Hyper-dimensional Gap Finite Elements for the Enforcement of Frictionless Contact Constraints

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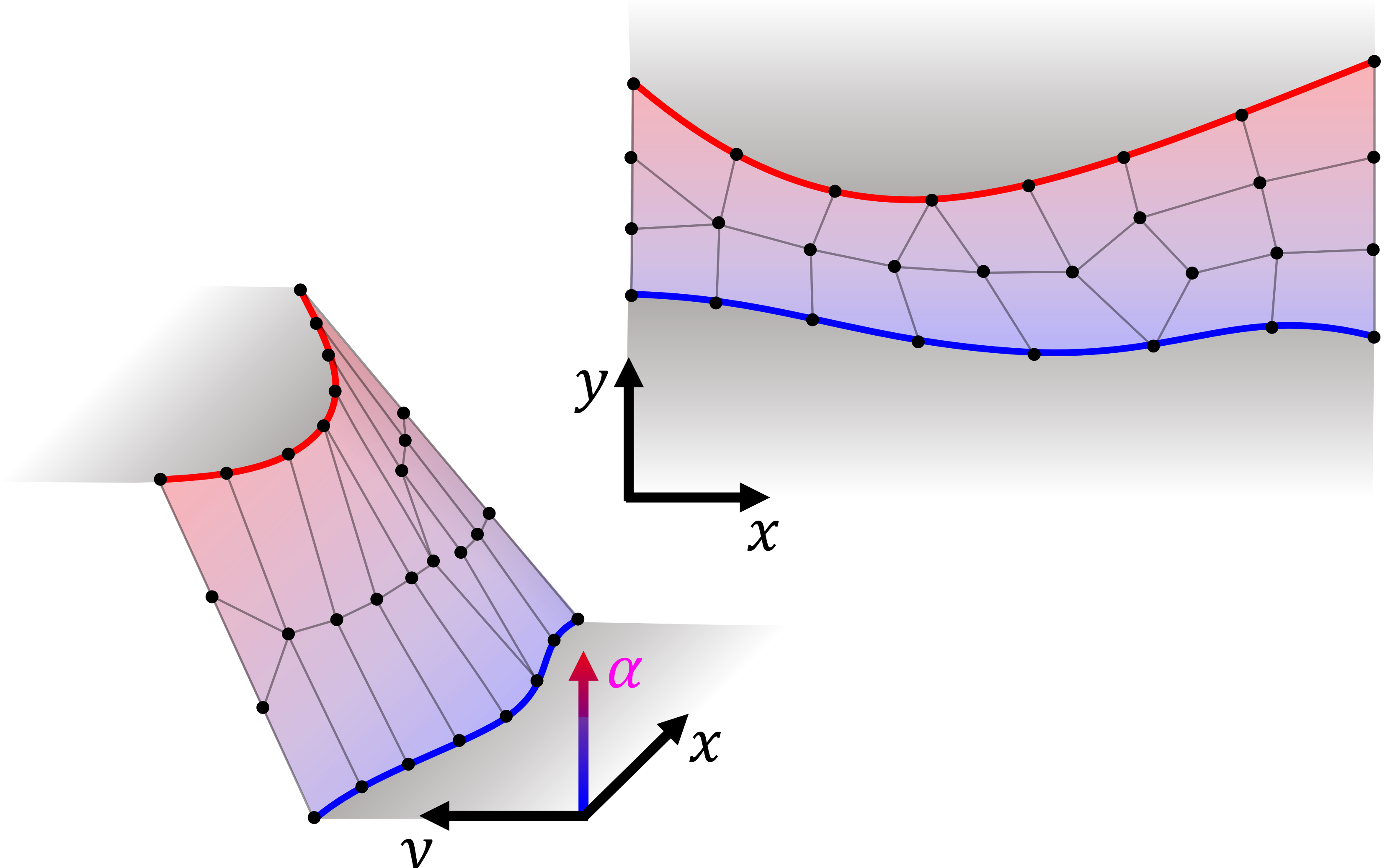
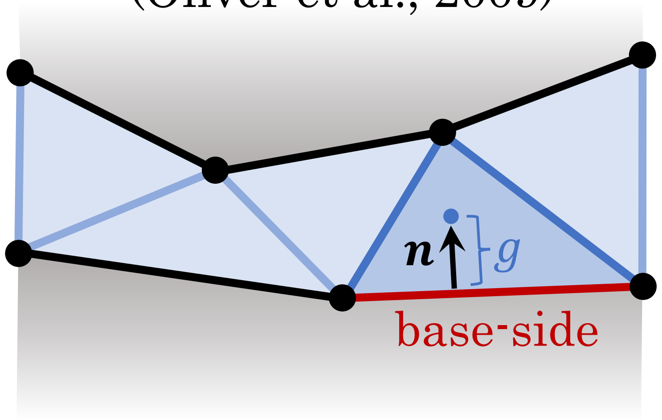
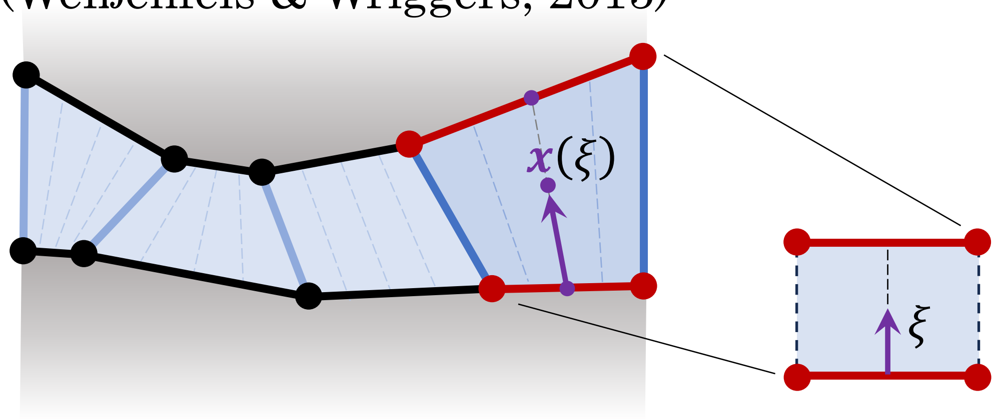


Comparison of volume-based contact discretization methods

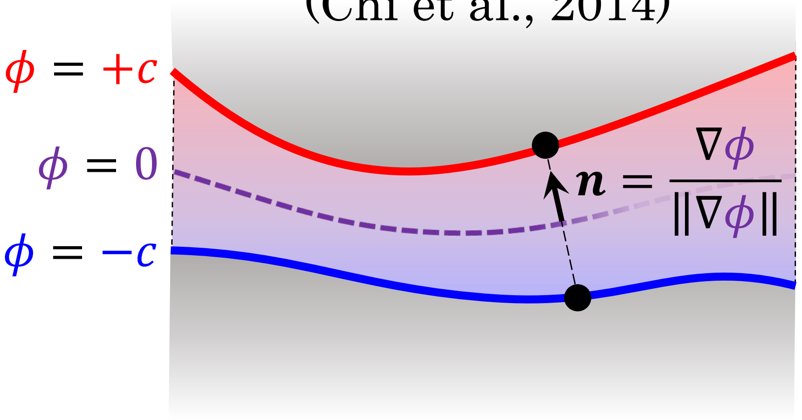
Contact domain method
(Oliver et al., 2009)



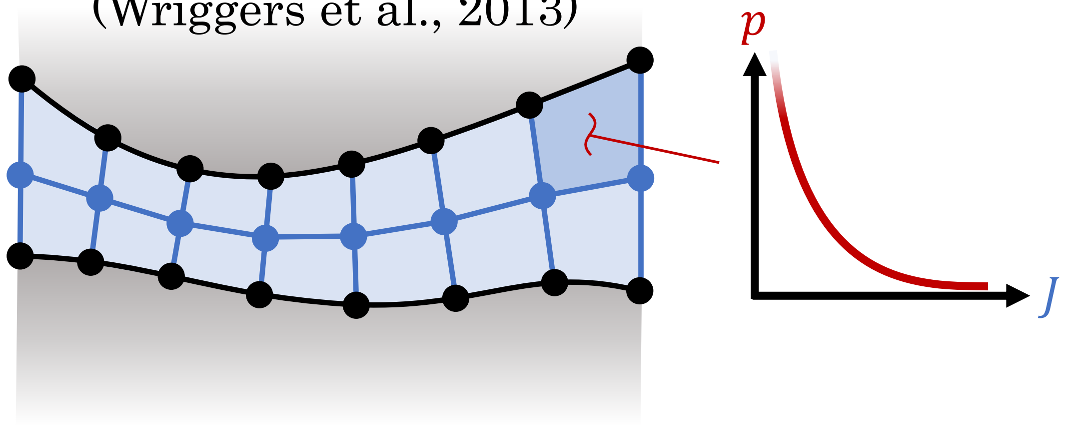
Contact layer elements
(Weißenfels & Wriggers, 2015)



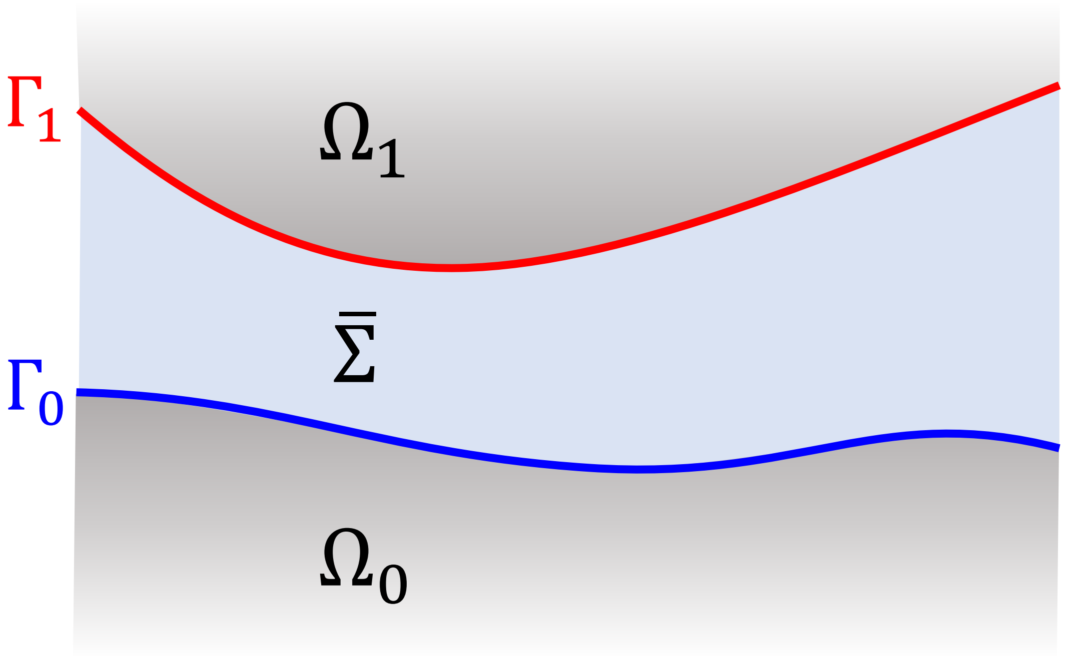
Level set method
(Chi et al., 2014)



