1 Weak Compatibility

An equivalent set of weakened boundary conditions may be written as

$$\int_{\partial\Omega_e} [u - \bar{u}]\, \lambda \, dA = 0 \quad \forall \lambda \in H^{1/2}(\partial\Omega_e), \tag{1}$$

where λ denotes a Lagrange multiplier field, and

$$H^{1/2}(\partial\Omega_e) \equiv \left\{ \lambda \in L^2(\partial\Omega_e) \,|\, \exists\, \lambda' \in H^1(\Omega_e) : \lambda = \mathrm{tr}(\lambda') \right\}. \tag{2}$$

A weak statement of the above equations can be written as

$$\int_{\Omega_e} \nabla_X u \cdot \nabla_X v \, dV = 0 \quad \forall v \in H^1_0(\Omega_e). \tag{3}$$

If we partition Ω_e into cells $\cup_i K_i$, we observe that

$$\int_{\Omega_e} \nabla_X u \cdot \nabla_X v \, dV = \sum_j \int_{\Gamma_j} v [\![\nabla_X u \cdot \mathbf{N}]\!] \, dA = 0 \quad \forall v \in H^1_0(\Omega_e). \tag{4}$$

If $[\![\nabla_X u \cdot \mathbf{N}]\!] = 0$, the above conditions are trivially satisfied. By contrast, to trivially satisfy compatibility in a weak sense, we insist that $[\![u]\!] = 0$ everywhere. We can consider enforcing both of these conditions in a weighted sense, resulting in the common PEM formulation in current usage:

$$\mathcal{F} = \sum_j \int_{\Gamma_j} \alpha_j [\![u]\!]^2 \, dA + \sum_j \int_{\Gamma_j} \beta_j [\![\nabla_X u \cdot \mathbf{N}]\!]^2 \, dA. \tag{5}$$

If $\alpha_j \gg \beta_j$, we obtain a field which satisfies compatibility preferrentially, whereas if $\alpha_j \ll \beta_j$, we satisfy Laplace's equation. The importance of compatibility is to guarantee that the shape functions span a set of admissible solutions to the original BVP, whereas satisfaction of Laplace's equation guarantees uniqueness of the resulting shape functions.

The generalized patch test requires that

$$\lim_{h \to 0} \sum_e \int_{\Omega_e} \psi \, u \, \mathbf{n} \, dA = 0 \quad \forall \psi \in C^\infty_0. \tag{6}$$

Let us momentarily consider the case when each element is bounded by some $d-1$ dimensional manifold, upon which we presume that the shape functions are well-defined as $\bar{u}$. Upon any given element domain, we insist that

$$\lim_{h \to 0} \int_{\Omega_e} \psi \, [\![u]\!] \, \mathbf{n} \, dA = 0 \quad \forall \psi \in C^\infty \tag{7}$$

must hold for the generalized patch test to be satisfied, where $[\![u]\!] = u - \bar{u}$. Suppose we define projection operators $\Pi : C^\infty(\Omega_e) \to P^k(\Omega_e)$ and $\pi : C^\infty(\Omega_e) \to C^\infty(\Omega_e) \backslash P^k(\Omega_e)$ such that

$$u = \Pi u + \pi u, \quad \psi = \Pi\psi + \pi\psi, \tag{8}$$

then

$$\int_{\Omega_e} \psi \, [\![u]\!] \, \mathbf{n} \, dA = \int_{\Omega_e} \Pi\psi \, \Pi[\![u]\!] \, \mathbf{n} \, dA + \int_{\Omega_e} \Pi\psi \, \pi[\![u]\!] \, \mathbf{n} \, dA \tag{9}$$

$$+ \int_{\Omega_e} \pi\psi \, \Pi[\![u]\!] \, \mathbf{n} \, dA + \int_{\Omega_e} \pi\psi \, \pi[\![u]\!] \, \mathbf{n} \, dA. \tag{10}$$

It can be shown that

$$\int_{\Omega_e} \Pi\psi \, \Pi[\![u]\!] \, \mathbf{n} \, dA = 0 \quad \forall \psi \in C^\infty \tag{11}$$

if the function defined on the element's boundary is sufficiently smooth, and can represent polynomial fields up to order k. Consequently,

$$\int_{\Omega_e} \psi \, [\![u]\!] \, \mathbf{n} \, dA = \int_{\Omega_e} \Pi\psi \, \pi[\![u]\!] \, \mathbf{n} \, dA \tag{12}$$

$$+ \int_{\Omega_e} \pi\psi \, \Pi[\![u]\!] \, \mathbf{n} \, dA + \int_{\Omega_e} \pi\psi \, \pi[\![u]\!] \, \mathbf{n} \, dA. \tag{13}$$

We effectively need to demonstrate passage of the E-1 and M-1 tests for convergence of nonconforming methods. Will need estimates for boundary integrals, or can demonstrate this numerically (preferred).

