FINITE ELEMENT : TECHNOLOGY

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Gauss integration
- Definition
- 4-node quadrilateral
- Quality of the integration

Patch test

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Gauss integration

r-point Gauss integration on a [-1:+1] segment:

$$\int_{-1}^{+1} f(\xi)d\xi \approx \sum_{1}^{r} w_i f(\xi_i)$$

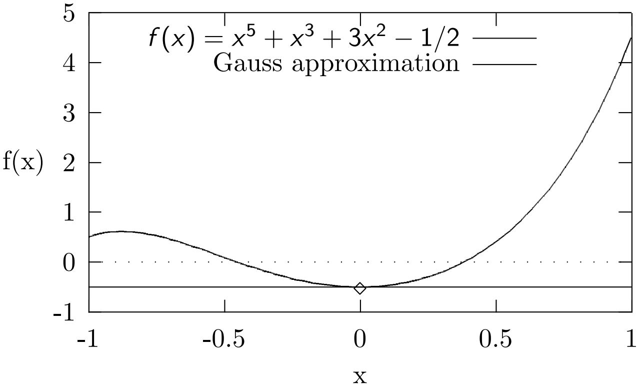
gives exact result for a $(2r-1)$ order polynom
Evaluation at *sampling points* ξ_i, combined with *weigthing coefficients* w_i
Example, order 2:

$$\begin{array}{lcll} f(\xi)=1: & 2 & = & w_1+w_2 \\ f(\xi)=\xi: & 0 & = & w_1\xi_1+w_2\xi_2 \\ f(\xi)=\xi^2: & 2/3 & = & w_1\xi_1^2+w_2\xi_2^2 \\ f(\xi)=\xi^3: & 0 & = & w_1\xi_1^3+w_2\xi_2^3 \end{array}$$

then $w_1 = w_2 = 1$, and $\xi_1 = -\xi_2 = 1/\sqrt{3}$

Onedimensional Gauss integration

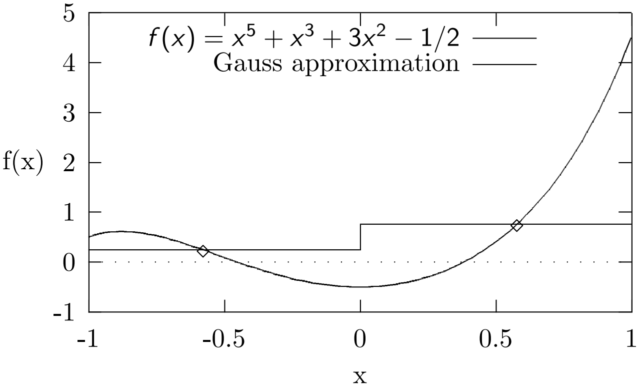
One integration point



$$\int_{-1}^{1} f(\xi) d\xi \approx 2 \; f(0)$$

Onedimensional Gauss integration

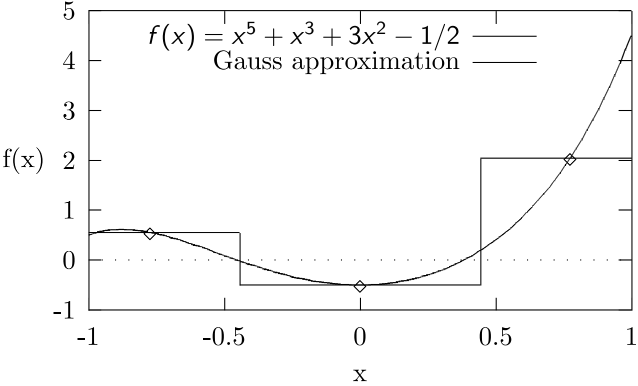
Two integration points



$$\int_{-1}^{1} f(\xi)d\xi \approx 1 \times f(-1/\sqrt{3}) + 1 \times f(1/\sqrt{3})$$

Onedimensional Gauss integration

Three integration points



$$\int_{-1}^{1} f(\xi)d\xi = \frac{5}{9}f(-\sqrt{3/5}) + \frac{8}{9}f(0) + \frac{5}{9}f(\sqrt{3/5})$$

