Plate Bending with 8-Node Hexahedral Continuum Elements

M. M. Rashid[a,*]

[a] *Department of Civil and Environmental Engineering, University of California at Davis, Davis, CA 95616 USA*

Abstract

Keywords: finite element method, plate bending, continuum elements, enhanced assumed strain method

1. Introduction

[? ?]

2. Locking in Thin Plate-Like Configurations

Generically, the phenomenon identified as "locking" manifests most obviously as nodal displacements that become far too small, often even approaching zero, as some limiting problem feature is approached. The problem feature of present interest is thinness of the body in a single local direction. Our particular focus here is the prospect of effectively analyzing thin plate- or shell-like bodies using what are nominally conventional continuum finite elements; i.e., displacement-based elements without rotational degrees of freedom. However, for such an analysis approach to be practical, the continuum elements must provide locking-free, robustly-accurate approximations with only a single element through the thickness. Practical utility further mandates that these qualities persist even when the reference configuration of the element is distorted.

The most general conception of locking – though perhaps the least illuminating with regard to devising strategies to avoid it – identifies locking as the result of an approximation subspace that, though it may be of high dimension, is nevertheless nearly orthogonal to the exact solution, under the energy inner product. A more context-specific understanding can be had by considering particular example problems, such as the following. Consider a thin, flat, circular, elastic plate

*corresponding author

Email address: mmrashid@ucdavis.edu (M. M. Rashid)

(thickness h, radius a) discretized into continuum finite elements, with one element through the thickness. Subjecting this plate to a uniform bending moment on its edge, with no transverse loading on the faces of the plate, produces a uniform equi-biaxial bending moment throughout the plate, with zero transverse shear resultants and zero twisting moments. For a thin plate suffering small deformations, the top and bottom faces deform into spherical surfaces, while normal material fibers remain normal to the mid-surface. The shear strains are zero throughout, while the in-plane normal strains are equi-biaxial, and linearly varying through the thickness. The transverse normal strain e_{33} (where the 3-direction of a Cartesian coordinate frame is normal to the plane of the plate) also varies linearly through the thickness, in such a manner that the corresponding stress component T_{33} is zero throughout. The in-plane stresses are equi-biaxial, i.e. $T_{11} = T_{22}$ and $T_{12} = 0$, and vary as $(a/h)^3$ with plate thickness. In this pure-bending scenario, $T_{i3} = 0$; for other problems loaded by transverse normal traction and/or edge forces or moments, these components behave as T_{13}, $T_{23} \sim (a/h)$; $T_{33} \sim 1$.

From the foregoing observations, it can be concluded that, for a thin plate loaded by transverse normal traction on its faces, transverse forces on its edges, and/or applied bending moments on its edges, the small-strain components are characterized by:

$$
\begin{aligned}
e_{11} &= (a/h)^3 x_3 b_{11}, \\
e_{22} &= (a/h)^3 x_3 b_{22}, \\
e_{12} &= (a/h)^3 x_3 b_{12}, \\
e_{13} &= (a/h) b_{13}, \\
e_{23} &= (a/h) b_{23}, \\
e_{33} &= b_{33} - (\nu/(1+\nu))(a/h)^3 x_3 (b_{11} + b_{22}),
\end{aligned}
\tag{1}
$$

where the b_{ij} that appear in 1 are functions of in-plane coordinates x_1, x_2 only, and are independent of (a/h). The (x_1, x_2) plane is taken to coincide with the midsurface of the plate, and a is an in-plane characteristic dimension of the plate. Here, a Riessner-Mindlin-type kinematic assumption has been adopted, and isotropic linear elasticity has been assumed. Both of these assumptions, while not completely general, are suited to the illucidative purposes at hand.

It might now be asked whether or not a given continuum element, in a thin configuration, is capable of conforming to 1 in the limit as $(a/h) \to \infty$. Specifically, the element should be able to approximate, under in-plane mesh refinement, the strain distribution 1 for given $b_{ij}(x_1, x_2)$, while retaining a single element through the thickness, and while maintaining the noted variation with (a/h) at every level of mesh refinement. If this occurs, then locking-free behavior can be expected, at least for the case of isotropic linear elasticity.

In fact, conventional isoparametric element formulations generally do not possess this capability. In fact, special measures must be taken to achieve the above-described pattern of strain; this is the subject of the next section. Our immediate aim here is to parse the strain distributions into distinct locking "mechanisms," and then assess some common element formulations against these measures. To this end, the parameters E_b, E_s, and E_n are defined by

$$E_b^2 = \int_{\mathcal{B}} (T_{11}^2 + T_{22}^2 + 2T_{12}^2) dv, \quad E_s^2 = \int_{\mathcal{B}} 2(T_{13}^2 + T_{23}^2) dv, \quad E_n^2 = \int_{\mathcal{B}} T_{33}^2 dv, \tag{2}$$

where $\mathcal{B}$ is the three-dimensional domain occupied by the plate. It is noted that $(E_b^2 + E_s^2 + E_n^2)^{1/2}$ is the L^2 norm of the stress field on the body, which is an equivalent norm to the energy norm. With these parameters in hand, the two dimensionless parameters

$$\lambda_s = (a/h)\frac{E_s}{E_b}, \quad \lambda_n = (a/h)^2 \frac{E_n}{E_b}, \tag{3}$$

are defined for purposes of assessing the locking tendency of particular element types. The parameter λ_s measures the magnitude of the transverse shear stress as compared to the bending stress, while λ_n compares the through-thickness normal stress to the bending stress. Both parameters are scaled by the thickness ratio in such a manner that they should not vary with h/a for thin-plate problems loaded by normal pressure and/or transverse edge-loads that themselves are constant with h/a.

