Rotating annulus problem

Consider an annulus whose inner radius at $r = R_i$ is fixed, and whose outer radius $r = R_o$ rigidly rotates through a specified total angle. The displacement field for this motion is described by

$$u_r = u_z = 0, \quad u_\theta = r\,\phi(r) \quad u_\theta(r = R_i) = 0,\; u_\theta(r = R_o) = R_o\,\phi(R_o), \tag{1}$$

or alternatively as

$$u_1 = r\left[\cos\left(\theta + \phi\right) - \cos\theta\right] = -2r\sin(\theta + \phi/2)\sin(\phi/2), \tag{2}$$

$$u_2 = r\left[\sin\left(\theta + \phi\right) - \sin\theta\right] = 2r\cos(\theta + \phi/2)\sin(\phi/2), \tag{3}$$

based upon a coordinate transformation of the form:

$$x_1 = r\cos\theta, \qquad x_2 = r\sin\theta, \tag{4}$$

equivalently:

$$r = \sqrt{x_1^2 + x_2^2}, \qquad \tan\theta = \frac{x_2}{x_1}. \tag{5}$$

If we suppose that $f(r, \theta)$ represents an arbitrary function of r and θ, then we may invoke the chain rule of differentiation to obtain expressions for $f_{,1}$ and $f_{,2}$, i.e.

$$f_{,1} = f_{,r}r_{,1} + f_{,\theta}\theta_{,1}, \tag{6}$$

$$f_{,2} = f_{,r}r_{,2} + f_{,\theta}\theta_{,2}. \tag{7}$$

We proceed in writing out the expressions for $r_{,i}$ and $\theta_{,i}$ by differentiating the relationships in (5) (once again employing the chain rule):

$$r_{,1} = \left[\sqrt{x_1^2 + x_2^2}\right]_{,1} = \frac{2x_1}{2\sqrt{x_1^2 + x_2^2}} = \frac{r\cos\theta}{r} = \cos\theta, \tag{8}$$

$$r_{,2} = \left[\sqrt{x_1^2 + x_2^2}\right]_{,2} = \frac{2x_2}{2\sqrt{x_1^2 + x_2^2}} = \frac{r\sin\theta}{r} = \sin\theta, \tag{9}$$

$$[\tan\theta]_{,i} = [\tan\theta]_{,\theta}\,\theta_{,i} = \left[\sec^2\theta\right]\theta_{,i} = \frac{\theta_{,i}}{\cos^2\theta}, \tag{10}$$

$$\frac{\theta_{,1}}{\cos^2\theta} = [\tan\theta]_{,1} = \left[\frac{x_2}{x_1}\right]_{,1} = -\frac{x_2}{x_1^2}, \tag{11}$$

$$\theta_{,1} = -\frac{\cos^2\theta\, r\sin\theta}{r^2\cos^2\theta} = -\frac{\sin\theta}{r}, \tag{12}$$

$$\frac{\theta_{,2}}{\cos^2\theta} = [\tan\theta]_{,2} = \left[\frac{x_2}{x_1}\right]_{,2} = \frac{1}{x_1} \tag{13}$$

$$\theta_{,2} = \frac{\cos^2\theta}{r\cos\theta} = \frac{\cos\theta}{r}. \tag{14}$$

Given these expressions for $r_{,i}$ and $\theta_{,i}$, we may now rewrite equations (6) and (7) as:

$$f_{,1} = f_{,r}\cos\theta - f_{,\theta}\frac{\sin\theta}{r}, \tag{15}$$

$$f_{,2} = f_{,r}\sin\theta + f_{,\theta}\frac{\cos\theta}{r}. \tag{16}$$

After some algebraic manipulation, we may write out expressions for $u_{\alpha,\beta}\,\forall\alpha,\beta = 1,2$:

$$u_{1,1} = -1 + \cos\phi - r\phi_{,r}\cos\theta\sin(\theta+\phi), \tag{17}$$

$$u_{1,2} = -\sin\phi - r\phi_{,r}\sin\theta\sin(\theta+\phi), \tag{18}$$

$$u_{2,1} = +\sin\phi + r\phi_{,r}\cos\theta\cos(\theta+\phi), \tag{19}$$

$$u_{2,2} = -1 + \cos\phi + r\phi_{,r}\sin\theta\cos(\theta+\phi). \tag{20}$$

If we define the following terms:

$$\theta' = \theta + \phi, \tag{21}$$

$$\mathbf{R}_\phi = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

$$\mathbf{e}_{\theta'} = \left\{ \begin{array}{c} -\sin\theta' \\ \cos\theta' \\ 0 \end{array} \right\}, \quad \mathbf{e}_r = \left\{ \begin{array}{c} \cos\theta \\ \sin\theta \\ 0 \end{array} \right\}, \tag{23}$$

$$\mathbf{W}_\phi = \mathbf{e}_{\theta'} \otimes \mathbf{e}_r, \tag{24}$$

then we may express the deformation gradient $\mathbf{F}$ as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{R}_\phi + r\phi_{,r}\mathbf{W}_\phi. \tag{25}$$

Using the matrix determinant lemma, it is easy to verify that this deformation is volume preserving (i.e. $\det(\mathbf{F}) = 1$.)