Gauss integration in a N-dimensional space

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta)d\xi d\eta d\zeta = \int_{-1}^{1} d\xi \int_{-1}^{1} d\eta \int_{-1}^{1} f(\xi,\eta,\zeta)d\zeta$$

Respectively r_1, r_2, r_3 Gauss points in each direction, so that:

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta,\zeta)d\xi d\eta d\zeta = \sum_{i=1}^{r_1}\sum_{j=1}^{r_2}\sum_{k=1}^{r_3} w_i w_j w_k f(\xi_i,\eta_j,\zeta_k)$$

- Usually, $r_1 = r_2 = r_3$
- Special integration rules for triangles

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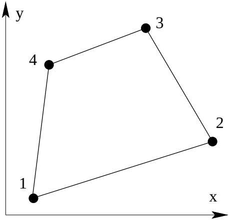
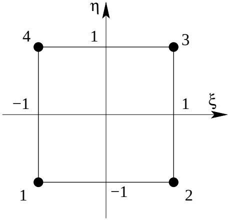
4-node quadrilateral (1)





- Bilinear interpolation of the geometry and of the unknown function

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4 \qquad y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$

$$u_x = N_1 q_{x1} + N_2 q_{x2} + N_3 q_{x3} + N_4 q_{x4} \qquad u_y = N_1 q_{y1} + N_2 q_{y2} + N_3 q_{y3} + N_4 q_{y4}$$

- Shape functions

$$N_1(\xi,\eta) = (1-\xi)(1-\eta)/4 \qquad N_2(\xi,\eta) = (1+\xi)(1-\eta)/4$$
$$N_1(\xi,\eta) = (1+\xi)(1+\eta)/4 \qquad N_2(\xi,\eta) = (1-\xi)(1+\eta)/4$$

- Jacobian matrix

$$[J] = \frac{1}{4}\begin{pmatrix} -x_1 + x_2 + x_3 - x_4 + \eta(x_1 - x_2 + x_3 - x_4) & -y_1 + y_2 + y_3 - y_4 + \eta(y_1 - y_2 + y_3 - y_4) \\ -x_1 - x_2 + x_3 + x_4 + \xi(x_1 - x_2 + x_3 - x_4) & -y_1 - y_2 + y_3 + y_4 + \xi(y_1 - y_2 + y_3 - y_4) \end{pmatrix}$$

4-node quadrilateral (2)

- Determinant of the Jacobian matrix: *terms in ξ and η*

$$
\begin{aligned}
8J = & (y_4 - y_2)(x_3 - x_1) - (y_3 - y_1)(x_4 - x_2) \\
& + ((y_3 - y_4)(x_2 - x_1) - (y_2 - y_1)(x_3 - x_4))\,\xi \\
& + ((y_4 - y_1)(x_3 - x_2) - (y_3 - y_2)(x_4 - x_1))\,\eta
\end{aligned}
$$

- Inverse of the jacobian matrix: *homographic function in ξ and η*

$$
[J]^{-1} = \frac{4}{J} \begin{pmatrix} -y_1 - y_2 + y_3 + y_4 + \xi(y_1 - y_2 + y_3 - y_4) & -x_1 - x_2 + x_3 + x_4 + \xi(x_1 - x_2 + x_3 - x_4) \\ +y_1 - y_2 - y_3 + y_4 - \eta(y_1 - y_2 + y_3 - y_4) & +x_1 - x_2 - x_3 + x_4 - \eta(x_1 - x_2 + x_3 - x_4) \end{pmatrix}
$$

- Derivative of the shape functions:

$$
\begin{pmatrix} \partial N_1/\partial x \\ \partial N_1/\partial y \end{pmatrix} = [J]^{-1} \begin{pmatrix} \partial N_1/\partial \xi \\ \partial N_1/\partial \eta \end{pmatrix} = [J]^{-1} \begin{pmatrix} -(1-\eta)/4 \\ -(1-\xi)/4 \end{pmatrix}
$$

