THE PATCHWORK DIVERGENCE THEOREM

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The divergence theorem in its usual form applies only to suitably smooth vector fields. For vector fields which are merely piecewise smooth, as is natural at a boundary between regions with different physical properties, one must patch together the divergence theorem applied separately in each region. We give an elegant derivation of the resulting *patchwork divergence theorem* which is independent of the metric signature in either region, and which is thus valid if the signature changes.

I. INTRODUCTION

In previous work [1] we discussed the failure, in the presence of signature change, of the standard conservation laws normally derived from Einstein's equations. In the process, we derived a form of the divergence theorem which is applicable when the signature changes, and related it to the usual divergence theorem across a boundary surface, i.e. for piecewise smooth vector fields. We here give a combined treatment of these results which emphasizes their similarity.

After first establishing our notation in Section II, we discuss the divergence theorem in the absence of a metric and relate it to the more usual formulation when a metric is given. In Section III we apply the divergence theorem to piecewise smooth vector fields, obtaining the *patchwork divergence theorem.* In Section IV we show how the patchwork divergence theorem generalizes standard results on boundary surfaces to our previous results on signature change. Finally, in Section V, we discuss some subtle issues related to the choice of differentiable structure in the presence of signature change.

II. THE USUAL DIVERGENCE THEOREM

The divergence theorem is usually stated in the presence of a (nondegenerate) metric. However, there is an alternate formulation which only requires a volume element, which we now summarize.

A *volume element* on an n-dimensional oriented manifold M is a nowhere vanishing n-form which is compatible with the orientation on M. In local coordinates x^i, an orientation is determined by choosing one of $\pm dx^1 \wedge ... \wedge dx^n$, and a volume element ω can then be obtained by multiplying this by any strictly positive function ω_0. Assuming a suitable ordering of the coordinates, we thus have

$$\omega = \omega_0 \, dx^1 \wedge ... \wedge dx^n \tag{1}$$

Given a volume element ω and a vector field X on M, we define the *divergence of X* by

$$\text{div}(X) \, \omega := \text{\pounds}_X \omega \tag{2}$$

Using the standard expression for the Lie derivative $\text{\pounds}_X \omega$ of ω in terms of components with respect to local coordinates x^i (see e.g. [2]), namely

$$(\text{\pounds}_X \omega)_{ab...e} = X^c \, \partial_c \omega_{ab...e} + \omega_{cb...e} \, \partial_a X^c + \omega_{ac...e} \, \partial_b X^c + ... + \omega_{ab...c} \, \partial_e X^c \tag{3}$$

this can be written

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$$\text{div}(X) = \frac{X^c \partial_c \omega_0}{\omega_0} + \partial_c X^c \tag{4}$$

Note that if a (nondegenerate) metric g is given, if ω is assumed to be the metric volume element (so $\omega_0 = \sqrt{|\det(g)|}$), and if ∇ denotes the Levi-Civita connection determined by g, then our definition agrees with the standard one, namely

$$\text{div}(X) = \nabla_a X^a \tag{5}$$

This follows since the torsion-free property of ∇ allows us to replace partial derivatives by covariant ones in (4), while metric-compatibility means that $\nabla_a(\det g)$ vanishes. We will refer to this as the *physical divergence* because of the important role played by the metric when making physical measurements.

Given an open region W of M bounded by S, Stokes' theorem says that

$$\oint_S \alpha = \int_W d\alpha \tag{6}$$

for any (suitably smooth) $(n-1)$-form α, where S must have the orientation induced by W. Using the identity relating Lie differentiation to exterior differentiation d and the interior product i, namely (see e.g. [2])

$$\pounds_X \alpha = d(i_X \alpha) + i_X(d\alpha) \tag{7}$$

for any differential form α, and noting that $d\omega = 0$, leads to the following preliminary form of the divergence theorem

$$\int_W \text{div}(X)\, \omega = \oint_S i_X \omega \tag{8}$$

In order to relate this to the usual divergence theorem, we need to rewrite the RHS in terms of the induced volume element on S.