A simple example problem will serve to illustrate the varied locking behaviors of some common continuum element types. Figure **??** shows a square plate of dimension 2×2, one-quarter of which is discretized into 16 continuum elements, with one element through the thickness. The plate is loaded by a uniform transverse pressure of unit magnitude, while the edge BCs correspond to "simple supports." Specifically, nodal displacements in the x_3-direction are restrained for the nodes along $x_1 = 1, x_3 = 0, 0 \leq x_2 \leq 1$ and along $x_2 = 1, x_3 = 0, 0 \leq x_1 \leq 1$. Appropriate symmetry BCs are enforced for nodes on the surfaces $x_1 = 0$ and $x_2 = 0$. An analytic solution is available for this problem when the plate is modeled using the Kirchhoff-Love theory of plate bending. Summing the series solution to achieve 6 digits of accuracy, the transverse displacement at the center of the plate is given by

$$u_{max} = 0.779972\,\frac{1-\nu^2}{Eh^3}, \tag{4}$$

where E is the elastic modulus and ν is the Poisson ratio, taken to be 1×10^7 and ν respectively.

Figure **??** shows the transverse displacement at the center of the plate, normalized by the exact value (4), for four element types:

E1: standard 8-node isoparametric hexahedron

E2: 8-node hex with element-averaged dilatation

E3: 8-node hex with element-averaged transverse shear strains

E4: standard 20-node isoparametric hex.

Full integration is used in all cases, i.e. $2 \times 2 \times 2$ Gaussian quadrature for elements E1, E2, and E3; and $3 \times 3 \times 3$ for E4. Element E2 is identical to E1, except that the normal components of strain at each integration point are additively adjusted such that the dilatation matches the element-averaged dilatation value. This adjustment is effective in mitigating *volumetric locking* associated with nearly incompressible material response, but is also often cited as improving element performance for "bending-dominated" situations. E3 is also similar to E1, but with the integration-point values of e_{13} and e_{23} replaced by their element-averaged values. Both E2 and E3 are examples of "enhanced assumed strain" formulations.

Figure **??** presents the error in the transverse displacement at the center of the 2×2 plate, for 11 different plate thicknesses ranging from $h/a = 0.05$ to 5×10^{-5} ($a = 2$ is the edge-length of the plate). In all cases, the 16-element mesh pictured in Figure **??** was used to generate the solutions. The displacement results clearly indicate locking, particularly for element types E1, E2, and E4, for which the error approaches unity as $h/a \to 0$. Whether locking occurs for element type E3 is less clear; the error is constant, though rather large, as the thickness ratio decreases. The locking parameters λ_s and λ_n can provide some insight into the specific mechanisms, if any, of locking that operate for particular element types and BVPs. These are plotted vs. thickness ratio in Figure **??**. Keeping in mind that $\lambda_s,\ \lambda_n$ have been defined so that they are constant with thickness ratio in the exact case, "locking" might reasonably be defined as λ_s and/or $\lambda_n \sim (h/a)^{\alpha},\ \alpha < 0$. Further, the two parameters serve to distinguish between the specific mechanism of locking: "shear locking" in the case of λ_s and "thickness locking" in the case of λ_n.

Based on this terminology, it is evident from Figure **??** that all element types E1 - E4 suffer from thickness locking, while element types E1, E2, and E4 exhibit shear locking. The severe shear locking that occurs with the standard 8-node hex element (type E1) is readily understood from Figure **??**, which depicts the cross-sections of hex-8 and hex-20 elements whose nodal displacements have been set consistent with a pure-bending deformation. It is evident from the figure that the standard hex-8 element produces transverse shear strains that are of order of the bending strain, as an artifact of any bending-type deformation. This lack of "independent control," via the nodal displacements, of the transverse-shear and bending strains is the root cause of the hex-8 element's severe shear-locking behavior, as has been noted by many other authors ( ).

What is somewhat surprising is that the hex-20 element (type E4), which can exactly represent pure-bending deformation with zero transverse shear strain (at least in undistorted configurations), still exhibits shear locking, though with a lower divergence rate than the hex-8 element. This reduction in divergence rate of the transverse shear strain likely accounts for the good deflection-solution accuracy in moderately-thin configurations (see Figure **??**). As the thickness ratio decreases further, however, this element, too, succombs to the effect of shear-locking. The reasons are more subtle than in the case of the hex-8 element, and relate to the fact that the nodal displacements required for satisfaction of in-plane equilibrium produce "unwanted" transverse shear strains. Whereas the bending strains and the transverse shear strains are not as tighly-coupled as they are in the hex-8 element, they are still coupled.

