

# Homework Assignment 2

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**Note:**  $\{e_t\}$  always denotes a sequence of independent, identically distributed random variables with mean zero and constant variance  $\sigma^2$

**1. If  $Y_t = e_t - e_{t-7}$ , show that  $\{Y_t\}$  is (weakly) stationary and that, for  $k > 0$ , its autocorrelation function is nonzero only for lag  $k = 7$ .**

**Answer:**

For stochastic process  $\{Y_t\}$  to be weakly stationary, two properties must hold: The mean function must be zero, and the variance must be constant for any time window,  $Y_{t,t-k}$ , for all  $t, k$ .

It is trivial to show that the mean function,  $\mu_t = E[Y_t]$  is zero, due to independence.

$$\begin{aligned} E[Y_t] &= E[e_t - e_{t-7}] \\ &= E[e_t] - E[e_{t-7}] \\ &= 0 - 0 = 0 \end{aligned}$$

To show constant variance, we show the following identity:

$$\begin{aligned} \gamma_{t,s} &= Cov(Y_t, Y_s) \\ &= Cov(e_t - e_{t-7}, e_s - e_{s-7}) \\ &= Cov(e_t, e_s) - Cov(e_t, e_{s-7}) - Cov(e_{t-7}, e_s) + Cov(e_{t-7}, e_{s-7}) \end{aligned}$$

If  $t = s$ , we have

$$\begin{aligned} \gamma_{s,s} &= Cov(Y_s, Y_s) \\ &= Cov(e_s, e_s) - Cov(e_s, e_{s-7}) - Cov(e_{s-7}, e_s) + Cov(e_{s-7}, e_{s-7}) \\ &= Var(e_s) - 0 - 0 + Var(e_{s-7}) \\ &= \sigma^2 + \sigma^2 \\ &= 2\sigma^2 \end{aligned}$$

And if  $t = s \pm 7$ , we have

$$\begin{aligned}
\gamma_{s+7,s} &= Cov(Y_{s+7}, Y_s) \\
&= Cov(e_{s+7}, e_s) - Cov(e_{s+7}, e_{s-7}) - Cov(e_{s+7-7}, e_s) + Cov(e_{s+7-7}, e_{s-7}) \\
&= 0 - 0 - Var(e_s, e_s) - 0 \\
&= -Var(e_s, e_s) \\
&= -\sigma^2 \\
\gamma_{s-7,s} &= Cov(Y_{s-7}, Y_s) \\
&= Cov(e_{s-7}, e_s) - Cov(e_{s-7}, e_{s-7}) - Cov(e_{s-7-7}, e_s) + Cov(e_{s-7-7}, e_{s-7}) \\
&= 0 - 0 - Var(e_{s-7}, e_{s-7}) - 0 \\
&= -Var(e_{s-7}, e_{s-7}) \\
&= -\sigma^2
\end{aligned}$$

Thus, we identify three scenarios,

$$\begin{cases} 2\sigma^2 & \text{for } t = s \\ -\sigma^2 & \text{for } t = s \pm 7 \\ 0 & \text{otherwise} \end{cases}$$

To show that the autocorrelation function is non-zero only for lag  $k = 7$ , we'll first show that it is in fact non-zero for lag  $k = 7$ . We again have three scenarios, as was shown previously.

First, we'll find  $\gamma_{t,t}$  and  $\gamma_{s,s}$ .

$$\begin{aligned}
\gamma_{t,t} &= Cov(Y_t, Y_t) = Var(Y_t) \\
&= Var(e_t - e_{t-7}) \\
&= E[(e_t - e_{t-7})(e_t - e_{t-7})] - (E[e_t - e_{t-7}])^2 \\
&= E[e_t^2 - e_t e_{t-7} + e_{t-7} e_t + e_{t-7}^2] \\
&= E[e_t^2 + e_{t-7}^2] \\
&= E[e_t^2] + E[e_{t-7}^2] \\
&= 2\sigma^2
\end{aligned}$$

The result is the same for  $\gamma_{s,s}$ . For  $t = s$  we have

$$\begin{aligned}
\rho_{t,s} &= \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} \\
&= \frac{2\sigma^2}{\sqrt{2\sigma^2 2\sigma^2}} \\
&= \frac{2\sigma^2}{2\sigma^2} \\
&= 1
\end{aligned}$$

For  $t = s \pm 7$ , we have

$$\begin{aligned}
\rho_{t,s} &= \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} \\
&= \frac{-\sigma^2}{\sqrt{2\sigma^2 2\sigma^2}} \\
&= \frac{-\sigma^2}{2\sigma^2} \\
&= -\frac{1}{2}
\end{aligned}$$

