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New analytical solution for bending problem of uniformly loaded rectangular plate supported on corner points

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Presented herein is a new analytical solution for the bending problem of a uniformly loaded, thin rectangular elastic plate supported on corner points. The solution is given in the form of a trigonometric series where the coefficients of the series form a fully regular infinite system of linear equations. Accurate numerical values for deflection, moments and shear forces for various aspect ratios of the plate are given. The results are compared with those obtained by other researchers.

Keywords: elasticity; plate theory; thin plates; rectangular plates; analytical solution

Introduction

In this article, we derive an exact analytical solution for the bending problem of a uniformly loaded, rectangular, elastic thin plate supported on corner points with all edges free (FFFF plate). Historically, the first analytical solution for the bending of an FFFF plate was given by Galerkin in 1915 (Galerkin 1933, 1952, Timoshenko and Woinowsky-Krieger 1959). He considers a rectangular plate having elastically supported edges which in a limiting case, when the flexural rigidity of edge support vanishes, is tantamount to an FFFF plate. Galerkin assumed the deflection of the plate to be in the form of a Fourier series and as a solution of the problem he obtained an infinite system of linear equations for coefficients of the series. He provided no analysis of the obtained solution and due to lack of computational facilities at that time, he calculated some numerical values only for a square plate by keeping only four terms in the series for the deflection. Nadai (1922) gave another analytical solution for an FFFF plate where he, for the particular solution of the governed biharmonic equation for deflection, used a symmetric fourth-order polynomial which satisfies the zero bending condition on free edges. By the method of superposition with two Fourier series solutions of homogeneous biharmonic equation, which also satisfies the zero bending moment condition, each of them giving a constant effective shear force along one pair of opposite edges, Nadai approximately satisfied the zero effective shear forces condition on free edges. He provided some numerical values for deflection of a quadratic plate. Lee and Ballesteros (1960) published a simple approximate solution of the FFFF plate problem by using a deflection function in the form of a symmetric polynomial of the fourth order. The polynomial coefficients were determined by the condition that they satisfy the governed biharmonic equation and zero effective shear force boundary condition at the free edges. Since such a polynomial cannot satisfy zero bending moment conditions at free edges, they replaced this condition with the condition that the average bending moments on free edges are zero. Wang *et al.* (2002) provided a critical review of the previous works on the FFFF thin plate problem and concluded that 'earlier works show that the focus is put on determining the deflection accurately' and that previous analytical solutions 'produce rather inaccurate stress resultants distributions'. They then considered a shear deformable FFFF plate and approximated the deflection function and the rotations of plate cross-sections by complete two-dimensional polynomials of arbitrary orders. For determination of polynomial coefficients from the energy functional, they used the Ritz method and in order to satisfy the natural boundary conditions they utilised the Lagrange multiplier method. By post-processing results of calculation with a smoothing technique, they obtained accurate results that are in close agreement with the results from the ABAQUS FE program in which an extremely fine mesh was used. Recently, the analytical solution of the FFFF plate problem was

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given by Lim *et al.* (2008) by using the symplectic method. Essentially, they provided a solution for the deflection of a plate in the form of a series of eigenfunction where eigenvalues are obtained from a transcendental equation and the coefficients corresponding to the eigenfunctions' expansion are obtained from the linear system of equations that results from the boundary conditions. They presented some accurate numerical values for deflection, moments and shearing forces. However, it must be noted that the method leads to a numerical solution of a transcendental equation which has an infinite number of complex roots and the coefficients of series expansion are defined by an infinite system of linear equations so that a large amount of numerics are involved in their solution. Beside the numerical calculation, the authors give no analysis of the obtained solution, only the observation that by increasing the number of term in a series 'theoretically speaking, absolute convergence is achieved'.

In what follows, we present a new solution of the problem of the bending of an FFFF plate. First, the problem is formulated and then its solution is derived by the method used by Galerkin. The derivation is a bit longer, necessary only in order to demonstrate that the solution may be obtained in a completely rational manner, without any special assumptions. The obtained solution is then analysed in some detail and at the end some numerical data, figures and comparison studies with results obtained by other researchers are given.

Definition of plate problem, governing equation and boundary conditions

We consider the bending problem of a homogeneous and isotropic, elastic, thin, rectangular plate with side lengths $2a$ and 2b, which is supported on corner points and subjected to a uniformly distributed load p. In order to preserve the symmetry of the problem, the Cartesian coordinate system (Oxy) has the origin at the centre of the plate in the way that the plate occupies the region bounded by $-a \leq x \leq a$ and $-b \leq y \leq$ b.

According to the classical thin plate theory, the bending of the considered plate is governed by the biharmonic differential equation (Timoshenko and Woinowsky-Krieger 1959, Chapter 4).

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \tag{1}$$

where w is deflection of the plate and $D \equiv \frac{h^3 E}{12(1-\nu^2)}$ is the bending rigidity, h the plate thickness, E the elastic modulus and ν Poisson's ratio. The bending moments M_x, M_y, twisting moment M_{xy}, transverse shearing forces Q_x, Q_y and effective shearing forces V_x, V_y, all defined by unit of length, are given in terms of w by the following expressions (Timoshenko and Woinowsky-Krieger 1959)

$$\begin{aligned} M_x &= -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) \\ M_y &= -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) \\ M_{xy} &= -(1-\nu)D\frac{\partial^2 w}{\partial x \partial y} \end{aligned} \tag{2}$$

$$Q_x = -D\frac{\partial \Delta w}{\partial x} \qquad Q_y = -D\frac{\partial \Delta w}{\partial y} \tag{3}$$

$$\begin{aligned} V_x &= -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right] \\ V_y &= -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right]. \end{aligned} \tag{4}$$

For the present problem, Equation (1) is to be solved subject to the following boundary conditions

$$w(\pm a, \pm b) = 0 \tag{5}$$

$$M_x(\pm a, y) = 0 \qquad V_x(\pm a, y) = 0 \tag{6}$$

$$M_y(x, \pm b) = 0 \qquad V_y(x, \pm b) = 0. \tag{7}$$

Note that as usual in the classical thin plate theory, we replace the Poisson boundary conditions, which assumes that at the free edge there no bending and twisting moments and no shearing forces, with the Kirchhoff conditions, which require that bending moments and the effective shear forces vanish along the free boundaries (Timoshenko and Woinowsky-Krieger 1959).