- Terms in

$$
\begin{pmatrix} \partial N_1/\partial x \\ \partial N_1/\partial y \end{pmatrix} = \frac{1}{J} \begin{pmatrix} (1,\xi) & (1,\xi) \\ (1,\eta) & (1,\eta) \end{pmatrix} \begin{pmatrix} -(1-\eta)/4 \\ -(1-\xi)/4 \end{pmatrix} = \begin{pmatrix} \dfrac{(1,\xi,\eta,\xi\eta,\xi^2)}{(1,\xi,\eta)} \\ \dfrac{(1,\xi,\eta,\xi\eta,\eta^2)}{(1,\xi,\eta)} \end{pmatrix}
$$

4-node quadrilateral (3)

- The jacobian is an homographic function for a generic quad.
- $[K]$ is obtained by Gauss integration, $[K] = \int_\Omega [B]^T[D][B]d\Omega$

$$[K] = \int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]Jd\xi d\eta = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j J((\xi_i,\eta_j)[B]^T(\xi_i,\eta_j)[D][B](\xi_i,\eta_j)$$

  - The stiffness matrix includes terms like $\dfrac{(1,\xi,\eta,\xi\eta,\xi^2,...,\eta^4,\xi^4,\xi^2\eta^2)}{(1,\xi,\eta)}$
  - *For a generic quad, the integration of $[K]$ is never exact*
- The internal forces are computed as:

$$[F_{int}] = \int_\Omega [B]^T[\sigma]d\Omega = \sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j J((\xi_i,\eta_j)[B]^T(\xi_i,\eta_j)[\sigma(\xi_i,\eta_j)]$$

  - The determinant at the denominator of $[B]$ vanishes with the determinant due to elementary volume
  - The internal forces (σ constant) include terms like $(1,\xi,\eta,\xi\eta,\xi^2,...,\eta^2,\xi^2,\xi^2\eta,\xi\eta^2)$
  - *Internal forces are integrated with a 2×2 rule*

4-node quadrilateral (4)

- If the "real world" quad is a parallelogram, the relation (x, y)–ξ, η are linear and not bilinear, so that the partial derivatives $\partial x/\partial\xi$, etc... are constant. The jacobian is also constant.
- The following terms are present in the interpolation functions and their derivatives:

$[N]$	1	ξ	η	$\xi\eta$
$[\partial N/\partial\xi]$	0	1	0	η
$[\partial N/\partial\eta]$	0	0	1	ξ

- $[K]$ is obtained by Gauss integration, using a constant J.
    - The product $[B]^T(\xi_i, \eta_i)[D][B](\xi_j, \eta_j)$, and also the stiffness matrix, include terms like $\xi^i\eta^j$ with $i + j \leqslant 2$
    - *$[K]$ is exactly integrated with a 2×2 rule*
- The internal forces present only linear terms $\times\sigma$
- *Only one Gauss point is needed for constant stress state, and 2×2 for linear stresses*

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Number of Gauss points for an exact integration (1)

The following terms are present in the interpolation functions and their derivatives:

- *For generic geometries*, the computation of $[B]$ involves derivatives of the shape functions and partial derivative of the coordinate of the physical space wrt the reference coordinates:

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \frac{\partial \xi}{\partial x}$$

- A typical space derivative term is .... $\dfrac{\partial \xi}{\partial x} = \dfrac{1}{J} \dfrac{\partial y}{\partial \eta}$
- A typical term of $[B]$ is then .... $\dfrac{1}{J} \dfrac{\partial N_i}{\partial \xi} \dfrac{\partial y}{\partial \eta}$
- A typical term of $[K]$ is then .... $\dfrac{1}{J} \left(\dfrac{\partial N_i}{\partial \xi} \dfrac{\partial y}{\partial \eta} \right)^2$
- A typical term of $[F_{int}]$ is then .... $\dfrac{\partial N_i}{\partial \xi} \dfrac{\partial y}{\partial \eta}$

Number of Gauss points for an exact integration (2)

- *For linear geometries*, and constant jacobian matrix (introducing the constant a):
$$\frac{\partial N_i}{\partial x} = a\frac{\partial N_i}{\partial \xi}$$
- A typical term of $[B]$ is then .... $\dfrac{\partial N_i}{\partial \xi}$
- A typical term of $[K]$ is then .... $\left(\dfrac{\partial N_i}{\partial \xi}\right)^2$
- A typical term of $[F_{int}]$ is then .... $\dfrac{\partial N_i}{\partial \xi}$