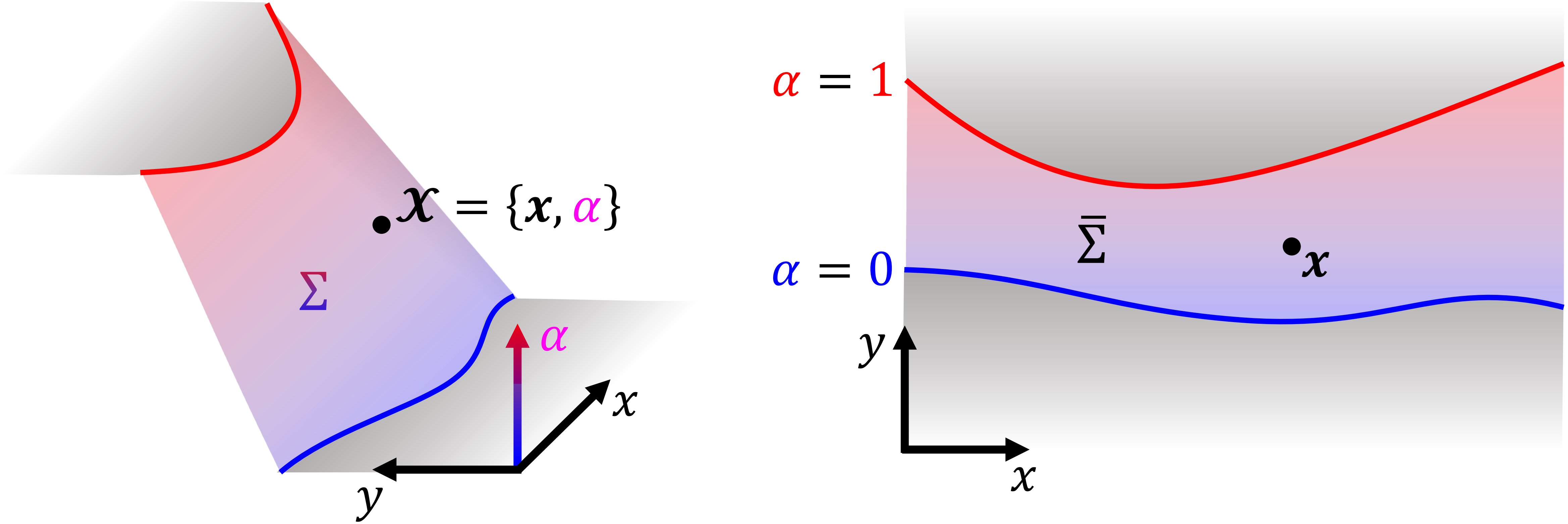
Third medium contact
(Wriggers et al., 2013)



Define an intermediate volume $\bar{\Sigma}$ between two bodies in unilateral contact



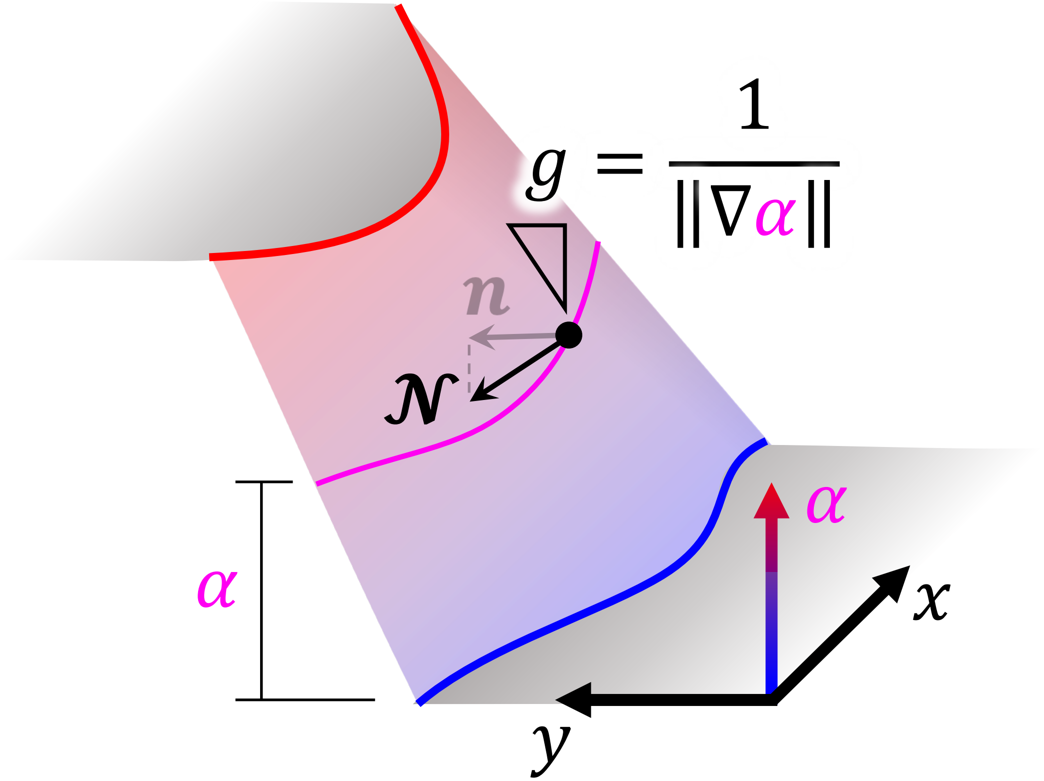
Introduce an additional coordinate $\alpha \in [0,1]$ such that $\Sigma \subset \mathbb{R}^{d+1}$ represents a d-dimensional manifold with hyper-dimensional coordinates $\boldsymbol{X} = \{\boldsymbol{x}, \alpha\} \in \Sigma$

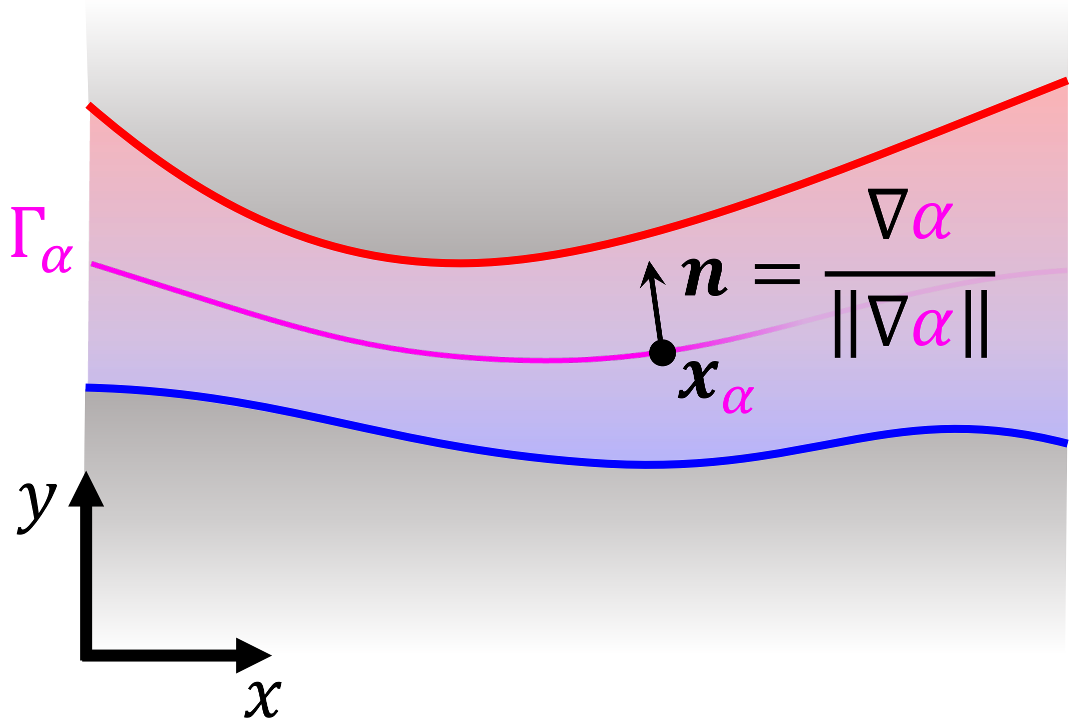


The hypersurface normal vector $\mathcal{N} \in \mathbb{R}^{d+1}$ is defined such that its orthographic projection $\boldsymbol{n} \in \mathbb{R}^d$ is the unit vector normal to the level surface $\Gamma_\alpha \subset \mathbb{R}^d$

Hypersurface normal:

$$\boxed{\boldsymbol{\mathcal{N}} = \{\boldsymbol{n}, -g\}}$$

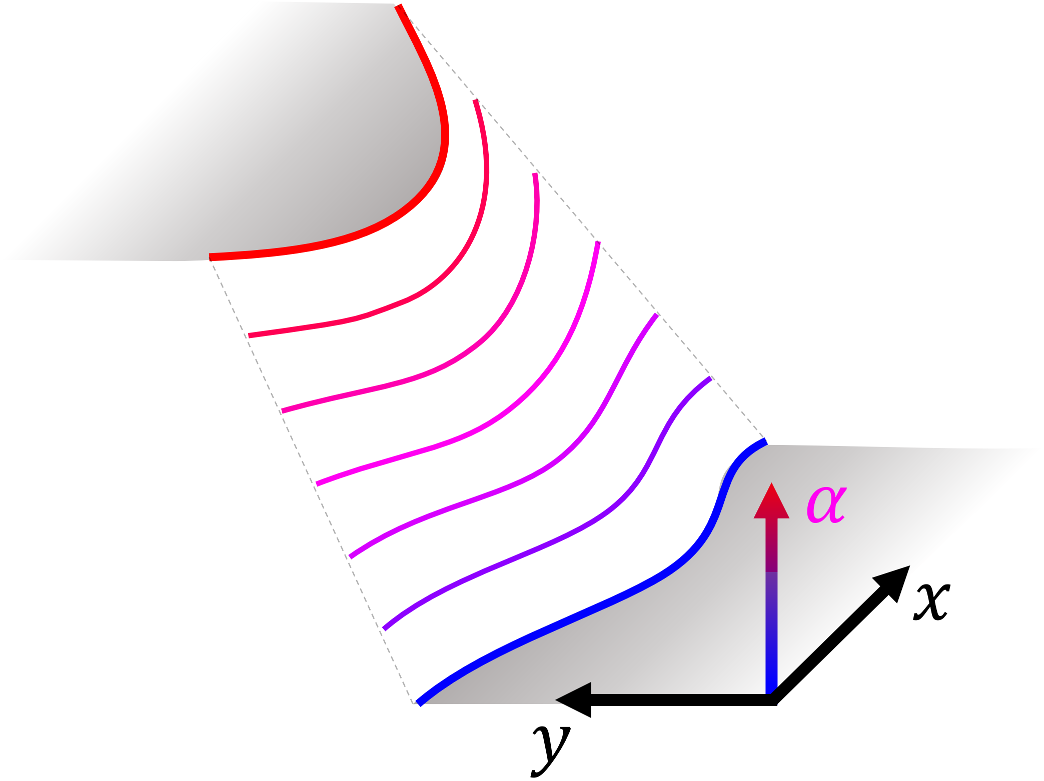


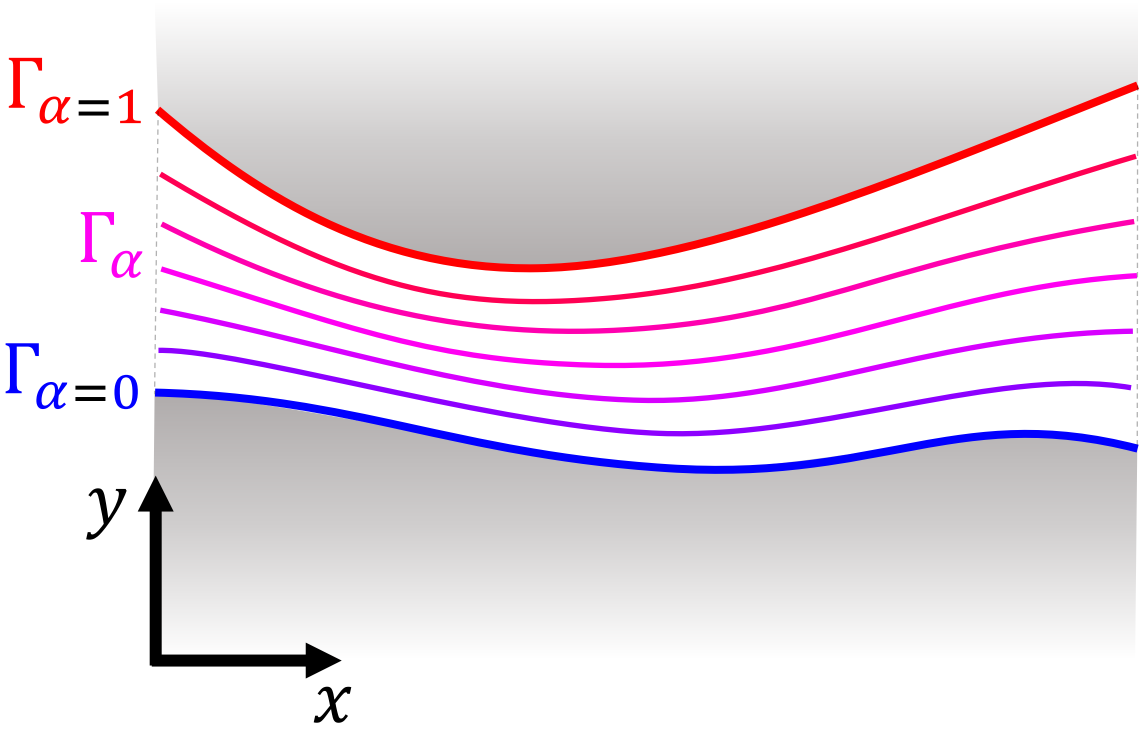


The family of surfaces Γ_α parameterized by α defines a *foliation* of the intermediate volume

Co-area formula:
(Federer, 1959)

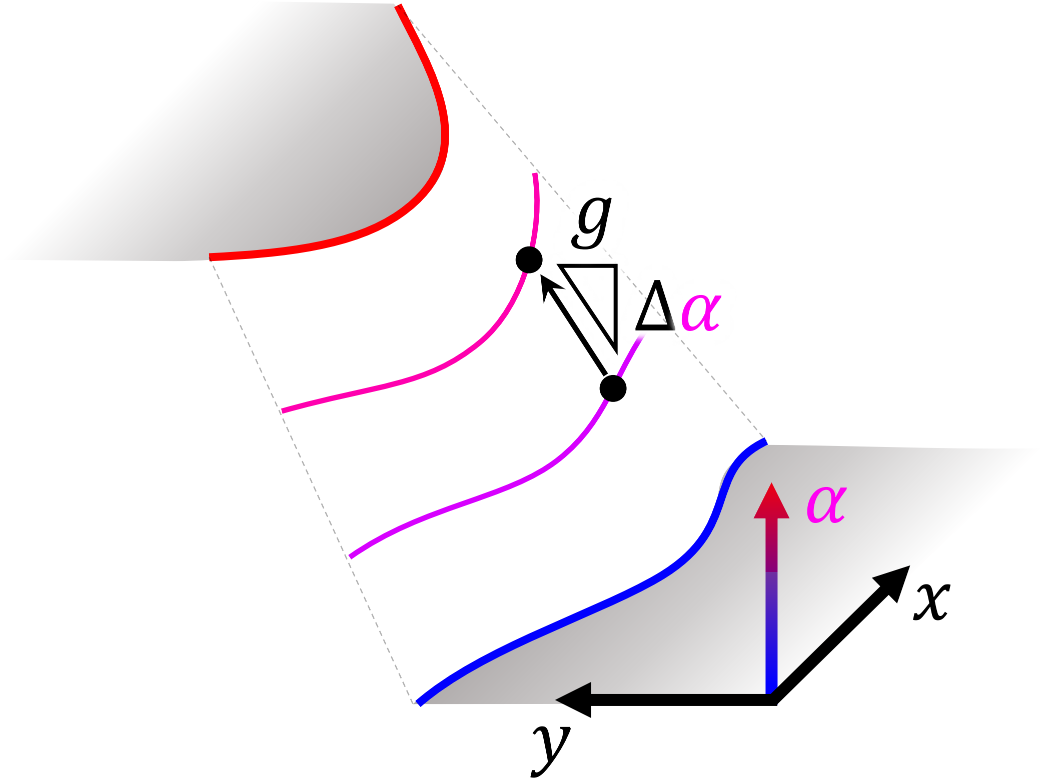
$$\boxed{\int_{\bar{\Sigma}} f(\boldsymbol{x}) \|\nabla\alpha\| \, \mathrm{d}V = \int_{\alpha=0}^{\alpha=1} \left(\int_{\Gamma_\alpha} f(\boldsymbol{x}_\alpha) \, \mathrm{d}\Gamma_\alpha \right) \mathrm{d}\alpha}$$

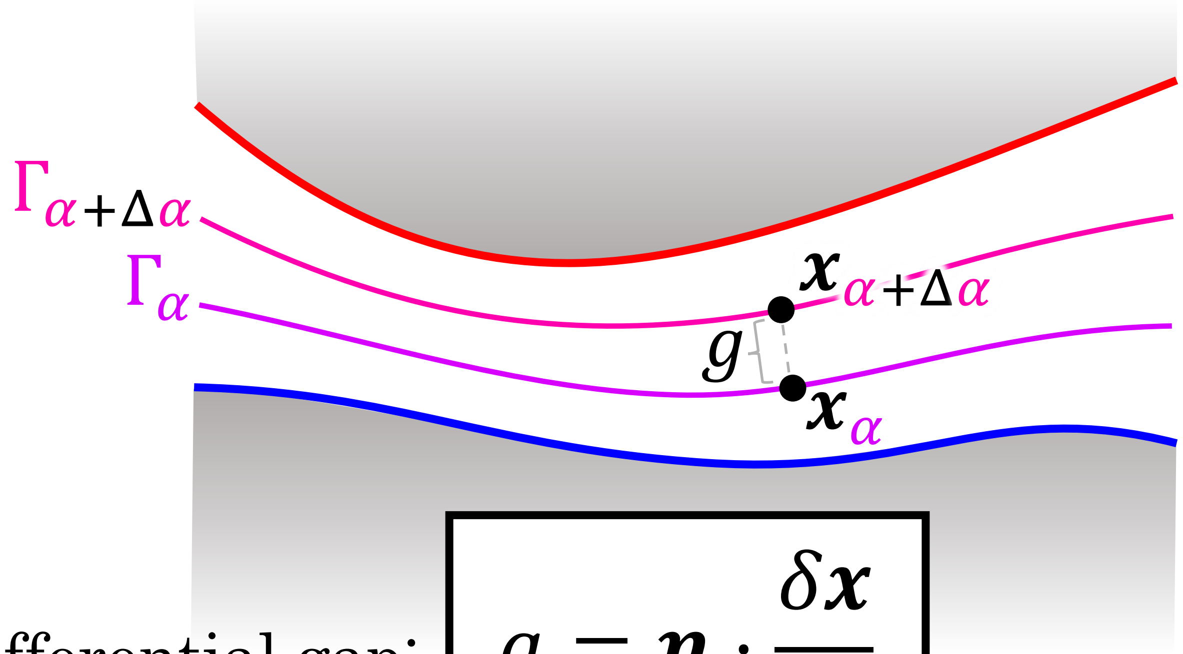




Definition of the $\delta/\delta\alpha$-derivative (Hadamard, 1903) via the calculus of moving surfaces (Grinfeld, 2010)

$\delta/\delta\alpha$-derivative:
$$\boxed{\frac{\delta f}{\delta \alpha} = \lim_{\Delta\alpha \to 0} \frac{f(\boldsymbol{x}_{\alpha+\Delta\alpha}) - f(\boldsymbol{x}_{\alpha})}{\Delta\alpha}}$$





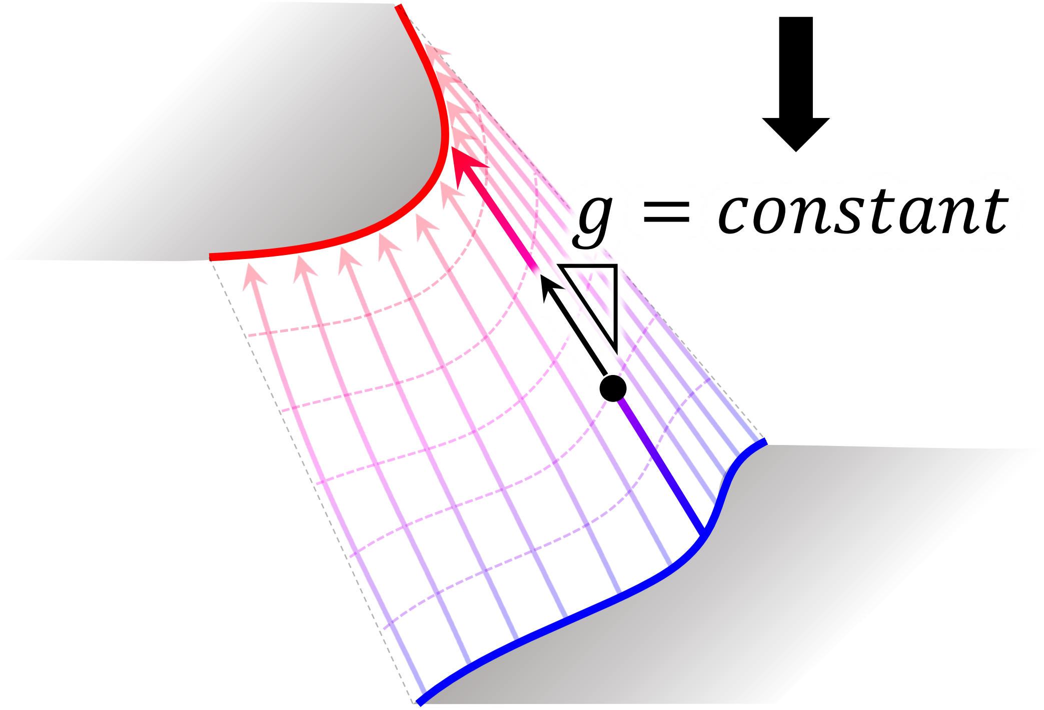
Differential gap:
$$\boxed{g = \boldsymbol{n} \cdot \frac{\delta \boldsymbol{x}}{\delta \alpha}}$$

