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A modified weak Galerkin finite element method

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ABSTRACT

In this paper we introduce a new discrete weak gradient operator and a new weak Galerkin (WG) finite element method for second order Poisson equations based on this new operator. This newly defined discrete weak gradient operator allows us to use a single stabilizer which is similar to the one used in the discontinuous Galerkin (DG) methods without having to worry about choosing a sufficiently large parameter. In addition, we will establish the optimal convergence rates and validate the results with numerical examples.

1. Introduction

In this paper we describe a finite element method that is a combination of processes used in the weak Galerkin (WG) [1] and discontinuous Galerkin (DG) [2] methods for second order Poisson equations. First, we introduce a new discrete weak gradient operator. This newly defined discrete weak gradient operator allows us to draw from the strengths of both WG and DG. The bilinear form defined here, $a(u, v)$, contains only one stabilizer term. In contrast to DG, once this method is implemented there is no need to experiment when applying it to different problems in order to find an α large enough to obtain convergence. On the other hand, the freedom to introduce a positive number as a parameter makes it possible to further optimize this method. The convergence rates of the present method are optimal.

We consider as our model problem the boundary value problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subseteq \mathbb{R}^d$ $(d = 2, 3)$ is a polygonal region. We seek an approximate solution to this problem by applying a finite element method. Since the main ideas are the same, we focus on the case $d = 2$. We begin with some notational definitions.

Let $\mathscr{T}_h$ be a shape regular triangulation of Ω with elements T and edges e. Let h_T be the diameter of T and $h = \max_{T\in\mathscr{T}_h} h_T$. Given two elements T_1 and T_2, with a common edge e, we denote by $\{v\}$ the function defined on e that is the average $(v_o|_{T_1} + v_o|_{T_2})/2$. We denote by $[\![v]\!]$ the function on e that is the *jump* experienced by v across the edge e: $[\![v]\!] = v_o|_{T_1}\mathbf{n}_1 + v_o|_{T_2}\mathbf{n}_2$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the outward unit normal vectors on edge e for elements T_1 and T_2, respectively.

Our weak formulation will use the following vector spaces of functions on Ω:

$$\begin{aligned} V_h &= \{v : v|_T \in P_k(T) \text{ for } T \in \mathscr{T}_h\}, \\ V_h^0 &= \{v \in V_h : v|_e = 0 \text{ for } e \in \partial\Omega\}, \quad \text{and} \\ W_k &= \{\mathbf{q} : \mathbf{q}|_T \in [P_k(T)]^2 \text{ for } T \in \mathscr{T}_h\}. \end{aligned}$$

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The notation $e \in \partial\Omega$ means that e is an edge in the boundary of Ω.

Define $(v, w)_D = \int_D vw\, dA$ and $(v, w)_{\partial D} = \int_{\partial D} vw\, ds$.

The projection operators Q_o and Q_b, defined piecewise on the interior and boundary of each of the elements of $\mathcal{T}_h$, are the L^2-projection onto $P_k(T)$, and L^2-projection onto $P_k(e)$, respectively. Let R_h be the projection operator onto $[P_{k-1}(T)]^2$ whose components are each the L^2-projection onto $P_{k-1}(T)$.

We now define the new discrete gradient used in the modified WG method.

Definition 1.1. Given a partition $\mathcal{T}_h$ of Ω and a piecewise smooth function u on Ω, for all $T \in \mathcal{T}_h$, the *discrete gradient* of u on T is the unique element $\nabla_w u|_T$ in $[P_{k-1}(T)]^2$ such that

$$(\nabla_w u, \mathbf{q})_T = -(u, \nabla \cdot \mathbf{q})_T + (\{u\}, \mathbf{q} \cdot \mathbf{n})_{\partial T} \quad \text{for all } \mathbf{q} \in [P_{k-1}(T)]^2,$$

where $\mathbf{n}$ denotes the outward facing unit normal perpendicular to an $e \in \partial T$.

