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# ON ONE-PARAMETER DESCRIPTION OF DAMAGE STATE FOR BRITTLE MATERIAL

C. L. CHOW and F. YANG†

Mechanical Engineering, Southern Illinois University at Edwardsville, Edwardsville, Illinois 62026-1805, U.S.A.

Department of Mechanical Engineering, University of Hong Kong, Hong Kong

**Abstract**—A simple approach for the description of material damage state within the framework of the continuum damage mechanics is presented in this paper. A one-parameter damage variable and a concept of damage surface are developed to enable the derivation of a generalized constitutive equation of elasticity coupled with damage. A special form of damage surface is chosen to demonstrate its applicability for which a set of material coefficients is selected to represent appropriately the effects of anisotropic material damage. The validity of the model is verified for an isotropic case and the comparisons between the predicted and measured results are illustrated using a typical brittle material, Polymethylmethacrylate (PMMA).

## 1. INTRODUCTION

IT IS WELL established that the deformation of most engineering materials is often accompanied by irreversible changes in their internal structures. Among these internal microstructural changes, the nucleation and the growth of distributed microscopic cavities and cracks will not only induce the occurrence of macro-cracks, but also lead to the deterioration of material properties, such as reductions in strength, rigidity and fracture toughness, or decrease in the remaining lifetime. This phenomenon of progressive degradation of the material prior to its final fracture is generally referred to as damage. Proper understanding and knowledge of the damaging process and its effects on the macroscopic behavior of materials are important pre-requisites for the analysis of structural integrity of practical problems. A systematic approach to these problems can be provided by the recently developed theory of continuum damage mechanics. Based on the thermodynamics of irreversible process and the internal state variable concept, the theory employs appropriate mechanical variables to model the damage state and then formulates their evolution equations to describe the progressive response of the damaging material.

The continuum damage theory pioneered by Kachanov[1] has become a very active research subject in the last decade and is used to study metal fatigue[2, 3], creep[4, 5] and creep–fatigue interaction[6, 7]. Recently, this method has been applied to various materials such as concrete and rock[8–11], ceramics[12, 13] and composite materials[14–17]. In addressing the problems of ductile fracture by introducing a second order damage tensor through the concepts of effective stress and effective strain using the hypothesis of energy equivalence, an anisotropic damage theory has been established[18–20], yielding satisfactory results in characterizing the crack initiation, propagation and fracture load.

This paper presents the development of a constitutive equation which can easily account for damage in brittle materials. This is achieved by introducing a damage surface[21] which is similar to the yield function in the conventional theory of plasticity[22]. The method adopted here parallels those formulated in ref. [23]. A special form of damage surface is constructed to illustrate the application of the model. Under certain specified conditions, the variation of material properties based on the proposed model is obtained and used to compare with the experimental data for PMMA (Polymethylmethacrylate), a typical brittle material.

## 2. DESCRIPTION OF DAMAGE STATE

In order to establish the model in a relatively general form, we employ the concept of internal state variable and phenomenological method to describe the state of a material element and, for

†On leave from Huazhong University of Science & Technology, Wuhan, China

simplicity, only consider the rate-independent behavior of the element in an isothermal process. Consequently, time and temperature will not appear in the formulation.

Suppose that a virgin material element (i.e. without prior loading and damaging history) is continuous and homogeneous. After having subjected it to a certain loading process, its state changes to become damaged as the material element undergoes irreversible deformation generated by internal dissipative mechanisms. Since we only consider the macroscopical response of the element, it is reasonable to hypothesize that we may neglect some fine details created during the damaging process as long as damage in the element is statistically homogeneous, or, the scale of stress and strain nonuniformities due to micro-cracks, voids, etc. is assumed small compared to the size of the material element under consideration. Thus, we can use an equivalent continuous and homogeneous element to replace it for which the two have the same mechanical behavior. Through the phenomenological theory stemming from the general principles of thermodynamics on the supposition that the response of the element is dependent merely upon the current state of the material and its irreversibly damaged process can be approximated by a series neighboring equilibrium states which are correspondingly described by various values of a finite set of internal variables, the element state can be modeled by the values of state variables and of internal variables. The state variables here are the components of stress $\sigma_{ij}$ and internal variable is taken to be a scalar $D$, but more could obviously be introduced if necessary. Because the emphasis of this investigation is on the elastic behavior with damage, it is further assumed that the material is elastic when the damage is constant during a particular damaging process, i.e. if an ideal element is deformed with increasing damage, and then unloaded, it will recover elastically. During unloading, the effective elastic modulus, determined from the initial slope of an unloading curve, remains invariant. Moreover, damage of the element will not progress unless its initial unloading state is surpassed when reloaded. This problem may be solved when a damage surface may be constructed in the stress space. It is a function of the state of stress and the internal variable $D$. The elastic region is defined as the locus of stress paths along which the damage does not change and is separated from the rest of the space by the damage surface.