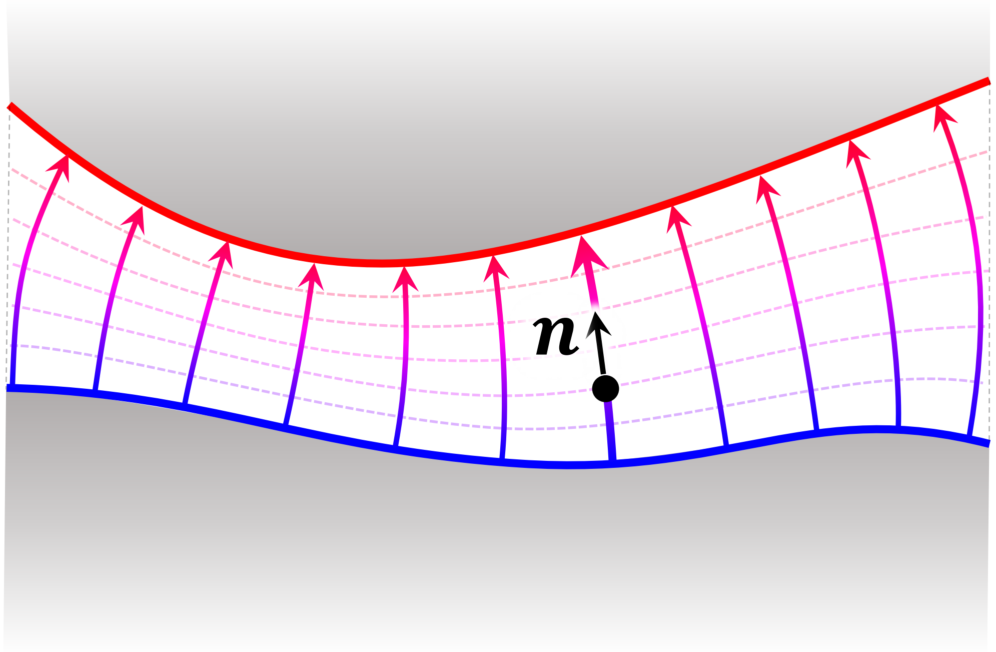
The hypersurface is uniquely defined such that g is *constant* along all gradient flow curves tangent to n

Zero normal curvature: $\boxed{\dfrac{\delta g}{\delta \alpha} = 0}$ $\Longleftrightarrow$ $$\min_{\boldsymbol{x}} \int_{\alpha=0}^{\alpha=1} \left(\int_{\Gamma_\alpha} g^2 \, \mathrm{d}\Gamma_\alpha \right) \mathrm{d}\alpha$$

$\Downarrow$

$g = constant$

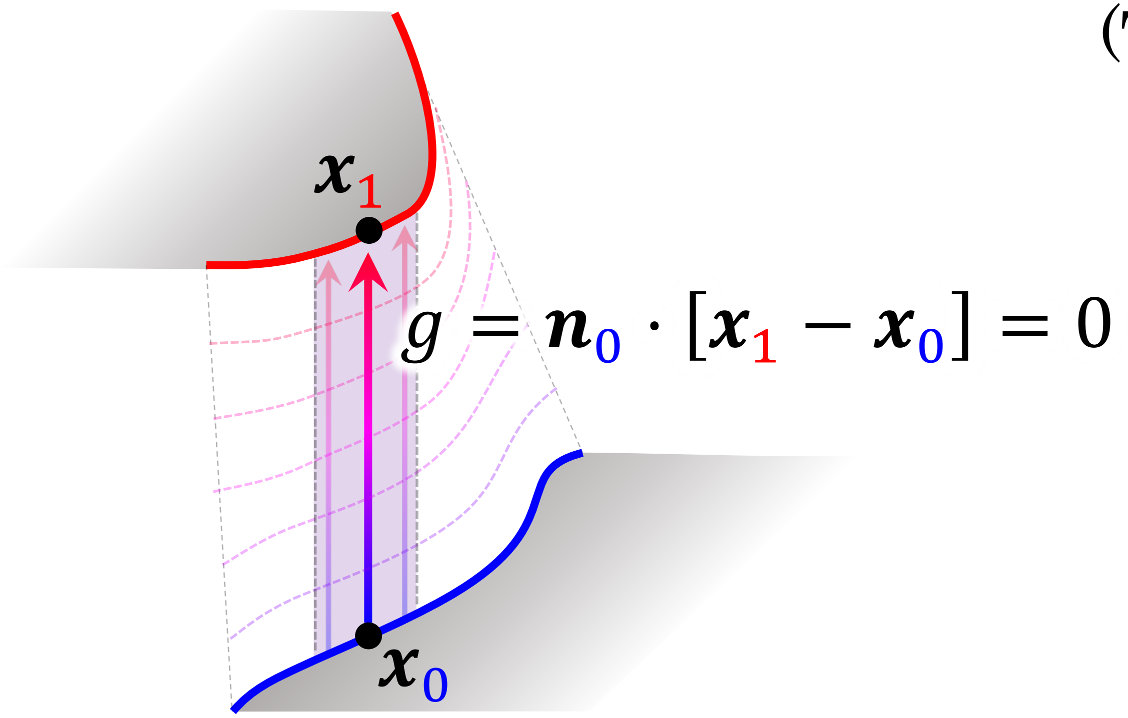


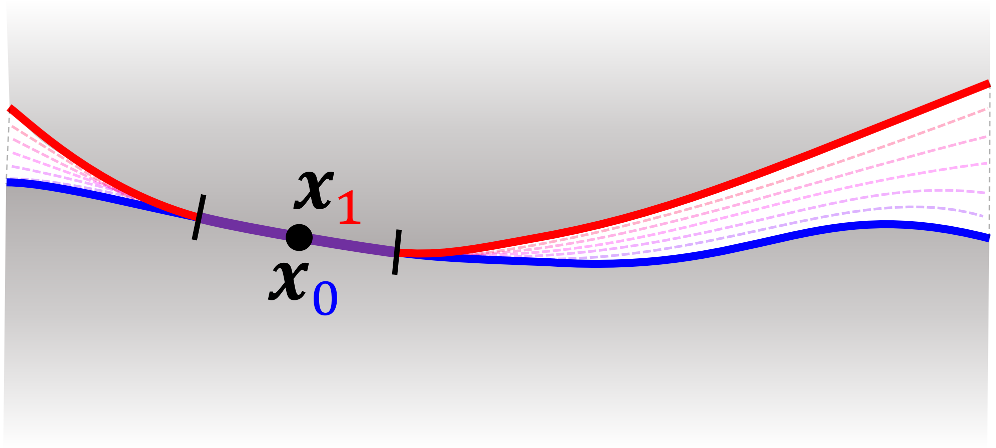


If $g = 0$ over a subset of Σ, projected point pairs on opposing surfaces will be joined by *straight* geodesic curves traversing the (locally "ruled") hypersurface

If $\quad g = 0 \quad \Longrightarrow \quad \dfrac{\delta \boldsymbol{n}}{\delta \alpha} = \boldsymbol{0}$ gradient flow curves have *zero curvature*

(TY Thomas, 1957)

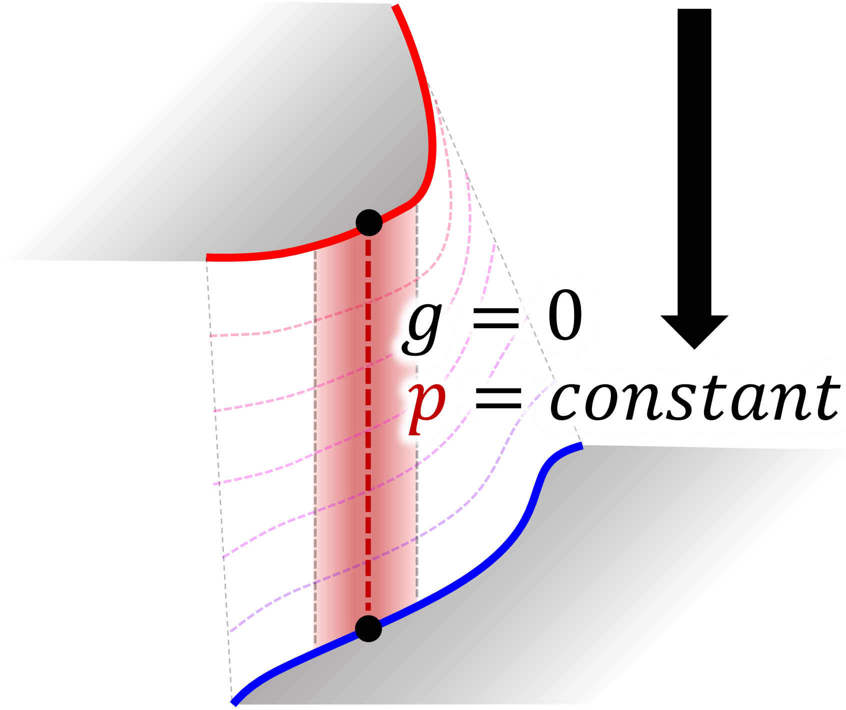


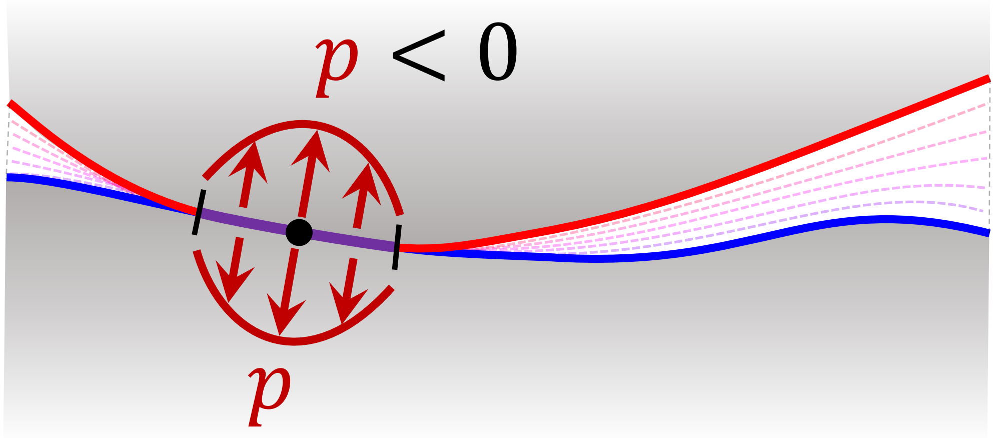


Introduce an extended *contact pressure field* p defined over the hypersurface, subject to the *Signorini conditions*, and *regularity constraints*

