On the Sensitivity of Finite Elements to Mesh Distortions

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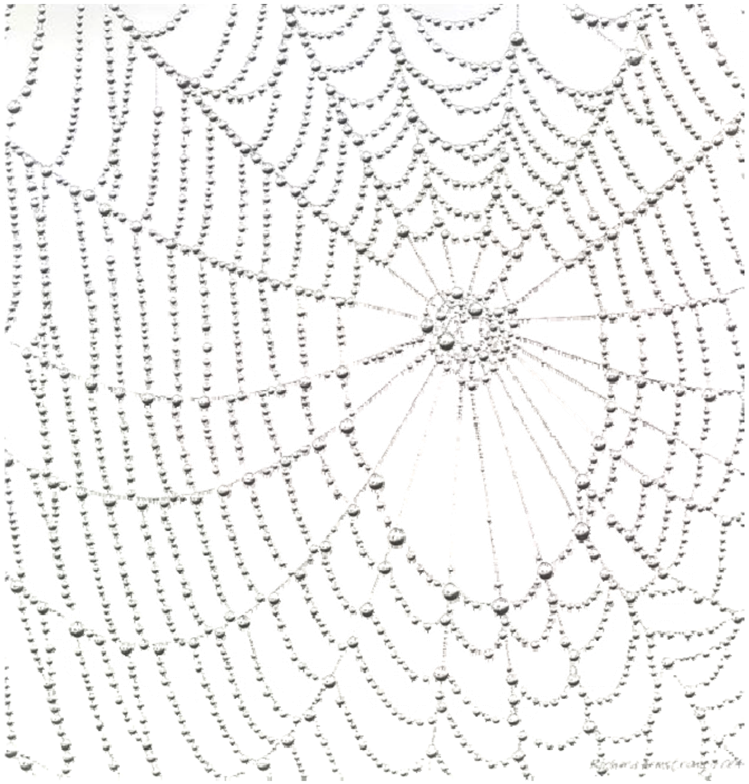

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motivation

- locking phenomena in structural and solid finite elements
- “optimal” technologies available for rectangular element shapes
- different technologies are identical for rectangular element shapes

$\Rightarrow$ sensitivity to mesh distortions is one of the last open problems (further challenges in non-linear analysis and 3d-shells)

How to Deal with this Problem

1. **avoid mesh distortions**

   complex geometries
   adaptive (re-) meshing

How to Deal with this Problem

1. **avoid mesh distortions**

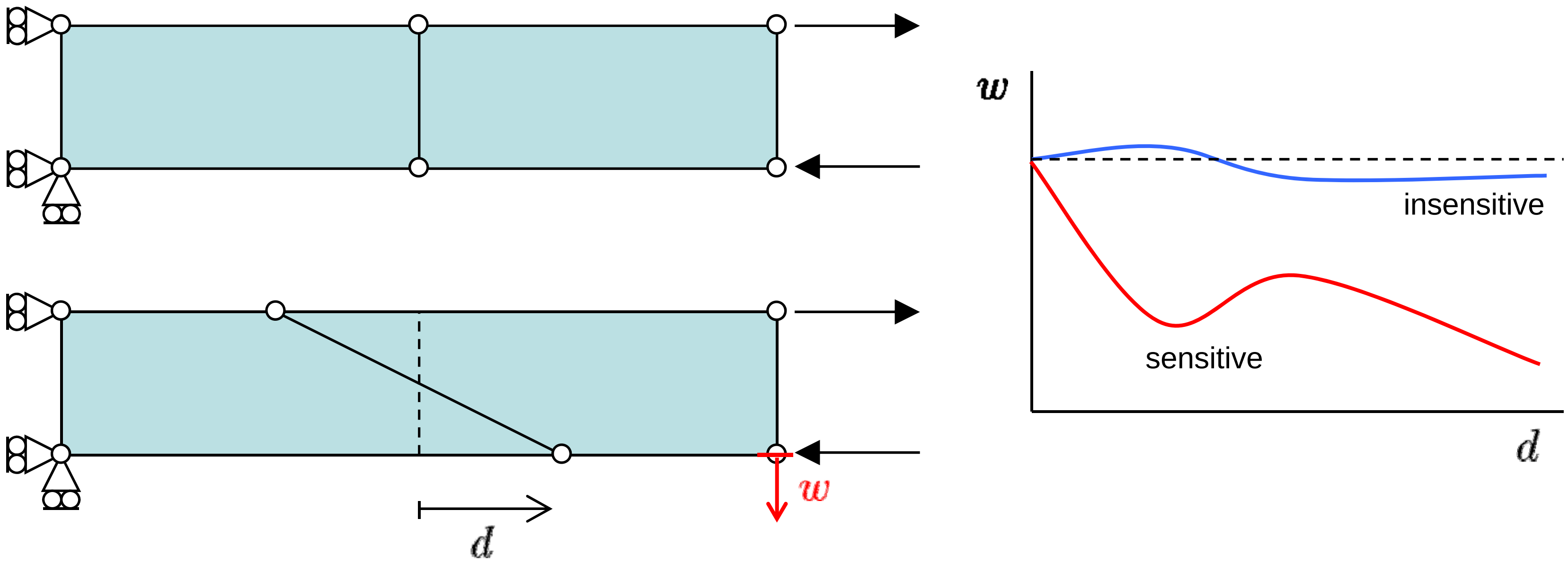
   complex geometries
   adaptive (re-) meshing

2. **develop finite elements which are insensitive to mesh distortions**

   two-dimensional elasticity
   four-node elements

Measuring Distortion Sensitivity

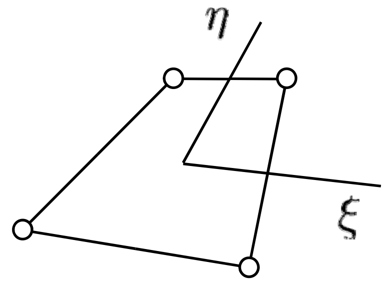
the most popular test: bending of a cantilever



various versions in the literature

Element Types under Investigation

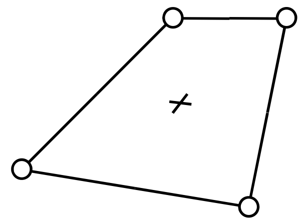
four-node plane stress elements



Q1

standard Galerkin formulation

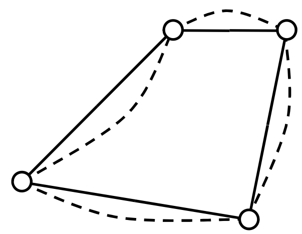
suffers from shear locking and volumetric locking



Q1-SRI

selective reduced integration of shear part

no shear locking, o.k. for small Poisson's ratio



Qm6

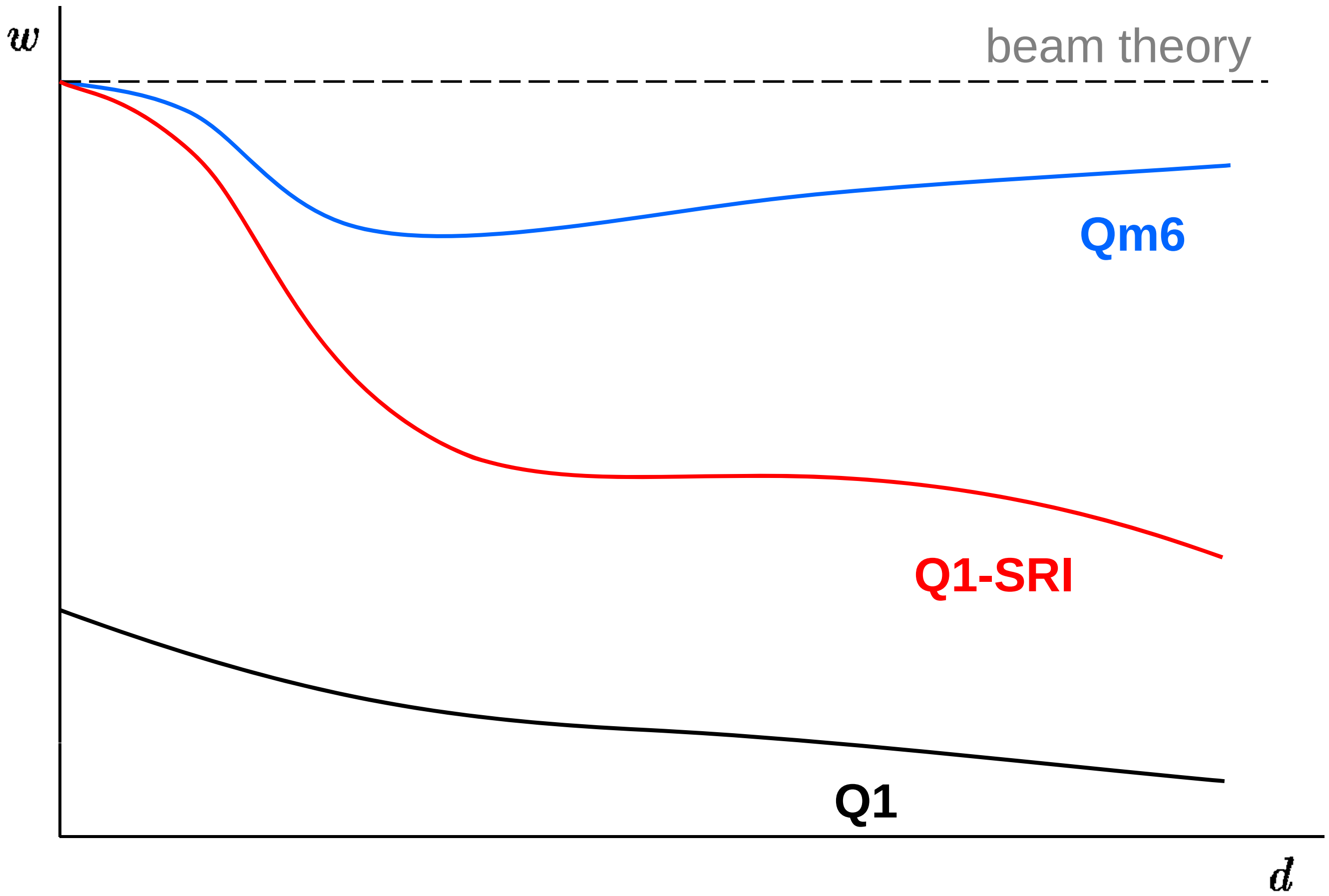
method of incompatible modes (= Q1-E4)

locking-free (?)

Taylor, Wilson (1973, 1976), Simo et al. (1990)

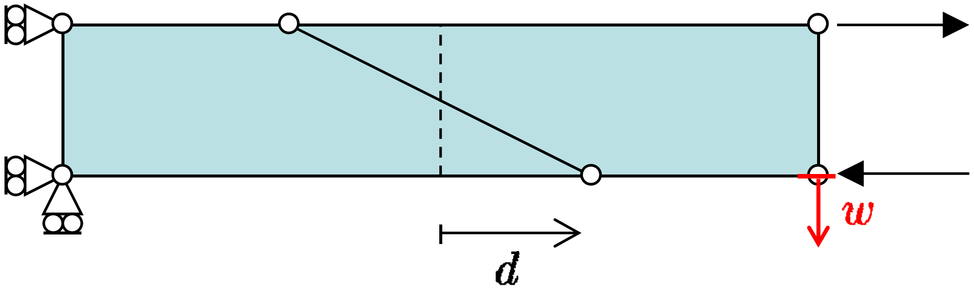
Two Element Bending Test

typical results



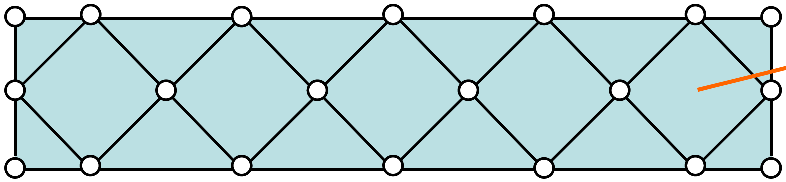
Two Element Bending Test

how meaningful is this test setup?

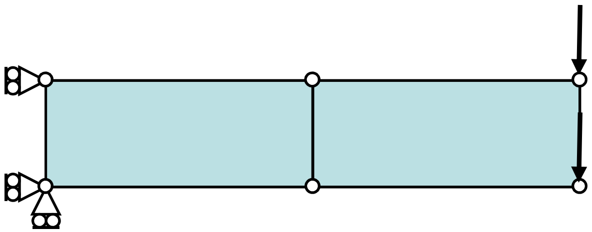


stresses in quadrilaterals
are identically zero!

is this a distorted mesh?

