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## A Mixture Theory for Microstructured Porous Media

*A porous continuum with very large lacunae, partially saturated by a fluid inclusion, is described by means of a new immiscible mixture which consists of a medium with ellipsoidal microstructure and of a classical fluid.*

### 1. Introduction

Some experimental researchers raised doubts about the capability of 'voids' theories in [1], or [2], to describe the behaviour of porous elastic solids with large vacuous interstices (see, for example, [3] and [4]); in particular, voids theories predict that size effects occur in bending of bars but not in the torsion in an isotropic material, while the polyurethane and syntactic foams displays size effects both in the bending and in the torsion, as well as the (anisotropic) human bone. Really, those theories can only represent solids with very small lacunae finely dispersed in the matrix, in which the volume fraction of the pores is the sole adjunctive dynamical variable of interest.

Elastic solids with large pores were, thus, modelled in [5] and [6] as continua with ellipsoidal structure for which the microdeformation is a pure strain: this means that each pore has different microdeformation along principal axes, but rotate locally with the matrix of the material; thus they may contract or expand, but they have no rotary inertia. Such changes are measured by the microstructural variable  $U$  that is a symmetric tensor field with positive determinant ( $U \in \text{Sym}^+$ ): the microstructure is ellipsoidal and the microdeformation is a pure microstretch; here lies the complementarity of our model with the Cosserat brother's theory that was sometime used to study porous media: they chose a microrotation tensor with no connection to the quoted volume fraction theory [1], in order to describe microdeformations.

When the voids are partially saturated by a fluid inclusion, the material is composed of more than one constituent; thus we need to use a mixture theory in which we could distinguish from each other the parts filled by different constituents on a scale which is large in comparison with molecular dimensions: so we construct a continuum theory of an immiscible mixture made of a continuum with ellipsoidal microstructure (the porous elastic solid) and of a compressible perfect fluid (the inclusion).

Following Biot [7], we consider virtual mass effects due to diffusion; also we introduce the microinertia associated with the rates of change of the local densities of the constituents, as well as that due to the deformation of the pores in the vicinity of their boundaries.

Finally, in the case that the lacunae are small and finely dispersed (and so microstrains along principal axes can be supposed equal), the symmetric tensor  $U$  reduces to a spherical tensor and we recover, as a particular case, the dynamic part of the theory of Passman and Batra [8], because the porous solid reduces to an elastic material with voids.

### 2. Preliminaries and Variational Principle

Notations and essential results which are required for the application of the variational method to the porous solids with a fluid inclusion are those presented in [9] and [10].

Precisely, the two components of the mixture are considered to be like interpenetrating continua that contemporaneously occupy a given region  $\mathcal{B}$  of the three-dimensional Euclidean space  $\mathcal{E}$ , in a certain time interval  $[\tau_0, \tau_1]$  during which the motion is observed. The portion of space occupied by the fluid is the volume fraction  $\beta$  of the fluid, while  $\beta_2$  is the volume fraction of the solid. The sum of them is less or equal to one:  $\beta + \beta_2 = 1 - \beta_v \leq 1$ , where  $\beta_v$  is the volume fraction of the part void of matter (see, also, [11]).

The fields that describe the motion are the velocities  $v_i$  and the bulk mass densities  $\rho_i$  ( $i = 1, 2$ ) of the constituents, the volume fraction of the inclusion  $\beta$  and the microstretch  $U$  (the bulk densities  $\rho_i$  are tied to the true densities  $\gamma_i$  from the relations  $\rho_1 = \gamma_1\beta$  and  $\rho_2 = \gamma_2\beta_2$ ); indices 1 and 2 are used for point out fluid and solid components, respectively. Besides, the following kinematic conditions of compatibility apply on the boundary  $\partial\mathcal{B}$  of

$\mathcal{B}$ , if  $n$  is the outer normal of  $\partial\mathcal{B}$ :

$$v_1 \cdot n = 0 \quad \text{and} \quad v_2 \cdot n = 0. \quad (1)$$

If the saturation condition  $\beta_v = 0$  applies, the pores are completely filled by the inclusion and the volume fraction  $\beta_2$  is closely related to the determinant of the microstretch  $U$  of the solid matrix; here, we do not impose this constraint and  $\beta_v \in (0, 1)$ .