$$\frac{\delta g}{\delta \alpha} = 0 \qquad \frac{\delta p}{\delta \alpha} = 0$$

$$g \geq 0 \qquad p \leq 0 \qquad pg = 0$$





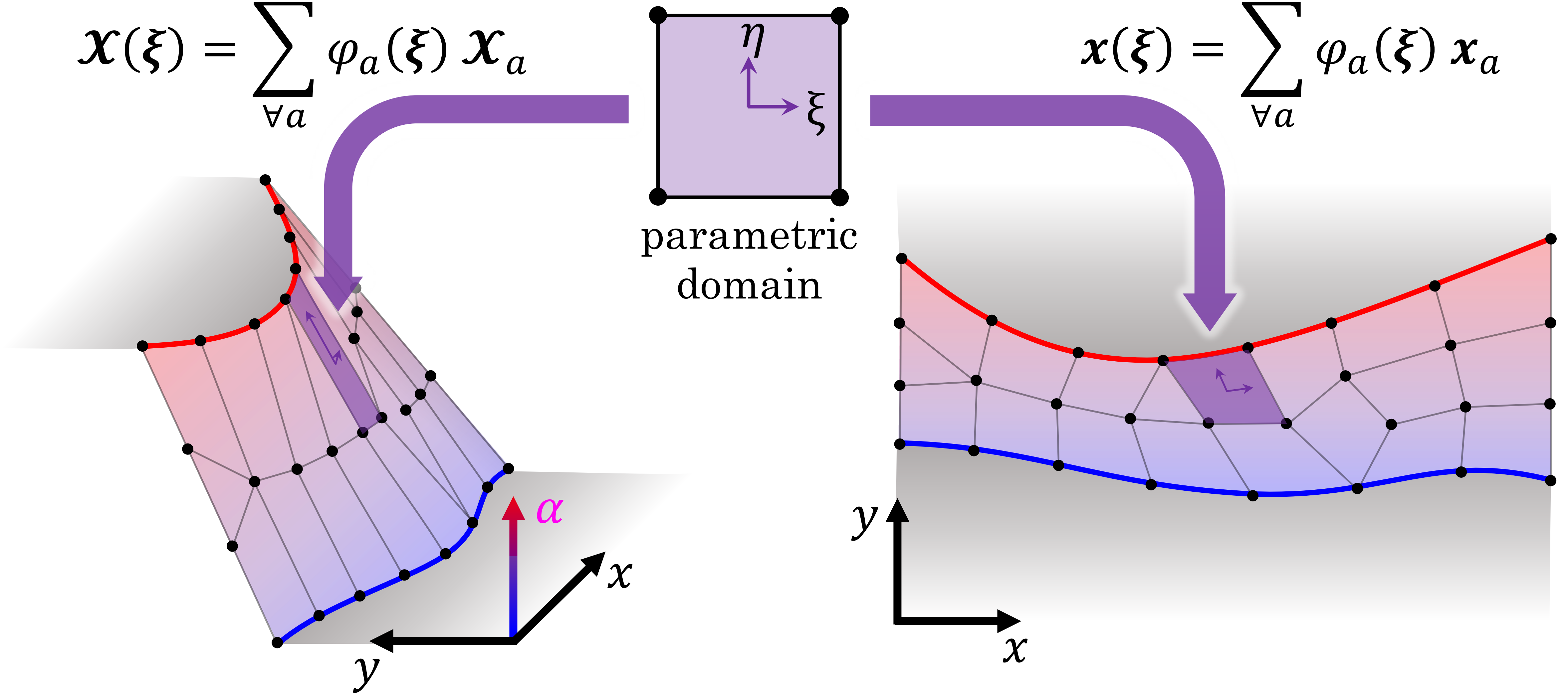
Hyper-dimensional formulations of frictionless contact are direct analogs of classical contact enforcement methods

	Integration over a single *reference surface* Γ_0		Integration over the *foliated hypersurface* $\Sigma = \{\Gamma_\alpha\}_{\alpha=0}^{\alpha=1}$
Gap function:	$g = \boldsymbol{n}_0 \cdot [\boldsymbol{x}_1 - \boldsymbol{x}_0]$	➡	$g = \boldsymbol{n} \cdot \dfrac{\delta \boldsymbol{x}}{\delta \alpha}$
Mortar method:	Lagrange multiplier $\Pi + \int_{\Gamma_0} p g \, \mathrm{d}\Gamma_0$	➡	$\Pi + \int_{\alpha=0}^{\alpha=1} \left(\int_{\Gamma_\alpha} p g \, \mathrm{d}\Gamma_\alpha \right) \mathrm{d}\alpha$
Penalty method:	penalty parameter $\Pi + \dfrac{1}{2} \int_{\Gamma_0} \kappa \langle -g \rangle^2 \mathrm{d}\Gamma_0$	➡	$\Pi + \dfrac{1}{2} \int_{\alpha=0}^{\alpha=1} \left(\int_{\Gamma_\alpha} \kappa \langle -g \rangle^2 \mathrm{d}\Gamma_\alpha \right) \mathrm{d}\alpha$

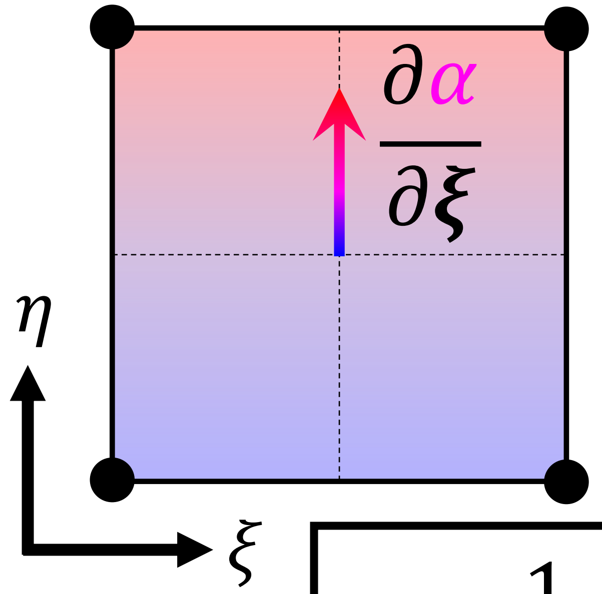
Let the hypersurface coordinates be expressed in terms of a set of *parametric coordinates* $\xi \subset \mathbb{R}^d$ (e.g., using isoparametric finite elements)

$$\boldsymbol{X}(\xi) = \sum_{\forall a} \varphi_a(\xi)\, \boldsymbol{X}_a$$

$$\boldsymbol{x}(\xi) = \sum_{\forall a} \varphi_a(\xi)\, \boldsymbol{x}_a$$

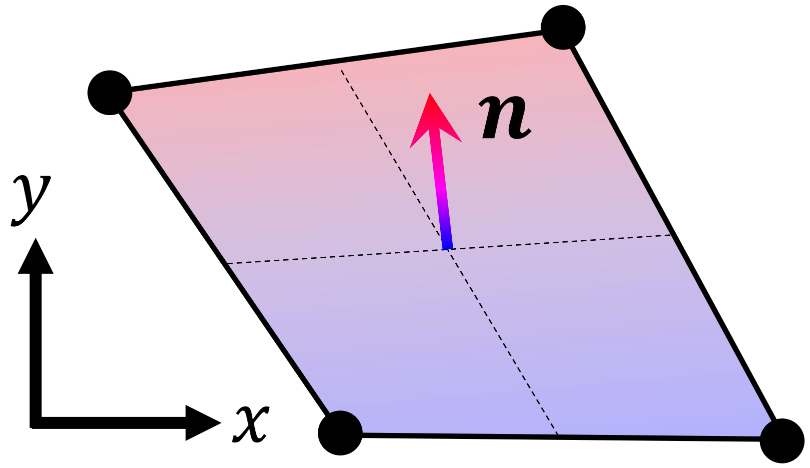


The Jacobian of the parametric map determines:
(1) the normal n, gap g, and $\delta/\delta\alpha$-derivative of ξ
(2) integration over the foliated hypersurface



$$\mathbf{J} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}}$$

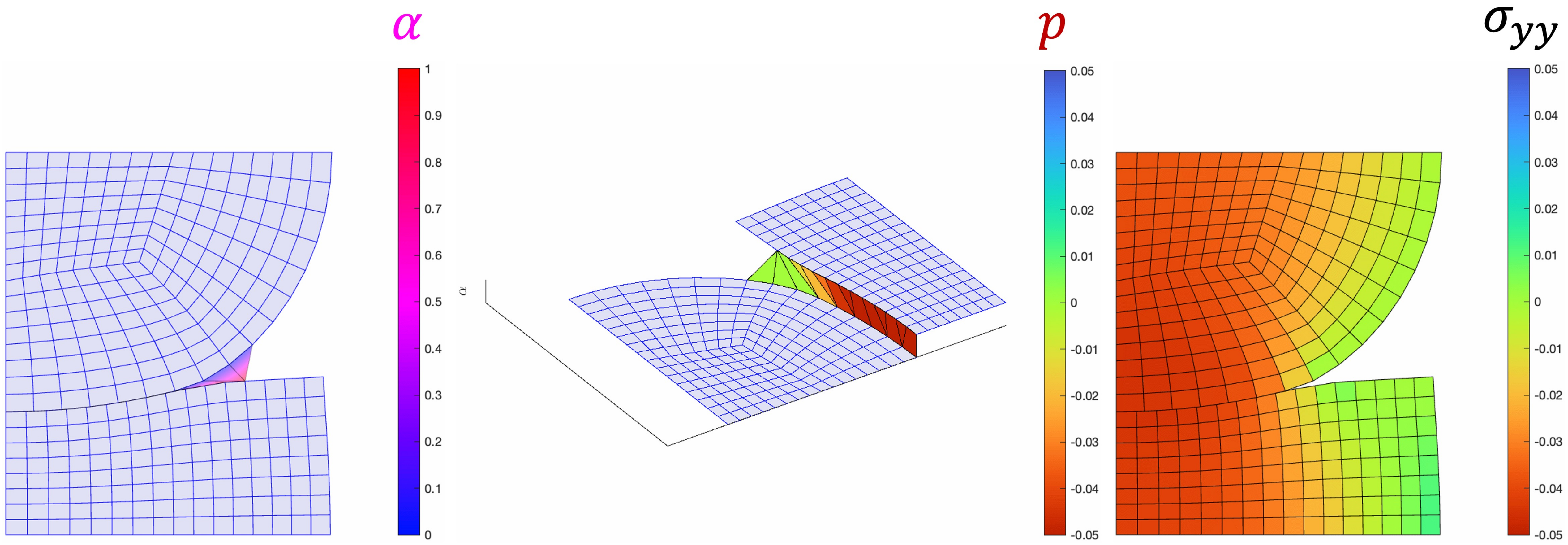
$$\mathcal{J} = \left\| \operatorname{cof}(\mathbf{J}) \cdot \frac{\partial \alpha}{\partial \boldsymbol{\xi}} \right\|$$