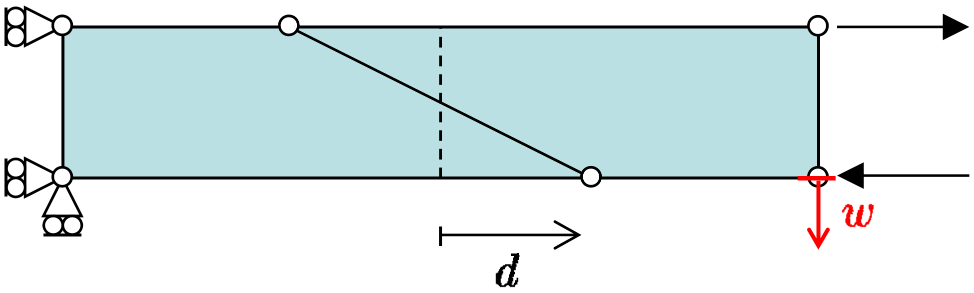


undistorted elements – different results



Two Element Bending Test

how meaningful is this test setup?



numerical experiment is restricted to

- specific loads and boundary conditions
- constant-linear stress distribution
- principal stresses aligned to edges
- “exact” solution = beam solution

may be a **hint** toward optimization of element technology
but not a **guideline**

Element Quality $\leftrightarrow$ Tendency to Locking

eigenvalues as "objective" measure of element quality

$$\boldsymbol{K} \cdot \boldsymbol{D} = \boldsymbol{F}$$

$$\boldsymbol{K} \cdot \boldsymbol{D_i} = \boldsymbol{\lambda_i} \cdot \boldsymbol{D_i}$$ (no sum on *i*)

$$\boldsymbol{K} \cdot \boldsymbol{D}_i - \lambda \cdot \boldsymbol{D}_i = (\boldsymbol{K} - \lambda_i \cdot \boldsymbol{I}) \cdot \boldsymbol{D}_i = \boldsymbol{0}$$

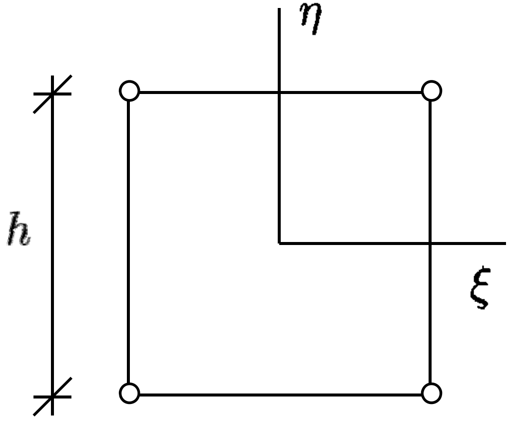
eigenvalue λ = stiffness

eigenvector **D** = deformation mode

- **no need to choose specific loads and boundary conditions**
- **eigenvalue spectrum = element deformation spectrum**

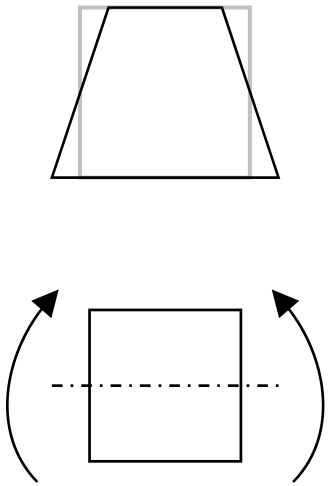
Eigenvalue Analysis

shear locking

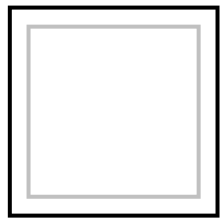


trapezoidal mode $\boldsymbol{D}_i = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

optimal: $\lambda_i = c \cdot h^3$



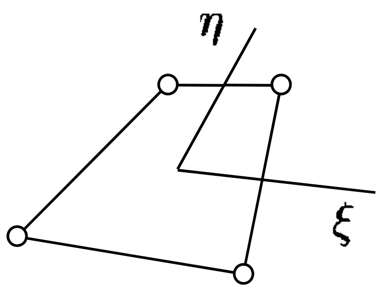
volumetric locking



optimal: only **one** eigenvalue with $\lim\limits_{\nu \to 0.5} \lambda_i \to \infty$

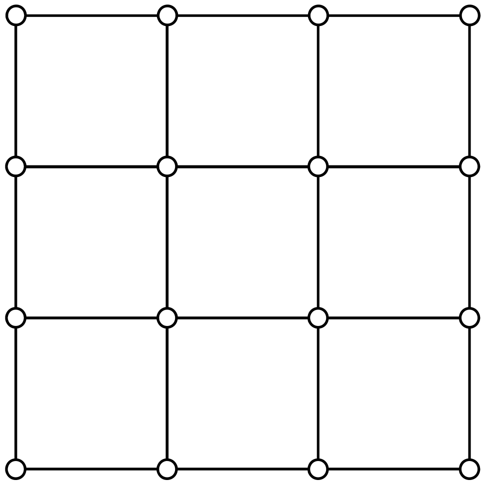
distortion sensitivity

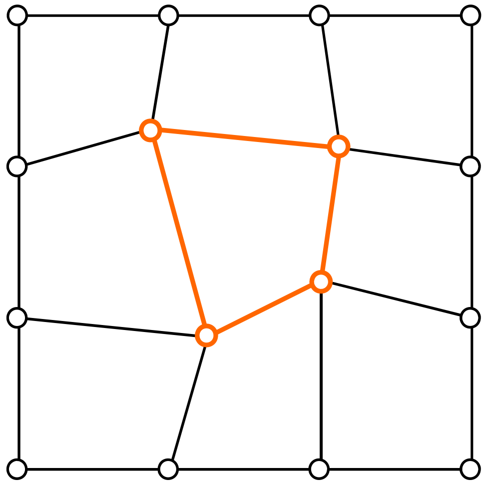
what are the “correct” values of $\lambda_1 \ldots \lambda_8$
for arbitrary element shapes?



Considering a Patch of Finite Elements

9 elements, 16 nodes, 32 d.o.f. $\Rightarrow$ spectrum of 32 eigenvalues





undistorted, “optimal” mesh
provides reference solution

$\lambda_1 \ldots \lambda_{32}$

$\boldsymbol{D}_1 \ldots \boldsymbol{D}_{32}$

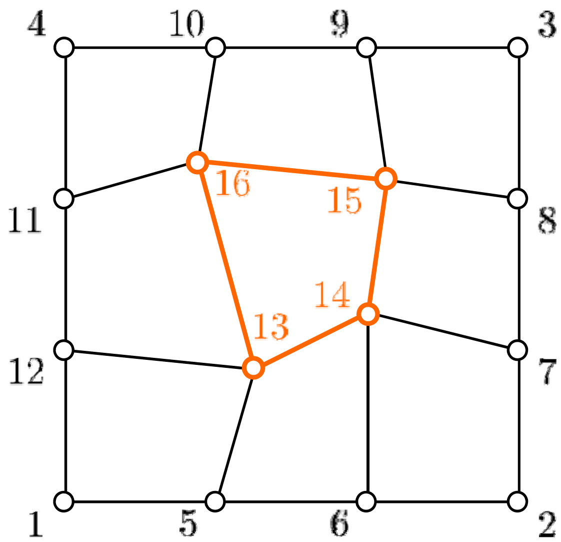
distorted mesh
perturbed eigenvalue spectrum

$\tilde{\lambda}_1 \ldots \tilde{\lambda}_{32}$

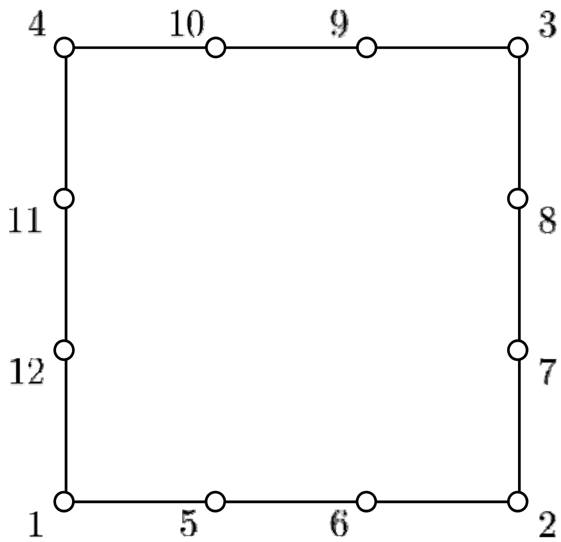
$\tilde{\boldsymbol{D}}_1 \ldots \tilde{\boldsymbol{D}}_{32}$

Distortion Patch Test

removing internal d.o.f. via static condensation



static condensation of node 13 – 16



$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{(1..24),(1..24)} & \boldsymbol{K}_{(1..24),(25..32)} \\ \boldsymbol{K}_{(25..32),(1..24)} & \boldsymbol{K}_{(25..32),(25..32)} \end{bmatrix}$$

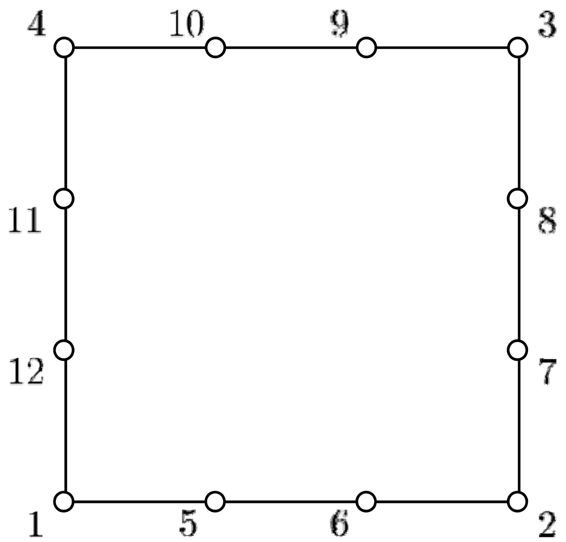
$$= \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_{22} \end{bmatrix}$$

macro element with

$$\boldsymbol{K}_{red} = \boldsymbol{K}_{11} - \boldsymbol{K}_{12} \cdot \boldsymbol{K}_{22}^{-1} \cdot \boldsymbol{K}_{21}$$

Distortion Patch Test

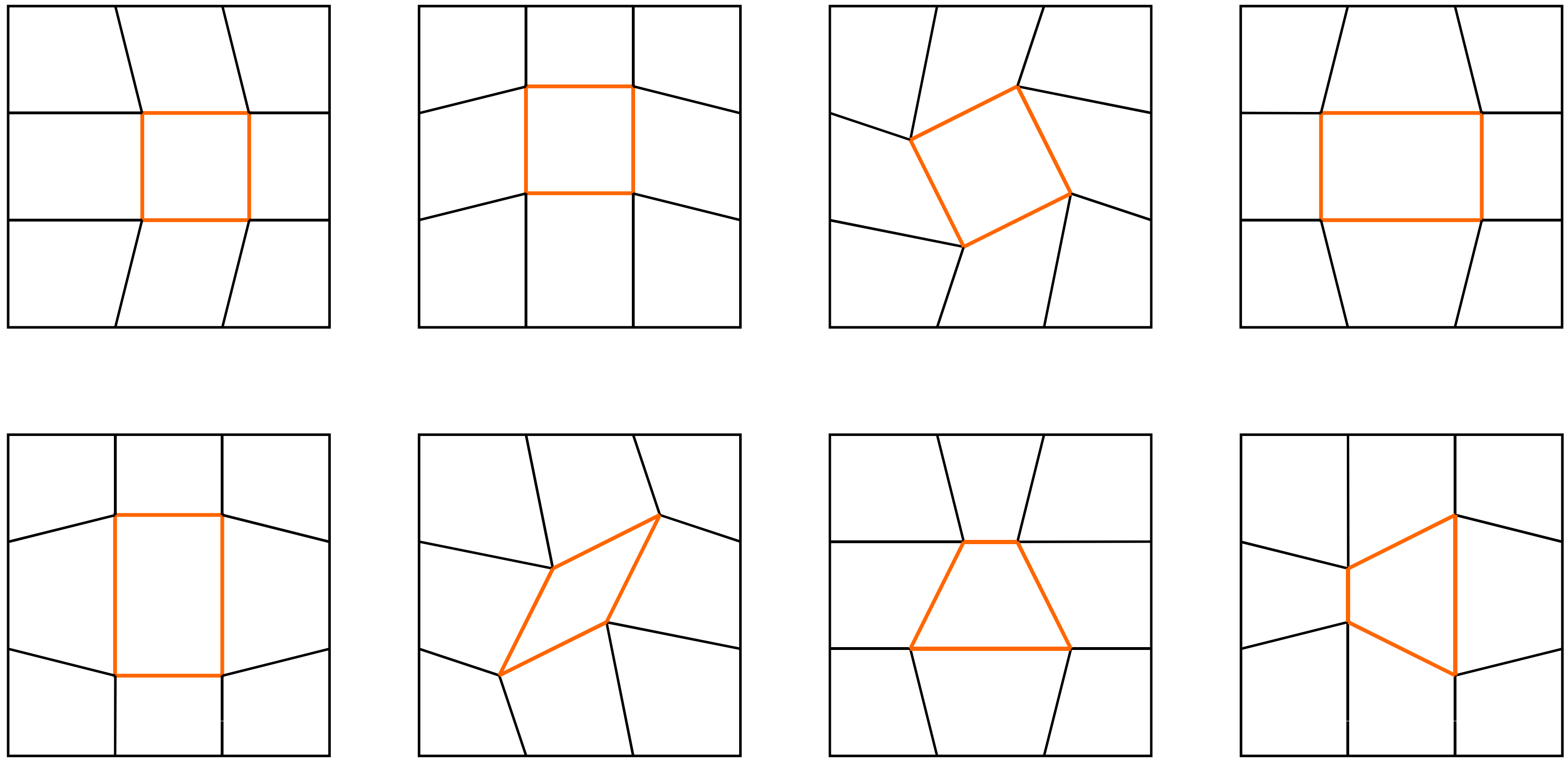
properties of macro element



- 24 eigenpairs $\boldsymbol{\lambda_1 \ldots \lambda_{24}, D_1 \ldots D_{24}}$
- $\boldsymbol{\tilde{\lambda}_1 \ldots \tilde{\lambda}_{24}, \tilde{D}_1 \ldots \tilde{D}_{24}}$ depend on locations of “invisible” nodes 13 – 16
- comparison of $\tilde{\lambda}_i$ and λ_i yields objective measure for distortion sensitivity

