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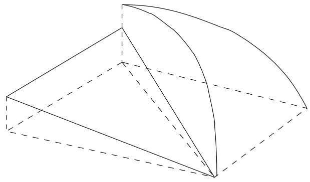

Chapter 5
Linear elasticity

This chapter illustrates the use of primal DG methods for a simple solid mechanics problem, namely the linear elasticity problem. We show that the DG scheme is very similar to the one obtained for elliptic problems.

5.1 Preliminaries

5.1.1 Strain and stress tensors

Let $\boldsymbol{u}(\boldsymbol{x})$ be the displacement vector at a point $\boldsymbol{x}$ of a homogeneous elastic body $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. The strain (or deformation) tensor $\boldsymbol{\epsilon}(\boldsymbol{u}) = (\epsilon_{kl}(\boldsymbol{u}))_{1\le k,l\le d}$ is defined by

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T),$$

or equivalently

$$\forall 1 \le k, l \le d, \quad \epsilon_{kl}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right),$$

since $\nabla \boldsymbol{u} = (\frac{\partial u_k}{\partial x_l})_{k,l}$. The stress tensor is denoted by $\boldsymbol{\sigma}(\boldsymbol{u}) = (\sigma_{ij}(\boldsymbol{u}))_{1\le i,j\le d}$ such that $\sigma_{ii}(\boldsymbol{u})$ is the normal stress in the direction x_i and $\sigma_{ij}(\boldsymbol{u})$ for $i \ne j$ are the shear stresses. The stress tensor satisfies the constitutive relationship:

$$\forall 1 \le i, j \le d, \quad \sigma_{ij}(\boldsymbol{u}) = \sum_{k,l=1}^{d} D_{ijkl}\epsilon_{kl}(\boldsymbol{u}), \tag{5.1}$$

where $\boldsymbol{D} = (D_{ijkl})_{ijkl}$ is a fourth order tensor satisfying some symmetry properties:

$$D_{ijkl} = D_{jikl} = D_{ijlk} = D_{klij}. \tag{5.2}$$

We assume that $\boldsymbol{D}$ is positive definite and piecewise constant in Ω, i.e,

$$\forall (\gamma_{ij})_{ij} \ne 0, \quad 0 < D_0 \sum_{ij} \gamma_{ij}^2 \le \sum_{ijkl} \gamma_{ij} D_{ijkl} \gamma_{kl} \le D_1 \sum_{ij} \gamma_{ij}^2. \tag{5.3}$$

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For example, in 2D, by Hooke's law, the stress tensor can be written as

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{pmatrix} = \begin{pmatrix} \lambda + 2\mu & \lambda & 0 \\ \lambda & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{pmatrix},$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé coefficients of the material.

5.1.2 Korn's inequalities

From the definition of the strain tensor, we immediately have

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d, \quad \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{L^2(\Omega)} \leq \|\nabla \boldsymbol{v}\|_{L^2(\Omega)}.$$

The reverse inequality is not true in general. If $\boldsymbol{v} \neq \boldsymbol{0}$ belongs to the space of rigid motions on Ω, then $\boldsymbol{\epsilon}(\boldsymbol{v}) = 0$. However, the reverse inequality is valid for functions vanishing on the boundary. This is Korn's first inequality [47, 29] in the usual Sobolev space $H_0^1(\Omega)^d$: there is a constant $C > 0$ such that

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d, \quad \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \leq C \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{L^2(\Omega)}.$$

In the Sobolev space $H^1(\Omega)^d$, the classical Korn inequality states that there is a constant $C > 0$ such that

$$\forall \boldsymbol{v} \in H^1(\Omega)^d, \quad \|\nabla \boldsymbol{v}\|_{L^2(\Omega)} \leq C(\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{L^2(\Omega)} + \|\boldsymbol{v}\|_{L^2(\Omega)}).$$

Korn's first inequality can be generalized to the broken Sobolev space $H^1(\mathcal{E}_h)^d$ (see [15]). Assume that Γ_{D} is a subset of the boundary $\partial\Omega$ with $|\Gamma_{\mathrm{D}}| > 0$. Then, there exists a positive constant C such that

$$\forall \boldsymbol{v} \in H^1(\mathcal{E}_h)^d, \quad |\!|\!|\nabla \boldsymbol{v}|\!|\!|_{H^0(\mathcal{E}_h)} \leq C \left(|\!|\!|\boldsymbol{\epsilon}(\boldsymbol{v})|\!|\!|^2_{H^0(\mathcal{E}_h)} + \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{1}{|e|^{\frac{1}{d-1}}} \|[\boldsymbol{v}]\|^2_{L^2(e)} \right)^{1/2}. \tag{5.4}$$