Rectangular C2D8 element

Uniform jacobian

The following terms are present in the interpolation functions and their derivatives:

$$
\begin{array}{ccccccccc}
[N] & 1 & \xi & \eta & \xi^2 & \xi\eta & \eta^2 & \xi^2\eta & \xi\eta^2 \\
[\partial N/\partial\xi] & 0 & 1 & 0 & 2\xi & \eta & 0 & 2\xi\eta & \eta^2 \\
[\partial N/\partial\eta] & 0 & 0 & 1 & 0 & \xi & 2\eta & \xi^2 & 2\xi\eta
\end{array}
$$

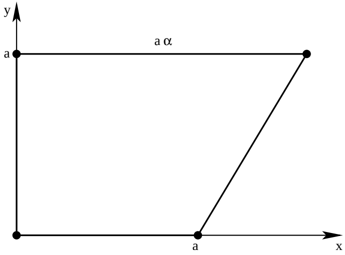
The product $[B]^T[D][B]$ includes terms like $\xi^i\eta^j$ with $i+j \leq 4$

- 3×3 points for a *full* integration (too much, exact until 5)
- 2×2 points are for *reduced* integration

Number of Gauss points needed

Element	Geometry	Loading	$[K]$	$[F_{int}]$
C2D4	Linear	Constant	4	1
C2D4	Bilinear	Constant	NO	1
C2D4	Linear	Linear	4	4
C2D4	Bilinear	Linear	NO	4
C2D8	Linear	Constant	4	4
C2D8	Bilinear	Constant	NO	4
C2D8	Bilinear	Linear	NO	4
C2D8	Generic	Constant	NO	4
C3D8	Linear	Constant	8	1
C3D8	Trilinear	Constant	NO	8
C3D20	Linear	Constant	27	8
C3D20	Trilinear	Constant	NO	8
C3D20	Trilinear	Linear	NO	27
C3D20	Generic	Constant	NO	27

Precision of the Gauss integration method



Compute $I = \int_{-1}^{+1} \int_{-1}^{+1} \frac{1}{J} \, d\xi \, d\eta$

Using a mapping on a square [0..1]:

$$x = (1 + (\alpha - 1)\eta) a\xi$$
$$y = a\eta$$

$$[J] = \begin{pmatrix} a(1 + (\alpha - 1)\eta) & a(\alpha - 1)\xi \\ 0 & a \end{pmatrix}$$

$$I = \frac{1}{a^2} \int_0^1 \frac{d\eta}{1 + (\alpha - 1)\eta}$$

Order	$\alpha = 2$	$\alpha = 5$	$\alpha = 10$
1	0.66666	0.33333	0.18182
2	0.69231	0.39130	0.23404
3	0.69312	0.40067	0.24962
exact	0.69315	0.40236	0.25584

Analytic expression:

$$I = \frac{\log \alpha}{a^2(\alpha - 1)}$$

(after Dhatt and Touzot)

Global algorithm

For each loading increment, do while $\|\{R\}_{iter}\| > EPSI$:
$iter = 0; iter < ITERMAX; iter++$

1. Update displacements: $\Delta\{u\}_{iter+1} = \Delta\{u\}_{iter} + \delta\{u\}_{iter}$
2. Compute $\Delta\{\varepsilon\} = [B].\Delta\{u\}_{iter+1}$ then $\Delta\underset{\sim}{\varepsilon}$ for each Gauss point
3. Integrate the constitutive equation: $\Delta\underset{\sim}{\varepsilon} \rightarrow \Delta\underset{\sim}{\sigma},\ \Delta\alpha_I,\ \dfrac{\Delta\underset{\sim}{\sigma}}{\Delta\underset{\sim}{\varepsilon}}$
4. Compute int and ext forces: $\{F_{int}(\{u\}_t + \Delta\{u\}_{iter+1})\}, \{F_e\}$
5. Compute the residual force: $\{R\}_{iter+1} = \{F_{int}\} - \{F_e\}$
6. New displacement increment: $\delta\{u\}_{iter+1} = -[K]^{-1}.\{R\}_{iter+1}$

Convergence

- Value of the residual forces $< R_\epsilon$, e.g.

