Weak Galerkin Finite Element Methods for Parabolic Equations*

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Received 13 June 2012; accepted 13 March 2013
Published online 25 May 2013 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/num.21786

A newly developed weak Galerkin method is proposed to solve parabolic equations. This method allows the usage of totally discontinuous functions in approximation space and preserves the energy conservation law. Both continuous and discontinuous time weak Galerkin finite element schemes are developed and analyzed. Optimal-order error estimates in both H^1 and L^2 norms are established. Numerical tests are performed and reported. Numer Methods Partial Differential Eq 29: 2004–2024, 2013

Keywords: finite element methods; parabolic equations; weak Galerkin methods

I. INTRODUCTION

In this article, we consider the initial-boundary value problem

$$\begin{aligned} u_t - \nabla\cdot(a\nabla u) &= f \quad \text{in } \Omega, \quad t \in J, \\ u &= g \quad \text{on } \partial\Omega, \quad t \in J, \\ u(\cdot,0) &= \psi \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where Ω is a polygonal or polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz-continuous boundary $\partial\Omega$, $J = (0, \bar{t}\,]$, and $a = (a_{ij}(x,t))_{d\times d} \in [L^\infty(\Omega\times\bar{J})]^{d^2}$ is a symmetric matrix-valued function satisfying the following property: there exists a constant $\alpha > 0$ such that

$$\alpha\xi^T\xi \le \xi^T a\xi \quad \forall \xi \in \mathbb{R}^d.$$

For simplicity, we shall concentrate on two-dimensional problems only (i.e., $d = 2$).

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*Any opinion, finding, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.
Contract grant sponsor: NSF IR/D program (to J.W. while working at the National Science Foundation)

For any nonnegative integer m, let $H^m(\Omega)$ be the standard Sobolov space, which is the collection of all real-valued functions defined on Ω with square integrable derivatives up to order m with seminorm

$$|\psi|_{s,\Omega} \equiv \left\{ \sum_{|\alpha|=s} \int_\Omega |\partial^\alpha \psi|^2 \, dx \right\}^{\frac{1}{2}},$$

and norm

$$\|\psi\|_{m,\Omega} \equiv \left(\sum_{j=0}^{m} |\psi|_{j,\Omega}^2 \right)^{\frac{1}{2}}.$$

For simplicity, we use $\|\cdot\|$ for the L^2 norm.

The parabolic problem can be written in a variational form as follows

$$\begin{aligned} (u_t, v) + (a\nabla u, \nabla v) &= (f, v) \quad \forall v \in H_0^1(\Omega),\ t \in J, \\ u(\cdot, 0) &= \psi, \end{aligned} \tag{2}$$

where u is called a weak solution if $u \in L^2(0, t; H^1(\Omega))$ and $u_t \in L^2(0, t; H^{-1}(\Omega))$, and if $u = g$ on $\partial\Omega$.

Parabolic problems have been treated by various numerical methods. For finite element methods, we refer to Refs. [1] and [2]. Discontinuous Galerkin finite element methods were studied in Refs. [3] and [4]. In Refs. [5] and [6], finite volume methods were presented, which were based on the integral conservation law rather than the differential equation. The integral conservation law was then enforced for small control volumes defined by the computational mesh.

The goal of this article is to consider a newly developed weak Galerkin (WG) finite element method for parabolic equation which is based on the definition of a discrete weak gradient operator proposed in Ref. [7]. In this numerical method, the analysis can be done by using the framework of Galerkin methods, and totally discontinuous functions are allowed to be used as the approximation space. Furthermore, the approximation results also satisfy the energy conservation law.

The rest of this article is organized as follows. In Section 2, we introduce some notation and establish a continuous time and discontinuous time WG finite element scheme for the parabolic initial boundary-value problem (1). In Section 3, we prove the energy conservation law of the WG approximation. Optimal-order error estimates in both H^1 and L^2 norms are proved in Section 4. The article is concluded with some numerical experiments to illustrate the theoretical analysis in Section 5.

II. THE WG METHOD

In this section, we design a continuous time and a discontinuous time WG finite element scheme for the initial-boundary value problem (1). We consider the space of discrete weak functions and the discrete weak operator introduced in Ref. [7]. Let $\mathcal{T}_h$ be a quasiuniform family of triangulations of a plane domain Ω, and T be each triangle element with ∂T as its boundary. For each $T \in \mathcal{T}_h$, let $P_j(T)$ be the set of polynomials on T with degree no more than j, and $P_l(\partial T)$ be the set of polynomials on ∂T with degree no more than l, respectively. Let $\hat{P}_j(T)$ be the set of homogeneous polynomials on T with degree j. The weak finite element space $S_h(j,l)$ is defined by

$$S_h(j,l) := \{v = \{v_0, v_b\} \,:\, v_0 \in P_j(T),\ v_b \in P_l(e) \quad \text{for all edge } e \subset \partial T,\ T \in \mathcal{T}_h\}.$$

Denote by $S_h^0(j,l)$ the subspace of $S_h(j,l)$ with vanishing boundary value on $\partial\Omega$; that is,

$$S_h^0(j,l) := \{v = \{v_0, v_b\} \in S_h(j,l),\ v_b|_{\partial T \cap \partial\Omega} = 0 \ \text{ for all } T \in \mathcal{T}_h\}.$$

Let $\sum_h = \{\mathbf{q} \in [L^2(\Omega)]^2 \,:\, \mathbf{q}|_T \in V(T,r) \text{ for all } T \in \mathcal{T}_h\}$, where $V(T,r)$ is a subspace of the set of vector-valued polynomials of degree no more than r on T. For each $v = \{v_0, v_b\} \in S_h(j,l)$, the discrete weak gradient $\nabla_d v \in \sum_h$ of v on each element T is given by the following equation:

$$\int_T \nabla_d v \cdot \mathbf{q}\, dT = -\int_T v_0 \nabla \cdot \mathbf{q}\, dT + \int_{\partial T} v_b \mathbf{q} \cdot \mathbf{n}\, ds \quad \forall \mathbf{q} \in V(T,r), \tag{3}$$

where $\mathbf{n}$ is the outward normal direction of ∂T. It is easy to see that this discrete weak gradient is well-defined.

To investigate the approximation properties of the discrete weak spaces $S_h(j,l)$ and $\sum_h$, we use three projections in this article. The first two are local projections defined on each triangle T: one is $Q_h u = \{Q_0 u, Q_b u\}$, the L^2 projection of $H^1(T)$ onto $P_j(T) \times P_l(\partial T)$ and another is $\mathbb{Q}_h$, the L^2 projection of $[L^2(T)]^2$ onto $V(T,r)$. The third projection Π_h is assumed to exist and satisfy the following property: for $\mathbf{q} \in H(div,\Omega)$ with mildly added regularity, $\Pi_h \mathbf{q} \in H(div,\Omega)$ such that $\Pi_h \mathbf{q} \in V(T,r)$ on each $T \in \mathcal{T}_h$, and

$$(\nabla \cdot \mathbf{q}, v_0)_T = (\nabla \cdot \Pi_h \mathbf{q}, v_0)_T \quad \forall v_0 \in P_j(T).$$

It is easy to see the following two useful identities:

$$\nabla_d(Q_h w) = \mathbb{Q}_h(\nabla w) \quad \forall w \in H^1(T), \tag{4}$$

and for any $\mathbf{q} \in H(div,\Omega)$

$$\sum_{T \in \mathcal{T}_h} (-\nabla \cdot \mathbf{q}, v_0)_T = \sum_{T \in \mathcal{T}_h} (\Pi_h \mathbf{q}, \nabla_d v)_T, \quad \forall v = \{v_0, v_b\} \in S_h^0(j,l). \tag{5}$$

The discrete weak spaces $S_h(j,l)$ and $\sum_h$ need to possess some good approximation properties to provide an acceptable finite element scheme. In Ref. [7], the following two criteria were given as a general guideline for their construction:

P1: For any $v \in S_h(j,l)$, if $\nabla_d v = 0$ on T, then one must have $v \equiv$ constant on T; that is, $v_0 = v_b =$ constant on T.

P2: For any $u \in H_0^1(\Omega) \cap H^{m+1}(\Omega)$, where $0 \le m \le j+1$, the discrete weak gradient of the L^2 projection $Q_h u$ of u in the discrete weak space $S_h(j,l)$ provides a good approximation of ∇u; that is, $\|\nabla_d(Q_h u) - \nabla u\| \le C h^m \|u\|_{m+1}$ holds true.

Examples of $S_h(j,l)$ and $\sum_h$ satisfying the conditions **P1** and **P2** can be found in Ref. [7]. For example, with $V(T, r = j+1) = [P_j(T)]^2 + \hat{P}_j(T)\mathbf{x}$ being the usual Raviart–Thomas element Ref. [8] of order j, one may take equal-order elements of order j for $S_h(j,l)$ in the interior and on the boundary of each element T. The key of using the Raviart–Thomas element for $V(T,r)$ is to ensure the condition **P1**. The condition **P2** was satisfied by the commutative property (4) which has been established in Ref. [7]. It was shown later in Refs. [9, 10] that the condition **P1** can be circumvented by a suitable stabilization term. Consequently, the selection of $V(T,r)$ and $S_h(j,l)$ becomes very flexible and robust in practical computation. It even allows the use of finite elements of arbitrary shape.