We will make the customary identification of the tangent and cotangent spaces of S with the corresponding subspaces of those of M, so that in particular we have

$$T_p S \subset T_p M \qquad \text{and} \qquad T_p^* S \subset T_p^* M \qquad (p \in S) \tag{9}$$

We define a 1-form $m \in T_p^* M$ for $p \in S$ to be *normal to S* if

$$m(Y) = 0 \qquad (\forall\, Y \in T_p S) \tag{10}$$

We note that if $m \neq 0$ is normal to S then each $Y \in T_p M$ such that $m(Y) \neq 0$ is not tangent to S, and hence is either inward or outward pointing. We further define m to be *outward pointing* if $m(Y) > 0$ whenever Y is outward pointing, and *inward pointing* if $m(Y) < 0$ for all such Y. These definitions extend directly from vectors and 1-forms at a point to the corresponding tensor fields. To give a simple and common example, if $W = \{f < 0\}$ and $S = \{f = 0\}$, then df is normal to S and outward pointing. In fact, the 1-forms which are normal to S and outward pointing are precisely the positive multiples of df.

Given a 1-form m which is normal to S, we can define an $(n-1)$-form σ on S via

$$m \wedge \sigma := \omega \tag{11}$$

The normality of m ensures that there is a unique σ on S satisfying this equation. Since ω is compatible with the orientation on M, σ is compatible with the induced orientation on S precisely when m is outward pointing. In this case, σ is the *volume element on S induced by m and ω*, or more simply the *induced volume element on S*. The interior product is a derivation, so that in particular

$$i_X(m \wedge \sigma) = (i_X m) \wedge \sigma - m \wedge (i_X \sigma) \tag{12}$$

Note that $i_X m = m(X)$ and that the pullback of m to S is zero.

Putting this all together, we finally obtain the *divergence theorem* in the form

$$\int_W \text{div}(X)\, \omega = \oint_S m(X)\, \sigma \tag{13}$$

where m is (any) outward pointing normal to S and σ is the induced volume element defined above.

In the presence of a metric and assuming that S is not null, letting m_a denote the (components of the) outward unit normal 1-form to S leads to the *physical divergence theorem*, namely

$$\int_W \nabla_a X^a \, \omega = \oint_S X^a m_a \, \sigma \tag{14}$$

where σ is the metric volume element on S. It is customary to write this as

$$\int_W \nabla_a X^a \, d^n W = \oint_S m^a X_a \, d^{n-1} S \tag{15}$$

where $d^n W = \omega$ and $d^{n-1} S = \sigma$ denote the metric volume elements on W and S, respectively, and m^a denotes the (components of the) vector field which is the metric dual of m_a; we will refer to this formulation as the *physical divergence theorem*. It is important to note that while m^a is indeed a unit vector normal to S, it is outward pointing only when it (m^a) is spacelike, and is instead inward pointing where it is timelike; m_a is of course always outward pointing.

III. THE PATCHWORK DIVERGENCE THEOREM

Consider now a boundary surface Σ, which divides W and S into two parts W^+ & W^- and S^+ & S^-, and let $S^0 = \Sigma \cap W$ be the enclosed region of Σ. [1] Suppose an outward-pointing 1-form m is given on S, and further suppose that outward-pointing 1-forms $m^\pm$ are given on the separate boundaries $\partial W^\pm = S^\pm \cup S^0$ of the two regions $W^\pm$ which agree with m on S and which are equal but opposite on Σ, i.e.

$$m^\pm|_{S^\pm} = m \tag{16}$$
$$m^-|_\Sigma = -m^+|_\Sigma =: l \tag{17}$$

Let ω and σ denote as usual the metric volume elements on M and S, and let $\sigma^\pm$ denote the induced volume elements on the boundaries $\partial W^\pm$, so that

$$\sigma^\pm|_S = \sigma \tag{18}$$
$$\sigma^-|_\Sigma = -\sigma^+|_\Sigma =: \sigma^0 \tag{19}$$

We give S^0 the orientation induced by l and ω, namely σ, which is the orientation it inherits as part of the boundary of W^- (and not W^+).

Now consider a vector field X which is piecewise smooth, so that the usual divergence theorem can be applied in each region. Adding the two resulting equations gives