$$||\{R\}||_n = \left(\sum_i R_i^n\right)^{1/n} \quad ; \quad ||\{R\}||_\infty = \max_i |R_i|$$

- Relative values:

$$\frac{||\{R\}_i - \{R\}_e||}{||\{R\}_e||} < \epsilon$$

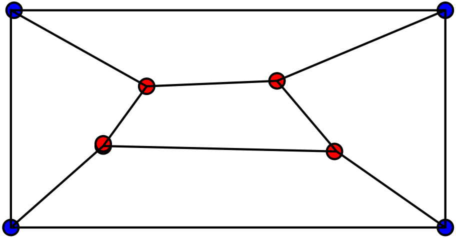
- Displacements

$$\left|\left|\{U\}_{k+1} - \{U\}_k\right|\right|_n < U_\epsilon$$

- Energy

$$\left[\{U\}_{k+1} - \{U\}_k\right]^T . \{R\}_k < W_\epsilon$$

The concept of *patch* (engineering version)



- Apply a given displacement field on the external (blue) nodes
- Check the results in the internal (red) nodes
- For instance, uniform strain; or shear, or bending
- Check with a bending displacement field: $u_x = xy$ and $u_y = -0.5(x^2 + \nu y^2)$, assuming bilinear geometry (with $\xi\eta$ term)
  - The resulting displacement for u_y should have terms like $(a + b\xi + c\eta + d\xi\eta)^2$. A nine node quad will pass, but not an eight-node quad (missing $\xi^2\eta^2$ term in the polynomial base).
  - The eight-node pass, provided the edges are straight
  - This demonstrates also the limitations of high order elements. For them, a complex shape (terms in $\xi^3\eta$, $\xi\eta^3$ for a cubic interpolation introduces terms like $\xi^6\eta^2$, etc. . . for a correct patch test simulation. They are not in the polynomial basis...

Rigid body mode (1)

Example of a 2D plane element

Zero strain:

$$u_{1,1} = 0 \quad ; \quad u_{2,2} = 0 \quad ; \quad u_{1,2} + u_{2,1} = 0$$

Possible displacement field:

$$u_1 = A - Cx_2 \quad ; \quad u_2 = B + Cx_1$$

3 rigid body modes:

2 translations $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$; 1 rotation $\begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$

Rigid body mode (2)

Example of a 2D axisymmetric element

Zero strain:

$$\varepsilon_r = u_{r,r} = 0 \quad ; \quad \varepsilon_\theta = \frac{u_r}{r} = 0 \quad ; \quad u_{z,z} = 0$$

Possible displacement field:

$$u_z = A$$

Only 1 rigid body modes: 1 translation $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Rank sufficiency/deficiency

No zero-energy mode other than rigid body modes

- r is the rank of the elementary stiffness matrix (number of evaluations)
- Check r with respect to $n_F - n_R$ (n_F is the number of element DOF, n_R is the number of rigid body modes)
- Rank sufficient element iff $r \geqslant n_F - n_R$
- *Rank deficiency,* d in the case $d = n_F - n_R - r \geqslant 0$
- Each Gauss point adds n_E to the rank of the matrix (n_E is the order of the stress-strain matrix, n_G the number of Gauss points), $r = n_E n_G$

RULE: $n_E n_G \geqslant n_F - n_R$

Rank-sufficient Gauss integration

Element	n	n_F	$n_F - n_R$	Min n_g	rule
3-node triangle	3	6	3	1	1-pt
6-node triangle	6	12	9	3	3-pt
4-node quadrilateral	4	8	5	2	2x2
8-node quadrilateral	8	16	13	5	3x3
9-node quadrilateral	9	18	15	5	3x3
8-node hexahedron	8	24	18	3	2x2x2
20-node hexahedron	20	60	54	9	3x3x3

Stiffness matrix of a rectangular element

rectangle $[\pm a, \pm a/2]$; E=96; ν=1/3

$$[K] = \begin{pmatrix} 42 & 18 & -6 & 0 & -21 & -18 & -15 & 0 \\ 18 & 78 & 0 & 30 & -18 & -39 & 0 & -69 \\ -6 & 0 & 42 & -18 & -15 & 0 & -21 & 18 \\ 0 & 30 & -18 & 78 & 0 & -69 & 18 & -39 \\ -21 & -18 & -15 & 0 & 42 & 18 & -6 & 0 \\ -18 & -39 & 0 & -69 & 18 & 78 & 0 & 30 \\ -15 & 0 & -21 & 18 & -6 & 0 & 42 & -18 \\ 0 & -69 & 18 & -39 & 0 & 30 & -18 & 78 \end{pmatrix}$$