Turning now to thickness-locking, the righthand graph in Figure **??** clearly indicates similar intensities of this effect for all elements tested, with quadratic divergence of λ_n as thickness decreases. Element type E3, which effectively mitigates shear-locking via element-averaging of the transverse shear strains, nevertheless exhibits full-strength thickness-locking. However, the deflection solution (Figure **??**) is not as badly harmed by thickness-locking as by shear-locking, the deflection error for element-type E3 remaining merely constant with thickness ratio (though rather high) rather than diverging. In this sense, it might be said that thickness-locking is not as serious as shear-locking, though this is of little consolation if the solution's stress state is of primary interest. Indeed, for all element types tested, the T_{33} component of stress is comparable, in magnitude, to the in-plane normal and shear stresses, whereas it should be several orders of magnitude smaller for the thinnest plates.

It is also worthy of note that the hex-20 element (type E4) offers no relief whatever from thickness locking. This is again somewhat surprising, considering that the higher-order shape functions of this element allow for a transverse normal strain (e_{33}) that varies linearly through the thickness. In the lower-order hex-8 element, e_{33} is constant through the thickness, whereas the in-plane normal strains e_{11}, e_{22} vary linearly in the thickness direction whenever bending is present. Zero T_{33} stress through the thickness is therefore not possible, due to Poisson contraction. The hex-20 element, with its linearly-varying e_{33} would seem to offer this potential, but in practice it does not. As with shear-locking in the hex-20 element, the reason is, once again, related to the coupling between the integration-point values of the strain components: in a multi-element problem, for any given element it is not possible to find, within the 60-dimensional space of nodal displacements, a set of values that produce $e_{33} = 0$ at all integration points, while simultaneously satisfying the discrete equations of equilibrium.

Some fairly general conclusions emerge from the foregoing simple plate-bending problem. First, low-order elements such as the conventional hex-8 are unable, for *any* set of nodal displacements, to produce strain distributions that resemble pure-bending deformation, i.e. zero transverse shear strain and a transverse normal strain that produces zero transverse normal stress. For such elements, severe locking is essentially assured. Less obviously, locking is also typically observed with higher-order elements, whose interpolants *are* capable of reproducing the kinematics of pure bending. This *single-element* kinematic capability is not, by itself, sufficient to avoid locking in multi-element problems. The reason is readily apparent when one considers that the integration-point values of all strain components are coupled together through the nodal displacements, where the latter are obviously shared by multiple elements. This observation strongly argues against the likelihood of discovering a conforming element displacement interpolant that exhibits locking-free behavior.

3. The Case of Thin Beam-Like Configurations

4. Alleviating Locking: Some Possible Approaches

A main conclusion of the previous section is that kinematic enhancements intended to render otherwise-conventional continuum elements effective in thin, bending-dominated scenarios are likely to be successful only if they are 1) non-conforming, and 2) specifically designed to encompass pure-

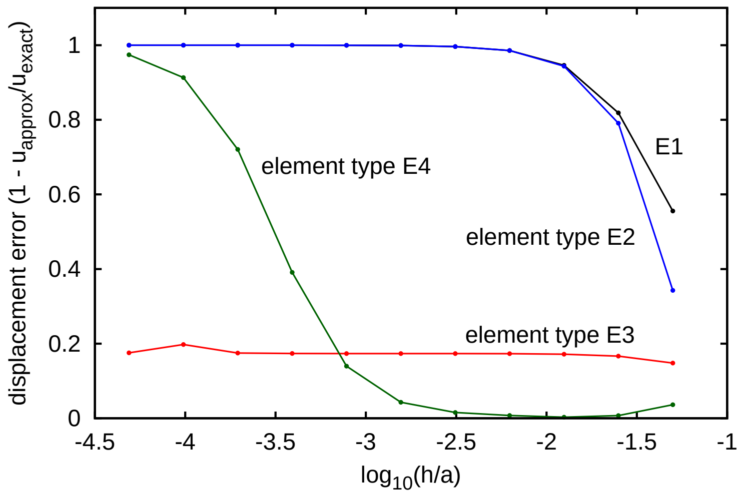
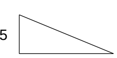


Figure 1: Normalized error of the transverse deflection at the center of the 2×2 plate of Figure **??**, vs. log of the thickness ratio. Results are shown for the four element types E1 - E4, which are described in the text.



