A LOCKING-FREE WEAK GALERKIN FINITE ELEMENT METHOD FOR ELASTICITY PROBLEMS IN THE PRIMAL FORMULATION

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Abstract. This paper presents an arbitrary order locking-free numerical scheme for linear elasticity on general polygonal/polyhedral partitions by using weak Galerkin (WG) finite element methods. Like other WG methods, the key idea for the linear elasticity is to introduce discrete weak strain and stress tensors which are defined and computed by solving inexpensive local problems on each element. Such local problems are derived from weak formulations of the corresponding differential operators through integration by parts. Locking-free error estimates of optimal order are derived in a discrete H^1-norm and the usual L^2-norm for the approximate displacement when the exact solution is smooth. Numerical results are presented to demonstrate the efficiency, accuracy, and the locking-free property of the weak Galerkin finite element method.

Key words. weak Galerkin, finite element methods, Korn's inequality, weak divergence, weak gradient, linear elasticity, locking-free, polygonal meshes, polyhedral meshes.

AMS subject classifications. Primary 65N30, 65N15, 74S05; Secondary 35J50, 74B05.

1. Introduction. In this paper, we are concerned with the development of efficient new numerical methods for linear elasticity equations by using the weak Galerkin finite element method recently developed in [21, 23, 22, 14]. Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be an open bounded and connected domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$ of an elastic body subject to an exterior force $\mathbf{f}$ and a given displacement boundary condition. The kinematic model of linear elasticity seeks a displacement vector field $\mathbf{u}$ satisfying

$$-\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f}, \qquad \text{in } \Omega, \tag{1.1}$$

$$\mathbf{u} = \widehat{\mathbf{u}}, \qquad \text{on } \Gamma, \tag{1.2}$$

where $\sigma(\mathbf{u})$ is the symmetric Cauchy stress tensor. For linear, homogeneous, and isotropic materials, the Cauchy stress tensor is given by

$$\sigma(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I},$$

where $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T)$ is the linear strain tensor, μ and λ are the Lamé constants. For linear plane strain, the Lamé constants are given by

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)},$$

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where E is the elasticity modulus and ν is Poisson's ratio.

Denote by $(\cdot,\cdot)$ the L^2-inner product in either $L^2(\Omega)$, $[L^2(\Omega)]^d$, or $[L^2(\Omega)]^{d\times d}$, as appropriate. A weak formulation for (1.1) in the primal form reads as follows: Find $\mathbf{u} \in [H^1(\Omega)]^d$ satisfying $\mathbf{u} = \widehat{\mathbf{u}}$ on Γ and

$$2(\mu\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})) + (\lambda\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \qquad \forall \mathbf{v} \in [H_0^1(\Omega)]^d, \tag{1.3}$$

where $H^1(\Omega)$ is the usual Sobolev space defined by

$$H^1(\Omega) = \{v :\ v \in L^2(\Omega),\ \nabla v \in L^2(\Omega)\},$$

and $H_0^1(\Omega)$ is the closed subspace of $H^1(\Omega)$ consisting of all the functions with vanishing boundary value.

The main objective of this paper is to study an application of the weak Galerkin finite element method [21, 23, 22, 20, 15, 19] to the linear elasticity problem (1.1)-(1.2) based on the primal formulation (1.3). The weak Galerkin (WG) refers to a generic finite element technique for partial differential equations where differential operators are approximated or reconstructed by solving inexpensive local problems on each element. Such local problems are often derived from weak formulations of the corresponding differential operators through integration by parts. Recent work on WG has revealed that the concept of discrete weak derivatives offers a new paradigm in the discretiztion of partial differential equations. The resulting numerical schemes often possess a great robustness in stability and convergence for which other competing methods are hard to achieve. For the linear elasticity problem (1.3), we shall demonstrate that the WG numerical approximations are not only accurate and robust with respect to the polygonal/polyhedral partition of the domain, but also naturally "locking-free" in terms of the Lamé constant λ. This is a result that the standard conforming finite element method does not have.

"Locking" refers to a phenomenon of numerical approximations for a certain problems whose mathematical formulations involve a parameter dependency. For the linear elasticity problem, the parameter is the Poisson ratio ν. For ν close to $\frac{1}{2}$ (i.e., when the material is nearly incompressible), it is well known that various finite element schemes, such as the continuous piecewise linear elements, results in poor observed convergence rates in the displacements. In 1983, Vogelious [18] showed absence of locking for the p-version of the finite element method on smooth domains. Later on, Scott and Vegelious [17] proved that no locking results when polynomials of degree $k \geq 4$ are used on triangular meshes. However, Babuška and Suri [2] found, for conforming methods, locking cannot be avoided on quadrilateral meshes for any polynomial of degree $k \geq 1$. In the discontinuous Galerkin context, Hansbo and Larson [13] proved that the numerical approximation arising from a discontinuous Galerkin method is locking-free for any values of $k \geq 1$, also see [24] for the case of $k = 1$. In [8], Daniele and Nicaise designed a locking-free discontinuous Galerkin method for composite materials featuring quasi-incompressible and compressible sections.

Locking occurs because for the limit case $\nu = \frac{1}{2}$ (or $\lambda = \infty$), the exact solution of the displacement must satisfy the constraint $\nabla\cdot\mathbf{u} = 0$. But locking is not a difficult issue to resolve for the linear elasticity equation. One possibility is the use of mixed methods by reformulating the linear elasticity equation as a generalized Stokes equation (see [4, 9, 1] and the references cited therein). Specifically, by introducing a pressure variable $p = \lambda\nabla\cdot\mathbf{u}$, the elasticity problem (1.1)-(1.2) can be reformulated as follows: Find $\mathbf{u} \in [H^1(\Omega)]^d$ and $p \in L^2(\Omega)$ satisfying $\mathbf{u} = \widehat{\mathbf{u}}$ on Γ, the compatibility condition $\int_\Omega \lambda^{-1} p dx = \int_\Gamma \widehat{\mathbf{u}} \cdot \mathbf{n} ds$, and the following equations

$$2(\mu\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})) + (\nabla \cdot \mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \qquad \forall \mathbf{v} \in [H_0^1(\Omega)]^d, \tag{1.4}$$

$$(\nabla \cdot \mathbf{u}, q) - (\lambda^{-1} p, q) = 0, \qquad \forall q \in L_0^2(\Omega). \tag{1.5}$$

Here $\mathbf{n}$ is the outward normal direction on Γ, and $L_0^2(\Omega)$ is the closed subspace of $L^2(\Omega)$ consisting of all functions with mean-value zero. Consequently, any finite elements that are stable for the Stokes problem would provide a locking-free approximation for the linear elasticity problem (1.1)-(1.2), but at the cost of solving a saddle-point problem with an additional pressure variable.

The weak Galerkin method can also be applied to the linear elasticity problem based on the mixed formulation (1.4)-(1.5). In principle, such applications should yield locking-free numerical approximations for the displacement variable. Surprisingly, we found that the weak Galerkin finite element method based on the primal formulation (1.3) is equivalent to the weak Galerkin when applied to the mixed formulation (1.4)-(1.5). This equivalence indicates that the weak Galerkin approximation arising from the primal formulation (1.3) might be locking-free. The main goal of this paper is to provide a rigorous mathematical justification to the locking-free nature of weak Galerkin. Specifically, we shall perform the following tasks in this study: (1) propose two weak Galerkin finite element schemes, one based on the primal formulation (1.3) and the other based on the mixed formulation (1.4)-(1.5); (2) show that the two weak Galerkin finite element schemes are equivalent; (3) establish a locking-free convergence theory for the WG scheme based on (1.4)-(1.5); and (4) numerically demonstrate the accuracy and the locking-free property of the proposed weak Galerkin finite element methods. The main mathematical challenge of this research lies in the error estimate for shape-regular finite element partitions consisting of arbitrary polygon or polyhedra. The corresponding technicality is represented by the inequality (9.16) in Lemma 9.10, which is a result of a creative use of the Korn's inequality and the domain inverse inequality [21].

Throughout the paper, we will follow the usual notation for Sobolev spaces and norms [5]. For any open bounded domain $D \subset \mathbb{R}^d$ with Lipschitz continuous boundary, we use $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ to denote the norm and seminorms in the Sobolev space $H^s(D)$ for any $s \geq 0$, respectively. The inner product in $H^s(D)$ is denoted by $(\cdot,\cdot)_{s,D}$. The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript D in the norm and inner product notation.

The paper is organized as follows. Section 2 is devoted to a discussion of weak divergence and weak gradient for vector-valued functions as well as their discrete analogues. In Section 3, we present a weak Galerkin finite element method for the linear elasticity problem based on the primal formulation (1.3). In Section 4, we describe another weak Galerkin finite element method based on the mixed formulation (1.4)-(1.5). It is also shown in Section 4 that the two weak Galerkin methods are equivalent. Section 5 is devoted to a discussion of stability conditions (i.e., the *inf-sup* condition and coercivity estimates) for the mixed weak Galerkin finite element method. In Section 6, we prepare ourselves for error estimates by deriving an identity. Section 7 is devoted to the establishment of an optimal order error estimate in a discrete H^1-norm. In Section 8, we use the usual duality argument to derive an optimal order error estimate in the L^2-norm for the displacement variable. In Section 9, we derive some supporting tools and inequalities useful for error analysis. Finally, in Section 10, we report some numerical results that demonstrate the accuracy and locking-free nature of the proposed weak Galerkin finite element methods for the linear elasticity problem.

2. Weak Divergence and Weak Gradient Operators. The divergence and gradient operators are two primary differential operators in the variational problem (1.4)-(1.5). The goal of this section is to review the definition and computation of weak gradient and weak divergence operators which have been introduced and studied in applications to other partial differential equations [23, 21, 22].

Let K be a polygon in 2D or polyhedra in 3D with boundary ∂K. Define the space of weak vector-valued functions in K as follows

$$V(K) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \ \mathbf{v}_0 \in [L^2(K)]^d, \mathbf{v}_b \in [L^2(\partial K)]^d\},$$

where $\mathbf{v}_0$ and $\mathbf{v}_b$ represent the values of $\mathbf{v}$ in K and on the boundary ∂K, respectively. Note that $\mathbf{v}_b$ is not necessarily related to the trace of $\mathbf{v}_0$ should it be well-defined. Denote by $\langle \cdot, \cdot \rangle_{\partial K}$ the standard inner-product in either $L^2(\partial K)$ or $[L^2(\partial K)]^d$, as appropriate.

Definition 2.1. *[21, 22](weak divergence) The weak divergence of $\mathbf{v} \in V(K)$, denoted by $\nabla_w \cdot \mathbf{v}$, is a bounded linear functional in the Sobolev space $H^1(K)$, so that its action on any $\phi \in H^1(K)$ is given by*

$$\langle \nabla_w \cdot \mathbf{v}, \phi \rangle_K := -(\mathbf{v}_0, \nabla \phi)_K + \langle \mathbf{v}_b \cdot \mathbf{n}, \phi \rangle_{\partial K},$$

where $\mathbf{n}$ is the unit outward normal direction on ∂K.

For any non-negative integer r, denote by $P_r(K)$ the set of polynomials with degree r or less on K.

Definition 2.2. *[21, 22](discrete weak divergence) The discrete weak divergence of $\mathbf{v} \in V(K)$, denoted by $\nabla_{w,r,K} \cdot \mathbf{v}$, is the unique polynomial in $P_r(K)$, satisfying*

$$(\nabla_{w,r,K} \cdot \mathbf{v}, \phi)_K = -(\mathbf{v}_0, \nabla \phi)_K + \langle \mathbf{v}_b \cdot \mathbf{n}, \phi \rangle_{\partial K}, \qquad \forall \phi \in P_r(K), \tag{2.1}$$

where $\mathbf{n}$ is the unit outward normal direction on ∂K.

Definition 2.3. *[23, 22](weak gradient) The weak gradient of $\mathbf{v} \in V(K)$, denoted by $\nabla_w \mathbf{v}$, is a matrix-valued bounded linear functional in the Sobolev space $[H^1(K)]^{d\times d}$, so that its action on any $\varphi \in [H^1(K)]^{d\times d}$ is given by*

$$\langle \nabla_w \mathbf{v}, \varphi \rangle_K := -(\mathbf{v}_0, \nabla \cdot \varphi)_K + \langle \mathbf{v}_b, \varphi \mathbf{n} \rangle_{\partial K},$$

where $\mathbf{n}$ is the unit outward normal direction on ∂K. Here the divergence $\nabla \cdot \varphi$ is applied to each row of φ, and $\varphi \mathbf{n}$ is the usual matrix-vector multiplication.

Definition 2.4. *[23, 22](discrete weak gradient) The discrete weak gradient of $\mathbf{v} \in V(K)$, denoted by $\nabla_{w,r,K} \mathbf{v}$, is the unique matrix-valued polynomial in $[P_r(K)]^{d\times d}$, satisfying*

$$(\nabla_{w,r,K} \mathbf{v}, \varphi)_K = -(\mathbf{v}_0, \nabla \cdot \varphi)_K + \langle \mathbf{v}_b, \varphi \mathbf{n} \rangle_{\partial K}, \quad \forall \varphi \in [P_r(K)]^{d\times d}, \tag{2.2}$$

where $\mathbf{n}$ is the unit outward normal direction on ∂K.

