constant for any time window, $Y_{t,t-k}$, for all t, k.

It is trivial to show that the mean function, $\mu_t = E[Y_t]$ is zero, due to independence.

$$
\begin{aligned}
E[Y_t] &= E[e_t - e_{t-7}] \\
&= E[e_t] - E[e_{t-7}] \\
&= 0 - 0 = 0
\end{aligned}
$$

To show constant variance, we show the following identity:

$$
\begin{aligned}
\gamma_{t,s} &= Cov(Y_t, Y_s) \\
&= Cov(e_t - e_{t-7}, e_s - e_{s-7}) \\
&= Cov(e_t, e_s) - Cov(e_t, e_{s-7}) - Cov(e_{t-7}, e_s) + Cov(e_{t-7}, e_{s-7})
\end{aligned}
$$

If $t = s$, we have

$$
\begin{aligned}
\gamma_{s,s} &= Cov(Y_s, Y_s) \\
&= Cov(e_s, e_s) - Cov(e_s, e_{s-7}) - Cov(e_{s-7}, e_s) + Cov(e_{s-7}, e_{s-7}) \\
&= Var(e_s) - 0 - 0 + Var(e_{s-7}) \\
&= \sigma^2 + \sigma^2 \\
&= 2\sigma^2
\end{aligned}
$$

And if $t = s \pm 7$, we have

$$
\begin{aligned}
\gamma_{s+7,s} &= Cov(Y_{s+7}, Y_s) \\
&= Cov(e_{s+7}, e_s) - Cov(e_{s+7}, e_{s-7}) - Cov(e_{s+7-7}, e_s) + Cov(e_{s+7-7}, e_{s-7}) \\
&= 0 - 0 - Var(e_s, e_s) - 0 \\
&= -Var(e_s, e_s) \\
&= -\sigma^2 \\
\gamma_{s-7,s} &= Cov(Y_{s-7}, Y_s) \\
&= Cov(e_{s-7}, e_s) - Cov(e_{s-7}, e_{s-7}) - Cov(e_{s-7-7}, e_s) + Cov(e_{s-7-7}, e_{s-7}) \\
&= 0 - 0 - Var(e_{s-7}, e_{s-7}) - 0 \\
&= -Var(e_{s-7}, e_{s-7}) \\
&= -\sigma^2
\end{aligned}
$$

Thus, we identify three scenarios,

$$
\begin{cases}
2\sigma^2 & \text{for } t = s \\
-\sigma^2 & \text{for } t = s \pm 7 \\
0 & \text{otherwise}
\end{cases}
$$

To show that the autocorrelation function is non-zero only for lag $k = 7$, we'll first show that it is in fact non-zero for lag $k = 7$. We again have three scenarios, as was shown previously.

First, we'll find $\gamma_{t,t}$ and $\gamma_{s,s}$.

$$
\begin{aligned}
\gamma_{t,t} &= Cov(Y_t, Y_t) = Var(Y_t) \\
&= Var(e_t - e_{t-7}) \\
&= E[(e_t - e_{t-7})(e_t - e_{t-7})] - \big(E[e_t - e_{t-7}]\big)^2 \\
&= E[e_t^2 - e_t e_{t-7} + e_{t-7} e_t + e_{t-7}^2] \\
&= E[e_t^2 + e_{t-7}^2] \\
&= E[e_t^2] + E[e_{t-7}^2] \\
&= 2\sigma^2
\end{aligned}
$$

The result is the same for $\gamma_{s,s}$. For $t = s$ we have

$$
\begin{aligned}
\rho_{t,s} &= \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} \\
&= \frac{2\sigma^2}{\sqrt{2\sigma^2 2\sigma^2}} \\
&= \frac{2\sigma^2}{2\sigma^2} \\
&= 1
\end{aligned}
$$

For $t = s \pm 7$, we have

$$\begin{aligned}
\rho_{t,s} &= \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} \\
&= \frac{-\sigma^2}{\sqrt{2\sigma^2 2\sigma^2}} \\
&= \frac{-\sigma^2}{2\sigma^2} \\
&= -\frac{1}{2}
\end{aligned}$$

However, these are the only values for which lag $k = 7$. For any other value of t, lag $k \neq 7$, and thus,

$$\begin{aligned}
\rho_{t,s} &= \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} \\
&= \frac{0}{\sqrt{2\sigma^2 2\sigma^2}} \\
&= \frac{0}{2\sigma^2} \\
&= 0
\end{aligned}$$

$\blacksquare$

2. Suppose that $\{Y_t\}$ is (weakly) stationary with autocovariance function γ_k.

(a) Show that $W_t = \nabla Y_t = Y_t - Y_{t-1}$ is (weakly) stationary by finding the mean and autocovariance function for $\{W_t\}$.