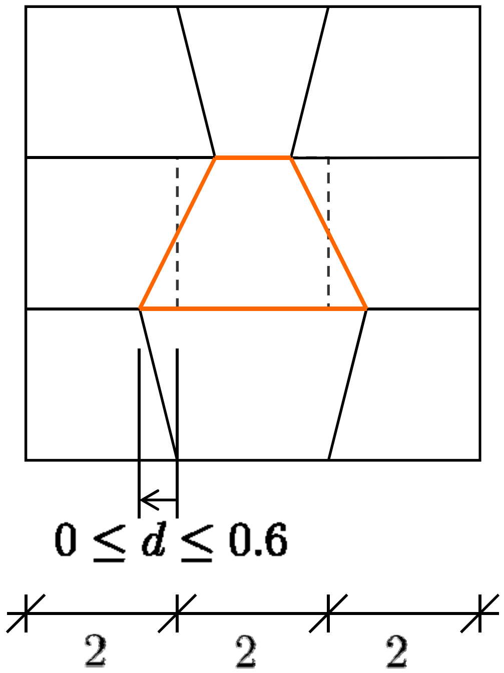
Element Distortion

8 generic distortion modes = 8 single element modes



Application of Distortion Patch Test

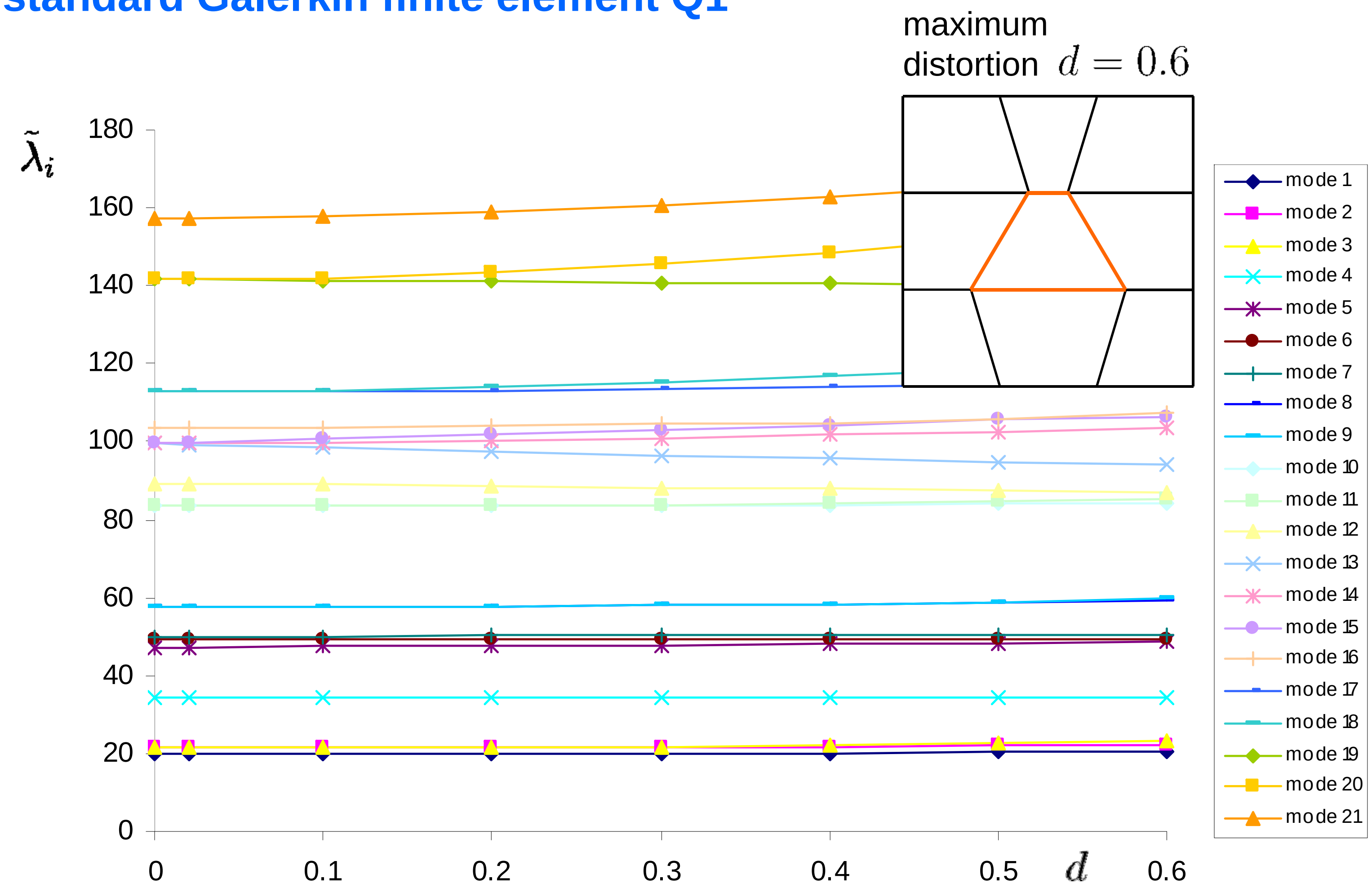
numerical experiments with Q1, Q1-SRI and Qm6



- trapezoidal type of distortion
- plotting $\tilde{\lambda}_i$ versus d
- comparing eigenvalue spectra for different values of d

