

# Homework 4

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1. Calculate and sketch the autocorrelation functions  $\rho_k$  for the following stationary processes.

(a)  $Y_t = -0.9Y_{t-1} + e_t$

**Answer:** For this AR(1) model, we let  $\phi = -0.9$ , such that

$$\begin{aligned} Y_t &= \phi Y_{t-1} + e_t \\ &= \phi(\phi Y_{t-2} + e_{t-1}) + e_t \\ &= \phi(\phi(Y_{t-3} + e_{t-2}) + e_{t-1}) + e_t \\ &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} \end{aligned}$$

Continuing this expansion indefinitely we get

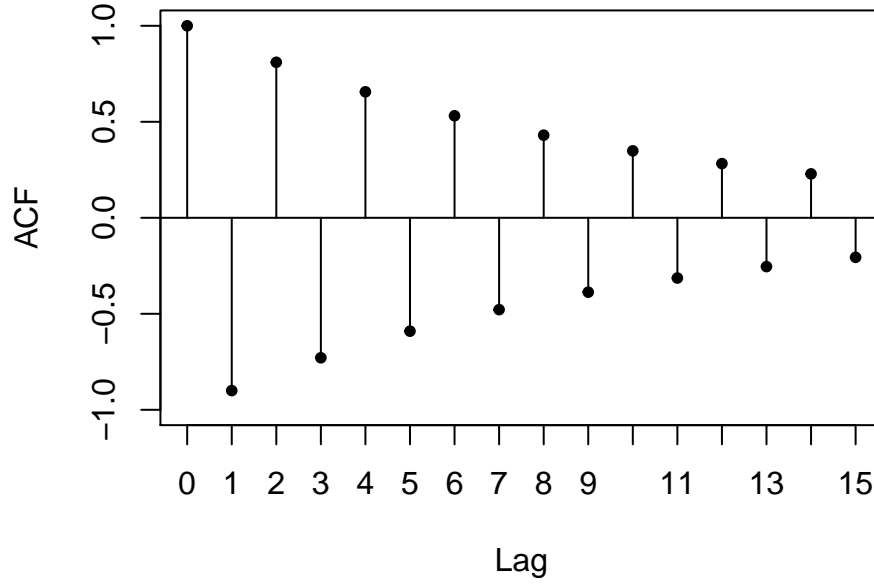
$$\begin{aligned} Y_t &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \dots \\ \rho_k &= \phi^k, s.t. |\rho_k| \leq 1 \end{aligned}$$

