Game Theory: Problem Set 1 Solutions

1 Basic understanding

For both of the games below, identify: (i) dominant strategies, (ii) dominated strategies, and (iii) Nash equilibria

1.

	C	D
A	$5, 2$	$9, 3$
B	$1, 12$	$6, 0$

2.

	L	M	R
U	$1, 0$	$2, 1$	$3, 3$
D	$3, 6$	$2, 2$	$1, 4$

Solution

Game 1

(i) A is a dominant strategy for the row player. There is no dominant strategy for the column player.

(ii) B is dominated by A for the row player. No strategy is dominated for the column player.

(iii) In order to find Nash equilibria, identify the best response for each player to each of the other player's strategies. Best response is indicated by underlined payoff in the figure below. (For example, if the row player plays A, the best response for the column player is to play D.)

	C	D
A	$\underline{5}, 2$	$\underline{9}, \underline{3}$
B	$1, \underline{12}$	$6, 0$

From the figure, it is evident that there is one set of strategies in which both are playing best response: (A, D). We conclude that (A, D) is a Nash equilibrium.

Game 2

(i) There is no dominant strategy for either player in this game.

(ii) No strategy is dominated for the row player. M is dominated by R for the column player.

(iii) To find Nash equilibria, we proceed as above and identify each player's best response to each of the other player's strategies:

$$\begin{array}{cccc} & L & M & R \\ U & 1,0 & \underline{2},1 & \underline{3},\underline{3} \\ D & \underline{3},\underline{6} & \underline{2},2 & 1,4 \end{array}$$

We conclude that there are two Nash equilibria: (D, L) and (U, R).

2 Simple advertising game

There are two firms and ten consumers. Each sale to a consumer generates \$1 in profits. All ten consumers buy from one firm or the other. Each firm chooses whether to advertise or not. Advertising costs \$4. If both firm choose the same level of advertising (i.e., both choose to advertise or both choose not to advertise), they split the ten consumers evenly. If one firm advertises and the other one does not, the advertising firm gets all ten consumers. You can assume that the firms payoff (utilities) simply equal their profits. Write this game in a matrix form and find all Nash equilibria. Is this game one of the games we have studied thus far in class?

Solution

If both advertise, the both have to pay \$4 but receive \$5 in profits from sales. Hence, they both receive a payoff $\$5 - \$4 = \$1$. If both don't advertise, they both receive a payoff of \$5 from sales. If one advertises and the other doesn't, the one that advertises pays \$4 but gets all the customers. The payoff for the one who advertises is $\$10 - \$4 = \$6$ and the payoff for the one who does not advertise is \$0. The associated game matrix is the following:

	Do not advertise	Advertise
Do not advertise	$5, 5$	$0, 6$
Advertise	$6, 0$	$1, 1$

In order to find the pure strategy Nash equilibria, we find each player's best response to each of the other player's strategies:

	Do not advertise	Advertise
Do not advertise	$5, 5$	$0, \underline{6}$
Advertise	$\underline{6}, 0$	$\underline{1}, \underline{1}$

There is one Nash equilibrium: $(Advertise, Advertise)$. This is a Prisoners' Dilemma: each player has one dominant one dominated strategy and both players would be better off if both players played the dominated strategy.

3 Splitting the pie

Player 1 and 2 are bargaining over how to split \$10. Each player i names an amount s_i, between 0 and 10, for herself. These numbers do not have to be in whole dollar units; in other words $s_i \in [0, 10]$. The two choices are made simultaneously. Each player's payoff is equal to the dollar amount she obtains. We will consider two possible games the players might play.

3.1 Excess demands destroy the whole pie

If $s_1 + s_2 \leqslant 10$, then player 1 gets s_1 and player 2 gets s_2. (If $s_1 + s_2 < 10$, the remaining $10 - s_1 - s_2$ is destroyed.) If, however, $s_1 + s_2 > 10$, then both players get zero and all of the money is destroyed. What are the Nash equilibria of this game?

Solution

Payoffs to players $i = 1, 2$ satisfy $u_i = s_i$ if $s_1 + s_2 \leqslant 10$ and $u_i = 0$ if $s_1 + s_2 > 10$. The best response of Player 1 to any action s_2 by Player 2 is to choose $s_1 = 10 - s_2$. Similarly, by symmetry Player 2 takes as given any action s_1 by Player 1 and best responds by choosing $s_2 = 10 - s_1$. Observe that the best response functions of the two players characterize the identical relationship: $s_1 + s_2 = 10$.