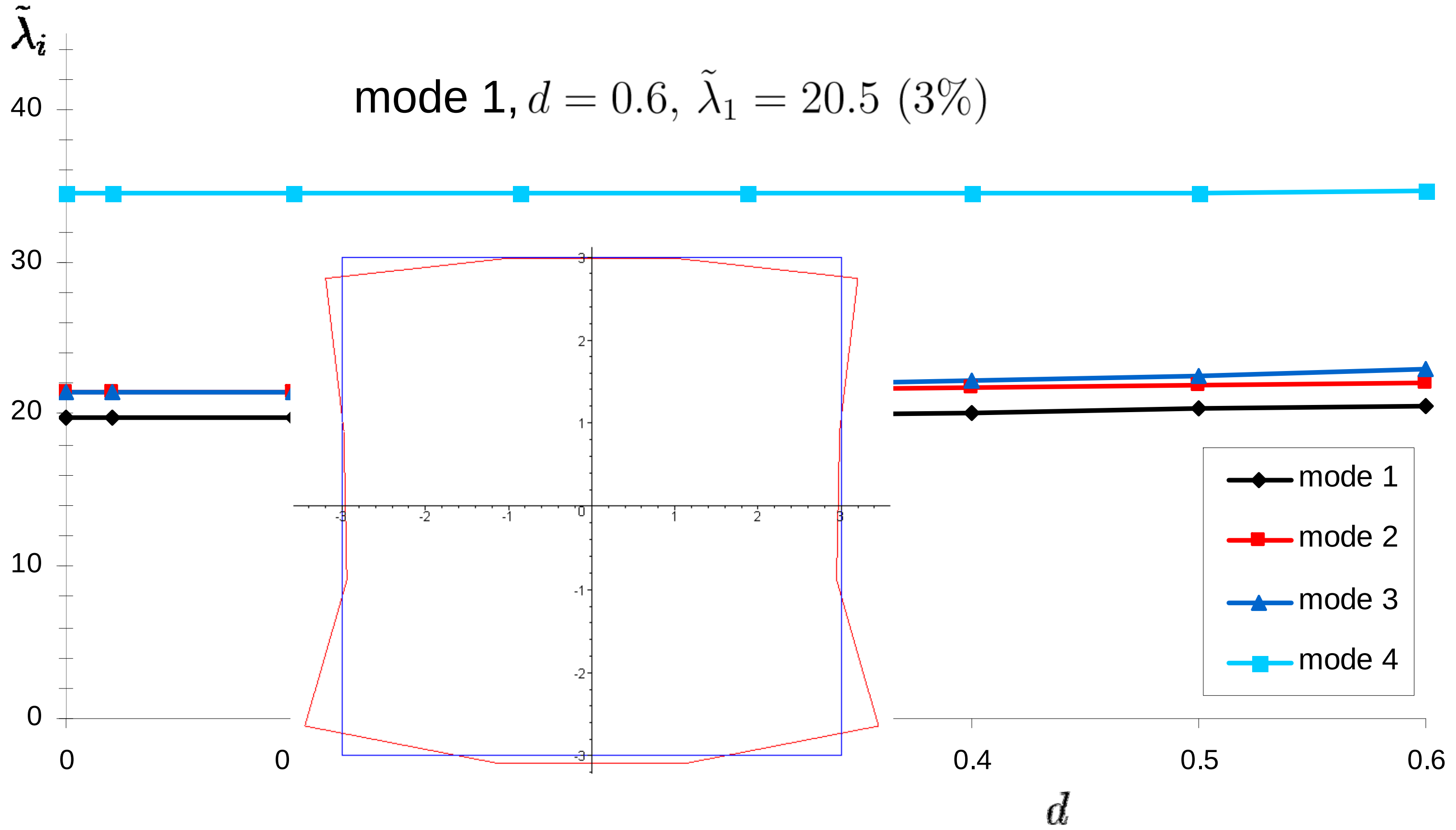
Eigenvalues versus Distortion Parameter

standard Galerkin finite element Q1



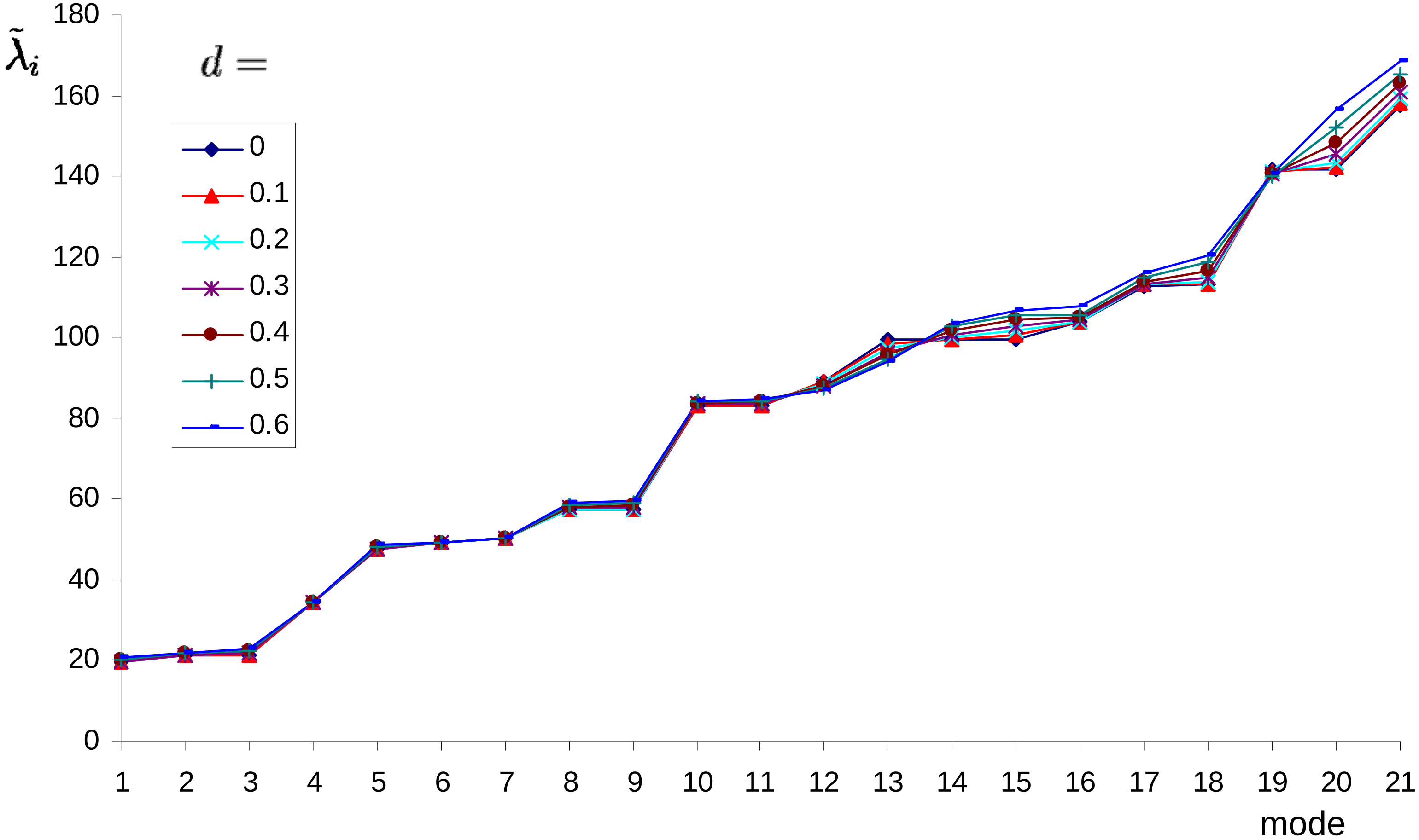
Eigenvalues versus Distortion Parameter

modes 1 – 4



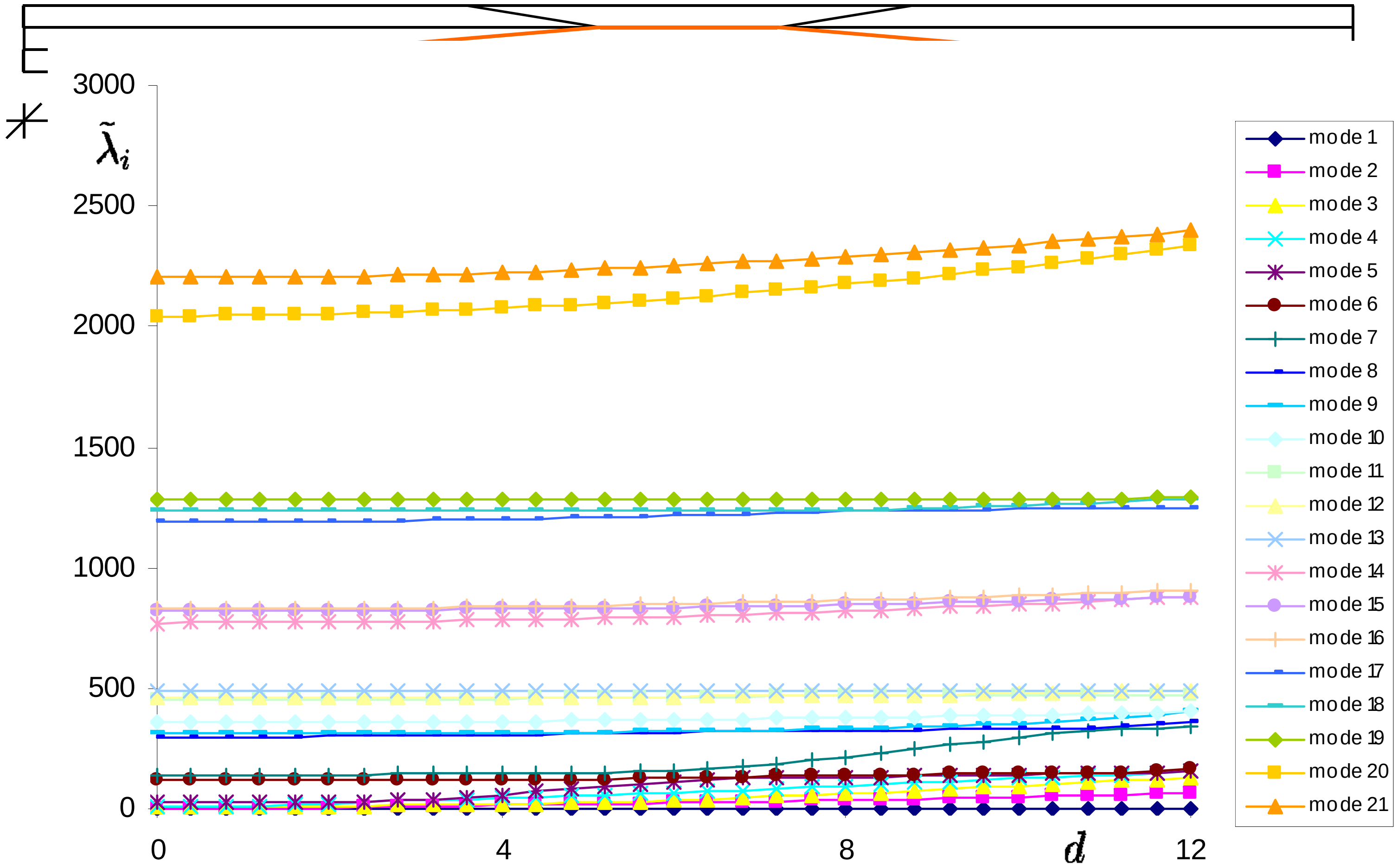
Eigenvalue Spectra

standard Galerkin finite element Q1



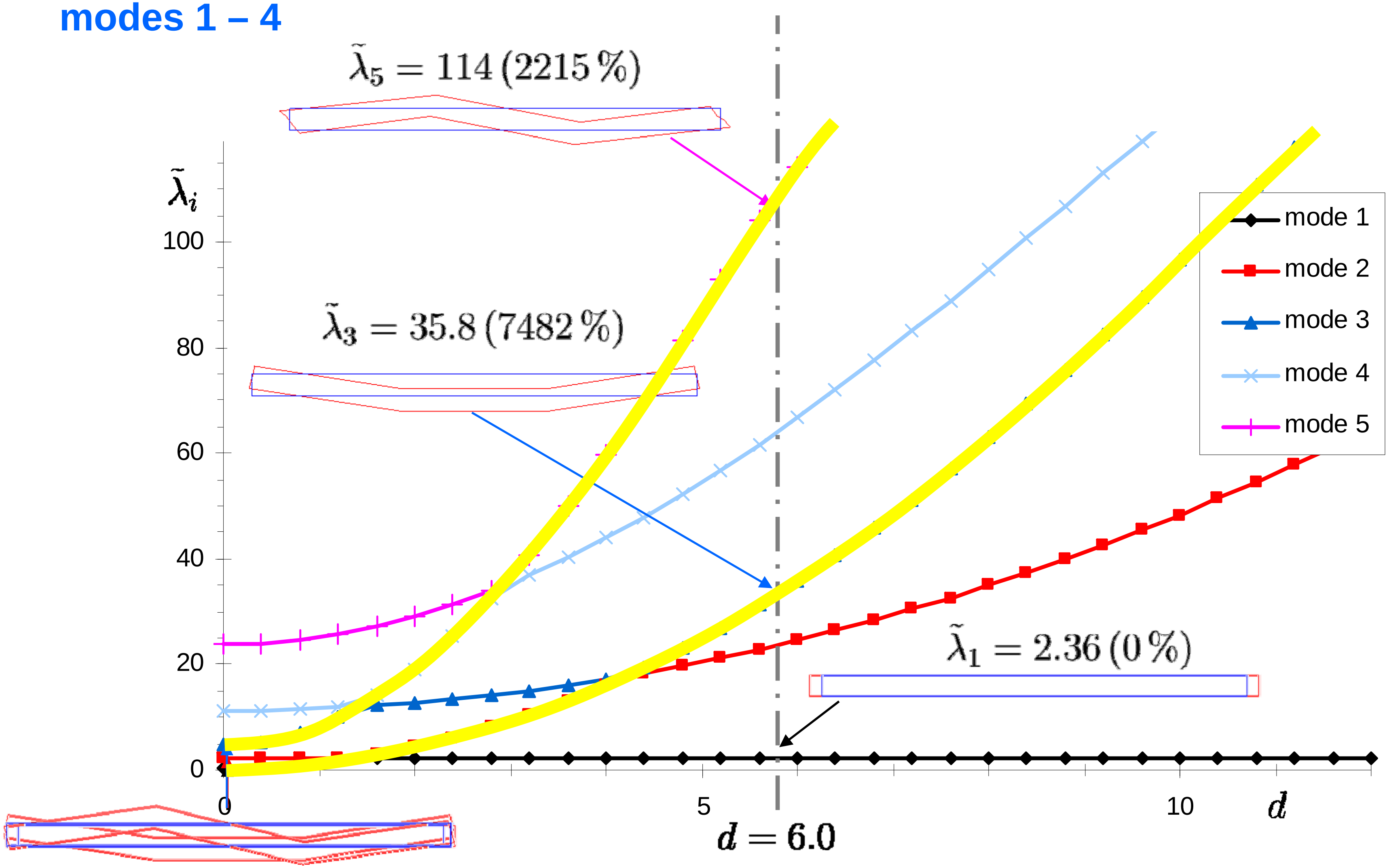
Application to “Thin” Structure

more sensitive to locking



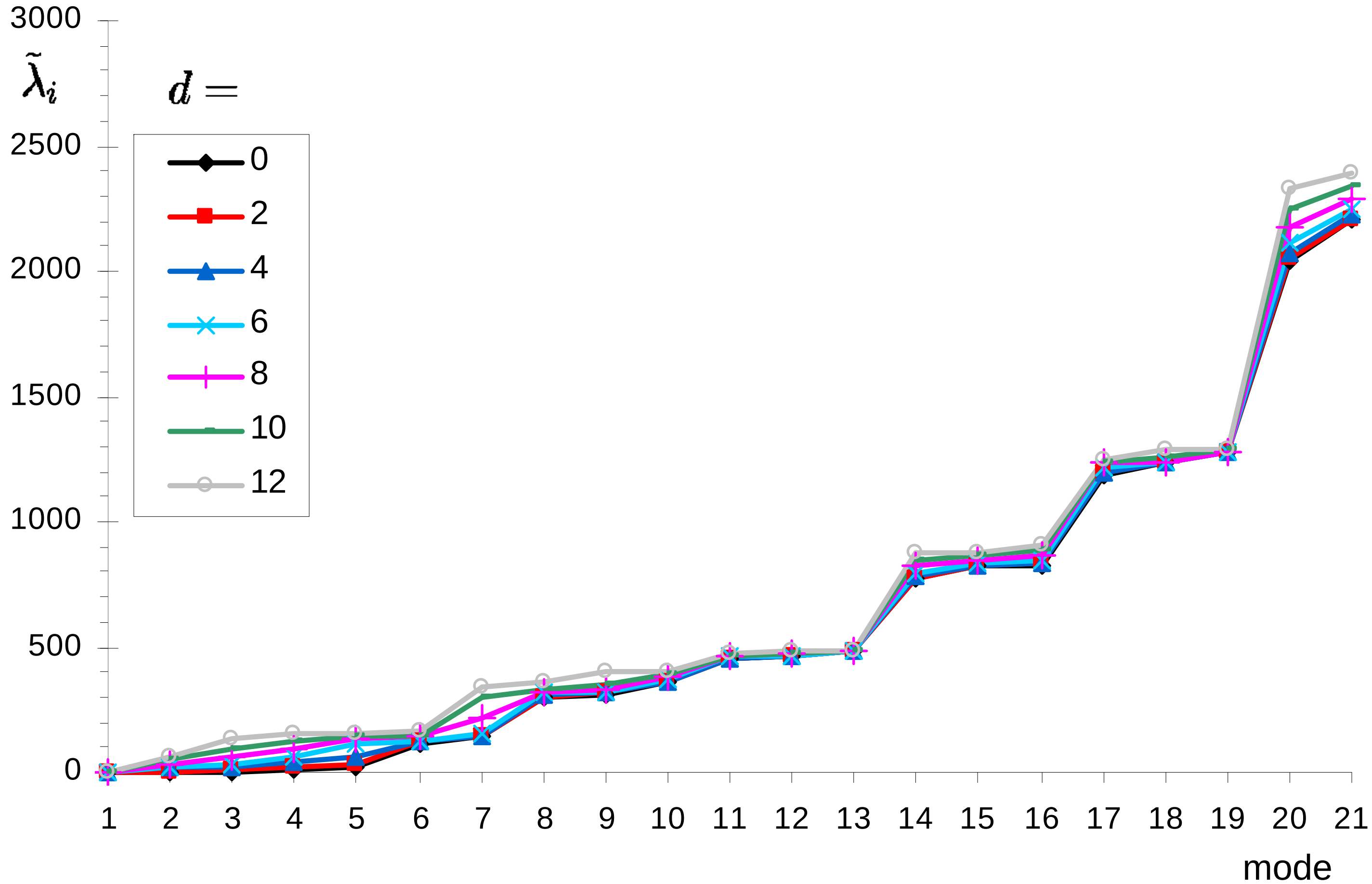
Application to "Thin" Structure

modes 1 – 4



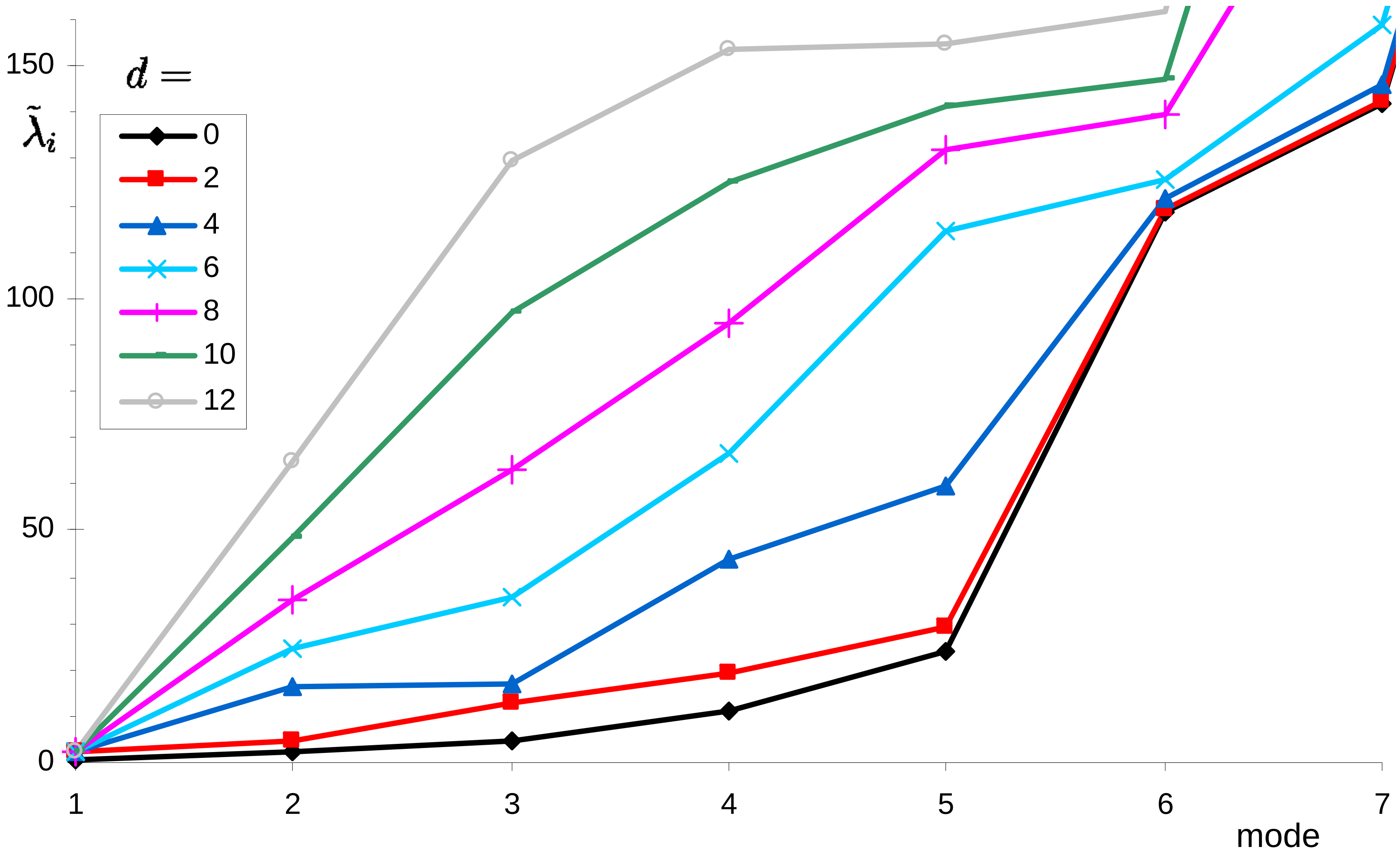