Eigenvalues = { 223.4 90 78 46.36 42 0 0 0 }
(three rigid body modes)

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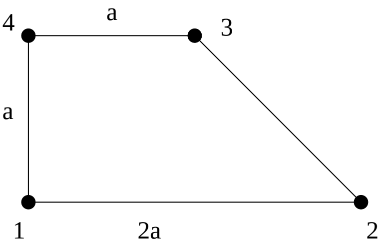
Stiffness matrix of a trapezoidal element



E=4206384; ν=1/3

Rule	Eigenvalues obtained with different Gauss rules (scaled by 10^{-6})							
1x1	8.77276	3.68059	2.26900	*0*	*0*	0	0	0
2x2	8.90944	4.09769	3.18565	2.64523	1.54678	0	0	0
3x3	8.91237	4.11571	3.19925	2.66438	1.56155	0	0	0
4x4	8.91246	4.11627	3.19966	2.66496	1.56199	0	0	0

Three rigid body modes, *but a rank deficiency by TWO is too few Gauss points are used*

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Analysis of the locking penomenon

For bad reasons, the element becomes too stiff

- Shear locking
- Volumetric locking
- Trapezoidal locking
- Locking in fields

Alias functions

Function which tries to mimic a given function in one element

Basis for a C2D3: $(1, \xi, \eta)$, for a C2D4: $(1, \xi, \eta, \xi\eta)$,
for a C2D8: $(1, \xi, \eta, \xi^2, \xi\eta, \eta^2, \xi^2\eta, \xi\eta^2)$.

Alias for various functions:

Function	ξ^2	$\xi\eta$	η^2	ξ^3	$\xi^2\eta$	$\xi\eta^2$	η^3
C2D3	ξ	0	η	ξ	0	0	η
C2D4	1	*OK*	1	ξ	η	ξ	η
C2D8	*OK*	*OK*	*OK*	ξ	*OK*	*OK*	η

Shear locking

C2D4, $-L \leq x_1 \leq L$, $-1 \leq x_2 \leq 1$

$$\text{Actual field} \quad \begin{array}{rcl} u_1 & = & x_1 x_2 \\ u_2 & = & -x_1^2/2 \end{array}$$

$$\text{Aliased field} \quad \begin{array}{rcl} u_1 & = & x_1 x_2 \\ u_2 & = & -L^2/2 \text{ !!} \end{array}$$

Computed shear for the alias, $\varepsilon_{12} = x/2$!! (actual solution: 0)
Computed stored elastic energy W_e for the real field and W_a for the alias:

$$\frac{W_a}{W_o} = 1 + \frac{1-\nu}{2} L^2$$

Solve the problem by computing shear on the middle of the element

Shear locking (2)

Analytic solution

$$
\begin{aligned}
\varepsilon_{11} &= u_{1,1} = x_2 & \sigma_{11} &= Ex_2/(1-\nu^2)\\
\varepsilon_{22} &= u_{2,2} = 0 & \sigma_{22} &= \nu Ex_2/(1-\nu^2)\\
2\varepsilon_{12} &= u_{2,1} + u_{1,2} = 0 & \sigma_{12} &= 0
\end{aligned}
$$

Solution with the *alias*

$$
\begin{aligned}
\varepsilon_{11} &= u_{1,1} = x_2 & \sigma_{11} &= Ex_2/(1-\nu^2)\\
\varepsilon_{22} &= u_{2,2} = 0 & \sigma_{22} &= \nu Ex_2/(1-\nu^2)\\
2\varepsilon_{12} &= u_{2,1} + u_{1,2} = x_1 & \sigma_{12} &= (1/2)Ex_1/(1+\nu)
\end{aligned}
$$

$$W_o = \frac{1}{2}\int_\Omega \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon} d\Omega$$