The main idea of the WG method is to use the discrete weak space $S_h(j,l)$ as testing and trial spaces and replace the classical gradient operator by the weak gradient operator ∇_d in (2).

First, we pose the continuous time WG finite element method, based on the weak gradient operator (3) and the variational formulation (2), which is to find $u_h(t) = \{u_0(\cdot,t), u_b(\cdot,t)\}$, belonging to $S_h(j,l)$ for $t \geq 0$, satisfying $u_b = Q_b g$ on $\partial\Omega, t > 0$ and $u_h(0) = Q_h\psi$ in Ω, and the following equation

$$((u_h)_t, v_0) + a(u_h, v) = (f, v_0) \quad \forall v = \{v_0, v_b\} \in S_h^0(j,l),\ t > 0, \tag{6}$$

where $a(\cdot,\cdot)$ is the weak bilinear form defined by

$$a(w, v) = (a\nabla_d w, \nabla_d v),$$

which is assumed to be bounded and coercive, that is, for constant $\alpha, \beta, \gamma > 0$

$$|a(u,v)| \leq \beta\|\nabla_d u\|\|\nabla_d v\|,$$
$$a(u,u) \geq \alpha\|\nabla_d u\|^2,$$

and that

$$|(a_t\nabla_d u, \nabla_d v)| \leq \gamma\|\nabla_d u\|\|\nabla_d v\|.$$

We now turn our attention to a discrete time WG finite element method. We introduce a time step k and the time levels $t = t_n = nk$, where n is a nonnegative integer, and denote by $U^n = U_h^n \in S_h(j,l)$ the approximation of $u(t_n)$ to be determined. The backward Euler WG method is defined by replacing the time derivative in (6) by a backward difference quotient, $\bar{\partial}U^n = (U^n - U^{n-1})/k$,

$$(\bar{\partial}U^n, v_0) + a(U^n, v) = (f(t_n), v_0) \quad \forall v = \{v_0, v_b\} \in S_h^0(j,l),\ n \geq 1,\ U^0 = Q_h\psi, \tag{7}$$

that is,

$$(U^n, v_0) + ka(U^n, v) = (U^{n-1} + kf(t_n), v_0) \quad \forall v = \{v_0, v_b\} \in S_h^0(j,l).$$

III. ENERGY CONSERVATION OF WG

This section investigates the energy conservation property of the WG finite element approximation u_h. The increase in internal energy in a small spatial region of the material K, that is, control volume, over the time period $[t - \Delta t, t + \Delta t]$ is given by

$$\int_K u(x, t+\Delta t)\,dx - \int_K u(x, t-\Delta t)\,dx = \int_{t-\Delta t}^{t+\Delta t}\int_K u_t\,dx\,dt.$$

Suppose that a body obeys the heat equation and, in addition, generates its own heat per unit volume at a rate given by a known function f varying in space and time, the change in internal energy in K is accounted for by the flux of heat across the boundaries together with the source energy. By Fourier's law, this is

$$-\int_{t-\Delta t}^{t+\Delta t}\int_{\partial K}\mathbf{q}\cdot\mathbf{n}\,ds\,dt+\int_{t-\Delta t}^{t+\Delta t}\int_{K} f\,dx\,dt,$$

where $\mathbf{q}=-a\nabla u$ is the flow rate of heat energy. The solution u of the problem (1) yields the following integral form of energy conservation:

$$\int_{t-\Delta t}^{t+\Delta t}\int_{K} u_t\,dx\,dt+\int_{t-\Delta t}^{t+\Delta t}\int_{\partial K}\mathbf{q}\cdot\mathbf{n}\,ds\,dt=\int_{t-\Delta t}^{t+\Delta t}\int_{K} f\,dx\,dt, \tag{8}$$

where the Green's formula was used. We claim that the numerical approximation from the WG finitel element method for (1) retains the energy conservation property (8).

In (6), we chose a test function $v=\{v_0, v_b=0\}$ so that $v_0=1$ on K and $v_0=0$ elsewhere. After integrating over the time period, we have

$$\int_{t-\Delta t}^{t+\Delta t}\int_{K} u_t\,dx\,dt+\int_{t-\Delta t}^{t+\Delta t}\int_{K} a\nabla_d u\cdot\nabla_d v\,dx\,dt=\int_{t-\Delta t}^{t+\Delta t}\int_{K} f\,dx\,dt. \tag{9}$$

Using the definition of operator $\mathbb{Q}_h$ and of the weak gradient ∇_d in (3), we arrive at

$$\begin{aligned}\int_K a\nabla_d u\cdot\nabla_d v\,dx&=\int_K \mathbb{Q}_h(a\nabla_d u_h)\cdot\nabla_d v\,dx=-\int_K \nabla\cdot\mathbb{Q}_h(a\nabla_d u_h)\,dx\\&=-\int_{\partial K}\mathbb{Q}_h(a\nabla_d u_h)\cdot\mathbf{n}\,ds.\end{aligned}$$

Then by substituting in (9), we have the energy conservation property

$$\int_{t-\Delta t}^{t+\Delta t}\int_{K} u_t\,dx\,dt+\int_{t-\Delta t}^{t+\Delta t}\int_{\partial K} -\mathbb{Q}_h(a\nabla_d u_h)\cdot\mathbf{n}\,ds\,dt=\int_{t-\Delta t}^{t+\Delta t}\int_{K} f\,dx\,dt,$$

which provides a numerical flux given by

$$\mathbf{q}_h\cdot\mathbf{n}=-\mathbb{Q}_h(a\nabla_d u_h)\cdot\mathbf{n}.$$

The numerical flux $\mathbf{q}_h\cdot\mathbf{n}$ is continuous across the edge of each element T, which can be verified by a selection of the test function $v=\{v_0, v_b\}$ so that $v_0\equiv 0$ and v_b arbitrary.

IV. ERROR ANALYSIS

In this section, we derive some error estimates for both continuous and discrete time WG finite element methods. The difference between the WG finite element approximation u_h and the L^2 projection $Q_h u$ of the exact solution u is measured. We first state a result concerning an approximation property as follows.

Lemma 4.1. *For $u \in H^{1+r}(\Omega)$ with $r > 0$, we have*

$$\|\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u)\| \le Ch^r\|u\|_{1+r}.$$

Proof. Since from (4) we have $\mathbb{Q}_h(\nabla u) = \nabla_d(Q_h u)$, then

$$\|\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u)\| = \|\Pi_h(a\nabla u) - a\nabla_d(Q_h u)\|.$$

Using the triangle inequality, the definition of Π_h and the second condition **P2** on the discrete weak space $S_h(j,l)$, we have

$$\|\Pi_h(a\nabla u) - a\nabla_d(Q_h u)\| \le \|\Pi_h(a\nabla u) - a\nabla u\| + \|a\nabla u - a\nabla_d(Q_h u)\| \le Ch^r\|u\|_{1+r}.$$

■

A. Continuous Time WG Finite Element Method

Our aim is to prove the following estimate for the error for the semidiscrete solution.

Theorem 4.1. *Let $u \in H^{1+r}(\Omega)$ and u_h be the solutions of (1) and (6), respectively. Denote by $e := u_h - Q_h u$ the difference between the WG approximation and the L^2 projection of the exact solution u. Then, there exists a constant C such that*

$$\|e(\cdot,t)\|^2 + \int_0^t \alpha\|\nabla_d e\|^2\,ds \le \|e(\cdot,0)\|^2 + Ch^{2r}\int_0^t \|u\|_{1+r}^2\,ds,$$

and

$$\int_0^t \|e_t\|^2\,ds + \frac{\alpha}{4}\|\nabla_d e(\cdot,t)\|^2 + \left(1+\frac{\gamma}{2\alpha}\right)\|e\|^2 \le \alpha\|\nabla_d e(\cdot,0)\|^2 + \left(1+\frac{\gamma}{2\alpha}\right)\|e(\cdot,0)\|^2$$
$$+ Ch^{2r}\left(\|u(\cdot,0)\|_{1+r}^2 + \|u\|_{1+r}^2 + \int_0^t \|u\|_{1+r}^2\,ds + \int_0^t \|u_t\|_{1+r}^2\,ds\right).$$

Proof. Let $v = \{v_0, v_b\} \in S_h^0(j,l)$ be the testing function. By testing (1) against v_0, together with $\mathbb{Q}_h(\nabla u) = \nabla_d(Q_h u)$ for $u \in H^1$ and $(Q_0 u_t, v_0) = (u_t, v_0)$, we obtain

$$\begin{aligned}(f, v_0) &= (u_t, v_0) + \sum_{T\in\mathcal{T}_h}(-\nabla\cdot(a\nabla u), v_0)_T \\ &= (u_t, v_0) + (\Pi_h(a\nabla u), \nabla_d v) \\ &= (Q_h u_t, v_0) + (\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u), \nabla_d v) + (a\nabla_d(Q_h u), \nabla_d v).\end{aligned}$$

As the numerical solution also satisfies Eq. (6), we have

$$(f, v_0) = ((u_h)_t, v_0) + a(u_h, v).$$

Combining the above two equations, we obtain

$$((u_h - Q_h u)_t, v_0) + a(u_h - Q_h u, v) = (\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u), \nabla_d v), \tag{10}$$

which shall be called the error equation for the WG method for the heat equation.