Answer:

Since we are given that Y_t is weakly stationary, then its mean function must be zero. Therefore,

$$\begin{aligned} E[W_t] &= E[\nabla Y_t] \\ &= E[Y_t - Y_{t-1}] \\ &= E[Y_t] - E[Y_{t-1}] \\ &= 0 - 0 = 0 \end{aligned}$$

We are given that Y_t has autocovariance function $\gamma_k = Cov(Y_t, Y_{t-k})$, so we can extend this to Y_{t-1} with autocovariance function $\gamma_{k-1} = Cov(Y_{t-1}, Y_{t-1-k})$.

$$\begin{aligned} Cov(Y_t, Y_{t-k}) &= Cov(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-1-k}) \\ &= Cov(Y_t, Y_{t-k}) - Cov(Y_t, Y_{t-1-k}) - Cov(Y_{t-1}, Y_{t-k}) + Cov(Y_{t-1}, Y_{t-1-k}) \\ &= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k \\ &= 2\gamma_k - \gamma_{k+1} - \gamma_{k-1} \end{aligned}$$

Thus, the variance is constant for equal time windows and W_t is weakly stationary.

(b) Show that $U_t = \nabla^2 Y_t = \nabla[\nabla Y_t] = \nabla[Y_t - Y_{t-1}] = Y_t - 2Y_{t-1} + Y_{t-2}$ is (weakly) stationary.

We have already shown that $W_t = \nabla Y_t$ is stationary. U_t is simply the difference of two W_t processes, and the difference of two stationary processes is also stationary.

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3. For a fixed positive integer s and a constant α, consider the stochastic process defined by $Y_t = e_t + \alpha e_{t-1} + \alpha^2 e_{t-2} + \cdots + \alpha^s e_{t-s}$.

(a) Show that the process is (weakly) stationary for any value of α.

The mean function is trivial due to independence,

$$\begin{aligned} E[Y_t] &= E[e_t] + \alpha E[e_{t-1}] + \alpha^2 E[e_{t-2}] + \cdots + \alpha^s E[e_{t-s}] \\ &= 0 + 0 + 0 + \cdots + 0 = 0 \end{aligned}$$

Autocovariance is given by

$$\begin{aligned} \gamma_{t,r} &= Cov(Y_t, Y_r) \\ &= Cov(e_t + \alpha e_{t-1} + \alpha^2 e_{t-2} + \cdots + \alpha^s e_{t-s}, e_r + \alpha e_{r-1} + \alpha^2 e_{r-2} + \cdots + \alpha^s e_{r-s}) \\ &= Cov\left(\sum_{i=0}^{s} \alpha^i e_{t-i}, \sum_{j=0}^{s} \alpha^j e_{t-j}\right) \\ &= \sum_{i=0}^{s}\sum_{j=0}^{s} \alpha^i \alpha^j Cov(e_{t-i}, e_{t-j}) \\ &= \sum_{i=0}^{s}\sum_{j=0}^{s} \alpha^i \alpha^j Cov(e_{t-i}, e_{t-j}) \end{aligned}$$

Now, when $i = j$ we get $Cov(e_{t-i}, e_{t-j}) = \sigma^2$, and when $i \neq j$, we get 0. Thus, we get the following matrix,

$$\begin{bmatrix} \sigma^2 & 0 & 0 & \dots & 0 & \\ 0 & \alpha^2\sigma^2 & 0 & \dots & 0 & \\ 0 & 0 & \alpha^4\sigma^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & \alpha^{2s}\sigma^2 & \end{bmatrix}$$

The summation over all of these leads to $\alpha^0\sigma^2 + \alpha^2\sigma^2 + \alpha^4\sigma^2 + \ldots + \alpha^{2s}\sigma^2 = \sigma^2\left(\sum_{i=0}^{s} \alpha^{2s}\right)$.

Thus for any value of α or s, the variance is the same for any time window t, r, and thus the process $\{Y_t\}$ is stationary.

(b) Find the autocorrelation function.

