

STAT 8700 Homework 3

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1. Suppose we have a population described by a Normal Distribution with known variance $\sigma^2 = 1600$ and unknown mean μ . 4 observations are collected from the population and the corresponding values were: 940, 1040, 910, and 990.

```
y.bar <- mean(940, 1040, 910, 990)
y.bar
```

```
## [1] 940
```

(a) If we choose to use a Normal(1000, 200^2) prior for θ , find the posterior distribution for θ by hand.

First, we'll derive the posterior for the single data point case, then for the general case.

Likelihood for a single data point

$$p(y|\theta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\theta)^2}{2\sigma^2}}$$

Normal Prior, s.t. $\theta \sim N(\mu_0, \tau_0^2)$

$$p(\theta) \propto e^{-\frac{(\theta-\mu_0)^2}{2\tau_0^2}}$$

Posterior for single observation

$$p(\theta) \propto e^{\left(-\frac{1}{2} \left(\frac{(y-\theta)^2}{\sigma^2} + \frac{(\theta-\mu_0)^2}{\tau_0^2} \right)\right)}$$
$$\theta|y \sim N(\mu_1, \tau_1^2), \text{ s.t. } \mu_1 = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{1}{\sigma^2}y}{\frac{1}{\tau_0^2} + \frac{1}{\sigma^2}}, \text{ and } \frac{1}{\tau_1^2} = \frac{1}{\tau_0^2} + \frac{1}{\sigma^2}$$

Now we are set up to extend this model to multiple observations. We will assume these four observations are *i.i.d.*, such that $y = (y_1, y_2, y_3, y_4)$.

Posterior density for multiple observations

$$\begin{aligned}
p(\theta|y) &\propto p(\theta)p(y|\theta) \\
&= p(\theta) \prod_{i=1}^n p(y_i|\theta) \\
&\propto e^{\left(-\frac{(\theta-\mu_0)^2}{2\tau_0^2}\right)} \prod_{i=1}^n e^{\left(-\frac{(y_i-\theta)^2}{2\sigma^2}\right)} \\
&\propto e^{\left(-\frac{1}{2}\left(\frac{(\theta-\mu_0)^2}{\tau_0^2} + \frac{1}{\sigma^2} \sum_{i=1}^n (y_i-\theta)^2\right)\right)}
\end{aligned}$$

After simplifying algebraically, we find that the posterior depends only on y by the sample mean, $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$, which means \bar{y} is a sufficient statistic. Now, since $\bar{y}|\theta, \sigma^2$, we can treat \bar{y} as a single observation and we get

$$p(\theta|y_1, y_2, y_3, y_4) = p(\theta|\bar{y}) = N(\theta|\mu_n, \tau_n^2), \text{ where } \mu_n = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \text{ and } \frac{1}{\tau_n^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2}$$

Substituting in our values, we have

$$\begin{aligned}
n &= 4 \\
\bar{y} &= 940 \\
\mu &= \theta \\
\sigma^2 &= 1600 \\
\tau_0^2 &= 200^2 \\
\frac{1}{\tau_4^2} &= \frac{1}{\tau_0^2} + \frac{n}{\sigma^2} \\
&= \frac{1}{200^2} + \frac{4}{1600} \\
&= \frac{1}{200^2} + \frac{1}{400} \\
&= 0.002525 \\
\mu_0 &= 1000 \\
\mu_4 &= \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \\
&= \frac{\frac{1}{200^2}1000 + \frac{4}{1600}940}{\frac{1}{200^2} + \frac{4}{1600}} \\
&= \frac{\frac{1}{40} + 2.35}{\frac{1}{400}} \\
&= 950 \\
p(\theta|y_1, y_2, y_3, y_4) &= p(\theta|\bar{y}) = N(\theta|\mu_4, \tau_4^2) \\
&= N(\theta|950, 396.03960396)
\end{aligned}$$

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(b) Find, by hand, a 95% credible interval for θ .

A 95% CI for θ is given by evaluating $p(y|\theta)$ at $y = 0.025$ and $y = 0.975$, with $\nu = 4$ degrees of freedom.

$$\begin{aligned}
 p(0.025; \theta) &= \frac{1}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} y^{-\left(\frac{\nu}{2}+1\right)} e^{-\frac{1}{2y}}, y > 0 \\
 &= \frac{1}{2^{\frac{4}{2}} \Gamma(\frac{4}{2})} (0.025)^{-\left(\frac{4}{2}+1\right)} e^{-\frac{1}{2(0.025)}} \\
 &= \frac{1}{4} (0.025)^{-3} e^{-\frac{1}{0.05}} \\
 &\approx 0.000032978457959 \\
 p(0.975; \theta) &= \frac{1}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} y^{-\left(\frac{\nu}{2}+1\right)} e^{-\frac{1}{2y}}, y > 0 \\
 &= \frac{1}{2^{\frac{4}{2}} \Gamma(\frac{4}{2})} (0.975)^{-\left(\frac{4}{2}+1\right)} e^{-\frac{1}{2(0.975)}} \\
 &= \frac{1}{4} 1.07891232152 e^{-\frac{1}{1.95}} \\
 &\approx 0.161514323478
 \end{aligned}$$

This gives us a 95% Credible Interval of (0.000032978457959, 0.161514323478).