Analytical solution

Since Equation (1) is linear, we may seek its solution in the form

$$w = w_0 + w_1 \tag{8}$$

where w_0 is a particular solution of Equation (1) and w_1 is a homogeneous solution of the homogenous equation $\Delta^2 w_1 = 0$. Note that because of the double symmetry of the problem both solutions must be even functions of x and of y.

A particular solution of Equation (1) may be sought in the form of a symmetrical polynomial of the fourth order in x and y.

$$w_0 = c_0 + c_1x^2 + c_2y^2 + c_3x^4 + c_4x^2y^2 + c_5y^4 \quad (9)$$

where c_k are unknown constants and where, in order to satisfy the biharmonic Equation (1), the following should hold

$$3c_3 + c_4 + 3c_5 = \frac{q}{8D}. \quad (10)$$

There are several possibilities for defining the polynomial constant, but it turns out that the solution process is most simplified if we require that the particular solution (9) satisfies the boundary conditions $w(\pm a, \pm b) = 0$, $M_x(\pm a, y) = 0$ and $M_y(x, \pm b) = 0$. These boundary conditions yield the particular solution which may be written as (Nadai 1922)

$$w_0 = \frac{p}{48(1-v)D}\Big[5\big(a^4 + b^4\big) - 6va^2b^2 - 6\big(a^2 - vb^2\big)x^2 - 6\big(b^2 - va^2\big)y^2 + x^4 - 6v\,x^2y^2 + y^4\Big]. \quad (11)$$

For the homogeneous biharmonic equation, the method of separation of variables (Vinson 1974, pp 27–31) yields the double symmetrical class of particular solutions.

$$\left\{\begin{matrix} \cosh\alpha_n y \\ y\sinh\alpha_n y \end{matrix}\right\}\cos\alpha_n x \qquad \left\{\begin{matrix} \cosh\beta_n x \\ x\sinh\beta_n x \end{matrix}\right\}\cos\beta_n y \quad (12)$$

where

$$\alpha_n \equiv \frac{(2n+1)\pi}{2a} \qquad \beta_n \equiv \frac{(2n+1)\pi}{2b} \qquad (n = 0, 1, 2, 3, \ldots). \quad (13)$$

From these, by linear combination, we form the homogeneous solution which may be written in the following convenient form

$$w_1 = \frac{p}{D}\sum_{n=0}^{\infty}(-1)^{n-1}\left(A_n\frac{\cosh\alpha_n y}{\cosh\alpha_n b} + B_n\frac{y\sinh\alpha_n y}{a\cosh\alpha_n b}\right)\times\cos\alpha_n x + \frac{p}{D}\sum_{n=0}^{\infty}(-1)^{n-1}\times\left(C_n\frac{\cosh\beta_n x}{\cosh\beta_n a} + D_n\frac{x\sinh\beta_n x}{b\cosh\beta_n a}\right)\cos\beta_n y. \quad (14)$$

By Equations (11) and (14), the solution (8) satisfies boundary condition $w(\pm a, \pm b) = 0$ and so for the determination of coefficients A_n, B_n, C_n, D_n (n = 0, 1, 2, ...) four boundary conditions (given by Equations (6)–(7)) remain.

From the boundary conditions $M_y(x, \pm b) = 0$ and $M_x(\pm a, y) = 0$, we find the following explicit expressions for coefficients A_n and C_n

$$A_n = -\frac{bB_n}{a}\left(\tanh\alpha_n b + \frac{2}{1-v}\frac{1}{\alpha_n b}\right)$$
$$C_n = -\frac{aD_n}{b}\left(\tanh\beta_n a + \frac{2}{1-v}\frac{1}{\beta_n a}\right) \quad (n = 0, 1, 2, \ldots). \quad (15)$$

Further, by expanding the boundary conditions $V_y(x, \pm b) = 0$ and $V_x(\pm a, y) = 0$ into a trigonometric series on cos α_nx and cos β_ny, respectively, we obtain the following pair of infinite systems of linear algebraic equations

$$B_n\left(\frac{3+v}{1-v}\tanh\alpha_n b - \frac{\alpha_n b}{\cosh^2\alpha_n b}\right) = \frac{4}{b}\sum_{m=0}^{\infty}D_m\frac{\alpha_n\beta_m^2}{\big(\alpha_n^2 + \beta_m^2\big)^2} + \frac{b}{\alpha_n^3} \quad (n = 0, 1, 2, \ldots) \quad (16)$$

$$D_n\left(\frac{3+v}{1-v}\tanh\beta_n a - \frac{\beta_n a}{\cosh^2\beta_n a}\right) = \frac{4}{a}\sum_{m=0}^{\infty}B_m\frac{\alpha_m^2\beta_n}{\big(\alpha_m^2 + \beta_n^2\big)^2} + \frac{a}{\beta_n^3} \quad (n = 0, 1, 2, \ldots). \quad (17)$$

The system of Equations (16)–(17) is very similar to those obtained by Hencky for the case of a plate of with all edges clamped (Meleshko 1997). In fact, the systems differ only by signs of coefficients and the form of diagonal terms on the right side of equations. So we may, as was done by Meleshko for a clamped plate, shows that that system of Equations (16)–(17) is fully regular, and since it has bounded free terms it therefore has a unique bounded solution which may be obtained by the method of successive approximation (Kantorovich and Krylov 1958). Observe that all terms in the system are positive so that we have

$$B_n > 0 \qquad D_n > 0 \qquad (n = 0, 1, 2, \ldots). \quad (18)$$

The solution of the truncated system will therefore give an approximation of the solution from below. Note that in the case of a square plate, we have $a = b$ and therefore, by Equation (13), $\alpha_n = \beta_n$. By subtracting Equations (16) and (17), we obtain a homogeneous system from which we find that $D_n = B_n$. Thus for determination of B_n we have to solve only Equation (16).