The left Cauchy-Green deformation tensor is then

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{1} + r\phi_{,r}(\mathbf{R}_\phi\mathbf{W}_\phi^T + \mathbf{W}_\phi\mathbf{R}_\phi^T) + r^2\phi_{,r}^2\mathbf{e}_{\theta'} \otimes \mathbf{e}_{\theta'}, \tag{26}$$

or

$$\mathbf{B} = \mathbf{1} + r\phi_{,r}(\mathbf{e}_{r'} \otimes \mathbf{e}_{\theta'} + \mathbf{e}_{\theta'} \otimes \mathbf{e}_{r'}) + r^2\phi_{,r}^2\mathbf{e}_{\theta'} \otimes \mathbf{e}_{\theta'}, \tag{27}$$

where $\mathbf{e}_{r'} = \left\{ \begin{array}{ccc} \cos\theta' & \sin\theta' & 0 \end{array} \right\}^T$. Further,

$$\mathbf{e}_{r'} \otimes \mathbf{e}_{\theta'} + \mathbf{e}_{\theta'} \otimes \mathbf{e}_{r'} = \left[\begin{array}{ccc} -\sin 2\theta' & \cos 2\theta' & 0 \\ \cos 2\theta' & \sin 2\theta' & 0 \\ 0 & 0 & 0 \end{array} \right], \tag{28}$$

and

$$\mathbf{e}_{r'} \otimes \mathbf{e}_{r'} = \frac{1}{2} \left[\begin{array}{ccc} 1 + \cos 2\theta' & \sin 2\theta' & 0 \\ \sin 2\theta' & 1 - \cos 2\theta' & 0 \\ 0 & 0 & 0 \end{array} \right], \tag{29}$$

yielding

$$\mathbf{B}_2 = \mathbf{1}_2 + r\phi_{,r} \left[\begin{array}{cc} -\sin 2\theta' & \cos 2\theta' \\ \cos 2\theta' & \sin 2\theta' \end{array} \right] + \frac{r^2\phi_{,r}^2}{2} \left[\begin{array}{cc} 1 + \cos 2\theta' & \sin 2\theta' \\ \sin 2\theta' & 1 - \cos 2\theta' \end{array} \right], \tag{30}$$

where $\mathbf{B}_2 = \mathbf{B} - \mathbf{e}_z \otimes \mathbf{e}_z$. It can be shown that the eigenvalues of $\mathbf{B}_2$ are

$$\lambda_1 = -\frac{1}{2}\left[-r^2\phi_{,r}^2 + \sqrt{r^4\phi_{,r}^4 + 4r^2\phi_{,r}^2} - 2 \right], \tag{31}$$

$$\lambda_2 = +\frac{1}{2}\left[r^2\phi_{,r}^2 + \sqrt{r^4\phi_{,r}^4 + 4r^2\phi_{,r}^2} + 2 \right], \tag{32}$$

with corresponding eigenvectors

$$\mathbf{v}_1 = \left\{ \begin{array}{cc} \frac{-\sqrt{r^4\phi_{,r}^4+4r^2\phi_{,r}^2}+r^2\phi_{,r}^2\cos 2\theta'-2r\phi_{,r}\sin 2\theta'}{2r\phi_{,r}\cos 2\theta'+r^2\phi_{,r}^2\sin 2\theta'} & 1 \end{array} \right\}, \tag{33}$$

$$\mathbf{v}_2 = \left\{ \begin{array}{cc} \frac{\sqrt{r^4\phi_{,r}^4+4r^2\phi_{,r}^2}+r^2\phi_{,r}^2\cos 2\theta'-2r\phi_{,r}\sin 2\theta'}{2r\phi_{,r}\cos 2\theta'+r^2\phi_{,r}^2\sin 2\theta'} & 1 \end{array} \right\}. \tag{34}$$

$$||\mathbf{v}_1|| = \frac{\sqrt{2(x^2+4+\sqrt{x^2+4}(2\sin y - x\cos y))}}{x\sin y + 2\cos y} \tag{35}$$