Let $e = u_h - Q_h u$ be the error. Use $v = e$ in the error equation, we obtain

$$(e_t, e) + a(e, e) = (\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u), \nabla_d e).$$

By Cauchy–Schwarz inequality and the coercivity of the bilinear form, we have

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 + \alpha\|\nabla_d e\|^2 \le \frac{1}{2\alpha}\|\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u)\|^2 + \frac{\alpha}{2}\|\nabla_d e\|^2.$$

It follows that

$$\frac{d}{dt}\|e\|^2 + \alpha\|\nabla_d e\|^2 \le \frac{1}{\alpha}\|\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u)\|^2,$$

and hence, after integration,

$$\|e\|^2 + \int_0^t \alpha\|\nabla_d e\|^2\,ds \le \|e(\cdot,0)\|^2 + \frac{1}{\alpha}\int_0^t \|\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u)\|^2\,dt. \tag{11}$$

Then by Lemma 4.1, we have the assertion.

To estimate $\nabla_d e$, we use the error equation with $v = (u_h - Q_h u)_t = e_t$. We obtain

$$\begin{aligned}(e_t, e_t) + a(e, e_t) &= (\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u), \nabla_d e_t)\\ &= \frac{d}{dt}(\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u), \nabla_d e) - (\Pi_h(a\nabla u_t) - a\mathbb{Q}_h(\nabla u_t), \nabla_d e)\\ &\quad - (\Pi_h(a_t\nabla u) - a_t\mathbb{Q}_h(\nabla u), \nabla_d e).\end{aligned}$$

By the Cauchy–Schwarz inequality, this shows that

$$\begin{aligned}\|e_t\|^2 + \frac{1}{2}\left(\frac{d}{dt}a(e,e) - (a_t\nabla_d e, \nabla_d e)\right) \le{}& \frac{d}{dt}(\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u), \nabla_d e)\\ &+ \frac{1}{2\alpha}\|\Pi_h(a\nabla u_t) - a\mathbb{Q}_h(\nabla u_t)\|^2 + \frac{\alpha}{2}\|\nabla_d e\|^2\\ &+ \frac{1}{2\alpha}\|\Pi_h(a_t\nabla u) - a_t\mathbb{Q}_h(\nabla u)\|^2 + \frac{\alpha}{2}\|\nabla_d e\|^2,\end{aligned}$$

that is,

$$\begin{aligned}\|e_t\|^2 + \frac{1}{2}\frac{d}{dt}a(e,e) \le{}& \frac{1}{2}(a_t\nabla_d e, \nabla_d e) + \frac{d}{dt}(\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u), \nabla_d e)\\ &+ \frac{1}{2\alpha}\|\Pi_h(a\nabla u_t) - a\mathbb{Q}_h(\nabla u_t)\|^2 + \frac{\alpha}{2}\|\nabla_d e\|^2\\ &+ \frac{1}{2\alpha}\|\Pi_h(a_t\nabla u) - a_t\mathbb{Q}_h(\nabla u)\|^2 + \frac{\alpha}{2}\|\nabla_d e\|^2.\end{aligned}$$

Thus, integrating with respect to t and together with the coercivity and boundedness yields

$$
\begin{aligned}
&\int_0^t \|e_t\|^2\,ds + \frac{\alpha}{2}\|\nabla_d e(\cdot,t)\|^2 \\
&\le \frac{\beta}{2}\|\nabla_d e(\cdot,0)\|^2 + (\Pi_h(a\nabla u(\cdot,t) - a\mathbb{Q}_h(\nabla u(\cdot,t)), \nabla_d e(\cdot,t)) \\
&\quad - (\Pi_h(a\nabla u(\cdot,0) - a\mathbb{Q}_h(\nabla u(\cdot,0)), \nabla_d e(\cdot,0)) + \frac{1}{2\alpha}\int_0^t \|\Pi_h(a\nabla u_t) - a\mathbb{Q}_h(\nabla u_t)\|^2\,ds \\
&\quad + \frac{1}{2\alpha}\int_0^t \|\Pi_h(a_t\nabla u) - a_t\mathbb{Q}_h(\nabla u)\|^2\,ds + \left(\alpha + \frac{\gamma}{2}\right)\int_0^t \|\nabla_d e\|^2\,ds.
\end{aligned}
$$

By adding $(\alpha+\gamma/2)/\alpha = 1 + \frac{\gamma}{2\alpha}$ times inequality (11) to the above inequality, we arrive at

$$
\begin{aligned}
&\int_0^t \|e_t\|^2\,ds + \frac{\alpha}{2}\|\nabla_d e(\cdot,t)\|^2 + \left(1 + \frac{\gamma}{2\alpha}\right)\|e\|^2 \\
&\le \frac{\beta}{2}\|\nabla_d e(\cdot,0)\|^2 + \left(1 + \frac{\gamma}{2\alpha}\right)\|e(\cdot,0)\|^2 \\
&\quad + (\Pi_h(a\nabla u(\cdot,t)) - a\mathbb{Q}_h(\nabla u(\cdot,t)), \nabla_d e(\cdot,t)) - (\Pi_h(a\nabla u(\cdot,0)) - a\mathbb{Q}_h(\nabla u(\cdot,0)), \nabla_d e(\cdot,0)) \\
&\quad + \frac{1}{2\alpha}\int_0^t \|\Pi_h(a\nabla u_t) - a\mathbb{Q}_h(\nabla u_t)\|^2\,ds + \frac{1}{2\alpha}\int_0^t \|\Pi_h(a_t\nabla u) - a_t\mathbb{Q}_h(\nabla u)\|^2\,ds \\
&\quad + \left(\frac{1}{\alpha} + \frac{\gamma}{2\alpha^2}\right)\int_0^t \|\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u)\|^2\,dt.
\end{aligned}
$$

Next, we use the Cauchy–Schwarz inequality to obtain

$$
\begin{aligned}
&\int_0^t \|e_t\|^2\,ds + \frac{\alpha}{4}\|\nabla_d e(\cdot,t)\|^2 + \left(1 + \frac{\gamma}{2\alpha}\right)\|e\|^2 \\
&\le \beta\|\nabla_d e(\cdot,0)\|^2 + \left(1 + \frac{\gamma}{2\alpha}\right)\|e(\cdot,0)\|^2 \\
&\quad + \frac{1}{\alpha}\|\Pi_h(a\nabla u(\cdot,t)) - a\mathbb{Q}_h(\nabla u(\cdot,t))\|^2 + \frac{1}{2\alpha}\|\Pi_h(a\nabla u(\cdot,0)) - a\mathbb{Q}_h(\nabla u(\cdot,0))\|^2 \\
&\quad + \frac{1}{2\alpha}\int_0^t \|\Pi_h(a\nabla u_t) - a\mathbb{Q}_h(\nabla u_t)\|^2\,ds + \frac{1}{2\alpha}\int_0^t \|\Pi_h(a_t\nabla u) - a_t\mathbb{Q}_h(\nabla u)\|^2\,ds \\
&\quad + \left(\frac{1}{\alpha} + \frac{\gamma}{2\alpha^2}\right)\int_0^t \|\Pi_h(a\nabla u) - a\mathbb{Q}_h(\nabla u)\|^2\,dt.
\end{aligned}
$$

Then by Lemma 4.1, the proof is completed. ■

B. Discrete Time WG Finite Element Method

We begin with the following lemma which provides a Poincaré-type inequality with the weak gradient operator.

Lemma 4.2. *Assume that* $\phi = \{\phi_0, \phi_b\} \in S_h^0(j, j)$, *then there exists a constant* C *such that*

$$\|\phi\| \le C\|\nabla_d \phi\|,$$

where $\nabla_d \phi \in V(T, r = j+1) = [P_j(T)]^2 + \hat{P}_j(T)\mathbf{x}$.