However, these are the only values for which lag  $k = 7$ . For any other value of  $t$ , lag  $k \neq 7$ , and thus,

$$\begin{aligned}
\rho_{t,s} &= \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} \\
&= \frac{0}{\sqrt{2\sigma^2 2\sigma^2}} \\
&= \frac{0}{2\sigma^2} \\
&= 0
\end{aligned}$$

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**2. Suppose that  $\{Y_t\}$  is (weakly) stationary with autocovariance function  $\gamma_k$ .**

**(a) Show that  $W_t = \nabla Y_t = Y_t - Y_{t-1}$  is (weakly) stationary by finding the mean and autocovariance function for  $\{W_t\}$ .**

**Answer:**

Since we are given that  $Y_t$  is weakly stationary, then its mean function must be zero. Therefore,

$$\begin{aligned} E[W_t] &= E[\nabla Y_t] \\ &= E[Y_t - Y_{t-1}] \\ &= E[Y_t] - E[Y_{t-1}] \\ &= 0 - 0 = 0 \end{aligned}$$

We are given that  $Y_t$  has autocovariance function  $\gamma_k = \text{Cov}(Y_t, Y_{t-k})$ , so we can extend this to  $Y_{t-1}$  with autocovariance function  $\gamma_{k-1} = \text{Cov}(Y_{t-1}, Y_{t-1-k})$ .

$$\begin{aligned} \text{Cov}(Y_t, Y_{t-k}) &= \text{Cov}(Y_t - Y_{t-1}, Y_{t-k} - Y_{t-1-k}) \\ &= \text{Cov}(Y_t, Y_{t-k}) - \text{Cov}(Y_t, Y_{t-1-k}) - \text{Cov}(Y_{t-1}, Y_{t-k}) + \text{Cov}(Y_{t-1}, Y_{t-1-k}) \\ &= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k \\ &= 2\gamma_k - \gamma_{k+1} - \gamma_{k-1} \end{aligned}$$

Thus, the variance is constant for equal time windows and  $W_t$  is weakly stationary.

**(b) Show that  $U_t = \nabla^2 Y_t = \nabla[\nabla Y_t] = \nabla[Y_t - Y_{t-1}] = Y_t - 2Y_{t-1} + Y_{t-2}$  is (weakly) stationary.**

We have already shown that  $W_t = \nabla Y_t$  is stationary.  $U_t$  is simply the difference of two  $W_t$  processes, and the difference of two stationary processes is also stationary.

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**3. For a fixed positive integer  $s$  and a constant  $\alpha$ , consider the stochastic process defined by  $Y_t = e_t + \alpha e_{t-1} + \alpha^2 e_{t-2} + \dots + \alpha^s e_{t-s}$ .**

**(a) Show that the process is (weakly) stationary for any value of  $\alpha$ .**

The mean function is trivial due to independence,

$$\begin{aligned} E[Y_t] &= E[e_t] + \alpha E[e_{t-1}] + \alpha^2 E[e_{t-2}] + \dots + \alpha^s E[e_{t-s}] \\ &= 0 + 0 + 0 + \dots + 0 = 0 \end{aligned}$$

Autocovariance is given by

$$\begin{aligned} \gamma_{t,r} &= Cov(Y_t, Y_r) \\ &= Cov(e_t + \alpha e_{t-1} + \alpha^2 e_{t-2} + \dots + \alpha^s e_{t-s}, e_r + \alpha e_{r-1} + \alpha^2 e_{r-2} + \dots + \alpha^s e_{r-s}) \\ &= Cov\left(\sum_{i=0}^s \alpha^i e_{t-i}, \sum_{j=0}^s \alpha^j e_{r-j}\right) \\ &= \sum_{i=0}^s \sum_{j=0}^s \alpha^i \alpha^j Cov(e_{t-i}, e_{r-j}) \\ &= \sum_{i=0}^s \sum_{j=0}^s \alpha^i \alpha^j Cov(e_{t-i}, e_{t-j}) \end{aligned}$$

Now, when  $i = j$  we get  $Cov(e_{t-i}, e_{t-j}) = \sigma^2$ , and when  $i \neq j$ , we get 0. Thus, we get the following matrix,

$$\begin{bmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \alpha^2 \sigma^2 & 0 & \dots & 0 \\ 0 & 0 & \alpha^4 \sigma^2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \alpha^{2s} \sigma^2 \end{bmatrix}$$

The summation over all of these leads to  $\alpha^0 \sigma^2 + \alpha^2 \sigma^2 + \alpha^4 \sigma^2 + \dots + \alpha^{2s} \sigma^2 = \sigma^2 \left( \sum_{i=0}^s \alpha^{2i} \right)$ .

Thus for any value of  $\alpha$  or  $s$ , the variance is the same for any time window  $t, r$ , and thus the process  $\{Y_t\}$  is stationary.

