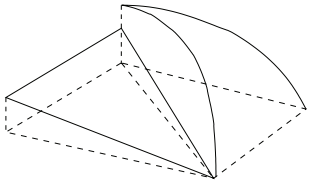

Chapter 1
One-dimensional problem

In this chapter, we define the primal DG methods for solving a two-point boundary value problem in one dimension.

1.1 Model problem

Let us consider the following two-point boundary value problem on the unit interval:

$$\forall x \in (0,1), \quad -(K(x)p'(x))' = f(x), \tag{1.1}$$

$$p(0) = 1, \tag{1.2}$$

$$p(1) = 0, \tag{1.3}$$

where $K \in \mathcal{C}^1(0,1)$ and $f \in \mathcal{C}^0(0,1)$. We also assume that there are two constants K_0 and K_1 such that

$$\forall x \in (0,1), \quad 0 < K_0 \le K(x) \le K_1.$$

We say that p is a solution of (1.1)–(1.3) if $p \in \mathcal{C}^2(0,1)$ and p satisfies the equations (1.1)–(1.3) pointwise.

1.2 A class of DG methods

Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition $\mathcal{E}_h$ of $(0,1)$, denote $I_n = (x_n, x_{n+1})$, and define

$$h_n = x_{n+1} - x_n, \quad h_{n-1,n} = \max(h_{n-1}, h_n), \quad h = \max_{0 \le n \le N-1} h_n.$$

Denote by $\mathcal{D}_k(\mathcal{E}_h)$ the space of piecewise discontinuous polynomials of degree k:

$$\mathcal{D}_k(\mathcal{E}_h) = \{v : v|_{I_n} \in \mathbb{P}_k(I_n) \ \ \forall j = 0, \ldots, N-1\},$$

where $\mathbb{P}_k(I_n)$ is the space of polynomials of degree k on the interval I_n. Let us denote $v(x_n^+) = \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} v(x_n + \epsilon)$ and $v(x_n^-) = \lim_{\substack{\epsilon \to 0 \\ \epsilon > 0}} v(x_n - \epsilon)$. Then we can define the jump and

average of v at the endpoints of I_n:

$$[v(x_n)] = v(x_n^-) - v(x_n^+), \quad \{v(x_n)\} = \frac{1}{2}(v(x_n^-) + v(x_n^+)) \quad \forall n = 1, \ldots, N-1.$$

By convention, we also extend the definition of jump and average at the endpoints of the unit interval:

$$[v(x_0)] = -v(x_0^+), \quad \{v(x_0)\} = v(x_0^+), \quad [v(x_N)] = v(x_N^-), \quad \{v(x_N)\} = v(x_N^-).$$

Next, we introduce jump terms of the solution and its derivative. Those terms are also referred to as penalty terms:

$$J_0(v, w) = \sum_{n=0}^{N} \frac{\sigma^0}{h_{n-1,n}}[v(x_n)][w(x_n)], \quad J_1(v, w) = \sum_{n=1}^{N-1} \frac{\sigma^1}{h_{n-1,n}}[v'(x_n)][w'(x_n)],$$

where σ^0 and σ^1 are two real nonnegative numbers. We note that J_0 penalizes the jump in the function v (or w), whereas J_1 penalizes the jump in the derivative of the function v (or w).

Let v be a function in $\mathcal{D}_k(\mathcal{E}_h)$. Let us multiply (1.1) by v and let us integrate by parts on each interval I_n:

$$\begin{aligned}&\int_{x_n}^{x_{n+1}} K(x)p'(x)v'(x)dx - K(x_{n+1})p'(x_{n+1})v(x_{n+1}^-) + K(x_n)p'(x_n)v(x_n^+)\\ &= \int_{x_n}^{x_{n+1}} f(x)v(x)dx, \quad n = 0, \ldots, N-1.\end{aligned}$$

By adding all N equations above, we obtain

$$\sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} K(x)p'(x)v'(x)dx - \sum_{n=0}^{N}[K(x_n)p'(x_n)v(x_n)] = \int_0^1 f(x)v(x)dx.$$

It is easy to check that for $1 \le n \le N-1$

$$[K(x_n)p'(x_n)v(x_n)] = \{K(x_n)p'(x_n)\}[v(x_n)] + \{v(x_n)\}[K(x_n)p'(x_n)]. \tag{1.4}$$

By applying (1.4) and by noting that the exact solution p satisfies $[K(x_n)p'(x_n)] = 0$ for all $1 \le n \le N-1$, we obtain

$$\sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} K(x)p'(x)v'(x)dx - \sum_{n=0}^{N}\{K(x_n)p'(x_n)\}[v(x_n)] = \int_0^1 f(x)v(x)dx.$$

We now note that the exact solution p is also continuous, i.e., $[p(x_n)] = 0$. Therefore, we see that if p is a solution of (1.1)–(1.3), then p satisfies

$$\begin{aligned}&\sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} K(x)p'(x)v'(x)dx - \sum_{n=0}^{N}\{K(x_n)p'(x_n)\}[v(x_n)] + \epsilon \sum_{n=0}^{N}\{K(x_n)v'(x_n)\}[p(x_n)]\\ &= \int_0^1 f(x)v(x)dx - \epsilon K(x_0)v'(x_0)p(x_0) + \epsilon K(x_N)v'(x_N)p(x_N)\\ &= \int_0^1 f(x)v(x)dx - \epsilon K(x_0)v'(x_0).\end{aligned}$$

Here, ϵ can be any real number, as the third term in the equation above is intrinsically zero. However, we restrict ourselves to the case $\epsilon \in \{-1, 0, -1\}$.

