STAT 8700 Final Question 2

Brian Detweiler

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2. (a) Consider a random sample $y_1, y_2, \ldots, y_n$ taken from a Normal population with mean $= \mu$ and known variance $= \sigma^2$. Show that the likelihood is equivalent to the likelihood of a single observation of $\overline{y}$ taken from a Normal population with a mean of μ' and $\sigma^{2'}$ where $\overline{y}$ is the mean of the y's. Find the appropriate expressions for μ' and $\sigma^{2'}$.

Given multiple observations of a Normal distribution with mean μ and known variance σ^2, the likelihood for observations $y_1, y_2, \ldots, y_n$ is given as

$$
\begin{aligned}
p(y_1, y_2, \ldots, y_i|\sigma^2) &= \prod_{i=1}^{n} p(y_i|\sigma^2) \\
&= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}}(y_i - \mu)^2 \\
&= \left[\frac{1}{\sigma\sqrt{2\pi}}\right]^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}
\end{aligned}
$$

where the data only appear in the exponential, $e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2}$. Using a trick from [1],

$$
\begin{aligned}
p(y_1, y_2, \ldots, y_i|\sigma^2) &\propto e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2} \\
&\propto e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[(y_i-\overline{y})-(\mu-\overline{y})\right]^2} \\
&\propto e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\overline{y})^2+\sum_{i=1}^{n}(\overline{y}-\mu)^2-2\sum_{i=1}^{n}(y_i-\overline{y})(\mu-\overline{y})} \\
&\propto e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\overline{y})^2+\sum_{i=1}^{n}(\overline{y}-\mu)^2-(\mu-\overline{y})\left(\left(\sum_{i=1}^{n} y_i\right)-n\overline{y}\right)} \\
&\propto e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\overline{y})^2+\sum_{i=1}^{n}(\overline{y}-\mu)^2-(\mu-\overline{y})(n\overline{y}-n\overline{y})} \\
&\propto e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\overline{y})^2+\sum_{i=1}^{n}(\overline{y}-\mu)^2} \\
&\propto e^{-\frac{1}{2\sigma^2}ns^2+n(\overline{y}-\mu)^2} \\
&\propto e^{-\frac{ns^2}{2\sigma^2}} e^{\frac{-n(\overline{y}-\mu)^2}{2\sigma^2}}
\end{aligned}
$$

And since σ^2 is constant, we can ignore it,

$$
p(y_1, y_2, \ldots, y_i|\sigma^2) \propto e^{\frac{-n(\overline{y}-\mu)^2}{2\sigma^2}}
$$

Now, we recognize this as a member of the exponential family of distributions, specifically a Gaussian,

$$p(y_1, y_2, \ldots, y_i|\sigma^2) \sim Normal\left(\overline{y}|\mu, \frac{\sigma^2}{n}\right)$$

This shows the equivalence to the likelihood for a single observation of $\overline{y}$, and hence, $\mu' = \mu$ and $\sigma^{2'} = \frac{\sigma^2}{n}$.

$\blacksquare$

(b) Suppose that y_i is Normally distributed with mean $= \mu$ and known variance $= \sigma_i^2$, for $i = 1, \ldots, n$. Show that if a uniform prior for μ is used then the posterior distribution of μ is Normal with mean $= \frac{\sum_{i=1}^n y_i/\sigma_i^2}{\sum_{i=1}^n 1/\sigma_i^2}$ and variance $= (\sum_{i=1}^n 1/\sigma_i^2)^{-1}$

Show all working.

Using a uniform prior on μ we have the improper prior,

$$p(\theta) \propto 1$$

Then the conditional distribution given in **3.2** is

$$\mu|\sigma^2, y \sim N(\overline{y}, \sigma^2/n)$$

which can be rewritten as

$$\mu|\sigma^2, y \sim N\left(\frac{1}{n}\sum_{i=1}^n y_i, \frac{1}{n}\sum_{i=1}^n \sigma_i^2\right)$$

Note that

$$\theta|\sigma_i^2, y_i \sim N\left(\frac{\sum_{i=1}^n y_i/\sigma_i^2}{\sum_{i=1}^n 1/\sigma_i^2}, \left(\sum_{i=1}^n 1/\sigma_i^2\right)^{-1}\right)$$

can be written as

$$\theta|\sigma_i^2, y_i \sim N\left(\sum_{i=1}^n y_i, \sum_{i=1}^n \sigma_i^2\right)$$

for $n = 1, 2, \ldots, n$.

Thus, the distribution given in **3.2** is a single observation of y of all y_i from $i = 1, 2, \ldots, n$.

$\blacksquare$

References

[1] Kevin P. Murphy
Conjugate Bayesian analysis of the Gaussian distribution
University of British Columbia, 2007
https://www.cs.ubc.ca/ murphyk/Papers/bayesGauss.pdf