Poroelasticity Notes

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1 List of Variables

1.1 General

B The spatial body within which the problem is defined

∂B Bounding surface of B

n_i Normal vector (on ∂B)

x_i Global position in cartesian coordinates (vector)

t Time (scalar)

g_i Gravitational body force (vector)

b_{ij} Biot's modulus (rank 2 tensor with non-zero components only on the diagonal; links total stress and fluid pressure increments)

1.2 Poroelastic Solid

u_i Displacement of the "skeleton"

ρ Overall mass density per unit of initial (undeformed) volume

σ_{ij} Total stress tensor

C_{ijkl} Elastic stiffness modulus of the "skeleton"

ϵ_{ij} "Skeleton" strain tensor

ρ^s Intrinsic matrix mass density

$\partial_u B$ The part of the boundary on which "skeleton" displacements are prescribed

$\bar{u}_i$ Prescribed "skeleton" displacements on $\partial_u B$

$\partial_t B$ The part of the boundary on which surface tractions are prescribed

$\bar{t}_i$ Prescribed traction vector on $\partial_t B$ (affects the total stress, σ_{ij})

1.3 Compressible Fluid

p Fluid (pore) pressure

$\mathcal{V}_i$ Volumetric fluid flux

ϕ Ratio of the pore volume in the present (deformed) configuration to the total RVE volume in the reference (undeformed) configuration (Lagrangian porosity)

ϕ_0 Initial pore volume in the reference (undeformed) configuration to the total RVE volume in the reference configuration

$\frac{1}{M}$ Inverse of Biot's modulus (scalar valued quantity that links pore pressure and porosity variation)

ρ^f Intrinsic fluid mass density

k_{ij} Hydraulic conductivity tensor

$\partial_p B$ The part of the boundary on which fluid pressure is prescribed

$\bar{p}$ Prescribed fluid pressure on $\partial_p B$

$\partial_{\mathcal{V}} B$ The part of the boundary on which normal volumetric flux is prescribed

$\bar{\mathcal{V}}$ Prescribed normal volumetric flux on $\partial_{\mathcal{V}} B$

2 Governing Equations: Strong Form

2.1 Equation of Equilibrium of the Poroelastic Solid

$$\sigma_{ij,j} + \rho g_i = 0 \qquad \forall x \in B \tag{1}$$

Boundary Conditions for the Poroelastic Solid:

$$u_i = \bar{u}_i \quad on \quad \partial_u B \tag{2}$$

$$\sigma_{ij} n_j = \bar{t}_i \quad on \quad \partial_t B \tag{3}$$

Initial Conditions for the Poroelastic Solid:

$$u_i(t = 0) = u_i^0 \tag{4}$$

2.2 Equation of Compressible Fluid Flow in a Porous Medium

$$\frac{\partial}{\partial t}\phi + \mathcal{V}_{i,i} = 0 \qquad \forall x \in B \tag{5}$$

Boundary Conditions for the Compressible Fluid:

$$p = \bar{p} \quad on \quad \partial_p B \tag{6}$$

$$\mathcal{V}_i n_i = \bar{\mathcal{V}} \quad on \quad \partial_{\mathcal{V}} B \tag{7}$$

Initial Conditions for the Compressible Fluid:

$$p(t = 0) = p^0 \tag{8}$$

2.3 Contitutive Relations for the Poroelastic Solid

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} - b_{ij} p \tag{9}$$

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{10}$$

2.3.1 Contitutive Relations for the Compressible Fluid

$$\phi = \phi_0 + b_{ij}\epsilon_{ij} + \frac{p}{M} \tag{11}$$

$$\mathcal{V}_i = -k_{ij}(p_{,j} - \rho^f g_j) \tag{12}$$

In practice: Set p^0 consistent with a hydrostatic pressure distribution as the initial condition, i.e. $p^0_{,j} = \rho^f g_j$. Then solve for the resulting u_i^0.