Substituting in our value of -0.9 for  $\phi$ , we get

$$\rho_k = -0.9^k$$

Such an autocorrelation function might look like this:

```
n <- 15
ACF <- ARMAacf(ar = c(-0.9), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



(b)  $Y_t = 8 + e_t - 0.75e_{t-1} + 0.5e_{t-2} - 0.25e_{t-3}$

**Answer:**

Looking at this MA(3) model, we have

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \theta_3 e_{t-3}$$

$$\rho_k = \begin{cases} 1 & \text{if } k = 0 \\ \frac{-\theta_1 + \theta_1 \theta_2 + \theta_2 \theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2} & \text{for } k \pm 1 \\ \frac{-\theta_2 + \theta_2 \theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2} & \text{for } k \pm 2 \\ \frac{-\theta_3}{1 + \theta_1^2 + \theta_2^2 + \theta_3^2} & \text{for } k \pm 3 \end{cases}$$

Letting  $\theta_1 = 0.75, \theta_2 = -0.5$  and  $\theta_3 = 0.25$ , we get

$$Y_t = e_t - 0.75e_{t-1} - (-0.5)e_{t-2} - 0.25e_{t-3}$$

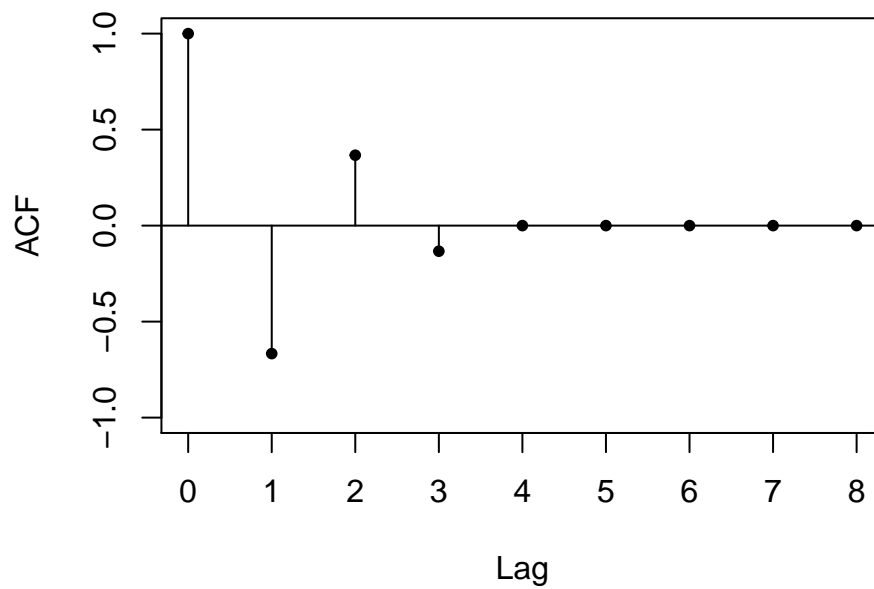
$$\rho_k = \begin{cases} 1 & \text{if } k = 0 \\ \frac{-0.75 + (0.75)(-0.5) + (-0.5)(0.25)}{1 + (0.75)^2 + (-0.5)^2 + (0.25)^2} = \frac{-1.25}{1.875} = -\frac{2}{3} & \text{for } k \pm 1 \\ \frac{-(-0.5) + (-0.5)(0.25)}{1 + (0.75)^2 + (-0.5)^2 + (0.25)^2} = \frac{0.375}{1.875} = \frac{1}{5} & \text{for } k \pm 2 \\ \frac{-(0.25)}{1 + (0.75)^2 + (-0.5)^2 + (0.25)^2} = \frac{-0.25}{1.875} = -\frac{2}{15} & \text{for } k \pm 3 \end{cases}$$

Such an autocorrelation function might look like this:

```

n <- 8
ACF <- ARMAacf(ma = c(-0.75, 0.5, -0.25), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)

```



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## 2. Verify that for an MA(1) process

$$\max_{-\infty < \theta < \infty} \rho_1 = 0.5 \text{ and } \min_{-\infty < \theta < \infty} \rho_1 = -0.5$$

**Answer:**

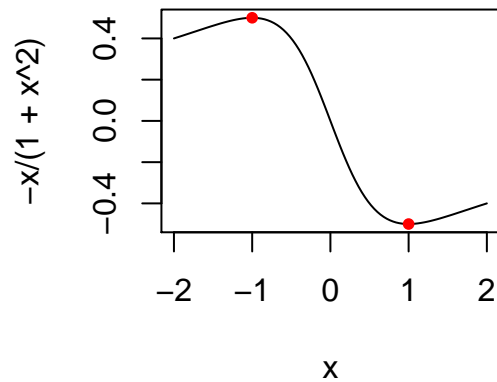
For MA(1), using  $k = 1$ , we know  $\rho_1 = \frac{-\theta}{1+\theta^2}$

We can find the global maxima and minima at the inflection points by taking the derivative and setting it equal to zero.

$$\begin{aligned} \frac{-\theta}{1+\theta^2} \frac{d}{d\theta} &= \frac{t^2 - 1}{(t^2 + 1)^2} \\ \frac{t^2 - 1}{(t^2 + 1)^2} &= 0 \\ \frac{t^2}{(t^2 + 1)^2} - \frac{1}{(t^2 + 1)^2} &= 0 \\ \frac{t^2}{(t^2 + 1)^2} &= \frac{1}{(t^2 + 1)^2} \\ t^2 &= 1 \\ t &= \pm 1 \end{aligned}$$

Now we just need to evaluate at  $t = \pm 1$  and we can see the global maximum and minimum:

$$\begin{aligned} \frac{-1}{1+1^2} &= \frac{-1}{2} \\ \frac{-(-1)}{1+(-1)^2} &= \frac{1}{2} \end{aligned}$$



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### 3. Consider the ARMA(1, 2) model

$$Y_t = 0.7Y_{t-1} + e_t + 0.8e_{t-1} - 0.6e_{t-2}$$

Assume that  $\{e_t\}$  is a white noise process with zero mean and unit variance ( $\sigma_e^2 = 1$ ). Find the numerical values of  $\rho_0, \rho_1$  and  $\rho_2$  by hand. Also find a recursive relationship between  $\rho_k$  and  $\rho_{k-1}$  for  $k > 2$ .

**Answer:**

As always,  $\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$ .

$$\begin{aligned}\rho_1 &= \frac{\gamma_1}{\gamma_0} \\ &= \end{aligned}$$

4. Consider a “AR(1)” process satisfying  $Y_t = \phi Y_{t-1} + e_t$ , where  $t > 0$ ,  $\phi$  can be any number and  $\{e_t\}$  is a white noise process with zero mean and variance  $\sigma_e^2$ . Let  $Y_0$  be a random variable with mean  $\mu$  and variance  $\sigma_0^2$ . Show that for  $t > 0$  we have

(a)  $Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots + \phi^{t-1} e_1 + \phi^t Y_0$

**Answer:**

Using recursion for a simple case, we have

$$\begin{aligned}Y_t &= \phi Y_{t-1} + e_t \\ &= \phi(\phi Y_{t-2} + e_{t-1}) + e_t \\ &= \phi(\phi(\phi Y_{t-3} + e_{t-2}) + e_{t-1}) + e_t \\ &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 Y_{t-3}\end{aligned}$$

Extending this down to  $Y_0$ , we get