Using the weak gradient, we may define the weak strain tensor as follows

$$\varepsilon_w(\mathbf{u}) = \frac{1}{2}(\nabla_w \mathbf{u} + \nabla_w \mathbf{u}^T). \tag{2.3}$$

Analogously, the weak stress tensor can be defined by

$$\sigma_w(\mathbf{u}) = 2\mu\varepsilon_w(\mathbf{u}) + \lambda(\nabla_w \cdot \mathbf{u})\mathbf{I}. \tag{2.4}$$

3. Numerical Algorithms. Let $\mathcal{T}_h$ be a finite element partition of the domain $\Omega \subset \mathbb{R}^d$ consisting of polygons in 2D or polyhedra in 3D which are shape regular interpreted as in [21]. For each $T \in \mathcal{T}_h$, denote by h_T the diameter of T. The mesh size of $\mathcal{T}_h$ is defined as $h = \max_{T\in\mathcal{T}_h} h_T$. Let $T \in \mathcal{T}_h$ be an element with e as an edge in 2D or a face in 3D. Denote by $\mathcal{E}_h$ the set of all edges or faces in $\mathcal{T}_h$ and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ the set of all interior edges or faces in $\mathcal{T}_h$.

On each element $T \in \mathcal{T}_h$, denote by $\mathsf{RM}(T)$ the space of rigid motions on T given by

$$\mathsf{RM}(T) = \{\mathbf{a} + \eta\mathbf{x} :\ \mathbf{a} \in \mathbb{R}^d,\ \eta \in so(d)\},$$

where $\mathbf{x}$ is the position vector on T and $so(d)$ is the space of skew-symmetric $d \times d$ matrices. The trace of the rigid motion on each edge $e \subset T$ forms a finite dimensional space denoted by

$$P_{RM}(e) = \{\mathbf{v} \in [L^2(e)]^d :\ \mathbf{v} = \tilde{\mathbf{v}}|_e \text{ for some } \tilde{\mathbf{v}} \in \mathsf{RM}(T),\ e \subset \partial T\}.$$

For any positive integer $k \geq 1$ and element $T \in \mathcal{T}_h$, we introduce a local weak finite element space as follows

$$V(k,T) = \left\{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} :\ \mathbf{v}_0 \in [P_k(T)]^d,\ \mathbf{v}_b \in V_{k-1}(e)\right\}, \tag{3.1}$$

where $V_{k-1}(e) \equiv [P_{k-1}(e)]^d + P_{RM}(e)$. Since $P_{RM}(e) \subset P_1(e)$, then the boundary component $V_{k-1}(e)$ is given by $[P_{k-1}(e)]^d$ for $k > 1$ and $P_{RM}(e)$ for $k = 1$.

The global weak finite element space V_h is given by patching the local spaces $V(k,T)$ with a single-valued component $\mathbf{v}_b$ on each interior element interface. All the weak finite element functions $\mathbf{v} \in V_h$ with vanishing boundary value $\mathbf{v}_b = 0$ on Γ form a subspace of V_h, which is denoted as

$$V_h^0 = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h :\ \mathbf{v}_b = 0 \text{ on } \Gamma\}. \tag{3.2}$$

For each $\mathbf{v} \in V_h$, the discrete weak divergence $\nabla_{w,k-1} \cdot \mathbf{v}$ and the discrete weak gradient $\nabla_{w,k-1}\mathbf{v}$ are computed by using (2.1) and (2.2) on each element $T \in \mathcal{T}_h$; i.e.,

$$\begin{aligned} (\nabla_{w,k-1} \cdot \mathbf{v})|_T =& \nabla_{w,k-1,T} \cdot (\mathbf{v}|_T), \qquad \mathbf{v} \in V_h, \\ (\nabla_{w,k-1} \mathbf{v})|_T =& \nabla_{w,k-1,T} (\mathbf{v}|_T), \qquad \mathbf{v} \in V_h. \end{aligned}$$

For simplicity of notation, we shall drop the subscript $k-1$ from the notation $\nabla_{w,k-1}$ and $\nabla_{w,k-1}\cdot$ in the rest of the paper. For each edge or face $e \in \mathcal{E}_h$, denote by Q_b the L^2 projection operator onto the space $V_{k-1}(e)$; i.e., the polynomial space $[P_{k-1}(e)]^2$ for $k > 1$ or the rigid motion space $P_{RM}(e)$ for $k = 1$.

Next, we introduce two bilinear forms

$$s(\mathbf{w},\mathbf{v}) = \sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b\mathbf{w}_0-\mathbf{w}_b, Q_b\mathbf{v}_0-\mathbf{v}_b\rangle_{\partial T}, \tag{3.3}$$

$$a_s(\mathbf{w},\mathbf{v}) = \sum_{T\in\mathcal{T}_h} 2(\mu\varepsilon_w(\mathbf{w}),\varepsilon_w(\mathbf{v}))_T + \sum_{T\in\mathcal{T}_h}(\lambda\nabla_w\cdot\mathbf{w},\nabla_w\cdot\mathbf{v})_T + s(\mathbf{w},\mathbf{v}), \tag{3.4}$$

where $\varepsilon_w(\mathbf{w})$ and $\nabla_w\cdot\mathbf{w}$ are computed by using the discrete weak gradient and weak divergence operators.

Weak Galerkin Algorithm 1. *For a numerical solution of the elasticity problem (1.3), find $\mathbf{u}_h=\{\mathbf{u}_0,\mathbf{u}_b\}\in V_h$ with $\mathbf{u}_b=Q_b\hat{\mathbf{u}}$ on Γ such that*

$$a_s(\mathbf{u}_h,\mathbf{v}) = (\mathbf{f},\mathbf{v}_0), \qquad \forall \mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h^0. \tag{3.5}$$

The rest of this section is devoted to a study of (3.5) on the solution existence and uniqueness.

Theorem 3.1. *There exists one and only one solution to the weak Galerkin finite element scheme (3.5).*

Proof. Since the number of equations equals the number of unknowns in (3.5), it suffices to prove the solution uniqueness. To this end, let $\mathbf{u}_h^{(j)}=\{\mathbf{u}_0^{(j)},\mathbf{u}_b^{(j)}\}\in V_h,\ j=1,2$, be two solutions of (3.5). It follows that $\mathbf{u}_b^{(j)}=Q_b\hat{\mathbf{u}}$ on Γ and

$$a_s(\mathbf{u}_h^{(j)},\mathbf{v}) = (\mathbf{f},\mathbf{v}_0), \qquad \forall\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h^0,\ j=1,2.$$

The difference of the two solutions, $\mathbf{w}=\mathbf{u}_h^{(1)}-\mathbf{u}_h^{(2)}$, satisfies $\mathbf{w}\in V_h^0$ and

$$a_s(\mathbf{w},\mathbf{v}) = 0, \qquad \forall\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h^0. \tag{3.6}$$

By letting $\mathbf{v}=\mathbf{w}$ in (3.6) we obtain

$$a_s(\mathbf{w},\mathbf{w}) = 0.$$

Using the definition of $a_s(\cdot,\cdot)$ we arrive at

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} 2(\mu\varepsilon_w(\mathbf{w}),\varepsilon_w(\mathbf{w}))_T &+ \sum_{T\in\mathcal{T}_h}(\lambda\nabla_w\cdot\mathbf{w},\nabla_w\cdot\mathbf{w})_T\\ &+\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b\mathbf{w}_0-\mathbf{w}_b, Q_b\mathbf{w}_0-\mathbf{w}_b\rangle_{\partial T}=0,\end{aligned}$$

which implies that

$$\varepsilon_w(\mathbf{w}) = 0, \qquad \text{in } T, \tag{3.7}$$

$$Q_b\mathbf{w}_0-\mathbf{w}_b = 0, \qquad \text{on } \partial T. \tag{3.8}$$

From the definition of the weak gradient, we have

$$\begin{aligned}(\nabla_w\mathbf{w},\tau)_T &= (\nabla\mathbf{w}_0,\tau)_T - \langle\mathbf{w}_0-\mathbf{w}_b,\tau\mathbf{n}\rangle_{\partial T}\\ &= (\nabla\mathbf{w}_0,\tau)_T - \langle Q_b\mathbf{w}_0-\mathbf{w}_b,\tau\mathbf{n}\rangle_{\partial T}\end{aligned}$$

for all $\tau \in [P_{k-1}(T)]^{d\times d}, k \geq 1$. It follows from (3.8) that $\nabla \mathbf{w}_0 = \nabla_w \mathbf{w}$ on each element T. Thus, with the help of (3.7),

$$\varepsilon(\mathbf{w}_0) = \varepsilon_w(\mathbf{w}) = 0,$$

which leads to $\mathbf{w}_0 \in \mathsf{RM}(T) \subset [P_1(T)]^d$. It follows that $\mathbf{w}_0|_e = Q_b \mathbf{w}_0 = \mathbf{w}_b$, and hence $\mathbf{w}_0$ is a continuous function in Ω with vanishing boundary value on Γ. From the second Korn's inequality (9.13), we obtain $\mathbf{w}_0 \equiv 0$ in Ω, and hence $\mathbf{w}_b \equiv 0$ from (3.8). This shows that $\mathbf{u}_h^{(1)} \equiv \mathbf{u}_h^{(2)}$, and hence the solution uniqueness and existence. We remark that the result holds true for any $\lambda \geq 0$. $\square$

4. An Equivalent Mixed Formulation. A strong form of the mixed formulation (1.4)-(1.5) reads as follows: Find $\mathbf{u}$ and p satisfying $\mathbf{u} = \widehat{\mathbf{u}}$ on Γ, the compatibility condition $\int_\Omega \lambda^{-1} p dx = \int_\Gamma \widehat{\mathbf{u}} \cdot \mathbf{n} ds$, and the following generalized Stokes equations

$$\begin{aligned} -\nabla \cdot (2\mu\varepsilon(\mathbf{u})) + \nabla p &= \mathbf{f}, && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= \lambda^{-1} p, && \text{in } \Omega. \end{aligned} \tag{4.1}$$

We assume that the generalized Stokes problem (4.1) has the $H^{1+s}(\Omega) \times H^s(\Omega)$-regularity for some $s \in (\frac{1}{2}, 1]$ in the sense that, for smooth data $\mathbf{f}$ and $\widehat{\mathbf{u}}$, the solution $\mathbf{u}$ and p of (4.1) satisfy $\mathbf{u} \in [H^{1+s}(\Omega)]^d$, $p \in H^s(\Omega)$, and the following a priori estimate

$$\|\mathbf{u}\|_{1+s} + \|p\|_s \leq C(\|\mathbf{f}\|_{s-1} + \|\widehat{\mathbf{u}}\|_{s+\frac{1}{2},\Gamma}) \tag{4.2}$$

for some constant C independent of the parameter λ. The regularity estimate (4.2) can be found in [10, 7] for convex polygonal domain with $s = 1$ when $\lambda = \infty$. For large values of λ, one may heuristically apply the result for the Stokes (by viewing $\lambda^{-1} p$ as a given function) to obtain

$$\|\mathbf{u}\|_{1+s} + \|p\|_s \leq C(\|\mathbf{f}\|_{s-1} + \|\widehat{\mathbf{u}}\|_{s+\frac{1}{2},\Gamma} + \lambda^{-1}\|p\|_s),$$

which implies (4.2) when λ is sufficiently large.

