Sample Midterm Solutions

Problem 1

List all the pure strategy Nash equilibria of this game:

	L	M	R
U	$20, 10$	$4, 5$	$2, 10$
M	$22, 3$	$5, 4$	$2, 3$
D	$15, 0$	$0, 10$	$0, 0$

The pure strategy Nash equilibria are (U, R) and (M, M).

Problem 2

Two firms decide simultaneously how much to invest in R&D to increase the quality of their product. They can choose a high or a low level of investment. If one invests more than the other one, consumers prefer its product so the payoff is 10 for the high investment firm and 5 for the other one. If they both invest the same, consumers are indifferent between both products. Each firm gets 8 if they both chose low investment, and 6 if they both choose high investment. Write down the payoffs matrix of the game and find the Nash equilibrium of the game. What type of a game is this?

	High	Low
High	6, 6	10,5
Low	5, 10	8, 8

This game is a Prisoner's dilemma. High is a dominant strategy for both players and the Nash equilibrium, $(High, High)$, is not efficient.

Problem 3

Part a

Consider the following simultaneous move game of a penalty kick. The kicker can shoot left or right. He is right-footed so he shoots more strongly when he shoots the left. The goalie knows this. The goalie can throw himself left or right. If the goalie throws himself in the wrong direction, the kicker always scores. If the goalie throws himself to the left and the kicker shoots to the left, the kicker scores 60% of the time. If the goalie throws himself to the right and the kicker shoots to the right, the kicker scores 40% of the time. (These numbers reflect the fact that the kicker is right-footed – goalie is more likely to defend successfully if the kicker shoots to his weaker side.)

If the kicker scores, kicker gets a payoff of 1 and goalie gets a payoff of 0.

If the kicker does not score, kicker gets a payoff of 0 and goalie gets a payoff of 1.

There is a unique mixed strategy equilibrium of this game. Find it.

<u>Solution:</u>

The payoff matrix is:

		Goalie	
		L	R
Kicker	L	0.6, 0.4	1, 0
	R	1,0	0.4, 0.6

Let k be the probability the kicker shoots to the left and g the probability the goalie throws himself to the left. Then the kicker is indifferent when

$$\begin{aligned} 0.6g + 1 - g &= g + 0.4(1-g) \\ g &= 0.6 \end{aligned}$$

And the goalie is indifferent when

$$\begin{aligned} 0.4k &= 0.6(1-k) \\ k &= 0.6 \end{aligned}$$

The unique mixed strategy equilibrium of the game is $((0.6L, 0.4R), (0.6L, 0.4R))$

Part b

Suppose you are the coach of the kicking team. You have access to two players. One is right-footed (as the kicker in Part a). The other one is ambidextrous. He shoots equally well to the right and to the left but his greater strength of one foot is compensated by the reduced strength of the other foot: a goalie who throws himself in the right direction defends his kicks 50% of the time. Which player should you have shoot the penalty kick? (Obviously, you want your team to score.) You should assume the goalie knows the players (so once you have chosen the player, the goalie knows whether it is the right-footed one or the ambidextrous one.)

Solution

For the right-footed player, the expected payoff is $0.6 \cdot 0.6 + 0.4 = 0.76$.

When the kicker is ambidextrous, the NE is $((0.5L, 0.5R), (0.5L, 0.5R))$. The kicker's expected payoff is $0.5 \cdot 0.5 + 0.5 = 0.75$.

Therefore, the coach prefers the right-footed player.

Problem 4

Suppose two workers simultaneously choose to contribute their efforts s_1 and s_2 to a joint project. Suppose the joint revenue is determined by the two workers' efforts s_1 and s_2 according to the following production function:

$$R(s_1, s_2) = s_1 s_2 + s_1 + s_2$$

The workers split the revenue equally, each receiving $\frac{1}{2}R(s_1, s_2)$. Effort is privately costly; in particular, exerting effort s costs $c(s) = s^2$. Suppose the workers can choose any non-negative effort $s_i > 0$.

Solve for the unique (pure strategy) Nash equilibrium of this game.

What are the workers' equilibrium payoffs?

If they could sign a binding enforceable contract on their efforts, how hard would they work and what would their payoffs be?

Solution

Worker 1 solves the problem

$$\begin{aligned} \max_{s_1} \Pi_1 &= \max_{s_1} \frac{1}{2}(s_1 s_2 + s_1 + s_2) - s_1^2 \\ \frac{\partial \Pi_1}{\partial s_1} &= \frac{s_2}{2} + \frac{1}{2} - 2s_1 = 0 \\ s_1 &= \frac{s_2}{4} + \frac{1}{4} \end{aligned}$$

Equivalently, the best response function for worker 2 is $s_2 = \frac{s_1}{4} + \frac{1}{4}$. Replacing this in worker 1's best response we get $s_1 = s_2 = \frac{1}{3}$.