Eigenvalue Spectrum of Thin Structure

standard Galerkin finite element Q1



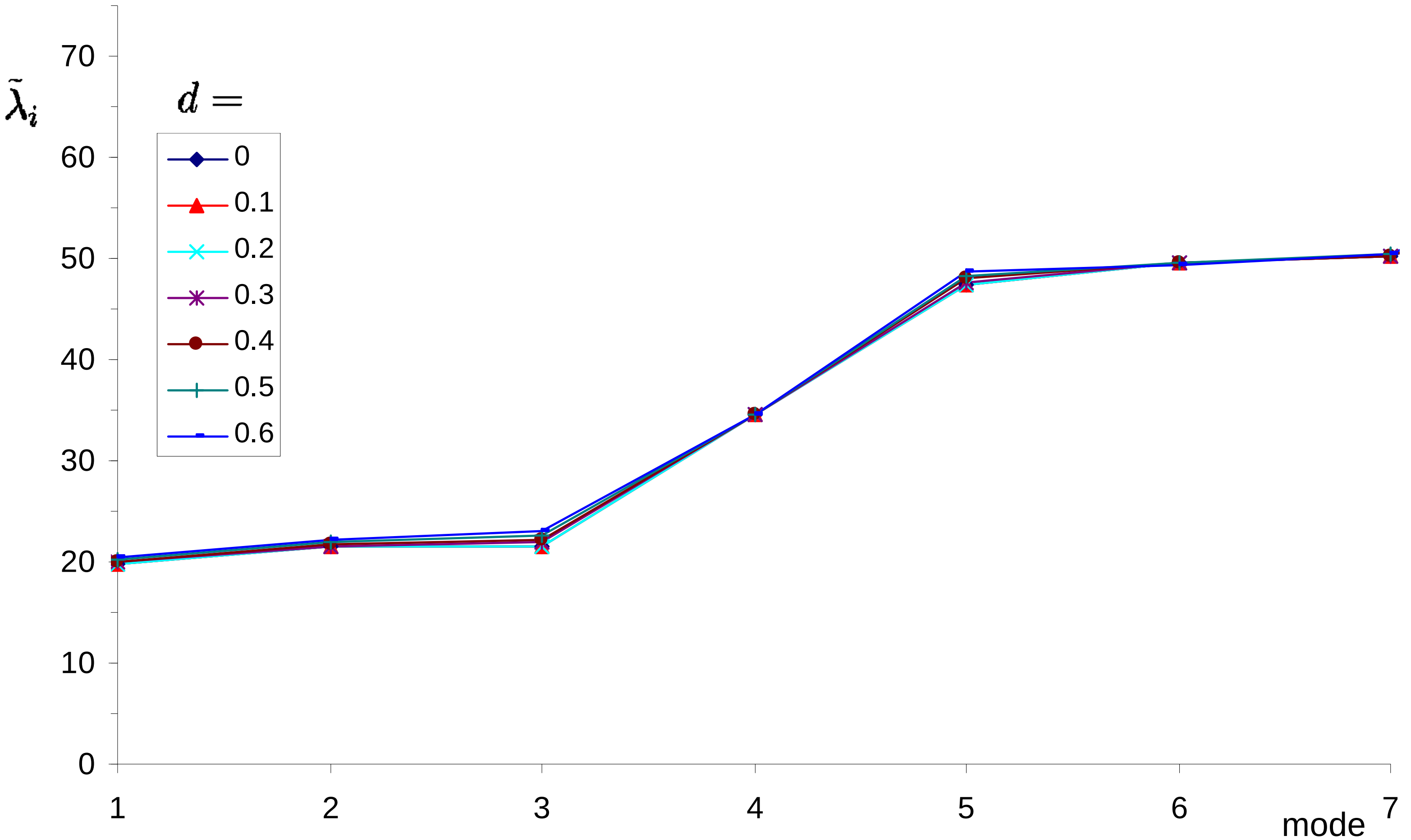
Eigenvalue Spectrum of Thin Structure

standard Galerkin finite element Q1



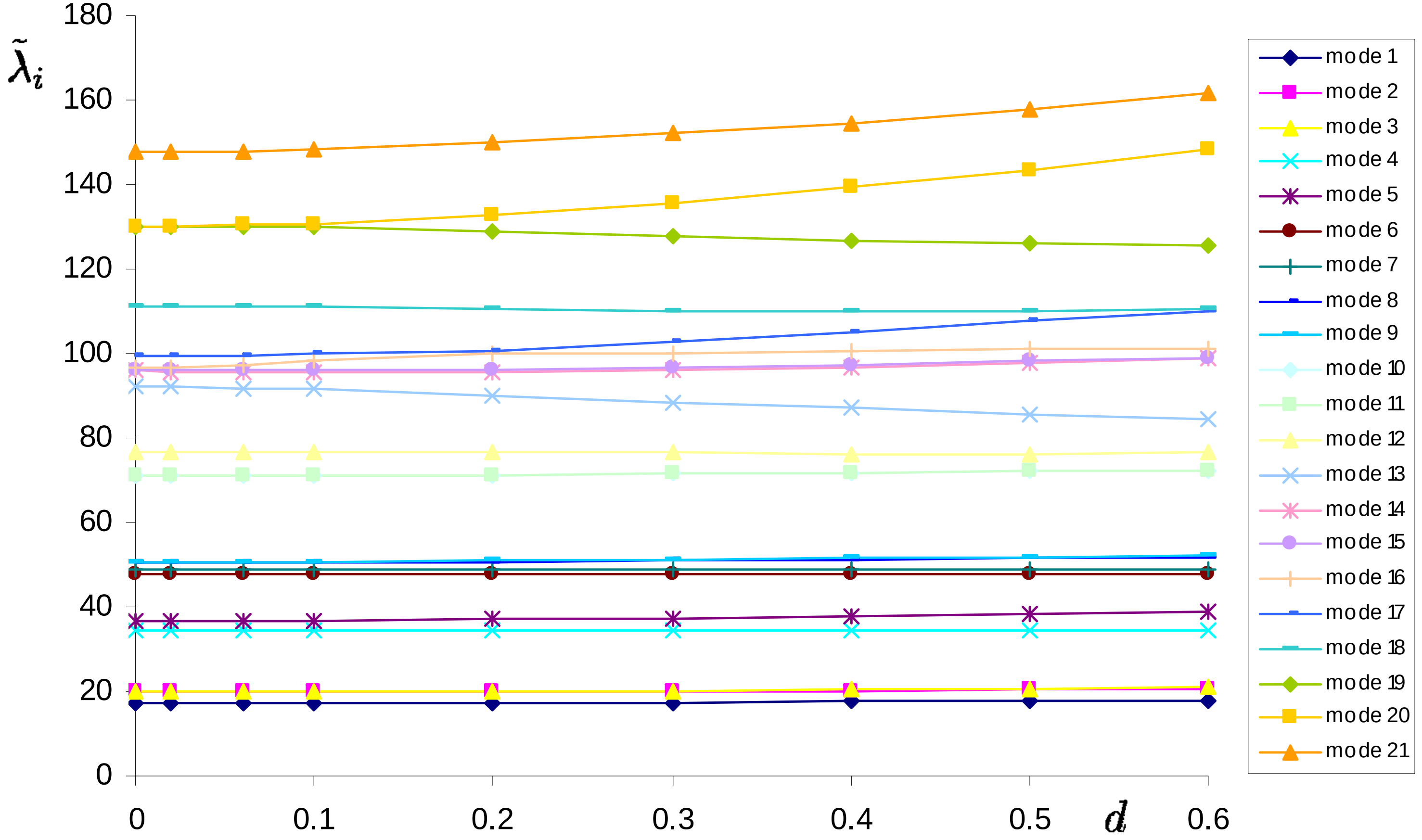
Eigenvalue Spectrum of Thick Structure

for comparison



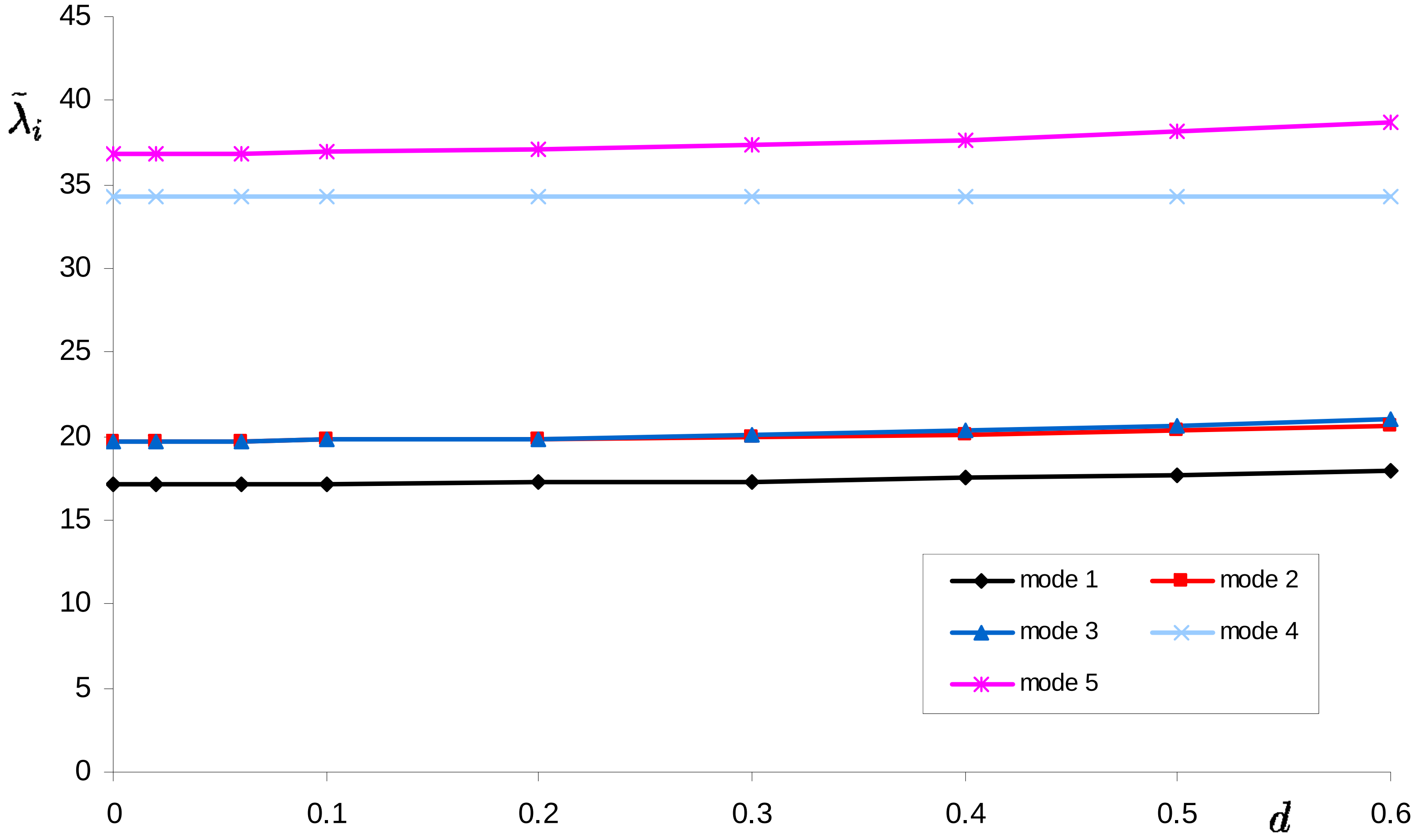
Selective Reduced Integration, Q1-SRI

eigenvalues for thick structure



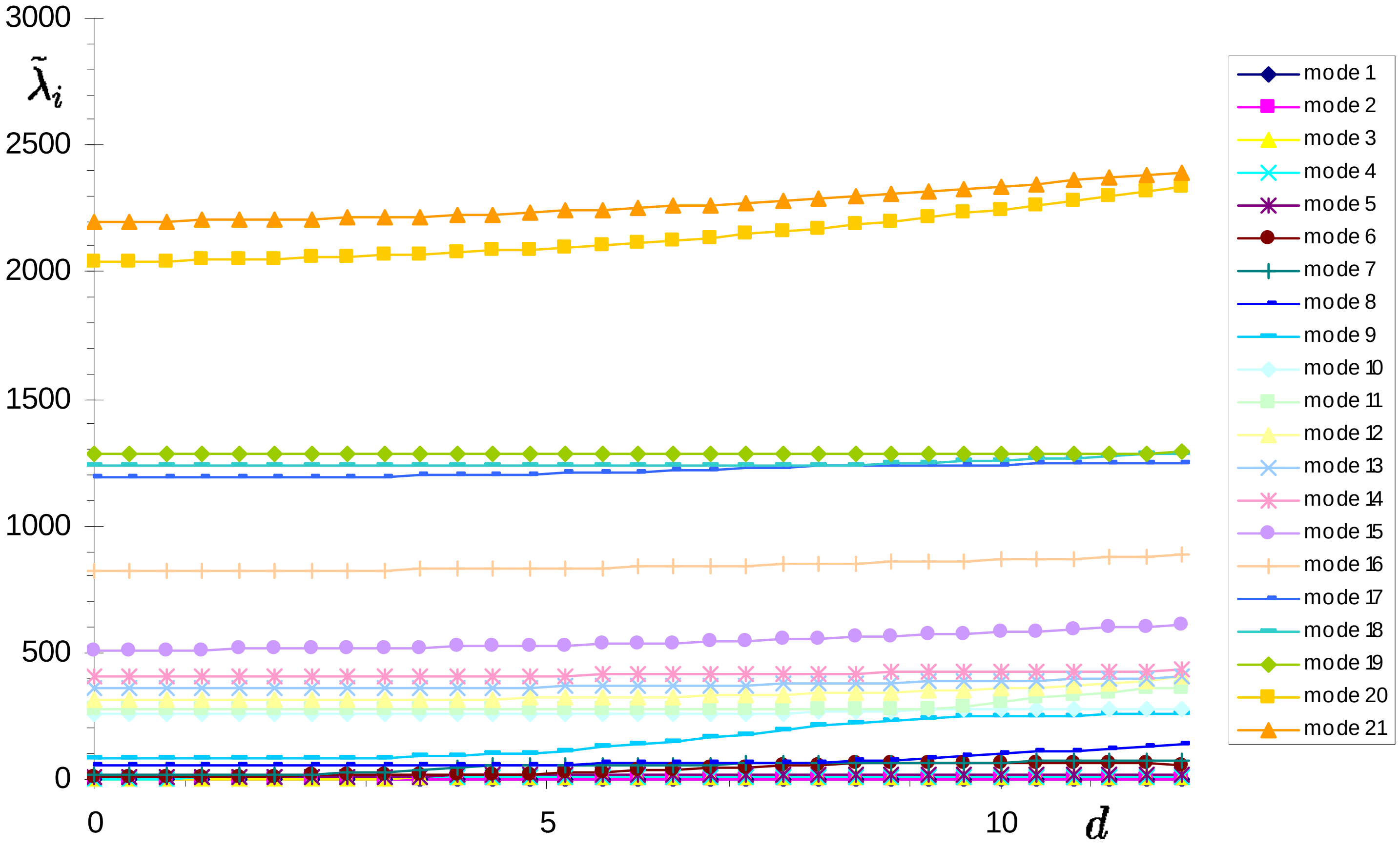
Selective Reduced Integration, Q1-SRI

eigenvalues for thick structure



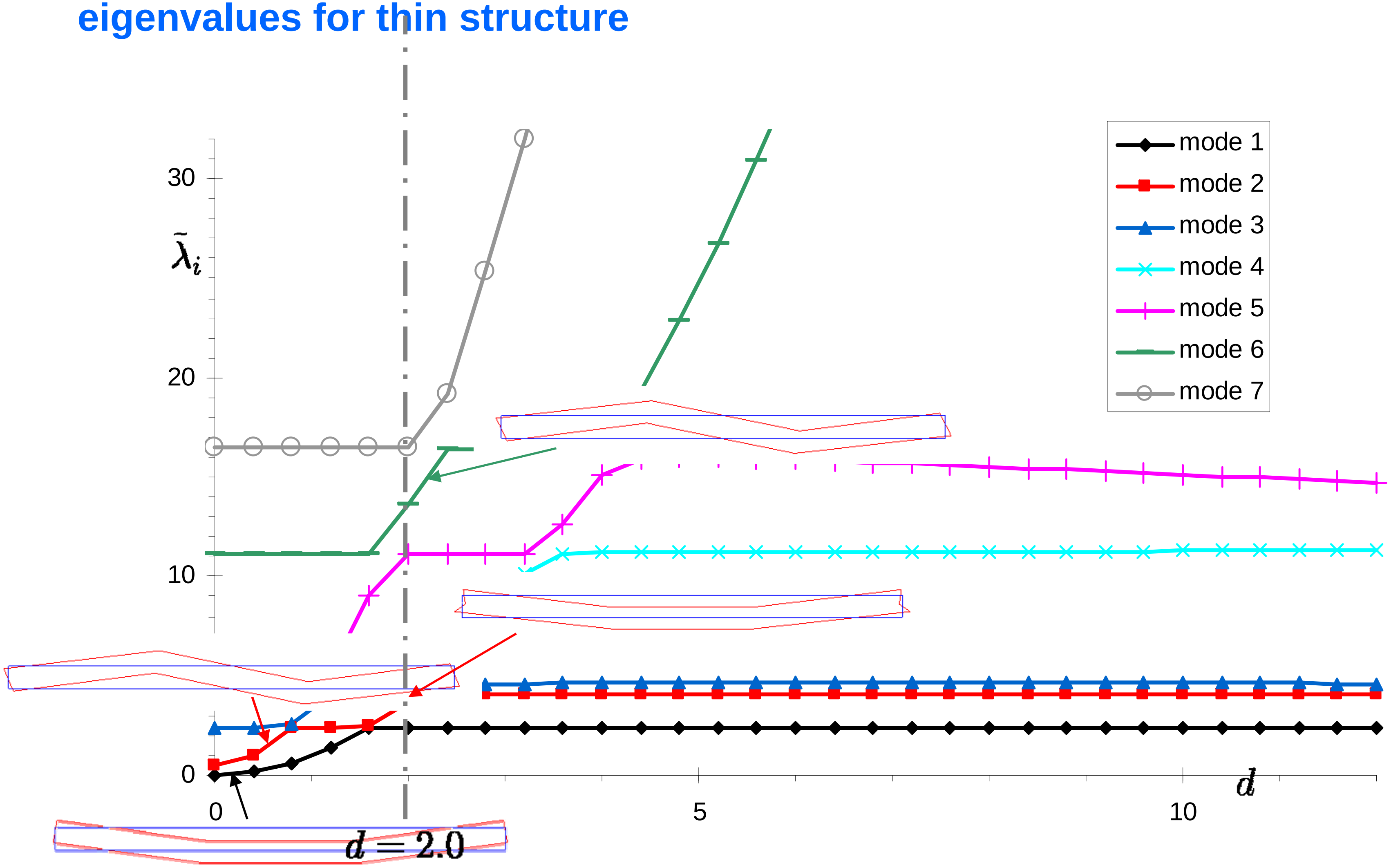