Definition 1.1. *Given a real vector space V, the function $a : V \times V \to \mathbb{R}$ is called a bilinear form if a is linear with respect to each of its arguments. In other words, for all $\alpha \in \mathbb{R}$, $v, v_1, v_2, w, w_1, w_2 \in V$, we have*

$$\begin{aligned} a(v_1 + v_2, w) &= a(v_1, w) + a(v_2, w), \\ a(\alpha v, w) &= \alpha a(v, w), \\ a(v, w_1 + w_2) &= a(v, w_1) + a(v, w_2), \\ a(v, \alpha w) &= \alpha a(v, w). \end{aligned}$$

We now define the DG bilinear form $a_\epsilon : \mathcal{D}_k(\mathcal{E}_h) \times \mathcal{D}_k(\mathcal{E}_h) \to \mathbb{R}$:

$$\begin{aligned} a_\epsilon(w, v) = &\sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} K(x) w'(x) v'(x) dx - \sum_{n=0}^{N} \{K(x_n) w'(x_n)\} [v(x_n)] \\ &+ \epsilon \sum_{n=0}^{N} \{K(x_n) v'(x_n)\} [w(x_n)] + J_0(w, v) + J_1(w, v). \end{aligned}$$

The DG bilinear form a_ϵ has the following properties:

- For $\epsilon = -1$, the form is symmetric, i.e.,

$$\forall v, w, \quad a_{-1}(v, w) = a_{-1}(w, v),$$

and we have

$$\begin{aligned} a_{-1}(v, v) = &\sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} K(x) (v'(x))^2 dx - 2 \sum_{n=0}^{N} \{K(x_n) v'(x_n)\} [v(x_n)] \\ &+ J_0(v, v) + J_1(v, v). \end{aligned}$$

- For $\epsilon \in \{0, +1\}$, the form is nonsymmetric, and we have

$$a_{+1}(v, v) = \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} K(x) (v'(x))^2 dx + J_0(v, v) + J_1(v, v) \geq 0, \tag{1.5}$$

$$\begin{aligned} a_0(v, v) = &\sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} K(x) (v'(x))^2 dx - \sum_{n=0}^{N} \{K(x_n) v'(x_n)\} [v(x_n)] \\ &+ J_0(v, v) + J_1(v, v). \end{aligned}$$

A class of DG methods for solving the boundary value problem (1.1)–(1.3) is as follows: Find $P^{\mathrm{DG}} \in \mathcal{D}_k(\mathcal{E}_h)$ such that

$$\forall v \in \mathcal{D}_k(\mathcal{E}_h), \quad a_\epsilon(P^{\mathrm{DG}}, v) = L(v), \tag{1.6}$$

where $L : \mathcal{D}_k(\mathcal{E}_h) \to \mathbb{R}$ is the linear form

$$L(v) = \int_0^1 f(x)v(x)dx - \epsilon K(x_0)v'(x_0) + \frac{\sigma^0}{h_{0,1}}v(x_0).$$

Remark: Problem (1.6) is an example of a finite-dimensional variational formulation. This concept is discussed in detail in the next chapter.
Remark: Depending on the choices of the parameters ϵ, σ^0, σ^1, we obtain several variations of DG methods that have appeared in the literature at different times.

- If $\epsilon = -1$, $\sigma^1 = 0$, and σ^0 is bounded below by a large enough constant, the resulting method is called the symmetric interior penalty Galerkin (SIPG) method, introduced in the late 1970s by Wheeler [109] and Arnold [1].

- If $\epsilon = -1$ and $\sigma^0 = \sigma^1 = 0$, the resulting method is called the global element method, introduced in 1979 by Delves and Hall [43]. However, the matrix associated with the bilinear form is indefinite, as the real parts of the eigenvalues are not all positive and thus the method is not stable.

- If $\epsilon = +1$, $\sigma^1 = 0$, and $\sigma^0 = 1$, the resulting method is called the nonsymmetric interior penalty Galerkin (NIPG) method, introduced in 1999 by Rivière, Wheeler, and Girault [95].

- If $\epsilon = +1$ and $\sigma^0 = \sigma^1 = 0$, the resulting method was introduced by Oden, Babuška, and Baumann in 1998 [84]. Throughout these notes, we will refer to this method as the NIPG 0 method, since it corresponds to the particular case of NIPG with $\sigma^0 = 0$.

- If $\epsilon = 0$, we obtain the incomplete interior penalty Galerkin (IIPG) method introduced by Dawson, Sun, and Wheeler [42] in 2004.

Remark: What if $\epsilon = 0$ and $\sigma^0 = \sigma^1 = 0$? Then the method is not convergent and not stable. One cannot even prove uniqueness and existence of the discrete solution.
Remark: It could be useful in practice to allow the penalty parameters to vary with each node. For instance a large value σ_n^0 yields a numerical solution with a small jump at the node x_n.

1.3 Existence and uniqueness of the DG solution

Since the problem is finite-dimensional, existence of a solution is equivalent to uniqueness. Let us assume that P^1 and P^2 are two solutions and let us define $\theta = P^1 - P^2$. Since both P^1 and P^2 satisfy (1.6), we have

$$\forall v \in \mathcal{D}_k(\mathcal{E}_h), \quad a_\epsilon(\theta, v) = 0.$$

Choosing in particular $v = \theta$ gives

$$a_\epsilon(\theta, \theta) = 0.$$

In the case of NIPG with $\sigma^0 > 0$, (1.5) yields

$$\forall n \geq 0, \quad \int_{x_n}^{x_{n+1}} K(x)(\theta'(x))^2 dx = 0, \quad \frac{\sigma^0}{h_{n-1,n}}[\theta(x_n)]^2 = 0.$$

Since K is strictly positive, the first equation implies that the function θ is equal to a constant k_n on each interval I_n. The second equation implies that all k_n's are equal to the same zero constant.

What should we do in the case of SIPG, IIPG, and NIPG 0? The proof is not as simple and is given in Chapter 2. In particular, some conditions on the penalty parameters need to be imposed in order to obtain uniqueness (hence existence) of the solution.