The mass of each constituent conserves, hence the continuity equations for each constituent in Eulerian, or Lagrangian, form apply

$$\frac{\partial \rho_i}{\partial \tau} + \operatorname{div}(\rho_i v_i) = 0 \quad \text{or} \quad \rho_i \iota_i = \rho_{i*}, \quad (2)$$

respectively, where  $\iota_i$  is the determinant of the deformation gradient  $F_i$  of the  $i^{\text{th}}$  constituent ( $\iota_i := \det F_i$ ), while starred quantities are values in a reference region  $\mathcal{B}_*$  of the body.

In order to have suggestions for the constitutive expression of microforces, we follow the idea of Biot [7] and give the expression of the kinetic energy of the mixture by taking into account all the phenomena of interest for the theory. We consider the effects of the virtual inertia of translation representing the local nonuniformities in the flow of constituents when they move with respect to one another, besides those due to the inertia of classical translation. In addition, we model local microvariations of the volume of the compressible inclusion, as well as those of pores that put the elastic matrix solid in motion, giving rise to effects of expansional inertia. Hence, the kinetic energy of the mixture has the following density per unit volume  $\kappa$ :

$$\begin{aligned} \kappa \left( \rho_1, \rho_2, \beta, \operatorname{grad} \beta, \frac{\partial \beta}{\partial \tau}, U, \operatorname{grad} U, \frac{\partial U}{\partial \tau}, v_1, v_2 \right) := & \frac{1}{2} \rho_1 v_1^2 + \frac{1}{2} \rho_2 v_2^2 + \\ & + \frac{1}{2} \rho_1 \mu(\beta) (\beta')^2 + \frac{1}{2} \rho_2 \nu(U) (U')^2 + \frac{1}{2} \rho_1 \phi(\beta) (v_2 - v_1)^2, \end{aligned} \quad (3)$$

where  $(\cdot)'$  ( $:= \frac{\partial(\cdot)}{\partial \tau} + [\operatorname{grad}(\cdot)]v_i$ ) denotes the material time derivative of  $(\cdot)$  following the motion of the  $i^{\text{th}}$  constituent. Expressions for functions  $\mu$  and  $\phi$ , as well as for  $\nu$ , depend on admissible expansional motion of the inclusion and of void boundaries, other than on the geometrical features of the pores themselves (see [6] and [10] for some particular cases). When the voids are not connected, we can suppose that  $v_2 = v_1$  and the virtual term disappears; moreover, if the pores expand or contract homogeneously in all directions,  $U$  is spherical and the related kinetic term reduces to  $\frac{1}{2} \rho_2 \tilde{\nu}(\beta_2) (\beta_2')^2$  ( $\tilde{\nu}$  related to  $\nu$ ): this term generates inertia specifications similar to classical immiscible mixtures (see [8] and [10]).

The potential energy of the mixture includes the energy of deformation of pores in the vicinity of their boundaries and those due to deformations of constituents; it is inserted as an energy of internal actions of density per unit volume  $\sigma := \sigma(\rho_1, F_2, \beta, \operatorname{grad} \beta, U, \operatorname{grad} U)$ .

