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A ROBUST QUADRILATERAL MEMBRANE FINITE ELEMENT WITH DRILLING DEGREES OF FREEDOM

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SUMMARY

A quadrilateral membrane finite element with drilling degrees of freedom is derived from variational principles employing an independent rotation field. Both displacement based and mixed approaches are investigated. The element exhibits excellent accuracy characteristics. When combined with a plate bending element, the element provides an efficient tool for linear analysis of shells.

1. INTRODUCTION

The need for membrane elements with drilling degrees of freedom (see Figure 1) arises in many practical engineering problems (e.g. in-filled frames, folded plates, etc.). When combined with a bending element, a membrane element of this kind provides a versatile tool for analysis of shells. While early efforts to construct an element of this type were unsuccessful, more recent works, including the independent approaches of Allman[1] and Bergan and Fellipa,[5] opened the prospectives for a successful solution. As a result, the revived interest of the engineering community in membrane elements with drilling degrees of freedom is manifested by a series of papers on the subject.[2,3,6,15,19,20] However, most of the proposed solutions are based on some form of 'free formulation', which concentrates solely on the choice of the finite element interpolation fields. The clever procedures of the finite element 'technology' are used to improve the performance of proposed elements.

A novel approach, which relies on a variational formulation employing an independent rotation field, has been presented recently by Hughes and Brezzi.[10] Reissner[17] was the first to suggest a variational formulation which utilized the skew-symmetric part of the stress tensor as a Lagrange multiplier to enforce the equality of independent rotations with the skew-symmetric part of the displacement gradient. A similar formulation is also given by de Veubeke.[8] In a significant contribution to the solution of this problem, Hughes and Brezzi[10] have extended Reissner's formulation by recognizing the instability of discrete approximations and suggested a way in which the discrete approximation can be stabilized. Some membrane elements with drilling degrees of freedom, derived from the displacement-type formulation of Hughes and Brezzi, are presented by Hughes *et al.*[11] Their work has assumed that the variational formulation employs an independent rotation field, i.e. strictly speaking it is based on the separate kinematics variables of displacement and rotation.

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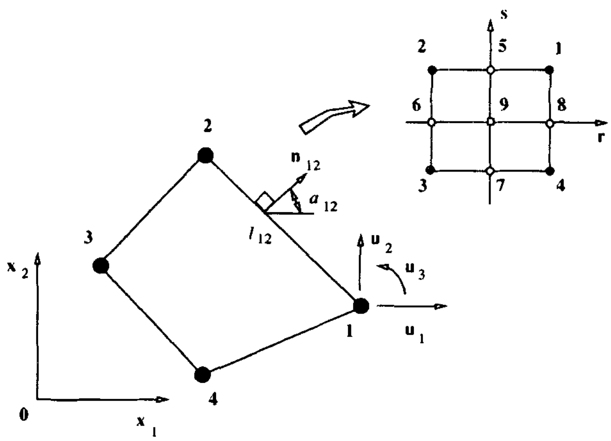

Figure 1. A quadrilateral element with drilling degrees of freedom

In this work, we extend the applications of Hughes and Brezzi[10] to combine an Allman-type interpolation for the displacement field with an independently interpolated rotation field. A mixed-type variational formulation is presented with the skew-symmetric part of the stress tensor introduced as a Lagrange multiplier to enforce the equality of independent rotations with the skew-symmetric components of the displacement gradient. A penalty displacement-type formulation with selective reduced integration is also employed. The displacement and rotation fields are the same in both formulations. In the mixed-type formulation the skew-symmetric part of stress is chosen constant over each element.

An outline of the paper is as follows. In Section 2 we give a short review for both the displacement-type and the mixed-type variational formulation. Our consideration follows closely the work of Hughes and Brezzi. The finite element interpolation and the discrete versions for both variational formulations are given in Section 3. Numerical evaluations of the derived membrane elements are presented in Section 4. Some closing remarks are given in Section 5.

2. VARIATIONAL FORMULATION

In this section we follow closely the work of Hughes and Brezzi.[10] The same index-free notation is utilized. For the sake of brevity, the discussion of boundary conditions is omitted. Inclusion of boundary conditions presents no difficulties for considerations to follow, and it can be handled in a standard manner (e.g. see Hughes[9]). We also limit ourselves to linear elastostatic problems.

Let Ω be a region occupied by a body. The boundary value problem under consideration is: For all $\mathbf{x} \in \Omega$

$$\operatorname{div} \boldsymbol{\sigma} + \mathbf{f} = \mathbf{0} \tag{1}$$

$$\operatorname{skew} \boldsymbol{\sigma} = \mathbf{0} \tag{2}$$

$$\boldsymbol{\psi} = \operatorname{skew} \nabla \mathbf{u} \tag{3}$$

$$\operatorname{symm} \boldsymbol{\sigma} = \mathbf{C} \cdot \operatorname{symm} \nabla \mathbf{u} \tag{4}$$

where (1) to (4) are, respectively, the equilibrium equations, the symmetry conditions for stress, the definition of rotation in terms of displacement gradient and the constitutive equations.