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2. The `normnp` function in the `Bolstad` package computes the posterior for the mean with a Normal prior. The function requires 4 inputs (in order): a vector containing the data, the prior mean, the prior standard deviation, and the population standard deviation. Suppose we consider a Normal population with a variance of 16, and we collect 15 observations from this population with values: 26.8, 26.3, 28.3, 28.5, 26.3, 31.9, 28.5, 27.2, 20.9, 27.5, 28.0, 18.6, 22.3, 25.0, 31.5.

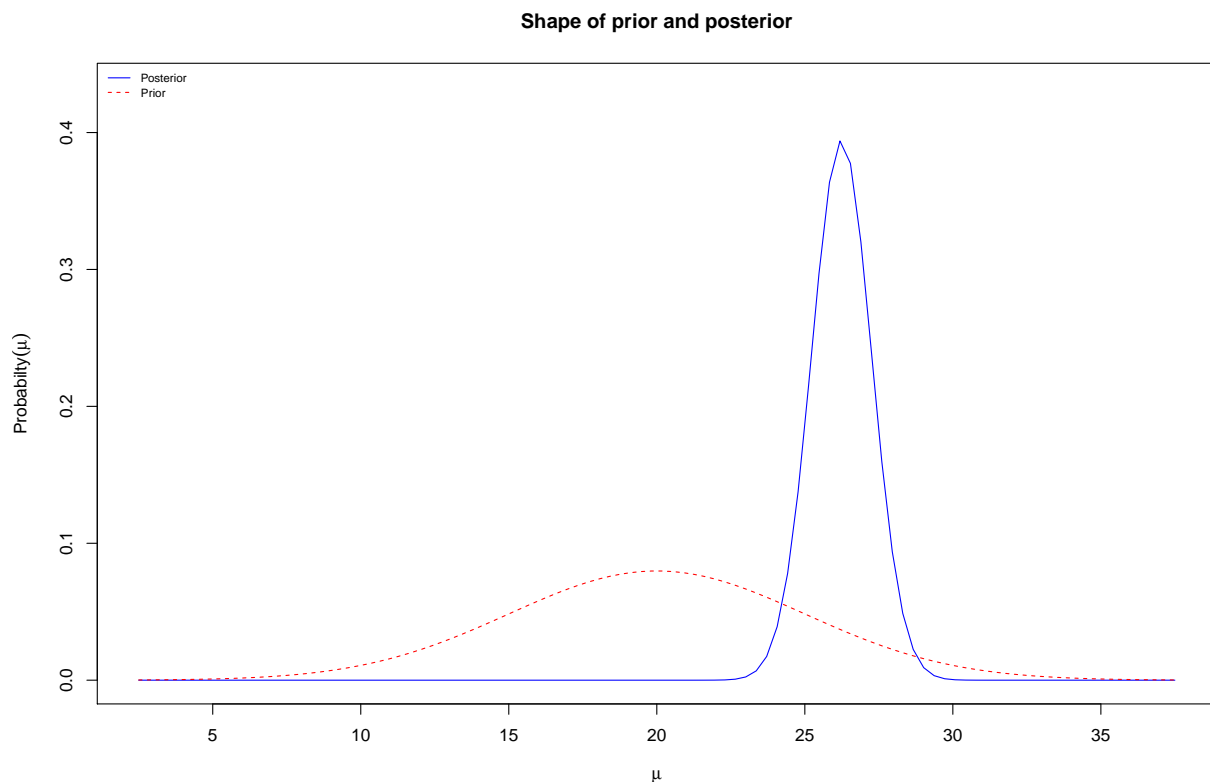
```
library(Bolstad)
var <- 16
obs <- c(26.8, 26.3, 28.3, 28.5, 26.3, 31.9, 28.5, 27.2, 20.9, 27.5, 28.0, 18.6, 22.3, 25.0, 31.5)
pop.st.dev <- sqrt(16)
```

(a) If we choose a $Normal(20, 25)$ prior, Use **R** to find the posterior distribution for the population mean.

```
prior.mu <- 20
prior.st.dev <- sqrt(25)

posterior <- normnp(obs, prior.mu, prior.st.dev, pop.st.dev)
```

```
## Known standard deviation :4
## Posterior mean           : 26.2404092
## Posterior std. deviation : 1.0114435
```



```
##
## Prob.    Quantile
## -----
## 0.005    23.6351035
## 0.010    23.8874398
## 0.025    24.2580164
## 0.050    24.5767327
## 0.500    26.2404092
## 0.950    27.9040857
## 0.975    28.2228020
## 0.990    28.5933786
## 0.995    28.8457149
```

(b) What are the posterior mean and variance?