**(b) Find the autocorrelation function.**

Autocorrelation is given by  $\rho_{t,r} = \frac{\gamma_{t,r}}{\sqrt{\gamma_{t,t} \gamma_{r,r}}}$ .

$$\begin{aligned} \gamma_{t,t} &= Var(Y_t) \\ &= Var(e_t + \alpha e_{t-1} + \alpha^2 e_{t-2} + \dots + \alpha^s e_{t-s}) \\ &= Var(e_t) + \alpha^2 Var(e_{t-1}) + \alpha^4 Var(e_{t-2}) + \dots + \alpha^{2s} Var(e_{t-s}) \\ &= \sigma^2 + \alpha^2 \sigma^2 + \alpha^4 \sigma^2 + \dots + \alpha^{2s} \sigma^2 \\ &= \sigma^2 \left( \sum_{i=0}^s \alpha^{2i} \right) \end{aligned}$$

The same holds for  $\gamma_{r,r}$ .

$$\text{So we have } \rho_{t,r} = \frac{\gamma_{t,r}}{\sqrt{\gamma_{t,t}\gamma_{r,r}}} \frac{\sigma^2 \left( \sum_{i=0}^s \alpha^{2s} \right) \sigma^2}{\sqrt{\sigma^2 \left( \sum_{i=0}^s \alpha^{2s} \right) \sigma^2 \left( \sum_{i=0}^s \alpha^{2s} \right)}} = 1.$$

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4. Let  $\{X_t\}$  be the time series of interest, however, due to measurement error we actually observe  $Y_t = X_t + e_t$ . We assume that  $\{X_t\}$  and  $\{e_t\}$  are independent processes. We call  $X_t$  the signal, and  $e_t$  the noise. If  $\{X_t\}$  is stationary with autocorrelation function  $\rho_k$ , show that  $\{Y_t\}$  is also stationary with  $Corr(Y_t, Y_{t-k}) = \frac{\rho_k}{(1 + \sigma_e^2/\sigma_X^2)}$ , where  $\sigma_e^2/\sigma_X^2$  is called the signal-to-noise ratio.

Since  $e_t$  has mean 0, then

$$\begin{aligned} E[Y_t] &= E[X_t + e_t] \\ &= E[X_t] + E[e_t] \\ &= E[X_t] + 0 = \mu_X \end{aligned}$$

Now, since  $\rho_k = \frac{\gamma_k}{\gamma_0}$ , we have

$$\begin{aligned} Corr(Y_t, Y_{t-k}) &= \frac{\rho_k}{(1 + \sigma_e^2/\sigma_X^2)} \\ &= \frac{\frac{\gamma_k}{\gamma_0}}{(1 + \sigma_e^2/\sigma_X^2)} \end{aligned}$$

Solving the gammas we find,

$$\begin{aligned} \gamma_0 &= Var(Y_t) \\ &= Var(X_t + e_t) \\ &= Var(X_t) + Var(e_t) \\ &= \sigma_X^2 + \sigma_e^2 \\ \gamma_k &= Cov(Y_t, Y_{t-k}) \\ &= Cov(X_t + e_t, X_{t-k} + e_{t-k}) \\ &= Cov(X_t, X_{t-k}) + Cov(X_t, e_{t-k}) + Cov(e_t, X_{t-k}) + Cov(e_t, e_{t-k}) \\ &= Cov(X_t, X_{t-k}) + 0 + 0 + Cov(e_t, e_{t-k}) \\ &= Cov(X_t, X_{t-k}) + Cov(e_t, e_{t-k}) \\ &= \sigma_X^2 + \sigma_e^2 \text{ for } k = 0, \sigma_X^2 \rho_k \text{ for } k > 0 \end{aligned}$$

Note that  $\rho_k = Corr(Y_t, Y_{t-k}) = \frac{Cov(Y_t, Y_{t-k})}{\sqrt{Var(Y_t)Var(Y_{t-k})}} = \frac{Cov(X_t, X_{t-k})}{\sigma_X^2 + \sigma_e^2}$ . Thus, we can write the Covariance as,

$$\begin{aligned} \gamma_k &= Cov(Y_t, Y_{t-k}) \\ &= (\sigma_X^2 + \sigma_e^2) \frac{Cov(Y_t, Y_{t-k})}{\sigma_X^2 + \sigma_e^2} \\ &= (\sigma_X^2 + \sigma_e^2) \rho_k \text{ for } k = 0, \sigma_X^2 \rho_k \text{ for } k > 0 \end{aligned}$$

Finally, we have two cases:

$$\begin{aligned}
Corr(Y_t, Y_{t-k}) &= \frac{Cov(Y_t, Y_{t-k})}{Var(Y_t)} \\
&= \frac{(\sigma_X^2 + \sigma_e^2)\rho_k}{\sigma_X^2 + \sigma_e^2} \\
&= \frac{\rho_k}{(1 + \sigma_e^2/\sigma_X^2)} \\
&= \rho_k, \text{ for } k = 0
\end{aligned}$$

and

$$\begin{aligned}
Corr(Y_t, Y_{t-k}) &= \frac{Cov(Y_t, Y_{t-k})}{Var(Y_t)} \\
&= \frac{\sigma_X^2 \rho_k}{\sigma_X^2 + \sigma_e^2} \\
&= \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_X^2}}, \text{ for } k > 0
\end{aligned}$$