As discussed above, the damage surfaces, which also represent initial and subsequent damage surface in stress space, may be expressed analytically by

$$F(\sigma_{ij}, D) = 0 \tag{2.1}$$

in which $D$ is a measure of damaged deformation and hence changes only when damage occurs. The damage function $F$ is assumed to be continuously differentiable with respect to its arguments. When the stress state is within $F$, the behavior of the material is elastic; if the stress point reaches the boundary $F$, i.e. if $F = 0$, a subsequent increment of stress may either cause purely elastic deformation or lead to the damage process depending on the direction of the increment as illustrated in Fig. 1.

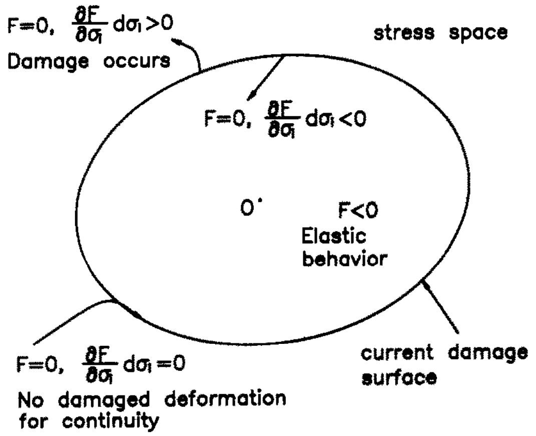


Fig. 1. Damage surface.

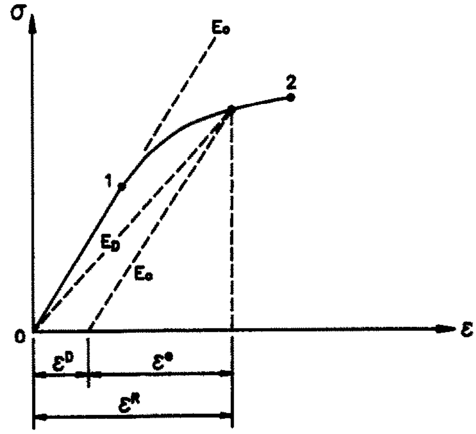


Fig. 2. Model for brittle material with damage.

Therefore, it is obvious that elastic loading or unloading without damage takes place when

$$F<0, \quad \text{or} \quad F=0 \quad \text{and} \quad \frac{\partial F}{\partial \sigma_{ij}} \mathrm{d}\sigma_{ij} \leqslant 0. \tag{2.2}$$

On the other hand, the damage process occurs when

$$F=0 \quad \text{and} \quad \frac{\partial F}{\partial \sigma_{ij}} \mathrm{d}\sigma_{ij} > 0. \tag{2.3}$$

Since eq. (2.2) must be satisfied during a damaging process the consistency condition is

$$\mathrm{d}F = 0. \tag{2.4}$$

This actually shows that during a damaging process the stress state $\sigma_{ij}$ always lies on the boundary of surface $F$.

A typical uniaxial tension stress–strain curve for the stable brittle material with damage consideration is shown in Fig. 2. Upon deformation, behavior of the material is elastic in the range 0–1. The damage process starts at the point 1 and further develops in the region 1–2. It is assumed that unloading from any point of the region 1–2 takes place at the modulus $E_D$ which is equal to the secant modulus of the diagram and permanent strain is considered negligible. The path of reloading is identical to that of unloading because hysteretic phenomena occurring during unloading and reloading are postulated to be very small. The deviation of the elastic response is a result of the nucleation of new micro-cracks and the growth of existing micro-defects.

For the case discussed above, the stress and strain relation, at any certain damage state, can be expressed as

$$\epsilon_{ij}^{R} = A_{ijkl}\sigma_{kl} \quad \text{(for fixed } D) \tag{2.5}$$

in which $A_{ijkl}(A_{ijkl} = A_{jikl} = A_{ijlk} = A_{klij})$ is the elastic compliance at the current state, and $\epsilon_{ij}^{R}$ are reversible strain components.