The Nash equilibrium is a pair of amounts (s_1^*, s_2^*) such that each player is choosing the best response to the other player's quantity, i.e., the two equations above are both satisfied. Thus, *every* pair of amounts $\left(s_1^*, s_2^*\right)$ satisfying $s_1^* + s_2^* = 10$ is a Nash equilibrium. Since both s_1 and s_2 can take values in the interval $[0, 10]$, there is a continuum (an infinite number) of Nash equilibria.

In addition to this continuum of equilibria, there in also one more, inefficient, equilibrium with $s_1 = s_2 = 10$. In this equilibrium, both players get a zero payoff but neither has a profitable deviation.

3.2 Allocation respects humility

Now consider an alternate game. As before, if $s_1 + s_2 \leqslant 10$, then player 1 gets s_1 and player 2 gets s_2. (If $s_1 + s_2 < 10$, the remaining $10 - s_1 - s_2$ is destroyed.) Now, however, if $s_1 + s_2 > 10$, the player who named the smaller amount gets that amount and the other person gets the remainder of the money. For example, if $s_1 + s_2 > 10$ and $s_1 < s_2$, then

$u_1 = s_1$ and $u_2 = 10 - s_1$. If $s_1 + s_2 > 10$ and it happens that $s_1 = s_2$, then both players get \$5, i.e., $u_1 = u_2 = 5$. What are the Nash equilibria of this game?

Solution

If player j chooses $s_j > 5$, then player i's best response is some number s_i arbitrarily smaller than 5 which will yield the payoff of $u_i = s_i > 5$. If player j chooses $s_i \leqslant 5$, then there are many responses but player i can obtain the payoff of $10 - s_i \geqslant 5$ (say by choosing $10 - s_i$, but also simply by choosing 10).

Putting these two observations together, we see that, if player i optimally responds to what player j is doing, she will obtain $u_i \geqslant 5$. But we can apply the same reasoning to player j and thus conclude that in equilbrium we also must have $u_j \geqslant 5$.

One way to get $u_i \geqslant 5$ and $u_j \geqslant 5$ is to set $s_1 = s_2 = 5$. It is easy to see this is an equilibrium. If the other player plays $s_j = 5$, setting $s_i = 5$ is a best response. Hence, $s_i = s_j = 5$ is an equilibrium.

We can also show that there are no other equilibria. If some player plays $s_i > 5$ in equilibrium, the other player will be choosing some s_j that satisfies $5 < s_j < s_i$ and thus we will have $u_i = 10 - s_j < 5$ in equilibrium, which we already said cannot happen.

Similarly, if some player plays $s_i < 5$ in equilibrium, she gets $u_i = s_i < 5$, which we already said cannot happen. Hence, $s_i = s_j = 5$ is the unique equilibrium.

Observe the striking difference between the results from sections 3.1 and 3.2. Small changes to the payoff structure for the players can dramatically reduce the number of Nash Equilibria from infinite to one.

4 Competing in quantities

Suppose two firms selling an identical product engage in Cournot competition. There are 100 potential customers and the industry demand is $100 - p$. Firms choose quantities $q_i \in [0, 50]$, which leads to a price that equalizes supply and demand. Firms maximize their profits. This applies to all four problems below.

4.1 Baseline game

Suppose the firms choose their quantities simultaneously. Each firm's marginal cost is \$10. Find the Nash equilibrium of this game.

Solution

Since demand is given by $Q = 100 - p$, where $Q = q_1 + q_2$, inverse demand is given by $p = 100 - q_1 - q_2$. Firm 1 chooses a $q_1 \in [0, 50]$ to maximize $q_1(100 - q_1 - q_2) - 10q_1$. The optimal q_1 is such that $MR = MC$: $100 - 2q_1 - q_2 = 10$. We obtain the best response function for firm 1 by solving for the optimal q_1 in terms of q_2 :

$$q_1 = 45 - \frac{1}{2}q_2$$

Since the firms have the same marginal cost, firm 2's best response function is

$$q_2 = 45 - \frac{1}{2}q_1$$

The Nash equilibrium is a pair of quantities (q_1^*, q_2^*) such that each firm is choosing the best response to the other player's quantity, i.e., the two equations above are both satisfied. Plugging the second equation into the first one, we get:

$$q_1^* = 45 - \frac{1}{2}\left(45 - \frac{1}{2}q_1^*\right) \quad \Rightarrow \quad q_1^* = 30$$

Again, by symmetry, $q_1^* = q_2^*$ so $q_2^* = 30$ as well.