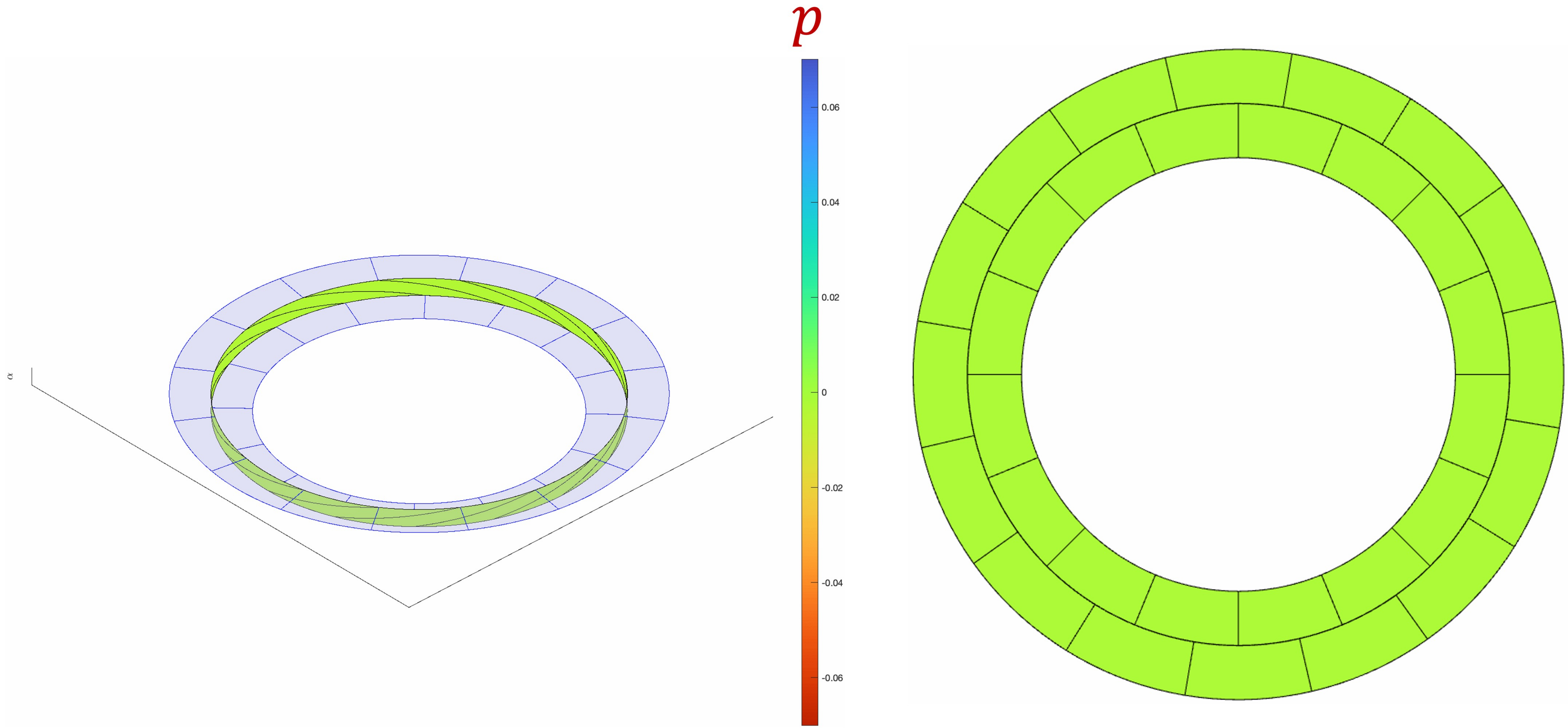
(1) $$\boldsymbol{n} = \frac{1}{\mathcal{J}} \operatorname{cof}(\mathbf{J}) \cdot \frac{\partial \alpha}{\partial \boldsymbol{\xi}} \qquad g = \frac{1}{\mathcal{J}} \det(\mathbf{J}) \qquad \frac{\delta \boldsymbol{\xi}}{\delta \alpha} = \boldsymbol{n} \cdot \frac{1}{\mathcal{J}} \operatorname{cof}(\mathbf{J})$$

(2) $$\int_{\alpha=0}^{\alpha=1} \left(\int_{\Gamma_\alpha} f(\boldsymbol{x}, \alpha)\, \mathrm{d}\Gamma_\alpha \right) \mathrm{d}\alpha = \int_{\square} f(\boldsymbol{\xi})\, \mathcal{J} \mathrm{d}\square$$

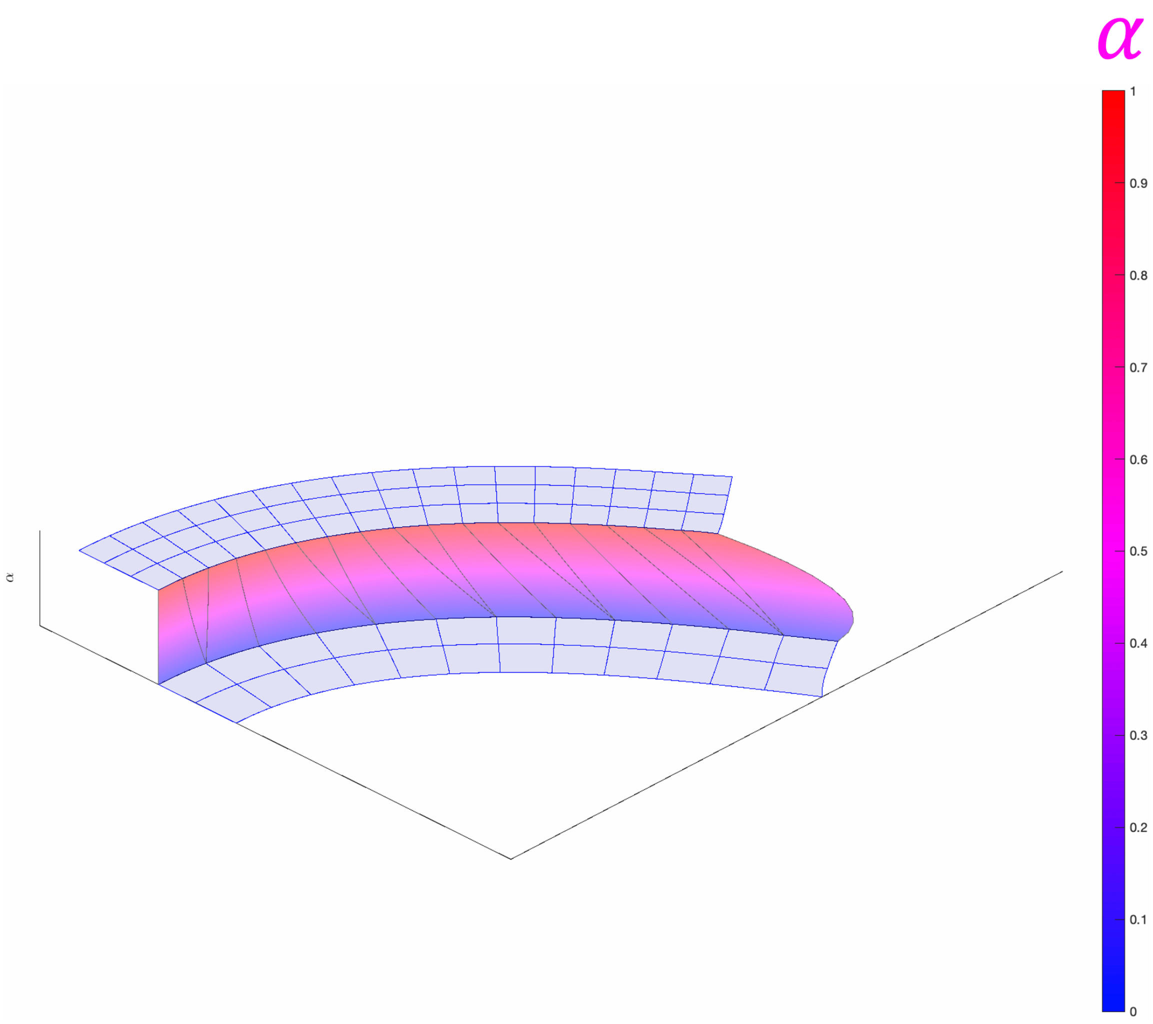
Demonstration of penalty-based hyper-dimensional frictionless contact enforcement

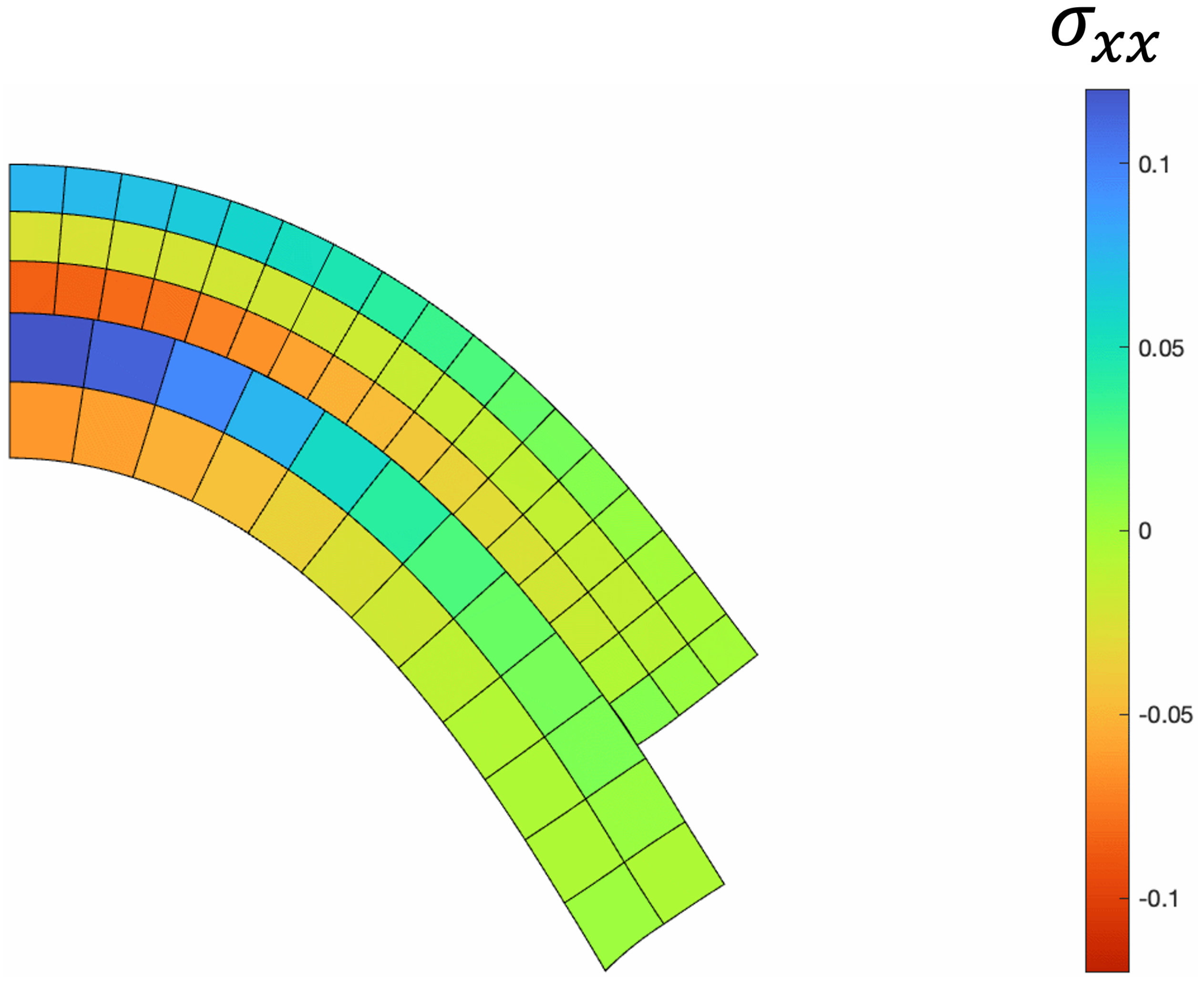


Demonstration of penalty-based hyper-dimensional frictionless contact enforcement