If total incremental strain at any stage of the deformation process is denoted by $\mathrm{d}\epsilon_{ij}^{R}$, within the limitation of small deformation, it may be linearly divided into elastic and damaged components as

$$\mathrm{d}\epsilon_{ij}^{R} = \mathrm{d}\epsilon_{ij}^{e} + \mathrm{d}\epsilon_{ij}^{D} \tag{2.6}$$

where the superscripts $e$ and $D$ refer to the elastic and damage strain components respectively. The elastic strain increment may be related to the stress increment $\mathrm{d}\sigma_{ij}$ through the generalized Hooke's law as

$$\mathrm{d}\epsilon_{ij}^{e} = A_{ijkl}\,\mathrm{d}\sigma_{kl}. \tag{2.7}$$

The incremental damaged strains are evaluated based on the damage function $F$, using the following equations, similar to the associative flow rule in plasticity theory:

$$\mathrm{d}\epsilon_{ij}^{D} = \begin{cases} \mathrm{d}\lambda \dfrac{\partial F}{\partial \sigma_{ij}}, & \mathrm{d}\lambda > 0, \quad \text{if } F=0 \quad \text{and} \quad \dfrac{\partial F}{\partial \sigma_{ij}} \mathrm{d}\sigma_{ij} > 0. \\ 0, & \text{otherwise} \end{cases} \tag{2.8}$$

where $\mathrm{d}\lambda$ is a scalar of proportionality and can be determined from the consistency condition (2.4).

While damaged deformation progresses, the stresses remain on the damage surface, the condition (2.4) gives

$$\frac{\partial F}{\partial \sigma_{ij}} \mathrm{d}\sigma_{ij} + \frac{\partial F}{\partial D} \mathrm{d}D = 0. \tag{2.9}$$

Letting $B = -(\partial F/\partial D)\,\mathrm{d}D\,(1/\mathrm{d}\lambda)$, which is a function of damage state, from eq. (2.9) one has

$$\mathrm{d}\lambda = \frac{1}{B} \frac{\partial F}{\partial \sigma_{ij}} \mathrm{d}\sigma_{ij}. \tag{2.10}$$

Then eqs (2.8) and (2.7) are substituted into eq. (2.6) to yield

$$\mathrm{d}\epsilon_{ij}^{R} = \left( A_{ijkl} + \frac{1}{B} \frac{\partial F}{\partial \sigma_{ij}} \frac{\partial F}{\partial \sigma_{kl}} \right) \mathrm{d}\sigma_{kl} \tag{2.11a}$$

or

$$\mathrm{d}\epsilon_{ij}^{R} = A_{ijkl}^{eD}\,\mathrm{d}\sigma_{kl} \tag{2.11b}$$

where $A_{ijkl}^{eD}$ is denoted as the elastic-damaged symmetric compliance coefficient for which $A_{ijkl}^{eD} = A_{jikl}^{eD} = A_{ijlk}^{eD} = A_{klij}^{eD}$.

The damaged stress–strain relations for some materials may be formulated only if the parameter $B$ defined in (2.10) is determined. For illustrative purposes, the damage surface is constructed as

$$F(\sigma_{ij}, D) = (R_{ijkl}\sigma_{ij}\sigma_{kl})^{1/2} - K(D) = 0 \tag{2.12}$$

where the coefficients $R_{ijkl}(R_{ijkl} = R_{jikl} = R_{ijlk} = R_{klij}$ is assumed) depict the influence of each stress component on the damage state of the material and can be determined from experimental data. The form of this function $F$ is different from those commonly adopted in plasticity, for instance, Von Mises yield function for isotropic materials and Hill-type yield function for the orthotropic case. The damage state is directly related to stress components not to the deviatoric stresses. It is not inherently assumed that the damaged deformation is incompressible and hydrostatic stress has no effect on damage strains.

It is convenient to introduce a parameter which reflects the loading stage or the position of the subsequent damage surface. This can be accomplished by defining an equivalent stress $\sigma_0$ as

$$\sigma_0^2 = R_{ijkl}\sigma_{ij}\sigma_{kl}. \tag{2.13}$$

For a certain value of $\sigma_0$, a corresponding subsequent damage surface on which the state of stress $\sigma_{ij}$ lies can be uniquely obtained. Any state of stresses which lies on the same surface will have the same value of equivalent stress $\sigma_0$.