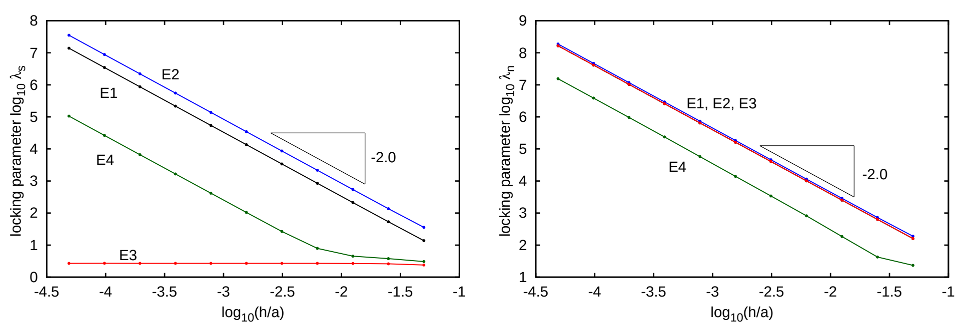


Figure 2: Shear-locking (left) and thickness-locking (right) parameters vs. plate thickness ratio for the problem of Figure **??**, in log-log format. The analytic solution corresponds to constant values with thickness for both parameters; a negative slope indicates divergence of the parameter as the plate thickness decreases, and hence locking.

bending deformation fields, regardless of the material response. Accordingly, two general approaches present themselves: *enhanced assumed strain methods*, and *incompatible modes*.

5. The Proposed 8-Node Element

The proposed element seeks to provide a number of desirable features, as follows. First, the element should be compatible with conventional continuum elements, with no constraints or other special measures required on common interfaces. This implies, principally, that only displacement degrees of freedom should appear in the element formulation. It is desired, in addition, that the bending element be compatible specifically with the standard 8-node hexahedral element, so that both may occur in the same mesh in arbitrary patterns. This precludes such feautres as mid-side nodes in the bending element. Second, we ask that the element formulation be free of dependence on special geometric requirements, such as parallelism of certain element faces to a mid-surface. In general, the formulation should be free of any reference to geometric features of the plate/shell or beam: only the geometry of the element itself is permitted to influence the formulation. And finally, we require that good performance be maintained in distorted configurations, including both in-plane and through-thickness distortion as described in the previous section. These desired qualities are in addition to the basic requirements that the element should pass standard patch tests, and should otherwise exhibit convergent behavior.

The above "wish list" is intended to permit meshing strategies that can easily accommodate complex geometries that involve both thin and monolithic parts, possibly with complicated junctions. It is worth noting that such geometries can be very time-consuming to mesh using combinations of conventional continuum and plate/shell elements. In part, this is because significant user-input is usually required to reduce 3D geometries to systems of 2D mid-surfaces.

The proposed element formulation takes the form of an "enhanced assumed strain" or "strain projection" modification of the standard 8-node hexahedral element. The basic idea is to use the "enhanced" strain field

$$\hat{\mathbf{e}} = \mathbf{e} + \tilde{\mathbf{e}} \tag{5}$$

in the constitutive evaluation, where $\mathbf{e} = \nabla_s \mathbf{u}$ is the kinematic strain – i.e. the symmetric part of the displacement gradient. The $\tilde{\mathbf{e}}$ term is the enhancement contribution, which is not required to derive from a compatible displacement field, nor even from a displacement field at all. As explained in [], enhanced assumed strain methods can be framed in the setting of the three-field Hu-Washizu variational principle, from which it emerges that the element-averages of $\tilde{\mathbf{e}}$ must vanish. As pointed out in [] and elsewhere, this condition also leads to satisfaction of engineering patch tests, and therefore to an expectation of convergent behavior. Apart from the zero-mean requirement on each element, $\tilde{\mathbf{e}}$ may vary arbitrarily. The strain enhancement is generally chosen to produce some desirable quality in the strain field that would be difficult to achieve via a compatible displacement interpolant.

The three-field framework put forward in [] encompasses all possible choices for the strain enhancement, including both those that do, and those that do not, depend on internal degrees of freedom. Though there is no theoretical distinction between these two cases, as a practical matter it is useful to distinguish between them. Specifically, when the strain enhancement depends on internal dof, the internal unknowns are generally eliminated at the element level through static condensation []. While doing this presents no essential difficulty, the computer implementation can be rather invasive in nonlinear codes where no structural provision has specifically been made to accommodate it. The classical "incompatible modes" element of Wilson [] and its derivatives [] are examples of enhanced-assumed-strain methods with internal dof. The term "incompatible modes" usually suggests that the strain enhancement $\tilde{\mathbf{e}}$ derives from an (incompatible) contribution to the displacement interpolant.

The element proposed herein is based on an enhanced-assumed-strain modification of the conventional isoparametric 8-node hex, in which *no* internal degrees of freedom appear. The term "strain-projection method" is therefore appropriate. Element types E2 and E3 in section 2 are of this type.

Considering now a typical element Ω in its physical configuration, the element formulation begins with computation of the element's inertia tensor

$$I_{ij} = \int_{\Omega} x_i x_j dv, \tag{6}$$

where x_i are global Cartesian coordinates. This is done in anticipation of the fact that the element's shape may be very "flat" or "thin" (i.e. plate-like or beam-like), whereas the design of the strain enhancement $\tilde{\mathbf{e}}$ will depend on the "short" direction(s). For the present purposes, it is entirely adequate to evaluate the above integral using the element's $2 \times 2 \times 2$ Gaussian quadrature rule.