5.2 Model problem

Assume that the boundary of the elastic body is divided into two disjoint sets Γ_{D} and Γ_{N} and assume that a system of body forces $\boldsymbol{f} : \Omega \to \mathbb{R}^d$ and surface tractions $\boldsymbol{g}_{\mathrm{N}} : \Gamma_{\mathrm{N}} \to \mathbb{R}^d$ act on the body. On the other part Γ_{D} of the boundary, the body is rigidly fixed in space. Under the assumption of small displacements [65], the displacement $\boldsymbol{u} = (u_i)_{1 \leq i \leq d}$ satisfies the following problem:

$$-\sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j}(\boldsymbol{u}) = f_i \quad \text{in} \quad \Omega \quad \forall i = 1, \ldots, d, \tag{5.5}$$

$$u_i = 0 \quad \text{on} \quad \Gamma_D \quad \forall i = 1, \ldots, d, \tag{5.6}$$

$$\sum_{j=1}^{d} \sigma_{ij}(\boldsymbol{u}) n_j = g_i \quad \text{on} \quad \Gamma_N \quad \forall i = 1, \ldots, d, \tag{5.7}$$

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where $\boldsymbol{n} = (n_i)_{1\le i\le d}$ is the unit outward normal to the boundary $\partial\Omega$ and f_i and g_i are the components of the forces $\boldsymbol{f}$ and $\boldsymbol{g}_{\mathrm{N}}$. Equations (5.5) represent the equilibrium equations. From [29], there exists a weak solution $\boldsymbol{u} \in H_0^1(\Omega)^d$ satisfying

$$\forall \boldsymbol{v} \in H_0^1(\Omega)^d, \quad \int_\Omega \sum_{ijkl} D_{ijkl}\epsilon_{kl}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \int_{\Gamma_N} \boldsymbol{g}_{\mathrm{N}}\cdot\boldsymbol{v}.$$

Besides, if $\Gamma_N = \emptyset$, then $\boldsymbol{u} \in H^2(\Omega)^d$.

5.3 DG scheme

Let $\mathcal{E}_h$ be a subdivision of Ω. The notation used here is the same as in Section 2.3. We consider the space of vector functions that generalizes the definition (2.29):

$$\boldsymbol{\mathcal{D}}_k(\mathcal{E}_h) = (\mathcal{D}_k(\mathcal{E}_h))^d.$$

The DG approximation $\boldsymbol{U}_h \in \boldsymbol{\mathcal{D}}_k(\mathcal{E}_h)$ satisfies the discrete variational problem

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_k(\mathcal{E}_h), \quad a_\eta(\boldsymbol{U}_h, \boldsymbol{v}) = L(\boldsymbol{v}), \tag{5.8}$$

where the bilinear form $a_\eta : \boldsymbol{\mathcal{D}}_k(\mathcal{E}_h) \times \boldsymbol{\mathcal{D}}_k(\mathcal{E}_h) \to \mathbb{R}$ is given by

$$\begin{aligned} a_\eta(\boldsymbol{w}, \boldsymbol{v}) &= \sum_{E\in\mathcal{E}_h} \int_E \sum_{ijkl} D_{ijkl}\epsilon_{kl}(\boldsymbol{w})\epsilon_{ij}(\boldsymbol{v}) \\ &- \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \sum_{ijkl} \{D_{ijkl}\epsilon_{kl}(\boldsymbol{w})n_j^e\}[v_i] + \eta \sum_{e\in\Gamma_h\cup\Gamma_D} \int_e \sum_{ijkl} \{D_{ijkl}\epsilon_{kl}(\boldsymbol{v})n_j^e\}[w_i] \\ &+ \sum_{i=1}^d \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{\delta_e}{|e|^\beta} \int_e [w_i][v_i], \end{aligned}$$

and the linear form $L : \boldsymbol{\mathcal{D}}_k(\mathcal{E}_h) \to \mathbb{R}$ is defined by

$$L(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \int_{\Gamma_N} \boldsymbol{g}_{\mathrm{N}}\cdot\boldsymbol{v}.$$

To avoid any confusion with the strain tensor, the parameter that yields a symmetric or nonsymmetric bilinear form is denoted here by $\eta \in \{-1, 0, 1\}$. The last term in the bilinear form a_η is the penalty term with two additional parameters: the penalty value $\delta_e > 0$ that can vary from face to face and the power β that is usually taken equal to $(d-1)^{-1}$ but can be larger for a superpenalized DG method. The variable n_e^j denotes the jth component of $\boldsymbol{n}_e$.