From Equations (16)–(17), we may also extract some information regarding the rate of convergence of coefficients B_n and D_n. By multiplying Equation (16) by α_n^2 and Equation (17) by β_n^2 and taking the limit, we obtain

$$\lim_{n\to\infty} \alpha_n^2 B_n = 0 \qquad \lim_{n\to\infty} \beta_n^2 D_n = 0. \tag{19}$$

On that basis, we may establish the following inequalities

$$B_n < \frac{C}{\alpha_n^{2+\gamma}} \qquad D_n < \frac{C}{\beta_n^{2+\gamma}} \tag{20}$$

where C and γ are some positive constants. By numerical solution of the Equations (16)–(17) we find that at least for $b/a > 1/50$ we have approximately $\gamma < 0.8$. Similarly, by multiplying Equation (16) by α_n^3 and Equation (17) by β_n^3 and taking the limit, we obtain

$$\begin{aligned}\lim_{n\to\infty} \alpha_n^3 B_n &= \frac{1-\nu}{3+\nu} b\left(\frac{4}{b^2}\sum_{m=0}^{\infty}\beta_m^2 D_m - 1\right)\\ \lim_{n\to\infty} \beta_n^3 D_n &= \frac{1-\nu}{3+\nu} a\left(\frac{4}{a^2}\sum_{m=0}^{\infty}\alpha_m^2 B_m - 1\right).\end{aligned} \tag{21}$$

With regard to previous consideration, we conclude that for a plate that is not too narrow we have $\lim_{n\to\infty}\alpha_n^3 B_n = \infty$ and $\lim_{n\to\infty}\beta_n^3 D_n = \infty$ and hence the series in Equation (21) diverges:

$$\sum_{m=0}^{\infty}\beta_m^2 D_m = \infty \qquad \sum_{m=0}^{\infty}\alpha_m^2 B_m = \infty. \tag{22}$$

To that we add the following useful sums

$$\sum_{n=0}^{\infty} B_n\,\alpha_n b \tanh\alpha_n b = \sum_{n=0}^{\infty} D_n \beta_n a \tanh\beta_n a \tag{23}$$

$$\begin{aligned}&\sum_{n=0}^{\infty} B_n\alpha_n b\left(\tanh\alpha_n b - \frac{1-\nu}{1+\nu}\frac{\alpha_n b}{\cosh^2\alpha_n b}\right)\\ &\quad + \sum_{n=0}^{\infty} D_n\beta_n a\left(\tanh\beta_n a - \frac{1-\nu}{1+\nu}\frac{\beta_n a}{\cosh^2\beta_n a}\right)\\ &\quad = -\frac{1-\nu}{1+\nu}\frac{a^2b^2}{2}\end{aligned} \tag{24}$$

which are obtained by summing up each of the equations of system (16)–(17) and then subtracting and adding the results.

Case of a very narrow strip. Introducing a small parameter $\varepsilon \equiv b/a$ and expanding the system (16)–(17) with respect to ε we obtain

$$\begin{aligned}B_n &= \frac{1}{2}\frac{1-\nu}{1+\nu}\frac{a^4}{\kappa_n^4} + O(\varepsilon^3)\\ D_n &= O(\varepsilon^3) \qquad (n = 0,1,2,\ldots)\end{aligned} \tag{25}$$

where $\kappa_n \equiv \frac{(2n+1)\pi}{2}$. By substituting these coefficients into Equation (15) we get

$$\begin{aligned}A_n &= -\frac{1}{1+\nu}\frac{a^4}{\kappa_n^5} + O(\varepsilon^3)\\ C_n &= O(\varepsilon^3) \quad (n = 0,1,2,\ldots).\end{aligned} \tag{26}$$

Introducing Equations (25) and (26) into Equation (14) and taking into account that $y = O(\varepsilon)$ yields the homogeneous solution in the form

$$\begin{aligned}w_1 &= \frac{1}{1+\nu}\frac{pa^4}{D}\sum_{n=0}^{\infty}\frac{(-1)^n}{\kappa_n^5}\cos\alpha_n x + O(\varepsilon^2)\\ &= \frac{1}{1+\nu}\frac{pa^4}{D}\frac{5-6(x/a)^2+(x/a)^4}{48} + O(\varepsilon^2).\end{aligned} \tag{27}$$

In view of Equations (27) and (11), the final deflection defined by Equation (8) becomes

$$w = \frac{p(2a)^4}{(1-\nu^2)D}\frac{5-6(x/a)^2+(x/a)^4}{384} + O(\varepsilon^2). \tag{28}$$

This is a well-known solution for deflection of a simply supported beam. Note that in the case of a very narrow strip the problem becomes essentially one-dimensional so the boundary conditions (7) become meaningless.