Selective Reduced Integration, Q1-SRI

eigenvalues for thin structure



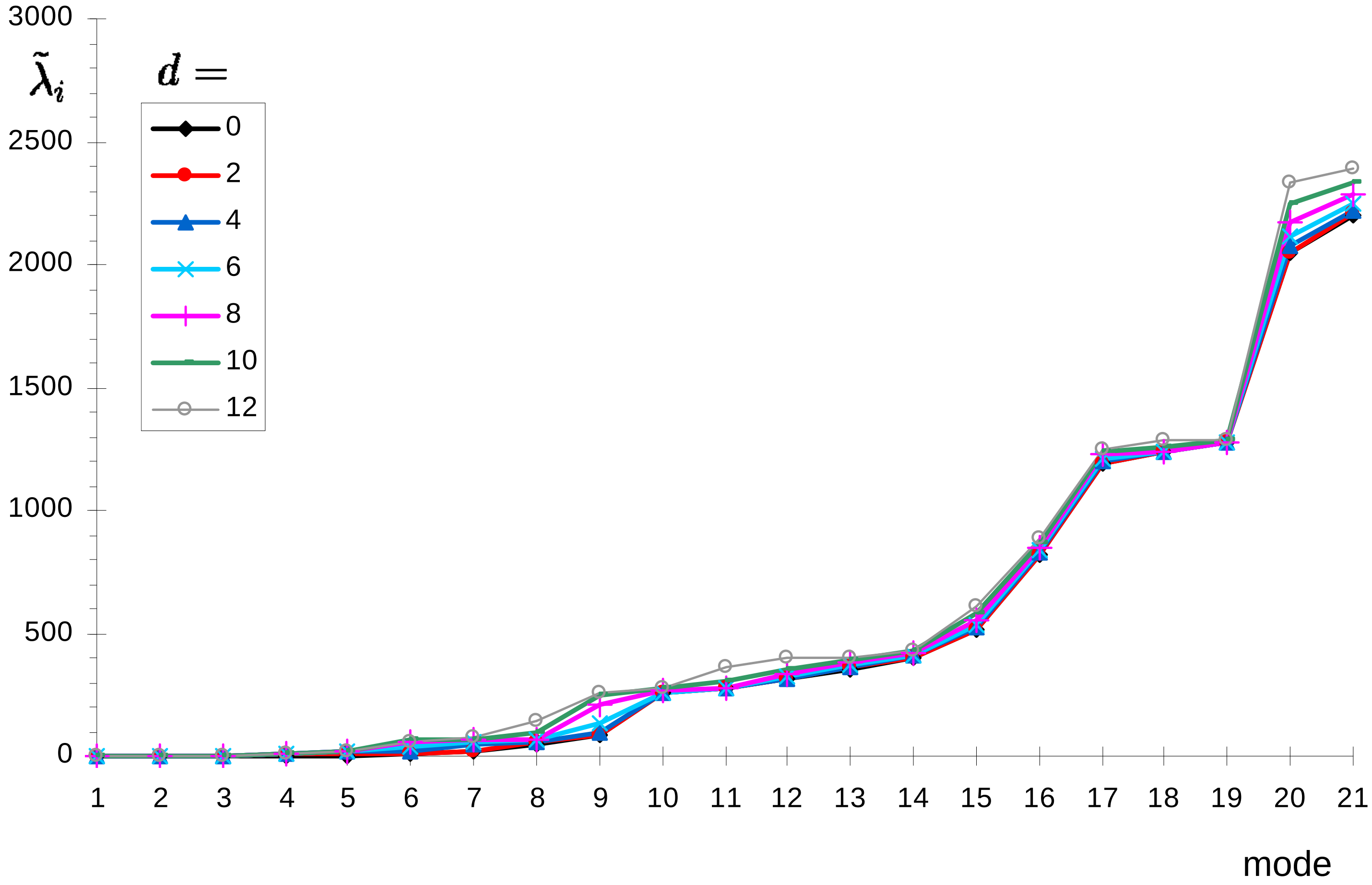
Selective Reduced Integration, Q1-SRI

eigenvalues for thin structure



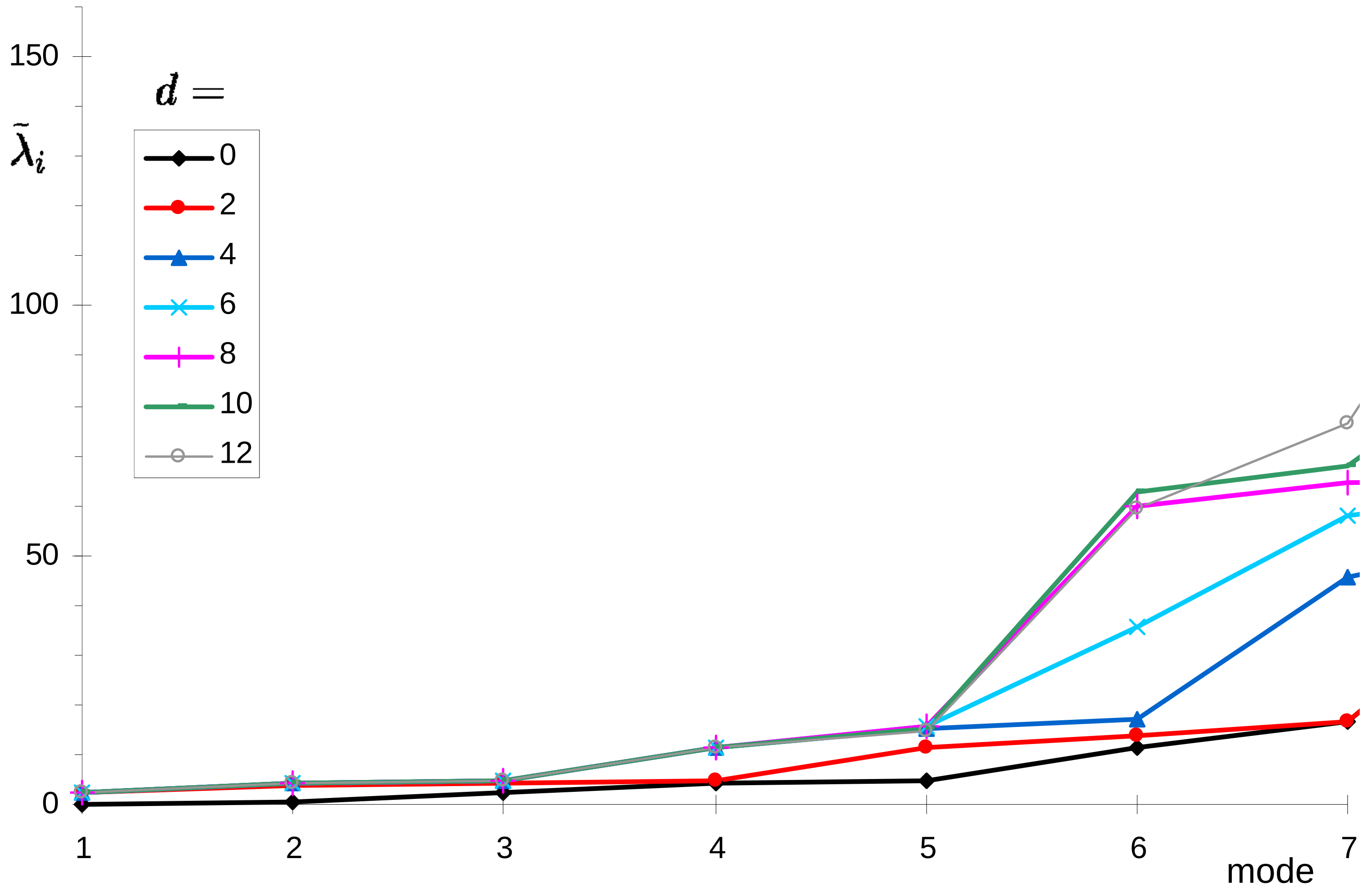
Selective Reduced Integration, Q1-SRI

eigenvalue spectrum



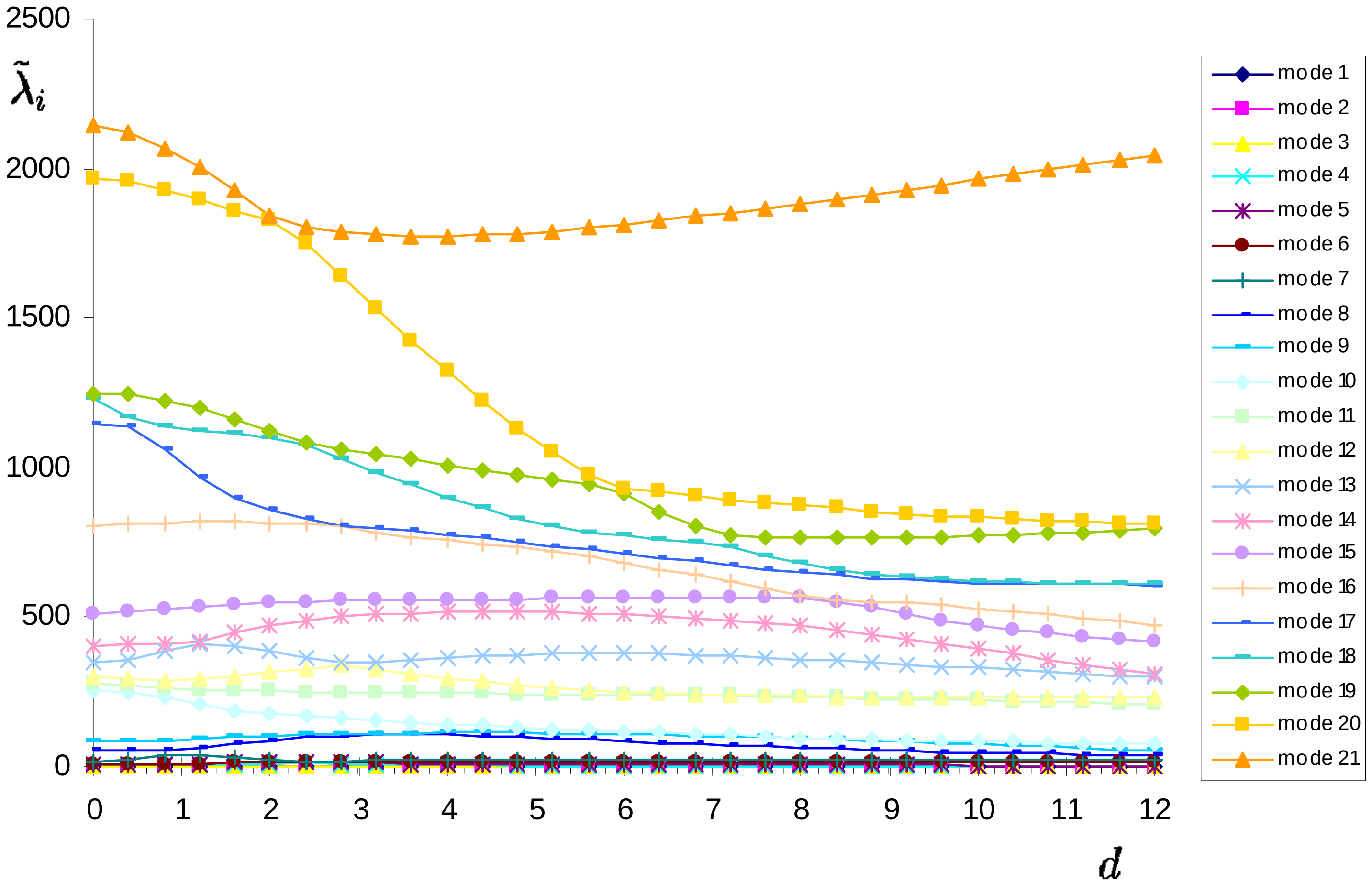
Selective Reduced Integration, Q1-SRI

eigenvalue spectrum



Method of Incompatible Modes, Qm6 (=Q1-E4)

eigenvalues for thin structure



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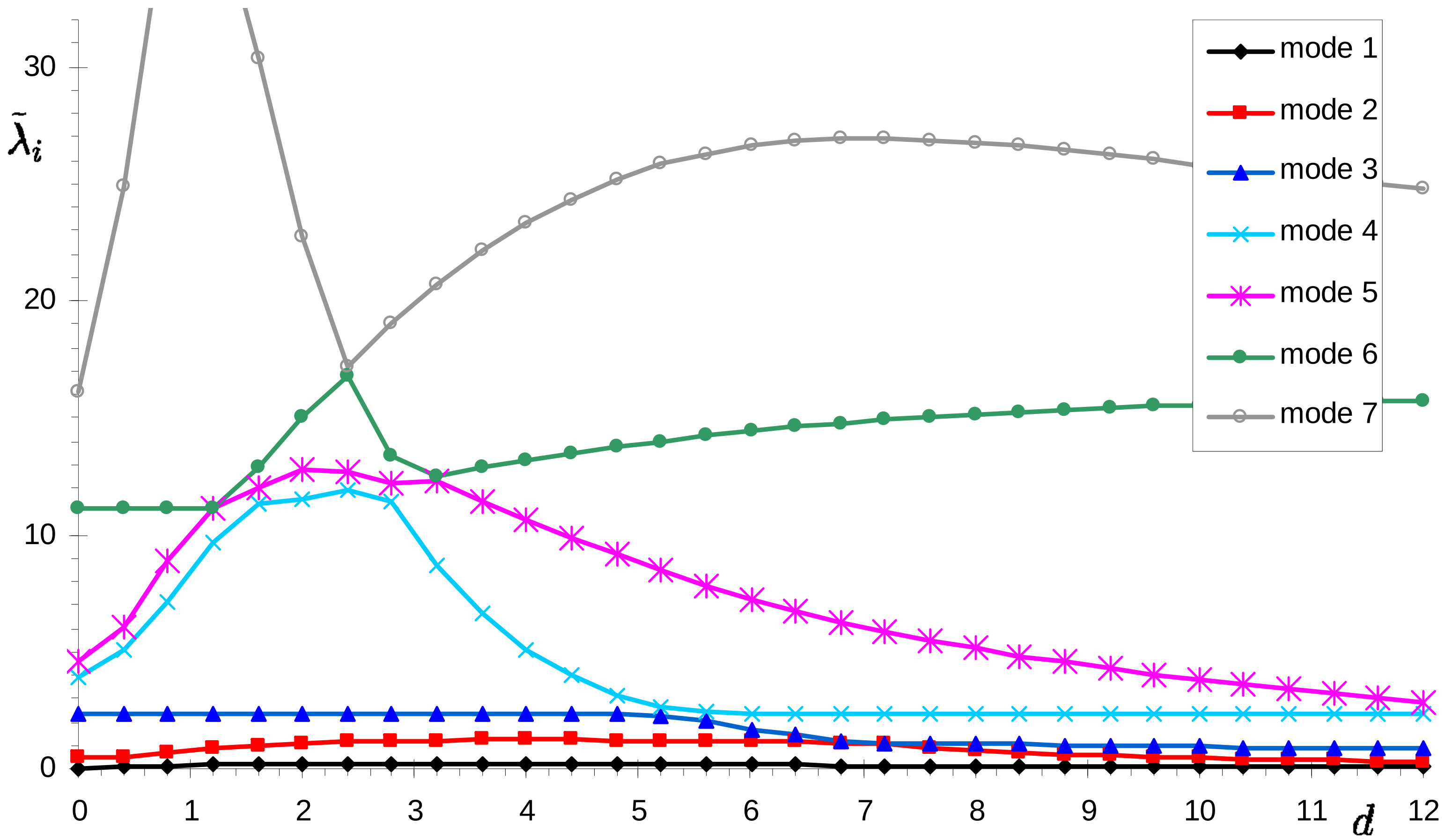