The triple of (principal vector, principal value) pairs $(\mathbf{m}^i, \mu^i)$ is then computed, and an *element-local coordinate system* is established that is aligned with the principal triad. In what follows, the components of tensors and vectors – in particular, strain tensors – will be resolved on the element-local coordinate system. Ordering the principal values as $\mu^3 \leq \mu^2 \leq \mu^1$, a pair of aspect ratios is defined by

$$\rho_1 = \left(\frac{\mu^3}{\mu^1}\right)^{1/2}, \quad \rho_2 = \frac{\sqrt{\mu^2} - \sqrt{\mu^3}}{\sqrt{\mu^1} - \sqrt{\mu^3}}, \tag{7}$$

thus $\rho_1 \in (0, 1]$ and $\rho_2 \in [0, 1]$. The "thinness" of the element is controlled by ρ_1, and the plate/beam character by ρ_2: if $\rho_1 \approx 0$ the element is thin, while $\rho_2 \approx 0$ corresponds to a beam and $\rho_2 \approx 1$ to a plate-like configuration. In the latter case, $\mathbf{m}^3$ lies normal to the plane of the plate, while for the beam-like case, $\mathbf{m}^1$ defines the long axis of the beam. In what follows, ρ_1 and ρ_2 will be used in the design of the strain enhancement $\tilde{\mathbf{e}}$.

The quadrature rule on the physical-configuration element Ω is taken to be the standard $2 \times 2 \times 2$ mapped Gaussian product rule, with weights given by the Jacobian-determinant values J_α ($\alpha = 1, \ldots, 8$) of the isoparametric mapping. Let V be an 8-dimensional linear space equipped with the weighted inner product

$$\langle u, v \rangle = \sum_{\alpha=1}^{8} J_\alpha u^\alpha v^\alpha, \quad u, v \in V. \tag{8}$$

This inner product can be thought of as an approximate $L_2(\Omega)$-inner product, where the elements of V are vectors of integration-point values of square-integrable functions defined on Ω. In view of this duality, the same symbol – e.g. $x_2, x_3, 1$ – will generally be used for both explicit functions of spatial position, and the corresponding elements of V, it being clear from the context which is meant. In particular, we define a "unit constant" element of V by $\mathfrak{i} = 1/\|1\| = 1/|\Omega|^{1/2}$, and denote by V' the 7-dimensional subspace $V' \subset V$ that is orthogonal to $\mathfrak{i}$. The projection of any $u \in V$ onto V' is given by $u - \langle u, \mathfrak{i} \rangle \mathfrak{i}$. All elements of V' obviously have zero mean (quadrature) value on Ω.

Returning now to the relation (5), we seek a strain enhancement $\tilde{\mathbf{e}}$ that will alleviate both shear-locking and thickness-locking when the element takes plate- or beam-like configurations. To this end, we denote the space of symmetric rank-two tensors by $\mathcal{S}$, and introduce the basis

$$\mathcal{S} = \operatorname{span}\left\{ 1/2(\mathbf{m}^i \otimes \mathbf{m}^j) \mid i, j = 1, 2, 3 \right\}. \tag{9}$$

It will prove useful to decompose $\mathcal{S}$ into three orthogonal subspaces as $\mathcal{S} = \mathcal{S}^t \oplus \mathcal{S}^r \oplus \mathcal{S}^n$. This decomposition will be used to seggregate the strain and stress into tangential, transverse, and normal components, respectively. The decomposition takes one of two different forms, depending on whether the element's shape is plate-like or beam-like. Specifically, for $\rho_2 > \bar{\rho}$, where $\bar{\rho}$ is a fixed parameter, the element is deemed to be plate-like, and the subspaces are defined as

$$\begin{aligned}
\mathcal{S}^t &= \operatorname{span}\left\{\mathbf{m}^1 \otimes \mathbf{m}^1,\ \mathbf{m}^2 \otimes \mathbf{m}^2,\ {}^1\!/\!_2(\mathbf{m}^1 \otimes \mathbf{m}^2 + \mathbf{m}^2 \otimes \mathbf{m}^1)\right\}, \\
\mathcal{S}^r &= \operatorname{span}\left\{{}^1\!/\!_2(\mathbf{m}^1 \otimes \mathbf{m}^3 + \mathbf{m}^3 \otimes \mathbf{m}^1),\ {}^1\!/\!_2(\mathbf{m}^2 \otimes \mathbf{m}^3 + \mathbf{m}^3 \otimes \mathbf{m}^2)\right\}, \\
\mathcal{S}^n &= \operatorname{span}\left\{\mathbf{m}^3 \otimes \mathbf{m}^3\right\}.
\end{aligned} \tag{10}$$