Dilatational locking

C2D4, $-L \leq x_1 \leq L$, $-1 \leq x_2 \leq 1$

$$\text{Actual field} \quad \begin{array}{rcl} u_1 & = & x_1 x_2 \\ u_2 & = & -\dfrac{x_1^2}{2} - \dfrac{\nu}{2(1-\nu)} x_2^2 \end{array}$$

$$\text{Aliased field} \quad \begin{array}{rcl} u_1 & = & x_1 x_2 \\ u_2 & = & -\dfrac{L^2}{2} - \dfrac{\nu}{2(1-\nu)} \ !! \end{array}$$

The computed stored elastic energy W_a for the alias tends to infinity if ν tends to 0.5

$$W_a = \frac{E}{2(1+\nu)} \left(\frac{1-\nu}{1-2\nu} x_2^2 + \frac{x_1^2}{2} \right) \qquad \text{instead of:} \qquad W_e = \frac{E}{2(1-\nu^2)} x_2^2$$

Solve the problem by adding a non conform. displacement $(x_1^2 - L^2, x_2^2 - 1)$

Dilatational locking (2)

Analytic solution

$$\begin{aligned} \varepsilon_{11} &= u_{1,1} = x_2 & \sigma_{11} &= Ex_2/(1-\nu^2) \\ \varepsilon_{22} &= u_{2,2} = -\nu x_2/(1-\nu) & \sigma_{22} &= 0 \\ \varepsilon_{33} &= u_{3,3} = 0 & \sigma_{33} &= \\ 2\varepsilon_{12} &= u_{2,1} + u_{1,2} = 0 & \sigma_{12} &= 0 \end{aligned}$$

Solution with the *alias*

$$\begin{aligned} \varepsilon_{11} &= u_{1,1} = x_2 & \sigma_{11} &= E(1-\nu)x_2/(1+\nu)(1-2\nu) \\ \varepsilon_{22} &= u_{2,2} = 0 & \sigma_{22} &= \nu Ex_2/(1+\nu)(1-2\nu) \\ 2\varepsilon_{12} &= u_{2,1} + u_{1,2} = x_1 & \sigma_{12} &= (1/2)Ex_1/(1+\nu) \end{aligned}$$

$$W_o = \frac{1}{2}\int_\Omega \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon} d\Omega$$

Locking of the 8-node rectangle

Consider a rectangle of length 2Λ and width 2 (with $\Lambda > 1$)

- Displacement basis: 1, ξ, η, ξ^2, $\xi\eta$, η^2, $\xi^2\eta$, $\xi\eta^2$
- Try $u_1 = x_1^2 x_2$, $u_2 = -x_1^3/3$, so that $\varepsilon_{11} = 2x_1x_2$, $\varepsilon_{22} = 0$, $\varepsilon_{12} = 0$.
- In fact, u_2 represented by its alias, $-x_1\Lambda^2/3$, so that the shear is $x_1^2 - \Lambda^2/3$
- *No locking* if the shear is evaluated at the second order Gauss point ($x_1 = \pm 1/\sqrt{3}$)
- Underintegration is a good remedy to locking

Locking of the 8-node rectangle (2)

Analytic solution

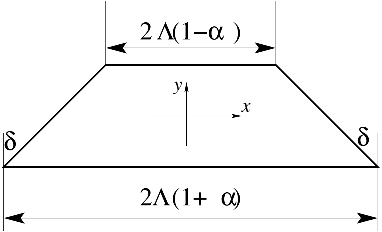
$$
\begin{aligned}
\varepsilon_{11} &= u_{1,1} = 2x_1x_2 & \sigma_{11} &= 2Ex_1x_2 \\
\varepsilon_{22} &= u_{2,2} = 0 & \sigma_{22} &= 0 \\
\varepsilon_{33} &= u_{3,3} = 0 & \sigma_{33} &= \\
2\varepsilon_{12} &= u_{2,1} + u_{1,2} = 0 & \sigma_{12} &= 0
\end{aligned}
$$