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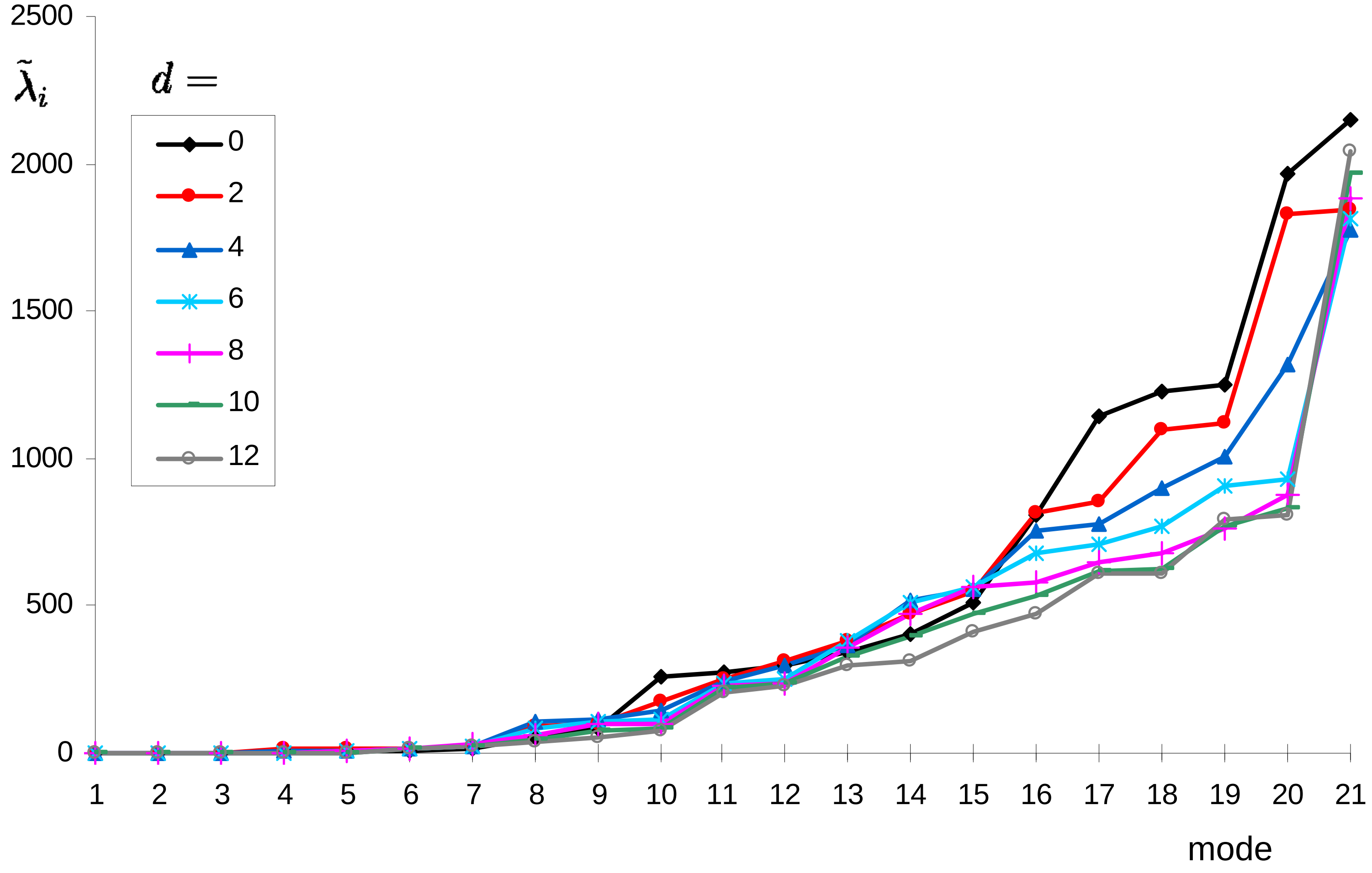

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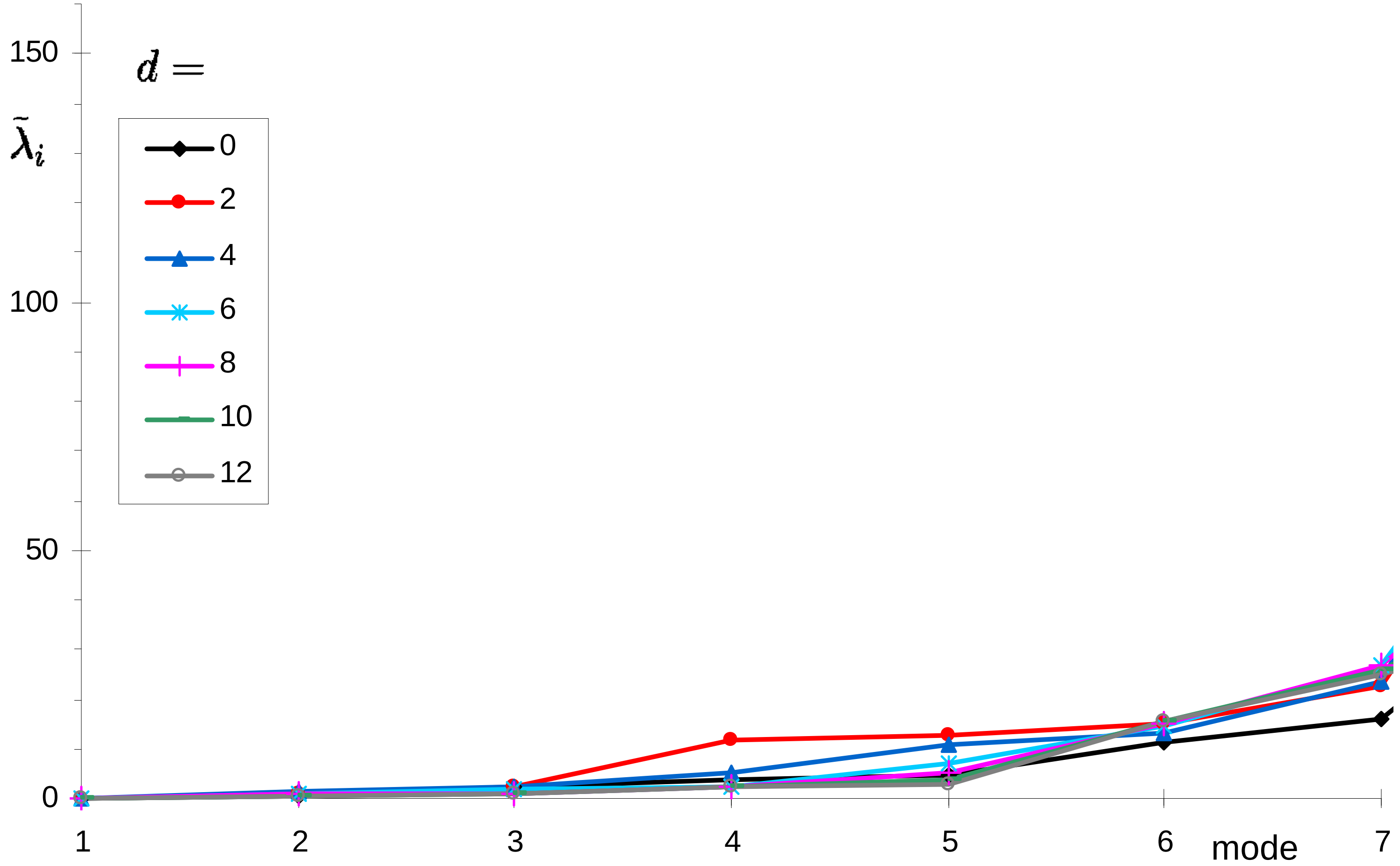
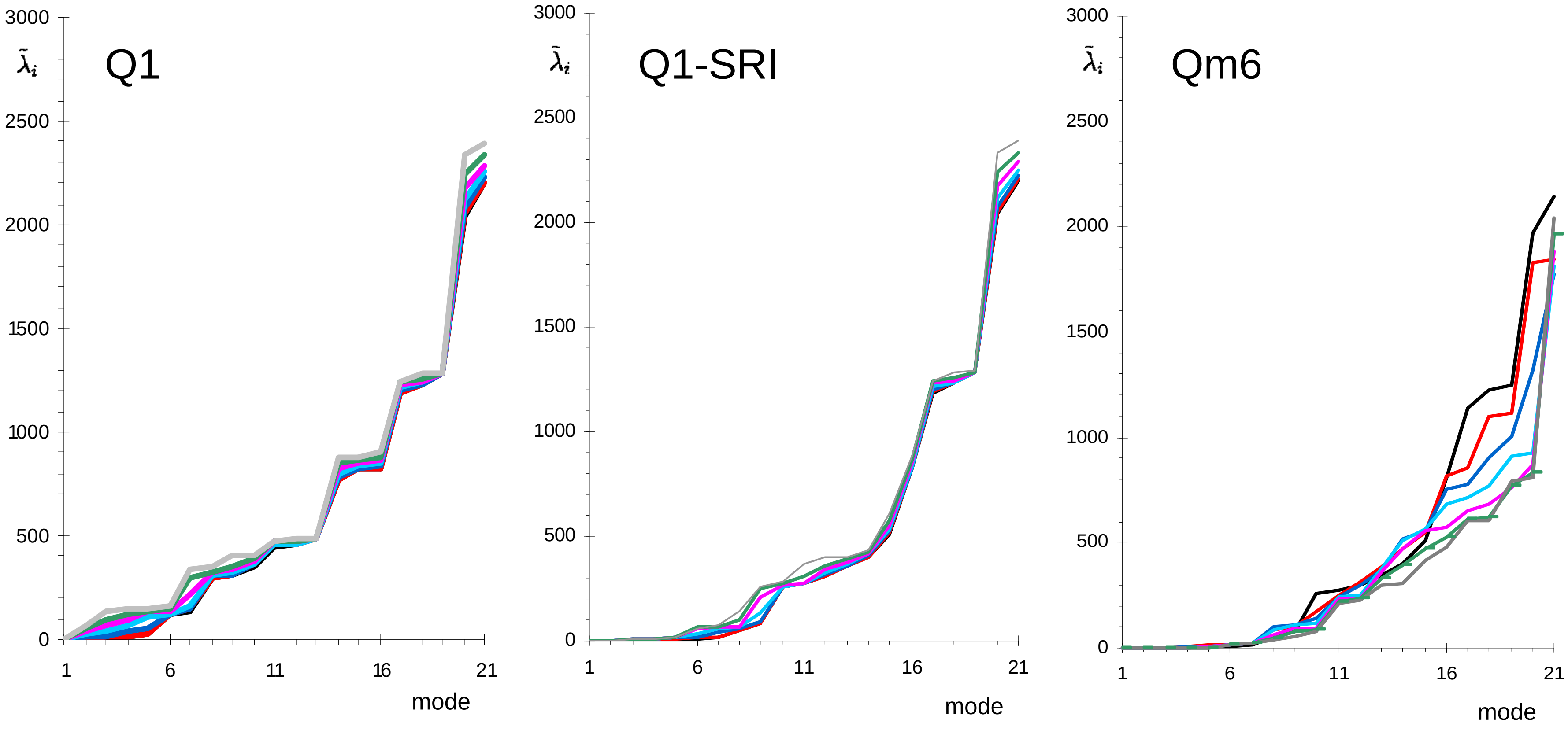