The idea of weak Galerkin can be applied to the mixed formulation (1.4)-(1.5) for the linear elasticity problem (1.1)-(1.2). This application requires an additional finite element space that approximates the auxiliary variable p. More precisely, we introduce

$$W_h = \{q : \ q|_T \in P_{k-1}(T), \ T \in \mathcal{T}_h\}, \qquad W_h^0 = W_h \cap L_0^2(\Omega)$$

and the following bilinear forms

$$\begin{aligned} a(\mathbf{w}, \mathbf{v}) &= 2(\mu\varepsilon_w(\mathbf{w}), \varepsilon_w(\mathbf{v}))_h + s(\mathbf{w}, \mathbf{v}), \\ b(\mathbf{v}, q) &= (\nabla_w \cdot \mathbf{v}, q)_h, \\ d(p, q) &= \lambda^{-1}(p, q), \end{aligned}$$

where $\mathbf{w}, \mathbf{v} \in V_h$, $p, q \in W_h$, and

$$\begin{aligned} (\mu\varepsilon_w(\mathbf{w}), \varepsilon_w(\mathbf{v}))_h &= \sum_{T\in\mathcal{T}_h} (\mu\varepsilon_w(\mathbf{w}), \varepsilon_w(\mathbf{v}))_T, \\ (\nabla_w \cdot \mathbf{v}, q)_h &= \sum_{T\in\mathcal{T}_h} (\nabla_w \cdot \mathbf{v}, q)_T. \end{aligned}$$

Weak Galerkin Algorithm 2. *For a numerical solution of the linear elasticity problem (1.4)-(1.5), find* $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\} \in V_h$ *and* $p_h \in W_h$ *satisfying* $\mathbf{u}_b = Q_b \hat{\mathbf{u}}$ *on* Γ*, the compatibility condition* $(\lambda^{-1} p_h, 1) = \int_\Gamma \widehat{\mathbf{u}} \cdot \mathbf{n} ds$*, and the following equations*

$$a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}_0), \qquad \forall \mathbf{v} \in V_h^0, \tag{4.3}$$

$$b(\mathbf{u}_h, q) - d(p_h, q) = 0, \qquad \forall q \in W_h^0. \tag{4.4}$$

Lemma 4.1. *The weak Galerkin Algorithms 1 and 2 are equivalent in the sense that the solution* $\mathbf{u}_h$ *from (3.5) and (4.3)-(4.4) are identical to each other.*

Proof. Assume that $\tilde{\mathbf{u}}_h$ and $\tilde{p}_h$ solves (4.3)-(4.4). Note that the equation (4.4) can be rewritten as

$$(\nabla_w \cdot \tilde{\mathbf{u}}_h, q)_T - \lambda^{-1} (\tilde{p}_h, q)_T = 0, \qquad \forall q \in P_{k-1}(T).$$

Since $\nabla_w \cdot \tilde{\mathbf{u}}_h \in P_{k-1}(T)$, then $\tilde{p}_h$ can be solved from the above equation as

$$\tilde{p}_h = \lambda \nabla_w \cdot \tilde{\mathbf{u}}_h. \tag{4.5}$$

Substituting (4.5) into (4.3) yields

$$a(\tilde{\mathbf{u}}_h, \mathbf{v}) + \lambda b(\mathbf{v}, \nabla_w \cdot \tilde{\mathbf{u}}_h) = (\mathbf{f}, \mathbf{v}_0), \qquad \forall \mathbf{v} \in V_h^0.$$

Note that

$$a(\mathbf{w}, \mathbf{v}) + \lambda b(\mathbf{v}, \nabla_w \cdot \mathbf{w}) = a_s(\mathbf{w}, \mathbf{v}), \qquad \forall \mathbf{w},\ \mathbf{v} \in V_h.$$

Thus, we have

$$a_s(\tilde{\mathbf{u}}_h, \mathbf{v}) = (\mathbf{f}, \mathbf{v}_0), \qquad \forall \mathbf{v} \in V_h^0,$$

which is the same as (3.5). It then follows from the solution uniqueness that $\tilde{\mathbf{u}}_h$ is identical with the numerical solution arising from the weak Galerkin Algorithm 1.

A similar argument can be applied to show that if $\mathbf{u}_h$ solves (3.5), then the pair $(\mathbf{u}_h; \lambda \nabla_w \cdot \mathbf{u}_h)$ is a solution of (4.3)-(4.4). Details are left to interested readers as an exercise. ∎

The reformulation (1.4)-(1.5) of the elasticity problem as a generalized Stokes system with nonzero divergence constraint often leads to numerical approximations which are locking-free as $\lambda \to \infty$. Due to the equivalence between the primal formulation (3.5) and the mixed formulation (4.3)-(4.4) in the WG finite element method, the WG scheme in the primal formulation (3.5) is locking-free if the corresponding WG finite element method for (1.4)-(1.5) can be proved to be stable and accurate in terms of the parameter λ. The rest of the paper is devoted to a stability analysis for the mixed weak Galerkin finite element scheme (4.3)-(4.4), which in turn implies a locking-free convergence for the linear elasticity problem in the displacement formulation (3.5).

5. Stability Conditions. In the weak finite element space V_h, we introduce the following semi-norm

$$|\!|\!|\mathbf{v}|\!|\!| = \Big(\sum_{T \in \mathcal{T}_h} \|\varepsilon(\mathbf{v}_0)\|_T^2 + h_T^{-1} \|Q_b \mathbf{v}_0 - \mathbf{v}_b\|_{\partial T}^2 \Big)^{\frac{1}{2}}, \quad \mathbf{v} \in V_h. \tag{5.1}$$

Lemma 5.1. *The semi-norm $|||\cdot|||$, as defined by (5.1), is truly a norm in the linear space V_h^0.*

Proof. We shall only verify the positivity property for $|||\cdot|||$. To this end, assume that $|||\mathbf{v}|||=0$ for some $\mathbf{v}\in V_h^0$. It follows that $\varepsilon(\mathbf{v}_0)=0$ on each element $T\in\mathcal{T}_h$ and $Q_b\mathbf{v}_0=\mathbf{v}_b$ on ∂T. Thus, $\mathbf{v}_0\in \mathsf{RM}(T)$ satisfies $\mathbf{v}_0|_e=Q_b(\mathbf{v}_0|_e)=\mathbf{v}_b$, which implies the continuity of $\mathbf{v}_0$ in the whole domain Ω. The boundary condition $\mathbf{v}_b=\mathbf{0}$ on Γ implies that $\mathbf{v}_0=\mathbf{0}$ on Γ. From the second Korn's inequality (9.13), we obtain $\mathbf{v}_0=\mathbf{0}$ in Ω. The fact that $Q_b\mathbf{v}_0=\mathbf{v}_b$ on ∂T gives $\mathbf{v}_b=\mathbf{0}$ on ∂T. Thus, $\mathbf{v}\equiv\mathbf{0}$ in Ω, which completes the proof of the lemma. □

Lemma 5.2. *There exist positive constants α_1 and α_2 such that*

$$\alpha_1|||\mathbf{v}|||^2\le a(\mathbf{v},\mathbf{v})\le\alpha_2|||\mathbf{v}|||^2,\qquad\forall\ \mathbf{v}\in V_h^0. \tag{5.2}$$

Proof. From (2.2) and the integration by parts, we have

$$\begin{aligned}(\varepsilon_w(\mathbf{v}),\varphi)_T=&(\nabla\mathbf{v}_0,\frac{1}{2}(\varphi+\varphi^T))_T-\langle\mathbf{v}_0-\mathbf{v}_b,\frac{1}{2}(\varphi+\varphi^T)\mathbf{n}\rangle_{\partial T}\\ =&(\varepsilon(\mathbf{v}_0),\varphi)_T-\langle Q_b\mathbf{v}_0-\mathbf{v}_b,\frac{1}{2}(\varphi+\varphi^T)\mathbf{n}\rangle_{\partial T}\end{aligned} \tag{5.3}$$

for any $\varphi\in[P_{k-1}(T)]^{d\times d}$. Thus,

$$|(\varepsilon_w(\mathbf{v}),\varphi)_T|\le\|\varepsilon(\mathbf{v}_0)\|_T\|\varphi\|_T+\|Q_b\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}\|\frac{1}{2}(\varphi+\varphi^T)\mathbf{n}\|_{\partial T}. \tag{5.4}$$

From the trace inequality (9.2) and the usual inverse inequality we have

$$\begin{aligned}\|\frac{1}{2}(\varphi+\varphi^T)\mathbf{n}\|_{\partial T}&\le C\left(h_T^{-1}\|\varphi\|_T^2+h_T\|\nabla\varphi\|_T^2\right)^{\frac{1}{2}}\\ &\le Ch_T^{-\frac{1}{2}}\|\varphi\|_T.\end{aligned}$$

Substituting the above into (5.4) yields

$$|(\varepsilon_w(\mathbf{v}),\varphi)_T|\le\left(\|\varepsilon(\mathbf{v}_0)\|_T+Ch_T^{-\frac{1}{2}}\|Q_b\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}\right)\|\varphi\|_T,$$

which leads to

$$\|\varepsilon_w(\mathbf{v})\|_T^2\le 2\|\varepsilon(\mathbf{v}_0)\|_T^2+Ch_T^{-1}\|Q_b\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2. \tag{5.5}$$

By representing $(\varepsilon(\mathbf{v}_0),\varphi)_T$ in terms of other two terms in (5.3), we can derive the following analogy of (5.5)

$$\|\varepsilon(\mathbf{v}_0)\|_T^2\le 2\|\varepsilon_w(\mathbf{v})\|_T^2+Ch_T^{-1}\|Q_b\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2. \tag{5.6}$$

The left inequality in (5.2) is a result of (5.6) by summing over all the element $T\in\mathcal{T}_h$, and the right inequality can be obtained by summing (5.5) over $T\in\mathcal{T}_h$. □

In the finite element space W_h^0, we introduce the following norm

$$|||q|||_0^2=|q|_{0,h}^2+h^2\|\nabla q\|_{0,h}^2,\qquad q\in W_h,$$

where

$$|q|_{0,h}^2 = h \sum_{e\in\mathcal{E}_h^0} \|[\![q]\!]_e\|_e^2, \qquad \|\nabla q\|_{0,h}^2 = \sum_{T\in\mathcal{T}_h} \|\nabla q\|_T^2.$$

LEMMA 5.3. *There exists a constant $\beta > 0$ such that*

$$\sup_{\mathbf{v}\in V_h^0, \mathbf{v}\neq 0} \frac{b(\mathbf{v}, q)}{|\!|\!|\mathbf{v}|\!|\!|} \geq \beta |\!|\!|q|\!|\!|_0, \qquad \forall\ q \in W_h^0. \tag{5.7}$$

Proof. From the definition of the discrete weak divergence (2.1), we have

$$b(\mathbf{v}, q) = \sum_{T\in\mathcal{T}_h} (\nabla_w \cdot \mathbf{v}, q)_T = \sum_{T\in\mathcal{T}_h} \left\{-(\mathbf{v}_0, \nabla q)_T + \langle \mathbf{v}_b \cdot \mathbf{n}, q\rangle_{\partial T}\right\}. \tag{5.8}$$

By setting $\mathbf{v} = \mathbf{v}_q := \{-h^2\nabla q, h[\![q]\!]_e\mathbf{n}\}$ in (5.8), where $e \in \mathcal{E}_h^0$ and $\mathbf{n}$ is the unit outward normal direction on e, we arrive at

$$b(\mathbf{v}_q, q) = h^2 \sum_{T\in\mathcal{T}_h} \|\nabla q\|_T^2 + h \sum_{e\in\mathcal{E}_h^0} \|[\![q]\!]_e\|_e^2 = |\!|\!|q|\!|\!|_0^2. \tag{5.9}$$

Furthermore, it is not hard to see that there exists a constant C_0 such that

$$|\!|\!|\mathbf{v}_q|\!|\!| \leq C_0 |\!|\!|q|\!|\!|_0. \tag{5.10}$$

Thus, we have

$$\sup_{\mathbf{v}\in V_h^0, \mathbf{v}\neq 0} \frac{b(\mathbf{v}, q)}{|\!|\!|\mathbf{v}|\!|\!|} \geq \frac{b(\mathbf{v}_q, q)}{|\!|\!|\mathbf{v}_q|\!|\!|} = \frac{|\!|\!|q|\!|\!|_0^2}{|\!|\!|\mathbf{v}_q|\!|\!|} \geq C_0^{-1} |\!|\!|q|\!|\!|_0,$$

which proves the inf-sup condition (5.7). ∎

6. Preparation for Error Estimates. For each element $T \in \mathcal{T}_h$, let Q_0 be the L^2 projection onto $[P_k(T)]^d$. For each edge/face $e \subset \partial T$, recall that Q_b is the L^2 projection onto the finite element space on e. Denote by $Q_h\mathbf{u}$ the L^2 projection onto the weak finite element space V_h such that on each element $T \in \mathcal{T}_h$,

$$Q_h\mathbf{u} := \{Q_0\mathbf{u}, Q_b\mathbf{u}\}.$$

Furthermore, let $\mathcal{Q}_h$ and $\mathbf{Q}_h$ be the L^2 projection onto $P_{k-1}(T)$ and $[P_{k-1}(T)]^{d\times d}$, respectively.