Note that, when u is continuous on Ω, $\{u\} = u$ on ∂T. Thus

$$(\nabla_w u, \mathbf{q})_T = -(u, \nabla \cdot \mathbf{q})_T + (u, \mathbf{q} \cdot \mathbf{n})_{\partial T} = (\nabla u, \mathbf{q})_T \tag{2}$$

for all $\mathbf{q} \in [P_{k-1}(T)]^2$. Note also that the discrete gradient defined here is different from the one defined in [1].

The weak formulation for our boundary value problem can now be given. Find $u_h \in V_h$ such that $u_h|_e = Q_b g$ for $e \in \partial\Omega$, and

$$a(u_h, v) = \sum_T (\nabla_w u_h, \nabla_w v)_T + \sum_e h^{-1}([\![u_h]\!], [\![v]\!])_e = (f, v) \quad \text{for all } v \in V_h^0. \tag{3}$$

Theorem 1.2. *There is a unique $u_h \in V_h^0$ satisfying $a(u_h, v) = (f, v)$ for all $v \in V_h^0$.*

Proof. Let $f = g = 0$. Since V_h^0 is finite dimensional, it suffices to show that $a(u_h, v) = 0$ for all $v \in V_h^0$ implies that $u_h = 0$ on Ω. Under this assumption, let $v = u_h$. Then

$$a(u_h, u_h) = \sum_T (\nabla_w u_h, \nabla_w u_h)_T + \sum_e h^{-1}([\![u_h]\!], [\![u_h]\!])_e = 0.$$

Since all the terms in this sum are nonnegative, they must be zero. Since $([\![u_h]\!], [\![u_h]\!])_e = 0$, and hence $[\![u_h]\!] = 0$, on each edge e, u_h is continuous on Ω. It follows from $(\nabla_w u_h, \nabla_w u_h)_T = 0$ on each $T \in \mathcal{T}_h$ that $\nabla u_h = \nabla_w u_h = 0$ on each T. Therefore u_h is constant on each T. Since $[\![u_h]\!] = 0$, u_h has the same constant value on each T. Now, since $g = 0$ on $\partial\Omega$, $u_h = 0$ on any boundary element T. Therefore, $u_h = 0$ on Ω. $\square$

The $|\!|\!| v |\!|\!|$ is defined by $|\!|\!| v |\!|\!|^2 = a(v, v)$. The positivity property of $|\!|\!| \cdot |\!|\!|$ follows from the above theorem. That $|\!|\!| v |\!|\!|$ has the additional properties of a norm is easily verified.

2. Properties of the discrete gradient

In this section, we establish properties of the discrete gradient and the interaction with the projection operators that will be used in the error analysis.

Lemma 2.1. *For all $u \in H^1(\Omega)$, $\nabla_w u = R_h \nabla u$.*

Proof. We establish the result on each element $T \in \mathcal{T}_h$. Let $\mathbf{q} \in [P_{k-1}(T)]^2$. It follows from (2) that $(\nabla_w u, \mathbf{q})_T = (\nabla u, \mathbf{q})_T = (R_h \nabla u, \mathbf{q})_T$. Therefore $(\nabla_w u - R_h \nabla u, \mathbf{q})_T = 0$ for all $\mathbf{q} \in [P_{k-1}(T)]^2$. In particular, letting $\mathbf{q} = \nabla_w u - R_h \nabla u \in [P_{k-1}(T)]^2$, yields $(\nabla_w u - R_h \nabla u, \nabla_w u - R_h \nabla u)_T = 0$. Therefore, $\nabla_w u = R_h \nabla u$ on T for all $T \in \mathcal{T}_h$. $\square$

Lemma 2.2. *For all $u \in H^{k+1}(\Omega)$,*

$$\left(\sum_T \|\nabla_w u - \nabla Q_o u\|_T^2 \right)^{\frac{1}{2}} = \left(\sum_T \|R_h \nabla u - \nabla Q_o u\|_T^2 \right)^{\frac{1}{2}} \le ch^k |u|_{k+1}.$$

Proof. The first equality follows from Lemma 2.1.