In (1) to (4) the Euclidian decomposition of second-rank tensors is employed, e.g.

$$\boldsymbol{\sigma} = \operatorname{symm} \boldsymbol{\sigma} + \operatorname{skew} \boldsymbol{\sigma} \tag{5}$$

where

$$\operatorname{symm} \boldsymbol{\sigma} = \tfrac{1}{2} (\boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{T}}) \tag{6}$$

$$\operatorname{skew} \boldsymbol{\sigma} = \tfrac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{T}}) \tag{7}$$

For the isotropic case and plane stress state, the constitutive modulus tensor $\mathbf{C} = \{C_{ijkl}\}$ has the form

$$C_{ijkl} = \lambda\, \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) \qquad i,j,k,l \in \{1,2\} \tag{8}$$

where

$$\lambda = \frac{vE}{(1 - v^2)} \tag{9}$$

$$\mu = \frac{E}{2(1 + v)} \tag{10}$$

while E and v are Young's modulus and Poisson's ratio, respectively.

Reissner[17] presented a variational formulation for the boundary value problem (1) to (4). This principle leads to a formulation which is inappropriate for numerical applications. Essentially, too many parameters for the skew-symmetric part of $\boldsymbol{\sigma}$ exist and the numerical problem fails the LBB conditions as well as the counts for the mixed patch test (e.g. see Zienkiewicz and Taylor[24]). Hughes and Brezzi modified variational problem of Reissner in order to preserve the stability of the discrete problem. The modification (see Hughes and Brezzi[10]) preserves (1) to (4) as the Euler–Lagrange equations. In addition, the symmetrical components of stress are eliminated using the constitutive equations (4) to give

Problem (M)

$$\begin{aligned}\Pi_\gamma(\mathbf{v}, \boldsymbol{\omega}, \operatorname{skew} \boldsymbol{\tau}) = {} & \frac{1}{2} \int_\Omega \operatorname{symm} (\nabla \mathbf{v}) \cdot \mathbf{C} \cdot (\operatorname{symm} \nabla \mathbf{v})\, \mathrm{d}\Omega + \int_\Omega \operatorname{skew} \boldsymbol{\tau}^{\mathrm{T}} \cdot (\operatorname{skew} \nabla \mathbf{v} - \boldsymbol{\omega})\, \mathrm{d}\Omega \\ & - \frac{1}{2}\gamma^{-1} \int_\Omega |\operatorname{skew} \boldsymbol{\tau}|^2\, \mathrm{d}\Omega - \int_\Omega \mathbf{v} \cdot \mathbf{f}\, \mathrm{d}\Omega \end{aligned} \tag{11}$$

where $\mathbf{v} \in \mathbf{V}$, $\boldsymbol{\omega} \in \mathbf{W}$, $\boldsymbol{\tau} \in \mathbf{T}$ are spaces of trial displacements, rotations and stresses. This variational formulation requires that the rotations $\boldsymbol{\omega}$ and stresses $\boldsymbol{\tau}$, together with the displacement generalized derivatives $\nabla \mathbf{v}$, belong to the space of square-integrable functions over the region Ω.

The variational equation which results from variations on (11)

$$\begin{aligned} 0 = \mathrm{D}\Pi_\gamma (\mathbf{u}, \boldsymbol{\psi}, \operatorname{skew} \boldsymbol{\sigma}) \cdot (\mathbf{v}, \boldsymbol{\omega}, \operatorname{skew} \boldsymbol{\tau}) = {} & \int_\Omega (\operatorname{symm} \nabla \mathbf{v}) \cdot \mathbf{C} \cdot (\operatorname{symm} \nabla \mathbf{u})\, \mathrm{d}\Omega \\ + \int_\Omega \operatorname{skew} \boldsymbol{\tau}^{\mathrm{T}} \cdot (\operatorname{skew} \nabla \mathbf{u} - \boldsymbol{\psi})\, \mathrm{d}\Omega & + \int_\Omega (\operatorname{skew} \nabla \mathbf{v}^{\mathrm{T}} \cdot \operatorname{skew} \boldsymbol{\sigma} - \boldsymbol{\omega}^{\mathrm{T}} \cdot \operatorname{skew} \boldsymbol{\sigma})\, \mathrm{d}\Omega \\ - \gamma^{-1} \int_\Omega \operatorname{skew} \boldsymbol{\tau}^{\mathrm{T}} \cdot \operatorname{skew} \boldsymbol{\sigma}\, \mathrm{d}\Omega & - \int_\Omega \mathbf{v} \cdot \mathbf{f}\, \mathrm{d}\Omega \end{aligned} \tag{12}$$