LEMMA 6.1. *[22] For any $\mathbf{v} \in [H^1(\Omega)]^d$, the following identities hold true for the projection operators Q_h, $\boldsymbol{Q}_h$, and $\mathcal{Q}_h$*

$$\nabla_w \cdot (Q_h\mathbf{v}) = \mathcal{Q}_h(\nabla \cdot \mathbf{v}), \tag{6.1}$$

$$\nabla_w(Q_h\mathbf{v}) = \boldsymbol{Q}_h(\nabla\mathbf{v}). \tag{6.2}$$

Consequently, one has

$$\varepsilon_w(Q_h\mathbf{v}) = \boldsymbol{Q}_h\varepsilon(\mathbf{v}). \tag{6.3}$$

Proof. We outline a proof for (6.1) only; a similar approach can be adopted to prove (6.2). From (2.1) and the integration by parts we have

$$\begin{aligned}(\nabla_w \cdot (Q_h \mathbf{v}), \varphi)_T &= -(Q_0 \mathbf{v}, \nabla\varphi)_T + \langle Q_b \mathbf{v}\cdot\mathbf{n}, \varphi\rangle_{\partial T}\\ &= -(\mathbf{v}, \nabla\varphi)_T + \langle \mathbf{v}\cdot\mathbf{n}, \varphi\rangle_{\partial T}\\ &= (\nabla\cdot\mathbf{v}, \varphi)_T\\ &= (\mathcal{Q}_h(\nabla\cdot\mathbf{v}), \varphi)_T,\end{aligned}$$

for all $\varphi \in P_{k-1}(T)$. Since by construction $\nabla_w \cdot (Q_h \mathbf{v}) \in P_{k-1}(T)$, then (6.1) follows. □

Lemma 6.2. *Assume that $(\mathbf{w};\rho) \in [H^{1+\gamma}(\Omega)]^d \times H^1(\Omega)$, $\gamma > \frac{1}{2}$, satisfies the following equation*

$$2\nabla\cdot(\mu\varepsilon(\mathbf{w})) + \nabla\rho = -\boldsymbol{\eta}, \qquad \text{in } \Omega. \tag{6.4}$$

Let $(Q_h\mathbf{w};\mathcal{Q}_h\rho)$ be the L^2 projection of $(\mathbf{w};\rho)$ in the finite element space $V_h \times W_h$. Then, we have

$$2(\mu\varepsilon_w(Q_h\mathbf{w}), \varepsilon_w(\mathbf{v}))_h + (\nabla_w\cdot\mathbf{v}, \mathcal{Q}_h\rho)_h = (\boldsymbol{\eta}, \mathbf{v}_0) + \ell_{\mathbf{w}}(\mathbf{v}) + \theta_\rho(\mathbf{v}), \tag{6.5}$$

for all $\mathbf{v} \in V_h^0$, where $\ell_{\mathbf{w}}$ and θ_ρ are two functionals in the linear space V_h^0 given by

$$\ell_{\mathbf{w}}(\mathbf{v}) = 2\sum_{T\in\mathcal{T}_h} \langle \mathbf{v}_0 - \mathbf{v}_b, \mu(\varepsilon(\mathbf{w}) - \boldsymbol{Q}_h\varepsilon(\mathbf{w}))\mathbf{n}\rangle_{\partial T}, \tag{6.6}$$

$$\theta_\rho(\mathbf{v}) = \sum_{T\in\mathcal{T}_h} \langle \mathbf{v}_0 - \mathbf{v}_b, (\rho - \mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T}. \tag{6.7}$$

Proof. From (6.3), (2.2) and the integration by parts, we get

$$\begin{aligned}&2\mu(\varepsilon_w(Q_h\mathbf{w}), \varepsilon_w(\mathbf{v}))_T\\ =&2\mu(\mathbf{Q}_h\varepsilon(\mathbf{w}), \varepsilon_w(\mathbf{v}))_T\\ =&-2\mu(\mathbf{v}_0, \nabla\cdot(\mathbf{Q}_h\varepsilon(\mathbf{w})))_T + 2\mu\langle \mathbf{v}_b, \mathbf{Q}_h\varepsilon(\mathbf{w})\mathbf{n}\rangle_{\partial T}\\ =&2\mu(\nabla\mathbf{v}_0, \mathbf{Q}_h\varepsilon(\mathbf{w}))_T - 2\mu\langle \mathbf{v}_0 - \mathbf{v}_b, \mathbf{Q}_h\varepsilon(\mathbf{w})\mathbf{n}\rangle_{\partial T}\\ =&2\mu(\nabla\mathbf{v}_0, \varepsilon(\mathbf{w}))_T - 2\mu\langle \mathbf{v}_0 - \mathbf{v}_b, \mathbf{Q}_h\varepsilon(\mathbf{w})\mathbf{n}\rangle_{\partial T}.\end{aligned} \tag{6.8}$$

Next, we have from (2.1), the integration by parts, and $\sum_{T\in\mathcal{T}_h}\langle \mathbf{v}_b, \rho\mathbf{n}\rangle_{\partial T}=0$ that

$$
\begin{aligned}
(\nabla_w\cdot\mathbf{v}, \mathcal{Q}_h\rho)_h &= \sum_{T\in\mathcal{T}_h}(\nabla_w\cdot\mathbf{v}, \mathcal{Q}_h\rho)_T\\
&= \sum_{T\in\mathcal{T}_h}\{-(\mathbf{v}_0, \nabla(\mathcal{Q}_h\rho))_T+\langle \mathbf{v}_b, (\mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T}\}\\
&= \sum_{T\in\mathcal{T}_h}\{(\nabla\cdot\mathbf{v}_0, \mathcal{Q}_h\rho)_T-\langle \mathbf{v}_0-\mathbf{v}_b, (\mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T}\}\\
&= \sum_{T\in\mathcal{T}_h}\{(\nabla\cdot\mathbf{v}_0, \rho)_T-\langle \mathbf{v}_0-\mathbf{v}_b, (\mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T}\}\\
&= \sum_{T\in\mathcal{T}_h}\{-(\mathbf{v}_0, \nabla\rho)_T+\langle \mathbf{v}_0, \rho\mathbf{n}\rangle_{\partial T}-\langle \mathbf{v}_0-\mathbf{v}_b, (\mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T}\}\\
&= \sum_{T\in\mathcal{T}_h}\{-(\mathbf{v}_0, \nabla\rho)_T+\langle \mathbf{v}_0-\mathbf{v}_b, \rho\mathbf{n}\rangle_{\partial T}-\langle \mathbf{v}_0-\mathbf{v}_b, (\mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T}\}\\
&= -(\mathbf{v}_0, \nabla\rho)+\sum_{T\in\mathcal{T}_h}\langle \mathbf{v}_0-\mathbf{v}_b, (\rho-\mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T},
\end{aligned}
$$

which leads to

$$
(\mathbf{v}_0, \nabla\rho) = -(\nabla_w\cdot\mathbf{v}, \mathcal{Q}_h\rho)_h+\sum_{T\in\mathcal{T}_h}\langle \mathbf{v}_0-\mathbf{v}_b, (\rho-\mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T}. \tag{6.9}
$$

Now testing (6.4) by using the component $\mathbf{v}_0$ of $\mathbf{v}=\{\mathbf{v}_0, \mathbf{v}_b\}\in V_h^0$ yields

$$
-2(\nabla\cdot(\mu\varepsilon(\mathbf{w})), \mathbf{v}_0)-(\nabla\rho, \mathbf{v}_0)=(\boldsymbol{\eta}, \mathbf{v}_0). \tag{6.10}
$$

From the integration by parts, we can rewrite (6.10) as

$$
2\sum_{T\in\mathcal{T}_h}(\mu\varepsilon(\mathbf{w}), \nabla\mathbf{v}_0)_T-2\sum_{T\in\mathcal{T}_h}\langle \mu\varepsilon(\mathbf{w})\mathbf{n}, \mathbf{v}_0\rangle_{\partial T}-(\nabla\rho, \mathbf{v}_0)=(\boldsymbol{\eta}, \mathbf{v}_0). \tag{6.11}
$$

Substituting (6.8) and (6.9) into (6.11) yields

$$
\begin{aligned}
&2\mu\sum_{T\in\mathcal{T}_h}\{(\varepsilon_w(Q_h\mathbf{w}), \varepsilon_w(\mathbf{v}))_T+\langle \mathbf{v}_0-\mathbf{v}_b, \mathbf{Q}_h\varepsilon(\mathbf{w})\mathbf{n}\rangle_{\partial T}-\langle \mathbf{v}_0, \varepsilon(\mathbf{w})\mathbf{n}\rangle_{\partial T}\}\\
&+(\nabla_w\cdot\mathbf{v}, \mathcal{Q}_h\rho)_h-\sum_{T\in\mathcal{T}_h}\langle \mathbf{v}_0-\mathbf{v}_b, (\rho-\mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T}=(\boldsymbol{\eta}, \mathbf{v}_0),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&2\mu\sum_{T\in\mathcal{T}_h}\{(\varepsilon_w(Q_h\mathbf{w}), \varepsilon_w(\mathbf{v}))_T-\langle \mathbf{v}_0-\mathbf{v}_b, (\varepsilon(\mathbf{w})-\mathbf{Q}_h\varepsilon(\mathbf{w}))\mathbf{n}\rangle_{\partial T}-\langle \mathbf{v}_b, \varepsilon(\mathbf{w})\mathbf{n}\rangle_{\partial T}\}\\
&+(\nabla_w\cdot\mathbf{v}, \mathcal{Q}_h\rho)_h-\sum_{T\in\mathcal{T}_h}\langle \mathbf{v}_0-\mathbf{v}_b, (\rho-\mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T}=(\boldsymbol{\eta}, \mathbf{v}_0).
\end{aligned}
$$

Using the boundary condition $\mathbf{v}_b=0$ we obtain

$$
\begin{aligned}
&2(\mu\varepsilon_w(Q_h\mathbf{w}), \varepsilon_w(\mathbf{v}))_h+(\nabla_w\cdot\mathbf{v}, \mathcal{Q}_h\rho)_h\\
=&\sum_{T\in\mathcal{T}_h}\{2\langle \mathbf{v}_0-\mathbf{v}_b, \mu(\varepsilon(\mathbf{w})-\mathbf{Q}_h\varepsilon(\mathbf{w}))\mathbf{n}\rangle_{\partial T}+\langle \mathbf{v}_0-\mathbf{v}_b, (\rho-\mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T}\}\\
&+(\boldsymbol{\eta}, \mathbf{v}_0),
\end{aligned} \tag{6.12}
$$

which is precisely the equation (6.5). This completes the proof. $\square$

7. Error Estimate in a Discrete H^1-Norm. For the weak Galerkin finite element solution $(\mathbf{u}_h; p_h) = (\{\mathbf{u}_0, \mathbf{u}_b\}; p_h) \in V_h \times W_h$ arising from (4.3)-(4.4) for the linear elasticity problem (1.1)-(1.2), we define its error functions $\mathbf{e}_h$ and ζ_h by

$$\mathbf{e}_h = \{\mathbf{e}_0, \mathbf{e}_b\} = \{Q_0\mathbf{u} - \mathbf{u}_0, Q_b\mathbf{u} - \mathbf{u}_b\}, \tag{7.1}$$

$$\zeta_h = \mathcal{Q}_h p - p_h, \tag{7.2}$$

where $(\mathbf{u}; p)$ is the exact solution of the variational problem (1.4)-(1.5). It is clear that $\mathbf{e}_h \in V_h^0$ and $\zeta_h \in W_h^0$.

Lemma 7.1. *The error functions $\mathbf{e}_h$ and ζ_h defined in (7.1)-(7.2) satisfy the following error equations*

$$a(\mathbf{e}_h, \mathbf{v}) + b(\mathbf{v}, \zeta_h) = \varphi_{\mathbf{u},p}(\mathbf{v}), \qquad \forall \mathbf{v} \in V_h^0, \tag{7.3}$$

$$b(\mathbf{e}_h, q) - d(\zeta_h, q) = 0, \qquad \forall q \in W_h^0, \tag{7.4}$$

where

$$\varphi_{\mathbf{u},p}(\mathbf{v}) = \ell_{\mathbf{u}}(\mathbf{v}) + \theta_p(\mathbf{v}) + s(Q_h\mathbf{u}, \mathbf{v}).$$

Proof. Observe that the exact solution $(\mathbf{u}; p)$ satisfies the equation (6.4) with $\boldsymbol{\eta} = \mathbf{f}$. Thus, from Lemma 6.2 we have

$$2(\mu\varepsilon_w(Q_h\mathbf{u}), \varepsilon_w(\mathbf{v}))_h + (\nabla_w \cdot \mathbf{v}, \mathcal{Q}_h p)_h = (\mathbf{f}, \mathbf{v}_0) + \ell_{\mathbf{u}}(\mathbf{v}) + \theta_p(\mathbf{v}),$$

which leads to

$$a(Q_h\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \mathcal{Q}_h p) = (\mathbf{f}, \mathbf{v}_0) + \ell_{\mathbf{u}}(\mathbf{v}) + \theta_p(\mathbf{v}) + s(Q_h\mathbf{u}, \mathbf{v}). \tag{7.5}$$

Subtracting (4.3) from (7.5) gives the equation (7.3).

To derive (7.4), using (6.1) we have for any $q \in W_h$

$$\begin{aligned} (\nabla_w \cdot (Q_h\mathbf{u}), q) - \lambda^{-1}(\mathcal{Q}_h p, q) &= (\mathcal{Q}_h(\nabla \cdot \mathbf{u}), q) - \lambda^{-1}(\mathcal{Q}_h p, q) \\ &= (\nabla \cdot \mathbf{u}, q) - \lambda^{-1}(p, q) = 0, \end{aligned} \tag{7.6}$$

where we have used (1.5) in the second line. The difference of (7.6) and (4.4) yields (7.4). $\square$

We are now in a position to derive an error estimate for the weak Galerkin finite element approximation $(\mathbf{u}_h; p_h)$.