The energy norm for the linear elasticity problem is defined below:

$$\|\boldsymbol{v}\|_{\mathcal{E}} = \left(\sum_{E\in\mathcal{E}_h} \|\boldsymbol{\epsilon}(\boldsymbol{v})\|^2_{L^2(E)} + \sum_{e\in\Gamma_h\cup\Gamma_D} \frac{\delta_e}{|e|^\beta} \|[\boldsymbol{v}]\|^2_{L^2(e)} \right)^{1/2}.$$

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5.3.1 Consistency

Let $\boldsymbol{u}$ be the solution of (5.5)–(5.7). Then, following a similar argument as in Section 2.4.1, one can obtain that $\boldsymbol{u}$ satisfies (5.8). The proof requires the additional result, which is an easy consequence of the symmetry of the stress tensor:

$$\sum_{1\le i,j\le d} \sigma_{ij}(\boldsymbol{u})\frac{\partial v_i}{\partial x_j} = \sum_{1\le i,j\le d} \sigma_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v}).$$

Indeed, we have

$$\begin{aligned}
\sum_{1\le i,j\le d} \sigma_{ij}(\boldsymbol{u})\frac{\partial v_i}{\partial x_j} &= \sum_{1\le i,j\le d} \frac{1}{2}\sigma_{ij}(\boldsymbol{u})\frac{\partial v_i}{\partial x_j} + \sum_{1\le i,j\le d} \frac{1}{2}\sigma_{ij}(\boldsymbol{u})\frac{\partial v_i}{\partial x_j}\\
&= \sum_{1\le i,j\le d} \frac{1}{2}\sigma_{ij}(\boldsymbol{u})\frac{\partial v_i}{\partial x_j} + \sum_{1\le j,i\le d} \frac{1}{2}\sigma_{ji}(\boldsymbol{u})\frac{\partial v_i}{\partial x_j}\\
&= \sum_{1\le i,j\le d} \frac{1}{2}\sigma_{ij}(\boldsymbol{u})\frac{\partial v_i}{\partial x_j} + \sum_{1\le i,j\le d} \frac{1}{2}\sigma_{ij}(\boldsymbol{u})\frac{\partial v_j}{\partial x_i}\\
&= \sum_{1\le i,j\le d} \sigma_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v}).
\end{aligned}$$

5.3.2 Local equilibrium

The analogous to local mass conservation for the elliptic problem (see Section 2.7.3) is here called local equilibrium, as the discretization of (5.5) is satisfied on each mesh element.

Lemma 5.1. *Fix a mesh element $E \in \mathcal{E}_h$, with outward normal $\boldsymbol{n}^E = (n_i^E)_{1\le i\le d}$. Let $\mathcal{N}(e;E)$ denote the element that shares the edge (or face) e with the element E and let U_h^i denote the ith component of $\boldsymbol{U}_h$:*

$$\begin{aligned}
\forall 1\le i\le d, \quad \int_E f_i = &-\sum_{jkl}\int_{\partial E\setminus\Gamma_{\mathrm{N}}} \{D_{ijkl}\epsilon_{kl}(\boldsymbol{U}_h)n_j^E\} - \int_{\partial E\cap\Gamma_{\mathrm{N}}} g_i\\
&+ \sum_{e\in\partial E\setminus\Gamma_{\mathrm{N}}} \frac{\delta_e}{|e|^\beta}\int_e (U_h^i|_E - U_h^i|_{\mathcal{N}(e;E)}).
\end{aligned}$$

Proof. For a fixed $1 \le i \le d$, choose the test function $\boldsymbol{v} = (v_j)_j$ in (5.8) such that $v_i = 1$ on E and zero elsewhere, and $v_j = 0$ for $j \ne i$. $\square$

5.3.3 Coercivity

Lemma 5.2. *If the parameter η is equal to -1 or 0, assume that the penalty value δ_e is sufficiently large and that $\beta \ge (d-1)^{-1}$. There exists a positive constant κ independent of h such that*

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_k(\mathcal{E}_h), \quad \kappa\|\boldsymbol{v}\|_{\mathcal{E}}^2 \le a_\eta(\boldsymbol{v},\boldsymbol{v}).$$