Deflection, moments and shearing forces

On the basis of Equations (11), (14) and (15), we may now express the deflection of a plate in the form

$$\begin{aligned}w &= \frac{p}{48(1-\nu)D}\Big[5(a^4+b^4) - 6\nu a^2b^2 - 6(a^2-\nu b^2)x^2\\ &\quad - 6(b^2-\nu a^2)y^2 + x^4 - 6\nu x^2y^2 + y^4\Big]\\ &\quad + \frac{p}{D}\sum_{n=0}^{\infty}(-1)^n B_n\left[\left(\tanh\alpha_n b + \frac{2}{1-\nu}\frac{1}{\alpha_n b}\right)\right.\\ &\quad \left.\times\frac{b}{a}\frac{\cosh\alpha_n y}{\cosh\alpha_n b} - \frac{y}{a}\frac{\sinh\alpha_n y}{\cosh\alpha_n b}\right]\cos\alpha_n x\\ &\quad + \frac{p}{D}\sum_{n=0}^{\infty}(-1)^n D_n\left[\left(\tanh\beta_n a + \frac{2}{1-\nu}\frac{1}{\beta_n a}\right)\right.\\ &\quad \left.\times\frac{a}{b}\frac{\cosh\beta_n x}{\cosh\beta_n a} - \frac{x}{b}\frac{\sinh\beta_n x}{\cosh\beta_n a}\right]\cos\beta_n y.\end{aligned} \tag{29}$$

By substituting Equation (29) into Equations (2) and (3) we obtain the corresponding expressions for moments and shearing forces

$$M_x = \frac{1+v}{4}p\left(a^2 - x^2\right) + (1-v)p\sum_{n=0}^{\infty}(-1)^n\alpha_n^2 B_n \times \left[\left(\tanh\alpha_n b + \frac{1+v}{1-v}\frac{2}{\alpha_n b}\right)\frac{b}{a}\frac{\cosh\alpha_n y}{\cosh\alpha_n b} - \frac{y}{a}\frac{\sinh\alpha_n y}{\cosh\alpha_n b}\right]\cos\alpha_n x - (1-v)p\sum_{n=0}^{\infty}(-1)^n\beta_n^2 D_n \times \left(\tanh\beta_n a\frac{a}{b}\frac{\cosh\beta_n x}{\cosh\beta_n a} - \frac{x}{b}\frac{\sinh\beta_n x}{\cosh\beta_n a}\right)\cos\beta_n y \tag{30}$$

$$M_y = \frac{1+v}{4}p\left(b^2 - y^2\right) - (1-v)p\sum_{n=0}^{\infty}(-1)^n\alpha_n^2 B_n \times \left(\tanh\alpha_n b\frac{b}{a}\frac{\cosh\alpha_n y}{\cosh\alpha_n b} - \frac{y}{a}\frac{\sinh\alpha_n y}{\cosh\alpha_n b}\right)\cos\alpha_n x + (1-v)p\sum_{n=0}^{\infty}(-1)^n\beta_n^2 D_n \times \left[\left(\tanh\beta_n a + \frac{1+v}{1-v}\frac{2}{\beta_n a}\right)\frac{a}{b}\frac{\cosh\beta_n x}{\cosh\beta_n a} - \frac{x}{b}\frac{\sinh\beta_n x}{\cosh\beta_n a}\right]\cos\beta_n y \tag{31}$$

$$M_{xy} = \frac{vp}{2}xy + (1-v)p\sum_{n=0}^{\infty}(-1)^n\alpha_n^2 B_n \times \left[\left(\tanh\alpha_n b + \frac{1+v}{1-v}\frac{1}{\alpha_n b}\right)\frac{b}{a}\frac{\sinh\alpha_n y}{\cosh\alpha_n b} - \frac{y}{a}\frac{\cosh\alpha_n y}{\cosh\alpha_n b}\right]\sin\alpha_n x + (1-v)p\sum_{n=0}^{\infty}(-1)^n\beta_n^2 D_n \times \left[\left(\tanh\beta_n a + \frac{1+v}{1-v}\frac{1}{\beta_n a}\right)\frac{a}{b}\frac{\sinh\beta_n x}{\cosh\beta_n a} - \frac{x}{b}\frac{\cosh\beta_n x}{\cosh\beta_n a}\right]\sin\beta_n y \tag{32}$$

$$\begin{aligned} Q_x &= -\frac{p}{2}x - \frac{2p}{a}\sum_{n=0}^{\infty}(-1)^n\alpha_n^2 B_n\frac{\cosh\alpha_n y}{\cosh\alpha_n b}\sin\alpha_n x + \frac{2p}{b}\sum_{n=0}^{\infty}(-1)^n\beta_n^2 D_n\frac{\sinh\beta_n x}{\cosh\beta_n a}\cos\beta_n y \\ Q_y &= -\frac{p}{2}y + \frac{2p}{a}\sum_{n=0}^{\infty}(-1)^n\alpha_n^2 B_n\frac{\sinh\alpha_n y}{\cosh\alpha_n b}\cos\alpha_n x - \frac{2p}{b}\sum_{n=0}^{\infty}(-1)^n\beta_n^2 D_n\frac{\cosh\beta_n x}{\cosh\beta_n a}\sin\beta_n y. \end{aligned} \tag{33}$$

From the foregoing expressions, we see that the deflection w and the bending moments M_x, M_y are even functions of both coordinates x and y, the shearing force Q_x is an even function of y but an odd function of x, while the shearing force Q_y is an even function of x and an odd function of y. The twisting moment M_{xy} has no such symmetrical properties.

By taking into consideration the asymptotic expansions

$$\frac{\cosh\beta_n x}{\cosh\beta_n a} \sim e^{-\beta_n a\left(1-\frac{|x|}{a}\right)} \qquad \frac{\sinh\beta_n x}{\cosh\beta_n a} \sim \operatorname{sgn}(x)e^{-\beta_n a\left(1-\frac{|x|}{a}\right)} \tag{34}$$

(similar for y cases), we may see that for all points inside the plate the coefficients in the series in the expressions (29)–(33) decrease exponentially. Hence all these series are uniformly and absolutely convergent for all points inside the plate and therefore represent continuous functions.