The posterior mean is 26.2404092, and variance is 1.0230179.

(c) Find a 95% credible interval for the population mean.

A 95% credible interval for the population mean is found at the 0.025 and 0.975 quantiles, (24.2580164, 28.222802).

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3. Suppose $y|\theta \sim \text{Poisson}(\theta)$, find the Jeffreys' prior density for θ . Find α and β for which the $\text{Gamma}(\alpha, \beta)$ density is a close match to the Jeffreys' prior.

Jeffrey's prior is given by $J(\theta) = \sqrt{I(\theta)}$, where $I(\theta) = -E\left[\frac{\partial^2}{\partial \theta^2} \ln p(y|\theta)\right]$.

The Poisson distribution we are interested in, is $p(y_n|\theta) = \theta^{\sum_{i=1}^n y_i} e^{-n\theta} \prod_{i=1}^n \frac{1}{y_i!}$.

So working through this by parts, we start with the natural log,

$$\begin{aligned} \ln \theta^{\sum_{i=1}^n y_i} e^{-n\theta} \prod_{i=1}^n \frac{1}{y_i!} &= \ln \frac{1}{y!} - \theta + y \ln \theta \\ &= \sum_{i=1}^n y_i \ln \theta - n\theta - \ln \sum_{i=1}^n y_i! \end{aligned}$$

Taking the first derivative with respect to θ , we get

$$\begin{aligned} \frac{\partial}{\partial \theta} \ln p(y_n|\theta) &= \frac{\partial}{\partial \theta} \sum_{i=1}^n y_i \ln \theta - n\theta - \ln \sum_{i=1}^n y_i! \\ &= \sum_{i=1}^n \frac{y_i}{\theta} - n - 0 \end{aligned}$$

Taking the second derivative with respect to θ , we get

$$\begin{aligned} \frac{\partial^2}{\partial \theta^2} \ln p(y_n|\theta) &= \frac{\partial}{\partial \theta} \sum_{i=1}^n \frac{y_i}{\theta} \\ &= - \sum_{i=1}^n y_i \frac{1}{\theta^2} \end{aligned}$$

Taking expectations,

$$\begin{aligned} -E\left[-\frac{y}{\theta^2} \middle| \theta\right] &= \frac{n\theta}{\theta^2} \\ &= \frac{n}{\theta} \end{aligned}$$

Finally, taking the square root to get the Jeffrey's prior, $J(I)$, we have

$$\begin{aligned} \sqrt{I(\theta)} &= \sqrt{\frac{n}{\theta}} \\ &\propto \sqrt{\frac{1}{\theta}} \\ &= \theta^{\frac{1}{2}} \end{aligned}$$

This comes closest to $\lim_{\beta \rightarrow 0} \text{Gamma}(\frac{1}{2}, \beta)$, though it is not a proper distribution.

■

4. Suppose we have multiple independent observations y_1, y_2, \dots, y_n from a $Poisson(\theta)$ distribution.

(a) Consider the conjugate Gamma prior. What values of the hyperparameters would lead to a flat (improper) prior distribution for θ ?

With a Gamma prior, we have

$$p(\theta) \propto e^{-\beta\theta} \theta^{\alpha-1}$$

So to get a flat prior out of this, we need the hyperparameters that result in $p(\theta) \propto 1$, so we have

$$\begin{aligned} p(\theta) &\propto e^{-\beta\theta} \theta^{\alpha-1} \\ &= e^{-0\theta} \theta^{1-1} \\ &= e^0 \theta^0 \\ &\propto 1 \\ \theta &\sim \text{Gamma}(\alpha = 1, \beta = 0) \end{aligned}$$

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(b) Using a general $Gamma(\alpha, \beta)$ prior, derive the posterior distribution for θ . What is the required sufficient statistic needed from the data?

$$\begin{aligned}
 p(\theta|y) &\propto p(y|\theta)p(\theta) \\
 &\propto e^{-n\theta} \theta^{\sum y_i} e^{-\beta\theta} \theta^{\alpha-1} \\
 &= e^{-[\theta(n+\beta)]} \theta^{\sum y_i + \alpha - 1} \\
 &= e^{-\theta(n+\beta)} \theta^{n\bar{y} + \alpha - 1}
 \end{aligned}$$

So we have $Gamma(\alpha + n\bar{y}, \beta + n)$.