Proof. Let $\bar{\phi}_0$ be a piecewise constant function with the cell average of ϕ on each element T. Let $\phi_I \in H_0^1(\Omega)$ be a continuous piecewise polynomial with vanishing boundary value lifted from ϕ as follows. Let $G_j(T)$ be the set of all Lagrangian nodal points for $P_{j+1}(T)$. At all internal Lagrangian nodal points $x_i \in G_j(T)$, we set $\phi_I(x_i) = \bar{\phi}_0$. At boundary Lagrangian points $x_i \in G_j(T) \cap \partial T$, we let $\phi_I(x_i)$ be the trace of $\bar{\phi}_0$ from either side of the boundary. At global Lagrangian points $x_i \in \partial T \cap \partial\Omega$, we set $\phi_I(x_i) = 0$. Let $[\![\phi_0]\!]_e$ denote the jump of ϕ_0 on the edge e; i.e.,

$$[\![\phi_0]\!]_e = \phi_0|_{T_1} - \phi_0|_{T_2} = (\phi_0|_{T_1} - \phi_b) - (\phi_0|_{T_2} - \phi_b). \tag{12}$$

By the classical Poincaré inequality for ϕ_I, we have

$$\begin{aligned} \|\phi\| &\le \|\phi - \phi_I\| + \|\phi_I\| \\ &\le \left(\sum_T \|\phi - \phi_I\|_T^2\right)^{\frac{1}{2}} + C\|\nabla\phi_I\|. \end{aligned} \tag{13}$$

From Lemma 4.3 in Ref. [11], we have

$$\|\phi - \phi_I\|_T^2 \le \sum_{T' \in \mathcal{T}(T)} h_{T'}^2 \|\nabla\phi_0\|_{T'}^2 + \sum_{e \in \varepsilon(T)} h_e \|[\![\phi_0]\!]\|_e^2 \quad \forall T \in \mathcal{T}_h, \tag{14}$$

where $\mathcal{T}(T)$ denotes the set of all triangles in $\mathcal{T}$ having a nonempty intersect with T, including T itself, and $\varepsilon(T)$ denotes the set of all edges having a nonempty intersection with T. Note the elementary fact that

$$\|\nabla\phi_I\|^2 \le C \sum_T |\phi_I(x_i) - \phi_I(x_k)|^2, \tag{15}$$

where x_i and x_k run through all the Lagrangian nodal points on T. By construction, $\phi_I(x_i)$ is the cell average of ϕ_0 on either the element T or an adjacent element T_* which shares with T a common edge or a vertex point. Thus, $\phi_I(x_i) - \phi_I(x_k)$ is either zero or the difference of the cell average of ϕ_0 on two adjacent elements T and T_*. For the later case, assume that x_e is a point shared by T and T_*. It is not hard to see that

$$\left|\bar{\phi}_0|_T - \phi_0(x_e)\right|^2 \le C \int_T |\nabla\phi_0|^2 dx.$$

Thus, we have

$$\begin{aligned} |\phi_I(x_i) - \phi_I(x_k)|^2 &= |\bar{\phi}_0|_T - \bar{\phi}_0|_{T_*}|^2 \\ &= \left|\bar{\phi}_0|_T - \phi_0|_T(x_e) + [\![\phi_0]\!](x_e) + \phi_0|_{T_*}(x_e) - \bar{\phi}_0|_{T_*}\right|^2 \\ &\le C \sum_{e \subset \partial T} h_e^{-1} \|[\![\phi_0]\!]\|_e^2 + C \int_{T \cup T_*} |\nabla\phi_0|^2 dx. \end{aligned}$$

Substituting the above estimate into (15) yields

$$\|\nabla\phi_I\|^2 \le C\sum_T \left(\|\nabla\phi_0\|_T^2 + h^{-1}\|[\![\phi_0]\!]\|_{\partial T}^2\right). \tag{16}$$

By combining (13) with (14) and (16), we obtain

$$\|\phi\|^2 \le C\sum_T \left(\|\nabla\phi_0\|_T^2 + h^{-1}\|\phi_0-\phi_b\|_{\partial T}^2\right), \tag{17}$$

where (12) has been applied to estimate the jump of ϕ_0 on each edge.

Next, we want to bound the two terms on the right hand side of (17) by $\|\nabla_d\phi\|$. Let us recall that the weak gradient $\nabla_d\phi$ is defined by

$$\int_T \nabla_d\phi\cdot\mathbf{q}dx = -\int_T \phi_0\nabla\cdot\mathbf{q}dx + \int_{\partial T}\phi_b\mathbf{q}\cdot\mathbf{n}ds \quad \forall\mathbf{q}\in V(T,j+1),$$

and by Green's formula, we have

$$\int_T \nabla_d\phi\cdot\mathbf{q}dx = \int_T \nabla\phi_0\cdot\mathbf{q}dx - \int_{\partial T}(\phi_0-\phi_b)\mathbf{q}\cdot\mathbf{n}ds \quad \forall\mathbf{q}\in V(T,j+1). \tag{18}$$

Let $\mathbf{q}$ in (18) satisfy the following

$$\int_T \mathbf{q}\cdot\mathbf{r}\,dx = \int_T \nabla\phi_0\cdot\mathbf{r}\,dx \quad \forall\mathbf{r}\in[P_j(T)]^2,$$
$$\int_{\partial T}\mathbf{q}\cdot\mathbf{n}\,\mu\,ds = -h^{-1}\int_{\partial T}(\phi_0-\phi_b)\mu\,ds \quad \forall\mu\in P_{j+1}[\partial T],$$

where by Lemma 5.1 in Ref. [12], $\mathbf{q}$ and ϕ have the norm equivalence of

$$\|\mathbf{q}\|_{L^2(T)} \approx \|\nabla\phi_0\|_{L^2(T)} + h^{-\frac{1}{2}}\|\phi_0-\phi_b\|_{L^2(\partial T)}. \tag{19}$$

Then from (18), we have

$$\begin{aligned}\int_T \nabla_d\phi\cdot\mathbf{q}dx &= \int_T \nabla\phi_0\cdot\mathbf{q}dx - \int_{\partial T}(\phi_0-\phi_b)\mathbf{q}\cdot\mathbf{n}ds\\ &= \int_T \nabla\phi_0\cdot\nabla\phi_0\,dx + h^{-1}\int_{\partial T}(\phi_0-\phi_b)^2\,ds\\ &= \|\nabla\phi_0\|_T^2 + h^{-1}\|\phi_0-\phi_b\|_{\partial T}^2.\end{aligned}$$

Using (18) and (19), we have

$$\begin{aligned}\|\nabla_d\phi\|_T \ge \frac{1}{\|\mathbf{q}\|}(\nabla_d\phi,\mathbf{q})_T &= \frac{1}{\|\mathbf{q}\|}(\|\nabla\phi_0\|_T^2 + h^{-1}\|\phi_0-\phi_b\|_{\partial T}^2)\\ &\ge C(\|\nabla\phi_0\|_T^2 + h^{-1}\|\phi_0-\phi_b\|_{\partial T}^2),\end{aligned}$$

then together with (17), we have the assertion. ■

With the results established in Lemma 4.2, we are ready to derive an error estimate for the discrete time WG approximation u_h as the following theorem.

Theorem 4.2. *Let $u \in H^{1+r}(\Omega)$ and U^n be the solutions of (1) and (7), respectively. Denote by $e^n := U^n - Q_h u(t_n)$ the difference between the backward Euler WG approximation and the L^2 projection of the exact solution u. Then, there exists a constant C such that*

$$\|e^n\|^2 + \sum_{j=1}^{n} \alpha k \|\nabla_d e^j\|^2 \leq \|e^0\|^2 + C(h^{2r}\|u\|_{1+r}^2 + k^2 \int_0^{t_n} \|u_{tt}\|^2\, ds), \quad \textit{for } n \geq 0,$$

and

$$\|\nabla_d e^n\|^2 \leq C\left(\|e^0\|^2 + \|\nabla_d e^0\|^2 + h^{2r}(\|u\|_{1+r}^2 + \|u_t\|_{1+r}^2) + k^2 \int_0^{t_n} \|u_{tt}\|^2 ds\right),$$

where $\|u\|_{1+r} = \max_{j=1,\cdots,n}\{\|u(t_j)\|_{1+r}\}$ and $\|u_t\|_{1+r} = \max_{j=1,\cdots,n}\{\|u_t(t_j)\|_{1+r}\}$.