Weak Compatibility

Consider the necessary and sufficient conditions for compatibility of a deformation gradient field $\mathbf{F}$ stated in equation (K), reiterated:

$$\nabla_X \times \mathbf{F} = \mathbf{0} \quad \forall \mathbf{X} \in \mathcal{B}_0. \tag{14}$$

If the deformation gradient is computed from a set of basis functions $\{\varphi_a\}_{a=1}^N$ for the displacement field via

$$\mathbf{F} = \sum_{a=1}^{N} \mathbf{x}_a \otimes \nabla_X \varphi_a, \tag{15}$$

we arrive at a set of equivalent conditions on the basis functions:

$$\nabla_X \times \nabla_X \varphi = \mathbf{0} \quad \forall \mathbf{X} \in \mathcal{B}_0,\, \varphi \in \{\varphi_a\}_{a=1}^{N}. \tag{16}$$

These equations are trivially satisfied when the basis functions are $C^2(\mathcal{B}_0)$ continuous, and may therefore be used to construct admissible trial solutions to the strong form statement of equilibrium described in section N.

In general, the construction of basis functions $\varphi \in C^2(\mathcal{B}_0)$ is not trivial. Moreover, we seek solutions to the weak form such that $\mathbf{u} \in H^1(\Omega)$. In what follows, we shall consider the space of functions $\varphi \in C^0(\mathcal{B}_0)$ satisfying a weakened statement of compatibility:

$$\int_{\mathcal{B}_0} (\nabla_X \times \nabla_X \varphi) \cdot \mathbf{v}\, dV = 0 \quad \forall \mathbf{v} \in C_0^\infty(\mathcal{B}_0). \tag{17}$$

Through repeated application of integration by parts, the condition that the test functions satisfy compatibility ($\nabla_X \times \nabla_X \mathbf{v} = \mathbf{0}\, \forall \mathbf{v} \in C_0^\infty(\mathcal{B}_0)$), and the divergence theorem, we determine

$$\int_{\mathcal{B}_0} \nabla_X \cdot (\nabla_X \varphi \times \mathbf{v})\, dV + \int_{\mathcal{B}_0} \nabla_X \varphi \cdot (\nabla_X \times \mathbf{v})\, dV = 0, \tag{18}$$

$$\int_{\mathcal{B}_0} \nabla_X \cdot (\nabla_X \times (\varphi \mathbf{v}))\, dV = \int_{\partial \mathcal{B}_0} (\nabla_X \times (\varphi \mathbf{v})) \cdot d\mathbf{A} = 0. \tag{19}$$

If we consider a partition $\mathcal{T}^h = \{\Omega_e\}_{e=1}^{N_e}$ of $\mathcal{B}_0$ into polytopal elements $\Omega_e \subset \mathcal{B}_0$ whose shape functions are assumed to be piece-wise smooth and continuous on element interiors, then we may impose the equivalent condition:

$$\sum_{e=1}^{N_e} \int_{\partial \Omega_e} (\nabla_X \times (\varphi|_{\Omega_e} \mathbf{v})) \cdot d\mathbf{A} = 0 \quad \forall \varphi,\, \mathbf{v} \in C_0^\infty(\mathcal{B}_0). \tag{20}$$

It it evident that if $[\![\varphi]\!] = \varphi|_{\Omega_a} - \varphi|_{\Omega_b} = 0\, \forall \partial\Omega_a \cap \partial\Omega_b \neq \emptyset$, then the above condition is trivially satisfied. Such is the case for $C^0(\mathcal{B}_0)$ conforming finite element methods.