Autocorrelation is given by $\rho_{t,r} = \frac{\gamma_{t,r}}{\sqrt{\gamma_{t,t}\gamma_{r,r}}}$.

$$\begin{aligned} \gamma_{t,t} &= Var(Y_t) \\ &= Var(e_t + \alpha e_{t-1} + \alpha^2 e_{t-2} + \ldots + \alpha^s e_{t-s}) \\ &= Var(e_t) + \alpha^2 Var(e_{t-1}) + \alpha^4 Var(e_{t-2}) + \ldots + \alpha^{2s} Var(e_{t-s}) \\ &= \sigma^2 + \alpha^2\sigma^2 + \alpha^4\sigma^2 + \ldots + \alpha^{2s}\sigma^2 \\ &= \sigma^2\left(\sum_{i=0}^{s} \alpha^{2s}\right) \end{aligned}$$

The same holds for $\gamma_{r,r}$.

So we have $\rho_{t,r} = \frac{\gamma_{t,r}}{\sqrt{\gamma_{t,t}\gamma_{r,r}}} \frac{\sigma^2\left(\sum_{i=0}^{s}\alpha^{2s}\right)\sigma^2}{\sqrt{\sigma^2\left(\sum_{i=0}^{s}\alpha^{2s}\right)\sigma^2\left(\sum_{i=0}^{s}\alpha^{2s}\right)}} = 1.$

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4. Let $\{X_t\}$ be the time series of interest, however, due to measurement error we actually observe $Y_t = X_t + e_t$. We assume that $\{X_t\}$ and $\{e_t\}$ are independent processes. We call X_t the signal, and e_t the noise. If $\{X_t\}$ is stationary with autocorrelation function ρ_k, show that $\{Y_t\}$ is also stationary with $Corr(Y_t, Y_{t-k}) = \frac{\rho_k}{(1+\sigma_e^2/\sigma_X^2)}$, where σ_e^2/σ_X^2 is called the signal-to-noise ratio.

Since e_t has mean 0, then

$$\begin{aligned} E[Y_t] &= E[X_t + e_t] \\ &= E[X_t] + E[e_t] \\ &= E[X_t] + 0 = \mu_X \end{aligned}$$

Now, since $\rho_k = \frac{\gamma_k}{\gamma_0}$, we have

$$\begin{aligned} Corr(Y_t, Y_{t-k}) &= \frac{\rho_k}{(1+\sigma_e^2/\sigma_X^2)} \\ &= \frac{\frac{\gamma_k}{\gamma_0}}{(1+\sigma_e^2/\sigma_X^2)} \end{aligned}$$

Solving the gammas we find,

$$\begin{aligned} \gamma_0 &= Var(Y_t) \\ &= Var(X_t + e_t) \\ &= Var(X_t) + Var(e_t) \\ &= \sigma_X^2 + \sigma_e^2 \end{aligned}$$

$$\begin{aligned} \gamma_k &= Cov(Y_t, Y_{t-k}) \\ &= Cov(X_t + e_t, X_{t-k} + e_{t-k}) \\ &= Cov(X_t, X_{t-k}) + Cov(X_t, e_{t-k}) + Cov(e_t, X_{t-k}) + Cov(e_t, e_{t-k}) \\ &= Cov(X_t, X_{t-k}) + 0 + 0 + Cov(e_t, e_{t-k}) \\ &= Cov(X_t, X_{t-k}) + Cov(e_t, e_{t-k}) \\ &= \sigma_X^2 + \sigma_e^2 \text{ for } k = 0, \sigma_X^2 \text{ for } k > 0 \end{aligned}$$

Note that $\rho_k = Corr(Y_t, Y_{t-k}) = \frac{Cov(Y_t, Y_{t-k})}{\sqrt{Var(Y_t)Var(Y_{t-k})}} = \frac{Cov(X_t, X_{t-k})}{\sigma_X^2 + \sigma_e^2}$. Thus, we can write the Covariance as,

$$\begin{aligned} \gamma_k &= Cov(Y_t, Y_{t-k}) \\ &= (\sigma_X^2 + \sigma_e^2)\frac{Cov(Y_t, Y_{t-k})}{\sigma_X^2 + \sigma_e^2} \\ &= (\sigma_X^2 + \sigma_e^2)\rho_k \text{ for } k = 0, \sigma_X^2 \rho_k \text{ for } k > 0 \end{aligned}$$

Finally, we have two cases:

$$\begin{aligned} Corr(Y_t, Y_{t-k}) &= \frac{Cov(Y_t, Y_{t-k})}{Var(Y_t)} \\ &= \frac{(\sigma_X^2 + \sigma_e^2)\rho_k}{\sigma_X^2 + \sigma_e^2} \\ &= \frac{\rho_k}{(1 + \sigma_e^2/\sigma_X^2)} \\ &= \rho_k, \text{ for } k = 0 \end{aligned}$$

and

$$\begin{aligned} Corr(Y_t, Y_{t-k}) &= \frac{Cov(Y_t, Y_{t-k})}{Var(Y_t)} \\ &= \frac{\sigma_X^2 \rho_k}{\sigma_X^2 + \sigma_e^2} \\ &= \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_X^2}}, \text{ for } k > 0 \end{aligned}$$