$$||\mathbf{v}_2|| = \frac{\sqrt{2(x^2+4-\sqrt{x^2+4}(2\sin y - x\cos y))}}{x\sin y + 2\cos y} \tag{36}$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{2\sqrt{x^2+4}(x\cos y - 2\sin y + \sqrt{x^2+4})}} \left\{ \begin{array}{c} x\cos y - 2\sin y + \sqrt{x^2+4} \\ 2\cos y + x\sin y \end{array} \right\}. \tag{37}$$

$$\lambda_1 = \frac{\sqrt{x^2+4}}{2}\left[\sqrt{x^2+4} - x\right], \tag{38}$$

$$\lambda_2 = \frac{\sqrt{x^2+4}}{2}\left[\sqrt{x^2+4} + x\right], \tag{39}$$

Consequently,

$$\mathbf{B}_2 = \lambda_1 \mathbf{v}_1 \otimes \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 \otimes \mathbf{v}_2, \tag{40}$$

and the in-plane Hencky strain is

$$\mathbf{h}_2 = \frac{1}{2}\left[\ln(\lambda_1)\mathbf{v}_1 \otimes \mathbf{v}_1 + \ln(\lambda_2)\mathbf{v}_2 \otimes \mathbf{v}_2\right], \tag{41}$$

where

For this special motion, it can be shown that the Hencky model of elasticity results in the following expressions for the in-plane principal stresses:

$$\sigma_1 = \mu \ln(\lambda_1), \quad \sigma_2 = \mu \ln(\lambda_2), \tag{42}$$

and it suffices to show that

$$\sigma_{1,r}(\cos\theta' + \sin\theta') + \frac{\sigma_{1,\theta'}}{r}(\cos\theta' - \sin\theta') = 0, \tag{43}$$

$$\sigma_{2,r}(\cos\theta' + \sin\theta') + \frac{\sigma_{2,\theta'}}{r}(\cos\theta' - \sin\theta') = 0. \tag{44}$$

Since the above equations hold irrespective of which value is chosen for θ, we may examine the trival case of $\theta = 0$, and thus

$$\sigma_{1,r}(\cos\phi + \sin\phi) + \frac{\sigma_{1,\phi}}{r}(\cos\phi - \sin\phi) = 0, \tag{45}$$

$$\sigma_{2,r}(\cos\phi + \sin\phi) + \frac{\sigma_{2,\phi}}{r}(\cos\phi - \sin\phi) = 0, \tag{46}$$

1 Alternative Approach

Consider the form of the stress divergence equations in cylindrical polar coordinates:

$$\nabla \cdot \boldsymbol{\sigma} = \left\{ \begin{array}{c} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} \\ \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\sigma_{r\theta}}{r} + \frac{\sigma_{\theta r}}{r} + \frac{\partial \sigma_{z\theta}}{\partial z} \\ \frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} \end{array} \right\} = \mathbf{0}. \tag{47}$$

By the assumptions of plane strain and axisymmetry, we rationalize that $\boldsymbol{\sigma}(r)$ is a function of r, alone, and therefore,

$$\nabla \cdot \boldsymbol{\sigma} = \left\{ \begin{array}{c} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\sigma_{rz}}{r} \end{array} \right\} = \mathbf{0}. \tag{48}$$

Furthermore, by the assumptions of plane strain, we observe that $\sigma_{rz} = 0$ and $\sigma_{\theta z} = 0$. Additionally, if we impose the incompressibility condition $\nabla \cdot \mathbf{v} = \mathrm{tr}(\mathbf{D}) = 0$, then for a linear hypoelastic material model of grade zero obeying

$$\dot{\boldsymbol{\sigma}} = \mathbb{C} : \mathbf{D} + \mathbf{W}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{W}, \tag{49}$$

we find that

$$\mathrm{tr}(\dot{\boldsymbol{\sigma}}) = 0 \;\Rightarrow\; \mathrm{tr}(\boldsymbol{\sigma}) = 0 \,\forall t. \tag{50}$$

The deformation necessitates that $\sigma_{zz} = 0$, and therefore $\sigma_{\theta\theta} = -\sigma_{rr}$. Consequently, we are left with 2 governing differential equations for σ_{rr} and $\sigma_{r\theta}$:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2\sigma_{rr}}{r} = 0, \quad \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0, \tag{51}$$

whose solutions are of the form

$$\sigma_{rr} = \frac{a}{r^2}, \quad \sigma_{r\theta} = \frac{b}{r^2}. \tag{52}$$

Consequently,

$$\left\{\begin{array}{c}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{r\theta}\end{array}\right\} = r^{-2}\left\{\begin{array}{c}+a\\ -a\\ b\end{array}\right\}. \tag{53}$$