If $\varphi \not\subset C^0(\mathcal{B}_0)$, we require

$$\lim_{h \to 0} \sum_{e=1}^{N_e} \int_{\partial \Omega_e} (\nabla_X \times (\varphi|_{\Omega_e} \mathbf{v})) \cdot d\mathbf{A} = 0 \quad \forall \varphi,\, \mathbf{v} \in C_0^\infty(\mathcal{B}_0), \tag{21}$$

which is equivalent to

$$\lim_{h\to 0}\sum_{e=1}^{N_e}\int_{\partial\Omega_e}(\varphi|_{\Omega_e}(\nabla_X\times\mathbf{v})+\nabla_X\varphi|_{\Omega_e}\times\mathbf{v})\cdot d\mathbf{A}=0\quad\forall\varphi,\,\mathbf{v}\in C_0^\infty(\mathcal{B}_0).\tag{22}$$

Equivalently, we may enforce this condition over each individual face $\Gamma = \Omega_a\cap\Omega_b\neq\emptyset$, such that

$$\lim_{h\to 0}\int_\Gamma([\![\varphi]\!](\nabla_X\times\mathbf{v})+\nabla_X[\![\varphi]\!]\times\mathbf{v})\cdot d\mathbf{A}=0\quad\forall\Gamma,\,\mathbf{v}\in C_0^\infty(\mathcal{B}_0).\tag{23}$$

We may expand $[\![\varphi]\!]=a_0+\mathbf{a}_1\cdot\mathbf{X}+O(\mathbf{X}^2)$ and $\mathbf{v}=\mathbf{b}_0+\mathbf{B}_1\mathbf{X}+O(\mathbf{X}^2)$ in terms of low-order polynomials to illustrate

$$\lim_{h\to 0}\int_\Gamma((a_0+\mathbf{a}_1\cdot\mathbf{X})\mathbf{B}_1+\mathbf{a}_1\times(\mathbf{b}_0+\mathbf{B}_1\mathbf{X}))\cdot d\mathbf{A}=0\quad\forall\Gamma,\,\mathbf{v}\in C_0^\infty(\mathcal{B}_0).\tag{24}$$

$$\lim_{h\to 0}\sum_{e=1}^{N_e}\int_{\partial\Omega_e}(\varphi|_{\Omega_e}(\nabla_X\times\mathbf{v}))\cdot d\mathbf{A}-\sum_{e=1}^{N_e}\int_{\Omega_e}\nabla_X\varphi|_{\Omega_e}\cdot(\nabla_X\times\mathbf{v})\,dV=0\quad\forall\varphi,\,\mathbf{v}\in C_0^\infty(\mathcal{B}_0),\tag{25}$$

and because $\nabla_X\times : C_0^\infty(\mathcal{B}_0)\mapsto C_0^\infty(\mathcal{B}_0)$, we have

$$\lim_{h\to 0}\sum_{e=1}^{N_e}\int_{\partial\Omega_e}\varphi|_{\Omega_e}\mathbf{v}\cdot d\mathbf{A}-\sum_{e=1}^{N_e}\int_{\Omega_e}\mathbf{v}\cdot\nabla_X\varphi|_{\Omega_e}\,dV=0\quad\forall\varphi,\,\mathbf{v}\in C_0^\infty(\mathcal{B}_0).\tag{26}$$

Recognizing that

$$\int_{\Omega_e}\mathbf{v}\cdot\nabla_X\varphi|_{\Omega_e}\,dV\leq||\nabla_X\varphi|_{\Omega_e}||^2_{\Omega_e},\tag{27}$$

If we insist on

$$\sum_{e=1}^{N_e}\int_{\partial\Omega_e}(\nabla_X\times(\varphi|_{\Omega_e}\mathbf{v}))\cdot d\mathbf{A}=0\quad\forall\varphi,\,\mathbf{v}\in P^1(\mathcal{B}_0),\tag{28}$$

i.e. if we expand $\mathbf{v}=\mathbf{v}_0+\mathbf{V}_1\mathbf{x}$ and

$$\sum_{e=1}^{N_e}\int_{\partial\Omega_e}(\nabla_X\times(\varphi|_{\Omega_e}\mathbf{v}))\cdot d\mathbf{A}=0\quad\forall\varphi,\,\mathbf{v}\in P^1(\mathcal{B}_0),\tag{29}$$