If, on the other hand, $\rho_2 \leq \bar{\rho}$, the element is beam-like in form, and the subspaces are instead given by

$$\begin{aligned}
\mathcal{S}^t &= \operatorname{span}\left\{\mathbf{m}^1 \otimes \mathbf{m}^1\right\}, \\
\mathcal{S}^r &= \operatorname{span}\left\{{}^1\!/\!_2(\mathbf{m}^1 \otimes \mathbf{m}^2 + \mathbf{m}^2 \otimes \mathbf{m}^1),\ {}^1\!/\!_2(\mathbf{m}^1 \otimes \mathbf{m}^3 + \mathbf{m}^3 \otimes \mathbf{m}^1)\right\}, \\
\mathcal{S}^n &= \operatorname{span}\left\{\mathbf{m}^2 \otimes \mathbf{m}^2,\ \mathbf{m}^3 \otimes \mathbf{m}^3,\ {}^1\!/\!_2(\mathbf{m}^2 \otimes \mathbf{m}^3 + \mathbf{m}^3 \otimes \mathbf{m}^2)\right\}.
\end{aligned} \tag{11}$$

A reasonable value for $\bar{\rho}$ to distinguish "plate-like" from "beam-like" elements is 0.2. It is worth remarking that the above decompositions remain unique and meaningful in the degenerate cases of an element in the form of a square plate or square-section beam. These special cases correspond to $\mu^1 = \mu^2$ and $\mu^2 = \mu^3$ respectively, such that the $\mathbf{m}^1$ and $\mathbf{m}^2$ directions (respectively, the $\mathbf{m}^2$ and $\mathbf{m}^3$ directions) are defined only up to within an arbitrary rotation about the remaining principal direction. Regardless of the choice of this rotation, however, the decompositions (10) and (11) remain invariant.

The utility of the above decompositions is best explained with reference to the Cauchy stress tensor $\mathbf{T}$. The stress can be decomposed as $\mathbf{T}^t + \mathbf{T}^r + \mathbf{T}^n$, where the superscripts denote membership in the corresponding subspaces defined above. Bending deformations, as induced by transverse loading, generally cause a large tangential component $\mathbf{T}^t$ that scales with $(a/h)^3$, i.e. the cube of the thickness ratio of the plate or beam (see eqn. 1 and preceding discussion). The transverse component $\mathbf{T}^r$, on the other hand, is usually far smaller, and is only linear in (a/h). Finally, the normal component $\mathbf{T}^n$, away from any locally large transverse loads and for flat/straight structural elements, is typically smaller still, and is constant with thickness ratio. These relative magnitudes of the stress components are compelled by equilibrium; they require that their conjugate strain components $\mathbf{e}^t$, $\mathbf{e}^r$, and $\mathbf{e}^n$ adjust themselves accordingly. However, standard continuum finite elements generally do not possess the requisite kinematic freedom needed to accommodate such adjustments. This is the fundamental source of locking.

Our locking-mitigation strategy relies upon designing an enhancement strain $\tilde{\mathbf{e}}$ which renders the element capable of achieving the above-described stress distributions. In what follows, superscripts $(\ )^t$, $(\ )^r$, $(\ )^n$ will again be used to indicate tensor components that lie in the corresponding subspaces of $\mathcal{S}$. The tangential component of the enhancement strain is, in all cases, taken to be zero:

$$\tilde{\mathbf{e}}^t = \mathbf{0}. \tag{12}$$

Thus, $\hat{\mathbf{e}}^t = \mathbf{e}^t$, i.e. the tangential component of the constitutive input is taken to be simply the corresponding kinematic strain. The transverse and normal components $\tilde{\mathbf{e}}^r$ and $\tilde{\mathbf{e}}^n$, on the other hand, are set so that the corresponding stress components are capable of achieving zero values, regardless of the value of $\hat{\mathbf{e}}^t$. This reflects the fact that $\mathbf{T}^r$ and $\mathbf{T}^n$ are usually far smaller than $\mathbf{T}^t$ in typical plate or beam problems. Yet, the kinematics of conventional continuum elements generally lead to comparable magnitudes across all three stress components, leading to locking. Proceeding in this direction, a strain tensor $\bar{\mathbf{e}} \in \mathcal{S}^r \oplus \mathcal{S}^n$ is defined such that

$$\hat{\mathbf{e}} = \mathbf{e}^t + \bar{\mathbf{e}} \ \Rightarrow\ \mathbf{T}^r = \mathbf{T}^n = \mathbf{0}\ \forall\ \mathbf{e}^t \in \mathcal{S}^t. \tag{13}$$

The "$\Rightarrow$" symbol here refers to evaluation of the prevailing constitutive equations. The zero-stress condition in (13) is akin to the plane-stress condition; selection of $\bar{\mathbf{e}}$ to enforce it requires, in general, iterative evaluation of the constitutive model. For isotropic linear elasticity, simple closed-form expressions are easily obtained:

$$\bar{e} = \tag{14}$$

In any case, computation of $\bar{\mathbf{e}}$ is of course a local to each integration point of the element.