3 Weak Form

3.1 Generalized Weak Form Problem Statement

Define:

$$u_i \in \mathcal{S} = \{u_i | u_i \in H^1(B), u_i = \bar{u}_i \quad on \quad \partial_u B\} \tag{13}$$

$$v_i \in V = \{v_i | v_i \in H^1(B), v_i = 0 \quad on \quad \partial_u B\} \tag{14}$$

and

$$p_i \in \mathcal{T} = \{p | p \in H^1(B), p = \bar{p} \quad on \quad \partial_p B\} \tag{15}$$

$$q_i \in \mathcal{Q} = \{q | q \in H^1(B), q = 0 \quad on \quad \partial_p B\} \tag{16}$$

Find $u_i \in \mathcal{S}$ and $p \in \mathcal{T}$ such that:

$$\int_B \sigma_{ij} v_{i,j} dv = \int_{\partial_t B} \bar{t}_i v_i da + \int_B \rho g_i v_i dv \quad \forall v_i \in V \tag{17}$$

and

$$\int_B \mathcal{V}_i q_{,i} dv = \int_{\partial_\mathcal{V} B} \bar{\mathcal{V}} q da + \int_B \frac{\partial \phi}{\partial t} q dv \quad \forall q \in \mathcal{Q} \tag{18}$$

3.2 Derivation of the Discrete-in-Time Equations

Substitute the constitutive relations for the fluid and the solid into each integral statement

$$\int_B \left[C_{ijkl} \{ \frac{1}{2}(u_{k,l} + u_{l,k}) \} - b_{ij} p \right] v_{i,j} dv = \int_{\partial_t B} \bar{t}_i v_i da + \int_B \rho g_i v_i dv \quad \forall v_i \in V \tag{19}$$

$$\int_B \left[-k_{ij}(p_{,j} - \rho^f g_j) \right] q_{,i} dv = \int_{\partial_\mathcal{V} B} \bar{\mathcal{V}} q da + \int_B \frac{\partial}{\partial t}(\phi_0 + b_{ij}\epsilon_{ij} + \frac{p}{M}) q dv \quad \forall q \in \mathcal{Q} \tag{20}$$

simplifying,

$$\int_B v_{i,j} C_{ijkl} u_{k,l} dv - \int_B v_{i,j} b_{ij} p dv = \int_{\partial_t B} v_i \bar{t}_i da + \int_B v_i \rho g_i dv \quad \forall v_i \in V \tag{21}$$

$$-\int_B q_{,i} k_{ij} p_{,j} dv - \int_B q \frac{\partial}{\partial t}(b_{ij} u_{i,j} + \frac{p}{M}) dv = \int_{\partial_\mathcal{V} B} q \bar{\mathcal{V}} da - \int_B q_{,i} k_{ij} \rho^f g_j dv \quad \forall q \in \mathcal{Q} \tag{22}$$

Integrate each of the above equations with respect to time from t_m to t_{m+1}

$$\int_{t_m}^{t_{m+1}} \left[\int_B v_{i,j} C_{ijkl} u_{k,l} dv - \int_B v_{i,j} b_{ij} p dv \right] dt ...$$

$$... = \int_{t_m}^{t_{m+1}} \left[\int_{\partial_t B} v_i \bar{t}_i da + \int_B v_i \rho g_i dv \right] dt \quad \forall v_i \in V \tag{23}$$

$$\int_{t_m}^{t_{m+1}} \left[-\int_B q_{,i} k_{ij} p_{,j} dv - \int_B q \frac{\partial}{\partial t}(b_{ij} u_{i,j} + \frac{p}{M}) dv \right] dt ...$$
$$... = \int_{t_m}^{t_{m+1}} \left[\int_{\partial_{\mathcal{V}} B} q \bar{\mathcal{V}} da - \int_B q_{,i} k_{ij} \rho^f g_j dv \right] dt \quad \forall q \in \mathcal{Q} \tag{24}$$

and define

$$\Delta t = t_{m+1} - t_m \tag{25}$$

We propose an approximate integration scheme (generalized trapezoidal rule) as follows for a general function of time, $f(t)$

$$\int_{t_m}^{t_{m+1}} f(t) dt \approx \left[\frac{1}{2}(1+\theta) f^{(m+1)} + \frac{1}{2}(1-\theta) f^{(m)} \right] \Delta t = f^{(m,\theta)} \Delta t \tag{26}$$