1.4 Linear system

In this section, we derive the linear system obtained from the DG scheme in the simpler case where K is the unit constant and $\sigma^1 = 0$. We also consider the case where discontinuous piecewise quadratic polynomials are used, namely $k = 2$. Let us choose for local basis functions of $\mathbb{P}_2(I_n)$ the monomial basis functions, translated from the interval $(-1, 1)$:

$$\mathbb{P}_2(I_n) = \text{span}\{\phi_0^n, \phi_1^n, \phi_2^n\}$$

with

$$\phi_0^n(x) = 1, \quad \phi_1^n(x) = 2\frac{x - x_{n+1/2}}{x_{n+1} - x_n}, \quad \phi_2^n(x) = 4\frac{(x - x_{n+1/2})^2}{(x_{n+1} - x_n)^2},$$

and $x_{n+1/2} = \frac{1}{2}(x_n + x_{n+1})$ is the midpoint of the interval I_n. To further simplify the computation, let us assume that there is a positive integer N such that

$$x_n = x_0 + nh, \quad h = \frac{1}{N}.$$

Thus, the local basis functions and their derivatives are simply

$$\phi_0^n(x) = 1, \quad \phi_1^n(x) = \frac{2}{h}(x - (n + 1/2)h), \quad \phi_2^n(x) = \frac{4}{h^2}(x - (n + 1/2)h)^2, \quad (1.7)$$

$$\phi_0^{n\prime}(x) = 0, \quad \phi_1^{n\prime}(x) = \frac{2}{h}, \quad \phi_2^{n\prime}(x) = \frac{8}{h^2}(x - (n + 1/2)h). \quad (1.8)$$

The global basis functions $\{\Phi_i^n\}$ for the space $\mathcal{D}_2(\mathcal{E}_h)$ are obtained from the local basis functions by extending them by zero:

$$\Phi_i^n(x) = \begin{cases} \phi_i^n(x), & x \in I_n, \\ 0, & x \notin I_n. \end{cases}$$

We can then expand the DG solution as

$$\forall x \in (0, 1), \quad P^{\text{DG}}(x) = \sum_{m=0}^{N-1} \sum_{j=0}^{2} \alpha_j^m \Phi_m^j(x), \quad (1.9)$$

where the coefficients α_j^m are unknown real numbers to be solved for. Plugging this form of P^{DG} into the scheme (1.6), we have

$$\forall 0 \le n \le N-1, \quad \forall 0 \le i \le 2, \quad \sum_{m=0}^{N-1} \sum_{j=0}^{2} \alpha_j^m a_\epsilon(\Phi_m^j, \Phi_n^i) = L(\Phi_n^i).$$

We obtain a linear system of the form $\boldsymbol{A}\boldsymbol{\alpha} = \boldsymbol{b}$, where $\boldsymbol{\alpha}$ is the vector with components α_j^m, $\boldsymbol{A}$ is the matrix with entries $a_\epsilon(\Phi_m^j, \Phi_n^i)$, and $\boldsymbol{b}$ is the vector with components $L(\Phi_n^i)$.

1.4.1 Computing the matrix A

Because of the local support of the global basis functions, the entries of the global matrix $\boldsymbol{A}$ can be obtained by first computing and assembling local matrices.

In what follows, we first describe how to compute the local matrices. We will regroup the terms defining a_ϵ into three groups: the terms involving integrals over I_n, the terms involving the interior nodes x_n, and those involving the boundary nodes x_0, x_N.

First, we consider the term corresponding to the integrals over the intervals I_n. On each element I_n, the DG solution P^{DG} is a quadratic polynomial, and we can write

$$\forall x \in I_n, \quad P^{\mathrm{DG}}(x) = \alpha_0^n \phi_0^n(x) + \alpha_1^n \phi_1^n(x) + \alpha_2^n \phi_2^n(x). \tag{1.10}$$

Thus, using the expansion above and choosing $v = \phi_i^n$ for $i = 0, 1, 2$, we obtain

$$\forall i = 0, 1, 2, \quad \int_{I_n} (P^{\mathrm{DG}})'(x)(\phi_i^n)'(x)dx = \sum_{j=0}^{2} \alpha_j^n \int_{I_n} (\phi_j^n)'(x)(\phi_i^n)'(x)dx.$$

This linear system can be rewritten as $\boldsymbol{A}_n \boldsymbol{\alpha}^n$, where

$$\boldsymbol{\alpha}^n = \begin{pmatrix} \alpha_0^n \\ \alpha_1^n \\ \alpha_2^n \end{pmatrix}, \quad (\boldsymbol{A}_n)_{ij} = \int_{I_n} (\phi_i^n)'(x)(\phi_j^n)'(x)dx.$$

One can easily compute $\boldsymbol{A}_n$:

$$\boldsymbol{A}_n = \frac{1}{h} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \frac{16}{3} \end{pmatrix}.$$

Second, we consider the terms involving the interior nodes x_n. By expanding the average and jump terms, we can write

$$\begin{aligned} &-\{(P^{\mathrm{DG}})'(x_n)\}[v(x_n)] + \epsilon\{v'(x_n)\}[P^{\mathrm{DG}}(x_n)] + \frac{\sigma^0}{h}[P^{\mathrm{DG}}(x_n)][v(x_n)] \\ &= b_n + c_n + d_n + e_n, \end{aligned}$$

where the terms are defined below:

$$b_n = \frac{1}{2}(P^{\mathrm{DG}})'(x_n^+)v(x_n^+) - \frac{\epsilon}{2}P^{\mathrm{DG}}(x_n^+)v'(x_n^+) + \frac{\sigma^0}{h}P^{\mathrm{DG}}(x_n^+)v(x_n^+),$$
$$c_n = -\frac{1}{2}(P^{\mathrm{DG}})'(x_n^-)v(x_n^-) + \frac{\epsilon}{2}P^{\mathrm{DG}}(x_n^-)v'(x_n^-) + \frac{\sigma^0}{h}P^{\mathrm{DG}}(x_n^-)v(x_n^-),$$
$$d_n = -\frac{1}{2}(P^{\mathrm{DG}})'(x_n^+)v(x_n^-) - \frac{\epsilon}{2}P^{\mathrm{DG}}(x_n^+)v'(x_n^-) - \frac{\sigma^0}{h}P^{\mathrm{DG}}(x_n^+)v(x_n^-),$$
$$e_n = \frac{1}{2}(P^{\mathrm{DG}})'(x_n^-)v(x_n^+) + \frac{\epsilon}{2}P^{\mathrm{DG}}(x_n^-)v'(x_n^+) - \frac{\sigma^0}{h}P^{\mathrm{DG}}(x_n^-)v(x_n^+).$$