We use now results of [10] to obtain equations of motion for the body under consideration by defining a Hamiltonian functional for porous media  $\mathcal{H}$  on  $\Omega := \bar{\mathcal{B}} \times [\tau_0, \tau_1]$ ; if we suppose that the virtual work is done by conservative forces only, we put

$$\begin{aligned} \mathcal{H}[\rho_1, F_2, \beta, U, v_1, v_2] := & \int_{\Omega} \left[ \kappa \left( \rho_1, \rho_2, \beta, \operatorname{grad} \beta, \frac{\partial \beta}{\partial \tau}, U, \operatorname{grad} U, \frac{\partial U}{\partial \tau}, v_1, v_2 \right) - \right. \\ & \left. - \sigma(\rho_1, F_2, \beta, \operatorname{grad} \beta, U, \operatorname{grad} U) - \rho_1(\omega + \zeta\beta) - \rho_2(\omega + C \cdot U) \right], \end{aligned} \quad (4)$$

where  $\omega = \hat{\omega}(x)$ ,  $\zeta\beta = \hat{\zeta}(x)\beta$  and  $C \cdot U = \hat{C}(x) \cdot U$  are the density per unit mass of the potential energy of external body forces and microforces, respectively.

A sextuple of functions  $(\rho_1, F_2, \beta, U, v_1, v_2)$ , assigned at the times  $\tau_0$  and  $\tau_1$  and that satisfy kinematic conditions (1) on the boundary and continuity equations (2), represents a possible motion for the mixture. The variational principle states that, in the class of all possible motions of the mixture, the natural one is described by a sextuple which extremises the time integral of the sum of the kinetic energy and the virtual work, that is, for which the first variation of the functional  $\mathcal{H}$  vanishes. Thus, to obtain dynamic equations for our mixture, we need to evaluate this first variation of  $\mathcal{H}$  along different 'paths' of sextuples. Very long mathematical procedures, similar to those followed in §3 of [10] and applied to compute all necessary variations, permit us to find the searched equations of motion.

### 3. Dynamic Balance Equations for Microstructured Porous Media

An immiscible mixture composed of an elastic continuum with ellipsoidal microstructure and a compressible perfect

fluid must satisfied the following equations of motion:

$$\begin{aligned} \rho_1 a_1 = & -\rho_1 \text{grad } \omega - \rho_2 (\text{grad } C)^T U + \frac{\partial \sigma}{\partial \rho_1} \text{grad } \rho_1 + \frac{\partial \sigma}{\partial \beta} \text{grad } \beta + [\text{grad } (\text{grad } \beta)] \frac{\partial \sigma}{\partial \text{grad } \beta} + \\ & + \rho_1 [\phi(\beta)(v_2 - v_1)]^{i_1} + (\text{grad } v_2)^T [\rho_1 \phi(\beta)(v_2 - v_1)] - \text{div} \left( \rho_1 \frac{\partial \sigma}{\partial \rho_1} I + \text{grad } \beta \otimes \frac{\partial \sigma}{\partial \text{grad } \beta} \right), \end{aligned} \quad (5)$$

$$\begin{aligned} \rho_2 a_2 = & -\rho_2 \text{grad } \omega - \rho_1 \beta \text{grad } \zeta - \frac{\partial \sigma}{\partial \rho_1} \text{grad } \rho_1 - \frac{\partial \sigma}{\partial \beta} \text{grad } \beta - [\text{grad } (\text{grad } \beta)] \frac{\partial \sigma}{\partial \text{grad } \beta} - \\ & - \rho_1 [\phi(\beta)(v_2 - v_1)]^{i_1} - (\text{grad } v_2)^T [\rho_1 \phi(\beta)(v_2 - v_1)] + \\ & + \text{div} \left[ \sigma I + \frac{\partial \sigma}{\partial F_2} F_2^T - \text{grad } U \otimes \frac{\partial \sigma}{\partial \text{grad } U} - \rho_1 \phi(\beta)(v_2 - v_1) \otimes (v_2 - v_1) \right], \end{aligned} \quad (6)$$

$$\rho_1 \mu(\beta) (\beta^{i_1})^{i_1} + \frac{1}{2} \rho_1 \frac{d\mu}{d\beta} (\beta) (\beta^{i_1})^2 = \rho_1 \zeta + \frac{1}{2} \rho_1 \frac{d\phi}{d\beta} (\beta) (v_2 - v_1)^2 - \frac{\partial \sigma}{\partial \beta} + \text{div} \left( \frac{\partial \sigma}{\partial \text{grad } \beta} \right) \quad (7)$$