$$\begin{aligned} \left(\sum_T \|R_h \nabla u - \nabla Q_o u\|_T^2 \right)^{\frac{1}{2}} &\le \left(\sum_T \|R_h \nabla u - \nabla u\|_T^2 \right)^{\frac{1}{2}} + \left(\sum_T \|\nabla u - \nabla Q_o u\|_T^2 \right)^{\frac{1}{2}} \\ &\le ch^k |u|_{k+1}. \quad \square \end{aligned}$$

Lemma 2.3. *For all $u \in H^{k+1}(\Omega)$,*

$$\left(\sum_e \|Q_o u - u\|_e^2 \right)^{\frac{1}{2}} \le ch^{k+\frac{1}{2}} |u|_{k+1}.$$

Proof. It follows from the trace inequality that

$$\left(\sum_e \|Q_o u - u\|_e^2\right)^{\frac{1}{2}} \le c\left(\sum_T (h^{-1}\|Q_o u - u\|_T^2 + h\|\nabla(Q_o u - u)\|_T^2)\right)^{\frac{1}{2}} \le ch^{k+\frac{1}{2}}|u|_{k+1}. \quad \square$$

Lemma 2.4. *For all* $\mathbf{w} \in W_{k-1}$ *and for all* v *such that* $v|_T \in H^1(T)$,

$$\sum_T (\nabla v, \mathbf{w})_T = \sum_T (\nabla_w v, \mathbf{w})_T + \sum_e ([\![v]\!], \{\mathbf{w}\})_e.$$

Proof.

$$\sum_T (\nabla_w v, \mathbf{w})_T = -\sum_T (v, \nabla\cdot\mathbf{w})_T + \sum_T (\{v\}, \mathbf{w}\cdot\mathbf{n})_{\partial T}, \quad \text{and}$$
$$\sum_T (\nabla v, \mathbf{w})_T = -\sum_T (v, \nabla\cdot\mathbf{w})_T + \sum_T (v, \mathbf{w}\cdot\mathbf{n})_{\partial T}.$$

Subtracting the first of these equations from the second yields

$$\sum_T (\nabla v - \nabla_w v, \mathbf{w})_T = \sum_e ([\![v]\!], \{\mathbf{w}\})_e. \quad \square$$

Lemma 2.5. *If* u *is the solution to* (1), *then*

$$\sum_T (\nabla_w(u-u_h), \nabla_w v)_T + \sum_e ([\![v]\!], \{R_h\nabla u - \nabla u\} - h^{-1}[\![u_h]\!])_e = 0,$$

for all $v \in V_h^0$.

Proof. Test Eq. (1) with $v \in V_h^0$:

$$-(\Delta u, v) = (f, v).$$

Therefore,

$$\begin{aligned}(f, v) &= \sum_T [-(\Delta u, v)_T] \\ &= \sum_T (\nabla u, \nabla v)_T - \sum_T (\nabla u\cdot\mathbf{n}, v)_{\partial T} \\ &= \sum_T (R_h\nabla u, \nabla v)_T - \sum_T (\nabla u\cdot\mathbf{n}, v)_{\partial T} \\ &= \sum_T (R_h\nabla u, \nabla_w v)_T + \sum_e ([\![v]\!], \{R_h\nabla u\})_e - \sum_T (\nabla u\cdot\mathbf{n}, v)_{\partial T} \quad \text{by Lemma 2.4.}\end{aligned}$$

Reindexing the last sum over the edges of the triangulation and using Lemma 2.1 yields

$$\sum_T (\nabla_w v, \nabla_w u)_T + \sum_e ([\![v]\!], \{R_h\nabla u - \nabla u\})_e = (f, v). \tag{4}$$

From the weak formulation (3), we have

$$\sum_T (\nabla_w u_h, \nabla_w v)_T + \sum_e (h^{-1}[\![u_h]\!], [\![v]\!])_e = (f, v). \tag{5}$$

Subtracting (5) from (4) completes the proof:

$$\sum_T (\nabla_w(u-u_h), \nabla_w v)_T + \sum_e ([\![v]\!], \{R_h\nabla u - \nabla u\} - h^{-1}[\![u_h]\!])_e = 0. \quad \square$$

3. Error analysis

Now, let $\mathscr{E} = u - u_h$.