Again, with the expansion (1.10) and with the choice $v = \phi_i^n$, the four terms defined above will yield the local matrices $\boldsymbol{B}_n$, $\boldsymbol{C}_n$, $\boldsymbol{D}_n$, and $\boldsymbol{E}_n$, respectively. For instance, the entries of $\boldsymbol{B}_n$ and $\boldsymbol{D}_n$ are given by

$$(\boldsymbol{B}_n)_{ij} = \frac{1}{2}(\phi_j^n)'(x_n^+)\phi_i^n(x_n^+) - \frac{\epsilon}{2}\phi_j^n(x_n^+)(\phi_i^n)'(x_n^+) + \frac{\sigma^0}{h}\phi_j^n(x_n^+)\phi_i^n(x_n^+),$$
$$(\boldsymbol{D}_n)_{ij} = -\frac{1}{2}(\phi_j^n)'(x_n^+)\phi_i^{n-1}(x_n^-) - \frac{\epsilon}{2}\phi_j^n(x_n^+)(\phi_i^{n-1})'(x_n^-) - \frac{\sigma^0}{h}\phi_j^n(x_n^+)\phi_i^{n-1}(x_n^-).$$

Carefully examining the terms, we see that $\boldsymbol{B}_n$ (resp., $\boldsymbol{C}_n$) corresponds to the interactions of the local basis functions of the interval I_n (resp., I_{n-1}) with themselves, whereas the matrices $\boldsymbol{D}_n$ and $\boldsymbol{E}_n$ couple the intervals I_n and I_{n-1}. One can easily compute the following four 3×3 matrices, using the definitions (1.7) and (1.8):

$$\boldsymbol{B}_n = \frac{1}{h}\begin{pmatrix} \sigma^0 & 1-\sigma^0 & -2+\sigma^0 \\ -\epsilon-\sigma^0 & -1+\epsilon+\sigma^0 & 2-\epsilon-\sigma^0 \\ 2\epsilon+\sigma^0 & 1-2\epsilon-\sigma^0 & -2+2\epsilon+\sigma^0 \end{pmatrix},$$

$$\boldsymbol{C}_n = \frac{1}{h}\begin{pmatrix} \sigma^0 & -1+\sigma^0 & -2+\sigma^0 \\ \epsilon+\sigma^0 & -1+\epsilon+\sigma^0 & -2+\epsilon+\sigma^0 \\ 2\epsilon+\sigma^0 & -1+2\epsilon+\sigma^0 & -2+2\epsilon+\sigma^0 \end{pmatrix},$$

$$\boldsymbol{D}_n = \frac{1}{h}\begin{pmatrix} -\sigma^0 & -1+\sigma^0 & 2-\sigma^0 \\ -\epsilon-\sigma^0 & -1+\epsilon+\sigma^0 & 2-\epsilon-\sigma^0 \\ -2\epsilon-\sigma^0 & -1+2\epsilon+\sigma^0 & 2-2\epsilon-\sigma^0 \end{pmatrix},$$

$$\boldsymbol{E}_n = \frac{1}{h}\begin{pmatrix} -\sigma^0 & 1-\sigma^0 & 2-\sigma^0 \\ \epsilon+\sigma^0 & -1+\epsilon+\sigma^0 & -2+\epsilon+\sigma^0 \\ -2\epsilon-\sigma^0 & 1-2\epsilon-\sigma^0 & 2-2\epsilon-\sigma^0 \end{pmatrix}.$$

Finally, we compute the local matrices arising from the boundary nodes x_0 and x_N:

$$f_0 = (P^{\mathrm{DG}})'(x_0)v(x_0) - \epsilon v'(x_0)P^{\mathrm{DG}}(x_0) + \frac{\sigma^0}{h}P^{\mathrm{DG}}(x_0)v(x_0),$$
$$f_N = -(P^{\mathrm{DG}})'(x_N)v(x_N) + \epsilon v'(x_N)P^{\mathrm{DG}}(x_N) + \frac{\sigma^0}{h}P^{\mathrm{DG}}(x_N)v(x_N).$$

These two terms above yield the matrices $\boldsymbol{F}_0$ and $\boldsymbol{F}_N$:

$$\boldsymbol{F}_0 = \frac{1}{h}\begin{pmatrix} \sigma^0 & 2-\sigma^0 & -4+\sigma^0 \\ -2\epsilon-\sigma^0 & -2+2\epsilon+\sigma^0 & 4-2\epsilon-\sigma^0 \\ 4\epsilon+\sigma^0 & 2-4\epsilon-\sigma^0 & -4+4\epsilon+\sigma^0 \end{pmatrix},$$

$$\boldsymbol{F}_N = \frac{1}{h}\begin{pmatrix} \sigma^0 & -2+\sigma^0 & -4+\sigma^0 \\ 2\epsilon+\sigma^0 & -2+2\epsilon+\sigma^0 & -4+2\epsilon+\sigma^0 \\ 4\epsilon+\sigma^0 & -2+4\epsilon+\sigma^0 & -4+4\epsilon+\sigma^0 \end{pmatrix}.$$