Proof. A calculation corresponding to the error Eq. (10) yields

$$(\bar{\partial}(U^n - Q_h u(t_n)), v_0) + a(U^n - Q_h u(t_n)), v) \\ = (u_t(t_n) - \bar{\partial} u(t_n), v_0) + (\Pi_h(a\nabla u(t_n)) - a\mathbb{Q}_h(\nabla u(t_n)), \nabla_d v),$$

that is,

$$(\bar{\partial} e^n, v_0) + a(e^n, v) = (u_t(t_n) - \bar{\partial} u(t_n), v_0) + (\Pi_h(a\nabla u(t_n)) - a\mathbb{Q}_h(\nabla u(t_n)), \nabla_d v). \tag{20}$$

Let $w_1^n = u_t(t_n) - \bar{\partial} u(t_n)$, and $w_2^n = \Pi_h(a\nabla u(t_n)) - a\mathbb{Q}_h(\nabla u(t_n))$, and choosing $v = e^n$ in (20), we have

$$(\bar{\partial} e^n, e^n) + a(e^n, e^n) = (w_1^n, e^n) + (w_2^n, \nabla_d e^n).$$

By coercivity of the bilinear form and Cauchy–Schwarz inequality, we obtain

$$\|e^n\|^2 - (e^{n-1}, e^n) + \alpha k \|\nabla_d e^n\|^2 \leq k\|w_1^n\|\|e^n\| + k\|w_2^n\|\|\nabla_d e^n\|,$$

that is,

$$\|e^n\|^2 + \alpha k \|\nabla_d e^n\|^2 \leq \frac{1}{2}\|e^{n-1}\|^2 + \frac{1}{2}\|e^n\|^2 + k\|w_1^n\|\|e^n\| + k\|w_2^n\|\|\nabla_d e^n\|.$$

By the Poincaré inequality of Lemma 4.2, we have

$$\frac{1}{2}\|e^n\|^2 + \alpha k \|\nabla_d e^n\|^2 \leq \frac{1}{2}\|e^{n-1}\|^2 + k(C\|w_1^n\| + \|w_2^n\|)(\|\nabla_d e^n\|).$$

Then by triangle inequality, it follows

$$\frac{1}{2}\|e^n\|^2 + \frac{\alpha k}{2}\|\nabla_d e^n\|^2 \leq \frac{1}{2}\|e^{n-1}\|^2 + \frac{k}{2\alpha}(C\|w_1^n\| + \|w_2^n\|)^2,$$

so that

$$\|e^n\|^2 + \alpha k\|\nabla_d e^n\|^2 \leq \|e^{n-1}\|^2 + \frac{Ck}{\alpha}\|w_1^n\|^2 + \frac{k}{\alpha}\|w_2^n\|^2,$$

and, by repeated application,

$$\|e^n\|^2 + \sum_{j=1}^{n} \alpha k\|\nabla_d e^j\|^2 \leq \|e^0\|^2 + \frac{Ck}{\alpha}\sum_{j=1}^{n}\|w_1^j\|^2 + \frac{k}{\alpha}\sum_{j=1}^{n}\|w_2^j\|^2. \tag{21}$$

We write

$$kw_1^j = ku_t(t_j) - (u(t_j) - u(t_{j-1})) = \int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}(s)\,ds, \tag{22}$$

that is,

$$w_1^j = u_t(t_j) - \frac{u(t_j) - u(t_{j-1})}{k} = \frac{1}{k}\int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}(s)\,ds,$$

so that

$$\begin{aligned}
\|w_1^j\|^2 &= \int_\Omega \{\frac{1}{k}\int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}(s)\,ds\}^2\,dx \\
&\leq \frac{1}{k^2}\int_\Omega \int_{t_{j-1}}^{t_j} (s - t_{j-1})^2\,ds \int_{t_{j-1}}^{t_j} u_{tt}^2(s)\,ds\ dx \\
&\leq Ck\int_{t_{j-1}}^{t_j} \|u_{tt}\|^2\,ds.
\end{aligned} \tag{23}$$

Substitute (23) into (21) and together with Lemma 4.1, we have the error estimate for $\|e^n\|$.

To show an estimate for $\nabla_d e^n$, we may choose instead $v = \bar{\partial} e^n$ in error Eq. (20) to obtain the following identity

$$(\bar{\partial} e^n, \bar{\partial} e^n) + a(e^n, \bar{\partial} e^n) = (w_1^n, \bar{\partial} e^n) + (w_2^n, \nabla_d \bar{\partial} e^n).$$

The second term on the right hand side can be written as

$$(w_2^n, \nabla_d \bar{\partial} e^n) = \bar{\partial}(w_2^n, \nabla_d e^n) - ((w_2^n)_t - \bar{\partial} w_2^n, \nabla_d e^{n-1}) + ((w_2^n)_t, \nabla_d e^{n-1}).$$

Then, the error equation becomes

$$\begin{aligned}
k\|\bar{\partial} e^n\|^2 + (a\nabla_d e^n, \nabla_d e^n) &= (a\nabla_d e^n, \nabla_d e^{n-1}) + k(w_1, \bar{\partial} e^n) \\
&\quad + k\bar{\partial}(w_2^n, \nabla_d e^n) - k((w_2^n)_t - \bar{\partial} w_2^n, \nabla_d e^{n-1}) + k((w_2^n)_t, \nabla_d e^{n-1}).
\end{aligned}$$

By triangle inequality, we have

$$\begin{aligned}
k\|\bar{\partial} e^n\|^2 + (a\nabla_d e^n, \nabla_d e^n) &\leq \frac{1}{2}(a\nabla_d e^n, \nabla_d e^n) + \frac{1}{2}(a\nabla_d e^{n-1}, \nabla_d e^{n-1}) \\
&\quad + \frac{k}{4}\|w_1^n\|^2 + k\|\bar{\partial} e^n\|^2 + k\bar{\partial}(w_2^n, \nabla_d e^n) \\
&\quad + \frac{k}{2}\|(w_2^n)_t - \bar{\partial} w_2^n\|^2 + \frac{k}{2}\|\nabla_d e^{n-1}\|^2 + \frac{k}{2}\|(w_2^n)_t\|^2 + \frac{k}{2}\|\nabla_d e^{n-1}\|^2,
\end{aligned}$$

and, after cancelation and by repeated application,

$$\frac{1}{2}(a\nabla_d e^n, \nabla_d e^n) \leq \frac{1}{2}(a\nabla_d e^0, \nabla_d e^0) + \frac{k}{4}\sum_{j=1}^{n} \|w_1^j\|^2 + (w_2^n, \nabla_d e^n) - (w_2^0, \nabla_d e^0)$$
$$+ \frac{k}{2}\sum_{j=1}^{n} \|(w_2^n)_t - \bar{\partial} w_2^j\|^2 + \frac{k}{2}\sum_{j=1}^{n} \|(w_2^n)_t\|^2 + k\sum_{j=1}^{n} \|\nabla_d e^{j-1}\|^2,$$

which is

$$\frac{\alpha}{2}\|\nabla_d e^n\|^2 \leq \frac{\beta}{2}\|\nabla_d e^0\|^2 + \frac{k}{4}\sum_{j=1}^{n}\|w_1^j\|^2 + \frac{1}{\alpha}\|w_2^n\|^2 + \frac{\alpha}{4}\|\nabla_d e^n\|^2 + \frac{1}{2\beta}\|w_2^0\|^2 + \frac{\beta}{2}\|\nabla_d e^0\|^2$$
$$+ \frac{k}{2}\sum_{j=1}^{n}\|(w_2^n)_t - \bar{\partial} w_2^j\|^2 + \frac{k}{2}\sum_{j=1}^{n}\|(w_2^n)_t\|^2 + k\sum_{j=1}^{n}\|\nabla_d e^{j-1}\|^2, \tag{24}$$

followed by triangle inequality and boundness and coercivity of the bilinear form. By the similar process as in (23) and Lemma 4.1, we have

$$\|(w_2^n)_t - \bar{\partial} w_2^j\|^2 \leq Ck\int_{t_{j-1}}^{t_j} \|(w_2)_{tt}\|^2\, ds \leq Ckh^{2r}\int_{t_{j-1}}^{t_j} \|u_{tt}\|_{1+r}^2\, ds.$$

Then by substituting (23), (21), and the above inequality into (24), we have the error estimate for $\|\nabla_d e^n\|$ as the following

$$\|\nabla_d e^n\|^2 \leq C\left(\|e^0\|^2 + \|\nabla_d e^0\|^2 + h^{2r}(\|u\|_{1+r}^2 + \|u_t\|_{1+r}^2) + k^2\int_0^{t_n}\|u_{tt}\|^2 ds\right),$$

which completes the proof. ∎

C. Optimal Order of Error Estimation in L^2

To get an optimal order of error estimate in L^2, the idea, similar to Wheeler's projection as in Refs. [13] and [1], is used where an elliptic projection E_h onto the discrete weak space $S_h(j,l)$ is defined as the following: find $E_h v \in S_h(j,l)$ such that $E_h v$ is the L^2 projection of the trace of v on the boundary $\partial\Omega$ and

$$(a\nabla_d E_h v, \nabla_d \chi) = (-\nabla\cdot(a\nabla v), \chi) \quad \forall \chi \in S_h^0(j,l). \tag{25}$$

In view of the weak formulation of the elliptic problem,

$$\begin{aligned} -\nabla\cdot(a\nabla v) &= F \quad \text{in } \Omega, \\ v &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{26}$$

this definition may be expressed by saying that $E_h v$ is the WG finite element approximation of the solution of the corresponding elliptic problem with exact solution v. By the error estimate results in Ref. [7], we have the following error estimate for $E_h v$.

Lemma 4.3. *Assume that problem (26) has the H^{1+s} regularity ($s \in (0, 1]$). Let $v \in H^{1+r}$ be the exact solution of (26), and $E_h v$ be a WG approximation of v defined in (25). Let $Q_h v = \{Q_0 v, Q_b v\}$ be the L^2 projection of v in the corresponding discrete weak space. Then there exists a constant C such that*

$$\|Q_0 v - E_h v\| \le C(h^{s+1}\|F - Q_0 F\| + h^{r+s}\|v\|_{r+1}), \tag{27}$$

and

$$\|\nabla_d (Q_h v - E_h v)\| \le Ch^r \|v\|_{r+1}. \tag{28}$$

Throughout this section, the error for the WG approximation is written as a sum of two terms,

$$u_h(t) - Q_h u(t) = \theta(t) + \rho(t), \quad \text{where } \theta = u_h - E_h u, \quad \rho = E_h u - Q_h u, \tag{29}$$

which will be bounded separately. Notice that the second term is the error in an elliptic problem and then can be handled by applying the results in Lemma 4.3. Then, our main goal here is to bound the first term θ.