Solution with the *alias* ($u_1 = x_1^2 x_2$ and $u_2 = -x_1\Lambda^2/3$)

$$
\begin{aligned}
\varepsilon_{11} &= u_{1,1} = 2x_1x_2 & \sigma_{11} &= 2Ex_1x_2 \\
\varepsilon_{22} &= u_{2,2} = 0 & \sigma_{22} &= 0 \\
\varepsilon_{33} &= u_{3,3} = 0 & \sigma_{33} &= 0 \\
2\varepsilon_{12} &= u_{2,1} + u_{1,2} = x_1^2 - \Lambda^2/3 & \sigma_{12} &= (1/2)(x_1^2 - \Lambda^2/3)/(1+\nu)
\end{aligned}
$$

$$
W_o = \frac{1}{2}\int_\Omega \underset{\sim}{\sigma} : \underset{\sim}{\varepsilon}\, d\Omega
$$

Trapezoidal locking



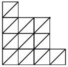
- Geometry: $x_1 = \Lambda\xi(1-\alpha\eta)$, and $x_2 = \eta$; $\xi = x_1/(1-\alpha x_2)\Lambda$.
- Displacement basis: $1,\ \xi,\ \eta,\ \xi^2,\ \xi\eta,\ \eta^2,\ \xi^2\eta,\ \xi\eta^2$
- Try $u_1 = x_1^2 x_2$, $u_2 = -x_1^2/2$, so that $\varepsilon_{11} = x_2$, $\varepsilon_{22} = 0$, $\varepsilon_{12} = 0$.
- In fact, the solution with the alias is:

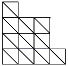
$$\varepsilon_{11} = \frac{\eta-\alpha}{1-\alpha\eta} \quad \varepsilon_{22} = \alpha\Lambda^2 \quad 2\varepsilon_{12} = \Lambda\xi\left(1+\frac{\alpha(\eta-\alpha)}{1-\alpha\eta}\right)$$

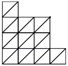
- *All components are affected*
- Error on shear component suppressed if the evaluation is made at $\xi = 0$ only.
- Error on ε_{22} cannot be easily suppressed

Dilatational locking on triangles

triangle C2D3

- Incompressible, $BL \rightarrow TR$ mesh....... 

- Incompressible, $TL \rightarrow BR$ mesh....... 

- Compressible, $BL \rightarrow TR$ mesh......... 

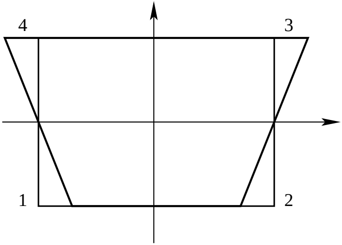
Spurious modes of a C2D4

Find a displacement field which does not produce any strain on the Gauss points

$$u_2^1 = u_2^2 = u_2^3 = u_2^4 = 0$$

$$u_1^1 = +a \quad ; \quad u_1^2 = -a$$

$$u_1^3 = +a \quad ; \quad u_1^4 = -a$$



C2D4 — C2D4r — C2D8 — C2D8r

Spurious modes

- An element has "internal degrees of freedom" which allow deformation process to occur in the element
- Rigid body mode $\{\Phi_o\}$ such as: $[K]\{\Phi_o\} = 0$ everywhere
- Spurious mode $\{\Phi_o\}$ such as: $[K]\{\Phi_o\} = 0$ in some places
- The number of independent states is given by the total number of dof in the element
- Number of independent states - Rigid body mode - Number of evaluation of the strain components = Number of spurious modes
- For an element of degree p, the number of strain evaluations is:
  $3p^2$ (reduced integration, 2D); $3(p+1)^2$ (full integration, 2D); $6p^2$ (reduced integration, 3D); $6p^2$ (reduced integration, 3D).
- The number of strain states is:
  $8p^3$ (Serendip, 2D); $2(p+1)^2$ (Lagrange, 2D); $36p-6$ (Serendip, 3D); $3(p+1)^3-6$ Lagrange, 3D).

Spurious modes

Polynomial degree p	Serendip 2D	Lagrange 2D	Serendip 3D	Lagrange 3D
1	2	2	12	12
2	1	3	6	27

For the 8-node underintegrated element, the following is a spurious mode:

$$u_1 = k_1 \xi (\eta^2 - 1/3)$$

$$u_2 = -k_2 \eta (\xi^2 - 1/3)$$