These local matrices are independent of the interval I_n. In the general case where the size of the intervals varies, the local matrices vary over all intervals. Once the local matrices have been computed, they are assembled into the global matrix. The assembling depends on the order of the unknowns α_i^n. Assuming that the unknowns are listed in the following order,

$$(\alpha_0^0, \alpha_1^0, \alpha_2^0, \alpha_0^1, \alpha_1^1, \alpha_2^1, \alpha_0^2, \alpha_1^2, \alpha_2^2, \ldots, \alpha_0^{N-1}, \alpha_1^{N-1}, \alpha_2^{N-1}),$$

we obtain a global matrix that is block tridiagonal:

$$\begin{pmatrix} \boldsymbol{M}_0 & \boldsymbol{D}_1 & & & & \\ \boldsymbol{E}_1 & \boldsymbol{M} & \boldsymbol{D}_2 & & & \\ & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & \boldsymbol{E}_{N-2} & \boldsymbol{M} & \boldsymbol{D}_{N-1} \\ & & & & \boldsymbol{E}_{N-1} & \boldsymbol{M}_N \end{pmatrix},$$

where

$$\boldsymbol{M} = \boldsymbol{A}_n + \boldsymbol{B}_n + \boldsymbol{C}_{n+1}, \quad \boldsymbol{M}_0 = \boldsymbol{A}_0 + \boldsymbol{F}_0 + \boldsymbol{C}_1, \quad \boldsymbol{M}_N = \boldsymbol{A}_{N-1} + \boldsymbol{F}_N + \boldsymbol{B}_{N-1}.$$

Remark: Since the penalty parameter is constant, the local matrices are independent of the subintervals. Thus, they can be defined before assembling the global matrix. Appendix B.1 contains a MATLAB® code that computes the global matrix.

1.4.2 Computing the right-hand side *b*

Each component of $\boldsymbol{b}$ is obtained by computing

$$L(\Phi_n^i) = \int_0^1 f(x)\Phi_n^i(x)dx - \epsilon K(x_0)(\Phi_n^i)'(x_0) + \frac{\sigma^0}{h}\Phi_n^i(x_0).$$

Because of the local support of Φ_n^i, the first term is reduced to

$$\int_0^1 f(x)\Phi_i^n(x)dx = \int_{x_n}^{x_{n+1}} f(x)\phi_i^n(x)dx.$$

After a change of variable, we obtain

$$\int_0^1 f(x)\Phi_i^n(x)dx = \frac{h}{2}\int_{-1}^1 f\left(\frac{h}{2}t + (n+1/2)h\right)t^i dt.$$

Because the integral cannot be computed exactly for most functions f, we rather compute an approximation by using a quadrature rule [6]. In particular, we choose the Gauss quadrature rule defined by a set of weights $(w_j)_{1\le j\le Q_G}$ and a set of nodes $(s_j)_{1\le j\le Q_G}$:

$$\int_{-1}^{1} v(t)dt \approx \sum_{j=1}^{Q_G} w_j v(s_j). \tag{1.11}$$

One can show that if v is a polynomial of degree $2Q_G - 1$, the Gauss quadrature rule is exact; i.e., the sign $\approx$ in (1.11) becomes an equality sign. Appendix A gives the sets of weights and nodes for different values of Q_G. We therefore have

$$\int_0^1 f(x)\Phi_i^n(x)dx \approx \frac{h}{2}\sum_{j=1}^{Q_G} w_j f\left(\frac{h}{2}s_j + (n+1/2)h\right)s_j^i.$$

We write the components of the vector $\boldsymbol{b}$ in an order consistent with the ordering of the unknowns α_i^n:

$$(b_0^0, b_1^0, b_2^0, b_0^1, b_1^1, b_2^1, b_0^2, b_1^2, b_2^2, \ldots, b_0^{N-1}, b_1^{N-1}, b_2^{N-1}),$$

where the first three components are

$$b_0^0 = \frac{h}{2}\sum_{j=1}^{Q_G} w_j f\left(\frac{h}{2}s_j + \frac{h}{2}\right) + \frac{\sigma^0}{h},$$

$$b_1^0 = \frac{h}{2}\sum_{j=1}^{Q_G} w_j f\left(\frac{h}{2}s_j + \frac{h}{2}\right)s_j - \epsilon K(x_0)\frac{2}{h} - \frac{\sigma^0}{h},$$

$$b_2^0 = \frac{h}{2}\sum_{j=1}^{Q_G} w_j f\left(\frac{h}{2}s_j + \frac{h}{2}\right)s_j^2 + \epsilon K(x_0)\frac{4}{h} + \frac{\sigma^0}{h},$$

and the last $3(N-1)$ components are

$$\forall 1 \le n \le N-1, \quad \forall 0 \le i \le 2, \quad b_i^n = \frac{h}{2}\sum_{j=1}^{Q_G} w_j f\left(\frac{h}{2}s_j + (n+1/2)h\right)s_j^i.$$

We remark that the first three components of $\boldsymbol{b}$ differ from the rest because of the nonzero Dirichlet boundary condition at x_0. The MATLAB code given in Appendix B.1 shows how to build $\boldsymbol{b}$ and also how to solve for $\boldsymbol{\alpha}$.

1.4.3 Imposing boundary conditions strongly

So far, we have imposed the boundary conditions (1.2), (1.3) weakly, through the addition of the terms $-\epsilon v'(x_0)p(x_0) + \frac{\sigma^0}{h}v(x_0)p(x_0)$. One can, however, impose them strongly by restricting the approximation space to

$$\mathcal{D}_k^0(\mathcal{E}_h) = \{v \in \mathcal{D}_k(\mathcal{E}_h) : v(0) = 0, v(1) = 0\}$$

and by writing

$$P^{\mathrm{DG}} = P_0^{\mathrm{DG}} + P_1$$

with P_1 being a continuous piecewise polynomial of degree k satisfying $P_1(0) = 1$ and $P_1(1) = 0$ and with $P_0^{\mathrm{DG}} \in \mathcal{D}_k^0(\mathcal{E}_h)$ a solution of the modified DG scheme:

$$\begin{aligned}
&\forall v \in \mathcal{D}_k^0(\mathcal{E}_h), \quad \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} K(x)(P_0^{\mathrm{DG}})'(x)v'(x)dx - \sum_{n=1}^{N-1} \{K(x_n)(P_0^{\mathrm{DG}})'(x_n)\}[v(x_n)] \\
&+\epsilon \sum_{n=1}^{N-1} \{K(x_n)v'(x_n)\}[P_0^{\mathrm{DG}}(x_n)] + \sum_{n=1}^{N-1} \frac{\sigma^0}{h}[v(x_n)][P_0^{\mathrm{DG}}(x_n)] \\
&= \int_0^1 f(x)v(x)dx - \sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} K(x)P_1'(x)v'(x).
\end{aligned}$$