where $\theta = +1$ corresponds to the Backward Euler method, $\theta = -1$ corresponds to the Forward Euler method, and $\theta = 0$ corresponds to the Crank-Nicolson method. Applying this rule to our integral statements from before, we obtain the dicrete-in-time weak form equations

$$\int_B v_{i,j} C_{ijkl}^{(m,\theta)} u_{k,l}^{(m,\theta)} dv - \int_B v_{i,j} b_{ij}^{(m,\theta)} p^{(m,\theta)} dv ...$$
$$... = \int_{\partial_t B} v_i \bar{t}_i^{(m,\theta)} da + \int_B v_i \rho^{(m,\theta)} g_i dv \quad \forall v_i \in V \tag{27}$$

$$-\Delta t \int_B q_{,i} k_{ij}^{(m,\theta)} p_{,j}^{(m,\theta)} dv - \int_B q b_{ij}^{(m+1)} u_{i,j}^{(m+1)} dv - \int_B q \frac{p^{(m+1)}}{M^{(m+1)}} dv ...$$
$$... = \Delta t \left[\int_{\partial_{\mathcal{V}} B} q \bar{\mathcal{V}}^{(m,\theta)} da - \int_B q_{,i} k_{ij}^{(m,\theta)} \rho^{f(m,\theta)} g_j dv \right] ...$$
$$... - \int_B q b_{ij}^{(m)} u_{i,j}^{(m)} dv - \int_B q \frac{p^{(m)}}{M^{(m)}} dv \quad \forall q \in \mathcal{Q} \tag{28}$$

4 Galerkin Approximation

We now wish to find an approximate solution, $u_i^h \in \mathcal{S}^h \subset \mathcal{S}$ and $p^h \in \mathcal{T}^h \subset \mathcal{T}$ such that

$$\mathbf{u}^h = \sum_{a \in \eta_0} \Phi_a \mathbf{u}_a + \sum_{a \in \eta_u} \Phi_a \bar{\mathbf{u}}_a, \qquad p^h = \sum_{a \in \zeta_0} \hat{\Phi}_a p_a + \sum_{a \in \zeta_p} \hat{\Phi}_a \bar{p}_a \tag{29}$$

$$\mathbf{v}^h = \sum_{a \in \eta_0} \Phi_a \mathbf{v}_a, \qquad q^h = \sum_{a \in \zeta_0} \hat{\Phi}_a q_a \tag{30}$$

with the following definitions for the sets of nodes, a. Note that the sets η_0 and η_u are defined independently from ζ_0 and ζ_p.

η_0 The set of nodes without prescribed skeleton displacements

η_u The set of nodes with prescribed skeleton displacements, $\bar{\mathbf{u}}$

ζ_0 The set of nodes without prescribed fluid pressure

ζ_p The set of nodes with prescribed fluid pressure, $\bar{p}$

Henceforth, we shall adopt matrix and vector representations for all quantities, with the stress and strain vectors arranged according to Voigt notation. It therefore becomes of interest to investigate the following quantities

$$\epsilon = \sum_a \mathbf{B}_a \mathbf{u}_a, \qquad \nabla \cdot p = \sum_a \hat{\mathbf{B}}_a p_a \tag{31}$$

where we define $\mathbf{B}_a$ and $\hat{\mathbf{B}}_a$ (in three spatial dimensions) as follows

$$\mathbf{B}_a = \begin{bmatrix} \Phi_{a,1} & 0 & 0 \\ 0 & \Phi_{a,2} & 0 \\ 0 & 0 & \Phi_{a,3} \\ 0 & \Phi_{a,3} & \Phi_{a,2} \\ \Phi_{a,3} & 0 & \Phi_{a,1} \\ \Phi_{a,2} & \Phi_{a,1} & 0 \end{bmatrix} \qquad \hat{\mathbf{B}}_a = \begin{bmatrix} \hat{\Phi}_{a,1} \\ \hat{\Phi}_{a,2} \\ \hat{\Phi}_{a,3} \end{bmatrix} \tag{32}$$