The energy per unit volume dissipated during damaging process is

$$W^D = \int_0^{\epsilon_{ij}^{R}} \sigma_{ij}\,\mathrm{d}\epsilon_{ij} - \tfrac{1}{2}\sigma_{ij}\epsilon_{ij}^{R}. \tag{2.14}$$

Differentiating it, the energy dissipation increment can be obtained

$$\mathrm{d}W^D = \tfrac{1}{2}(\sigma_{ij}\,\mathrm{d}\epsilon_{ij}^{R} - \epsilon_{ij}^{R}\,\mathrm{d}\sigma_{ij}) \tag{2.15}$$

which may be expressed as

$$\mathrm{d}W^D = \tfrac{1}{2}\sigma_{ij}\mathrm{d}\epsilon_{ij}^{D} \tag{2.16}$$

by using eqs (2.5) and (2.7).

An equivalent damaged strain increment $\mathrm{d}\epsilon^D$ may be chosen in connection with $\mathrm{d}W^D$ such that

$$\mathrm{d}W^D = \tfrac{1}{2}\sigma_{ij}\,\mathrm{d}\epsilon_{ij}^{D} = \tfrac{1}{2}\sigma_0\,\mathrm{d}\epsilon^D. \tag{2.17}$$

From eq. (2.8)

$$\mathrm{d}\epsilon_{ij}^{D} = \mathrm{d}\lambda\,\frac{R_{ijkl}\sigma_{kl}}{\sigma_0}. \tag{2.18}$$

Combining eqs (2.17) and (2.18) gives

$$\mathrm{d}\lambda = \mathrm{d}\epsilon^D. \tag{2.19}$$

The condition of $R_{1111} = 1$ must be satisfied for $\sigma_0 = \sigma_{11}$ when all stress components vanish except for $\sigma_{11}$. The relationship between $\sigma_0$ and $\epsilon^D$ can be readily established by means of uniaxial test data.

When $\sigma_0$—$\epsilon^D$ curve is

$$\epsilon^D = f(\sigma_0) \tag{2.20}$$

and $\mathrm{d}\epsilon^D = f'(\sigma_0)\,\mathrm{d}\sigma_0, f'(\sigma_0)$ is the slope of a particular point on the curve, then the parameter $B$ becomes

$$B = \frac{1}{f'(\sigma_0)}. \tag{2.21}$$

Thus, eq. (2.11) may be written as

$$d\epsilon_{ij}^{R} = A_{ijkl}^{eD}\, d\sigma_{kl} \tag{2.22a}$$

$$A_{ijkl}^{eD} = A_{ijkl} + \frac{f'(\sigma_0)}{\sigma_0^2} R_{ijmn}\sigma_{mn} R_{pqkl}\sigma_{pq}. \tag{2.22b}$$

Obviously, $A_{ijkl}^{eD}$ is a function of stress state. Equation (2.22) is important in numerical analysis of damage problems, especially when employing the finite element method.

## 3. EXAMPLE

For verifying the model developed in the preceding section, an example for the case shown in Fig. 2 is considered, in which the compliance $A_{ijkl}$ in eq. (2.5) changes during loading process as it is accompanied by material damage.

In the damaging process, from eq. (2.5), the strain increment is

$$d\epsilon_{ij}^{R} = A_{ijkl}\, d\sigma_{kl} + dA_{ijkl}\sigma_{kl}. \tag{3.1}$$

Comparing this equation with eq. (2.6) and using eqs (2.8), (2.18) and (2.19), one has

$$dA_{ijkl}\sigma_{kl} = \frac{d\epsilon^{D}}{\sigma_0} R_{ijkl}\sigma_{kl}. \tag{3.2}$$

As discussed in the introduction section, it can be further assumed that the material coefficients $A_{ijkl}$ are merely functions of damage state $D$, independent from the damaging processes[24]. In other words, the dimensions of the damage surfaces in stress space are only dependent upon the maximum current equivalent stress $\sigma_0$. Hence, with eq. (2.20), eq. (3.2) leads to

$$dA_{ijkl} = \frac{f'(\sigma_0)\, d\sigma_0}{\sigma_0} R_{ijkl}. \tag{3.3}$$

If a power law

$$\epsilon^{D} = m\sigma_0^{n} \tag{3.4}$$

is taken, where $m$ and $n$ are positive constants, eq. (3.3) becomes

$$dA_{ijkl} = mn\sigma_0^{n-2}\, d\sigma_0 R_{ijkl}. \tag{3.5}$$

Based on such a model, eq. (3.4) allows damage initiation as soon as the stress is applied. However, the amount of damage strain is very small and can be ignored for low levels of stress because of the choice of the power law.