Following the above strategy, the error estimate for continuous time WG finite element method in L^2 and H^1 are provided in the next two theorems.

Theorem 4.3. *Under the assumption of Theorem 4.1 and the assumption that the corresponding elliptic problem has the H^{1+s} regularity ($s \in (0, 1]$), there exists a constant C such that*

$$\begin{aligned}\|u_h(t) - Q_h u(t)\| &\le \|u_h(0) - Q_h u(0)\| + Ch^{r+s}\left(\|\psi\|_{r+1} + \int_0^t \|u_t\|_{r+1}\, ds\right)\\ &\quad + Ch^{s+1}(\|f(0) - Q_0 f(0)\| + \|u_t(0) - Q_0 u_t(0)\|)\\ &\quad + Ch^{s+1}\left\{\int_0^t (\|f_t - Q_0 f_t\| + \|u_{tt} - Q_0 u_{tt}\|)\, ds\right\}.\end{aligned}$$

Proof. We write the error according to (29) and obtain the error bound for ρ easily by Lemma 4.3 as the following

$$\|\rho\| \le C\left(h^{s+1}(\|f - Q_0 f\| + \|u_t - Q_0 u_t\|) + h^{r+s}\left(\|\psi\|_{r+1} + \int_0^t \|u_t\|_{r+1}\, ds\right)\right). \tag{30}$$

To estimate θ, we note that by our definitions

$$\begin{aligned}(\theta_t, \chi) + a(\theta, \chi) &= ((u_h)_t, \chi) + a(u_h, \chi) - (E_h u_t, \chi) - a(E_h u, \chi)\\ &= (f, \chi) - (E_h u_t, \chi) - a(E_h u, \chi)\\ &= (f, \chi) + (\nabla \cdot (a\nabla u), \chi) - (E_h u_t, \chi)\\ &= (u_t, \chi) - (E_h u_t, \chi)\\ &= (Q_h u_t, \chi) - (E_h u_t, \chi)\\ &= -(\rho_t, \chi),\end{aligned}$$

which is

$$(\theta_t, \chi) + a(\theta, \chi) = -(\rho_t, \chi) \quad \forall \chi \in S_h^0(j,l),\ t > 0, \tag{31}$$

where we have used the fact that the operator E_h commutes with time differentiation. Because $\theta \in S_h^0(j,l)$, we may choose $\chi = \theta$ in (31) and obtain

$$(\theta_t, \theta) + a(\theta, \theta) = -(\rho_t, \theta), \quad t > 0.$$

Since

$$a(\theta, \theta) \geq \alpha \|\nabla_d \theta\|^2 > 0,$$

we have

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 = \|\theta\|\frac{d}{dt}\|\theta\| \leq \|\rho_t\|\|\theta\|,$$

and hence

$$\|\theta(t)\| \leq \|\theta(0)\| + \int_0^t \|\rho_t\|\, ds.$$

Using Lemma 4.3, we find

$$\begin{aligned} \|\theta(0)\| &= \|u_h(0) - E_h u(0)\| \\ &\leq \|u_h(0) - Q_h u(0)\| + \|E_h u(0) - Q_h u(0)\| \\ &\leq \|u_h(0) - Q_h u(0)\| \\ &\quad + C[h^{s+1}(\|f(0) - Q_0 f(0)\| + \|u_t(0) - Q_0 u_t(0)\|) + h^{r+s}\|\psi\|_{r+1}], \end{aligned} \tag{32}$$

and as

$$\|\rho_t\| = \|E_h u_t - Q_h u_t\| \leq C[h^{s+1}(\|f_t - Q_0 f_t\| + \|u_{tt} - Q_0 u_{tt}\|) + h^{r+s}\|u_t\|_{r+1}], \tag{33}$$

the desired bound for $\|\theta(t)\|$ now follows. ∎

Theorem 4.4. *Under the assumption of Theorem 4.3 and the assumption that the coefficient matrix a in (1) is independent of time t, there exists a constant C such that*

$$\begin{aligned} \|\nabla_d(u_h(t) - Q_h u(t))\|^2 &\leq 4\beta\|\nabla_d(u_h(0) - Q_h u(0))\|^2 + Ch^{2r}(\|\psi\|_{r+1}^2 + \|u\|_{r+1}^2) \\ &+ Ch^{2(s+1)}\int_0^t (\|f_t - Q_0 f_t\| + \|u_{tt} - Q_0 u_{tt}\|)^2\, ds + Ch^{2(r+s)}\int_0^t \|u_t\|_{r+1}^2\, ds. \end{aligned}$$

Proof. As in the proof of Theorem 4.3, we write the error in the form (29). Here by Lemma 4.3,

$$\|\nabla_d \rho(t)\| \leq Ch^r\|u\|_{r+1}. \tag{34}$$

To estimate $\nabla_d \theta$, we may choose $\chi = \theta_t$ in the Eq. (31) for θ. We obtain

$$(\theta_t, \theta_t) + a(\theta, \theta_t) = -(\rho_t, \theta_t).$$

As the coefficient matrix a in the bilinear form $a(\cdot,\cdot)$ is independent of time t, we have

$$\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}(a\nabla_d\theta, \nabla_d\theta) = -(\rho_t, \theta_t) \le \frac{1}{2}\|\rho_t\|^2 + \frac{1}{2}\|\theta_t\|^2,$$

so that

$$\frac{d}{dt}(a\nabla_d\theta, \nabla_d\theta) \le \|\rho_t\|^2.$$

Then by integrating with respect to time t and using the coercivity and boundedness of the bilinear form, we obtain

$$\begin{aligned}\alpha\|\nabla_d\theta\|^2 &\le (a\nabla_d\theta, \nabla_d\theta) \le (a\nabla_d\theta(0), \nabla_d\theta(0)) + \int_0^t \|\rho_t\|^2\, ds \le \beta\|\nabla_d\theta(0)\|^2 + \int_0^t \|\rho_t\|^2\, ds \\ &\le \beta(\|\nabla_d(u_h(0) - Q_hu(0))\| + \|\nabla_d(E_hu(0) - Q_hu(0))\|)^2 + \int_0^t \|\rho_t\|^2\, ds.\end{aligned}$$

Hence, in view of Lemma 4.3 and (33), we have

$$\begin{aligned}\|\nabla_d\theta(t)\|^2 &\le 2\beta\|\nabla_d(u_h(0) - Q_hu(0))\|^2 + Ch^{2r}\|\psi\|_{r+1}^2 \\ &\quad + Ch^{2(s+1)}\int_0^t (\|f_t - Q_0f_t\| + \|u_{tt} - Q_0u_{tt}\|)^2\, ds + Ch^{2(r+s)}\int_0^t \|u_t\|_{r+1}^2\, ds,\end{aligned}$$

which completes the proof. ∎

Next, we derive an error estimate for the backward Euler WG method.

Theorem 4.5. *Let $u \in H^{1+r}(\Omega)$ and U^n be the solutions of (1) and (7), respectively. And let Q_hu be the L^2 projection of the exact solution u. Then, there exists a constant C such that*

$$\begin{aligned}\|U^n - Q_hu(t_n)\| &\le \|U^0 - Q_hu(0)\| + Ch^{r+s}\left(\|\psi\|_{r+1} + \int_0^{t_n} \|u_t\|_{r+1}\, ds\right) \\ &\quad + Ch^{s+1}(\|f(0) - Q_0f(0)\| + \|u_t(0) - Q_0u_t(0)\|) \\ &\quad + Ch^{s+1}(\|f(t_n) - Q_0f(t_n)\| + \|u_t(t_n) - Q_0u_t(t_n)\|) \\ &\quad + Ch^{s+1}\left\{\int_0^{t_n} (\|f_t - Q_0f_t\| + \|u_{tt} - Q_0u_{tt}\|)\, ds\right\} + Ck\int_0^{t_n} \|u_{tt}\|\, ds.\end{aligned}$$