$$\begin{aligned}Y_t &= \phi Y_{t-1} + e_t \\ &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \dots + \phi^{t-1} e_1 + \phi^t Y_0\end{aligned}$$

(b)  $E[Y_t] = \phi^t \mu$ .

Using the result from (a), we get

$$\begin{aligned}
E[Y_t] &= E[\phi Y_{t-1} + e_t] \\
&= E[e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \dots + \phi^{t-1} e_1 + \phi^t Y_0] \\
&= E[e_t] + E[\phi e_{t-1}] + E[\phi^2 e_{t-2}] + E[\phi^3 e_{t-3}] + \dots + E[\phi^{t-1} e_1] + E[\phi^t Y_0] \\
&= 0 + \phi \cdot 0 + \phi^2 \cdot 0 + \phi^3 \cdot 0 + \dots + \phi^{t-1} \cdot 0 + \phi^t E[Y_0] \\
&= \phi^t E[Y_0] \\
&= \phi^t \mu
\end{aligned}$$

(c)

$$\text{Var}(Y_t) = \begin{cases} \frac{1-\phi^{2t}}{1-\phi^2} \sigma_e^2 + \phi^{2t} \sigma_0^2 & \text{for } \phi \neq 1 \\ t\sigma_e^2 + \sigma_0^2 & \text{for } \phi = 1 \end{cases}$$

Similarly,

$$\begin{aligned}
\text{Var}(Y_t) &= \text{Var}(\phi Y_{t-1} + e_t) \\
&= \text{Var}(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \dots + \phi^{t-1} e_1 + \phi^t Y_0) \\
&= \text{Var}(e_t) + \text{Var}(\phi e_{t-1}) + \text{Var}(\phi^2 e_{t-2}) + \text{Var}(\phi^3 e_{t-3}) + \dots + \text{Var}(\phi^{t-1} e_1) + \text{Var}(\phi^t Y_0) \\
&= \text{Var}(e_t) + \phi^2 \text{Var}(e_{t-1}) + \phi^4 \text{Var}(e_{t-2}) + \phi^6 \text{Var}(e_{t-3}) + \dots + \phi^{2(t-1)} \text{Var}(e_1) + \phi^{2t} \text{Var}(Y_0) \\
&= \sigma_e^2 + \phi^2 \sigma_e^2 + \phi^4 \sigma_e^2 + \phi^6 \sigma_e^2 + \dots + \phi^{2(t-1)} \sigma_e^2 + \phi^{2t} \sigma_0^2
\end{aligned}$$

Letting  $\phi = 1$ , it is clear that  $\text{Var}(Y_t) = t\sigma_e^2 + \sigma_0^2$ .

If  $\phi \neq 1$ , we can see that  $\text{Var}(Y_t) = \sigma_e^2(1 + \phi^2 + \phi^4 + \dots + \phi^{2(t-1)}) + \phi^{2t} \sigma_0^2$ . The expanded series identity for  $(1 + \phi^2 + \phi^4 + \dots + \phi^{2(t-1)})$  is  $\frac{1-\phi^{2t}}{1-\phi^2}$ , and thus, for  $\phi \neq 1$ , we have  $\text{Var}(Y_t) = \frac{1-\phi^{2t}}{1-\phi^2} \sigma_e^2 + \phi^{2t} \sigma_0^2$ .

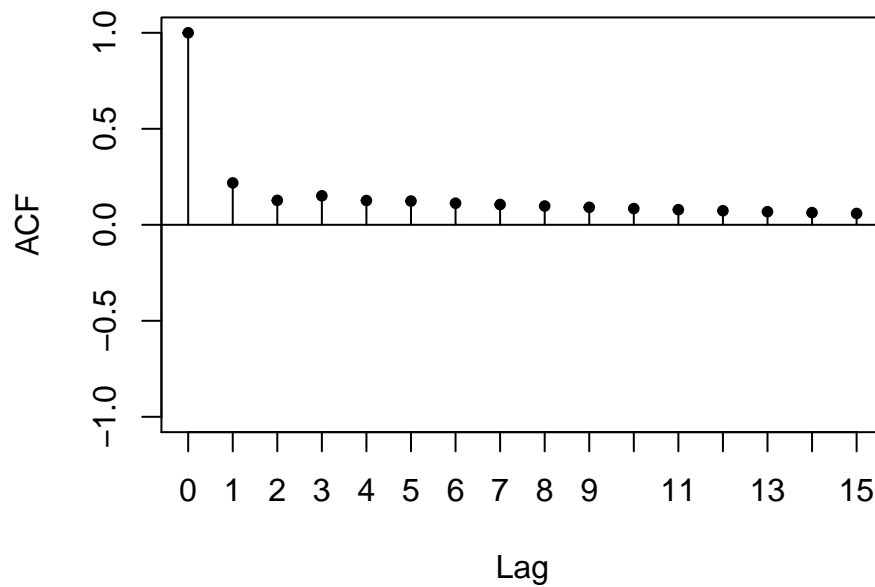
(d) Suppose  $\mu = 0$ . Show that if  $\{Y_t\}$  is stationary, then  $\text{Var}(Y_t) = \frac{\sigma_e^2}{1-\phi^2}$ .

**5. The following command in R will plot the theoretical autocorrelation function of an ARMA(2, 2) model  $Y_t = 0.5Y_{t-1} + 0.4Y_{t-2} + e_t - 0.7e_{t-1} - 0.6e_{t-2}$  for the first 15 lags:**

