For the element-level calculations, we require:

$$\mathbf{f}_a = \int_{\Omega_0} \mathbf{P} \cdot \nabla \varphi_a dV = \sum_{q=1}^{N_{qp}} w_q \mathbf{P}^{(q)} \cdot \nabla \varphi_a^{(q)} \tag{1}$$

Abstractly, we could consider the step-wise procedure of constructing $\mathbf{P}$:

$$\mathbf{P} = J \boldsymbol{\sigma} \mathbf{F}^{-T} \tag{2}$$

where

$$J = |\mathbf{F}| \tag{3}$$

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \mathbf{B}}(\mathbf{B}) \tag{4}$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T \tag{5}$$

$$\mathbf{F} = \mathbf{I} + \sum_{a=1}^{N_{nd}} \mathbf{u}_a \otimes \nabla \varphi_a \tag{6}$$

Suppose that we are interested in the compressible neo-hookean hyperelastic constitutive model:

$$W = \frac{\mu}{2}(I_1 - 3) - \mu \ln J + \frac{\lambda}{2}(\ln J)^2 \tag{7}$$

where

$$\boldsymbol{\sigma} = \frac{\mu}{J}(\mathbf{B} - \mathbf{I}) + \frac{\lambda}{J}(\ln J)\mathbf{I} \tag{8}$$

and

$$\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{B}} = -\frac{\mu}{J^2}\frac{\partial J}{\partial \mathbf{B}}\mathbf{B} + \frac{\mu}{J}\mathbf{I} \otimes \mathbf{I} - \frac{\lambda}{J^2}\frac{\partial J}{\partial \mathbf{B}}(\ln J)\mathbf{I} + \frac{\lambda + \mu}{J^2}\frac{\partial J}{\partial \mathbf{B}}\mathbf{I} \tag{9}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{B}} = \frac{\mu}{J}\mathbf{I} \otimes \mathbf{I} - \frac{1}{J^2}\frac{\partial J}{\partial \mathbf{B}}\Big((\lambda + \mu)\mathbf{I} - \mu\mathbf{B} - \lambda(\ln J)\mathbf{I}\Big) \tag{10}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{B}} = \frac{\mu}{J}\mathbf{I} \otimes \mathbf{I} + \frac{1}{J^2}\Big(\mu(\mathbf{B} - \mathbf{I}) + \lambda(\ln J - 1)\mathbf{I}\Big)\frac{\partial J}{\partial \mathbf{B}} \tag{11}$$

$$\frac{\partial J}{\partial \mathbf{B}} = \frac{\partial J}{\partial \mathbf{F}}\frac{\partial \mathbf{F}}{\partial \mathbf{B}} = J\mathbf{F}^{-T}\frac{\partial \mathbf{F}}{\partial \mathbf{B}} \tag{12}$$

$$\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{B}} = \frac{\mu}{J}\mathbf{I} \otimes \mathbf{I} + \frac{1}{J}\Big(\mu(\mathbf{B} - \mathbf{I}) + \lambda(\ln J - 1)\mathbf{I}\Big)\mathbf{F}^{-T}\frac{\partial \mathbf{F}}{\partial \mathbf{B}} \tag{13}$$

$$\frac{\partial \mathbf{F}}{\partial \mathbf{B}} = \left(\frac{\partial \mathbf{B}}{\partial \mathbf{F}}\right)^{-1} \tag{14}$$

$$\frac{\partial \mathbf{B}}{\partial \mathbf{F}} = \mathbf{I} \otimes \mathbf{F}^T + \mathbf{F} \otimes \mathbf{I} \tag{15}$$

$$\frac{\partial \sigma_{ij}}{\partial F_{kl}} = \frac{\mu}{J}(\delta_{ik}F_{jl} + \delta_{jk}F_{il}) + \frac{1}{J}\Big(\mu(B_{ij} - \delta_{ij}) + \lambda(\ln J - 1)\delta_{ij}\Big)F_{lk}^{-1} \tag{16}$$