Theorem 7.2. *Let the solution of (1.4)-(1.5) be sufficiently smooth such that $(\mathbf{u}; p) \in [H^{k+1}(\Omega)]^d \times H^k(\Omega)$ for some $k \geq 1$. For the weak Galerkin finite element solution $(\mathbf{u}_h; p_h) \in V_h \times W_h$ arising from (4.3)-(4.4), we have*

$$|||Q_h\mathbf{u} - \mathbf{u}_h||| + \lambda^{-\frac{1}{2}}\|\mathcal{Q}_h p - p_h\| + |||\mathcal{Q}_h p - p_h|||_0 \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k), \tag{7.7}$$

where C is a generic constant independent of $(\mathbf{u}; p)$. Consequently, the following error estimate holds true

$$|||\mathbf{u} - \mathbf{u}_h||| + \lambda^{-\frac{1}{2}}\|p - p_h\| + |||p - p_h|||_0 \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k). \tag{7.8}$$

Proof. By choosing $\mathbf{v} = \mathbf{e}_h$ in (7.3) and $q = \zeta_h$ in (7.4) we have

$$a(\mathbf{e}_h, \mathbf{e}_h) + \lambda^{-1}\|\zeta_h\|^2 = \varphi_{\mathbf{u},p}(\mathbf{e}_h).$$

By applying Lemma 9.11 to the term $\varphi_{\mathbf{u},p}(\mathbf{e}_h)$ we arrive at

$$a(\mathbf{e}_h, \mathbf{e}_h) + \lambda^{-1}\|\zeta_h\|^2 \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k)|||\mathbf{e}_h|||.$$

Next, using Lemma 5.2 and the above estimate we obtain

$$\alpha_1|||\mathbf{e}_h|||^2 + \lambda^{-1}\|\zeta_h\|^2 \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k)|||\mathbf{e}_h|||. \tag{7.9}$$

To derive an error estimate for the "pressure" variable p in a λ-independent norm, we use the *inf-sup* condition (5.7) to obtain

$$\beta|||\zeta_h|||_0 \leq \sup_{\mathbf{v}\in V_h^0, \mathbf{v}\neq 0} \frac{b(\mathbf{v}, \zeta_h)}{|||\mathbf{v}|||}. \tag{7.10}$$

From (7.3)

$$b(\mathbf{v}, \zeta_h) = -a(\mathbf{e}_h, \mathbf{v}) + \varphi_{\mathbf{u},p}(\mathbf{v}).$$

Thus, it follows from Lemma 5.2, the error estimate (7.9), and Lemma 9.11 that

$$\begin{aligned} |b(\mathbf{v}, \zeta_h)| &\leq \alpha_2|||\mathbf{e}_h|||\ |||\mathbf{v}||| + |\varphi_{\mathbf{u},p}(\mathbf{v})| \\ &\leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k)\ |||\mathbf{v}|||. \end{aligned}$$

Substituting the above estimate into (7.10) yields

$$|||\zeta_h|||_0 \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k). \tag{7.11}$$

Combining (7.9) with (7.11) gives rise to the error estimate (7.7). Finally, (7.8) stems from the usual triangle inequality, the estimate (7.7) and the error estimate for L^2 projections. ∎

8. Error Estimate in L^2. As usual, we use the duality argument to derive an L^2 error estimate for the weak Galerkin finite element method. To this end, consider the problem of seeking $\mathbf{\Phi} \in [H^1(\Omega)]^d$ and $\xi \in L_0^2(\Omega)$ satisfying

$$-\nabla \cdot (2\mu\varepsilon(\mathbf{\Phi}) + \xi I) = \mathbf{e}_0, \qquad \text{in } \Omega, \tag{8.1}$$

$$\nabla \cdot \mathbf{\Phi} - \lambda^{-1}\xi = 0, \qquad \text{in } \Omega, \tag{8.2}$$

$$\mathbf{\Phi} = \mathbf{0}, \qquad \text{on } \Gamma. \tag{8.3}$$

Assume that the dual problem (8.1)-(8.3) has the $[H^{1+s}(\Omega)]^d \times H^s(\Omega)$-regularity with $\frac{1}{2} < s \leq 1$ in the sense that the solution $(\mathbf{\Phi}; \xi) \in [H^{1+s}(\Omega)]^d \times H^s(\Omega)$ and satisfies the following a priori estimate:

$$\|\mathbf{\Phi}\|_{s+1} + \|\xi\|_s \leq C\|\mathbf{e}_0\|. \tag{8.4}$$

THEOREM 8.1. *Assume that the solutions of (4.1) are sufficiently smooth such that $(\mathbf{u}; p) \in [H^{k+1}(\Omega)]^d \times H^k(\Omega)$ for some integer $k \geq 1$. Let $(\mathbf{u}_h; p_h) \in V_h \times W_h$*

be the corresponding weak Galerkin finite element solution arising from (4.3)-(4.4). Then, under the regularity assumption (8.4), there exists a constant C, such that

$$\|Q_0\mathbf{u} - \mathbf{u}_0\| \le Ch^{k+s}\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big). \tag{8.5}$$

Moreover, it follows from the triangle inequality and the error estimate for the L^2 projection that

$$\|\mathbf{u} - \mathbf{u}_0\| \le Ch^{k+s}\big(\|\mathbf{u}\|_{k+1} + \|p\|_k\big). \tag{8.6}$$

Proof. Note that the solution $(\mathbf{\Phi};\xi)$ of (8.1)-(8.3) satisfies (6.4) with $\boldsymbol{\eta} = \mathbf{e}_0$. Thus, using Lemma 6.2, namely the identity (6.5), we have

$$2(\mu\varepsilon_w(Q_h\mathbf{\Phi}), \varepsilon_w(\mathbf{v}))_h + (\nabla_w\cdot\mathbf{v}, \mathcal{Q}_h\xi)_h = (\mathbf{e}_0, \mathbf{v}_0) + \ell_{\mathbf{\Phi}}(\mathbf{v}) + \theta_\xi(\mathbf{v}) \tag{8.7}$$

for all $\mathbf{v} \in V_h^0$.

By choosing $\mathbf{v} = \mathbf{e}_h$ in (8.7) we obtain

$$\|\mathbf{e}_0\|^2 = a(Q_h\mathbf{\Phi}, \mathbf{e}_h) + b(\mathbf{e}_h, \mathcal{Q}_h\xi) - \varphi_{\mathbf{\Phi},\xi}(\mathbf{e}_h), \tag{8.8}$$

where

$$\varphi_{\mathbf{\Phi},\theta}(\mathbf{e}_h) = \theta_\xi(\mathbf{e}_h) + \ell_{\mathbf{\Phi}}(\mathbf{e}_h) + s(Q_h\mathbf{\Phi}, \mathbf{e}_h).$$

Using (7.4) we have

$$b(\mathbf{e}_h, \mathcal{Q}_h\xi) - d(\mathcal{Q}_h\xi, \zeta_h) = 0. \tag{8.9}$$

From (6.1) and (8.2)

$$\begin{aligned} b(Q_h\mathbf{\Phi}, \zeta_h) &= \sum_{T\in\mathcal{T}_h} (\nabla_w\cdot Q_h\mathbf{\Phi}, \zeta_h)_T \\ &= \sum_{T\in\mathcal{T}_h} (\mathcal{Q}_h(\nabla\cdot\mathbf{\Phi}), \zeta_h)_T \\ &= d(\mathcal{Q}_h\xi, \zeta_h). \end{aligned}$$

Thus, by substituting the above into (8.9)

$$b(\mathbf{e}_h, \mathcal{Q}_h\xi) = b(Q_h\mathbf{\Phi}, \zeta_h).$$

Combining the last equation with (8.8) we obtain

$$\|\mathbf{e}_0\|^2 = a(Q_h\mathbf{\Phi}, \mathbf{e}_h) + b(Q_h\mathbf{\Phi}, \zeta_h) - \varphi_{\mathbf{\Phi},\xi}(\mathbf{e}_h),$$

which, together with the error equation (7.3), leads to

$$\|\mathbf{e}_0\|^2 = \varphi_{\mathbf{u},p}(Q_h\mathbf{\Phi}) - \varphi_{\mathbf{\Phi},\xi}(\mathbf{e}_h). \tag{8.10}$$

The rest of the proof shall deal with the two terms on the right-hand side of (8.10). The second term $\varphi_{\bar{\mathbf{u}},q}(\mathbf{e}_h)$ can be estimated by using Lemma 9.11, the error estimate (7.7), and the regularity assumption (8.4) as follows

$$\begin{aligned} |\varphi_{\mathbf{\Phi},\xi}(\mathbf{e}_h)| &\le Ch^s(\|\mathbf{\Phi}\|_{s+1} + \|\xi\|_s)\ |\!|\!|\mathbf{e}_h|\!|\!| \\ &\le Ch^{k+s}(\|\mathbf{u}\|_{k+1} + \|p\|_k)\ \|\mathbf{e}_0\|. \end{aligned} \tag{8.11}$$

The estimate for the first term $\varphi_{\mathbf{u},p}(Q_h\mathbf{\Phi})$ is a bit complicated, for which we detail as follows.

(i) For the term $s(Q_h\mathbf{u},Q_h\mathbf{\Phi})$ in $\varphi_{\mathbf{u},p}(Q_h\mathbf{\Phi})$, from the Cauchy-Schwarz inequality, (9.1) and (9.3), we have

$$
\begin{aligned}
|s(Q_h\mathbf{u},Q_h\mathbf{\Phi})| =&\Big|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_b(Q_0\mathbf{u})-Q_b\mathbf{u}, Q_b(Q_0\mathbf{\Phi})-Q_b\mathbf{\Phi}\rangle_{\partial T}\Big|\\
\le&\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\mathbf{u}-\mathbf{u}\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\mathbf{\Phi}-\mathbf{\Phi}\|_{\partial T}^2\Big)^{\frac12}\\
\le& Ch^{k+s}\|\mathbf{u}\|_{k+1}\ \|\mathbf{\Phi}\|_{s+1}.
\end{aligned}\tag{8.12}
$$

(ii) For the term $\ell_{\mathbf{u}}(Q_h\mathbf{\Phi})$, we have from the orthogonality of Q_b and the boundary condition (8.3) that

$$
\begin{aligned}
&2\mu\sum_{T\in\mathcal{T}_h}\langle Q_b\mathbf{\Phi}-\mathbf{\Phi},(\varepsilon(\mathbf{u})-\mathbf{Q}_h(\varepsilon(\mathbf{u})))\mathbf{n}\rangle_{\partial T}\\
=&2\mu\sum_{T\in\mathcal{T}_h}\langle Q_b\mathbf{\Phi}-\mathbf{\Phi},\varepsilon(\mathbf{u})\mathbf{n}\rangle_{\partial T}=0.
\end{aligned}\tag{8.13}
$$

Thus, it follows from the Cauchy-Schwarz inequality, (9.1), (9.3) and (9.4) that

$$
\begin{aligned}
|\ell_{\mathbf{u}}(Q_h\mathbf{\Phi})| =&\left|2\mu\sum_{T\in\mathcal{T}_h}\langle Q_0\mathbf{\Phi}-Q_b\mathbf{\Phi},(\varepsilon(\mathbf{u})-\mathbf{Q}_h\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}\right|\\
=&\left|2\mu\sum_{T\in\mathcal{T}_h}\langle Q_0\mathbf{\Phi}-\mathbf{\Phi},(\varepsilon(\mathbf{u})-\mathbf{Q}_h\varepsilon(\mathbf{u}))\mathbf{n}\rangle_{\partial T}\right|\\
\le& 2\mu\Big(\sum_{T\in\mathcal{T}_h} h_T\|\varepsilon(\mathbf{u})-\mathbf{Q}_h\varepsilon(\mathbf{u})\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\mathbf{\Phi}-\mathbf{\Phi}\|_{\partial T}^2\Big)^{\frac12}\\
\le& Ch^{k+s}\|\mathbf{u}\|_{k+1}\ \|\mathbf{\Phi}\|_{s+1}.
\end{aligned}\tag{8.14}
$$

(iii) As to the term $\theta_p(Q_h\mathbf{\Phi})$, we again use the orthogonality of Q_b and the boundary condition (8.3) combined with the Cauchy-Schwarz inequality, (9.1), (9.3) and (9.5) to obtain

$$
\begin{aligned}
|\theta_p(Q_h\mathbf{\Phi})| =&\left|\sum_{T\in\mathcal{T}_h}\langle Q_0\mathbf{\Phi}-Q_b\mathbf{\Phi},(p-\mathcal{Q}_hp)\mathbf{n}\rangle_{\partial T}\right|\\
=&\left|\sum_{T\in\mathcal{T}_h}\langle Q_0\mathbf{\Phi}-\mathbf{\Phi},(p-\mathcal{Q}_hp)\mathbf{n}\rangle_{\partial T}\right|\\
\le&\Big(\sum_{T\in\mathcal{T}_h} h_T\|p-\mathcal{Q}_hp\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\mathbf{\Phi}-\mathbf{\Phi}\|_{\partial T}^2\Big)^{\frac12}\\
\le& Ch^{k+s}\|p\|_k\ \|\mathbf{\Phi}\|_{s+1}.
\end{aligned}\tag{8.15}
$$

By combining the three estimates (8.12), (8.14), and (8.15) we arrive at

$$
\begin{aligned}
|\varphi_{\mathbf{u},p}(Q_h\mathbf{\Phi})| \le& Ch^{k+s}(\|\mathbf{u}\|_{k+1}+\|p\|_k)\|\mathbf{\Phi}\|_{s+1}\\
\le& Ch^{k+s}(\|\mathbf{u}\|_{k+1}+\|p\|_k)\|\mathbf{e}_0\|,
\end{aligned}\tag{8.16}
$$

where we have used the regularity assumption (8.4) in the second line. Finally, by substituting (8.11) and (8.16) into (8.10) we obtain the desired error estimate of (8.5). This completes the derivation of the L^2 error estimate. $\square$

9. Supporting Tools and Inequalities. In this section, we present some technical inequalities that support the error analysis established in previous sections.