Summary of Results

eigenvalue spectra



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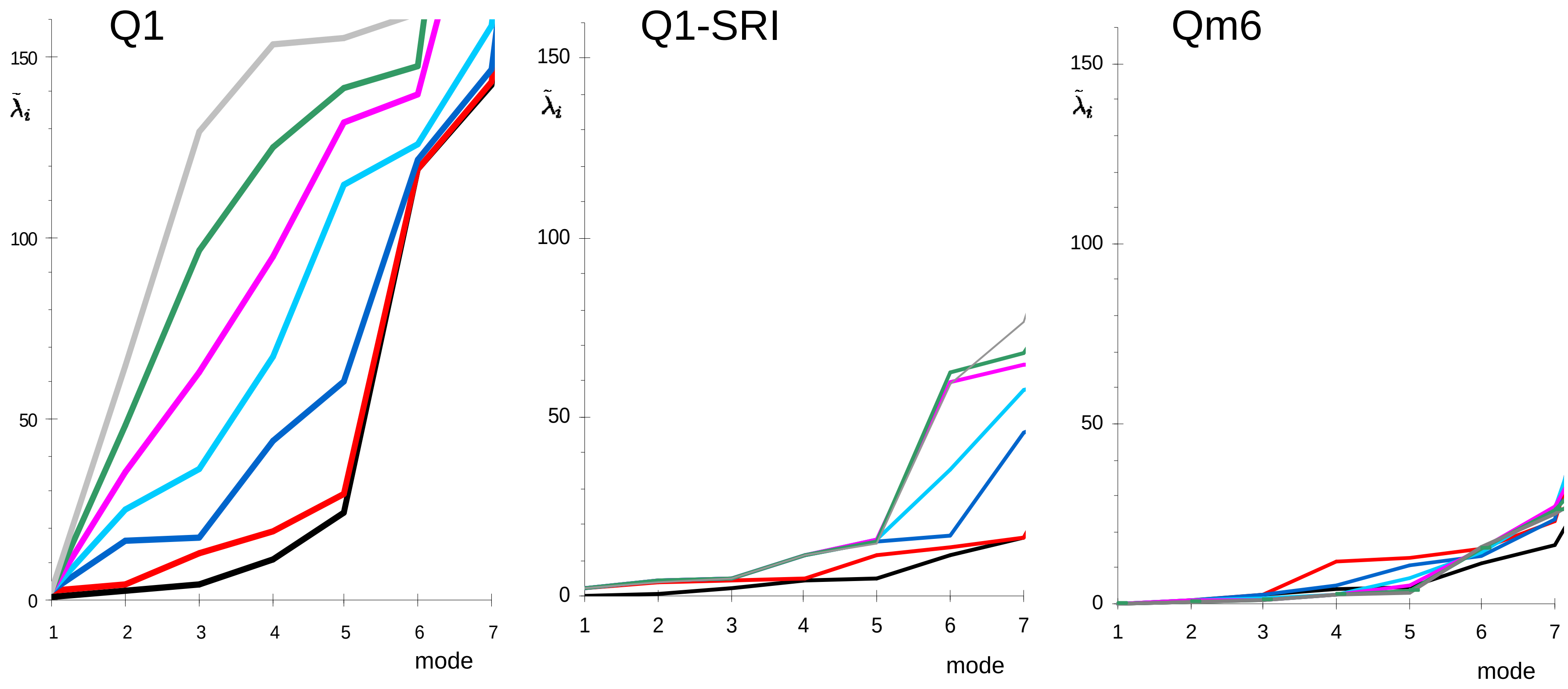
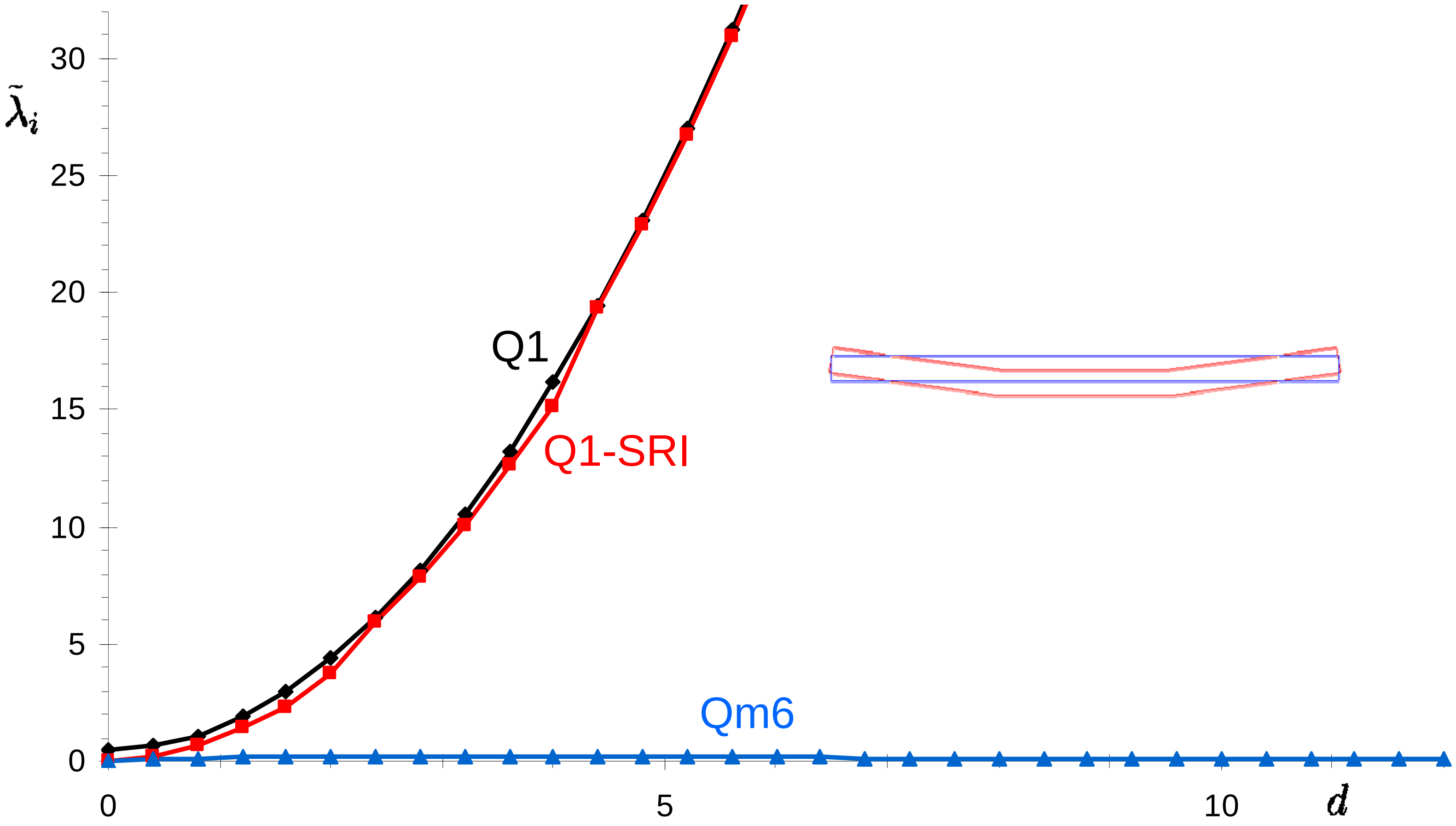


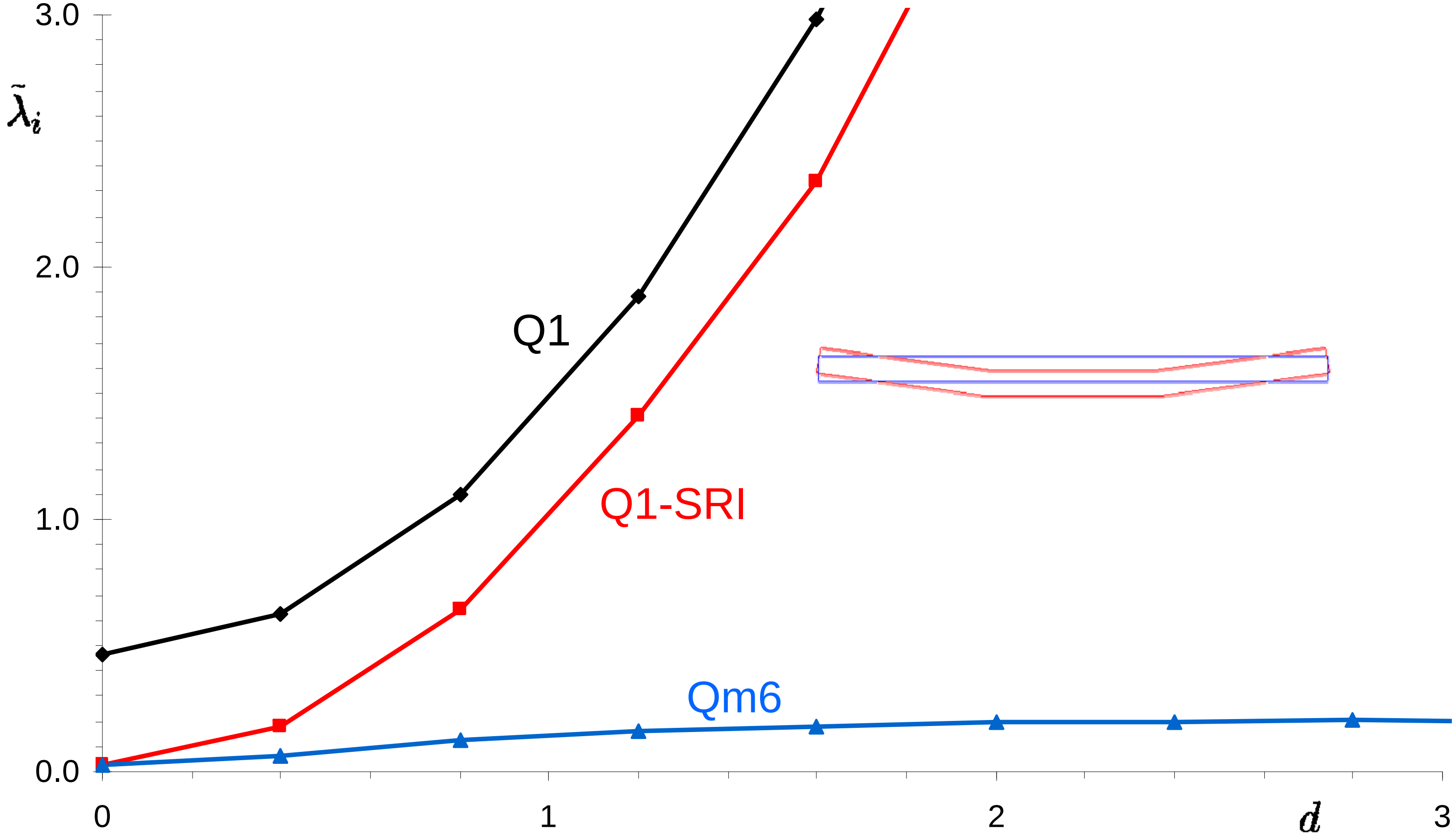
Summary of Results

eigenvalues of mode 1 in dependence of mesh distortion d



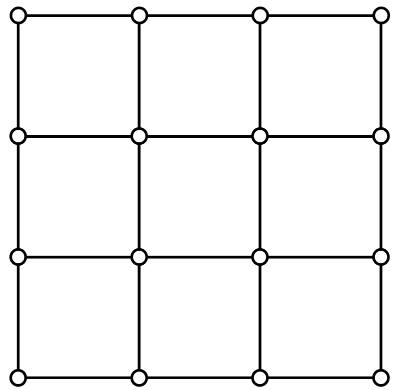
Summary of Results

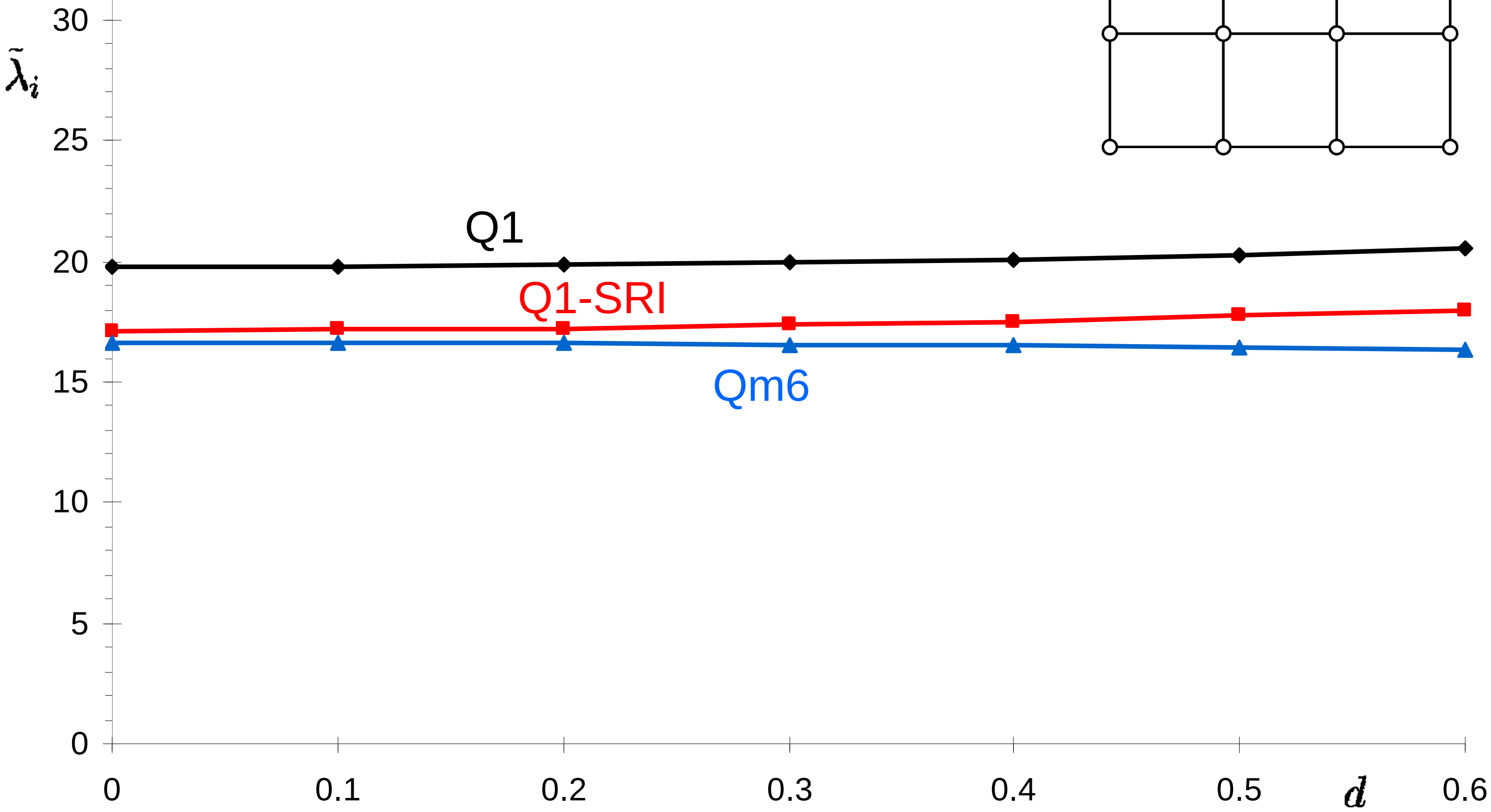
eigenvalues of mode 1 in dependence of mesh distortion d



Summary of Results

eigenvalues of mode , thick structure (square)





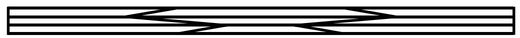
Conclusions

measuring distortion sensitivity

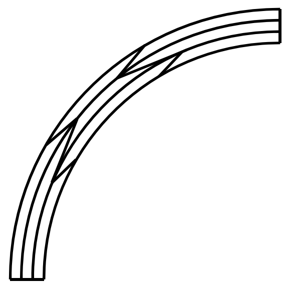
- bending test is not so bad
- eigenvalues may provide objective measure

newly proposed distortion patch test

- objective measure for distortion sensitivity
- universal locking test
- applicable to arbitrary problems
  (thin and curved structures, near incompressibility, etc.)



$\nu \to 0.5$



numerical results

- distortion sensitivity is related to locking
- Q1 and Q1-SRI equally sensitive to distortion
- Qm6 **significantly better** than Q1-SRI
  (more than bending test implies)