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\begin{aligned}
\int_{W^+} \mathrm{div}(X_+) \, \omega &+ \int_{W^-} \mathrm{div}(X_-) \, \omega \\
&= \oint_{\partial W^+} m^+(X) \, \sigma^+ + \oint_{\partial W^-} m^-(X) \, \sigma^- \\
&= \int_{S^+} m^+(X_+) \, \sigma^+ - \int_{S^0} m^+(X_+) \, \sigma^+ + \int_{S^-} m^-(X_-) \, \sigma^- + \int_{S^0} m^-(X_-) \, \sigma^- \\
&= \int_{S^+} m(X) \, \sigma^+ + \int_{S^-} m(X) \, \sigma^- - \int_{S^0} l([X]) \, \sigma^0
\end{aligned}
\tag{20}
$$

where the minus sign in the third line is due to the difference in orientation of S^0 and ∂W^+ and where

$$[Q] := \lim_{\to \Sigma^+} Q^+ - \lim_{\to \Sigma^-} Q^- \tag{21}$$

denotes the discontinuity in Q across Σ. We can rewrite this as

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[1] We emphasize that Σ is to be viewed as a hypersurface in a given manifold M, so that there are no complications in the manifold structure at or near Σ; this issue is discussed further in Section V.

$$\int_W \mathrm{div}(X)\,\omega = \oint_S m(X)\,\sigma - \int_{S^0} l([X])\,\sigma^0 \tag{22}$$

and this is the *patchwork divergence theorem.* In components we obtain

$$\int_W \nabla_a X^a\,\omega = \oint_S X^a m_a\,\sigma - \int_{S^0} [X^a]\,l_a\,\sigma^0 \tag{23}$$

In the presence of a metric, it is customary to assume that m and l are unit; note that the convention adopted here is that l points from W^- to W^+. If we introduce the (unit) vector field n^a which is normal to Σ and which points from W^- to W^+, we finally obtain

$$\int_W \nabla_a X^a\,d^n W = \oint_S m^a X_a\,d^{n-1}S - \epsilon \int_{S^0} n^a\,[X_a]\,d^{n-1}\Sigma \tag{24}$$

where $d^{n-1}\Sigma = \sigma^0$ denotes the metric volume element on Σ and where there is an important sign difference depending on whether n^a is spacelike ($\epsilon = 1$) or timelike ($\epsilon = -1$); $l_a n^a = 1$ in both cases.

IV. APPLICATIONS

A. Boundary Surface

A boundary surface in general relativity can be represented as a hypersurface in a Lorentzian manifold across which the matter model changes. A spacelike hypersurface corresponds to a change in the matter model at a particular time, and a timelike hypersurface corresponds to a change at a particular place, while a null hypersurface corresponds to a gravitational shock wave. We thus consider a manifold M with a Lorentzian metric g and a given non-null hypersurface Σ which divides M into 2 regions $M^\pm$. Let n denote the unit normal vector to Σ which points from M^- to M^+, and let $\epsilon = g(n, n) = \pm 1$. The extrinsic curvature K of Σ can be defined by

$$2K := \pounds_n g \tag{25}$$

so that in components

$$2K_{ab} = n^c \partial_c g_{ab} + g_{mb}\partial_a n^c + g_{am}\partial_b n^c \tag{26}$$

We shall assume here that g is C^1 across M, so that in particular the *Darmois junction conditions*

$$[h] = 0 = [K] \tag{27}$$

on the induced metric $h = g + \epsilon n \otimes n$ and extrinsic curvature K of Σ are satisfied. [2]

In the non-null case, Israel [3] used Einstein's equations and the Gauss-Codazzi relations between the curvature of M and that of Σ to relate the stress-energy tensor of the matter to the intrinsic and extrinsic curvatures of Σ. Clarke and Dray [4] generalized some of Israel's results to the null case. Corrected versions of some of Israel's result appear in [1], including

$$\rho := G_{ab}\,n^a n^b = \frac{1}{2}\left((K^c{}_c)^2 - K_{ab}K^{ab} - \epsilon\,\mathcal{R}\right) \tag{28}$$

where $\mathcal{R}$ denotes the scalar curvature of the intrinsic metric h on Σ. In the case where n^a is timelike, ρ can be interpreted as the energy density. If the Darmois junction conditions are satisfied, the RHS of this equation is continuous at Σ, and therefore so is the energy density, yielding

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[2] We reiterate that we are assuming that Σ is a hypersurface in a given manifold, so that local coordinates exist which span Σ. Equation (27) is thus to be interpreted as applying to n-dimensional tensors, obtained by projection into Σ. In practice, however, it is often convenient to consider the pullback of Equation (27) to Σ, which contains the same information but which only requires local coordinates *within* Σ.

$$[\rho] = \left[G_{ab}\, n^a n^b \right] = 0 \tag{29}$$

If we now apply the patchwork divergence theorem (24) to

$$X_a := G_{ab} n^b \tag{30}$$

then the term containing $l([X])$ vanishes, and we are left with the usual statement of the divergence theorem as though no boundary surface were present. If we instead consider

$$X_a := G_a{}^b l_b = l_b G^b{}_a \tag{31}$$

we obtain the same result. It is important to note that these results hold even though X may well not be C^1!