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5. Suppose

$$Y_t = \beta_0 + \sum_{i=1}^{k} [A_i cos(2\pi f_i t) + B_i sin(2\pi f_i t)]$$

where $\beta_0, f_1, f_2, \ldots, f_k$ are constants, and $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k$ are independent random variables with zero means and variances $Var(A_i) = Var(B_i) = \sigma_i^2$. Show that $\{Y_t\}$ is (weakly) stationary by finding the mean and autocovariance function.

Finding the mean, we have

$$\begin{aligned}
E[Y_t] &= E\Bigg[\beta_0 + \sum_{i=1}^{k} [A_i cos(2\pi f_i t) + B_i sin(2\pi f_i t)]\Bigg] \\
&= \beta_0 + E\Bigg[\sum_{i=1}^{k} A_i cos(2\pi f_i t) + B_i sin(2\pi f_i t)\Bigg] \\
&= \beta_0 + E\Bigg[\sum_{i=1}^{k} A_i cos(2\pi f_i t)\Bigg] + E\Bigg[\sum_{i=1}^{k} B_i sin(2\pi f_i t)\Bigg] \\
&= \beta_0 + \sum_{i=1}^{k} cos(2\pi f_i t) E[A_i] + \sum_{i=1}^{k} sin(2\pi f_i t) E[B_i] \\
&= \beta_0 + 0 + 0 = \beta_0
\end{aligned}$$

Thus, the mean is constant over time.

We now find the covariance,

$$\begin{aligned}
Cov(Y_t, Y_s) &= Cov\Bigg[\beta_0 + \sum_{i=1}^{k} [A_i cos(2\pi f_i t) + B_i sin(2\pi f_i t)], \beta_0 + \sum_{j=1}^{k} [A_j cos(2\pi f_j s) + B_j sin(2\pi f_j s)]\Bigg] \\
&= Cov\Bigg[\sum_{i=1}^{k} A_i cos(2\pi f_i t) + B_i sin(2\pi f_i t), \sum_{j=1}^{k} A_j cos(2\pi f_j s) + B_j sin(2\pi f_j s)\Bigg] \\
&= \sum_{i=1}^{k} \sum_{j=1}^{k} Cov\Bigg[A_i cos(2\pi f_i t) + B_i sin(2\pi f_i t), A_j cos(2\pi f_j s) + B_j sin(2\pi f_j s)\Bigg] \\
&= \sum_{i=1}^{k} \sum_{j=1}^{k} \Bigg[Cov(A_i cos(2\pi f_i t), A_j cos(2\pi f_j s)) + Cov(A_i cos(2\pi f_i t), B_j sin(2\pi f_j s)) \\
&+ Cov(B_i sin(2\pi f_i t), A_j cos(2\pi f_j s)) + Cov(B_i sin(2\pi f_i t), B_j sin(2\pi f_j s))\Bigg] \\
&= \sum_{i=1}^{k} \sum_{j=1}^{k} \Bigg[cos(2\pi f_i t) cos(2\pi f_j s) Cov(A_i, A_j) + cos(2\pi f_i t) sin(2\pi f_j s) Cov(A_i, B_j) \\
&+ sin(2\pi f_i t) cos(2\pi f_j s) Cov(B_i, A_j) + sin(2\pi f_i t) sin(2\pi f_j s) Cov(B_i, B_j)\Bigg]
\end{aligned}$$

If $i \neq j$, the covariance is zero, but if $i = j$ we have,

$$\begin{aligned}
Cov &= \sum_{i=1}^{k}\sum_{j=1}^{k}[cos(2\pi f_i t)cos(2\pi f_j s)\sigma^2 + cos(2\pi f_i t)sin(2\pi f_j s)\sigma^2 + sin(2\pi f_i t)cos(2\pi f_j s)\sigma^2 + sin(2\pi f_i t)sin(2\pi f_j s)\sigma^2] \\
&= \sigma^2 \sum_{i=1}^{k}\sum_{j=1}^{k}[cos(2\pi f_i t)cos(2\pi f_j s) + cos(2\pi f_i t)sin(2\pi f_j s) + sin(2\pi f_i t)cos(2\pi f_j s) + sin(2\pi f_i t)sin(2\pi f_j s)\Big] \\
&=
\end{aligned}$$