Recall that $\mathcal{T}_h$ is a shape-regular finite element partition of Ω. There exists a constant $C > 0$ such that, for any side $e \subset T$ with $T \in \mathcal{T}_h$, the following trace inequality holds true [21]

$$\|\phi\|_e^2 \leq C\big(h_T^{-1}\|\phi\|_T^2 + h_T\|\nabla\phi\|_T^2\big), \qquad \forall \phi \in H^1(T), \tag{9.1}$$

where h_T is the size of T. Furthermore, in the polynomial space $P_j(T)$, $j \geq 0$, we have from the inverse inequality that

$$\|\phi\|_e^2 \leq Ch_T^{-1}\|\phi\|_T^2, \qquad \forall \phi \in P_j(T),\ T \in \mathcal{T}_h. \tag{9.2}$$

LEMMA 9.1. *[21] Assume that $\mathcal{T}_h$ is a finite element partition of Ω satisfying the shape regularity assumption as defined in [21]. Let $k \geq 1$ be the order of the finite element method and $1 \leq r \leq k$. Let $\mathbf{w} \in [H^{r+1}(\Omega)]^d$, $\rho \in H^r(\Omega)$ and $0 \leq m \leq 1$. There holds*

$$\sum_{T\in\mathcal{T}_h} h_T^{2m}\|\mathbf{w} - Q_0\mathbf{w}\|_{T,m}^2 \leq Ch^{2(r+1)}\|\mathbf{w}\|_{r+1}^2, \tag{9.3}$$

$$\sum_{T\in\mathcal{T}_h} h_T^{2m}\|\varepsilon(\mathbf{w}) - \boldsymbol{Q}_h\varepsilon(\mathbf{w})\|_{T,m}^2 \leq Ch^{2r}\|\mathbf{w}\|_{r+1}^2, \tag{9.4}$$

$$\sum_{T\in\mathcal{T}_h} h_T^{2m}\|\rho - \mathcal{Q}_h\rho\|_{T,m}^2 \leq Ch^{2r}\|\rho\|_r^2. \tag{9.5}$$

9.1. Korn's inequality. Korn's inequality is a fundamental tool in the study of elasticity equations. The inequality can be found in many existing literature, see [6, 11, 16, 3] for example. For convenience, we provide a summary here for this useful inequality.

THEOREM 9.2. *(Korn's Inequality) Assume that the domain Ω is open bounded with Lipschitz continuous boundary. Then, there exists a constant C such that*

$$\|\nabla\mathbf{v}\|_0 \leq C\left(\|\varepsilon(\mathbf{v})\|_0 + \|\mathbf{v}\|_0\right), \qquad \forall\ \mathbf{v} \in [H^1(\Omega)]^d. \tag{9.6}$$

Proof. A proof can be given by using the following inequality [11]

$$\|\phi\|_0 \leq C(\|\nabla\phi\|_{-1} + \|\phi\|_{-1}), \qquad \forall\ \phi \in L^2(\Omega). \tag{9.7}$$

To this end, for any $\mathbf{v} \in [H^1(\Omega)]^d$, it is not hard to check that

$$\partial_j\partial_k v_i = \partial_j\varepsilon_{ik}(\mathbf{v}) + \partial_k\varepsilon_{ij}(\mathbf{v}) - \partial_i\varepsilon_{jk}(\mathbf{v}). \tag{9.8}$$

It follows that $\nabla v_i \in H^{-1}(\Omega)$ if $\varepsilon(\mathbf{v}) \in [L^2(\Omega)]^{d\times d}$. Moreover, from (9.7) and (9.8) one has

$$\begin{aligned} \|\nabla\mathbf{v}\|_0 &\leq C(\|\nabla\varepsilon(\mathbf{v})\|_{-1} + \|\nabla\mathbf{v}\|_{-1}) \\ &\leq C(\|\varepsilon(\mathbf{v})\|_0 + \|\mathbf{v}\|_0), \end{aligned}$$

which is the Korn's inequality (9.6). $\square$

The following is a characterization of the space of rigid motions as the kernel of the strain tensor operator.

LEMMA 9.3. *Let Ω be an open bounded and connected domain in $\mathbb{R}^d$. The space of rigid motions $\mathsf{RM}(\Omega)$ is identical to the kernel of the strain tensor operator; i.e., for $\mathbf{v} \in [H^1(\Omega)]^d$, there holds $\varepsilon(\mathbf{v}) = 0$ if and only if $\mathbf{v} \in \mathsf{RM}(\Omega)$.*

Proof. For any $\mathbf{v} \in \mathsf{RM}(\Omega)$, there exist $\mathbf{a} \in \mathbb{R}^d$ and a skew-symmetric $d \times d$ matrix η such that $\mathbf{u} = \mathbf{a} + \eta\mathbf{x}$. It is easy to check that $\varepsilon(\mathbf{v}) = 0$.

For any $\mathbf{v}$ satisfying $\varepsilon(\mathbf{v}) = 0$, we have from (9.8) that $\partial_j \partial_k v_i = 0$, and hence $\mathbf{v} \in [P_1(\Omega)]^d$. Thus, there exist $\mathbf{a} \in \mathbb{R}^{\mathbf{d}}$ and $\eta \in \mathbb{R}^{d \times d}$ such that $\mathbf{v} = \mathbf{a} + \eta\mathbf{x}$. Since $\varepsilon(\mathbf{v}) = 0$, we get

$$0 = \varepsilon(\mathbf{v}) = \varepsilon(\mathbf{a} + \eta\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2}}(\nabla(\mathbf{a} + \eta\mathbf{x}) + \nabla(\mathbf{a} + \eta\mathbf{x})^{\mathbf{T}}) = \frac{\mathbf{1}}{\mathbf{2}}(\eta + \eta^{\mathbf{T}}).$$

It follows that $\eta^T = -\eta$, which means that η is skew-symmetric, and hence $\mathbf{v} \in \mathsf{RM}(\Omega)$. $\square$

LEMMA 9.4. *Let $\mathbf{x}_i$, $i = 0, \ldots, d-1$, be d-points in Ω which form a $(d-1)-$dimensional hyperplane. If $\mathbf{v} \in \mathsf{RM}(\Omega)$ satisfies $\mathbf{v}(\mathbf{x}_i) = \mathbf{0}$, $i = 0, \ldots, d-1$, then we must have $\mathbf{v} \equiv \mathbf{0}$.*

Proof. Let $\mathbf{v} = \mathbf{a} + \eta\mathbf{x}$ be the rigid motion with $\mathbf{v}(\mathbf{x}_i) = \mathbf{0}$. Thus,

$$\eta(\mathbf{x}_i - \mathbf{x}_0) = \mathbf{0}, \qquad i = 1, \ldots, d-1. \tag{9.9}$$

Note that the set of vectors $\{\mathbf{x}_i - \mathbf{x}_0\}_{i=1}^{d-1}$ is linearly independent since they form a basis of a $(d-1)-$dimensional subspace of $\mathbb{R}^d$.

For $d = 2$, the skew-symmetric matrix η is either zero or invertible. From the equation (9.9), we see that η can be nothing except $\eta = 0$. For $d = 3$, the matrix η has eigenvalue $\lambda_0 = 0$. If $\eta \neq 0$, then the eigen-space corresponding to the eigenvalue $\lambda_0 = 0$ must have dimension 1. But the equation (9.9) indicates that the dimension for this eigen-space is no less than 2. Consequently, we must have $\eta = 0$. Finally, from $\mathbf{v}(\mathbf{x}_0) = \mathbf{0}$ we have $\mathbf{a} = 0$. This shows that $\mathbf{v} \equiv 0$. $\square$

THEOREM 9.5. *(Second Korn's Inequality) Assume that the domain Ω is connected, open bounded with Lipschitz continuous boundary. Let $\Phi : H^1(\Omega) \to \mathbb{R}^+$ be a semi-norm on $H^1(\Omega)$ satisfying*

$$\Phi(\mathbf{v}) = 0 \textit{ and } \mathbf{v} \in \mathsf{RM}(\Omega) \Rightarrow \mathbf{v} = 0.$$

Then, there exists a constant C such that

$$\|\mathbf{v}\|_1 \le C(\|\varepsilon(\mathbf{v})\|_0 + \Phi(\mathbf{v})), \tag{9.10}$$

for all $\mathbf{v} \in [H^1(\Omega)]^d$.

Proof. We verify the inequality (9.10) by a contradiction argument. To this end, assume that (9.10) does not hold true. Then, for each integer n, there exists $\mathbf{v}_n \in [H^1(\Omega)]^d$ such that

$$\|\mathbf{v}_n\|_1 > n(\|\varepsilon(\mathbf{v}_n)\|_0 + \Phi(\mathbf{v}_n)). \tag{9.11}$$

We may assume $\|\mathbf{v}_n\|_1 = 1$, and hence, there exists a subsequence $\{\mathbf{v}_{n_k}\}$ which is weakly convergent in H^1 and strongly convergent in $L^2(\Omega)$. From (9.11), we have

$$\|\varepsilon(\mathbf{v}_n)\|_0 + \Phi(\mathbf{v}_n) < n^{-1}.$$

Thus,

$$\|\varepsilon(\mathbf{v}_n)\|_0 + \Phi(\mathbf{v}_n) \to 0,$$

which, together with Korn's inequality (9.6), implies that $\{\mathbf{v}_{n_k}\}$ is a Cauchy sequence in $H^1(\Omega)$. Hence, there exists a function $\mathbf{v} \in H^1(\Omega)$ such that

$$\mathbf{v}_{n_k} \to \mathbf{v} \qquad \text{strongly in } H^1(\Omega).$$

Moreover, we have

$$\|\varepsilon(\mathbf{v})\|_0 + \Phi(\mathbf{v}) = \lim_{k\to\infty} (\|\varepsilon(\mathbf{v}_{n_k})\|_0 + \Phi(\mathbf{v}_{n_k})) = 0.$$

Thus,

$$\mathbf{v} \in \mathsf{RM}(\Omega) \text{ and } \Phi(\mathbf{v}) = 0,$$

which leads to $\mathbf{v} = 0$. This is a contradiction to the assumption that

$$\|\mathbf{v}\|_1 = \lim_{k\to\infty} \|\mathbf{v}_{n_k}\|_1 = 1.$$

This completes the proof. □

The following are two particular cases of the seminorm $\Phi(\cdot)$ that satisfy the conditions of Theorem 9.5.

COROLLARY 9.6. *Assume that the domain Ω is connected, open bounded with Lipschitz continuous boundary. Let $D \subset \Omega$ be a subdomain of Ω, and $Q_{D,RM}$ be the L^2 projection from $[L^2(\Omega)]^d$ onto the space of rigid motions $\mathsf{RM}(D)$. Then, there exists a constant C such that*

$$\|\mathbf{v}\|_1 \le C(\|\varepsilon(\mathbf{v})\|_0 + \|Q_{D,RM}\mathbf{v}\|_{0,D}), \tag{9.12}$$

for all $\mathbf{v} \in [H^1(\Omega)]^d$.

Proof. Define $\Phi(\mathbf{v}) = \|Q_{D,RM}\mathbf{v}\|_{0,D}$ which is clearly a semi-norm in $[H^1(\Omega)]^d$. The estimate (9.12) stems from Theorem 9.5 if the conditions of the theorem are verified for this semi-norm. To this end, for any $\mathbf{v}$ satisfying $\varepsilon(\mathbf{v}) = 0$, we have from Lemma 9.3 that $\mathbf{v} \in \mathsf{RM}(\Omega)$, and hence $Q_{D,RM}\mathbf{v} = \mathbf{v}$ as $D \subset \Omega$. If, in addition, $\Phi(\mathbf{v}) = 0$, then $\|\mathbf{v}\|_{0,D} = \|Q_{D,RM}\mathbf{v}\|_{0,D} = 0$. Hence, $\mathbf{v} \equiv 0$ in Ω. □

COROLLARY 9.7. *Assume that the domain Ω is connected, open bounded with Lipschitz continuous boundary. Let $\Gamma_1 \subset \partial\Omega$ be a nontrivial portion of the boundary $\partial\Omega$ with dimension $d-1$. For any fixed real number $1 \le p < \infty$, there exists a constant C such that*

$$\|\mathbf{v}\|_1 \le C(\|\varepsilon(\mathbf{v})\|_0 + \|\mathbf{v}\|_{L^p(\Gamma_1)}), \tag{9.13}$$

for all $\mathbf{v} \in [H^1(\Omega)]^d$.

Proof. In $[H^1(\Omega)]^d$, we define a semi-norm by

$$\Phi(\mathbf{v}) := \left(\int_{\Gamma_1} |\mathbf{v}|^p ds\right)^{\frac{1}{p}}, \qquad \mathbf{v} \in [H^1(\Omega)]^d.$$

We claim that this semi-norm satisfies the conditions of Theorem 9.5. In fact, if $\mathbf{v}$ satisfies $\Phi(\mathbf{v}) = 0$, then we have $\mathbf{v} = 0$ on Γ_1. Hence $\mathbf{v} = 0$ on d-points of a hyperplane of dimension $d-1$. If, in addition, $\mathbf{v} \in \mathsf{RM}(\Omega)$, then from Lemma 9.4 we must have $\mathbf{v} \equiv 0$ on Ω. ∎

9.2. Some technical inequalities. For any $\mathbf{x}_0 \in \mathbb{R}^d$, denote by $B(\mathbf{x}_0, r)$ the d-ball centered at $\mathbf{x}_0$ with radius r. The unit d-ball centered at the origin is denoted by $\widehat{B}$, called the reference d-ball. The reference d-ball can be identified with the d-ball $B(\mathbf{x}_0, r)$ using the following affine map

$$\mathbf{x} = F(\widehat{\mathbf{x}}) := \mathbf{x}_0 + r\widehat{\mathbf{x}}.$$

The inverse of the affine map F is given by

$$\widehat{\mathbf{x}} = F^{-1}(\mathbf{x}) := (\mathbf{x} - \mathbf{x}_0)/r.$$

Any function $\phi = \phi(\mathbf{x})$ on $B(\mathbf{x}_0, r)$ defines a function on the reference d-ball as follows

$$\widehat{\phi}(\widehat{\mathbf{x}}) = \phi(F(\widehat{\mathbf{x}})),$$

which shall be denoted as

$$\widehat{\phi}(\widehat{\mathbf{x}}) = \widehat{\phi(\mathbf{x})}.$$

It is clear that $\phi \in \mathsf{RM}(B(\mathbf{x}_0, r))$ if and only if $\widehat{\phi} \in \mathsf{RM}(\widehat{B})$.