Or

$$\sigma_{ik} = \frac{1}{J}\left(\mu(F_{il}F_{kl} - \delta_{ik}) + \lambda(\ln J)\delta_{ik}\right) \tag{17}$$

$$\frac{\partial \sigma_{ik}}{\partial F_{mn}} = \frac{1}{J}\left(\mu(\delta_{im}F_{kn} + F_{in}\delta_{km}) + \left[\mu(\delta_{ik} - F_{il}F_{kl}) + \lambda(1 - \ln J)\delta_{ik}\right] F^{-1}_{nm}\right) \tag{18}$$

So

$$P_{ij} = J\sigma_{ik}F^{-1}_{jk} \tag{19}$$

$$P_{ij} = \mu(F_{ij} - F^{-1}_{ji}) + \lambda(\ln J)F^{-1}_{ji} \tag{20}$$

$$\frac{\partial P_{ij}}{\partial F_{mn}} = J\sigma_{ik}(F^{-1}_{nm}F^{-1}_{jk} - F^{-1}_{jm}F^{-1}_{nk}) \tag{21}$$

$$+\left(\mu(\delta_{im}\delta_{jn} + F_{in}F^{-1}_{jm}) + \left[\mu(F^{-1}_{ji} - F_{ij}) + \lambda(1 - \ln J)F^{-1}_{ji}\right] F^{-1}_{nm}\right) \tag{22}$$

$$\frac{\partial P_{ij}}{\partial F_{mn}} = \left(\mu\left[(F_{ij} - F_{ji})F^{-1}_{nm} + (F_{ni} - F_{in})F^{-1}_{jm}\right] + \lambda(\ln J)(F^{-1}_{ji}F^{-1}_{nm} - F^{-1}_{ni}F^{-1}_{jm})\right) \tag{23}$$

$$+\left(\mu(\delta_{im}\delta_{jn} + F_{in}F^{-1}_{jm}) + \left[\mu(F^{-1}_{ji} - F_{ij}) + \lambda(1 - \ln J)F^{-1}_{ji}\right] F^{-1}_{nm}\right) \tag{24}$$

$$\frac{\partial P_{ij}}{\partial F_{mn}} = \mu(\delta_{im}\delta_{jn} + F^{-1}_{ni}F^{-1}_{jm}) + \lambda(F^{-1}_{ji}F^{-1}_{nm} - (\ln J)F^{-1}_{ni}F^{-1}_{jm}) \tag{25}$$

Ultimately:

$$P_{ij} = \mu(F_{ij} - F^{-1}_{ji}) + \lambda(\ln J)F^{-1}_{ji} \tag{26}$$

$$\frac{\partial P_{ij}}{\partial F_{mn}} = \mu(\delta_{im}\delta_{jn} + F^{-1}_{ni}F^{-1}_{jm}) + \lambda(F^{-1}_{ji}F^{-1}_{nm} - (\ln J)F^{-1}_{ni}F^{-1}_{jm}) \tag{27}$$

And

$$F_{mn} = \delta_{mn} + \sum_{a=1}^{N_{nd}} u_{ma}\varphi_{a,n} \tag{28}$$

$$P_{ij} = \mu(F_{ij} - F^{-1}_{ji}) + \lambda(\ln J)F^{-1}_{ji} \tag{29}$$

$$\frac{\partial F_{mn}}{\partial u_{kb}} = \delta_{mk}\varphi_{b,n} \tag{30}$$

$$\frac{\partial P_{ij}}{\partial u_{kb}} = \left[\mu(\delta_{ik}\delta_{jn} + F^{-1}_{ni}F^{-1}_{jk}) + \lambda(F^{-1}_{ji}F^{-1}_{nk} - (\ln J)F^{-1}_{ni}F^{-1}_{jk})\right]\varphi_{b,n} \tag{31}$$

Finally:

$$f_{ia} = \sum_q w_q P^{(q)}_{ij}\varphi^{(q)}_{a,j} \tag{32}$$

$$K_{iakb} = \sum_q w_q \frac{\partial P^{(q)}_{ij}}{\partial u_{kb}}\varphi^{(q)}_{a,j} \tag{33}$$