$$\text{and} \quad \rho_2 \nu(U) (U^{i_2})^{i_2} + \frac{1}{2} \rho_2 \frac{d\nu}{dU} (U) (U^{i_2})^2 = \rho_2 C - \frac{\partial \sigma}{\partial U} + \text{div} \left( \frac{\partial \sigma}{\partial \text{grad } U} \right), \quad (8)$$

where  $I$  is the identity tensor, the acceleration  $a_i$  of the  $i^{\text{th}}$  constituent is defined by  $a_i := \frac{\partial v_i}{\partial \tau} + (\text{grad } v_i) v_i$ ,  $i=1,2$ , and  $(\text{grad } U \otimes \frac{\partial \sigma}{\partial \text{grad } U})_{ij} := U_{hk,i} \frac{\partial \sigma}{\partial U_{hk,j}}$ .

Fluid and solid momentum equations (5) and (6), respectively, are, in some sense, classical, even if in the appearances of stress tensors  $T_i$  there are terms that account for the presence of the microstructure:

$$T_1 := -\rho_1 \frac{\partial \sigma}{\partial \rho_1} I - \text{grad } \beta \otimes \frac{\partial \sigma}{\partial \text{grad } \beta}, \quad (9)$$

$$T_2 := \sigma I + \frac{\partial \sigma}{\partial F_2} F_2^T - \text{grad } U \otimes \frac{\partial \sigma}{\partial \text{grad } U} - \rho_1 \phi(\beta)(v_2 - v_1) \otimes (v_2 - v_1), \quad (10)$$

where the term  $\text{grad } \beta \otimes \frac{\partial \sigma}{\partial \text{grad } \beta}$  allows, in the fluid inclusion, a difference in the diagonal elements in the direction of the volumetric fluid fraction gradient, while the additional Reynolds stress in the solid phase is related to virtual inertia; significant terms appearing with opposite sign in (5) and (6) represent the interaction force  $f$  between phases:

$$\begin{aligned} f := & \frac{\partial \sigma}{\partial \rho_1} \text{grad } \rho_1 + \frac{\partial \sigma}{\partial \beta} \text{grad } \beta + [\text{grad } (\text{grad } \beta)] \frac{\partial \sigma}{\partial \text{grad } \beta} + \\ & \rho_1 [\phi(\beta)(v_2 - v_1)]^{i_1} + (\text{grad } v_2)^T [\rho_1 \phi(\beta)(v_2 - v_1)]. \end{aligned} \quad (11)$$

Additional equations (7) and (8) answer to the fact that the fluid volume fraction  $\beta$  and the pore microstretch  $U$  can change without affecting the gross motion of constituents. Thus, those balance equations model the microstructural force systems operative in our two-phase mixture. Equation (7) is present in previous theories (quoted, for example, as the balance of 'equilibrated' force in [8]), even if it remains ambiguous the expression of the inertia term of those theories, because there does not appear the full Lagrangian derivative of the kinetic energy related to the volume fraction variable. Equation (8) generalizes equation (5) of [6] to the case in which the microinertia due to local changes of pores structure depend on actual values  $U$  of the microstretch; hence, on the left hand side, it occurs its complete Lagrangian variation while, on the right one, we can recognize the microstress  $\Sigma := \frac{\partial \sigma}{\partial \text{grad } U}$  and the internal microforce  $Z := \frac{\partial \sigma}{\partial U}$ , that arise from this conservative case and which are usually related to boundary microtractions, or to weakly non-local internal effects, and to interactive forces between the gross and fine structures, respectively.