Theorem 3.1. *There is a constant* $c \in \mathbb{R}$ *such that* $|\!|\!|\mathscr{E}|\!|\!| \le ch^k|u|_{k+1}$.

Proof. We begin with

$$\begin{aligned}
|||\mathscr{E}|||^2 &= a(\mathscr{E}, \mathscr{E}) \\
&= \sum_T (\nabla_w \mathscr{E}, \nabla_w \mathscr{E})_T + \sum_e ([\![\mathscr{E}]\!], h^{-1}[\![\mathscr{E}]\!])_e.
\end{aligned}$$

Letting $\tilde{\mathscr{E}} = Q_o u - u_h$,

$$\begin{aligned}
\sum_T (\nabla_w \mathscr{E}, \nabla_w \mathscr{E})_T &= \sum_T (R_h \nabla u - \nabla_w u_h, \nabla_w \mathscr{E})_T \quad \text{(by Lemma 2.1)} \\
&= \sum_T (\nabla_w Q_o u - \nabla_w u_h, \nabla_w \mathscr{E})_T + \sum_T (R_h \nabla u - \nabla_w Q_o u, \nabla_w \mathscr{E})_T \\
&= \sum_T (\nabla_w \tilde{\mathscr{E}}, \nabla_w \mathscr{E})_T + \sum_T (R_h \nabla u - \nabla_w Q_o u, \nabla_w \mathscr{E})_T,
\end{aligned}$$

and letting $v = \tilde{\mathscr{E}}$ in Lemma 2.5, we get

$$\sum_T (\nabla_w \tilde{\mathscr{E}}, \nabla_w \mathscr{E})_T = -\sum_e ([\![\tilde{\mathscr{E}}]\!], \{R_h \nabla u - \nabla u\} - h^{-1}[\![u_h]\!])_e.$$

Our estimation for $|||\mathscr{E}|||$ follows:

$$\begin{aligned}
|||\mathscr{E}|||^2 &= \sum_e ([\![\mathscr{E}]\!], h^{-1}[\![\mathscr{E}]\!])_e - \sum_e ([\![\tilde{\mathscr{E}}]\!], \{R_h \nabla u - \nabla u\} - h^{-1}[\![u_h]\!])_e + \sum_T (R_h \nabla u - \nabla_w Q_o u, \nabla_w \mathscr{E})_T \\
&= \sum_e ([\![\mathscr{E}]\!], -\{R_h \nabla u - \nabla u\} + h^{-1}([\![\mathscr{E}]\!] + [\![u_h]\!]))_e \\
&\quad + \sum_e ([\![\mathscr{E} - \tilde{\mathscr{E}}]\!], \{R_h \nabla u - \nabla u\} - h^{-1}[\![u_h]\!])_e + \sum_T (R_h \nabla u - \nabla_w Q_o u, \nabla_w \mathscr{E})_T \\
&= \sum_e ([\![\mathscr{E}]\!], -\{R_h \nabla u - \nabla u\} + h^{-1}[\![u]\!])_e + \sum_e ([\![u - Q_o u]\!], \{R_h \nabla u - \nabla u\} + h^{-1}[\![u - u_h]\!])_e \\
&\quad + \sum_T (R_h \nabla u - \nabla_w Q_o u, \nabla_w \mathscr{E})_T \qquad (6) \\
&= \sum_e ([\![\mathscr{E}]\!], -\{R_h \nabla u - \nabla u\} + h^{-1}[\![u - Q_o u]\!])_e + \sum_e ([\![u - Q_o u]\!], \{R_h \nabla u - \nabla u\})_e \\
&\quad + \sum_T (R_h \nabla u - \nabla_w Q_o u, \nabla_w \mathscr{E})_T \qquad (7) \\
&\le \left(\sum_e \|[\![\mathscr{E}]\!]\|_e^2\right)^{\frac{1}{2}} \left(\sum_e \|\{R_h \nabla u - \nabla u\}\|_e^2\right)^{\frac{1}{2}} + h^{-1} \left(\sum_e \|[\![\mathscr{E}]\!]\|_e^2\right)^{\frac{1}{2}} \left(\sum_e \|[\![u - Q_o u]\!]\|_e^2\right)^{\frac{1}{2}} \\
&\quad + \left(\sum_e \|[\![u - Q_o u]\!]\|_e^2\right)^{\frac{1}{2}} \left(\sum_e \|\{R_h \nabla u - \nabla u\}\|_e^2\right)^{\frac{1}{2}} + \left(\sum_T \|R_h \nabla u - \nabla_w Q_o u\|_T^2\right)^{\frac{1}{2}} \left(\sum_T \|\nabla_w \mathscr{E}\|_T^2\right)^{\frac{1}{2}}. \qquad (8)
\end{aligned}$$

