STAT 8700 Final Question 1

Brian Detweiler

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1. A random variable X is said to have a lognormal distribution with parameters μ and σ^2 if it has the following PDF:

$$\frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}} \text{ for } x > 0$$

Now suppose that data $x_1, \ldots, x_n$ are a random sample from a lognormal population with σ^2 known but μ unknown. Show that a $Normal(m, s^2)$ prior on μ is conjugate and find the parameters of the posterior distribution.

Show all working.

Using the PDF above, our likelihood is

$$\begin{aligned} p(x|\theta) &= \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln x - \theta)^2}{2\sigma^2}} \text{ for } x > 0 \\ &\propto e^{-\frac{(\ln x - \theta)^2}{2\sigma^2}} \end{aligned}$$

With $\theta \sim Normal(m, s^2)$, we have

$$\begin{aligned} p(\theta) &= \frac{1}{s\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2s^2}} \\ &\propto e^{-\frac{(x-m)^2}{2s^2}} \end{aligned}$$

Now, using Bayes' Rule, we can find our posterior distribution,

$$\begin{aligned} p(\theta|x) &\propto p(x|\theta) \cdot p(\theta) \\ &= e^{-\frac{(\ln x - \theta)^2}{2\sigma^2}} \cdot e^{-\frac{(\theta - m)^2}{2s^2}} \\ &= e^{-\frac{(\ln x - \theta)^2}{2\sigma^2} - \frac{(\theta - m)^2}{2s^2}} \\ &= e^{-\frac{1}{2\sigma^2}(\ln x^2 - 2\ln x\theta + \theta^2) - \frac{1}{2s^2}(\theta^2 - 2m\theta + m^2)} \\ &= e^{-\frac{1}{2\sigma^2}\ln x^2 - \frac{1}{2s^2}m^2 + \theta(\frac{1}{\sigma^2}\ln x + \frac{1}{s^2}m) + \theta^2(-\frac{1}{2\sigma^2} - \frac{1}{2s^2})} \end{aligned}$$

Let $\frac{1}{\tau_1^2} = \frac{1}{\sigma^2} + \frac{1}{s^2}$. Now, we have

$$
\begin{aligned}
p(\theta|x) &\propto p(x|\theta) \cdot p(\theta) \\
&= e^{\left[\frac{\tau_1^2}{\sigma^2} \ln x^2 + \frac{\tau_1^2}{s^2} m^2 - \theta\left(\frac{2\tau_1^2}{\sigma^2} \ln x + \frac{2\tau_1^2}{s^2}\right) + \theta^2\right] \frac{1}{2\tau_1^2}}
\end{aligned}
$$

Finally, let $\mu_1 = \frac{\frac{1}{\sigma^2} \ln x + \frac{1}{2s^2} m^2}{\frac{1}{\sigma^2} + \frac{1}{s^2}}$

Now we have

$$
p(\theta|y) \propto e^{-\frac{1}{2\tau_1^2}(\theta-\mu_1)^2}
$$

This implies that the posterior is $Normal(\mu_1, \tau_1^2)$.

$\blacksquare$