We shall also recast the "skeleton" modulus tensor, C_{ijkl}, as the canonical modulus matrix, $\mathbf{D}$. Further, we will rearrange the Biot modulus, b_{ij}, into the form of a column vector, $\mathbf{b}$, as shown below (in three spatial dimensions)

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

with these definitions, we may now express the Galerkin approximation to the weak form as follows

$$\begin{aligned} &\sum_{b\in\eta_0} \left(\int_B \mathbf{B}_a^T \mathbf{D}^{(m,\theta)} \mathbf{B}_b dv \right) \mathbf{u}_b^{(m,\theta)} + \sum_{b\in\eta_u} \left(\int_B \mathbf{B}_a^T \mathbf{D}^{(m,\theta)} \mathbf{B}_b dv \right) \bar{\mathbf{u}}_b^{(m,\theta)} ... \\ &... - \sum_{b\in\zeta_0} \left(\int_B \mathbf{B}_a^T \mathbf{b}^{(m,\theta)} \hat{\Phi}_b dv \right) p_b^{(m,\theta)} - \sum_{b\in\zeta_p} \left(\int_B \mathbf{B}_a^T \mathbf{b}^{(m,\theta)} \hat{\Phi}_b dv \right) \bar{p}_b^{(m,\theta)} ... \\ &... = \int_{\partial_t B} \Phi_a \bar{\mathbf{t}}^{(m,\theta)} da + \int_B \Phi_a \rho^{(m,\theta)} \mathbf{g} dv \quad \forall a \in \eta_0 \end{aligned} \tag{34}$$

$$
\begin{aligned}
\Delta t\Bigg[-\sum_{b\in\zeta_0}\left(\int_B \hat{\mathbf{B}}_a^T\mathbf{k}^{(m,\theta)}\hat{\mathbf{B}}_b dv\right)p_b^{(m,\theta)} - \sum_{b\in\zeta_p}\left(\int_B \hat{\mathbf{B}}_a^T\mathbf{k}^{(m,\theta)}\hat{\mathbf{B}}_b dv\right)\bar{p}_b^{(m,\theta)}\Bigg]...\\
...-\sum_{b\in\eta_0}\left(\int_B \hat{\Phi}_a\mathbf{b}^{T(m+1)}\mathbf{B}_b dv\right)\mathbf{u}_b^{(m+1)} - \sum_{b\in\eta_u}\left(\int_B \hat{\Phi}_a\mathbf{b}^{T(m+1)}\mathbf{B}_b dv\right)\bar{\mathbf{u}}_b^{(m+1)}...\\
...-\sum_{b\in\zeta_0}\left(\int_B \hat{\Phi}_a M^{-1(m+1)}\hat{\Phi}_b dv\right)p_b^{(m+1)} - \sum_{b\in\zeta_p}\left(\int_B \hat{\Phi}_a M^{-1(m+1)}\hat{\Phi}_b dv\right)\bar{p}_b^{(m+1)}...\\
...=\Delta t\left[\int_{\partial_{\mathcal{V}}B}\hat{\Phi}_a\bar{\mathcal{V}}^{(m,\theta)}da - \int_B \hat{\mathbf{B}}_a^T\mathbf{k}^{(m,\theta)}\rho^{f(m,\theta)}\mathbf{g}dv\right]...\\
...-\sum_{b\in\eta_0}\left(\int_B \hat{\Phi}_a\mathbf{b}^{T(m)}\mathbf{B}_b dv\right)\mathbf{u}_b^{(m)} - \sum_{b\in\eta_u}\left(\int_B \hat{\Phi}_a\mathbf{b}^{T(m)}\mathbf{B}_b dv\right)\bar{\mathbf{u}}_b^{(m)}...\\
...-\sum_{b\in\zeta_0}\left(\int_B \hat{\Phi}_a M^{-1(m)}\hat{\Phi}_b dv\right)p_b^{(m)} - \sum_{b\in\zeta_p}\left(\int_B \hat{\Phi}_a M^{-1(m)}\hat{\Phi}_b dv\right)\bar{p}_b^{(m)} \quad \forall a\in\zeta_0
\end{aligned}
\tag{35}
$$

