Qualifying Exam Review Questions

May 31, 2016

1.) Given that the traction vector $\mathbf{t}(\mathbf{x}, \mathbf{n})$ characterizes the distribution of force at position $\mathbf{x}$ acting upon an oriented surface with normal $\mathbf{n}$, prove that the traction depends *linearly* upon $\mathbf{n}$, giving rise to the Cauchy stress tensor. <u>Hint</u>: You will need to make use of Newton's third law: $\mathbf{t}(\mathbf{x}, -\mathbf{e}_i) = -\mathbf{t}(\mathbf{x}, \mathbf{e}_i)$.

2.) Prove that the Cauchy stress tensor is symmetric.

3.) Suppose that the Earth (of radius R) consists of a highly viscous, nearly-incompressible fluid of uniform density ρ, such that the stress state in the fluid is purely hydrostatic. Making use of Newton's universal law of gravitation ($F = G\frac{m_1 m_2}{r^2}$) write out an expression for the body force per unit mass $\mathbf{b}(r)$ as a function of the radial distance r from the center of the Earth. Then, utilize this expression to obtain an estimate for the pressure at the center of the Earth.

4.) Consider a ball of negligible radius which is attached to a rigid circular ring in 2-dimensions, such that the ring is described by $(x-a)^2 + (y-b)^2 = r^2$. The ball is permitted to slide freely on the ring. Suppose that an arbitrary body force $\mathbf{b}(\mathbf{x})$ (which varies with spatial position $\mathbf{x}$) acts upon the ball. Further, suppose that $\mathbf{b}(\mathbf{x})$ derives from a scalar potential field $f(\mathbf{x})$ such that $\mathbf{b}(\mathbf{x}) = \nabla f(\mathbf{x})$. Write out the system of equations necessary to solve for the stationary (equilibrium) position of the ball on the ring.

5.) Explain (in words) the meaning of "weak" as it relates to the *weak form* statement of elastostatics.

6.) Consider a bi-material bar with uniform cross-sectional area A and length L which is aligned with the $x-$axis. For $0 < x < L/2$, the bar consists of an elastic material with Young's modulus E_1. For $L/2 < x < L$, the bar consists of a *different* elastic material with Young's modulus E_2. The bar is fixed at its left end such that the axial displacement $u(x=0) = 0$, and loaded at its right end (at $x = L$) by an axial force P. Using the theory of minimum potential energy, derive the strong form statement of equilibrium for the bi-material bar. This should include: the point-wise statement of equilibrium, the boundary conditions for the bar, and the *interface* conditions at $x = L/2$.

7.) Write out the expressions for each of the following deformation measures in terms of the deformation gradient $\mathbf{F}$. Also, provide a physical interpretation of the following deformation measures:

- $\mathbf{E}$: The Lagrangian strain tensor
- $\mathbf{C}$: The Right (Green's) deformation tensor
- $\mathbf{B}$: The Left (Cauchy-Green) deformation tensor
- $\mathbf{A}$: The Almansi (Eulerian) strain tensor

8.) Demonstrate that the Lagrangian strain tensor $\mathbf{E}$ and the Right (Green's) deformation tensor $\mathbf{C}$ possess the same eigenvectors. Further, write out an expression which relates the eigenvalues of $\mathbf{E}$ and $\mathbf{C}$.

9.) Prove Namson's relation for area transformations: $\mathbf{n}\,da = J\mathbf{F}^{-T}\mathbf{N}\,dA$.

10.) Establish the relationship between the Cauchy stress tensor $\boldsymbol{\sigma}$ and the first Piola-Kirchhoff stress tensor $\mathbf{P}$.

11.) Demonstrate the following work conjugacies:

$$\frac{\boldsymbol{\sigma} : \mathbf{D}}{\rho} = \frac{\mathbf{P} : \dot{\mathbf{F}}}{\rho_0} = \frac{\mathbf{S} : \dot{\mathbf{E}}}{\rho_0}$$

12.) Derive the transport theorem for an advected scalar quantity $f(\mathbf{x})$:

$$\frac{d}{dt}\int_{\Omega(t)} f(\mathbf{x})\,dv = \int_{\Omega(t)} \frac{\partial}{\partial t} f(\mathbf{x})\,dv + \int_{\partial\Omega(t)} f(\mathbf{x})(\mathbf{v}\cdot\mathbf{n})\,da$$

13.) Suppose that we have a constitutive law described by $\boldsymbol{\sigma} = f(\mathbf{D})$. Under a superposed rigid-body rotation $\mathbf{Q}$ we should have $\hat{\boldsymbol{\sigma}} = f(\hat{\mathbf{D}})$, where it is *given* that $\hat{\mathbf{D}} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T$, and we further *require* that $\hat{\boldsymbol{\sigma}} = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T$. What conditions must be placed upon the constitutive relation $f(\mathbf{D})$ such that the aforementioned requirement is satisfied?

14.) Derive the path-integral expression for the J-integral in 2D as the rate of dissipation in a material region R surrounding a crack tip as it extends in a self-similar manner.

15.) From the path-integral expression for the J-integral, derive an *area*-integral expression for the J-integral.

16.) Explain where the major and minor symmetries of the elastic modulus tensor C_{ijkl} come from.

17.) Drucker's postulate claims that for any closed path in *stress-space*, the following relation holds: $\oint(\sigma_{ij} - \sigma_{ij}^0)d\varepsilon_{ij} \geq 0 \ \forall\sigma_{ij}^0$. What two main assumtions are necessary for Drucker's postulate to hold true?

18.) If $\mathbf{x}^T\mathbf{A}\mathbf{x} = 0 \ \forall\mathbf{x}$, what conclusions can we make about the specific form of $\mathbf{A}$?

19.) Write out the spectral respresentation of a symmetric tensor $\mathbf{S}$. Then, consider the action of $\mathbf{S}$ upon some arbitrary vector $\mathbf{v}$ (write out an expression for the resulting vector): what is the geometric interpretation of this expression?

20.) The (Hu-Washizu) three-field mixed-variational principle consists of the following potential energy functional $\Pi\left[\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}\right]$:

$$\Pi\left[u_i, \sigma_{ij}, \varepsilon_{ij}\right] = \int_\Omega \left[\sigma_{ij}\left(\frac{1}{2}(u_{i,j} + u_{j,i}) - \varepsilon_{ij}\right) + W(\varepsilon_{ij})\right] dv - \int_\Omega b_i u_i \, dv - \int_{\partial\Omega} \bar{t}_i u_i \, da$$

wherein the displacement, stress, and strain fields are represented independently of one another. Within this setting:

(a.) What are the corresponding weak/variational equations that arise from the principle of stationary potential energy, and what are the requirements on the function spaces which contain each of the three fields ($\mathbf{u}$, $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$)?

(b.) What are the corresponding strong form (Euler-Lagrange) equations?

21.) What does it mean for a function to be contained within the $H^1(\Omega)$ space of functions? Can you think of a function which is contained in $C^0(\Omega)$ which is not contained within $H^1(\Omega)$? Conversely, are there any functions which are contained in $H^1(\Omega)$ that are not contained within $C^0(\Omega)$?