```

n <- 15
ACF <- ARMAacf(ar = c(0.5, 0.4), ma = c(-0.7, -0.6), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)

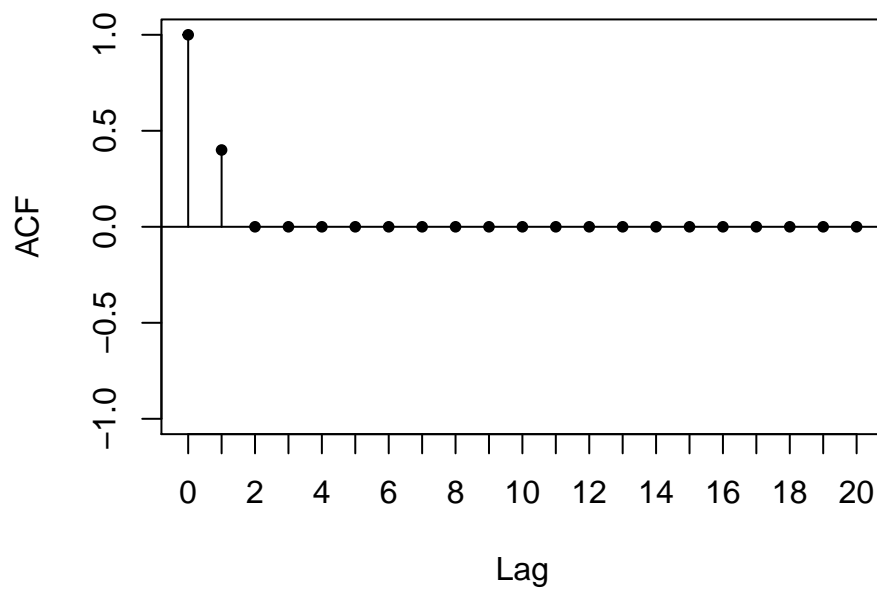
```



Modify the code to generate the theoretical autocorrelation functions up to 20 lags of the following ARMA processes:

(a) MA(1) with  $\theta = 0.5$

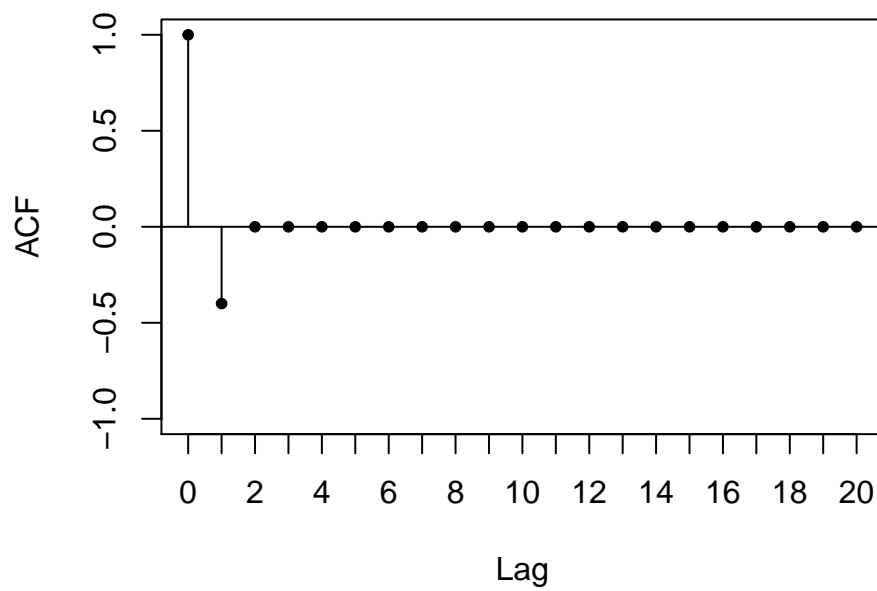
```
n <- 20
ACF <- ARMAacf(ma = c(0.5), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