4.2 Raising rival's costs

Now, suppose that firm 1 can spend \$X to increase the other firm's marginal cost to \$20 (leaving its own marginal cost unchanged). Then, given the new costs, the firms choose their quantities simultaneously. What is the most money (i.e., the largest X), that firm 1 would be willing to spend on this?

Solution

If firm 1 does not increase the marginal cost of firm 2, its profits are

$$\pi_1^0 = 30(100 - 30 - 30) - 10 \cdot 30 = 900$$

If firm 1 pays \$X to increase the marginal cost of firm 2, firm 2's best response function becomes

$$q_2 = 40 - \frac{1}{2}q_1$$

while firm 1's best response function remains the same as in the previous part:

$$q_1 = 45 - \frac{1}{2}q_2$$

Putting the two equations together we get:

$$q_1^* = 45 - \frac{1}{2}\left(40 - \frac{1}{2}q_1^*\right) \quad \Rightarrow \quad q_1^* = \frac{100}{3}$$

and the new profit is

$$\pi_1^{\text{invest}} = \frac{100}{3}\left(100 - \frac{100}{3} - \left(40 - \frac{100}{6}\right) - 10\right) = 1111.1$$

Hence, firm 1 would be willing to pay up to \$211.10 to raise its rival's marginal cost to \$20.

4.3 Covert information

As in the baseline game, each firm's marginal cost is \$10. Suppose that, **unbeknownst to firm 2**, firm 1 can observe the quantity chosen by firm 2 before it chooses its own quantity. (For example, firm 1 has a spy in firm 2's headquarters and firm 2 does not suspect this is even possible.) How do the profits of each firm change? Along with an exact numerical answer, provide an intuition for the direction of the change(s), if any.

Solution

Firm 2 does not know that firm 1 can observe the quantity it chooses. It will hence behave as if the firms were choosing quantities simultaneously and choose $q_2^* = 30$. Firm 1 will in turn choose the best response (given by the best response function from part 1) to this quantity, i.e. $q_1^* = 30$. So, the profits will not change. This is a general feature of any pure-strategy equilibrium of any static game. Since equilibrium already requires that each player's prediction is correct, covertly learning what the other player will do cannot change the outcome.

4.4 Overt information

As in the baseline game, each firm's marginal cost is \$10. Suppose that **everyone knows** that firm 1 can observe the quantity chosen by firm 2 before it chooses its own quantity. How do the profits of each firm change? Along with an exact numerical answer, provide an intuition for the direction of the change(s), if any.

Solution

Firm 2 knows that firm 1 will observe the quantity it chooses and respond accordingly. This means that firm 2 will be maximizing profits taking firm 1's best response function into account. In other words, firm 2 will maximize

$$\pi_2 = q_2\left(100 - q_2 - \left(45 - \frac{1}{2}q_2\right) - 10\right) = 45q_2 - \frac{1}{2}q_2^2$$

The optimality condition (FOC) is

$$45 - q_2^* = 0$$

and

$$q_2^* = 45$$

Firm 2's profits become

$$\pi_2 = 45^2 - \frac{1}{2}45^2 = \frac{2025}{2} = 1012.5$$

For firm 1, we know that

$$q_1 = 45 - \frac{1}{2}q_2 = 45 - \frac{1}{2}45 = \frac{45}{2} = 22.5$$

and

$$\pi_1 = 22.5\,(100 - 45 - 22.5) - 10 \cdot 22.5 = 506.25$$

So, the fact that firm 1 knows what the firm 2 does decreases the profit of firm 1 and increases the profit of firm 2. The basic intuition for this is that firm 1's observation of firm 2's quantity effectively allows firm 2 to commit to produce a high quantity and thus force firm 1 to produce less.