Thus, $n\bar{y}$ is sufficient because it is free of θ .

5. Derive the gamma posterior distribution (equation 2.15) for the Poisson model parameterized in terms of rate and exposure with conjugate prior distribution.

$$\begin{aligned}
p(\theta|y) &\propto p(y|\theta)p(\theta) \\
&\propto \left[\theta^{\left(\sum_{i=1}^n y_i\right)} e^{-(x_i)\theta} \right] \cdot \left[\frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \right] \\
&\propto \left[\theta^{\left(\sum_{i=1}^n y_i\right)} e^{-(x_i)\theta} \right] \cdot \left[\theta^{\alpha-1} e^{-\beta\theta} \right] \\
&= \theta^{\left(\alpha+\sum_{i=1}^n y_i-1\right)} e^{-\left(\beta+\sum_{i=1}^n x_i\right)\theta}
\end{aligned}$$

And thus we have the posterior as $\theta|y \sim \text{Gamma}(\alpha + \sum_{i=1}^n y_i, \beta + \sum_{i=1}^n x_i)$.

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6. The table at the end of the assignment gives the number of fatal accidents and deaths on scheduled airline flights per year over a ten year period from 1976 to 1985.

```
years <- c(1976:1985)
fatal.accidents <- c(24, 25, 31, 31, 22, 21, 26, 20, 16, 22)
passenger.deaths <- c(734, 516, 754, 877, 814, 362, 764, 809, 223, 1066)
death.rate <- c(0.19, 0.12, 0.15, 0.16, 0.14, 0.06, 0.13, 0.13, 0.03, 0.15)

airline.deaths <- as.data.frame(cbind(years, fatal.accidents, passenger.deaths, death.rate))
airline.deaths
```

##	years	fatal.accidents	passenger.deaths	death.rate
## 1	1976	24	734	0.19
## 2	1977	25	516	0.12
## 3	1978	31	754	0.15
## 4	1979	31	877	0.16
## 5	1980	22	814	0.14
## 6	1981	21	362	0.06
## 7	1982	26	764	0.13
## 8	1983	20	809	0.13
## 9	1984	16	223	0.03
## 10	1985	22	1066	0.15

(a) Assume that the number of fatal accidents in each year are independent with a $Poisson(\theta)$ distribution. Using a flat prior for θ , find the posterior distribution for θ based on the the 10 years of provided data. If you have a $Gamma(\alpha, \beta)$ distribution then the function `qgamma(q, shape=a, rate=b)` will return the q th quantile of the $Gamma(\alpha, \beta)$ distribution. Use this to find the ‘symmetric’ 95% credible interval for θ .

Using a flat prior, we have $\theta \sim Gamma(1, 0)$. So our posterior distribution becomes $\theta|y \sim Gamma(1 + \sum_{i=1}^n y_i, \sum_{i=1}^n x_i)$.

```
theta.given.y <- qgamma(c(0.025, 0.975),
                        shape=(1 + sum(airline.deaths$fatal.accidents)),
                        rate=(sum(airline.deaths$death.rate)))
```

The symmetric 95% credible interval is (166.3949791, 214.4730647).

■

(b) Now assume that the number of fatal accidents in each year follow independent Poisson distributions with a constant rate and an exposure in each year proportional to the number of passenger miles flown. Again using a flat prior distribution for θ , determine the posterior distribution based on the data. (Estimate the number of passenger miles flown in each year by dividing the appropriate columns of table and ignoring round-off errors, death rate is per 100 million miles.) Give a 95% predictive interval for the number of fatal accidents in 1986 under the assumption that 8×10^{11} passenger miles are flown that year.

```
miles.flown <- (airline.deaths$passenger.deaths / airline.deaths$death.rate) * 100000000
miles.flown
```

```
## [1] 386315789474 430000000000 502666666667 548125000000 581428571429
## [6] 603333333333 587692307692 622307692308 743333333333 710666666667
```

```
airline.deaths$miles.flown <- miles.flown
```

(c) Repeat (a) above, replacing ‘fatal accidents’ with ‘passenger deaths.’

```
theta.given.y <- qgamma(c(0.025, 0.975),  
                        shape=(1 + sum(airline.deaths$passenger.deaths)),  
                        rate=(airline.deaths$death.rate))
```

The symmetric 95% credible interval is $(3.5567929 \times 10^4, 5.9033231 \times 10^4)$.

■

(d) Repeat (b) above, replacing ‘fatal accidents’ with ‘passenger deaths.’

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