In that case, the global matrix is still block tridiagonal, but the blocks $\boldsymbol{M}_0$ and $\boldsymbol{M}_N$ are of size 2×2, the blocks $\boldsymbol{E}_1$ and $\boldsymbol{D}_{N-1}$ of size 3×2, and the blocks $\boldsymbol{E}_{N-1}$ and $\boldsymbol{D}_1$ of size 2×3.

1.5 Convergence of the DG method

One can show that if the exact solution is smooth enough, the numerical error decreases as one increases the number of intervals, i.e., as one decreases the mesh size h. We define the numerical error obtained on the mesh $\mathcal{E}_h$ by

$$e_h = p - P^{\mathrm{DG}}.$$

Definition 1.2. *Given a space V, the function $\|\cdot\| : V \to \mathbb{R}$ is called a norm if for all $v, w \in V$ and $t \in \mathbb{R}$, we have*

$$\begin{aligned}
&\text{(i)}\ \|v\| \geq 0, \\
&\text{(ii)}\ \|v\| = 0 \Leftrightarrow v = 0, \\
&\text{(iii)}\ \|tv\| = |t|\|v\|, \\
&\text{(iv)}\ \|v + w\| \leq \|v\| + \|w\|.
\end{aligned}$$

The function $\|\cdot\|$ is called a seminorm if only properties (i)*,* (iii)*, and* (iv) *are satisfied.*

Define the energy norm of the error by

$$\|e_h\|_{\mathcal{E}} = \left(\sum_{n=0}^{N-1} \int_{x_n}^{x_{n+1}} K(x)(e_h'(x))^2 dx + J_0(e_h, e_h) \right)^{1/2} \tag{1.12}$$

and the L^2 norm of the error by

$$\|e_h\|_{L^2(0,1)} = \left(\int_0^1 (e_h(x))^2 dx \right)^{1/2}.$$

Table 1.1. *Convergence rates of primal DG method for uniform meshes in one dimension.*

Method	β_1	β_2
NIPG $\sigma^0 \geq 0$	k	$k+1$ if k odd k if k even
SIPG $\sigma^0 > \sigma_*^0$	k	$k+1$
IIPG $\sigma^0 > \sigma_*^0$	k	$k+1$ if k odd k if k even

Table 1.2. *Convergence rates of primal DG method for nonuniform meshes in one dimension.*

Method	β_1	β_2
NIPG $\sigma^0 \geq 0$	k	k
SIPG $\sigma^0 > \sigma_*^0$	k	$k+1$
IIPG $\sigma^0 > \sigma_*^0$	k	k

One can show that $\|e_h\|_{\mathcal{E}} = Ch^{\beta_1}$ and $\|e_h\|_{L^2(0,1)} = Ch^{\beta_2}$, where C is a constant independent of h (see Chapter 2). The convergence rate of the method in the energy norm (resp., L^2 norm) is then defined to be the power β_1 (resp., β_2). Assuming that the solution is smooth, the mesh is uniform ($h_n = h$ for all n), and discontinuous piecewise polynomials of degree k are used, the convergence rates are summarized in Table 1.1. These rates can be proved theoretically, and they are obtained numerically for h sufficiently small by applying the formulas

$$\beta_1 = \frac{1}{\ln(2)} \ln\left(\frac{\|e_h\|_{\mathcal{E}}}{\|e_{h/2}\|_{\mathcal{E}}}\right), \quad \beta_2 = \frac{1}{\ln(2)} \ln\left(\frac{\|e_h\|_{L^2(0,1)}}{\|e_{h/2}\|_{L^2(0,1)}}\right). \tag{1.13}$$

If the meshes are nonuniform, the rates are suboptimal in L^2 norm for the NIPG and IIPG methods (see Table 1.2). The MATLAB code given in Appendix B.1 computes both the energy norm and the L^2 norm of the error.

1.6 Numerical experiments

In the case where the solution is given by the expression

$$p(x) = (1-x)e^{-x^2},$$

convergence results are obtained for both energy norm and L^2 norm of the error. We vary the parameter ϵ in $\{-1, 0, 1\}$, the penalty values σ^0, and the polynomial approximations from linear to quartic. We first consider uniform meshes of size h. The convergence rates are given in Tables 1.3–1.6 for $k = 1$, 2, and 3, respectively, and for different penalty values. We note that for $k = 1$, the choice $\sigma^0 = 0$ yields indefinite matrices, and thus the system cannot be solved.

Table 1.3. *Numerical errors and convergence rates for piecewise linear approximation.*

Method	h	$\|e_h\|_{\mathcal{E}}$	β_1	$\|e_h\|_{L^2(0,1)}$	β_2
NIPG $\sigma^0 = 1$	1/2	2.5300×10^{-1}		7.3161×10^{-2}	
	1/4	1.1630×10^{-1}	1.1211	1.9453×10^{-2}	1.9110
	1/8	5.4024×10^{-2}	1.1067	4.9477×10^{-3}	1.9752
	1/16	2.5720×10^{-2}	1.0706	1.2416×10^{-3}	1.9945
	1/32	1.2498×10^{-2}	1.0411	3.1061×10^{-4}	1.9990
SIPG $\sigma^0 = 2$	1/2	5.8471×10^{-1}		8.9892×10^{-2}	
	1/4	2.0222×10^{-1}	1.5317	1.7327×10^{-2}	2.3751
	1/8	8.5447×10^{-2}	1.2428	3.5659×10^{-3}	2.2806
	1/16	3.5828×10^{-2}	1.2539	7.3340×10^{-4}	2.2816
	1/32	1.5448×10^{-2}	1.2136	1.5613×10^{-4}	2.2318
IIPG $\sigma^0 = 1$	1/2	3.4091×10^{-1}		9.2456×10^{-2}	
	1/4	1.2112×10^{-1}	1.4929	2.5039×10^{-2}	1.8845
	1/8	5.1662×10^{-2}	1.2292	6.5011×10^{-3}	1.9454
	1/16	2.4615×10^{-2}	1.0695	1.6553×10^{-3}	1.9735
	1/32	1.2155×10^{-2}	1.0179	4.1755×10^{-4}	1.9871