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## 5. Suppose

$$Y_t = \beta_0 + \sum_{i=1}^k [A_i \cos(2\pi f_i t) + B_i \sin(2\pi f_i t)]$$

where  $\beta_0, f_1, f_2, \dots, f_k$  are constants, and  $A_1, A_2, \dots, A_k, B_1, B_2, \dots, B_k$  are independent random variables with zero means and variances  $\text{Var}(A_i) = \text{Var}(B_i) = \sigma_i^2$ . Show that  $\{Y_t\}$  is (weakly) stationary by finding the mean and autocovariance function.

Finding the mean, we have

$$\begin{aligned} E[Y_t] &= E\left[\beta_0 + \sum_{i=1}^k [A_i \cos(2\pi f_i t) + B_i \sin(2\pi f_i t)]\right] \\ &= \beta_0 + E\left[\sum_{i=1}^k A_i \cos(2\pi f_i t) + B_i \sin(2\pi f_i t)\right] \\ &= \beta_0 + E\left[\sum_{i=1}^k A_i \cos(2\pi f_i t)\right] + E\left[\sum_{i=1}^k B_i \sin(2\pi f_i t)\right] \\ &= \beta_0 + \sum_{i=1}^k \cos(2\pi f_i t) E[A_i] + \sum_{i=1}^k \sin(2\pi f_i t) E[B_i] \\ &= \beta_0 + 0 + 0 = \beta_0 \end{aligned}$$

Thus, the mean is constant over time.

We now find the covariance,

$$\begin{aligned} \text{Cov}(Y_t, Y_s) &= \text{Cov}\left[\beta_0 + \sum_{i=1}^k [A_i \cos(2\pi f_i t) + B_i \sin(2\pi f_i t)], \beta_0 + \sum_{j=1}^k [A_j \cos(2\pi f_j s) + B_j \sin(2\pi f_j s)]\right] \\ &= \text{Cov}\left[\sum_{i=1}^k A_i \cos(2\pi f_i t) + B_i \sin(2\pi f_i t), \sum_{j=1}^k A_j \cos(2\pi f_j s) + B_j \sin(2\pi f_j s)\right] \\ &= \sum_{i=1}^k \sum_{j=1}^k \text{Cov}\left[A_i \cos(2\pi f_i t) + B_i \sin(2\pi f_i t), A_j \cos(2\pi f_j s) + B_j \sin(2\pi f_j s)\right] \\ &= \sum_{i=1}^k \sum_{j=1}^k \left[ \text{Cov}(A_i \cos(2\pi f_i t), A_j \cos(2\pi f_j s)) + \text{Cov}(A_i \cos(2\pi f_i t), B_j \sin(2\pi f_j s)) \right. \\ &\quad \left. + \text{Cov}(B_i \sin(2\pi f_i t), A_j \cos(2\pi f_j s)) + \text{Cov}(B_i \sin(2\pi f_i t), B_j \sin(2\pi f_j s)) \right] \\ &= \sum_{i=1}^k \sum_{j=1}^k \left[ \cos(2\pi f_i t) \cos(2\pi f_j s) \text{Cov}(A_i, A_j) + \cos(2\pi f_i t) \sin(2\pi f_j s) \text{Cov}(A_i, B_j) \right. \\ &\quad \left. + \sin(2\pi f_i t) \cos(2\pi f_j s) \text{Cov}(B_i, A_j) + \sin(2\pi f_i t) \sin(2\pi f_j s) \text{Cov}(B_i, B_j) \right] \end{aligned}$$

If  $i \neq j$ , the covariance is zero, but if  $i = j$  we have,

$$\begin{aligned}
Cov &= \sum_{i=1}^k \sum_{j=1}^k [\cos(2\pi f_i t) \cos(2\pi f_j s) \sigma^2 + \cos(2\pi f_i t) \sin(2\pi f_j s) \sigma^2 + \sin(2\pi f_i t) \cos(2\pi f_j s) \sigma^2 + \sin(2\pi f_i t) \sin(2\pi f_j s) \sigma^2] \\
&= \sigma^2 \sum_{i=1}^k \sum_{j=1}^k [\cos(2\pi f_i t) \cos(2\pi f_j s) + \cos(2\pi f_i t) \sin(2\pi f_j s) + \sin(2\pi f_i t) \cos(2\pi f_j s) + \sin(2\pi f_i t) \sin(2\pi f_j s)] \\
&=
\end{aligned}$$