Demonstration of sliding-only hyper-dimensional frictionless mortar contact





Define $\frac{\delta}{\delta x} = \left\{\overline{\overline{\nabla}}, \frac{\delta}{\delta \alpha}\right\}$ as the $d+1$ decomposition of the *hypersurface gradient operator*

$\delta/\delta\alpha$-derivative:

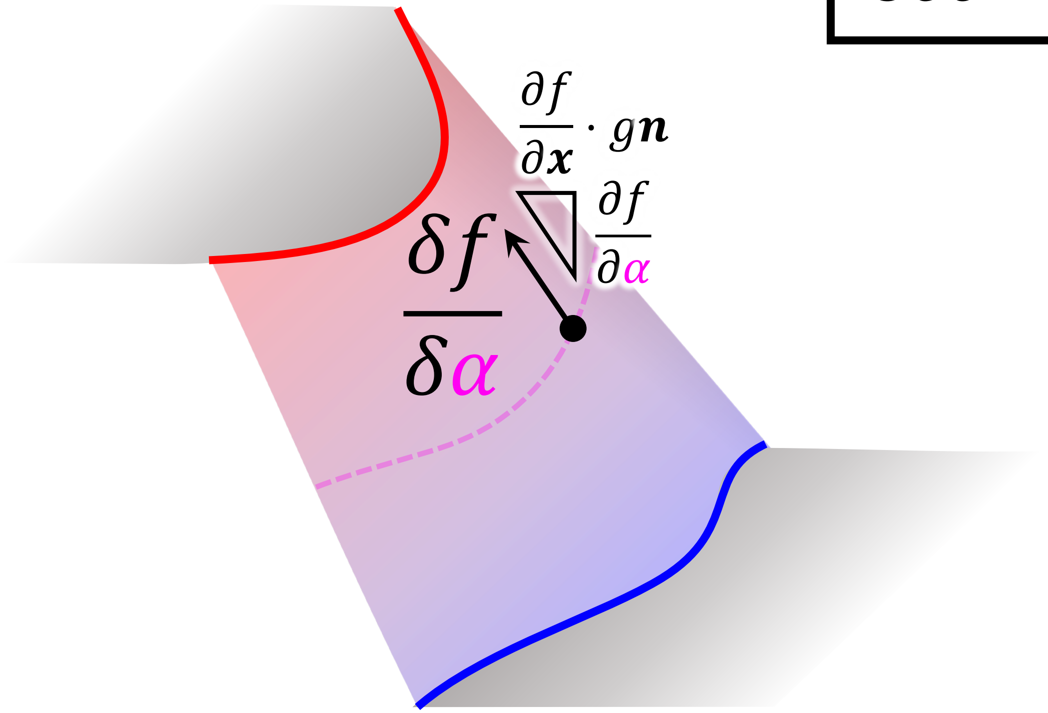
$$\frac{\delta f}{\delta \alpha} = \frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial \boldsymbol{x}} \cdot g\boldsymbol{n}$$

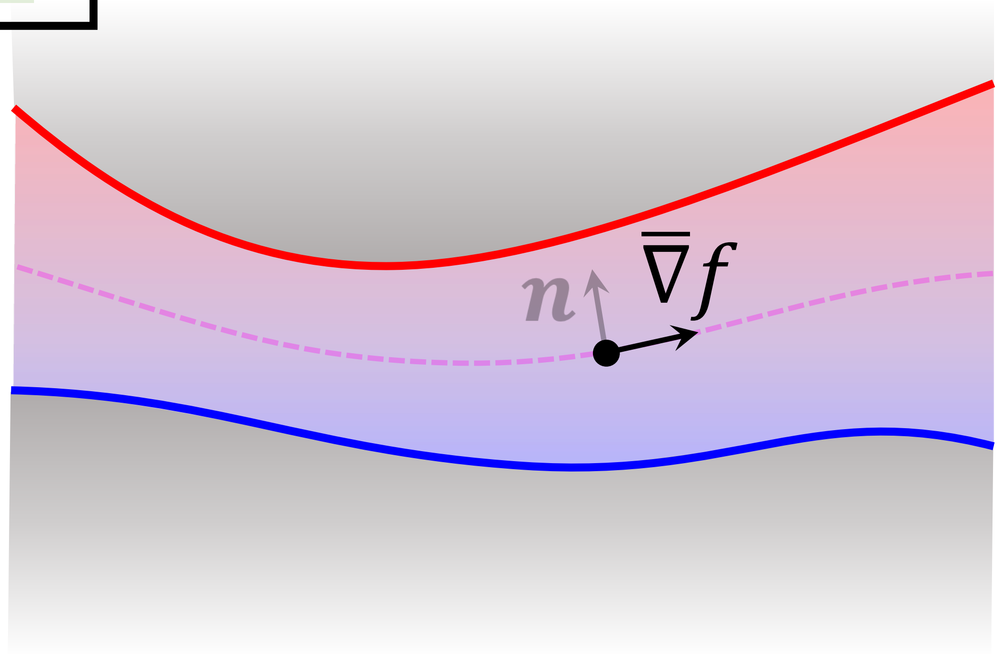
Hypersurface gradient

$$\boxed{\frac{\delta f}{\delta \boldsymbol{x}} = \left\{\overline{\overline{\nabla}} f, \frac{\delta f}{\delta \alpha}\right\}}$$

Surface gradient:

$$\overline{\overline{\nabla}} f = \frac{\partial f}{\partial \boldsymbol{x}} \cdot [\mathbf{1} - \boldsymbol{n} \otimes \boldsymbol{n}]$$

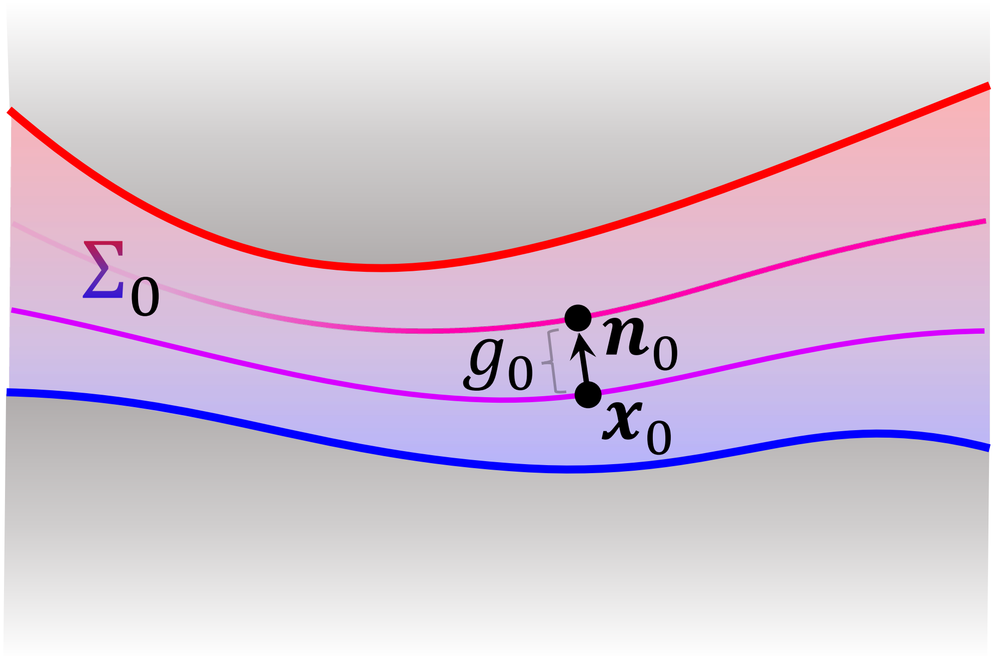




Suppose there exists a mapping $\chi(t) : \Sigma_0 \mapsto \Sigma(t)$ relating the *undeformed* coordinates $\mathcal{X}_0 \in \Sigma_0$ to the *deformed* coordinates $\mathcal{X}(t) \in \Sigma(t)$ of the hypersurface