With $\bar{\mathbf{e}}$ in hand, the essential idea is to enhance $\hat{\mathbf{e}}$ so that $\hat{\mathbf{e}}^r + \hat{\mathbf{e}}^n$ reproduces $\bar{\mathbf{e}}$, plus the element-average of $\mathbf{e}^r + \mathbf{e}^n$. It is tempting to simply replace $\mathbf{e}^r + \mathbf{e}^n$ with $\bar{\mathbf{e}}$, thus producing, after constitutive evaluation, $\mathbf{T}^r + \mathbf{T}^n = \mathbf{0}$ in the actual problem solution. However, this would not work, for two reasons. First, even though $\mathbf{T}^r$ and $\mathbf{T}^n$ are usually small in thin structures, equilibrium would nevertheless not hold, in general, unless they can take nonzero values. Second, approximability of the overall method, and hence convergence, depends upon the condition that the enhanced strain $\hat{\mathbf{e}}$ have the same element average as the kinematic strain $\mathbf{e}$. This condition would fail, in general, with such a simple replacement. So, instead we take the following approach. For plate-like elements ($\rho_2 > \bar{\rho}$), the integration-point values of the enhanced (i.e. constitutive-input) strain are taken to be:

$$\begin{aligned}
\hat{\mathbf{e}}^t &= \mathbf{e}^t, \\
\hat{\mathbf{e}}^r &= (1-\beta)\langle \mathbf{e}^r, \mathrm{i}\rangle \mathrm{i} + \beta \mathbf{e}^r + \Big[\bar{\mathbf{e}}^r - \langle \bar{\mathbf{e}}^r, \mathrm{i}\rangle \mathrm{i}\Big], \\
\hat{\mathbf{e}}^n &= (1-\beta)\langle \mathbf{e}^n, \mathrm{i}\rangle \mathrm{i} + \beta \mathbf{e}^n + \Big[\bar{\mathbf{e}}^n - \langle \bar{\mathbf{e}}^n, \mathrm{i}\rangle \mathrm{i}\Big].
\end{aligned} \tag{15}$$

Here, β is a function of the aspect ratio ρ_1, and should vanish as the element becomes infinitely thin (i.e. as $\rho_1 \to 0$). All of the numerical results presented in the next section were obtained using

$$\beta = 0.5\rho_1^3. \tag{16}$$

The purpose of this decomposition is to segregate the part of the strain that is related to bending- and twisting-type deformation, which should be preserved in its “kinematic form,” from the part which is, in a sense, incidental to bending and twisting. The later components of strain are generally responsible for shear and thickness locking, and are the proper targets for modification via the enhancement term $\tilde{\mathbf{e}}$. In any given bending and/or twisting-type deformation, the strain component $\mathbf{e}^t$ In essence, with

The individual components of the strain must be treated differently; for this purpose it is convenient to resolve all strain tensors into components (e.g. e_{ij}) with respect to the principal triad. Considering the shear components first, we take the strain enhancement to be a negative factor times the projection of the kinematic strain component onto V':

$$\begin{aligned}
\tilde{e}_{12} &= -(1-f(\rho_2))f(\rho_1)\big[e_{12} - \langle e_{12}, \mathrm{i}\rangle \mathrm{i}\big], \\
\tilde{e}_{23} &= -f(\rho_1)\big[e_{23} - \langle e_{23}, \mathrm{i}\rangle \mathrm{i}\big] \\
\tilde{e}_{13} &= -f(\rho_1)\big[e_{13} - \langle e_{13}, \mathrm{i}\rangle \mathrm{i}\big].
\end{aligned} \tag{17}$$

The smooth function $f(\cdot)$ is set so that $f(0) = 1$ and $f(1) = 0$. A form for $f(\cdot)$ that has been found to give good performance in practice is $f(\rho) = 1 - \rho^4$. All illustrative computations presented in the next section use this form for $f(\cdot)$. The shear components of the enhancement strain are seen to vary in such a manner that the corresponding constitutive input $\hat{e}_{ij}$ varies between the kinematic strain itself, and its element-mean value, depending on the element's aspect ratios. Specifically, The $\hat{e}_{13}$ and $\hat{e}_{23}$ components, which correspond to transverse shear strains in the case of plate-like configurations, approach their element-mean values the thinner the element becomes; i.e. as $\rho_1 \to 0$. In order that $\hat{e}_{12}$ be subject to significant enhancement, it is further necessary that ρ_2 approach unity; i.e. the element should be not only thin, but also beam-like.

Turning now to the normal components of the strain enhancement, it is first noted that $\tilde{e}_{11}$ is taken to be zero under all circumstances, regardless of the values of the aspect ratios. The $\tilde{e}_{22},\ \tilde{e}_{33}$ components, on the other hand, will depend on *auxiliary values* $\varepsilon_1, \varepsilon_2, \varepsilon_3$ which are defined as follows. Consider the set of strain components $\{e_{11},\ e_{22},\ \varepsilon_1,\ e_{12}+\tilde{e}_{12},\ e_{23}+\tilde{e}_{23},\ e_{13}+\tilde{e}_{13}\}$, which consists of "kinematic" values in the 11 and 22 positions, "enhanced" shear components, and the auxiliary value ε_1 in place of the 33 entry. The latter value is to be computed such that the corresponding end-step stress $T_{33} = 0$. In the case of isotropic linear elasticity, this plane-stress condition is easily seen to give $\varepsilon_1 = -\nu(e_{11}+e_{22})/(1-\nu)$, where ν is the Poisson ratio. The $\varepsilon_2, \varepsilon_3$ values are computed in a similar manner: we consider the strains $\{e_{11},\ \varepsilon_2,\ \varepsilon_3,\ e_{12}+\tilde{e}_{12},\ e_{23}+\tilde{e}_{23},\ e_{13}+\tilde{e}_{13}\}$, and set $\varepsilon_2, \varepsilon_3$ so that $T_{22} = T_{33} = 0$. In the case of isotropic linear elasticity, we have $\varepsilon_2 = \varepsilon_3 = -\nu e_{11}$. For nonlinear materials, computation of the auxiliary values generally entails iterative evaluation of the constitutive update. With these values in hand, a pair of "composite" auxiliary values are computed as