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Proof. If $\eta = 1$, we simply have from (5.3)

$$a_1(\boldsymbol{v}, \boldsymbol{v}) \geq D_0 \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_0^2 + \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{\delta_e}{|e|^\beta} \|[\boldsymbol{v}]\|_{L^2(e)}^2 \geq \min(D_0, 1) \|\boldsymbol{v}\|_{\mathcal{E}}^2.$$

If $\eta = -1$ or $\eta = 0$, we have

$$\begin{aligned} a(\boldsymbol{v}, \boldsymbol{v}) \geq\ & D_0 \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_0^2 + \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{\delta_e}{|e|^\beta} \|[\boldsymbol{v}]\|_{L^2(e)}^2 \\ & -(1-\eta) \sum_{e \in \Gamma_h \cup \Gamma_D} \sum_{ijkl} \int_e \{D_{ijkl} \epsilon_{kl}(\boldsymbol{v}) n_j^e\}[v_i]. \end{aligned}$$

It suffices to bound the last term of the inequality above. This is done using the trace inequality (2.5) and following a similar argument as in Section 2.7.1. □

A consequence of Korn's inequality (5.4) and the coercivity of the bilinear form a_η is the following lemma.

Lemma 5.3. *There exists a unique solution $\boldsymbol{U}_h$ to problem* (5.8).

Proof. It suffices to prove uniqueness. Denoting by $\boldsymbol{w}_h$ the difference of two solutions $\boldsymbol{U}_h^1$ and $\boldsymbol{U}_h^2$ to problem (5.8), we have

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_k(\mathcal{E}_h), \quad a_\eta(\boldsymbol{w}_h, \boldsymbol{v}) = 0.$$

Choosing $\boldsymbol{v} = \boldsymbol{w}_h$ and using Lemma 5.2, we have

$$\|\boldsymbol{w}_h\|_{\mathcal{E}} = 0.$$

Thus, $\boldsymbol{w}_h = \boldsymbol{0}$ from (5.4). □

5.4 Error analysis

A priori error estimates in the energy norm are given in the following theorem.

Theorem 5.4. *Let $k \geq 1$. Assume that $\beta = (d-1)^{-1}$ if the mesh contains quadrilaterals or hexahedra; otherwise assume that $\beta \geq (d-1)^{-1}$. Assume that the solution $\boldsymbol{u}$ of* (5.5)–(5.7) *belongs to $H^s(\mathcal{E}_h)^d$ for $s \geq 3/2$. Then, under the assumptions of Lemma* 5.2, *there is a constant C independent of h such that*

$$\|\boldsymbol{U}_h - \boldsymbol{u}\|_{\mathcal{E}} \leq C h^{\min(k+1,s)-1} \|\boldsymbol{u}\|_{H^s(\mathcal{E}_h)}. \tag{5.9}$$

Proof. The proof follows closely the proof of the error estimates for the elliptic problem (see Section 2.8). First, we obtain an orthogonality equation by using the consistency of the method:

$$\forall \boldsymbol{v} \in \boldsymbol{\mathcal{D}}_k(\mathcal{E}_h), \quad a_\eta(\boldsymbol{U}_h - \boldsymbol{u}, \boldsymbol{v}) = 0.$$

Let $\tilde{\boldsymbol{u}}$ be an approximation of $\boldsymbol{u}$ satisfying (2.10). Define $\boldsymbol{\chi} = \boldsymbol{U}_h - \tilde{\boldsymbol{u}}$ and choose $\boldsymbol{v} = \boldsymbol{\chi}$ in the equation above,

$$a_\eta(\boldsymbol{\chi}, \boldsymbol{\chi}) = a_\eta(\boldsymbol{u} - \tilde{\boldsymbol{u}}, \boldsymbol{\chi}),$$

or from the coercivity of a_η:

$$\begin{aligned}\kappa\|\boldsymbol{\chi}\|_{\mathcal{E}}^2 \leq &\sum_{E\in\mathcal{E}_h}\sum_{ijkl} D_{ijkl}\epsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\epsilon_{ij}(\boldsymbol{\chi}) - \sum_{e\in\Gamma_h\cup\Gamma_D}\sum_{ijkl}\int_e \{D_{ijkl}\epsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})n_j^e\}[\chi_i] \\ &+\eta\sum_{e\in\Gamma_h\cup\Gamma_D}\sum_{ijkl}\int_e\{D_{ijkl}\epsilon_{kl}(\boldsymbol{\chi})n_j^e\}[u_i-\tilde{u}_i] + \sum_{e\in\Gamma_h\cup\Gamma_D}\frac{\delta_e}{|e|^\beta}\int_e[\boldsymbol{u}-\tilde{\boldsymbol{u}}]\cdot[\boldsymbol{\chi}] \\ &= T_1+\cdots+T_4. \end{aligned} \tag{5.10}$$