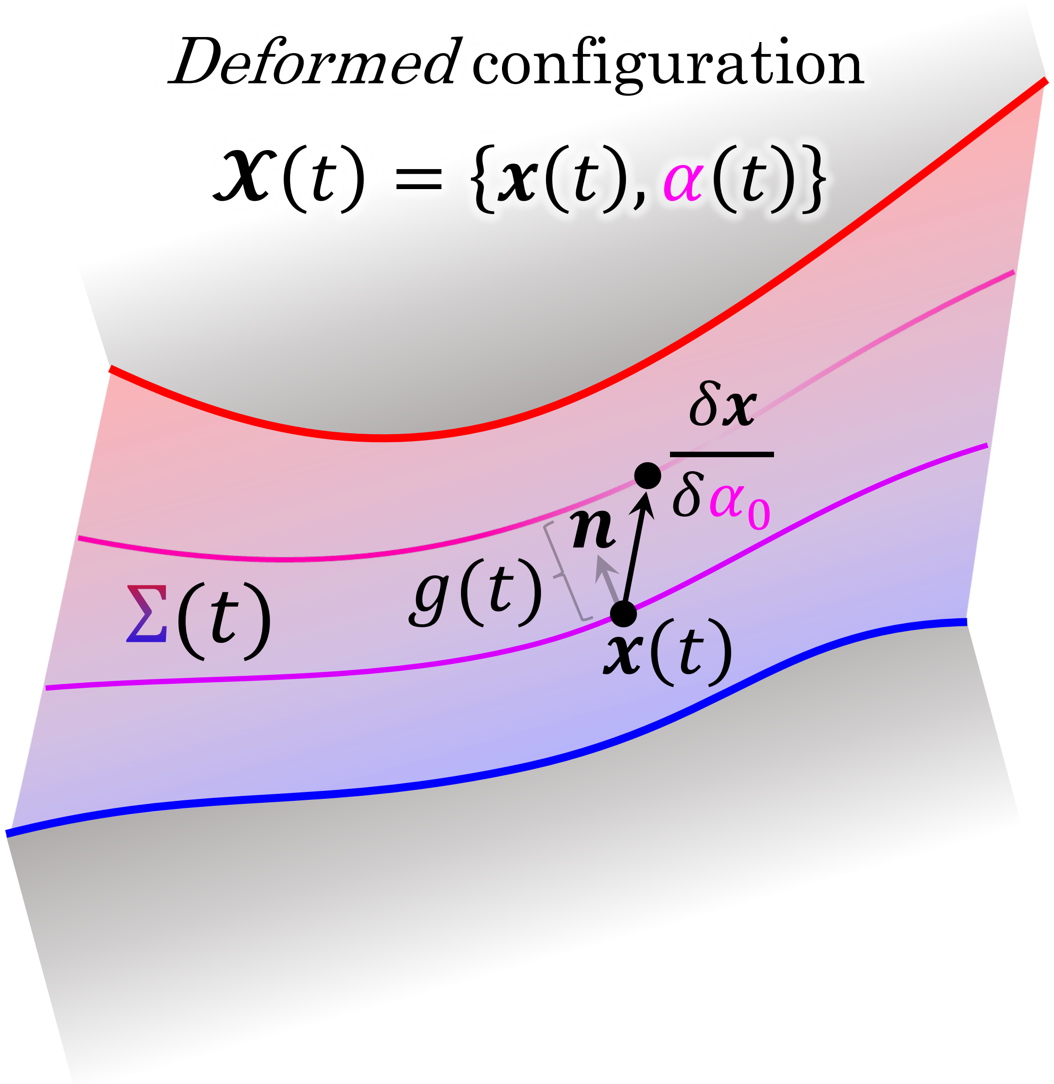
Undeformed configuration

$$\boldsymbol{\mathcal{X}}_0 = \{\boldsymbol{x}_0, \alpha_0\}$$



Deformed configuration

$$\boldsymbol{\mathcal{X}}(t) = \{\boldsymbol{x}(t), \alpha(t)\}$$

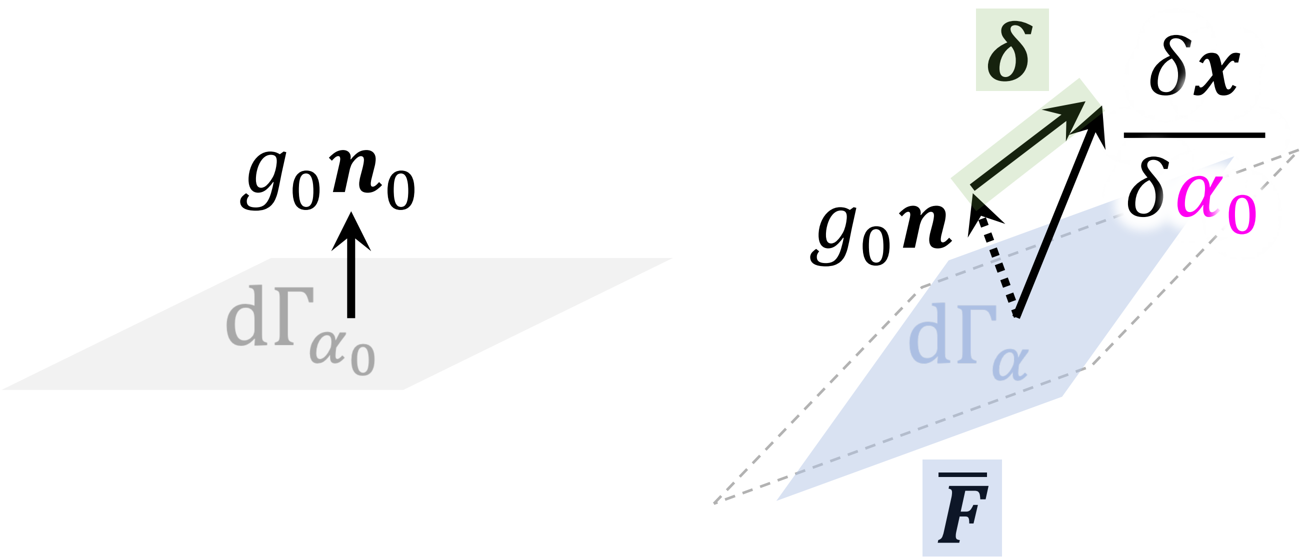


Define the *hypersurface deformation gradient*, decomposed in a block-structured format:

Hypersurface deformation gradient:

$$\boxed{\boldsymbol{\mathcal{F}} = \frac{\delta \boldsymbol{\mathcal{X}}}{\delta \boldsymbol{\mathcal{X}}_0} + \boldsymbol{n} \otimes \boldsymbol{\mathcal{N}}_0} \quad \Longrightarrow \quad \boldsymbol{\mathcal{F}} = \left[\begin{array}{c:c} \overline{\boldsymbol{F}} & \boldsymbol{\delta} \\ \hdashline \overline{\nabla}_0 \alpha & \delta\alpha / \delta\alpha_0 \end{array}\right]$$

Undeformed: *Deformed:*



Differential displacement vector:

$$\boldsymbol{\delta} = \frac{\delta \boldsymbol{x}}{\delta \alpha_0} - g_0 \boldsymbol{n}$$

Surface deformation gradient:

$$\overline{\boldsymbol{F}} = \overline{\nabla}_0 \boldsymbol{x} + \boldsymbol{n} \otimes \boldsymbol{n}_0$$

The *hypersurface velocity gradient* is similarly decomposed in a block-structured format:

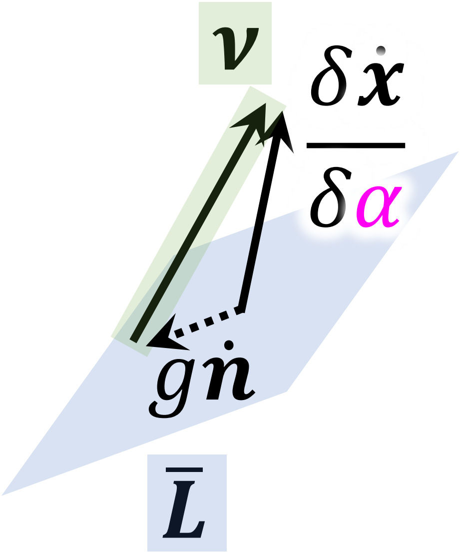
Hypersurface velocity gradient:

$$\boxed{\mathcal{L} = \frac{\delta \dot{\mathcal{X}}}{\delta \mathcal{X}} + \dot{\boldsymbol{n}} \otimes \mathcal{N}}$$

$$\mathcal{L} = \left[\begin{array}{c:c} \bar{\boldsymbol{L}} & \boldsymbol{\nu} \\ \hdashline \bar{\nabla}\dot{\alpha} & \delta\dot{\alpha}/\delta\alpha \end{array}\right]$$

$$\dot{\mathcal{N}} = -\mathcal{N} \cdot \frac{\delta \dot{\mathcal{X}}}{\delta \mathcal{X}}$$

(TY Thomas, 1957)



Differential velocity vector:

$$\boldsymbol{\nu} = \frac{\delta \dot{\boldsymbol{x}}}{\delta \alpha} - g\dot{\boldsymbol{n}}$$

Surface velocity gradient:

$$\bar{\boldsymbol{L}} = \bar{\nabla}\dot{\boldsymbol{x}} + \dot{\boldsymbol{n}} \otimes \boldsymbol{n}$$

The *hyper-dimensional Jacobian* and its inverse ease the computation of the hypersurface deformation and velocity gradients

Hyper-dimensional Jacobian:

$$\boxed{\mathcal{J} = \frac{\partial \boldsymbol{X}}{\partial \xi} + \boldsymbol{n} \otimes \boldsymbol{e}_\alpha} \quad \longrightarrow \quad \mathcal{J} = \begin{bmatrix} \partial \boldsymbol{x}/\partial \xi & \boldsymbol{n} \\ \partial \alpha/\partial \xi & 0 \end{bmatrix}$$

$$\boxed{\mathcal{J}^{-1} = \frac{\delta \xi}{\delta \boldsymbol{X}} + \boldsymbol{e}_\alpha \otimes \boldsymbol{\mathcal{N}}} \quad \longrightarrow \quad \mathcal{J}^{-1} = \begin{bmatrix} \bar{\nabla} \xi & \delta \xi/\delta \alpha \\ \boldsymbol{n} & -g \end{bmatrix}$$

$$\boxed{\boldsymbol{\mathcal{F}} = \mathcal{J}\mathcal{J}_0^{-1}} \qquad \boxed{\boldsymbol{\mathcal{L}} = \dot{\mathcal{J}}\mathcal{J}^{-1}} \qquad \frac{\delta \varphi_a}{\delta \boldsymbol{X}} = \frac{\partial \varphi_a}{\partial \xi} \cdot \mathcal{J}^{-1} = \left\{ \bar{\nabla} \varphi_a, \frac{\delta \varphi_a}{\delta \alpha} \right\}$$

Constitutive models for frictional sliding are formulated in terms of *hypersurface deformation and stress measures*

Hypersurface Green-Lagrangian strain:

$$\boxed{\boldsymbol{\mathcal{E}} = \frac{1}{2}(\boldsymbol{\mathcal{F}}^T\boldsymbol{\mathcal{F}} - \boldsymbol{I})}$$

$$\xrightarrow[(\dot{\alpha} = 0)]{} \quad \boldsymbol{\mathcal{E}} = \frac{1}{2}\left[\begin{array}{c:c} 2\overline{\boldsymbol{E}} & \boldsymbol{\Delta} \\ \hdashline \boldsymbol{\Delta} & \boldsymbol{\delta}\cdot\boldsymbol{\delta} \end{array}\right]$$

Frame invariant elastic potential: $\psi(\boldsymbol{\mathcal{E}})$

$$\Downarrow$$

Hypersurface Kirchhoff stress:

$$\boxed{\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{F}}\frac{\partial\psi}{\partial\boldsymbol{\mathcal{E}}}\boldsymbol{\mathcal{F}}^T}$$

Co-rotational differential displacement:

$$\boldsymbol{\Delta} = \overline{\boldsymbol{F}}^T \cdot \boldsymbol{\delta}$$

Surface Green-Lagrangian strain:

$$\overline{\boldsymbol{E}} = \frac{1}{2}(\overline{\boldsymbol{F}}^T\overline{\boldsymbol{F}} - \boldsymbol{I})$$

Variational consistency necessitates the use of hyper-dimensional stress and deformation measures to determine nodal forces

Contact stress power: $\dot{\psi} = \boldsymbol{\mathcal{T}} : \boldsymbol{\mathcal{L}}$

Contact (virtual) work: $\dot{W} = \int_{\alpha=0}^{\alpha=1} \left(\int_{\Gamma_\alpha} \dot{\psi} \, \mathrm{d}\Gamma_\alpha \right) \mathrm{d}\alpha = \sum_{\forall a} \boldsymbol{f}_a \cdot \dot{\boldsymbol{x}}_a$

Variationally consistent nodal forces:

$$\boxed{\boldsymbol{f}_a = \int_{\square} \left(\boldsymbol{\mathcal{T}} \cdot \frac{\delta \varphi_a}{\delta \boldsymbol{X}} - \boldsymbol{\mathcal{N}}\{\bar{\bar{\nabla}} \varphi_a \cdot \boldsymbol{\mathcal{T}} \cdot \boldsymbol{\mathcal{N}}\} \right) \mathcal{J} \mathrm{d}\square}$$

Ongoing and future work

- *Frictional sliding*
- Expanded theoretical framework (publication in preparation)
- Alternative enforcement methods (dual pass formulation)
- Alternative discretization methods (meshfree, IGA, BEM, etc.)
- Connections to the mathematics of general relativity:
    - ADM formalism (Arnowitt, Deser and Misner, 1962)

Questions?