In the next section the variational equation (12) is used to construct a mixed-type discrete formulation. It is possible to eliminate the skew-symmetric part of the stress tensor (see Hughes and Brezzi[10]) by substituting

$$\gamma^{-1}\,\text{skew}\,\boldsymbol{\sigma} = \text{skew}\,\nabla\mathbf{u} - \boldsymbol{\psi} \tag{13}$$

into *Problem* (M) to obtain

Problem (D)

$$\tilde{\Pi}_\gamma(\mathbf{v},\boldsymbol{\omega}) = \frac{1}{2}\int_\Omega \text{symm}\,(\nabla\mathbf{v})\cdot\mathbf{C}\cdot(\text{symm}\,\nabla\mathbf{v})\,\mathrm{d}\Omega + \frac{1}{2}\gamma\int_\Omega |\text{skew}\,\nabla\mathbf{v} - \boldsymbol{\omega}|^2\,\mathrm{d}\Omega - \int_\Omega \mathbf{v}\cdot\mathbf{f}\,\mathrm{d}\Omega \tag{14}$$

The corresponding variational equation now is

$$0 = \mathrm{D}\tilde{\Pi}_\gamma(\mathbf{u},\boldsymbol{\psi})\cdot(\mathbf{v},\boldsymbol{\omega}) = \int_\Omega (\text{symm}\,\nabla\mathbf{v})\cdot\mathbf{C}\cdot(\text{symm}\,\nabla\mathbf{u})\,\mathrm{d}\Omega + \gamma\int_\Omega (\text{skew}\,\nabla\mathbf{v} - \boldsymbol{\omega})^{\mathrm{T}}\cdot(\text{skew}\,\nabla\mathbf{u} - \boldsymbol{\psi})\,\mathrm{d}\Omega - \int_\Omega \mathbf{v}\cdot\mathbf{f}\,\mathrm{d}\Omega \tag{15}$$

The variational equation (15) is taken as the basis for constructing the displacement-type discrete formulation which is presented in the next section.

The parameter γ, which appears in the formulations is problem dependent (see Hughes and Brezzi[10]). For isotropic elasticity and Dirichlet boundary value problems Hughes *et al.*[11] suggest γ be taken as the shear modulus value, i.e. $\gamma = \mu$. However, the numerical studies we have performed have shown that the formulation is insensitive to the value of γ used (at least for several orders of magnitude which bound the shear modulus).

3. FINITE ELEMENT INTERPOLATION

The particular choice for finite dimensional spaces $\mathbf{V}^h$, $\mathbf{W}^h$, $\mathbf{T}^h$ (subspaces of $\mathbf{V}$, $\mathbf{W}$, $\mathbf{T}$, respectively) is presented in this section along with the resulting discrete formulations.

We first consider the discrete version of *Problem* (M)

Problem (M^h)

$$0 = \int_{\Omega^h} (\text{symm}\,\nabla\mathbf{v}^h)^{\mathrm{T}}\cdot\mathbf{C}\cdot(\text{symm}\,\nabla\mathbf{u}^h)\,\mathrm{d}\Omega + \int_{\Omega^h} \text{skew}\,\boldsymbol{\tau}^{h\mathrm{T}}\cdot(\text{skew}\,\nabla\mathbf{u}^h - \psi^h)\,\mathrm{d}\Omega + \int_{\Omega^h} (\text{skew}\,\nabla\mathbf{v}^{h\mathrm{T}}\cdot\text{skew}\,\boldsymbol{\sigma}^h - \omega^{h\mathrm{T}}\cdot\text{skew}\,\boldsymbol{\sigma}^h)\,\mathrm{d}\Omega - \gamma^{-1}\int_{\Omega^h} \text{skew}\,\boldsymbol{\tau}^{h\mathrm{T}}\cdot\text{skew}\,\boldsymbol{\sigma}^h\,\mathrm{d}\Omega - \int_{\Omega^h} \mathbf{v}^{h\mathrm{T}}\cdot\mathbf{f}\,\mathrm{d}\Omega \tag{16}$$

We consider a 4-node quadrilateral element with degrees of freedom shown in Figure 1. The independent rotation field is interpolated as a standard bilinear field over each element. Accordingly

$$u_3 \equiv \psi^h = \sum_e \sum_{I=1}^{4} N_I^e(r,s)\,\psi_I \tag{17}$$

where (e.g. see Zienkiewicz and Taylor[24])

$$N_I^e(r, s) = \tfrac{1}{4}(1 + r_I r)(1 + s_I s); \quad I = 1, 2, 3, 4 \tag{18}$$

The in-plane displacement approximation is taken as an Allman-type interpolation (see Figure 1)

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbf{u}^h = \sum_e \sum_{I=1}^{4} N_I^e(r, s)\, \mathbf{u}_I + \sum_e \sum_{I=5}^{8} NS_I^e(r, s) \frac{l_{JK}}{8} (\psi_K - \psi_J)\, \mathbf{n}_{JK}$$

$$+ \sum_e NB_9^e(r, s)\, \Delta\mathbf{u}_9 \tag{19}$$

where l_{JK} and $\mathbf{n}_{JK}$ are the length and the outward unit normal vector on the element side associated with the corner nodes J and K, i.e.