We now bound the terms T_i, using Cauchy–Schwarz's, Young's inequalities, and the approximation bounds:

$$\begin{aligned} T_1 &\leq \frac{\kappa}{8}\sum_{E\in\mathcal{E}_h}\int_E D_{ijkl}\epsilon_{kl}(\boldsymbol{\chi})\epsilon_{ij}(\boldsymbol{\chi}) + C\sum_{E\in\mathcal{E}_h}\int_E D_{ijkl}\epsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})\epsilon_{ij}(\boldsymbol{u}-\tilde{\boldsymbol{u}}) \\ &\leq \frac{\kappa}{8}\|\boldsymbol{\chi}\|_{\mathcal{E}}^2 + Ch^{2\min(k+1,s)-2}|||\boldsymbol{u}|||_{H^s(\mathcal{E}_h)}^2. \end{aligned}$$

The second term is bounded as follows:

$$\begin{aligned} T_2 &\leq \sum_{e\in\Gamma_h\cup\Gamma_D}\sum_{ijkl}\left(\frac{|e|^\beta}{\delta_e}\right)^{\frac{1}{2}-\frac{1}{2}}\|\{D_{ijkl}\epsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})n_j^e\}\|_{0,e}\|[\chi_i]\|_{0,e} \\ &\leq \frac{\kappa}{8}\|\boldsymbol{\chi}\|_{\mathcal{E}}^2 + C\sum_{e\in\Gamma_h\cup\Gamma_D}\sum_{ijkl}\frac{|e|^\beta}{\delta_e}\|\{D_{ijkl}\epsilon_{kl}(\boldsymbol{u}-\tilde{\boldsymbol{u}})n_j^e\}\|_{0,e}^2 \\ &\leq \frac{\kappa}{8}\|\boldsymbol{\chi}\|_{\mathcal{E}}^2 + Ch^{2\min(k+1,s)-3+\beta(d-1)}|||\boldsymbol{u}|||_{H^s(\mathcal{E}_h)}^2. \end{aligned}$$

Thus, if $\beta(d-1)\geq 1$, we have

$$T_2 \leq \frac{\kappa}{8}\|\boldsymbol{\chi}\|_{\mathcal{E}}^2 + Ch^{2\min(k+1,s)-2}|||\boldsymbol{u}|||_{H^s(\mathcal{E}_h)}^2.$$

The terms T_3 and T_4 vanish if the approximation $\tilde{\boldsymbol{u}}$ is continuous. Such an approximation can be constructed in $\mathcal{D}_k(\mathcal{E}_h)$ if the mesh contains only triangular elements or tetrahedral elements. Otherwise, these two terms can be bounded using the trace inequalities (2.1), (2.5) and the approximation results (2.10). We remark that the bound for T_4 is valid if $\beta\leq(d-1)^{-1}$:

$$|T_3+T_4| \leq \frac{\kappa}{4}\|\boldsymbol{\chi}\|_{\mathcal{E}}^2 + Ch^{2\min(k+1,s)-2}|||\boldsymbol{u}|||_{H^s(\mathcal{E}_h)}^2.$$

From the bounds above, we conclude that

$$\|\chi\|_{\mathcal{E}} \leq Ch^{\min(k+1,s)-1}|||\boldsymbol{u}|||_{H^s(\mathcal{E}_h)},$$

which, with the triangle inequality, yields (5.9). □

Remark: If one wants to use a larger discrete space on quadrilaterals or hexahedra, namely the space $\mathbb{Q}_k$, then the a priori error estimate is valid for all $\beta\geq(d-1)^{-1}$.

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5.5 Bibliographical remarks

Primal DG methods for linear elasticity have been studied in [111, 110, 66, 90]. Mixed DG methods are considered in [71].

Exercises

5.1. Prove that if $\boldsymbol{u} \in H^2(\Omega)^d$ satisfies problem (5.5)–(5.7), then $\boldsymbol{u}$ satisfies problem (5.8).

5.2. Complete the proof of the coercivity of the bilinear form a_η.

5.3. Derive an L^2 estimate for the numerical error for the SIPG method.