Table 1.4. *Numerical errors and convergence rates for piecewise quadratic approximation.*

Method	h	$\|e_h\|_{\mathcal{E}}$	β_1	$\|e_h\|_{L^2(0,1)}$	β_2
NIPG $\sigma^0 = 0$	1/2	9.3544×10^{-2}		2.0713×10^{-2}	
	1/4	2.5299×10^{-2}	1.8865	7.9581×10^{-3}	1.3800
	1/8	6.6182×10^{-3}	1.9345	2.4210×10^{-3}	1.7168
	1/16	1.6804×10^{-3}	1.9775	6.4211×10^{-4}	1.9147
	1/32	4.2196×10^{-4}	1.9936	1.6305×10^{-4}	1.9774
NIPG $\sigma^0 = 1$	1/2	7.0690×10^{-2}		1.5754×10^{-2}	
	1/4	1.7289×10^{-2}	2.0315	5.0566×10^{-3}	1.6395
	1/8	4.2388×10^{-3}	2.0281	1.3419×10^{-3}	1.9138
	1/16	1.0472×10^{-3}	2.0171	3.3533×10^{-4}	2.0006
	1/32	2.6026×10^{-4}	2.0085	8.3121×10^{-5}	2.0123
SIPG $\sigma^0 = 2$	1/2	1.7472×10^{-1}		1.6963×10^{-2}	
	1/4	5.7965×10^{-2}	1.5917	2.8754×10^{-3}	2.5605
	1/8	9.8399×10^{-3}	2.5584	2.5109×10^{-4}	3.5174
	1/16	2.2901×10^{-3}	2.1032	2.9131×10^{-5}	3.1075
	1/32	5.6312×10^{-4}	2.0238	3.5624×10^{-6}	3.0316
IIPG $\sigma^0 = 1$	1/2	1.3032×10^{-1}		3.9401×10^{-2}	
	1/4	2.5275×10^{-2}	2.3663	7.7062×10^{-3}	2.3541
	1/8	5.6861×10^{-3}	2.1522	1.6145×10^{-3}	2.2548
	1/16	1.3649×10^{-3}	2.0586	3.6500×10^{-4}	2.1451
	1/32	3.3563×10^{-4}	2.0238	8.6547×10^{-5}	2.0763

Table 1.5. *Numerical errors and convergence rates for piecewise cubic approximation.*

Method	h	$\|e_h\|_{\mathcal{E}}$	β_1	$\|e_h\|_{L^2(0,1)}$	β_2
NIPG $\sigma^0 = 0$	1/2	5.6180×10^{-3}		1.2627×10^{-3}	
	1/4	6.4343×10^{-4}	3.1262	6.7644×10^{-5}	4.2224
	1/8	7.5240×10^{-5}	3.0962	3.8391×10^{-6}	4.1391
	1/16	9.0807×10^{-6}	3.0506	2.2809×10^{-7}	4.0730
	1/32	1.1151×10^{-6}	3.0255	1.3892×10^{-8}	4.0373
NIPG $\sigma^0 = 1$	1/2	5.2783×10^{-3}		1.0881×10^{-3}	
	1/4	6.2018×10^{-4}	3.0893	5.8542×10^{-5}	4.2162
	1/8	7.3930×10^{-5}	3.0683	3.3513×10^{-6}	4.1266
	1/16	9.0061×10^{-6}	3.0373	1.9968×10^{-7}	4.0689
	1/32	1.1107×10^{-6}	3.0193	1.2170×10^{-8}	4.0362
SIPG $\sigma^0 = 1$	1/2	6.4024×10^{-3}		4.4484×10^{-4}	
	1/4	6.8810×10^{-4}	3.2179	2.1387×10^{-5}	4.3784
	1/8	7.7514×10^{-5}	3.1501	1.1225×10^{-6}	4.2519
	1/16	9.2066×10^{-6}	3.0737	6.3845×10^{-8}	4.1360
	1/32	1.1225×10^{-6}	3.0359	3.7981×10^{-9}	4.0712
IIPG $\sigma^0 = 1$	1/2	7.3848×10^{-3}		4.3616×10^{-3}	
	1/4	6.5496×10^{-4}	3.4950	2.2715×10^{-4}	4.2631
	1/8	7.3054×10^{-5}	3.1643	1.3096×10^{-5}	4.1164
	1/16	8.8549×10^{-6}	3.0444	7.9424×10^{-7}	4.0434
	1/32	1.0983×10^{-6}	3.0111	4.9047×10^{-8}	4.0173

We now consider a nonuniform mesh constructed as follows. The unit interval is first divided into N intervals of length $1/N$. Each subinterval is then divided into three nonuniform subintervals of length $1/(7N)$, $1/(2N)$, and $5/(14N)$, respectively. Numerical errors and convergence rates are shown in Table 1.9. The rates are suboptimal in the L^2 norm for the NIPG method and for polynomials of degree one or two.