Eq. (6) follows from the fact that $[\![u]\!] = [\![u - u_h]\!] + [\![u_h]\!] = [\![\mathscr{E}]\!] + [\![u_h]\!]$, and Eq. (7) follows from $[\![u]\!] = 0$.

We will deal with each of the four terms in (8) separately. For the first term we have

$$\begin{aligned}
\left(\sum_e \|[\![\mathscr{E}]\!]\|_e^2\right)^{\frac{1}{2}} \left(\sum_e \|\{R_h \nabla u - \nabla u\}\|_e^2\right)^{\frac{1}{2}} &= \left(\sum_e h^{-1} \|[\![\mathscr{E}]\!]\|_e^2\right)^{\frac{1}{2}} \left(\sum_e h \|\{R_h \nabla u - \nabla u\}\|_e^2\right)^{\frac{1}{2}} \\
&\le |||\mathscr{E}||| h^{\frac{1}{2}} c \left(\sum_T h^{-1} \|R_h \nabla u - \nabla u\|_T^2 + \sum_T h \|\nabla (R_h \nabla u - \nabla u)\|_T^2\right)^{\frac{1}{2}} \\
&\le |||\mathscr{E}||| c h^k |u|_{k+1}. \qquad (9)
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
h^{-1} \left(\sum_e \|[\![\mathscr{E}]\!]\|_e^2\right)^{\frac{1}{2}} \left(\sum_e \|[\![u - Q_o u]\!]\|_e^2\right)^{\frac{1}{2}} &= h^{-1} \left(\sum_e h^{-1} \|[\![\mathscr{E}]\!]\|_e^2\right)^{\frac{1}{2}} \left(\sum_e h \|[\![u - Q_o u]\!]\|_e^2\right)^{\frac{1}{2}} \\
&\le |||\mathscr{E}||| c h^k |u|_{k+1} \quad \text{(by Lemma 2.3)}.
\end{aligned}$$

For the third term,

$$\begin{aligned}
&\left(\sum_e \|[\![u - Q_o u]\!]\|_e^2\right)^{\frac{1}{2}} \left(\sum_e \|\{R_h \nabla u - \nabla u\}\|_e^2\right)^{\frac{1}{2}} \\
&\leq ch^{k+\frac{1}{2}}|u|_{k+1}\left(\sum_e \|\{R_h \nabla u - \nabla u\}\|_e^2\right)^{\frac{1}{2}} \quad \text{(by Lemma 2.3)} \\
&\leq ch^{k+\frac{1}{2}}|u|_{k+1}\left(\sum_T h^{-1}\|R_h \nabla u - \nabla u\|_T^2 + \sum_T h\|\nabla(R_h \nabla u - \nabla u)\|_T^2\right)^{\frac{1}{2}} \\
&\leq ch^{2k}|u|_{k+1}^2. \qquad (10)
\end{aligned}$$

For the fourth term,

$$\begin{aligned}
\left(\sum_T \|\nabla_w \mathcal{E}\|_T^2\right)^{\frac{1}{2}} \left(\sum_T \|R_h \nabla u - \nabla_w Q_o u\|_T^2\right)^{\frac{1}{2}} &= \left(\sum_T \|\nabla_w \mathcal{E}\|_T^2\right)^{\frac{1}{2}} \left(\sum_T \|R_h \nabla u - \nabla Q_o u\|_T^2\right)^{\frac{1}{2}} \\
&\leq |\!|\!|\mathcal{E}|\!|\!| ch^k |u|_{k+1} \quad \text{(by Lemma 2.2)}. \qquad (11)
\end{aligned}$$