Under the assumption of incompressibility, We may characterize the deformation via the velocity field $v_r = 0$, $v_z = 0$, and $v_\theta(r) = r\dot{\phi}(r)$, yielding the velocity gradient (in cylindrical polar coordinates):

$$\mathbf{L} = \nabla\mathbf{v} = \left[\begin{array}{ccc} v_{r,r} & v_{r,\theta}/r - v_\theta/r & v_{r,z}\\ v_{\theta,r} & v_{\theta,\theta}/r + v_r/r & v_{\theta,z}\\ v_{z,r} & v_{z,\theta}/r & v_{z,z}\end{array}\right] = \left[\begin{array}{ccc} 0 & -\dot{\phi} & 0\\ \dot{\phi} + r\dot{\phi}_{,r} & 0 & 0\\ 0 & 0 & 0\end{array}\right], \tag{54}$$

and the corresponding rate of deformation and spin tensors:

$$\mathbf{D} = \frac{r\dot{\phi}_{,r}}{2}\left[\begin{array}{ccc} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{array}\right], \quad \mathbf{W} = \frac{2\dot{\phi} + r\dot{\phi}_{,r}}{2}\left[\begin{array}{ccc} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{array}\right]. \tag{55}$$

The resulting stress rate equations are

$$\left\{\begin{array}{c}\dot{\sigma}_{rr}\\ \dot{\sigma}_{\theta\theta}\\ \dot{\sigma}_{r\theta}\end{array}\right\} = \left\{\begin{array}{c}0\\ 0\\ \mu r\dot{\phi}_{,r}\end{array}\right\} + (2\dot{\phi} + r\dot{\phi}_{,r})\left\{\begin{array}{c}-\sigma_{r\theta}\\ \sigma_{r\theta}\\ \sigma_{rr}\end{array}\right\}, \tag{56}$$

or

$$\left\{\begin{array}{c}\dot{a}\\ \dot{b}\end{array}\right\} = \left\{\begin{array}{c}0\\ \mu r^3\dot{\phi}_{,r}\end{array}\right\} + (2\dot{\phi} + r\dot{\phi}_{,r})\left[\begin{array}{cc} 0 & -1\\ 1 & 0\end{array}\right]\left\{\begin{array}{c}a\\ b\end{array}\right\}, \tag{57}$$

which must be valid $\forall r$, t. If we assume that $\dot{\phi} = f(r)$ (and not of t), then we obtain the condition

$$3f_{,r} + rf_{,rr} = 0, \tag{58}$$

implying $f_{,r} = Br^{-3}$, and $\phi(r,t) = (A - Br^{-2}/2)t$, thus

$$\left\{\begin{array}{c}\dot{a}\\ \dot{b}\end{array}\right\} = \left\{\begin{array}{c}0\\ \mu B\end{array}\right\} + 2A\left[\begin{array}{cc} 0 & -1\\ 1 & 0\end{array}\right]\left\{\begin{array}{c}a\\ b\end{array}\right\}, \tag{59}$$

and

$$a(t) = -\frac{\mu B}{2A} - C_2\sin(2At) + C_1\cos(2At), \tag{60}$$

$$b(t) = C_1\sin(2At) + C_2\cos(2At). \tag{61}$$

Imposing the initial conditions $a(0) = b(0) = 0$ results in:

$$a(t) = \frac{\mu B}{2A} \left[\cos(2At) - 1\right], \quad b(t) = \frac{\mu B}{2A} \sin(2At). \tag{62}$$

Imposing the boundary conditions $\phi(R_i, t) = 0 \; \forall t$, $\phi(R_o, t) = \Phi \, t \; \forall t$ yields:

$$A = \frac{\Phi R_i^{-2}}{R_i^{-2} - R_o^{-2}}, \quad B = \frac{2\Phi}{R_i^{-2} - R_o^{-2}}. \tag{63}$$

The final analytical solution for the displacement and stress fields is presented below:

$$u_r = 0, \quad u_\theta = r\Phi \frac{R_i^{-2} - r^{-2}}{R_i^{-2} - R_o^{-2}} t, \quad u_z = 0, \tag{64}$$

$$\sigma_{rr} = -\sigma_{\theta\theta} = \mu \frac{r^{-2}}{R_i^{-2}} \left[\cos\left(2 \frac{\Phi R_i^{-2}}{R_i^{-2} - R_o^{-2}} t\right) - 1\right], \tag{65}$$

$$\sigma_{r\theta} = \mu \frac{r^{-2}}{R_i^{-2}} \sin\left(2 \frac{\Phi R_i^{-2}}{R_i^{-2} - R_o^{-2}} t\right). \tag{66}$$