We can simplify these expressions by introducing the following notation for the integral statements

$$
\mathbf{K_{uu}}_{ab}^{(m)} = \int_B \mathbf{B}_a^T\mathbf{D}^{(m)}\mathbf{B}_b dv \qquad (3\times 3) \tag{36}
$$

$$
\mathbf{K_{up}}_{ab}^{(m)} = \int_B \mathbf{B}_a^T\mathbf{b}^{(m)}\hat{\Phi}_b dv \qquad (3\times 1) \tag{37}
$$

$$
\mathbf{K_{pu}}_{ab}^{(m)} = \int_B \hat{\Phi}_a\mathbf{b}^{T(m)}\mathbf{B}_b dv \qquad (1\times 3) \tag{38}
$$

$$
K^{1\ (m)}_{pp_{ab}} = \int_B \hat{\mathbf{B}}_a^T\mathbf{k}^{(m)}\hat{\mathbf{B}}_b dv \qquad (1\times 1) \tag{39}
$$

$$
K^{2\ (m)}_{pp_{ab}} = \int_B \hat{\Phi}_a M^{-1(m)}\hat{\Phi}_b dv \qquad (1\times 1) \tag{40}
$$

this yields

$$
\begin{aligned}
\sum_{b\in\eta_0}\mathbf{K_{uu}}_{ab}^{(m,\theta)}\mathbf{u}_b^{(m,\theta)} - \sum_{b\in\zeta_0}\mathbf{K_{up}}_{ab}^{(m,\theta)}p_b^{(m,\theta)}...\\
...=\int_{\partial_t B}\Phi_a\bar{\mathbf{t}}^{(m,\theta)}da + \int_B \Phi_a\rho^{(m,\theta)}\mathbf{g}dv...\\
...-\sum_{b\in\eta_u}\mathbf{K_{uu}}_{ab}^{(m,\theta)}\bar{\mathbf{u}}_b^{(m,\theta)} + \sum_{b\in\zeta_p}\mathbf{K_{up}}_{ab}^{(m,\theta)}\bar{p}_b^{(m,\theta)} \quad \forall a\in\eta_0
\end{aligned}
\tag{41}
$$

$$-\sum_{b\in\eta_0}\mathbf{K_{pu}}_{ab}^{(m+1)}\mathbf{u}_b^{(m+1)}-\Delta t\sum_{b\in\zeta_0}K^{1\ (m,\theta)}_{pp_{ab}}p_b^{(m,\theta)}-\sum_{b\in\zeta_0}K^{2\ (m+1)}_{pp_{ab}}p_b^{(m+1)}...$$

$$...=\Delta t\left[\int_{\partial_{\mathcal{V}}B}\hat{\Phi}_a\bar{\mathcal{V}}^{(m,\theta)}da-\int_B\hat{\mathbf{B}}_a^T\mathbf{k}^{(m,\theta)}\rho^{f(m,\theta)}\mathbf{g}dv\right]...$$

$$...+\sum_{b\in\eta_u}\mathbf{K_{pu}}_{ab}^{(m+1)}\bar{\mathbf{u}}_b^{(m+1)}+\Delta t\sum_{b\in\zeta_p}K^{1\ (m,\theta)}_{pp_{ab}}\bar{p}_b^{(m,\theta)}+\sum_{b\in\zeta_p}K^{2\ (m+1)}_{pp_{ab}}\bar{p}_b^{(m+1)}...$$

$$-\sum_{b\in\eta_0}\mathbf{K_{pu}}_{ab}^{(m)}\mathbf{u}_b^{(m)}-\sum_{b\in\eta_u}\mathbf{K_{pu}}_{ab}^{(m)}\bar{\mathbf{u}}_b^{(m)}-\sum_{b\in\zeta_0}K^{2\ (m)}_{pp_{ab}}p_b^{(m)}-\sum_{b\in\zeta_p}K^{2\ (m)}_{pp_{ab}}\bar{p}_b^{(m)}\quad\forall a\in\zeta_0 \tag{42}$$