Lines (9) and (10) result from applications of the trace inequality. Now combining our results for the four terms in (8) yields the estimate

$$|\!|\!|\mathcal{E}|\!|\!|^2 \leq |\!|\!|\mathcal{E}|\!|\!| ch^k|u|_{k+1} + ch^{2k}|u|_{k+1}^2.$$

Now, using the inequality $2ab \leq a^2 + b^2$, yields

$$|\!|\!|\mathcal{E}|\!|\!| \leq ch^k|u|_{k+1}. \quad \square$$

To obtain an L^2 estimate of the error $\mathcal{E} = u - u_h$ consider

$$-\Delta\phi = \mathcal{E} \quad \text{on } \Omega, \qquad (12)$$

where $\phi \in H_0^2(\Omega)$. Assume, for some $c > 0$,

$$\|\phi\|_2 \leq c\|\mathcal{E}\|.$$

Now, testing (12) with $\mathcal{E}$, we have

$$\begin{aligned}
\|\mathcal{E}\|^2 &= (-\Delta\phi, \mathcal{E}) \\
&= \sum_T (\nabla\phi, \nabla\mathcal{E})_T - \sum_T (\nabla\phi \cdot \mathbf{n}, \mathcal{E})_{\partial T} \\
&= \sum_T (\nabla\phi, \nabla\mathcal{E})_T - \sum_e (\nabla\phi, [\![\mathcal{E}]\!])_e \\
&= \sum_T (R_h\nabla\phi, \nabla\mathcal{E})_T - \sum_e (\nabla\phi, [\![\mathcal{E}]\!])_e + \sum_T (\nabla\phi - R_h\nabla\phi, \nabla\mathcal{E})_T \\
&= \sum_T (R_h\nabla\phi, \nabla_w\mathcal{E})_T + \sum_e ([\![\mathcal{E}]\!], \{R_h\nabla\phi\})_e - \sum_e (\nabla\phi, [\![\mathcal{E}]\!])_e + \sum_T (\nabla\phi - R_h\nabla\phi, \nabla\mathcal{E})_T \\
&= \sum_T (\nabla_w\phi, \nabla_w\mathcal{E})_T + \sum_e ([\![\mathcal{E}]\!], \{R_h\nabla\phi - \nabla\phi\})_e \\
&\quad + \sum_T (\nabla\phi - R_h\nabla\phi, R_h\nabla\mathcal{E})_T + \sum_T (\nabla\phi - R_h\nabla\phi, \nabla\mathcal{E} - R_h\nabla\mathcal{E})_T \\
&= \sum_T (\nabla_w Q_o\phi, \nabla_w\mathcal{E})_T + \sum_T (\nabla_w(\phi - Q_o\phi), \nabla_w\mathcal{E})_T \\
&\quad + \sum_e ([\![\mathcal{E}]\!], \{R_h\nabla\phi - \nabla\phi\})_e + \sum_T (\nabla\phi - R_h\nabla\phi, \nabla\mathcal{E} - R_h\nabla\mathcal{E})_T. \qquad (13)
\end{aligned}$$

We will bound each term in (13).