$$\mathbf{n}_{JK} = \begin{Bmatrix} n_1 \\ n_2 \end{Bmatrix} = \begin{Bmatrix} \cos\alpha_{JK} \\ \sin\alpha_{JK} \end{Bmatrix} \qquad l_{JK} = ((x_{K1} - x_{J1})^2 + (x_{K2} - x_{J2})^2)^{1/2} \tag{20}$$

and a FORTRAN-like definition of adjacent corner nodes

$$J = 1 - 4; \quad K = \operatorname{mod}(I, 4) + 1 \tag{21}$$

In (19) we also employ serendipity shape functions defined by (see Zienkiewicz and Taylor[24])

$$NS_I^e(r, s) = \tfrac{1}{2}(1 - r^2)(1 + s_I s); \quad I = 5, 7 \tag{22}$$

$$NS_I^e(r, s) = \tfrac{1}{2}(1 + r_I r)(1 - s^2); \quad I = 6, 8 \tag{23}$$

To reflect the superior performance of the 9-node Lagrangian element over that for the 8-node serendipity element, a hierarchical bubble function interpolation is added in (19) where

$$NB_9^e(r, s) = (1 - r^2)(1 - s^2) \tag{24}$$

The terms in the element stiffness matrix arising from this interpolation may be eliminated at the element level by static condensation (see Wilson[23]).

The skew-symmetric stress field is chosen constant over the element, i.e.

$$\operatorname{skew} \tau^h = \sum_e \tau_0^e \tag{25}$$

We further define matrix notation

$$\operatorname{symm} \nabla \mathbf{u}^e = \mathbf{B}_I^e \mathbf{u}_I + \mathbf{G}_I^e \psi_I \tag{26}$$

where $\mathbf{u}_I$ and ψ_I are nodal values of the displacement and the rotation fields, respectively. The $\mathbf{B}_I^e$ matrix in (26) has the standard form

$$\mathbf{B}_I^e = \begin{bmatrix} N_{I,x_1}^e & 0 \\ 0 & N_{I,x_2}^e \\ N_{I,x_2}^e & N_{I,x_1}^e \end{bmatrix}; \qquad I = 1, 2, 3, 4 \tag{27}$$

and the part of the displacement interpolation associated with the rotation defines

$$\mathbf{G}_I^e = \frac{1}{8} \begin{bmatrix} (l_{IJ}\cos\alpha_{IJ}\, NS_{L,x_1}^e - l_{IK}\cos\alpha_{IK}\, NS_{M,x_1}^e) \\ (l_{IJ}\sin\alpha_{IJ}\, NS_{L,x_2}^e - l_{IK}\sin\alpha_{IK}\, NS_{M,x_2}^e) \\ (l_{IJ}\cos\alpha_{IJ}\, NS_{L,x_2}^e - l_{IK}\cos\alpha_{IK}\, NS_{M,x_2}^e) + (l_{IJ}\sin\alpha_{IJ}\, NS_{L,x_1}^e - l_{IK}\sin\alpha_{IK}\, NS_{M,x_1}^e) \end{bmatrix} \tag{28}$$

where, in (28) above

$$I = 1, 2, 3, 4;\ M = I + 4;\ L = M - 1 + 4\,\mathrm{aint}(1/I);\ K = \mathrm{mod}(M, 4) + 1;\ J = L - 4 \tag{29}$$

Furthermore, we denote

$$\mathrm{skew}\,\nabla\mathbf{u}^e - \psi^e = \mathbf{b}_I^e \mathbf{u}_I + g_I^e \psi_I \tag{30}$$

where

$$\mathbf{b}_I^e = \langle -\tfrac{1}{2} N_{I,x_2}^e; \tfrac{1}{2} N_{I,x_1}^e \rangle, \quad I = 1, 2, 3, 4 \tag{31}$$

and

$$\begin{aligned} g_I^e = [& -\tfrac{1}{16}(l_{IJ} \cos\alpha_{IJ}\, NS_{L,x_2}^e - l_{IK} \cos\alpha_{IK}\, NS_{M,x_2}^e) \\ & + \tfrac{1}{16}(l_{IJ} \sin\alpha_{IJ}\, NS_{L,x_1}^e - l_{IK} \sin\alpha_{IK}\, NS_{M,x_1}^e) - N_I^e]; \quad I = 1, 2, 3, 4 \end{aligned} \tag{32}$$

with indices J, K, L, M again defined by (29).

The first term in the discrete formulation (16) of *Problem* (M^h) gives rise to the element stiffness matrix

$$\mathbf{K}^e = \int_{\Omega^e} [\mathbf{B}^e \quad \mathbf{G}^e]^{\mathrm{T}}\, \mathbf{C}\, [\mathbf{B}^e \quad \mathbf{G}^e]\, \mathrm{d}\Omega \tag{33}$$

The second term in (16) is denoted

$$\mathbf{h}^e = \int_{\Omega^e} \langle \mathbf{b}^e; \mathbf{g}^e \rangle^{\mathrm{T}}\, \mathrm{d}\Omega \tag{34}$$