$$\begin{aligned} \bar{\varepsilon}_2 &= (1 - f(\rho_2))\varepsilon_2, \\ \bar{\varepsilon}_3 &= (1 - f(\rho_2))\varepsilon_3 + f(\rho_2)\varepsilon_1. \end{aligned} \tag{18}$$

For plate-like configurations, $\bar{\varepsilon}_2 \approx 0$ and $\bar{\varepsilon}_3 \approx \varepsilon_1$, whereas $\bar{\varepsilon}_{2,3}$ approach $\varepsilon_{2,3}$ as the element becomes more beam-like. These composite values are now used to form the normal components of the strain enhancement:

$$\begin{aligned} \tilde{e}_{11} &= 0, \\ \tilde{e}_{22} &= -(1 - f(\rho_2))f(\rho_1)\big[e_{22} - \langle e_{22}, \mathfrak{i}\rangle\mathfrak{i} - \bar{\varepsilon}_2 + \langle\bar{\varepsilon}_2, \mathfrak{i}\rangle\mathfrak{i}\big], \\ \tilde{e}_{33} &= -f(\rho_1)\big[e_{33} - \langle e_{33}, \mathfrak{i}\rangle\mathfrak{i} - \bar{\varepsilon}_3 + \langle\bar{\varepsilon}_3, \mathfrak{i}\rangle\mathfrak{i}\big]. \end{aligned} \tag{19}$$

The meaning of this strain enhancement can be understood most easily by considering some special cases of element geometry. Firstly, for a thin, square, plate-like element, $f(\rho_1) \approx 1$ and $f(\rho_2) = 1$. In this case, $\hat{e}_{11}$ and $\hat{e}_{22}$ are very nearly the corresponding kinematic strains, while the transverse normal strain has a significant enhancement. This enhancement serves, essentially, to replace the kinematic e_{33} component with its element-average plus the projection of $\bar{\varepsilon}_3$ onto V'. The element-average of $\hat{e}_{33}$ is therefore equal to that of its kinematic counterpart, as required. But the variation of $\hat{e}_{33}$ among the element's integration points is now such that it is within an additive constant of the value that causes $T_{33} = 0$ at all integration points. This additive constant is controlled by the element-average of the kinematic e_{33}.

As a second special case, consider a slender beam: $f(\rho_1) \approx 1$ and $f(\rho_2) \approx 0$. In view of (18), $\bar{\varepsilon}_{2,3} \approx \varepsilon_{2,3}$ in this case. Were these values to be used directly as $\hat{e}_{22}$ and $\hat{e}_{33}$, then $T_{22} = T_{33} = 0$ would result. Instead, (19) provides for enhancements which cause $\hat{e}_{22}$ and $\hat{e}_{33}$ to equal the corresponding element-averaged kinematic strain, plus the projection of $\bar{\varepsilon}_2$ (respectively $\bar{\varepsilon}_3$ onto V'. Thus, the constitutive-input strains $\hat{e}_{22}$ and $\hat{e}_{33}$ are each within additive constants of the integration-point distributions that cause $T_{22} = T_{33} = 0$ at the end of the step. As in the case of the plate-like element, these additive constants are controlled by the element-average of the kinematic strains.

The above discussion is useful for purposes of explaining the purpose of (18, 19), but the conclusions strictly hold only in the limiting case of vanishing plate thickness or infinite beam slenderness. For finite but small thicknesses, $f(\rho_1)$ is slighly less than unity. This leads to a small, but important, presence of the projection of e_{22} and/or e_{33} onto V', in the corresponding constitutive-input strains. These contributions to the constitutive-input strains are needed to "stabilize" the element, as their absence can lead to a rank-defficiency in the assembled global stiffness matrix. However, if these components are too large, then thickness-locking results. This balance is the basis for the selection of the function $f(\rho) = 1 - \rho^4$, which will be seen, in the next section, to lead to favorable performance across a range of BVPs and mesh configurations, with element thickness ratios down to $\rho_1 = 10^{-4}$. At the other end of the thickness scale, the enhancement strains (17, 19) are designed to vanish as the element approaches a cube, thus recovering the standard 8-node hexahedral element.

6. Performance Assessment

7. Concluding Remarks