(b) MA(1) with  $\theta = -0.5$

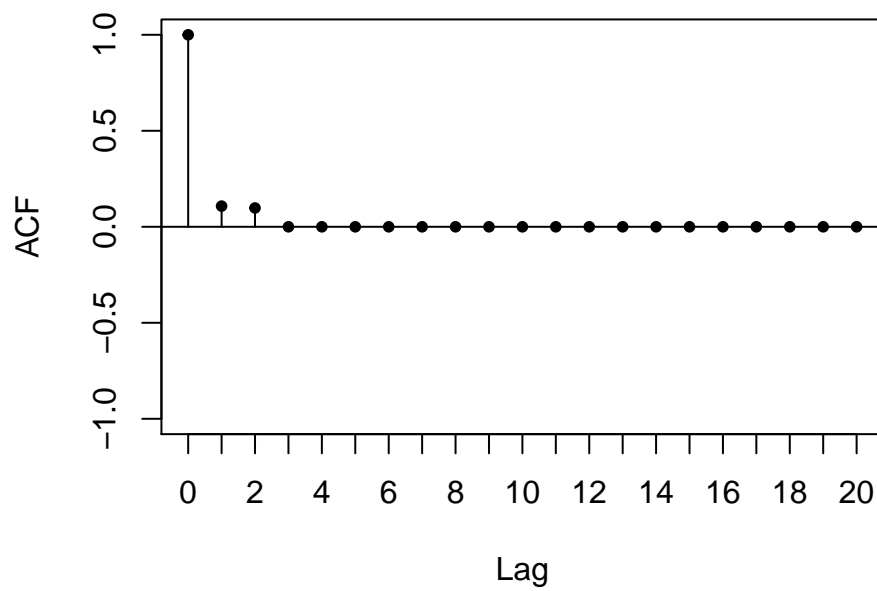
```
n <- 20
ACF <- ARMAacf(ma = c(-0.5), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```





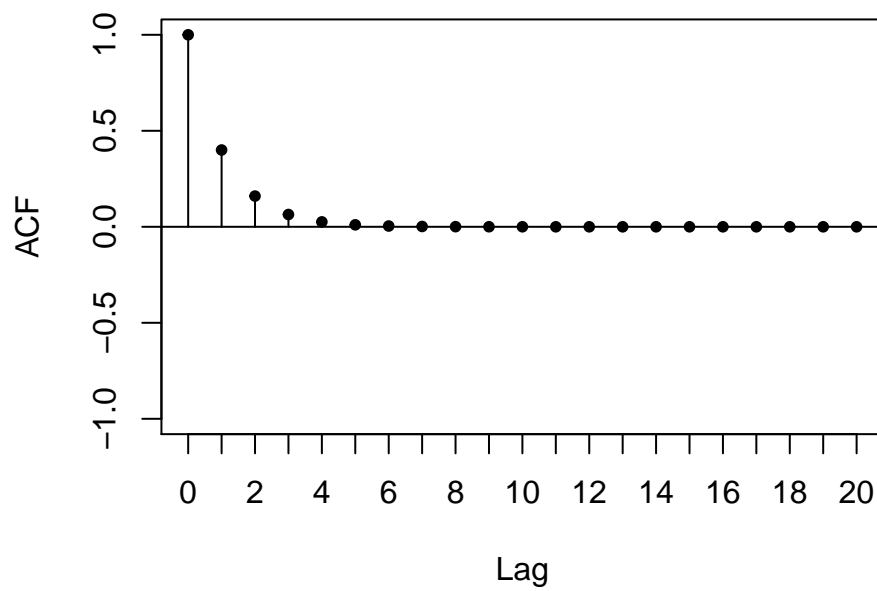
(c) MA(2) with  $\theta_1 = \theta_2 = 0.1$

```
n <- 20
ACF <- ARMAacf(ma = c(0.1, 0.1), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



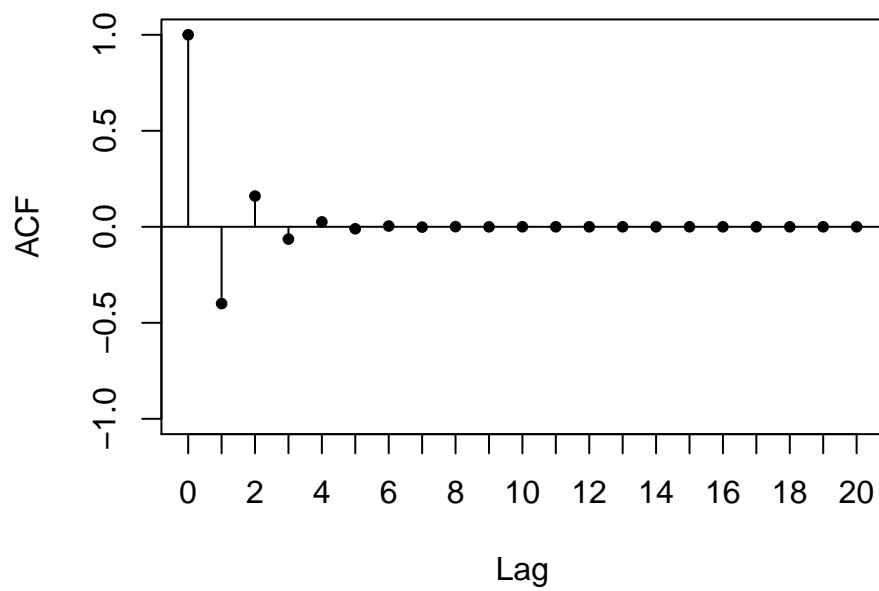
(d) **AR(1)** with  $\phi = 0.4$

```
n <- 20
ACF <- ARMAacf(ar = c(0.4), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



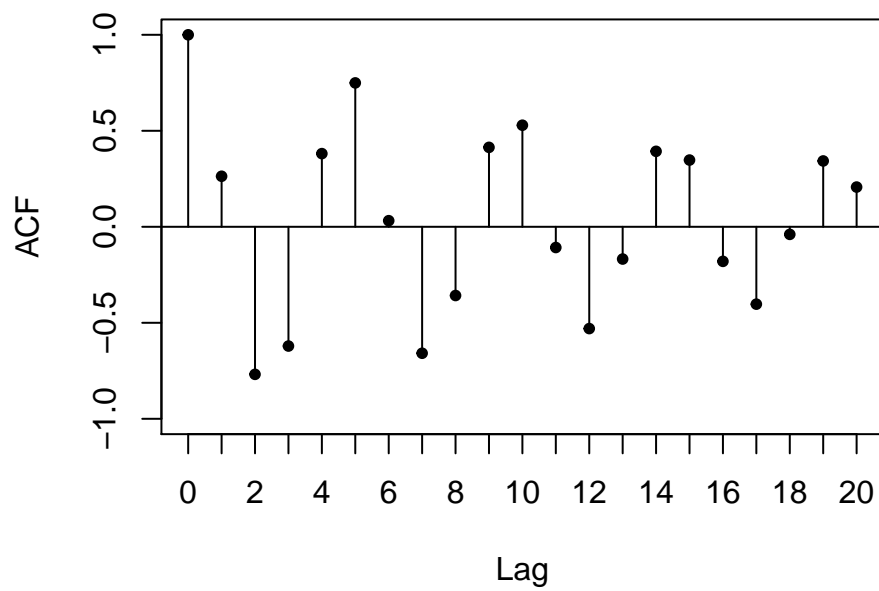
(e) AR(1) with  $\phi = -0.4$