With this notation at hand, the discrete mixed-type formulation can be rewritten as

$$\begin{bmatrix} \mathbf{K}^e & \mathbf{h}^e \\ \mathbf{h}^{e\mathrm{T}} & -\gamma^{-1}\Omega^e \end{bmatrix} \begin{Bmatrix} \mathbf{a} \\ \tau_0^e \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ 0 \end{Bmatrix}; \quad \mathbf{a} = \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\psi} \end{Bmatrix} \tag{35}$$

Since the skew-symmetric part of the stress is interpolated independently in each element, the corresponding part of the stiffness matrix (35), may be eliminated at the element level using static condensation to yield

$$\hat{\mathbf{K}}^e \mathbf{a} = \mathbf{f}; \quad \hat{\mathbf{K}}^e = \mathbf{K}^e + \frac{\gamma}{\Omega^e} \mathbf{h}^e \mathbf{h}^{e\mathrm{T}} \tag{36}$$

In a completely analogous manner we can construct an approximation for the displacement-type variational formulation. The discrete version of *Problem* (D) follows from (15)

Problem (D^h)

$$\begin{aligned} 0 = & \int_{\Omega^h} (\mathrm{symm}\,\nabla\mathbf{v}^h)^{\mathrm{T}} \cdot \mathbf{C} \cdot (\mathrm{symm}\,\nabla\mathbf{u}^h)\, \mathrm{d}\Omega \\ & + \gamma \int_{\Omega^h} (\mathrm{skew}\,\nabla\mathbf{v}^h - \omega^h)^{\mathrm{T}} \cdot (\mathrm{skew}\,\nabla\mathbf{u}^h - \psi^h)\, \mathrm{d}\Omega - \int_{\Omega^h} \mathbf{v}^{h\mathrm{T}} \cdot \mathbf{f}\, \mathrm{d}\Omega \end{aligned} \tag{37}$$

The rotation and displacement fields are again interpolated by (17) and (19), respectively.

The first term in the displacement-type formulation (37) produces the same element stiffness matrix $\mathbf{K}^e$ defined by (33). The second term in (37), however, is different. Note that using the interpolations for displacement (19) and rotation (17), this term can be directly obtained via (30):

$$\mathbf{P}^e = \gamma \int_{\Omega^e} \begin{Bmatrix} \mathbf{b}^e \\ \mathbf{g}^e \end{Bmatrix} \langle \mathbf{b}^e;\, \mathbf{g}^e \rangle \, d\Omega \tag{38}$$

Hence the matrix counterpart of (37) for one element in a displacement-type formulation is

$$[\mathbf{K}^e + \mathbf{P}^e]\,\mathbf{a} = \mathbf{f}; \qquad \mathbf{a} = \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\psi} \end{Bmatrix} \tag{39}$$

The parts of the element stiffness matrix $\mathbf{K}^e$ and $\mathbf{h}^e$ in (35) and (39) are computed using 3×3 Gaussian quadrature. The matrix $\mathbf{P}^e$ in (39) is integrated by a single point Gaussian quadrature. By fully integrating $\mathbf{K}^e$ and combining with $\mathbf{P}^e$ or $\mathbf{h}^e\,\mathbf{h}^{e\mathrm{T}}\,\gamma/\Omega^e$ the spurious zero energy modes are prevented; and no additional devices are needed (see, e.g. MacNeal and Harder[15]). The 'equivalence theorem' of Malkus and Hughes[16] may be used to show that this approach of selective reduced integration is equivalent to the mixed formulation (36). The only difference in our case, however, occurs owing to hierarchical interpolation of the displacement field by the bubble function (see (19)). For selective reduced integration of the displacement-type interpolation the bubble function gives no contribution to the penalty stiffness $\mathbf{P}^e$, as opposed to its analogue of a rank-one update ($\mathbf{h}^e\mathbf{h}^{e\mathrm{T}}\,\gamma/\Omega^e$) in the mixed-type approach. This difference occurs only for skewed elements, and, as demonstrated by the numerical examples, it is of minor importance.

4. NUMERICAL EVALUATION

Several numerical examples are presented to demonstrate the accuracy of the membrane element presented herein. The element is also combined with the well known DKQ plate bending element[4] and used to solve a spherical shell with a hole, one of the problems in the set posed by MacNeal and Harder.[14] To avoid the membrane locking the modification suggested by Taylor[19] is performed. In addition, the 8-point integration rule on $\mathbf{K}^e$ is used. However, the modification of Jetteur[13] for a warped element is not needed.

Both mixed-type (36) and displacement-type formulations (39) are evaluated. In results to follow they are denoted as M-type and D-type, respectively.

4.1. The patch test

First a patch test[21] is performed on a one-element test. This will not only test the coding for our elements but can also detect any spurious modes which may exist in the elements. A skewed element (see Figure 2) is fixed with a minimum number of constraints and exposed to uniform tension. Both displacement-type and mixed-type formulations pass the patch test.

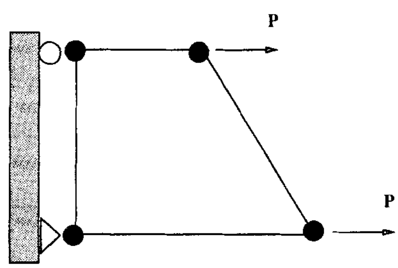

Figure 2. The patch test—One-element test

Similar formulations with an Allman-type interpolation field,[1, 20] however, do not pass the one-element patch test. The reason is the presence of a spurious mode which occurs for constant values of nodal rotations.