Lemma 9.8. *Let $Q_{B(\mathbf{x}_0,r)}$ and $\widehat{Q}_{\widehat{B}}$ be the L^2 projections onto the space of rigid motions $\mathsf{RM}(B(\mathbf{x}_0, r))$ and $\mathsf{RM}(\widehat{B})$, respectively. Then, for any $\mathbf{v} \in [L^2(B(\mathbf{x}_0, r))]^d$ we have*

$$\widehat{Q_{B(\mathbf{x}_0,r)}\mathbf{v}} = \widehat{Q}_{\widehat{B}}\widehat{\mathbf{v}}. \tag{9.14}$$

Proof. From the definition of the L^2 projection, we have for any $\phi \in \mathsf{RM}(B(\mathbf{x}_0, r))$

$$\int_{B(\mathbf{x}_0,r)} Q_{B(\mathbf{x}_0,r)}\mathbf{v}\ \phi dx = \int_{B(\mathbf{x}_0,r)} \mathbf{v}\ \phi dx = r^d \int_{\widehat{B}} \widehat{\mathbf{v}}\ \widehat{\phi} d\widehat{x} = r^d \int_{\widehat{B}} \widehat{Q}_{\widehat{B}}\widehat{\mathbf{v}}\ \widehat{\phi} d\widehat{x}.$$

Also by changing the domain from $B(\mathbf{x}_0, r)$ to $\widehat{B}$,

$$\int_{B(\mathbf{x}_0,r)} Q_{B(\mathbf{x}_0,r)}\mathbf{v}\ \phi dx = r^d \int_{\widehat{B}} \widehat{Q_{B(\mathbf{x}_0,r)}\mathbf{v}}\ \widehat{\phi} d\widehat{x}.$$

It follows that

$$\int_{\widehat{B}} \widehat{Q_{B(\mathbf{x}_0,r)}\mathbf{v}}\ \widehat{\phi} d\widehat{x} = \int_{\widehat{B}} \widehat{Q}_{\widehat{B}}\widehat{\mathbf{v}}\ \widehat{\phi} d\widehat{x},$$

which leads to (9.14). $\square$

Lemma 9.9. *There exists a constant C such that, for any $\mathbf{w} \in [H^1(B(\mathbf{x}_0, r))]^d$, we have*

$$\|\mathbf{w} - Q_{B(\mathbf{x}_0,r)}\mathbf{w}\|_{0,B(\mathbf{x}_0,r)} \leq Cr\|\varepsilon(\mathbf{w})\|_{0,B(\mathbf{x}_0,r)}.$$

Proof. Let $\mathbf{w}^\perp = \mathbf{w} - Q_{B(\mathbf{x}_0,r)}\mathbf{w}$. It follows from Lemma 9.8 that

$$\widehat{\mathbf{w}^\perp} = \widehat{\mathbf{w}} - \widehat{Q}_{\widehat{B}}\widehat{\mathbf{w}}.$$

It is also easy to see the following identities

$$\widehat{Q}_{\widehat{B}}\widehat{\mathbf{w}^\perp} = 0, \quad \varepsilon(\widehat{\mathbf{w}^\perp}) = \varepsilon(\widehat{\mathbf{w}}). \tag{9.15}$$

By mapping to the reference d-ball, we have

$$\|\mathbf{w}^\perp\|^2_{0,B(\mathbf{x}_0,r)} = r^d\|\widehat{\mathbf{w}^\perp}\|^2_{0,\widehat{B}}.$$

Using the second Korn's inequality (9.12) with $\Omega = \widehat{B}$ and $D = \widehat{B}$, we obtain

$$\begin{aligned}
\|\widehat{\mathbf{w}^\perp}\|^2_{0,\widehat{B}} &\leq C(\|\varepsilon(\widehat{\mathbf{w}^\perp})\|^2_{0,\widehat{B}} + \|\widehat{Q}_{\widehat{B}}\widehat{\mathbf{w}^\perp}\|^2_{0,\widehat{B}}) \\
&\leq C\|\varepsilon(\widehat{\mathbf{w}})\|^2_{0,\widehat{B}} \\
&\leq Cr^{2-d}\|\varepsilon(\mathbf{w})\|^2_{0,B(\mathbf{x}_0,r)},
\end{aligned}$$

where we have used (9.15) in the second line. Thus, one has

$$\|\mathbf{w} - Q_{B(\mathbf{x}_0,r)}\mathbf{w}\|^2_{0,B(\mathbf{x}_0,r)} = \|\mathbf{w}^\perp\|^2_{0,B(\mathbf{x}_0,r)} \leq Cr^2\|\varepsilon(\mathbf{w})\|^2_{0,B(\mathbf{x}_0,r)}.$$

This completes the proof of the lemma. $\square$

Lemma 9.10. *Let $\mathcal{T}_h$ be a shape regular finite element partition of Ω. There exists a constant C independent of $T \in \mathcal{T}_h$ such that*

$$\|\mathbf{v}_0 - Q_b\mathbf{v}_0\|^2_{\partial T} \leq Ch_T\|\varepsilon(\mathbf{v}_0)\|^2_T, \qquad \forall\ \mathbf{v}_0 \in [P_k(T)]^d. \tag{9.16}$$

Proof. From the shape regularity assumption, there exists a d-ball $B(\mathbf{x}_0, r) \subset T$ for which the radius r is proportional to h_T; i.e., $r = \lambda_0 h_T$ with a constant λ_0 bounded away from 0. For any $\mathbf{v}_0 \in [P_k(T)]^d$, consider the projection $Q_{B(\mathbf{x}_0,r)}\mathbf{v}_0$ which is naturally extended to T. Since $(Q_{B(\mathbf{x}_0,r)}\mathbf{v}_0)|_{\partial T}$ belongs to the finite element space on ∂T and Q_b is the L^2 projection onto this finite element space, then

$$\|\mathbf{v}_0 - Q_b\mathbf{v}_0\|_{\partial T} \leq \|\mathbf{v}_0 - Q_{B(\mathbf{x}_0,r)}\mathbf{v}_0\|_{\partial T}. \tag{9.17}$$

Using the trace and inverse inequality (9.1)-(9.2) we have

$$\begin{aligned}
\|\mathbf{v}_0 - Q_{B(\mathbf{x}_0,r)}\mathbf{v}_0\|^2_{\partial T} &\leq C(h_T^{-1}\|\mathbf{v}_0 - Q_{B(\mathbf{x}_0,r)}\mathbf{v}_0\|^2_T + h_T\|\nabla(\mathbf{v}_0 - Q_{B(\mathbf{x}_0,r)}\mathbf{v}_0)\|^2_T) \\
&\leq Ch_T^{-1}\|\mathbf{v}_0 - Q_{B(\mathbf{x}_0,r)}\mathbf{v}_0\|^2_T \\
&\leq Ch_T^{-1}\|\mathbf{v}_0 - Q_{B(\mathbf{x}_0,r)}\mathbf{v}_0\|^2_{B(\mathbf{x}_0,r)},
\end{aligned}$$

where we have applied the domain inverse inequality (see the Appendix in [21]) in the third line. Now applying Lemma 9.9 to the term $\|\mathbf{v}_0 - Q_{B(\mathbf{x}_0,r)}\mathbf{v}_0\|^2_{B(\mathbf{x}_0,r)}$ we obtain

$$\|\mathbf{v}_0 - Q_{B(\mathbf{x}_0,r)}\mathbf{v}_0\|^2_{\partial T} \le Ch_T\lambda_0^2\|\varepsilon(\mathbf{v}_0)\|^2_{B(\mathbf{x}_0,r)} \le Ch_T\|\varepsilon(\mathbf{v}_0)\|^2_T.$$

Combining (9.17) with the above estimate gives (9.16). This completes the proof. □

LEMMA 9.11. *Assume that the finite element partition $\mathcal{T}_h$ of Ω is shape regular, and the finite element space V_h is constructed as in Section 3 with $k \ge 1$. Let $1 \le r \le k$. Then, for any $\mathbf{w} \in [H^{r+1}(\Omega)]^d$, $\rho \in H^r(\Omega)$ and $\mathbf{v} \in V_h$, the following estimates hold true*

$$|s(Q_h\mathbf{w}, \mathbf{v})| \le Ch^r\|\mathbf{w}\|_{r+1}|||\mathbf{v}|||, \tag{9.18}$$
$$|\ell_{\mathbf{w}}(\mathbf{v})| \le Ch^r\|\mathbf{w}\|_{r+1}|||\mathbf{v}|||, \tag{9.19}$$
$$|\theta_\rho(\mathbf{v})| \le Ch^r\|\rho\|_r|||\mathbf{v}|||, \tag{9.20}$$

where $\ell_{\mathbf{w}}(\cdot)$ and $\theta_\rho(\cdot)$ are given as in (6.6) and (6.7).

Proof. To derive (9.18), we use (3.3), the Cauchy-Schwarz inequality, and the estimates (9.1) and (9.3) to obtain

$$\begin{aligned}
|s(Q_h\mathbf{w}, \mathbf{v})| =& \Big|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_bQ_0\mathbf{w} - Q_b\mathbf{w}, Q_b\mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T}\Big| \\
=& \Big|\sum_{T\in\mathcal{T}_h} h_T^{-1}\langle Q_0\mathbf{w} - \mathbf{w}, Q_b\mathbf{v}_0 - \mathbf{v}_b\rangle_{\partial T}\Big| \\
\le& \Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_0\mathbf{w} - \mathbf{w}\|^2_{\partial T}\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_b\mathbf{v}_0 - \mathbf{v}_b\|^2_{\partial T}\Big)^{\frac{1}{2}} \\
\le& C\Big(\sum_{T\in\mathcal{T}_h} h_T^{-2}\|Q_0\mathbf{w} - \mathbf{w}\|^2_T + \|\nabla(Q_0\mathbf{w} - \mathbf{w})\|^2_T\Big)^{\frac{1}{2}}|||\mathbf{v}||| \\
\le& Ch^r\|\mathbf{w}\|_{r+1}|||\mathbf{v}|||.
\end{aligned}$$

To prove (9.19), we use the Cauchy-Schwarz inequality, the estimates (9.1) and (9.4) to get

$$\begin{aligned}
|\ell_{\mathbf{w}}(\mathbf{v})| =& \left|2\mu\sum_{T\in\mathcal{T}_h}\langle\mathbf{v}_0 - \mathbf{v}_b, (\varepsilon(\mathbf{w}) - \mathbf{Q}_h\varepsilon(\mathbf{w}))\mathbf{n}\rangle_{\partial T}\right| \\
\le& C\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{v}_0 - \mathbf{v}_b\|^2_{\partial T}\Big)^{\frac{1}{2}}\Big(\sum_{T\in\mathcal{T}_h} h_T\|\varepsilon(\mathbf{w}) - \mathbf{Q}_h\varepsilon(\mathbf{w})\|^2_{\partial T}\Big)^{\frac{1}{2}} \\
\le& Ch^r\|\mathbf{w}\|_{r+1}\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{v}_0 - \mathbf{v}_b\|^2_{\partial T}\Big)^{\frac{1}{2}}.
\end{aligned} \tag{9.21}$$

For the term $\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2$, from Lemma 9.10 we have

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2\\
\leq& 2\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{v}_0-Q_b\mathbf{v}_0\|_{\partial T}^2+2\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_b\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2\\
\leq& C\sum_{T\in\mathcal{T}_h}\|\varepsilon(\mathbf{v}_0)\|_T^2+2\sum_{T\in\mathcal{T}_h} h_T^{-1}\|Q_b\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2\\
\leq& C|\!|\!|\mathbf{v}|\!|\!|^2.
\end{aligned} \tag{9.22}
$$

Substituting (9.22) into (9.21) yields

$$
|\ell_{\mathbf{w}}(\mathbf{v})|\leq Ch^r\|\mathbf{w}\|_{r+1}|\!|\!|\mathbf{v}|\!|\!|.
$$

As to (9.20), we use the Cauchy-Schwarz inequality, the estimates (9.5) and (9.22) to obtain

$$
\begin{aligned}
|\theta_\rho(\mathbf{v})|=&\left|\sum_{T\in\mathcal{T}_h}\langle \mathbf{v}_0-\mathbf{v}_b,(\rho-\mathcal{Q}_h\rho)\mathbf{n}\rangle_{\partial T}\right|\\
&\leq\Big(\sum_{T\in\mathcal{T}_h} h_T^{-1}\|\mathbf{v}_0-\mathbf{v}_b\|_{\partial T}^2\Big)^{\frac12}\Big(\sum_{T\in\mathcal{T}_h} h_T\|(\rho-\mathcal{Q}_h\rho)\mathbf{n}\|_{\partial T}^2\Big)^{\frac12}\\
&\leq Ch^r\|\rho\|_r|\!|\!|\mathbf{v}|\!|\!|.
\end{aligned}
$$

This completes the proof of the lemma. □

10. Numerical Results. In this section, we shall report some numerical results for the weak Galerkin finite element scheme proposed and analyzed in previous sections. In our numerical investigation, we considered the linear elasticity equation (1.1) in the two dimensional square domain $\Omega=(0,1)^2$. The weak Galerkin approximations are obtained by using the lowest order finite elements. More precisely, the finite element space is given either by

$$
V_h=\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}:\ \mathbf{v}_0\in[P_1(T)]^2,\mathbf{v}_b\in P_{RM}(e),\ e\subset\partial T,\ T\in\mathcal{T}_h\} \tag{10.1}
$$

or by

$$
\bar{V}_h=\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}:\ \mathbf{v}_0\in[P_1(T)]^2,\mathbf{v}_b\in P_1(e),\ e\subset\partial T,\ T\in\mathcal{T}_h\}. \tag{10.2}
$$

Since $P_{RM}(e)\subset P_1(e)$ for each edge $e\subset\partial T$, then V_h is clearly a subspace of $\bar{V}_h$. It can be seen that all the theoretical results developed for V_h can be extended to $\bar{V}_h$ without any difficulty.

