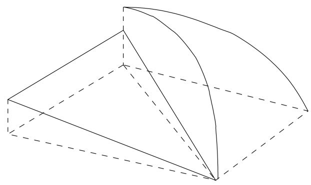

Appendix A
Quadrature rules

Let $\mathcal{O}$ be a domain in $\mathbb{R}^d$. A quadrature rule is defined by a set of nodes $(\boldsymbol{x}_i)_{i \in I}$ in $\mathcal{O}$ and a set of weights $(w_i)_{i \in I}$ in $\mathbb{R}$. The general form of a quadrature rule is

$$\int_{\mathcal{O}} f \approx \sum_{i \in I} w_i f(\boldsymbol{x}_i).$$

In this chapter, we define quadrature rules for an interval, for the reference triangle, and for the reference quadrilateral.

A.1 Gauss quadrature rule on intervals

With a change of variable, we can transform any given integral on the interval (a, b) into an integral on the interval $(-1, 1)$:

$$\int_a^b f(s)ds = \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}t + \frac{a+b}{2}\right) dt.$$

Therefore, we consider only the Gauss quadrature rule on the interval $(-1, 1)$:

$$\int_{-1}^1 f(s)ds \approx \sum_{i=1}^{Q_{\mathrm{G}}} w_i f(s_i).$$

The Gauss quadrature rule with Q_{G} nodes is exact for polynomials of degree less than or equal to $2Q_{\mathrm{G}}-1$. Table A.1 lists the quadrature nodes and weights for several values of Q_{G}.

A.2 Quadrature rules on the reference triangle

We present numerical quadrature rules [44] for computing $\int_{\hat{E}} \phi(x, y)dxdy$, where $\hat{E}$ is the reference triangle defined in Section 2.5.1. The quadrature rule is

$$\int_{\hat{E}} \hat{v} \approx \sum_{i=1}^{Q_{\mathrm{D}}} w_i \hat{v}(s_{x,i}, s_{y,i}).$$

Table A.1. *Gauss quadrature nodes and weights on the interval* $(-1, 1)$.

Q_G	k	w_j	s_j
1	1	2.000000000000	0.000000000000
2	3	1.000000000000	−0.577350269189
		1.000000000000	0.577350269189
3	5	0.555555555555	−0.774596669241
		0.888888888888	0.000000000000
		0.555555555555	0.774596669241
4	7	0.347854845137	−0.861136311594
		0.652145154862	−0.33998104358
		0.652145154862	0.339981043584
		0.347854845137	0.861136311594

Table A.2. *Quadrature weights and points for reference triangle.*

Q_D	k	w_i	$s_{x,i}$	$s_{y,i}$
1	1	0.500000000000	0.333333333333	0.333333333333
3	2	0.166666666666	0.666666666667	0.166666666667
		0.166666666666	0.166666666667	0.166666666667
		0.166666666666	0.166666666667	0.666666666667
4	3	−0.281250000000	0.333333333333	0.333333333333
		0.260416666666	0.200000000000	0.200000000000
		0.260416666666	0.600000000000	0.200000000000
		0.260416666666	0.200000000000	0.600000000000
6	4	0.1116907948390	0.108103018168	0.445948490915
		0.1116907948390	0.445948490915	0.445948490915
		0.1116907948390	0.445948490915	0.108103018168
		0.0549758718276	0.816847572980	0.091576213509
		0.0549758718276	0.091576213509	0.091576213509
		0.0549758718276	0.091576213509	0.816847572980
7	5	0.112500000000	0.333333333333	0.333333333333
		0.062969590272	0.101286507323	0.101286507323
		0.062969590272	0.797426985353	0.101286507323
		0.062969590272	0.101286507323	0.797426985353
		0.066197076394	0.470142064105	0.470142064105
		0.066197076394	0.059715871789	0.470142064105
		0.066197076394	0.470142064105	0.059715871789
12	6	0.058393137863	0.501426509658	0.249286745170
		0.058393137863	0.249286745170	0.249286745170
		0.058393137863	0.249286745170	0.501426509658
		0.025422453185	0.873821971016	0.063089014491
		0.025422453185	0.063089014491	0.063089014491
		0.025422453185	0.063089014491	0.873821971016
		0.041425537809	0.053145049844	0.310352451033

Table A.2. *Continued*

Q_D	k	w_i	$s_{x,i}$	$s_{y,i}$
		0.041425537809	0.310352451033	0.053145049844
		0.041425537809	0.053145049844	0.636502499123
		0.041425537809	0.636502499123	0.053145049844
		0.041425537809	0.636502499123	0.310352451033
		0.041425537809	0.310352451033	0.636502499123
13	7	−0.074785022233	0.333333333333	0.333333333333
		0.087807628716	0.479308067841	0.260345966079
		0.087807628716	0.260345966079	0.260345966079
		0.087807628716	0.260345966079	0.479308067841
		0.026673617804	0.869739794195	0.065130102902
		0.026673617804	0.065130102902	0.065130102902
		0.026673617804	0.065130102902	0.869739794195
		0.038556880445	0.048690315425	0.312865496004
		0.038556880445	0.312865496004	0.048690315425
		0.038556880445	0.048690315425	0.638444188569
		0.038556880445	0.638444188569	0.048690315425
		0.038556880445	0.638444188569	0.312865496004
		0.038556880445	0.312865496004	0.638444188569

Let k denote the polynomial degree for which this rule is exact. Table A.2 gives the values of the quadrature nodes $(s_{x,i}, s_{y,i})$ and weights w_i for several values of k and Q_D.

A.3 Quadrature rule on the reference quadrilateral

Let $\hat{E}$ be the reference quadrilateral: $\hat{E} = (-1, 1)^2$. We apply the one-dimensional Gauss quadrature rule in each direction:

$$\int_{-1}^{1}\int_{-1}^{1} f(\hat{x}, \hat{y})\,d\hat{x}\,d\hat{y} \approx \sum_{i=1}^{Q_G}\sum_{j=1}^{Q_G} w_i w_j f(s_i, s_j).$$