Proof. In analogy with (29), we write

$$U^n - Q_hu(t_n) = (U^n - E_hu(t_n)) + (E_hu(t_n) - Q_hu(t_n)) = \theta^n + \rho^n,$$

where $\rho^n = \rho(t_n)$ is bounded as shown in (30). To bound θ^n, we use

$$\begin{aligned}(\bar{\partial}\theta^n, \chi) + a(\theta^n, \chi) &= (\bar{\partial}U^n, \chi) + a(U^n, \chi) - (\bar{\partial}E_h u(t_n), \chi) - a(E_h u(t_n), \chi)\\ &= (f(t_n), \chi) - (\bar{\partial}E_h u(t_n), \chi) - a(E_h u(t_n), \chi)\\ &= (f(t_n), \chi) + (\nabla\cdot(a\nabla u(t_n)), \chi) - (\bar{\partial}E_h u(t_n), \chi)\\ &= (u_t(t_n), \chi) - (\bar{\partial}E_h u(t_n), \chi)\\ &= (u_t(t_n) - \bar{\partial}u(t_n), \chi) + (\bar{\partial}u(t_n) - \bar{\partial}E_h u(t_n), \chi),\end{aligned}$$

that is,

$$(\bar{\partial}\theta^n, \chi) + a(\theta^n, \chi) = (w^n, \chi), \tag{35}$$

where

$$w^n = (u_t(t_n) - \bar{\partial}u(t_n)) + (\bar{\partial}u(t_n) - \bar{\partial}E_h u(t_n)) = w_1^n + w_3^n.$$

By choosing $\chi = \theta^n$ in (35) and the coercivity of the bilinear form, we have

$$(\bar{\partial}\theta^n, \theta^n) \le \|w^n\|\|\theta^n\|,$$

or

$$\|\theta^n\|^2 - (\theta^{n-1}, \theta^n) \le k\|w^n\|\|\theta^n\|,$$

so that

$$\|\theta^n\| \le \|\theta^{n-1}\| + k\|w^n\|,$$

and, by repeated application, it follows

$$\|\theta^n\| \le \|\theta^0\| + k\sum_{j=1}^n \|w^j\| \le \|\theta^0\| + k\sum_{j=1}^n \|w_1^j\| + k\sum_{j=1}^n \|w_3^j\|.$$

As in (32), $\theta^0 = \theta(0)$ is bounded as desired. By using the representation in (22), we obtain

$$k\sum_{j=1}^n \|w_1^j\| \le \sum_{j=1}^n \left\| \int_{t_{j-1}}^{t_j} (s - t_{j-1})u_{tt}(s)\,ds \right\| \le k\int_0^{t_n} \|u_{tt}\|\,ds.$$

We write

$$w_3^j = \bar{\partial}u(t_n) - E_h\bar{\partial}u(t_n) = (E_h - I)k^{-1}\int_{t_{j-1}}^{t_j} u_t\,ds = k^{-1}\int_{t_{j-1}}^{t_j} (E_h - I)u_t\,ds, \tag{36}$$

and, by (33) we have

$$\begin{aligned}k\sum_{j=1}^n \|w_3^j\| &\le \sum_{j=1}^n \int_{t_{j-1}}^{t_j} C[h^{s+1}(\|f_t - Q_0 f_t\| + \|u_{tt} - Q_0 u_{tt}\|) + h^{r+s}\|u_t\|_{r+1}]\,ds\\ &= C\left[h^{s+1}\int_0^{t_n} (\|f_t - Q_0 f_t\| + \|u_{tt} - Q_0 u_{tt}\|)\,ds + h^{r+s}\int_0^{t_n} \|u_t\|_{r+1}\,ds\right].\end{aligned}$$

Thus, together with the estimate of ρ in (30), we have the assertion. ∎

Theorem 4.6. *Under the assumption of Theorem 4.5, and the assumption that the coefficient matrix a is independent of time t, there exists a constant C such that*

$$\begin{aligned}\|\nabla_d(U^n - Q_h u(t_n))\|^2 \le 2\|\nabla_d(U^0 - Q_h u(0))\|^2 + Ch^{2r}(\|\psi\|_{r+1}^2 + \|u\|_{r+1}^2) \\ + Ch^{2(s+1)}\int_0^{t_n}(\|f_t - Q_0 f_t\| + \|u_{tt} - Q_0 u_{tt}\|)^2\,ds \\ + Ch^{2(r+s)}\int_0^{t_n}\|u_t\|_{r+1}^2\,ds + Ck^2\int_0^{t_n}\|u_{tt}\|^2\,ds.\end{aligned}$$

Proof. It is sufficient to estimate $\nabla_d\theta^n$. To this end, we choose $\chi = \bar{\partial}\theta^n$ in (35), and it is easily seen that

$$(\bar{\partial}\theta^n, \bar{\partial}\theta^n) + a(\theta^n, \bar{\partial}\theta^n) = \|\bar{\partial}\theta^n\|^2 + \frac{1}{2}\bar{\partial}a(\theta^n, \theta^n) + \frac{k}{2}a(\bar{\partial}\theta^n, \bar{\partial}\theta^n) = (w^n, \bar{\partial}\theta^n),$$

so that

$$\bar{\partial}a(\theta^n, \theta^n) \le \|w^n\|^2.$$

By repeating the application, we have

$$\begin{aligned}a(\theta^n, \theta^n) &\le a(\theta^0, \theta^0) + k\sum_{j=0}^{n}\|w^j\|^2 \\ &\le a(\theta^0, \theta^0) + 2k\sum_{j=0}^{n}\|w_1^j\|^2 + 2k\sum_{j=0}^{n}\|w_3^j\|^2.\end{aligned}$$

As in (36), we obtain

$$k\|w_3^j\|^2 = k\int_\Omega\left(k^{-1}\int_{t_{j-1}}^{t_j}\rho_t\,ds\right)^2 dx \le \int_\Omega\left(\int_{t_{j-1}}^{t_j}\rho_t^2\,ds\right)dx \le \int_{t_{j-1}}^{t_j}\|\rho_t\|^2\,ds.$$

Together with (23), (33) and (34), we have the assertion. ∎

V. NUMERICAL EXPERIMENTS

In section 2, we mentioned that the discrete weak space $S_h(j,l)$ and $\sum_h$ in the WG method need to satisfy two conditions. In Ref. [7], the authors proposed several possible combinations of $S_h(j,l)$ and $\sum_h$. Through out this section, we use a uniform triangular mesh $\mathcal{T}_h$, the discrete weak space $S_h(0,0)$, that is, space consisting of piecewise constants on the triangles and edges, respectively, and $\sum_h$ with $V(T,1)$ to be the lowest order Raviart-Thomas element $RT_0(T)$ in the WG method, which were used in Ref. [14] for the numerical studies of the WG method for second-order elliptic problems. We also adopt the various norms used in Ref. [14] to present the numerical results of the error e_h between the L^2 projection $Q_h u$ of the exact solution and the numerical solution u_h.

TABLE I. Convergence rate for heat equation with inhomogeneous Dirichlet boundary condition with $k = h$.

h	$\|e_h\|_{\{\infty,T\}}$	$\|e_h\|_{\{\infty,\partial T\}}$	$\|\nabla_d e_h\|$	$\|e_h\|_{\{L^2,T\}}$	$\|e_h\|_{\{L^2,\partial T\}}$
1/8	1.38e-01	1.44e-01	2.41e-01	4.90e-02	8.78e-02
1/16	6.97e-02	7.26e-02	8.34e-02	2.20e-02	4.03e-02
1/32	3.47e-02	3.56e-02	2.97e-02	1.05e-02	1.94e-02
1/64	1.72e-02	1.75e-02	1.10e-02	5.16e-03	9.54e-03
1/128	8.59e-03	8.65e-03	4.27e-03	2.56e-03	4.74e-03
$O(h^r)\ r =$	1.0012	1.0138	1.4550	1.0643	1.0533

Example 5.1. *As the first example, we consider the following heat equation in $\Omega = (0,1) \times (0,1)$,*

$$\begin{aligned} u_t - \nabla \cdot (a\nabla u) &= f \quad \text{in } \Omega, \quad \text{for } t > 0, \\ u &= g \quad \text{on } \partial\Omega, \quad \text{for } t > 0, \\ u(\cdot, 0) &= \psi \quad \text{in } \Omega, \end{aligned} \tag{37}$$

where $a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ *and f, g, and ψ are determined by setting the exact solution* $u = \sin(2\pi(t^2+1) + \pi/2)\sin(2\pi x + \pi/2)\sin(2\pi y + \pi/2)$.

For this inhomogeneous Dirichlet boundary condition, with a uniform triangular mesh $\mathcal{T}_h$, we chose the approximation space

$$S_h = \left\{ v = \{v_0, v_b\} \ : \ \begin{array}{r} v_0 \in P_0(T), \quad \text{for all } T \in \mathcal{T}_h, \\ v_b \in P_0(e) \text{ for all } T \in \mathcal{T}_h \text{ and } e \subset \partial T \notin \partial\Omega, \\ v_b = g_h \quad \text{for all } T \in \mathcal{T}_h \text{ and } \partial T \in \partial\Omega, \end{array} \right\},$$

where g_h is the L^2 projection of g in the piecewise constant finite element space on the boundary $\partial\Omega$. In the test, $k = h$ and $k = h^2$ are used to check the order of convergency with respect to time step size k and mesh size h, respectively, because the convergence rate of the error is dominated by that of the time step size k when $k = h$, and the convergence rate of the error is dominated by that of the mesh size h when $k = h^2$. The results are shown in Tables I and II. Because the exact solution is smooth, we observe the optimal convergence rates in both L^2 and discrete H^1 norms for the Dirichlet boundary data-type initial boundary value problem, which is consistent with the theoretical results shown in Sections 4 and 5.