Consider next the boundary of the plate. On the boundaries $y = \pm b$ the expression for deflection (29) simplifies to

$$w = w_0(x, \pm b) + \frac{2}{1-v}\frac{p}{aD}\sum_{n=0}^{\infty}(-1)^n\frac{B_n}{\alpha_n}\cos\alpha_n x. \tag{35}$$

By using the inequality (20) , where we assume that $0 < \gamma < 1$, we see that the coefficients of the above series satisfy the condition $\frac{B_n}{\alpha_n} < \frac{C}{\alpha_n^{3+\gamma}}$. Therefore the series is absolute and uniformly convergent and represents a function which is two times continuously differentiable and has a piecewise continuous third-order derivative (Courant 1965, Chapter 8). This is true also for boundaries $x = \pm a$.

In a similar way to deflection it may be shown that the series in expressions for moments (30)–(32) are convergent at all points on the boundary and they represent a continuous function that has piecewise continuous first-order derivatives. As an example consider the bending moment M_x given by Equation (30). At the boundaries $x = \pm a$ we have $M_x(\pm a, y) = 0$ exactly and at the bounders $y = \pm b$ we have

$$M_x = \frac{1+v}{4}p\left[\left(a^2 - x^2\right) + \frac{4}{a}\sum_{n=0}^{\infty}(-1)^n\alpha_n B_n\cos\alpha_n x\right]. \tag{36}$$

By using Equation (20), we may see that the coefficients of the series in this expression obey the inequality $\alpha_n B_n < \frac{C}{\alpha_n^{1+\gamma}}$ from which follows the above conclusions.

The twisting moment at the corners of the plate requires special attention. For example, at a corner with coordinates $x = a$ and $y = b$ the twisting moment given by Equation (32) becomes

$$M_{xy}(a,b) = \frac{vp}{2}ab + \frac{(1+v)p}{a}\sum_{n=0}^{\infty}\alpha_n B_n \times \left(\tanh\alpha_n b - \frac{1-v}{1+v}\frac{\alpha_n b}{\cosh^2\alpha_n b}\right) + \frac{(1+v)p}{b}\sum_{n=0}^{\infty}\beta_n D_n \times \left(\tanh\beta_n a - \frac{1-v}{1+v}\frac{\beta_n a}{\cosh^2\beta_n a}\right). \quad (37)$$

By using Equation (24), the twisting moment simplifies to

$$M_{xy}(a,b) = \frac{p}{2}ab. \quad (38)$$

Now, as it is well known (Timoshenko and Woinowsky-Krieger 1959) that a twisting moment at corners produces the corner force $F = 2M_{xy}$. From the static equilibrium of the plate, we have $F = pad$ and thus $M_{xy} = pad/2$ which matches Equation (38). We note that Equation (38) represents the difference between the present solution and the symplectic solution (Lim *et al.* 2008), where this value was not derived from an expression for twisting moment but only from the static equilibrium of the plate.

On the boundary, from Equation (33), we obtain the following expressions for the shearing force Q_x

$$\begin{aligned} Q_x(x,\pm b) &= -\frac{p}{2}\left[x + \frac{4}{a}\sum_{n=0}^{\infty}(-1)^n\alpha_n^2 B_n \sin\alpha_n x\right] \\ Q_x(\pm a,y) &= \mp\frac{p}{2}\left[a + \frac{4}{a}\sum_{n=0}^{\infty}\alpha_n^2 B_n\frac{\cosh\alpha_n y}{\cosh\alpha_n b} - \frac{4}{b}\sum_{n=0}^{\infty}(-1)^n\beta_n^2 D_n\tanh\beta_n a\cos\beta_n y\right]. \end{aligned} \quad (39)$$

On the basis of Equation (20), we have estimations $\alpha_n^2 B_n < \frac{C}{\alpha_n^{\gamma}}$ and $\beta_n^2 D_n < \frac{C}{\beta_n^{\gamma}}$. Hence for $-a < x < a$ and $-b < y < b$ the above series still converge, but we may expect a very slow convergence rate. The same is true for Q_y. At the corners of the plate we have

$$\begin{aligned} Q_x(\pm a,b) &= \mp\frac{p}{2}\left(a + \frac{4}{a}\sum_{n=0}^{\infty}\alpha_n^2 B_n\right) \\ Q_y(a,\pm b) &= \mp\frac{p}{2}\left(b + \frac{4}{b}\sum_{n=0}^{\infty}\beta_n^2 D_n\right). \end{aligned} \quad (40)$$

So if the plate is not too narrow, the convergence, by Equation (22), fails. In this case the corners of the plate are singular points for shearing force

$$\begin{aligned} Q_x(\pm a,b) &= \mp\infty \\ Q_y(a,\pm b) &= \mp\infty. \end{aligned} \quad (41)$$

This result may be justified also on the physical basis by noting the presence of concentrated reaction forces at the corners of the plate. By using Equation (23), the reaction forces along the sides of the plate are

$$\begin{aligned} F_x &= \int_{-b}^{b} Q_x(a,y)\,dy = -pab - \frac{4p}{ab}\left(b\sum_{n=0}^{\infty}\alpha_n B_n\tanh\alpha_n b - a\sum_{n=0}^{\infty}\beta_n D_n\tanh\beta_n a\right) \\ &= -pab \\ F_y &= \int_{-a}^{a} Q_y(x,b)\,dx = -pab - \frac{4p}{ab}\left(b\sum_{n=0}^{\infty}\alpha_n B_n\tanh\alpha_n b - a\sum_{n=0}^{\infty}\beta_n D_n\tanh\beta_n a\right) \\ &= -pab. \end{aligned} \quad (42)$$

Therefore, the sum of forces is $2F_x + 2F_y = -4\ pad$ which satisfies the equilibrium condition of the plate.