The results of Dray and Padmanabhan on piecewise Killing vectors [5] can be viewed as application of the patchwork divergence theorem in this setting.

B. Signature Change

Consider now a situation similar to the above, but where the metric is Lorentzian only in M^-, and Riemannian in M^+, so that the metric is now either discontinuous or zero at Σ. Such signature-changing models were introduced by Dray *et al.* [6–8]; similar models have since been used in a cosmological setting [9–13].

We can impose the Darmois junction conditions by means of 1-sided limits to Σ. In order to apply the patchwork divergence theorem we need a volume element on Σ, and we cannot now use the metric volume element there. One possibility is to work in normal coordinates and note that both sides induce the same volume element; as suggested in [1]. We discuss this issue further in the next section, and here assume that a suitable choice has been made.

On either manifold-with-boundary $M^\pm$, let n^a denote the unit normal vector as described just before and after (24). [3] Even though ϵ now changes sign between the two regions, ϵn^a is continuous across Σ, so that the RHS of (24) remains well-defined. However, Israel's results such as (29) no longer hold, and one obtains instead [1]

$$\left[G_{ab}\, n^a n^b \right] = -\mathcal{R} \tag{32}$$
$$\left[G^a{}_b\, n^b l_a \right] = (K^c{}_c)^2 - K_{ab} K^{ab} \tag{33}$$

The patchwork divergence theorem now contributes a surface term at Σ, and for $X_a = G_{ab} n^b$ as above we now obtain [4]

$$\int_W \nabla_a X^a \, d^n W = \oint_S m^a X_a \, d^{n-1} S - \int_{S^0} \left((K^c{}_c)^2 - K_{ab} K^{ab} \right) d^{n-1} \Sigma \tag{34}$$

If, however, we instead set $X_a = G_a{}^b l_b$, we now obtain a different conservation law, namely

$$\int_W \nabla_a X^a \, d^n W = \oint_S m^a X_a \, d^{n-1} S + \int_{S^0} \mathcal{R} \, d^{n-1} \Sigma \tag{35}$$

These are two of the main results of [1].

V. DISCUSSION

In the *normal coordinate* approach, $M^\pm$ are viewed as disjoint manifolds-with-boundary, with metrics of different signatures, which are being identified via an isometry of their boundaries $\Sigma^\pm$. To make the result a manifold, a differentiable structure must be specified at the identified boundary Σ. This can naturally be done by requiring

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[3] We will assume throughout this section that these normal vectors make sense, and can be obtained as 1-sided limits from $M^\pm$. This important issue will be discussed in Section V, where one way of achieving this is described. As discussed there, what really matters is whether the scalars in the last term in (24) have limits, not whether the normal vectors n^a do.

[4] In deriving this result it is slightly easier to use (33) in (23) rather than using (24) directly.

that normal coordinates, defined separately on either side of Σ, be admissible coordinates. In these coordinates, a signature-changing metric will take the form

$$ds^2 = \epsilon\, d\tau^2 + h_{ij}\, dx^i dx^j \tag{36}$$

where τ denotes proper time/proper distance away from Σ, and $\{x^i : i = 1, ..., n-1\}$ are local coordinates on Σ. Although this metric is discontinuous at Σ, the metric volume elements

$$d^n V := \sqrt{|\det(g)|}\, d\tau \wedge dx^1 \wedge ... \wedge dx^{n-1} \tag{37}$$

obtained separately on Σ using 1-sided limits from $M^{\pm}$ are identical, so that

$$d^{n-1}\Sigma := \sqrt{|\det(g)|}\, dx^1 \wedge ... \wedge dx^{n-1} \tag{38}$$

can be taken to be the natural volume element on Σ. Furthermore, in this approach the normal vectors n^a can clearly be obtained as 1-sided limits to Σ. This leads to the results of the previous section.