Applying Lemma 2.5 to the first term in (13),

$$\begin{aligned}
\left|\sum_T(\nabla_w Q_o\phi, \nabla_w(u-u_h))_T\right| &= \left|\sum_e([\![Q_o\phi-\phi]\!], \{R_h\nabla u-\nabla u\}+h^{-1}[\![u_h-u]\!])_e\right| \\
&\le \left(\sum_e \|[\![Q_o\phi-\phi]\!]\|_e^2\right)^{\frac{1}{2}}\left(\sum_e\left(\|\{R_h\nabla u-\nabla u\}\|_e^2+h^{-2}\|[\![u_h-u]\!]\|_e^2\right)\right)^{\frac{1}{2}} \\
&\le c\left(\sum_T\left(h^{-1}\|Q_o\phi-\phi\|_T^2+h\|\nabla(Q_o\phi-\phi)\|_T^2\right)\right)^{\frac{1}{2}} \\
&\quad\cdot\left[\left(\sum_T\left(h^{-1}\|R_h\nabla u-\nabla u\|_T^2+h\|\nabla(R_h\nabla u-\nabla u)\|_T^2\right)\right)^{\frac{1}{2}}+\left(h^{-1}|\!|\!|\mathscr{E}|\!|\!|^2\right)^{\frac{1}{2}}\right] \\
&\le ch^{3/2}\|\phi\|_2 h^{k-\frac{1}{2}}|u|_{k+1} \\
&\le ch^{k+1}|u|_{k+1}\|\mathscr{E}\|.
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
\left|\sum_T(\nabla_w(\phi-Q_o\phi), \nabla_w\mathscr{E})_T\right| &\le \left(\sum_T\|\nabla_w(\phi-Q_o\phi)\|_T^2\right)^{\frac{1}{2}}\left(\sum_T\|\nabla_w\mathscr{E}\|_T^2\right)^{\frac{1}{2}} \\
&\le ch\|\phi\|_2|\!|\!|\mathscr{E}|\!|\!| \\
&\le ch^{k+1}|u|_{k+1}\|\mathscr{E}\|.
\end{aligned}$$

For the third term, we have

$$\begin{aligned}
\left|\sum_e([\![\mathscr{E}]\!], \{R_h\nabla\phi-\nabla\phi\})_e\right| &\le \left(\sum_e h^{-1}\|[\![\mathscr{E}]\!]\|_e^2\right)^{\frac{1}{2}}\left(h\sum_e\|\{R_h\nabla\phi-\nabla\phi\}\|_e^2\right)^{\frac{1}{2}} \\
&\le |\!|\!|\mathscr{E}|\!|\!|\left(h\sum_T\left(h^{-1}\|R_h\nabla\phi-\nabla\phi\|_T^2+h\|\nabla(R_h\nabla\phi-\nabla\phi)\|_T^2\right)\right)^{\frac{1}{2}} \\
&\le |\!|\!|\mathscr{E}|\!|\!|ch\|\phi\|_2 \\
&\le ch^{k+1}|u|_{k+1}\|\mathscr{E}\|.
\end{aligned}$$

For the fourth term, we have

$$\begin{aligned}
\sum_T(\nabla\phi-R_h\nabla\phi, \nabla\mathscr{E}-R_h\nabla\mathscr{E})_T &\le \left(\sum_T\|\nabla\phi-R_h\nabla\phi\|_T^2\right)^{\frac{1}{2}}\left(\sum_T\|\nabla\mathscr{E}-R_h\nabla\mathscr{E}\|_T^2\right)^{\frac{1}{2}} \\
&\le ch\|\phi\|_2h^k|\mathscr{E}|_{k+1} \\
&\le ch\|\mathscr{E}\|h^k|u|_{k+1} \\
&\le ch^{k+1}|u|_{k+1}\|\mathscr{E}\|,
\end{aligned}$$

using the fact that the $(k+1)^{\text{st}}$ derivative of $u_h=0$.

Combining these results, we have

$$\|\mathscr{E}\|^2\le ch^{k+1}\|\mathscr{E}\||u|_{k+1}.$$

This establishes

Theorem 3.2. *There is a constant $c\in\mathbb{R}$ such that $\|\mathscr{E}\|\le ch^{k+1}|u|_{k+1}$.*

4. Numerical experiments

In this section, we give three numerical examples using scheme (3) constructed in Section 1 to verify the error estimate in Theorem 3.2.

We construct triangular mesh as follows. First we partition the square domain $\Omega=[0,1]\times[0,1]$ into $N\times N$ sub-squares uniformly to obtain the square mesh. Next we divide each square element into two triangles by the diagonal line with a positive slope, completing the construction of a triangular mesh. Let $h=1/N(N=2,4,8,16,32,64)$ be mesh sizes for different triangular meshes. All of the examples given below will use these triangulations of Ω, and will apply the method to find a solution $u_h=\{u_o,u_b\}$ where $u_0|_T\in P_1(T)$, and $u_b|_e\in P_1(e)$.

Table 1
Discrete L^2-norm error and convergence rate for Example 4.1.