4.2. *A simple beam: The higher order patch test*

A simple beam with a length to height aspect ratio of 10 is subjected to a pure bending state. The beam is modelled by one row of six membrane elements with drilling degrees of freedom, as shown in Figure 3. No drilling degree of freedom is restrained; only a minimum number of restraints is imposed. Two load cases are considered. The first load case is a unit couple applied at the free end and represents a higher order patch test.[21] When a regular mesh is used, the solution is exact. For a distorted mesh (see Figure 3) the accuracy is still good. The second load case is, to our knowledge, a novel test. The loading is again a unit moment, but this time applied as a concentrated moment at the drilling degrees of freedom at both ends. The value of parameter γ set equal to the shear modulus yields excellent results. The difference from the exact solution, for a regular mesh, is due to the fact that a single concentrated moment at a drilling degree of freedom is not a consistent loading (which follows from displacement interpolation (19)). The results of the analysis can be compared with the beam theory exact solution of 1·5 for vertical displacement and 0·6 for end rotation.

The same analysis is repeated for the membrane element of Taylor and Simo.[20] To prevent the occurrence of the spurious mode, besides the minimum number of restraints (see Figure 3), one drilling degree of freedom is fixed as well. The results of this analysis are also presented in Table I. Note that, in this case, the computed rotations are wrong as dictated by the need to restrain the spurious singular rotation mode.

4.3. *A cantilever beam*

A shear-loaded cantilever beam is selected as a test problem by many authors.[3, 5, 11, 15] The elasticity solution (e.g. see Timoshenko and Goodier[22]) for the tip displacement is

$$u_2 = \frac{Pl^3}{3EI} + \frac{(4+5\nu)Pl}{2Eh} = 0{\cdot}3553$$

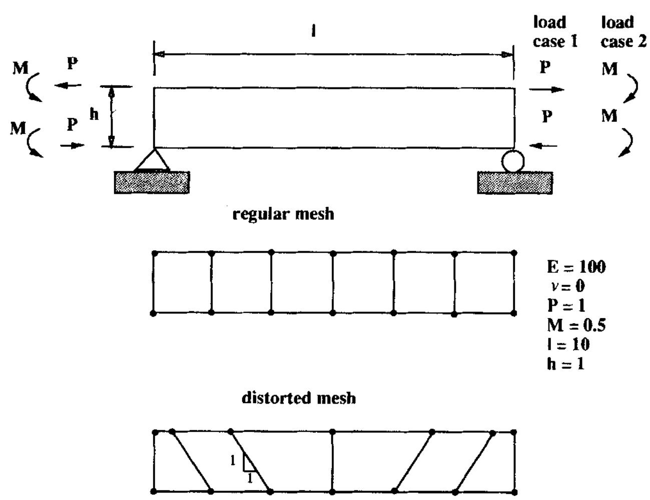

Figure 3. A simple beam

Table I. A simple beam (Figure 3)

Formulation	Mesh	Load case	Vert. displ.	End rot.
M-type	reg.	1	1·5	0·6
M-type	dist.	1	1·14185	0·57255
M-type	reg.	2	1·5	0·62070
M-type	dist.	2	1·39220	0·50612
D-type	reg.	1	1·5	0·6
D-type	dist.	1	1·14045	0·57247
D-type	reg.	2	1·5	0·62070
D-type	dist.	2	1·39200	0·49508
Taylor and Simo[20]	reg.	1	1·5	1·2
Taylor and Simo[20]	dist.	1	1·14195	1·10485
Taylor and Simo[20]	reg.	2	1·5	2·18980
Taylor and Simo[20]	dist.	2	1·39300	2·30490

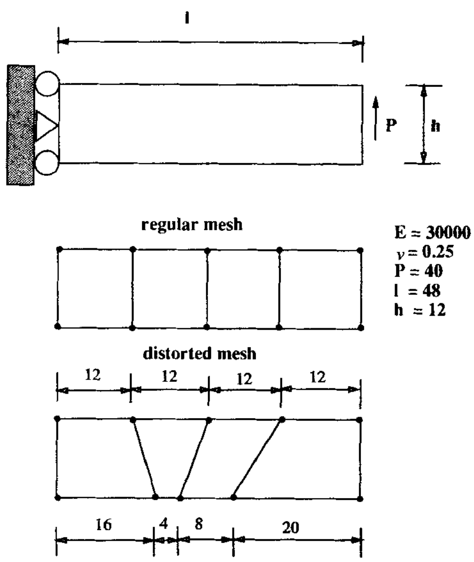

Figure 4. Short cantilever beam

Table II. Short cantilever beam (Figure 4)

Mesh	Allman[3]	MacNeal and Harder[15]	M-type	D-type
4 × 1	0·3026	0·3409	0·3445	0·3445
8 × 2	0·3394	—	0·3504	0·3504
16 × 4	0·3512	—	0·3543	0·3543
4 × 1*	—	0·2978	0·3066	0·3065