In the light of Schapery's method[24], an alternative formulation is selected

$$D = \frac{mn}{2(n+1)}\sigma_0^{n+1} \quad \text{for } d\sigma_0 \geq 0. \tag{3.6}$$

$A_{ijkl}$ can be expressed as

$$A_{ijkl}(D) = A_{ijkl}(0) + qD^{p}R_{ijkl} \tag{3.7}$$

where

$$p = \frac{n-1}{n+1}, \quad q = \frac{2(n+1)}{n-1}\left[\frac{mn}{2(n+1)}\right]^{2/(n+1)}$$

and $A_{ijkl}(0)$ are the elastic compliance components without damage. Equation (3.6) is different from that in Wang[25] where the damage variable always lies between zero and unity. Equation (3.6) shows that the magnitude of the damage variable can extend beyond unity. This is attributed to the difference in the definition and physical interpretation of the damage variable $D$. From eq. (3.7), it can be seen that the coefficients $R_{ijkl}$ reflect the anisotropic behavior induced by material damage.

During a damaging process, eq. (3.7) is employed to predict instantaneous $A_{ijkl}$ values through eq. (3.6). For unloading, the coefficients $A_{ijkl}$ remain constant and are determined from $\sigma_0$ at the

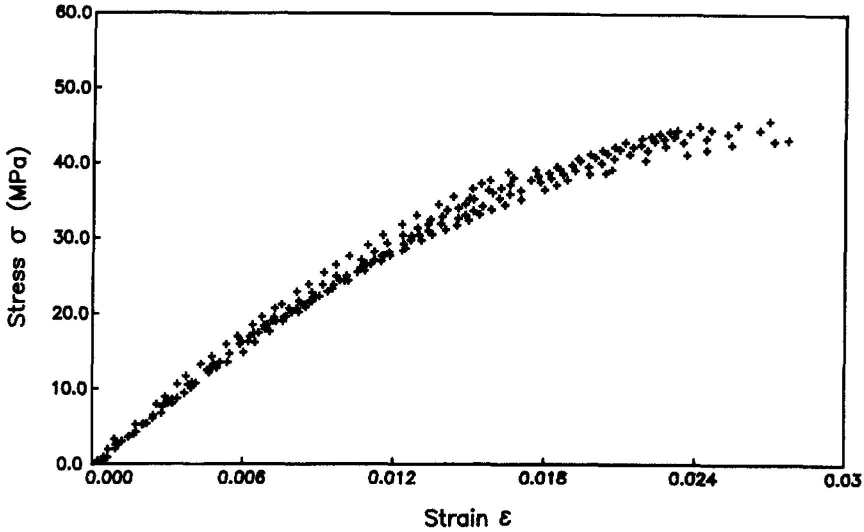


Fig. 3. $\sigma$–$\epsilon$ relation.

start of unloading. Upon reloading, $D$ as well as $A_{ijkl}$ do not change unless the equivalent stress $\sigma_0$ exceeds its preceding maximum value.

If the anisotropy induced by damage is not evident, the material behavior affected by damage can be regarded as isotropic and the expression of $A_{ijkl}$ in eq. (3.7) may be simplified. In this case, the non-zero values of $R_{ijkl}$ are as follows,

$$R_{1111} = R_{2222} = R_{3333} = 1, \quad R_{1122} = R_{1133} = R_{2233}, \quad R_{2323} = R_{1313} = R_{1212}$$

and they are not independent, for instance,

$$R_{1212} = 2(1 - R_{1122}). \tag{3.8}$$

Experimental data concerning the gradual degradation of material coefficients for tensile specimens of Polymethylmethacrylate(PMMA) are used to elucidate the model. The stress–strain relationship of the material is shown in Fig. 3. The initial material constants measured are, Young's modulus $E_0 = 2.67$ GPa, Poisson's ratio $\nu_0 = 0.385$. As the material is brittle and the deformation is small, the assumption that the material behavior after damage initiation still remains isotropic