```
n <- 20
ACF <- ARMAacf(ar = c(-0.4), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



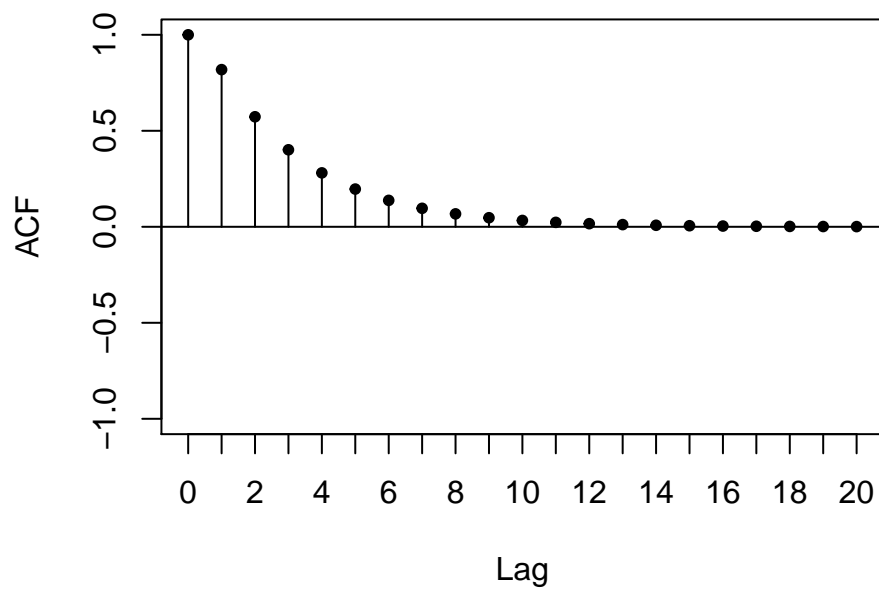
(f) AR(2) with  $\phi_1 = 0.5$  and  $\phi_2 = -0.9$

```
n <- 20
ACF <- ARMAacf(ar = c(0.5, -0.9), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



(g) ARMA(1, 1) with  $\phi = 0.7$  and  $\theta = 0.4$

```
n <- 20
ACF <- ARMAacf(ar = c(0.7), ma = c(0.4), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```



(h) ARMA(1, 2) given in Question 3

```
n <- 20
ACF <- ARMAacf(ar = c(0.7), ma = c(0.8, -0.6), lag.max = n)
plot(0:n, ACF, type = 'h', xlab = 'Lag', ylim = c(-1, 1), xaxp = c(0, n, n))
points(0:n, ACF, pch = 20)
abline(h = 0)
```