Numerical results

For the numerical calculations, the above theory was implemented into the special computer program where for all calculations we use the quad precision numerical model. For the solution of the system of Equations (16)–(17), we use the method of simple iterations which starts from a zero initial guess. The iteration process stops when the results of successive approximations match exactly for all unknowns. In this way the maximum possible precision of calculation is obtained. We found that the number of iterations is roughly independent of the number of unknowns and it was in all the cases less than, say, 30. Table 1 presents the convergence study for the solution of the system (16) for a square plate. As was expected on the basis of the regularity of the system, the successive solutions for particular coefficient forms a monotonically increasing sequence where by doubling the number of equations we gain approximately one decimal place more precise a result.

On the basis of the convergence study, we may expect slow convergence in the calculation of moments and especially shearing force on the boundary of the plate. A more serious problem is, however, the

Table 1. Convergence of coefficients B_n/a^4 calculated from the truncated system (16) for square plate with $v = 0.3$ for various number of equations N.

n/N	5	10	20	40	80	160
0	0.0699601968	0.0699607968	0.0699608694	0.0699608769	0.0699608776	0.0699608776
1	0.0031506594	0.0031517686	0.0031519069	0.0031519213	0.0031519226	0.0031519227
2	0.0007692731	0.0007707646	0.0007709646	0.0007709857	0.0007709877	0.0007709878
3	0.0003021032	0.0003037694	0.0003040181	0.0003040452	0.0003040478	0.0003040480
4	0.0001496564	0.0001513341	0.0001516188	0.0001516512	0.0001516544	0.0001516547
5	0.0000850753	0.0000866599	0.0000869685	0.0000870056	0.0000870093	0.0000870096

presence of the singularity of shearing forces at plate corners. As it is known (Lanczos 1956, Chapter 4), any presence of a discontinuity in a function represented by the Fourier series results in high-frequency oscillations around the true value and amplitudes decrease very slowly with an increasing number of terms. To somehow improve the convergence rate and reduce oscillations we use the σ method (Lanczos 1956, pp 219–229). By this method the coefficients obtained by solving the system (16)–(17) are corrected as

$$\begin{aligned} B_n &\leftarrow B_n \frac{\sin[\pi(2n+1)/2N]}{\pi(2n+1)/2N} \\ D_n &\leftarrow D_n \frac{\sin[\pi(2n+1)/2N]}{\pi(2n+1)/2N} \quad (n = 0, 1, 2, \ldots, N). \end{aligned} \tag{43}$$

We found that this smoothing has practically no effect on the calculation of deflection, some minor effect on calculation of moments near the boundary; but it approximately doubles the convergence rate in the calculation of shearing forces and also it noticeably smoothes the shear forces and equivalent shearing forces functions (see Figure 2).

In the calculation of deflection, moments and shearing forces by expressions (29)–(33), the termination criteria for series summation was taken to be $|s_{n+1} - s_n| < tol \times |s_n|$ where s_n is the value after summing n terms and where tol was taken as 10^{-9}, 10^{-7} and 10^{-6} for computing the deflections, moments and shear forces, respectively. In the tables below, we also provide information regarding the number of terms used in the calculation of the particular value of a quantity. In order to avoid problems with overflow in calculation of a combination of hyperbolic functions in expressions (29)–(33), we code such functions as

$$\begin{aligned} \frac{\cosh \beta_n x}{\cosh \beta_n a} &= e^{-\beta_n a\left(1-\frac{|x|}{a}\right)} \frac{1 + e^{-2\beta_n |x|}}{1 + e^{-2\beta_n a}} \\ \frac{\sinh \beta_n x}{\cosh \beta_n a} &= \operatorname{sgn}(x) e^{-\beta_n a\left(1-\frac{|x|}{a}\right)} \frac{1 - e^{-2\beta_n |x|}}{1 + e^{-2\beta_n a}}. \end{aligned} \tag{44}$$

In Tables 2 and 3, the numerical values of deflection and bending moments at the centre of a

Table 2. Deflection and bending moments in centre of a uniformly loaded rectangular plate supported on corner points.

	Centre: $x = y = 0$					
	$w = \alpha pL^4/D$		$M_x = \beta pL^2$		$M_y = \beta_1 pL^2$	
b/a	α	N	β	N	β_1	N
1	0.02550650	5	0.1117108	4	0.1117108	4
1.5	0.07982476	6	0.0979679	7	0.2689618	6
2	0.23116248	8	0.0854994	9	0.4893227	8
3	1.14678735	10	0.0691765	13	1.1166264	11
4	3.63048838	12	0.0619329	18	1.9926625	14
5	8.8837934	14	0.0591613	22	3.1180590	17
10	142.7959984	22	0.0576971	45	12.4932680	29

N is number of terms used in calculation of corresponding quantity. $v = 0.3$, $L = 2a$.

Table 3. Deflection, bending moments and shear force at middle of the edges in uniformly loaded rectangular plate supported on corner points.

	$x = 0$ $y = \pm b$					
	$w = \alpha_2 pL^4/D$		$M_x = \beta_1 pL^2$		$Q_y = \gamma_2 pL$	
b/a	α_1	N	β_1	N	γ_2	N
1	0.01774741	101	0.1504393	2947	−0.20166	4480
1.5	0.02155586	104	0.1814091	3083	−0.32125	4378
2	0.02595590	105	0.2176284	3150	−0.43829	4316
3	0.03536734	107	0.2954803	3196	−0.66969	4238
4	0.04496160	107	0.3749508	3198	−0.90037	4183
5	0.05457739	108	0.4546119	3183	−1.131	4140
10	0.10266849	109	0.8530248	3067	−2.283	4004
	$x = \pm a$ $y = 0$					
	$w = \alpha_1 pL^4/D$		$M_y = \beta_2 pL^2$		$Q_x = \gamma_1 pL$	
b/a	α_1	N	β_2	N	γ_1	N
1	0.01774741	101	0.1504393	2947	−0.20166	4480
1.5	0.08006698	98	0.3042747	2783	−0.18014	4602
2	0.24184153	96	0.5201808	2663	−0.16033	4691
3	1.18508931	92	1.1412534	2497	−0.13401	4793
4	3.7056776	90	2.0144928	2385	−0.12227	4841
5	9.0056600	88	3.1388186	2303	−0.11776	4868
10	143.3043651	82	12.5134633	2070	−0.11540	4919

N is number of terms in calculation of corresponding quantity. $v = 0.3$, $L = 2a$.