Each worker's payoff is ${}^1\!/_2 R\left({}^1\!/_3, {}^1\!/_3\right) - c\left({}^1\!/_3\right) = {}^1\!/_2 {}^7\!/_9 - {}^1\!/_9 = {}^5\!/_{18}$.

If they can jointly decide on their efforts, they will both pick $s = s_1 = s_2$ such that their joint revenue is maximized

$$\begin{aligned}
\max_s \Pi(s) &= \max_s s^2 + 2s - 2s^2 \\
\frac{\partial \Pi(s)}{\partial s} &= 2s + 2 - 4s = 0 \\
s &= 1
\end{aligned}$$

If they could enforce $s_1 = s_2 = 1$, each would get $^1\!/_2 R\,(1,1) - c(1) = {}^3\!/_2 - 1 = {}^1\!/_2$.

Problem 5

Consider two firms sequentially competing in quantities. Firm 1 enters first and chooses a quantity Q_1. Firm 2 enters second, observes the quantity Firm 1 has produced, and then chooses its quantity Q_2. The price is then determined by the inverse demand $P = 20 - (Q_1 + Q_2)$. Both firms have a marginal cost of 4. Find the unique Subgame Perfect Equilibrium of the game. Does the game have any Nash equilibria that are **not** Subgame Perfect? If so, give one example of such an equilibrium and discuss its interpretation.

Using backward induction, we first find the best response for Firm 2:

$$\begin{aligned}
\max_{Q_2} \Pi_2 &= (20 - Q_1 - Q_2 - 4)\, Q_2 \\
\frac{\partial \Pi_2}{\partial Q_2} &= 20 - Q_1 - 2Q_2 - 4 = 0 \\
Q_2(Q_1) &= \frac{16 - Q_1}{2}
\end{aligned}$$

Firm 1 knows that Firm 2's production will depend on what Firm 1 produces in the first period, because it observes Q_1 before producing. Consequently, Firm 1 includes Firm 2's best response in its profits

$$\begin{aligned}
\max_{Q_1} \Pi_1 &= (20 - Q_1 - Q_2 - 4)\, Q_1 = \left(20 - Q_1 - \frac{16 - Q_1}{2} - 4\right) Q_1 \\
\Pi_1 &= \left(8 - \frac{Q_1}{2}\right) Q_1 \\
\frac{\partial \Pi_1}{\partial Q_1} &= 8 - Q_1 = 0 \Rightarrow Q_1 = 8
\end{aligned}$$

Plugging $Q_1 = 8$ into Firm 2's best response, we get $Q_2 = 4$.

Problem 6

Part a

You are considering buying a company. The company is either *good* or *bad*. Based on your initial information, the probability the company is good is 30%. You commission an audit of the company. The audit firm is either *corrupt* or *honest*. If the audit firm is corrupt, it produces a positive report regardless of whether the company is good or bad. If the audit firm is honest, it produces a positive report if and only if the company is good. The chance that the audit firm is corrupt is 20%. The audit report has come back positive. What should be your updated belief that the company is good?

Solution

This is a standard application of Bayes' Rule.

$$\begin{aligned}
Pr\,(good|positive) &= \frac{Pr\,(positive|good)\,Pr\,(good)}{Pr\,(positive|good)\,Pr\,(good) + Pr\,(positive|bad)\,Pr\,(bad)} \\
&= \frac{1 \times 0.3}{1 \times 0.3 + 0.2 \times 0.7} \\
&= \frac{.3}{.44} \\
&\simeq 0.68
\end{aligned}$$

Part b

Consider the exact same situation as above but now there are two audit firms. Each one is corrupt with probability 20%. Whether one is corrupt is independent of whether the other one is corrupt. You have commissioned a separate report from each one and **both** have come back with a positive report. What should be your updated belief that the company is good? (As before, your initial belief, before the reports, is that there is a 30% chance the company is good.)

Solution

Again we apply Bayes' Rule.

$$\begin{aligned}
Pr\,(good|2positive) &= \frac{Pr\,(2positive|good)\,Pr\,(good)}{Pr\,(2positive|good)\,Pr\,(good) + Pr\,(2positive|bad)\,Pr\,(bad)} \\
&= \frac{1 \times 0.3}{1 \times 0.3 + 0.2 \times 0.2 \times 0.7} \\
&= \frac{.3}{.328}
\end{aligned}$$

$$\simeq \quad 0.91$$