define contributions to the global forcing/residual vector as

$$\mathbf{F_u}_a^{(m)}=\int_{\partial_t B}\Phi_a\bar{\mathbf{t}}^{(m)}da+\int_B\Phi_a\rho^{(m)}\mathbf{g}dv-\sum_{b\in\eta_u}\mathbf{K_{uu}}_{ab}^{(m)}\bar{\mathbf{u}}_b^{(m)}+\sum_{b\in\zeta_p}\mathbf{K_{up}}_{ab}^{(m)}\bar{p}_b^{(m)}\qquad(3\times1) \tag{43}$$

$$F_{p\ a}^{1(m)}=\int_{\partial_{\mathcal{V}}B}\hat{\Phi}_a\bar{\mathcal{V}}^{(m)}da-\int_B\hat{\mathbf{B}}_a^T\mathbf{k}^{(m)}\rho^{f(m)}\mathbf{g}dv+\sum_{b\in\zeta_p}K^{1\ (m)}_{pp_{ab}}\bar{p}_b^{(m)}\qquad(1\times1) \tag{44}$$

$$F_{p\ a}^{2(m)}=\sum_{b\in\eta_u}\mathbf{K_{pu}}_{ab}^{(m)}\bar{\mathbf{u}}_b^{(m)}+\sum_{b\in\zeta_p}K^{2\ (m)}_{pp_{ab}}\bar{p}_b^{(m)}\qquad(1\times1) \tag{45}$$

substituting for the above expressions

$$\sum_{b\in\eta_0}\mathbf{K_{uu}}_{ab}^{(m,\theta)}\mathbf{u}_b^{(m,\theta)}-\sum_{b\in\zeta_0}\mathbf{K_{up}}_{ab}^{(m,\theta)}p_b^{(m,\theta)}=\mathbf{F_u}_a^{(m,\theta)}\quad\forall a\in\eta_0 \tag{46}$$

$$-\sum_{b\in\eta_0}\mathbf{K_{pu}}_{ab}^{(m+1)}\mathbf{u}_b^{(m+1)}-\Delta t\sum_{b\in\zeta_0}K^{1\ (m,\theta)}_{pp_{ab}}p_b^{(m,\theta)}-\sum_{b\in\zeta_0}K^{2\ (m+1)}_{pp_{ab}}p_b^{(m+1)}...$$

$$...=\Delta tF_{p\ a}^{1(m,\theta)}+F_{p\ a}^{2(m+1)}-F_{p\ a}^{2(m)}-\sum_{b\in\eta_0}\mathbf{K_{pu}}_{ab}^{(m)}\mathbf{u}_b^{(m)}-\sum_{b\in\zeta_0}K^{2\ (m)}_{pp_{ab}}p_b^{(m)}\quad\forall a\in\zeta_0 \tag{47}$$

and expanding the (m,θ) terms, we obtain

$$(1+\theta)\left[\sum_{b\in\eta_0}\mathbf{K_{uu}}_{ab}^{(m+1)}\mathbf{u}_b^{(m+1)}-\sum_{b\in\zeta_0}\mathbf{K_{up}}_{ab}^{(m+1)}p_b^{(m+1)}\right]...$$

$$...=(1+\theta)\mathbf{F_u}_a^{(m+1)}+(1-\theta)\left[\mathbf{F_u}_a^{(m)}-\sum_{b\in\eta_0}\mathbf{K_{uu}}_{ab}^{(m)}\mathbf{u}_b^{(m)}+\sum_{b\in\zeta_0}\mathbf{K_{up}}_{ab}^{(m)}p_b^{(m)}\right]\quad\forall a\in\eta_0 \tag{48}$$

$$
-\sum_{b\in\eta_0}\mathbf{K_{pu}}_{ab}^{(m+1)}\mathbf{u}_b^{(m+1)}-\sum_{b\in\zeta_0}\left((1+\theta)\frac{\Delta t}{2}K^{1\ (m+1)}_{pp_{ab}}+K^{2\ (m+1)}_{pp_{ab}}\right)p_b^{(m+1)}...
$$