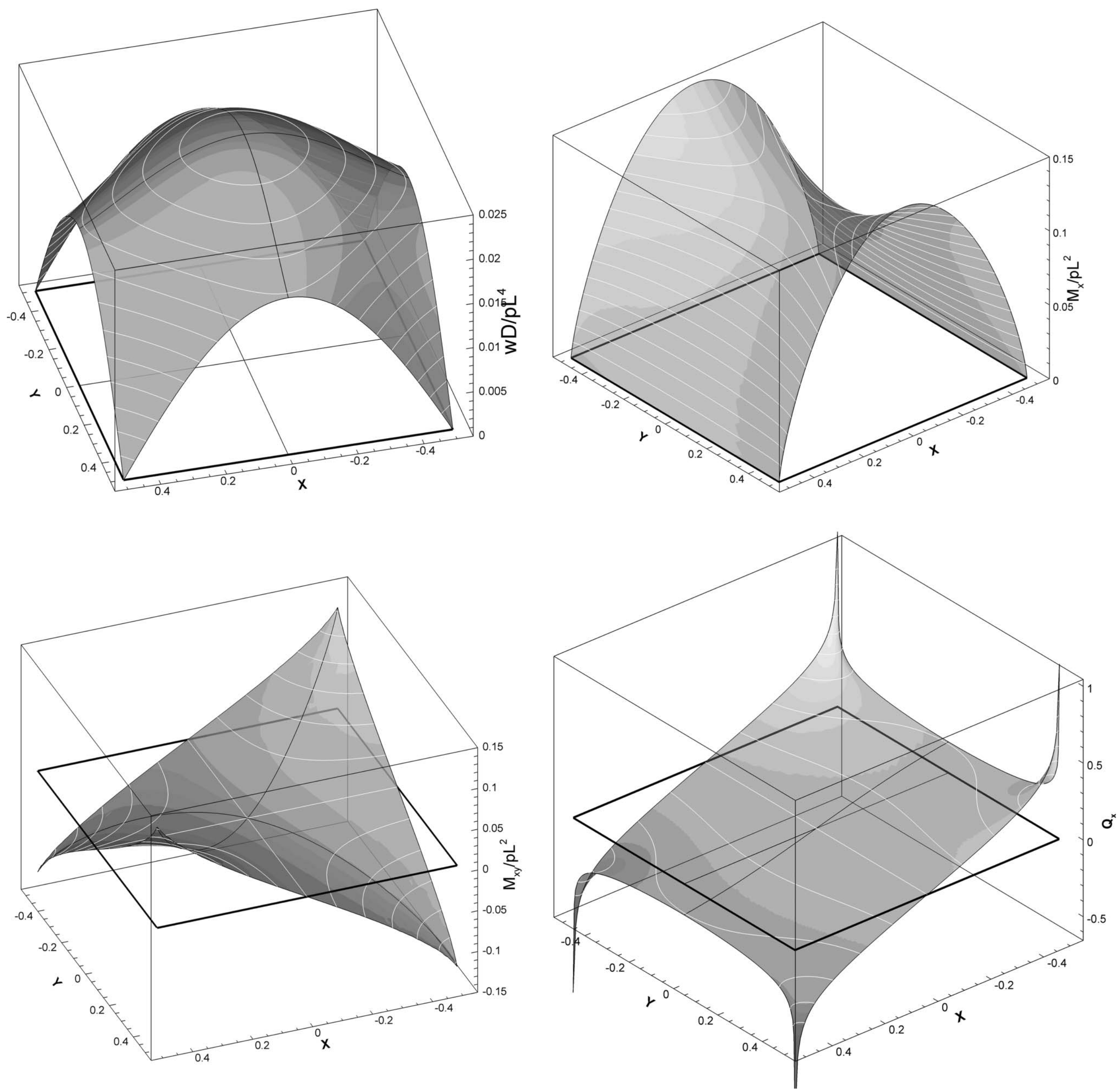

Figure 1. Distribution of deflection, bending moment, twisting moment and shearing force over uniformly loaded square plate supported on corner points.

plate and the middle of the edges of a plate for various values of the ration b/a are given. In Table 2, the values of shearing forces are also provided. As expected, the calculations of deflection and moments at the centre of a plate exhibit rapid convergence. At the edges of the plate, the convergence in the calculation of deflection is slower, and the calculation of bending moments and shearing forces to desired precision requires summation of several thousands of the terms of the series. Figure 1 shows the distribution of deflection w, bending moment M_x, twisting moment M_{xy} and shearing force Q_x over a square plate. The effect of the smoothing of coefficients (43) on calculations is illustrated in Figure 2 where distribution of the effective shearing force V_x is displayed.

In Table 4, we compare the values of twisting moment at plate corners calculated by Equation (32) and exact values given by Equation (38). It can be seen that the relative discrepancy between the calculated and exact values is less than 0.1%.