On the other hand, one can take the *continuous metric* approach, in which one starts with a manifold M on which a continuous, covariant tensor g of rank two is given which is assumed to be a metric on (the interior of) $M^{\pm}$. The only way for the signature of g to change is for it to be degenerate at Σ; we nonetheless refer to g as a *signature-changing metric*. Such a metric can be put in the form

$$ds^2 = N\, dt^2 + h_{ij}\, dx^i dx^j \tag{39}$$

where the function N is zero (only) on Σ. The limit of the metric volume element on M to Σ is now zero, so that it is not clear how to interpret (11) for the induced volume element. Nonetheless, Σ does still have a metric volume element due to the induced (nondegenerate) metric h, and this volume element agrees with $d^{n-1}\Sigma$ as defined using the normal coordinate approach.

In the continuous metric approach, however, there is no unit normal vector which can play the role of m in the divergence theorem. If we pick

$$m = \sqrt{|N|}\, dt \tag{40}$$

so that (11) is satisfied away from Σ, then $m = 0$ on Σ, and the RHS of the divergence theorem is identically zero (provided X is suitably smooth; see below). Contrast this with the normal coordinate approach, in which $m = d\tau$ is a basis 1-form and hence nonzero! In order to preserve a divergence theorem in this approach, we are thus led to pick an arbitrary, nonzero m, and then to define the volume element on M via (11) rather than using the metric volume element. The patchwork divergence theorem in the form (22) now holds, and can be used to derive results analogous to those in the previous section, although the volume integral over W is nonstandard.

Which of these approaches is relevant depends of course on what problem is being discussed. In particular, the very requirement that the tensors in the integrands of the divergence theorem be suitably smooth is different in the two approaches. For instance, **if** X^a is assumed to be smooth at Σ in the continuous metric approach, **then** $m_a X^a = 0$ in that approach. If, on the other hand, X^a is smooth in the normal coordinate approach, $m_a X^a$ can have a nonzero limit at Σ. [5] In a given physical situation with given smoothness of the physical fields, the same physical results will be obtained in either approach (provided suitable limits are taken). However, we feel that, due to the fundamental role played by unit normal derivatives in initial-value problems, the normal coordinate approach is more likely to automatically incorporate appropriate smoothness conditions at Σ.

Clarke & Dray [4] showed that the continuity of the induced metric h at the identified boundary of two manifolds-with-boundary leads to a unique differentiable structure for the resulting manifold. While their derivation assumed constant signature, this was only used in showing that the resulting n-dimensional metric is continuous. Thus, their result naturally extends to the case of signature change, where it results in the differentiable structure defined by the normal coordinate approach (with its discontinuous metric).

Their result can also be applied to the continuous metric approach as follows. Suppose that one is given two manifolds-with-boundary with (suitably controlled) degenerate metrics on the boundaries. Switching to the differentiable structures on each side induced by (1-sided) normal coordinates leads to a unique differentiable structure via

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[5] This problem is in addition to the necessity of specifying which index structure for a given tensor is fundamental, as shown e.g. by the difference between Equations (34) and (35).

the generalized result of Clarke & Dray. One can now reverse at least one of the changes in differentiable structure, obtaining a manifold structure on the glued-together manifold which agrees with at least one of the manifolds-with-boundary. If (and only if) the degeneracies on the two original boundaries are compatible in an appropriate sense (based essentially on the behavior of the metric when expanded as a power series in proper time/distance from Σ), a unique manifold structure compatible with both original manifolds-with-boundary is thus obtained. In particular, one can always construct several inequivalent manifold structures with continuous (degenerate) metrics, but at most one of these will agree with both of the original manifold structures on the two sides.

This issue can be largely ignored by adopting the *invariant approach* used in [1], which is closely related to the elegant approach used by Carfora and Ellis [14]. In this approach, one starts by identifying the disjoint manifolds-with-boundary $M^{\pm}$ as in the normal coordinate approach. With no further assumptions about the manifold structure at Σ, the 1-sided unit normal vectors still make sense, so that the (pullback of the) Darmois junction conditions can still be imposed.

In this approach, and further assuming that the 1-sided limits of $n^a X_a$ exist, one obtains the patchwork divergence theorem in the form

$$\int_W \nabla_a X^a \, d^n W = \oint_S m^a X_a \, d^{n-1} S - \int_{S^0} [\epsilon n^a X_a] \, d^{n-1} \Sigma \tag{41}$$

While in practice this is equivalent to (24), this form of the theorem emphasizes that it is the limits of physical quantities to Σ which determine the physics, not the choice (or even existence) of a manifold structure there.

The related issue of defining hypersurface distributions in the presence of signature change will be discussed elsewhere [15].

ACKNOWLEDGEMENTS

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