N	$\|u-u_h\|$	Order
2	3.1239E−02	
4	1.1339E−02	1.4621
8	3.3681E−03	1.7513
16	9.1461E−04	1.8807
32	2.3842E−04	1.9396
64	6.0878E−05	1.9695

Table 2
Discrete L^2-norm error and convergence rate Example 4.2.

N	$\|u-u_h\|$	Order
2	8.0959E−03	
4	1.8468E−03	2.1321
8	4.0990E−04	2.1717
16	8.8522E−05	2.2112
32	1.9811E−05	2.1598
64	4.6291E−06	2.0975

Table 3
Discrete L^2-norm error and convergence rate Example 4.3.

N	$\|u-u_h\|$	Order
2	1.6041E−02	
4	3.7817E−03	2.0846
8	8.2975E−04	2.1883
16	1.6945E−04	2.2918
32	3.4859E−05	2.2813
64	7.4966E−06	2.2172

4.1. Homogeneous boundary cases

We consider the following elliptic problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned} \tag{14}$$

In all three numerical examples, we choose source term $f(x)$ according to the corresponding analytical solution of each example.

Example 4.1. The analytical solution to (14) is

$$u = \sin(\pi x)\sin(\pi y).$$

The finite element method (3), with the different mesh sizes $h = 1/N(N = 2, 4, 8, 16, 32)$, is applied, and the corresponding discrete L^2-norms errors and convergence rates are listed in Table 1.

Example 4.2. The analytical solution to (14) is

$$u = x(1-x)y(1-y).$$

Numerical error results and convergence rate are listed in Table 2 and are based on the same mesh sizes as that of the first example.

Example 4.3. The analytical solution to (14) is

$$u = x(1-x)y(1-y)\exp(x-y).$$

Numerical error results and convergence rate are listed in Table 3 based on the same mesh sizes as that of the first example.

All the three numerical examples given above are in good agreement with the theoretical analysis in Section 3, which validates that the WG finite element method (3) is stable and convergent with second order convergence rate in discrete L^2 norm.

Table 4
Discrete L^2-norm error and convergence rate for Example 4.4.

N	$\|u - u_h\|$	Order
2	1.6701E−01	
4	5.4979E−02	1.6030
8	1.5817E−02	1.7974
16	4.3953E−03	1.8475
32	1.1840E−03	1.8922
64	3.0968E−04	1.9349

Table 5
Discrete L^2-norm error and convergence rate Example 4.5.

N	$\|u - u_h\|$	Order
2	1.2026E−02	
4	4.0292E−03	1.5776
8	1.1195E−03	1.8476
16	3.0289E−04	1.8860
32	8.0659E−05	1.9089
64	2.1013E−05	1.9405

Table 6
Discrete L^2-norm error and convergence rate Example 4.6.

N	$\|u - u_h\|$	Order
2	1.4564E−02	
4	5.2053E−03	1.4844
8	1.4748E−03	1.8195
16	3.9698E−04	1.8933
32	1.0504E−04	1.9181
64	2.7270E−05	1.9456

4.2. Nonhomogeneous boundary cases

In this subsection, we consider examples of the same elliptic problem (1) with a nonhomogeneous boundary condition.

Example 4.4. The analytical solution to (1) is

$$u = \sin(\pi x)\sin(\pi y) + x.$$

The finite element method (3), with the different mesh sizes $h = 1/N(N = 2, 4, 8, 16, 32)$ is applied, and the corresponding discrete L^2-norms errors and convergence rates are listed in Table 4.

Example 4.5. The analytical solution to (1) is

$$u = x(1 - x)y(1 - y) + x.$$

Numerical error results and convergence rate are listed in Table 5 based on the same mesh sizes as that of the first example.

Example 4.6. The analytical solution to (1) is

$$u = x(1 - x)y(1 - y)\exp(x - y) + x.$$

Numerical error results and convergence rate are listed in Table 6 based on the same mesh sizes as that of the first example.

All the three numerical examples given above are in good agreement with the theoretical analysis in Section 3, which validates that the WG finite element method (3) is stable and convergent with second order convergence rate in discrete L^2 norm for nonhomogeneous boundary case.

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