Table 5 presents a comparison study of the bending results obtained by various researchers. All the

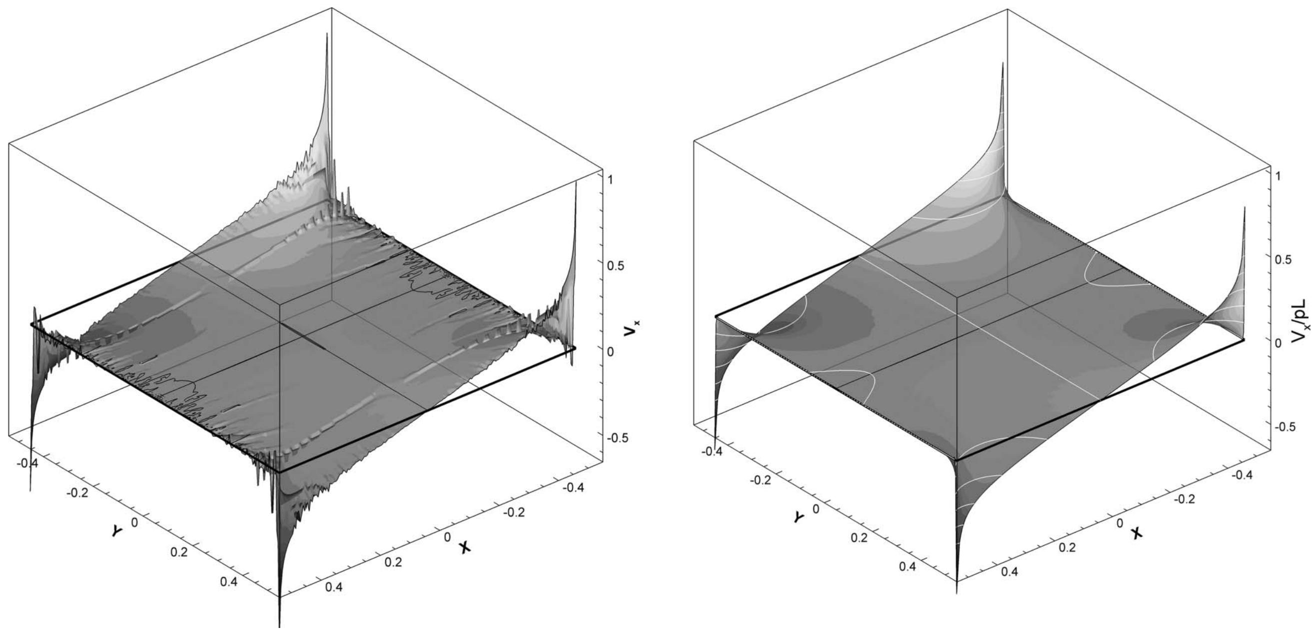

Figure 2. Distribution of effective shearing force V_x over uniformly loaded square plate supported on corner points: non-smoothed (left) and smoothed (right).

Table 4. Twisting moment at corner of uniformly loaded rectangular plate supported on corner points.

	$M_{xy}(a,b)/pL^2$			
b/a	*Exact*	*Series*	N	Relative error %
1	0.125	0.12493	3221	0.06
1.5	0.1875	0.18739	3282	0.06
2	0.25	0.24985	3393	0.06
3	0.375	0.37474	3623	0.07
4	0.5	0.49962	3825	0.08
5	0.625	0.62448	4003	0.08
10	1.25	1.24853	4667	0.12

N is number of terms used in calculation of corresponding quantity. $\nu = 0.3$, $L = 2a$

Table 5. Comparison of bending results of uniformly loaded square plate supported at the corners.

	x	y	FEM, Zienkiewicz and Taylor (2000)	Modified Ritz with smoothing[a], Wang *et al.* (2002)	ABAQUS[a], Wang *et al.* (2002)	Symplectic, Lim *et al.* (2008)	Present study	Factor
w	*0*	*0*	*0.0244*	–	*0.0257*	*0.0255065*	*0.0255065*	pL^4/D
w	*0*	*a*	*0.0173*	–	*0.01795*	*0.0177474*	*0.0177474*	pL^4/D
M_x	0	0	0.109	–	0.1115	0.111711	0.111711	pL^2
M_x	0	a	0.150	–	0.1499	0.150439	0.150439	pL^2
M_{xy}	−0.6a	−0.6a	–	0.0418	0.0419	0.041317	0.04132	pL^2
M_{xy}	−a	−0.6a	–	0.0611[b]	0.0631[b]	0.06349	0.06345	pL^2
Q_x	−0.8a	0	–	0.1756	0.1754	–	0.17374	pL
Q_x	−a	0	–	0.2001[b]	0.1989[b]	–	0.20166	pL
V_x	−0.8a	0	–	–	–	0.000272[c]	0.00027	pL
V_x	−a	0	–	–	–	0.001649[c]	0	pL

[a]shear deformable plate model with $h/a = 0.01$.
[b]calculated at $x = -0.977$a.
[c]in Lim *et al.* (2008) these values are attributed to Q_x). $\nu = 0.3$, $L = 2a$.

methods furnished very similar results for deflection and bending moments. We note that results for the Ritz method and ABAQUS (Wang *et al*. 2002) are for a thin shear deformable plate and so bending results for the twisting moment and shearing forces may not be directly compared. However, they are comparable at points very close to the boundary.

Conclusions

We present a new analytical solution for the bending problem of a corner-supported rectangular thin plate under a uniformly distributed load. The solution is in the form of the Fourier series where the coefficients of the series are defined by an infinite system of algebraic equations which is fully regular and thus may be approximately solved by the method of successive approximations. The coefficients are corrected by the σ-method in order to speed up convergence rates and to obtain smooth distributions of shearing forces over the plate.

Compared to the symplectic method (Lim *et al*. 2008), the present classical method is more direct and it does not require any special transformation of the plate equations. Also the computational efficiency of both methods is the same. There are, however, three differences to be mentioned. First, from the present solution we deduce the solution for a very narrow plate that yields the solution of a simply supported beam. Second, from the expression for the twisting moment, we deduce the exact value of the reaction force at plate corners. And third, the present solution predicts the singularity of shearing forces at corner points which are not observed in the symplectic solution.

Finally, the present analytical results are very accurate and are in good agreement with results available in literature. Thus they may be well used as benchmark solutions to check results obtained by various numerical methods.

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