TABLE II. Convergence rate for heat equation with inhomogeneous Dirichlet boundary condition with $k = h^2$.

h	$\|e_h\|_{\{\infty,T\}}$	$\|e_h\|_{\{\infty,\partial T\}}$	$\|\nabla_d e_h\|$	$\|e_h\|_{\{L^2,T\}}$	$\|e_h\|_{\{L^2,\partial T\}}$
1/8	7.11e-02	8.30e-02	7.81e-02	3.28e-02	5.39e-02
1/16	1.89e-02	2.28e-02	1.84e-02	8.53e-03	1.38e-02
1/32	4.79e-03	5.84e-03	4.52e-03	2.16e-03	3.49e-03
1/64	1.21e-03	1.48e-03	1.13e-03	5.43e-04	8.79e-04
1/128	3.64e-04	4.31e-04	2.88e-04	1.49e-04	2.47e-04
$O(h^r)\ r =$	1.9025	1.8994	2.0213	1.9454	1.9418

TABLE III. Convergence rate for heat equation with Robin boundary condition with $k = h$.

h	$\|e_h\|_{\{\infty,T\}}$	$\|e_h\|_{\{\infty,\partial T\}}$	$\|\nabla_d e_h\|$	$\|e_h\|_{\{L^2,T\}}$	$\|e_h\|_{\{L^2,\partial T\}}$
1/8	1.65e-01	1.72e-01	1.80e-01	9.86e-02	1.83e-01
1/16	1.00e-01	1.02e-01	8.55e-02	5.86e-02	1.09e-01
1/32	5.54e-02	5.59e-02	4.01e-02	3.22e-02	5.95e-02
1/64	2.92e-02	2.93e-02	1.91e-02	1.69e-02	3.13e-02
1/128	1.50e-02	1.50e-02	9.24e-03	8.67e-03	1.60e-02
$O(h^r)\ r =$	0.8656	0.8793	1.0713	0.8767	0.8789

TABLE IV. Convergence rate for heat equation with Robin boundary condition with $k = h^2$.

h	$\|e_h\|_{\{\infty,T\}}$	$\|e_h\|_{\{\infty,\partial T\}}$	$\|\nabla_d e_h\|$	$\|e_h\|_{\{L^2,T\}}$	$\|e_h\|_{\{L^2,\partial T\}}$
1/8	3.18e-02	3.90e-02	2.61e-02	1.88e-02	3.57e-02
1/16	8.29e-03	1.01e-03	6.28e-03	4.87e-03	9.22e-03
1/32	2.10e-03	2.54e-03	1.55e-03	1.23e-03	2.32e-03
1/64	5.24e-04	6.34e-04	3.88e-04	3.08e-04	5.83e-04
1/128	1.39e-04	1.62e-04	1.06e-04	8.79e-05	1.65e-04
$O(h^r)\ r =$	1.9591	1.9785	1.9875	1.9352	1.9383

Next, we consider this heat problem with a mixed boundary condition

$$\begin{cases} u = g & \text{on } \partial\Omega_1, \\ a\nabla u \cdot n + u = 0, & \text{on } \partial\Omega_2, \end{cases}$$

where $\partial\Omega_1 \cap \partial\Omega_2 = \emptyset$, and $\partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega$. The exact solution is set to be $u = \sin(2\pi(t^2+1)+\pi/2)\sin(\pi y)e^{-x}$, where $\partial\Omega_2$ is the boundary segment $x = 1$ and $\partial\Omega_1$ is the union of all other boundary segments. For the mixed boundary data type of initial boundary value problem, we also achieved the optimal convergence rates of the error in all norms as shown in Tables III and IV.

Example 5.2. *For the second example, we consider the parabolic problem (37) of full tensor with Dirichlet boundary condition and coefficient matrix $a = \begin{bmatrix} x^2+y^2+1 & xy \\ xy & x^2+y^2+1 \end{bmatrix}$, which is symmetric and positive definite, and f, g, and ψ are determined by setting the exact solution $u = \sin(2\pi(t^2+1)+\pi/2)\sin(2\pi x+\pi/2)\sin(2\pi y+\pi/2)$. The results are shown in Tables V and VI, which confirm the theoretical rates of convergence in L^2.*

TABLE V. Convergence rate for parabolic problem with inhomogeneous Dirichlet boundary condition with $k = h$.

h	$\|e_h\|_{\{\infty,T\}}$	$\|e_h\|_{\{\infty,\partial T\}}$	$\|\nabla_d e_h\|$	$\|e_h\|_{\{L^2,T\}}$	$\|e_h\|_{\{L^2,\partial T\}}$
1/8	1.21E-01	1.38E-01	3.57E-01	4.64E-02	8.14E-02
1/16	5.64E-02	6.10E-02	1.23E-01	1.87E-02	3.39E-02
1/32	2.67E-02	2.81E-02	4.34E-02	8.35E-03	1.54E-02
1/64	1.29E-02	1.33E-02	1.55E-02	3.95E-03	7.29E-03
1/128	6.33E-03	6.42E-03	5.61E-03	1.92E-03	3.55E-03
$O(h^r)\ r =$	1.0654	1.1076	1.4978	1.1487	1.1301

TABLE VI. Convergence rate for parabolic problem with inhomogeneous Dirichlet boundary condition with $k = h^2$.

h	$\|e_h\|_{\{\infty,T\}}$	$\|e_h\|_{\{\infty,\partial T\}}$	$\|\nabla_d e_h\|$	$\|e_h\|_{\{L^2,T\}}$	$\|e_h\|_{\{L^2,\partial T\}}$
1/8	7.41E-02	9.68E-02	1.24E-01	3.53E-02	5.74E-02
1/16	1.92E-02	2.56E-02	3.00E-02	9.14E-03	1.47E-02
1/32	4.83E-03	6.44E-03	7.45E-03	2.30E-03	3.70E-03
1/64	1.21E-03	1.62E-03	1.86E-03	5.78E-04	9.28E-04
1/128	3.35E-04	4.34E-04	4.67E-04	1.53E-04	2.48E-04
$O(h^r)\ r =$	1.9474	1.9500	2.0118	1.9637	1.9636

For the discrete H^1 norm, the numerical convergence is of order $\mathcal{O}(h^2)$ which is one order higher than the theoretical prediction. We believe that this suggests a superconvergence between the WG finite element approximation and the L^2 projection of the exact solution. Interested readers are encouraged to conduct a study on the superconvergence phenomena.

References

1. V. Thomée, Galerkin finite element methods for parabolic problems, Springer Series in Computational mathematics, Springer-Verlag, New York, Inc., Secaucus, NJ, 2006.
2. K. Eriksson and C. Johnson, Adaptive finite element methods for parabolic problems IV: nonlinear problems, SIAM J Numer Anal 32 (1995), 1729–1749.
3. S. Larsson, V. Thomée, and L. B. Wahlbin, Numerical solution of parabolic integrodifferential equations by the discontinuous Galerkin method, Math Comp 67 (1998), 45–71.
4. X. Makridakii and I. Babuška, On the stability of the discontinuous Galerkin method for the heat equation, SIAM J Num Anal 34 (1997), 389–401.
5. P. Chatzipantelidis, R. D. Lazarov, and V. Thomée, Error estimates for a finite volume element method for parabolic equations in convex polygonal domains, Numer Methods Partial Differential Equations 20 (2004), 650–674.
6. M. Yang and J. Liu, A quadratic finite volume element method for parabolic problems on quadrilateral meshes, IMA J Numer Anal 31 (2011), 1038–1061.
7. J. Wang and X. Ye, A weak Galerkin finite element method for elliptic problems, J Comput Appl Math 241 (2013), 103–115.
8. P. Raviart and J. Thomas, A mixed finite element method for second order elliptic problems, Mathematical Aspects of the Finite Element Method, I. Galligani and E. Magenes, editors, Lecture Notes in Math 606, Springer-Verlag, New York, 1977.
9. J. Wang and X. Ye, A weak Galerkin mixed finite element method for second-order elliptic problems, arXiv:1202.3655v2, 2012.
10. L. Mu, J. Wang, and X. Ye, Weak Galerkin finite element methods on polytopal meshes, arXiv:1204.3655v2, 2012.
11. J. Wang, Y. Wang, and X. Ye, Unified a posteriori error estimator for finite element methods for the Stokes equations, Int J Numer Anal Modeling, to appear.
12. L. Mu, J. Wang, and X. Ye, A new Galerkin finite element method for the Stokes equations, Preprint and private communication.
13. M. F. Wheeler, A priori L2 error estimates for Galerkin approximations to parabolic partial differential equations, SIAM J Numer Anal 10 (1973), 723–759.
14. L. Mu, J. Wang, Y. Wang, and X. Ye, A computational study of the weak Galerkin method for second-order elliptic equations, Numer Algor (2012), DOI: 10.1007/s11075-012-9651-1.