1.7 Bibliographical remarks

Stability and convergence of the NIPG method with zero penalty were obtained by Babuška, Baumann, and Oden [8] in one dimension. The analysis of the NIPG method with or without penalty was proved by Rivière, Wheeler, and Girault [96, 95] in any dimensions. The analysis of the IIPG method is almost identical to the analysis of the SIPG method, which can be obtained from Wheeler's work [109]. Using a standard lift argument, one can show suboptimal error estimates in the L^2 norm for both NIPG and IIPG on general meshes. In the case of uniform meshes in one dimension, Larsson and Niklasson [77] proved optimal convergence rates for polynomial degrees of even parity. The work of Cockburn, Gunzman, and Rivière [31] shows that for some nonuniform meshes, numerical rates remain suboptimal with a loss of one power of h.

Table 1.6. *Numerical errors and convergence rates for piecewise quartic approximation.*

Method	h	$\|e_h\|_{\mathcal{E}}$	β_1	$\|e_h\|_{L^2(0,1)}$	β_2
NIPG $\sigma^0 = 0$	1/2	7.4885×10^{-4}		1.1286×10^{-4}	
	1/4	5.0944×10^{-5}	3.8776	8.7699×10^{-6}	3.6859
	1/8	3.3003×10^{-6}	3.9482	5.9422×10^{-7}	3.8834
	1/16	2.0841×10^{-7}	3.9850	3.7975×10^{-8}	3.9678
	1/32	1.3061×10^{-8}	3.9960	2.3870×10^{-9}	3.9917
NIPG $\sigma^0 = 1$	1/2	7.2760×10^{-4}		1.0604×10^{-4}	
	1/4	4.8061×10^{-5}	3.9202	7.8791×10^{-6}	3.7504
	1/8	3.0614×10^{-6}	3.9726	5.2009×10^{-7}	3.9212
	1/16	1.9200×10^{-7}	3.9949	3.2856×10^{-8}	3.9845
	1/32	1.2001×10^{-8}	3.9998	2.0550×10^{-9}	3.9989
SIPG $\sigma^0 = 1$	1/2	7.8419×10^{-4}		3.7197×10^{-5}	
	1/4	5.5204×10^{-5}	3.8283	1.3837×10^{-6}	4.7485
	1/8	3.6525×10^{-6}	3.9178	4.6745×10^{-8}	4.8875
	1/16	2.3277×10^{-7}	3.9719	1.4983×10^{-9}	4.9634
	1/32	1.4642×10^{-8}	3.9907	4.7192×10^{-11}	4.9886
IIPG $\sigma^0 = 1$	1/2	1.4898×10^{-3}		8.3173×10^{-4}	
	1/4	6.7918×10^{-5}	4.4552	4.2646×10^{-5}	4.2856
	1/8	3.6617×10^{-6}	4.2132	2.2486×10^{-6}	4.2452
	1/16	2.1688×10^{-7}	4.0775	1.2739×10^{-7}	4.1416
	1/32	1.3289×10^{-8}	4.0286	7.5627×10^{-9}	4.0742

Table 1.7. *Numerical errors and convergence rates on nonuniform meshes.*

Method	N	k	$\|e_h\|_{\mathcal{E}}$	Rate	$\|e_h\|_{L^2(0,1)}$	Rate
NIPG $\sigma^0 = 1$	256	1	6.4397×10^{-4}		4.5206×10^{-6}	
	512	1	3.2190×10^{-4}	1.000	2.1943×10^{-6}	1.043
	256	2	6.6800×10^{-7}		1.5482×10^{-7}	
	512	2	1.6700×10^{-7}	2.000	3.8701×10^{-8}	2.000
SIPG $\sigma^0 = 1$	256	1	7.0270×10^{-4}		2.9266×10^{-7}	
	512	1	3.4453×10^{-4}	1.028	7.0062×10^{-8}	2.062
	256	2	4.8454×10^{-6}		1.8525×10^{-9}	
	512	2	1.2112×10^{-6}	2.000	2.3166×10^{-10}	3.000
IIPG $\sigma^0 = 1$	256	1	6.3022×10^{-4}		6.4539×10^{-7}	
	512	1	3.1511×10^{-4}	1.000	1.6133×10^{-7}	2.000
	256	2	7.8261×10^{-7}		1.7002×10^{-7}	
	512	2	1.9567×10^{-7}	2.000	4.2490×10^{-8}	2.000

Exercises

1.1. Show that the energy norm $||\cdot||_{\mathcal{E}}$ defined by (1.12) is indeed a norm for the space X_h:

$$X_h = \{v : v|_{I_n} \in \mathcal{C}^1(I_n), \quad n = 0, \ldots, N-1\}.$$

1.2. Derive the local matrices $\boldsymbol{A}_n$, $\boldsymbol{B}_n$, $\boldsymbol{C}_n$, $\boldsymbol{D}_n$, and $\boldsymbol{E}_n$ for polynomial degree $k = 4$.

1.3. Derive all the local matrices and the global matrix in the case of nonzero σ^1 and for polynomial degree $k = 2$.

1.4. Modify the code given in Appendix B.1 so that the energy norm of the error is computed. Run the code for the following exact solutions: (a) $p(x) = (1-x)^3$, (b) $p(x) = (1-x)\cos x$. Plot the numerical solution. By varying the number of intervals, compute the numerical convergence rates for both the energy norm and the L^2 norm of the error. Choose $\sigma^0 = 1$ and $\epsilon = 1$.

1.5. Define the DG method for solving (1.1) with $K = 1$ and with the following boundary conditions:

$$p(0) = 1,$$
$$p'(1) = 0.$$

Modify the code given in Appendix B.1 and run it for the exact solution: $p(x) = (1-x)^2 e^x$. Compute numerical errors obtained for the number of intervals $N = 4, 8, 16, 32$. Choose $\epsilon = -1$ and vary $\sigma^0 = 0.1, 1, 10$.

1.6. Implement the DG method in the case where the boundary condition $p(1) = 0$ is imposed strongly, as discussed in Section 1.4.3. Compute numerical convergence rates for the L^2 norm of the error for the exact solution $p(x) = (1-x)e^{-x}$. Compare the results obtained with NIPG, SIPG, and IIPG for $\sigma^0 = 1$, $k = 2$.