*Irregular mesh after MacNeal and Harder[15]

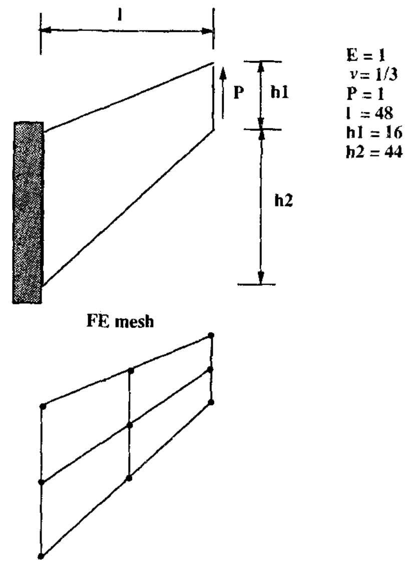

Figure 5. Cook's membrane

Table III. Cook's membrane (Figure 5)

Mesh	Allman[3]	Simo *et al.*[18]	M-type	D-type
1 × 1	—	16·743	14·066	14·065
2 × 2	20·27	21·124	20·683	20·682
4 × 4	22·78	23·018	22·993	22·984
8 × 8	23·56	23·685	23·668	23·626

for the properties selected (see Figure 4 for details, and also pages 219–220 and 254–255 of Hughes[9]).

The finite element solution is obtained for a coarse mesh of four square elements and also for finer meshes constructed by bisection. The results obtained are compared with some of the results available in the literature. All are presented in Table II.

4.4. *Cook's problem*

A trapezoidal membrane (Figure 5), suggested by Cook,[7] is another popular test problem.[3,5,18] Besides the shear dominated behaviour (similar to that of the previous test), it also displays the effects of mesh distortion. The results for the tip deflection (see Table III) can be compared to the reference value 23·91, obtained by numerical analysis for a refined model.

4.5. *Hemispherical shell with 18° hole*

The membrane presented herein is combined with a DKQ plate element[4] to construct a flat quadrilateral shell element. The performance of the shell element is evaluated on a standard test

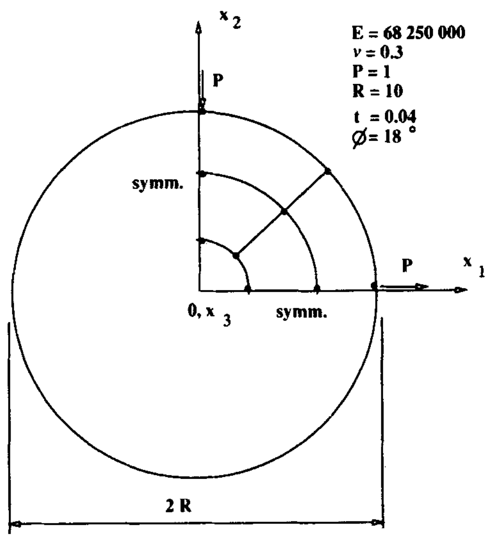

Figure 6. Pinched hemispherical shell with a hole

Table IV. Hemispherical shell (Figure 6)

Mesh	Taylor[19]	Simo *et al.*[18]	M-type	D-type
4 × 4	0·086524	0·093372	0·087548	0·087528
8 × 8	0·094153	0·092814	0·093714	0·093701
12 × 12	0·093679	—	0·093587	0·093584
16 × 16	0·093501	0·092907	0·093488	0·093487

Table V. Hemispherical shell (Figure 6)—Mesh 8 × 8

γ/μ	M-type	D-type
0.001	0·093967	0·093853
0.05	0·093813	0·093711
1·0	0·093714	0·093701
50·0	0·093700	0·093700
1000·0	0·093700	0·093700

problem of a hemispherical shell with a hole[14] (see Figure 6). It is important to establish that the proposed formulation causes no membrane locking when applied to shell analysis. The results of this analysis should be compared with the solution of 0·094 given by MacNeal and Harder[14] and the value of 0·093 suggested recently by Simo *et al.*[18] The performance of the shell element (see Table IV) is only slightly better than the one of Taylor,[19] since the difference consists only in the new membrane formulation. Even for a relatively coarse mesh, the accuracy of the element is comparable to the geometrically exact shell model of Simo *et al.*[18]

The analysis of this problem is repeated for different values of γ, other than $\gamma = \mu$ used to obtain the results presented in Table IV. The finite element model with the 8 × 8 mesh is used for this purpose. The results of the analysis are presented in Table V. Note that the formulation is rather insensitive to the chosen value of γ. For the higher values of γ the results exhibit an asymptotic behaviour. This is a consequence of enforcing the equality between the independent rotation field and skew-symmetric part of the displacement gradient.