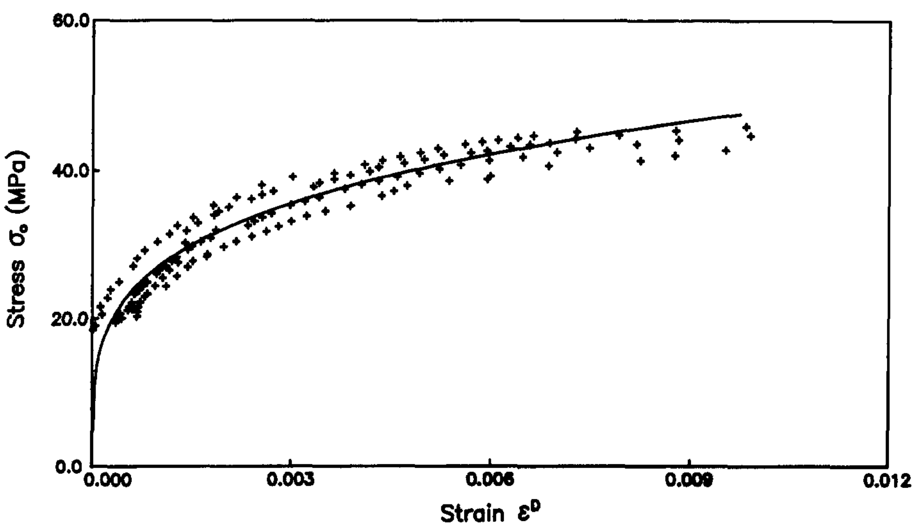


Fig. 4. $\sigma_0$–$\epsilon^D$ curve.

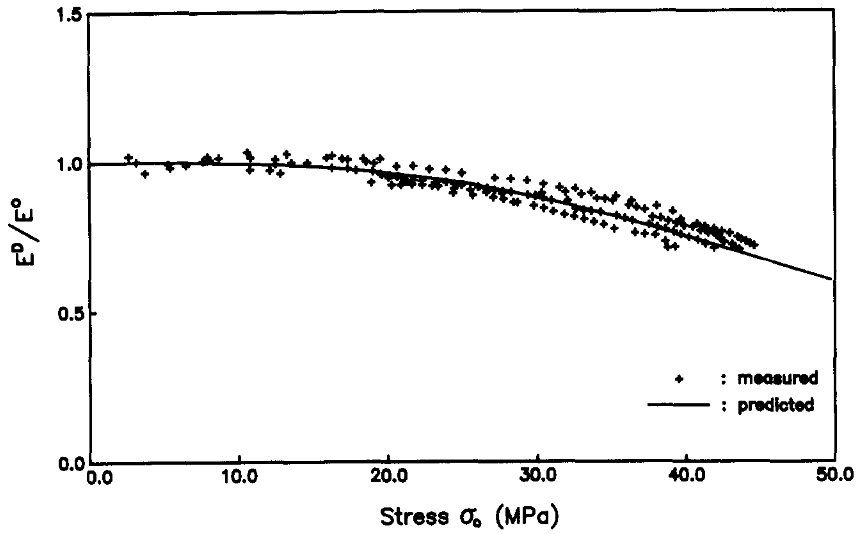


Fig. 5. $E^D/E^0$–$\sigma_0$ curve.

is considered acceptable. The relationship between $\epsilon^D$ and $\sigma_0$ is presented in Fig. 4 which can be expressed as

$$\epsilon^D = 1.37 \times 10^{-9} \sigma_0^{4.09} \quad (\sigma_0 \text{ in MPa}) \tag{3.9}$$

and

$$R_{1111} = 1, \quad R_{1122} = -0.3, \quad R_{1212} = 2.6.$$

The evolution law of the material coefficients can be derived from eq. (3.7) as

$$E_D = E_0(1 + E_0 q D^p)^{-1}$$

$$\nu_D = E_D\left(\frac{\nu_0}{E_0} + 0.3qD^p\right). \tag{3.10}$$

Figures 5 and 6 depict the predicted and measured values of $E_D/E_0$ and $\nu_D/\nu_0$ vs equivalent stress $\sigma_0$ respectively. It can be observed that the predicted values provide reasonable approximations to the experimental results. The damage evolution for the material under consideration