$$
...=(1+\theta)\frac{\Delta t}{2}F^{1(m+1)}_{p\ a}+F^{2(m+1)}_{p\ a}+(1-\theta)\frac{\Delta t}{2}F^{1(m)}_{p\ a}-F^{2(m)}_{p\ a}...
$$

$$
...-\sum_{b\in\eta_0}\mathbf{K_{pu}}_{ab}^{(m)}\mathbf{u}_b^{(m)}-\sum_{b\in\zeta_0}\left((\theta-1)\frac{\Delta t}{2}K^{1\ (m)}_{pp_{ab}}+K^{2\ (m)}_{pp_{ab}}\right)p_b^{(m)}\quad\forall a\in\zeta_0 \tag{49}
$$

For $\theta = 0$ (Crank-Nicolson method) we find

$$
\sum_{b\in\eta_0}\mathbf{K_{uu}}_{ab}^{(m+1)}\mathbf{u}_b^{(m+1)}-\sum_{b\in\zeta_0}\mathbf{K_{up}}_{ab}^{(m+1)}p_b^{(m+1)}...
$$

$$
...=\mathbf{F_u}_a^{(m+1)}+\mathbf{F_u}_a^{(m)}-\sum_{b\in\eta_0}\mathbf{K_{uu}}_{ab}^{(m)}\mathbf{u}_b^{(m)}+\sum_{b\in\zeta_0}\mathbf{K_{up}}_{ab}^{(m)}p_b^{(m)}\quad\forall a\in\eta_0 \tag{50}
$$

$$
-\sum_{b\in\eta_0}\mathbf{K_{pu}}_{ab}^{(m+1)}\mathbf{u}_b^{(m+1)}-\sum_{b\in\zeta_0}\left(\frac{\Delta t}{2}K^{1\ (m+1)}_{pp_{ab}}+K^{2\ (m+1)}_{pp_{ab}}\right)p_b^{(m+1)}...
$$

$$
...=\frac{\Delta t}{2}F^{1(m+1)}_{p\ a}+F^{2(m+1)}_{p\ a}+\frac{\Delta t}{2}F^{1(m)}_{p\ a}-F^{2(m)}_{p\ a}...
$$

$$
...-\sum_{b\in\eta_0}\mathbf{K_{pu}}_{ab}^{(m)}\mathbf{u}_b^{(m)}+\sum_{b\in\zeta_0}\left(\frac{\Delta t}{2}K^{1\ (m)}_{pp_{ab}}-K^{2\ (m)}_{pp_{ab}}\right)p_b^{(m)}\quad\forall a\in\zeta_0 \tag{51}
$$

The above equations may be cast in matrix form as:

$$
\begin{bmatrix}\mathbf{K_{uu}} & -\mathbf{K_{up}}\\ -\mathbf{K_{pu}} & -\left(\frac{\Delta t}{2}\mathbf{K^1_{pp}}+\mathbf{K^2_{pp}}\right)\end{bmatrix}^{(m+1)}\begin{bmatrix}\mathbf{u}\\ \mathbf{p}\end{bmatrix}^{(m+1)}...
$$

$$
...=\begin{bmatrix}-\mathbf{K_{uu}} & \mathbf{K_{up}}\\ -\mathbf{K_{pu}} & \left(\frac{\Delta t}{2}\mathbf{K^1_{pp}}-\mathbf{K^2_{pp}}\right)\end{bmatrix}^{(m)}\begin{bmatrix}\mathbf{u}\\ \mathbf{p}\end{bmatrix}^{(m)}...
$$

$$
...+\begin{bmatrix}\mathbf{F_u}\\ \left(\frac{\Delta t}{2}\mathbf{F^1_p}+\mathbf{F^2_p}\right)\end{bmatrix}^{(m+1)}+\begin{bmatrix}\mathbf{F_u}\\ \left(\frac{\Delta t}{2}\mathbf{F^1_p}-\mathbf{F^2_p}\right)\end{bmatrix}^{(m)} \tag{52}
$$