The discrete weak gradient and the discrete weak divergence are computed on each element $T\in\mathcal{T}_h$ according to their definition. More precisely, for any $\mathbf{v}\in V_h$ or $\mathbf{v}\in\bar{V}_h$, the discrete weak gradient $\nabla_w\mathbf{v}$ on the element T is given by

$$
(\nabla_w\mathbf{v},\varphi)_T=\langle\mathbf{v}_b,\varphi\cdot\mathbf{n}\rangle_{\partial T},\qquad \forall\,\varphi\in[P_0(T)]^{2\times 2}. \tag{10.3}
$$

The discrete weak divergence $\nabla_w\cdot\mathbf{v}$ on T is given by

$$
(\nabla_w\cdot\mathbf{v},\psi)_T=\langle\mathbf{v}_b,\psi\mathbf{n}\rangle_{\partial T},\qquad \forall\,\psi\in P_0(T). \tag{10.4}
$$

Denote by $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\}$ and $\mathbf{u}$ the solution to the weak Galerkin formulation (3.5) and the original equation (1.3), respectively. Define the error by $\mathbf{e}_h = Q_h\mathbf{u} - \mathbf{u}_h = \{\mathbf{e}_0, \mathbf{e}_b\}$ where $Q_h\mathbf{u}$ is the L^2 projection of the exact solution $\mathbf{u}$ in V_h or $\bar{V}_h$, as appropriate. The error for the weak Galerkin finite element solution is computed in three norms defined as follows

$$
\begin{aligned}
|||\mathbf{e}_h|||_*^2 &= \sum_{T\in\mathcal{T}_h} \left\{ 2\mu \int_T |\varepsilon_w(\mathbf{e}_h)|^2 dT + \lambda \int_T |\nabla_w \cdot \mathbf{e}_h|^2 dT + h_T^{-1} \int_{\partial T} |Q_b\mathbf{e}_0 - \mathbf{e}_b|^2 ds \right\}, \\
\|\mathbf{e}_0\|^2 &= \sum_{T\in\mathcal{T}_h} \int_T |\mathbf{e}_0|^2 dT, \\
\|\mathbf{e}_b\|^2 &= \sum_{T\in\mathcal{T}_h} h_T \int_{\partial T} |\mathbf{e}_b|^2 ds.
\end{aligned}
$$

It can be seen that $|||\mathbf{e}_h|||_*$ is an H^1-like norm for the error function, and $\|\mathbf{e}_0\|$ and $\|\mathbf{e}_b\|$ are typical L^2 norms for the error in the interior and on the boundary of each element.

10.1. Test Problem 1. Consider the elasticity equation (1.1) in the square domain $\Omega = (0,1)^2$ which is partitioned into uniform triangular mesh $\mathcal{T}_h$ with mesh size h. The right-hand side function $\mathbf{f}$ is chosen so that the exact solution is given by

$$
\mathbf{u} = \begin{pmatrix} \sin(x)\sin(y) \\ 1 \end{pmatrix}.
$$

Table 10.1 illustrates the computational result when the rigid motions are employed on the boundary of each element; i.e., the space V_h. Table 10.2 shows the result when linear functions are used on the boundary of each element; i.e., the space $\bar{V}_h$. Theoretically, these two methods have the same order of convergence which is confirmed by these two tables. Note that the WG method with V_h has less number of degrees of freedom than that of $\bar{V}_h$.

A numerical scheme is said to be convergent with order α if the error decreases proportionally to h^α, where h is the mesh parameter. From Tables 10.1 and 10.2 we see that the convergence of the weak Galerkin finite element scheme in the L^2-norm is of order 2 and that in the H^1-norm is of order 1. The numerical results are in consistency with the theoretical prediction.

Table 10.1

Problem 1: $\lambda = 1$, $\mu = 0.5$, and WG with V_h (the rigid motion space on element boundary).

1/h	$\|\mathbf{u}_0 - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$|||\mathbf{u}_h - Q_h\mathbf{u}|||_*$	order
2	0.0750	–	0.0424	–	0.3103	–
4	0.0192	1.97	0.0115	1.88	0.1566	0.99
8	0.0049	1.98	0.0031	1.87	0.0787	0.99
16	0.0012	1.99	0.0008	1.93	0.0394	1.00
32	0.0003	2.00	0.0002	1.97	0.0197	1.00

Table 10.2
Problem 1: $\lambda = 1$, $\mu = 0.5$, and WG with $\bar{V}_h$ (linear functions on element boundary).

1/h	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|\|$	order
2	0.0743	–	0.0424	–	0.3082	–
4	0.0190	1.96	0.0113	1.90	0.1555	0.99
8	0.0048	1.98	0.0031	1.88	0.0782	0.99
16	0.0012	1.99	0.0008	1.93	0.0392	1.00
32	0.0003	2.00	0.0002	1.97	0.0196	1.00

10.2. Test Problem 2. In the second test, the linear elasticity equation (1.1) is also defined on the unit square domain $\Omega = (0, 1)^2$, but the exact solution is given as follows

$$\mathbf{u} = \begin{pmatrix} (x+y)^2 \\ (x-y)^2 \end{pmatrix}.$$

The right-hand side function $\mathbf{f}$ is computed to match the exact solution. The numerical results, as shown in Tables 10.3-10.4, are again based on the uniform partition $\mathcal{T}_h$ for this domain. The numerical results confirm the theoretical prediction developed in previous sections.

Table 10.3
Problem 2: $\lambda = 1$, $\mu = 0.5$, and WG with V_h (the rigid motion space on element boundary).

1/h	$\|\mathbf{u}_0 - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|\|_*$	order
2	0.4334	–	0.2089	–	1.8410	–
4	0.1095	1.98	0.0528	1.98	0.9264	0.99
8	0.0275	2.00	0.0133	1.99	0.4644	1.00
16	0.0069	2.00	0.0034	2.00	0.2324	1.00
32	0.0017	2.00	0.0008	2.00	0.1162	1.00

Table 10.4
Problem 2: $\lambda = 1$, $\mu = 0.5$, and WG with $\bar{V}_h$ (linear functions on element boundary).

1/h	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|\|$	order
2	0.4310	–	0.2009	–	1.8355	–
4	0.1088	1.99	0.0502	2.00	0.9233	0.99
8	0.0273	2.00	0.0126	2.00	0.4628	1.00
16	0.0068	2.00	0.0032	1.99	0.2316	1.00
32	0.0017	2.00	0.0008	2.00	0.1158	1.00

10.3. Locking-Free Tests. In the locking-free investigation, the linear elasticity equation (1.1) has exact solutions given by

$$\mathbf{u} = \begin{pmatrix} \sin(x)\sin(y) \\ \cos(x)\cos(y) \end{pmatrix} + \lambda^{-1} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The right-hand side function $\mathbf{f}$ is computed to match the exact solution (note that it is λ-dependent). The numerical results are based on the same uniform partition

$\mathcal{T}_h$ for the unit square domain. The results, as shown in Tables 10.5-10.12, clearly indicate a locking-free convergence for the weak Galerkin finite element method in various norms, which is consistent with theory.

Table 10.5
WG with V_h based on the element of $\{P_1(T)/P_{RM}(e)\}$, $\mu = 0.5$, and $\lambda = 1$.

1/h	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|$	order
2	0.0352	–	0.0331	–	0.1544	–
4	0.0097	1.86	0.0120	1.46	0.0834	0.89
8	0.0026	1.91	0.0037	1.68	0.0433	0.94
16	0.0007	1.96	0.0010	1.87	0.0220	0.98
32	0.0002	1.98	0.0003	1.96	0.0110	0.99

Table 10.6
WG with V_h based on the element of $\{P_1(T)/P_{RM}(e)\}$, $\mu = 0.5$, and $\lambda = 100$.

1/h	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|$	order
2	0.0344	–	0.0291	–	0.1449	–
4	0.0100	1.79	0.0113	1.36	0.0774	0.90
8	0.0028	1.82	0.0038	1.59	0.0403	0.94
16	0.0008	1.91	0.0011	1.81	0.0205	0.97
32	0.0002	1.96	0.0003	1.93	0.0103	0.99

Table 10.7
WG with V_h based on the element of $\{P_1(T)/P_{RM}(e)\}$, $\mu = 0.5$, and $\lambda = 10,000$.

1/h	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|$	order
2	0.0344	–	0.0290	–	0.1447	–
4	0.0100	1.79	0.0113	1.36	0.0773	0.90
8	0.0028	1.82	0.0038	1.59	0.0403	0.94
16	0.0008	1.90	0.0011	1.81	0.0205	0.97
32	0.0002	1.96	0.0003	1.93	0.0103	0.99

Table 10.8
WG with V_h based on the element of $\{P_1(T)/P_{RM}(e)\}$, $\mu = 0.5$, and $\lambda = 1,000,000$.

1/h	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|$	order
2	0.0344	–	0.0290	–	0.1447	–
4	0.0100	1.79	0.0113	1.36	0.0773	0.90
8	0.0028	1.82	0.0038	1.59	0.0403	0.94
16	0.0008	1.90	0.0011	1.81	0.0205	0.97
32	0.0002	1.96	0.0003	1.93	0.0103	0.99

Table 10.9
WG with $\bar{V}_h$ based on the element of $\{P_1(T)/P_1(e)\}$, $\mu = 0.5$, and $\lambda = 1$.

1/h	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|\|$	order
2	0.0341	–	0.0313	–	0.1518	–
4	0.0093	1.87	0.0115	1.45	0.0816	0.90
8	0.0025	1.91	0.0036	1.67	0.0424	0.95
16	0.0006	1.96	0.0010	1.86	0.0215	0.98
32	0.0002	1.98	0.0003	1.95	0.0108	0.99

Table 10.10
WG with $\bar{V}_h$ based on the element of $\{P_1(T)/P_1(e)\}$, $\mu = 0.5$, and $\lambda = 100$.

1/h	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|\|$	order
2	0.0340	–	0.0281	–	0.1441	–
4	0.0098	1.79	0.0111	1.34	0.0769	0.91
8	0.0028	1.82	0.0037	1.58	0.0400	0.94
16	0.0007	1.90	0.0011	1.81	0.0204	0.97
32	0.0002	1.96	0.0003	1.93	0.0102	0.99

Table 10.11
WG with $\bar{V}_h$ based on the element of $\{P_1(T)/P_1(e)\}$, $\mu = 0.5$, and $\lambda = 10,000$.

1/h	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|\|$	order
2	0.0340	–	0.0280	–	0.1439	–
4	0.0098	1.79	0.0111	1.34	0.0768	0.91
8	0.0028	1.82	0.0037	1.58	0.0400	0.94
16	0.0007	1.90	0.0011	1.81	0.0203	0.97
32	0.0002	1.96	0.0003	1.93	0.0102	0.99

Table 10.12
WG with $\bar{V}_h$ based on the element of $\{P_1(T)/P_1(e)\}$, $\mu = 0.5$, and $\lambda = 1,000,000$.

1/h	$\|\mathbf{u}_h - Q_0\mathbf{u}\|$	order	$\|\mathbf{u}_b - Q_b\mathbf{u}\|$	order	$\|\|\|\mathbf{u}_h - Q_h\mathbf{u}\|\|\|$	order
2	0.0340	–	0.0280	–	0.1439	–
4	0.0098	1.79	0.0111	1.34	0.0768	0.91
8	0.0028	1.82	0.0037	1.58	0.0400	0.94
16	0.0007	1.90	0.0011	1.81	0.0203	0.97
32	0.0002	1.96	0.0003	1.93	0.0102	0.99

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