5. CLOSURE

We have presented a novel membrane element with drilling degrees of freedom based on a variational formulation which employs an independent rotation field. The equality of the independent rotation field and skew-symmetric part of the displacement gradient is enforced in both the displacement-type and the mixed-type formulations. This novelty enables a robust element performance in the problems where the loading is a concentrated moment directly applied at the drilling degree of freedom. It also provides a consistency in combining the element into the general finite element model for in-filled frames, folded plates or similar complex structural systems. The element exhibits excellent accuracy characteristics for both regular and distorted meshes. When the membrane element of this kind is combined with a plate bending element, a flat shell element is formed which also performs with high accuracy. The quadrilateral element formulation presented herein can be expanded to the adequate triangular element, so that both elements can be accommodated within a single computer program modulus. This we address in our forthcoming paper.[12]

REFERENCES

1. D. J. Allman, 'A compatible triangular element including vertex rotations for plane elasticity problems', *Comp. Struct.*, **19**, 1–8 (1984).
2. D. J. Allman, 'The constant strain triangle with drilling rotations: A simple prospect for shell analysis', in J. R. Whiteman (ed.). *Proceedings The Mathematics of Finite Elements and Applications*, Academic Press, New York, 1987, pp. 230–236.
3. D. J. Allman, 'A quadrilateral finite element including vertex rotations for plane elasticity problems', *Int. j. numer. methods eng.*, **26**, 717–739 (1988).
4. J. L. Batoz and M. B. Tahar, 'Evaluation of a new quadrilateral thin plate bending element', *Int. j. numer. methods eng.*, **18**, 1655–1677 (1982).
5. P. G. Bergan and C. A. Fellipa, 'A triangular membrane element with rotational degrees of freedom', *Comp. Methods Appl. Mech.*, **50**, 25–60 (1985).
6. N. Carpenter, H. Stolarski and T. Belytschko, 'A flat triangular shell element with improved membrane interpolation', *Commun. appl. numer. methods*, **1**, 161–168 (1985).
7. R. D. Cook, 'Improved two-dimensional finite element', *J. Struct. Div. ASCE*, **100**, 1851–1865 (1974).
8. F. de Veubeke, 'Stress function approach', *Proceedings World Congress on Finite Element Methods in Solid Mechanics, Vol. 1*, Bournemouth, England, 1975, pp.1–5.
9. T. J. R. Hughes, *The Finite Element Method: Linear Static and Dynamic Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1987.
10. T. J. R. Hughes and F. Brezzi, 'On drilling degrees of freedom', *Comp. Methods Appl. Mech. Eng.*, **72**, 105–121 (1989).
11. T. J. R. Hughes, F. Brezzi, A. Masud and I. Harari, 'Finite element with drilling degrees of freedom: Theory and numerical evaluation', *Preprint*, 1989.
12. A. Ibrahimbegovic, E. L. Wilson and R. L. Taylor, 'A unified formulation for quadrilateral and triangular flat shell elements with drilling degrees of freedom', *Commun. appl. numer. methods*, submitted.
13. P. Jetteur, 'Improvement of the quadrilateral 'jet' shell element for a particular class of problems', *IREM Internal Report 87/1*, Ecole Polytechnique Federale de Laussane, Laussane, 1987.
14. R. H. MacNeal and R. L. Harder, 'A proposed standard set of problems to test finite element accuracy', *J. Finite Elem. Anal. Des.*, **1**, 3–20 (1985).
15. R. H. MacNeal and R. L. Harder, 'A refined four-noded membrane element with rotational degrees of freedom', *Comp. Struct.*, **18**, 75–84 (1988).
16. D. S. Malkus and T. J. R. Hughes, 'Mixed finite element methods—Reduced and selective integration techniques: A unification of concepts', *Comp. Methods Appl. Mech. Eng.*, **15**, 68–81 (1978).
17. E. Reissner, 'A note on variational principles in elasticity', *Int. J. Solids Struct.*, **1**, 93–95 (1965).
18. J. C. Simo, D. D. Fox and M. S. Rifai, 'On a stress resultant geometrically exact shell model Part II: The linear theory; computational aspects', *Comp. methods Appl. Mech. Eng.*, **73**, 53–92 (1989).

19. R. L. Taylor, 'Finite element analysis of linear shell problems', in J. R. Whiteman (ed.), *Proceedings The Mathematics of Finite Elements and Applications*, Academic Press, New York, 1987, pp. 211–223.
20. R. L. Taylor and J. C. Simo, 'Bending and membrane elements for analysis of thick and thin shells', in J. Middelton and G. N. Pande (eds.), *Proceedings NUMETA 85*, 1985, pp. 587–591.
21. R. L. Taylor, J. C. Simo, O. C. Zienkiewicz and A. C. Chan, 'The patch test: A condition for assessing finite element convergence', *Int. j. numer. methods eng.*, **22**, 39–62 (1986).
22. S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, McGraw-Hill, New York, 1951.
23. E. L. Wilson, 'The static condensation algorithm', *Int. j. numer. methods eng.*, **8**, 199–203 (1974).
24. O. C. Zienkiewicz and R. L. Taylor, *The Finite Element Method: Basic Formulation and Linear Problems, Vol. I*, McGraw-Hill, New York, 1989.