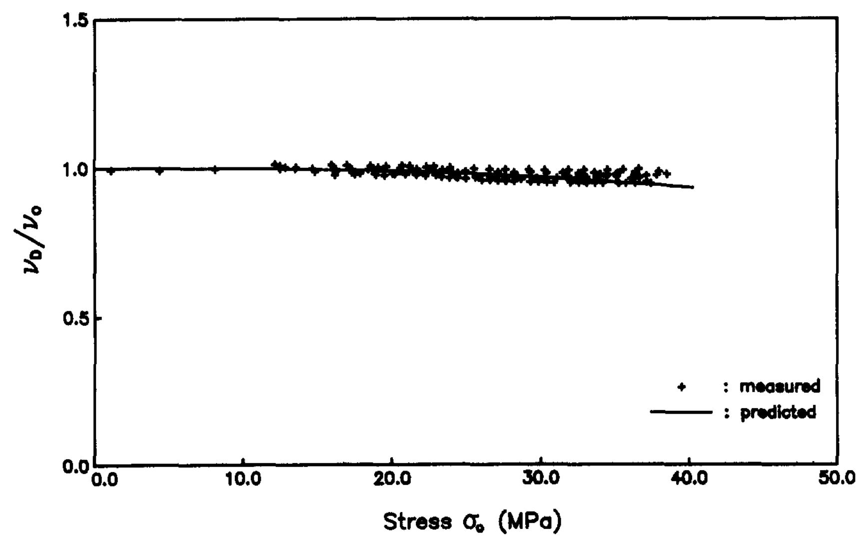


Fig. 6. $\nu_D/\nu_0$–$\sigma_0$ curve.

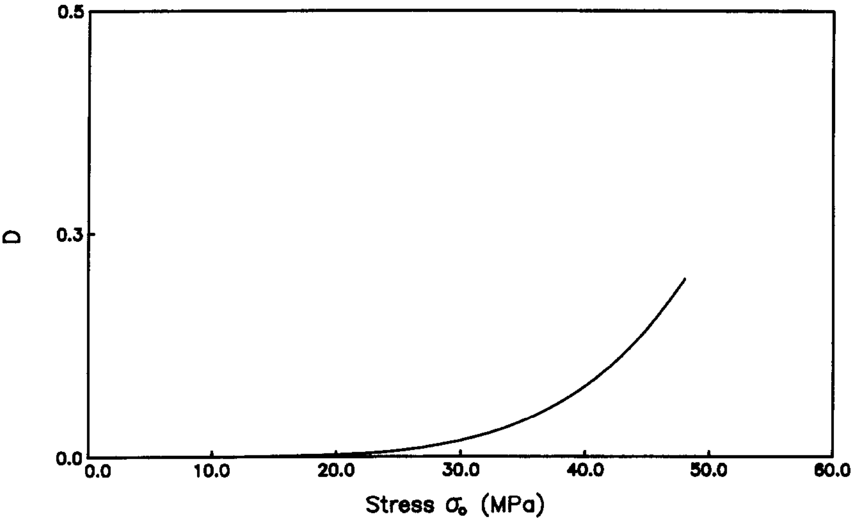


Fig. 7. $D$–$\sigma_0$ curve.

evaluated using eq. (3.6) is illustrated in Fig. 7. The critical value of damage $D$ defined as the threshold condition before failure is assessed at about 0.22.

## 4. SUMMARY AND CONCLUSION

The concept of continuum damage mechanics has been employed to develop a simple constitutive damage model for brittle materials. The approach is based on the hypothesis of damage surface which is similar to the yield function in plasticity theory and uses one scalar valued internal variable to represent the damage state. The stress and strain relation which takes into account the anisotropic effect induced by damage is established and it can be readily incorporated into the finite element analysis. It is also demonstrated that the method can be applied to predict the degraded material coefficients when the damaged material is linearly elastic. The reason that the present investigation is limited to the influence of damage on the elastic response of the material is that the validity of the theory can be directly and easily tested against experiment observation or numerical simulation. For example, in the one-dimensional case, the change of the elastic modulus is the simplest way of identifying progressive material damage. The present study offers a simplified method of analysis to solve the damage problems of brittle materials. With this approach, using similar method described in ref. [12], it is not difficult to investigate the asymptotic field for a stationary crack and the area near the crack, referred to as the damage zone, which is analogous to the plastic zone in ductile materials.

Although the comparison between the results of present work and those from experiment for isotropic material has been illustrated, the actual application of this method to cover a wide range of engineering brittle materials is complicated by the requirement for numerous experimentally determined damage parameters. However, this is unavoidable in the proper characterization of some special materials, for instance, brittle fiber-reinforced composites for which preliminary investigation has shown promise.

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