To complete the dynamic specifications for the mixture, we need to state the balances of moment of momentum for the two constituents. We use suggestions from [5] and [8] to formulate the following conditions:

$$\text{skw } T_1 = M \quad \text{and} \quad \text{skw } T_2 = \text{skw } (U Z^T + \text{grad } U \odot \Sigma) - M, \quad (12)$$

where  $M$  is the skew tensor of moment of momentum supply,  $\text{skw } A := \frac{1}{2}(A - A^T)$ , for each tensor  $A$ , and  $(\text{grad } U \odot \Sigma)_{ij} := U_{ih,k} \Sigma_{jhk}$ ; these relations for stress tensors specify that neither of two is 'a priori' symmetric. By inserting the above-explicit definitions for the fields and by eliminating the symmetric parts, we have:

$$-\text{skw} \left( \text{grad } \beta \otimes \frac{\partial \sigma}{\partial \text{grad } \beta} \right) = M \quad \text{and} \quad (13)$$

$$\text{skw} \left( \frac{\partial \sigma}{\partial F_2} F_2^T - \text{grad } U \otimes \frac{\partial \sigma}{\partial \text{grad } U} \right) = \text{skw} \left( U \frac{\partial \sigma}{\partial U} + \text{grad } U \odot \frac{\partial \sigma}{\partial \text{grad } U} \right) - M. \quad (14)$$

At the end, by summing previous two balance equations, we obtain the balance of angular momentum for all the mixture:

$$\text{skw} \left( -\frac{\partial \sigma}{\partial F_2} F_2^T + \text{grad } \beta \otimes \frac{\partial \sigma}{\partial \text{grad } \beta} + U \frac{\partial \sigma}{\partial U} + \text{grad } U \odot \frac{\partial \sigma}{\partial \text{grad } U} + \text{grad } U \otimes \frac{\partial \sigma}{\partial \text{grad } U} \right) = 0; \quad (15)$$

this relation represents the condition of material objectivity for the scalar potential  $\sigma$ ; in fact,  $\sigma$  is a scalar function and its value must be invariant, if the observer is changed (see, also, §3 of [6]).

Methaphysical principles of Truesdell for classical mixtures apply to our mixture also (see page 221 of [12]), but we must observe that constitutive valuations for dependent fields are affected by microstructural variables other than macro ones, as it could be easily seen in equations (9) and (10).

#### 4. Spherical Microstretch

Our theory includes, as a peculiar case, porous solids with inclusion in which the lacunae in the matrix material are spherical, very small, and finely distributed. In this approximation, microstrains along principal axes of voids can be supposed equal and the microstructural variable  $U$  reduces to a spherical tensor, that is,

$$U = \left[ \frac{(1 - \beta_2)(\rho_{1*} + \rho_{2*})}{(1 - \beta_{2*})(\rho_1 + \rho_2)} \right]^{\frac{1}{3}} I. \quad (16)$$

With this condition on  $U$ , we perform developments similar to previous general instance (see, also, §5 of [5]) and obtain the following equations for the porous solid phase which substitute equations (6) and (8) (balances (5) and (7) have not substantial changes):

$$\begin{aligned} \rho_2 a_2 = & -\rho_2 \text{grad } \omega - \rho_1 \beta \text{grad } \zeta - \frac{\partial \tilde{\sigma}}{\partial \rho_1} \text{grad } \rho_1 - \frac{\partial \tilde{\sigma}}{\partial \beta} \text{grad } \beta - [\text{grad } (\text{grad } \beta)] \frac{\partial \tilde{\sigma}}{\partial \text{grad } \beta} - \\ & - \rho_1 [\phi(\beta)(v_2 - v_1)]'^1 - (\text{grad } v_2)^T [\rho_1 \phi(\beta)(v_2 - v_1)] + \\ & + \text{div} \left[ \sigma I + \frac{\partial \tilde{\sigma}}{\partial F_2} F_2^T - \text{grad } \beta_2 \otimes \frac{\partial \tilde{\sigma}}{\partial \text{grad } \beta_2} - \rho_1 \phi(\beta)(v_2 - v_1) \otimes (v_2 - v_1) \right] \end{aligned} \quad (17)$$

$$\text{and} \quad \rho_2 \tilde{\nu}(\beta_2) (\beta_2')'^2 + \frac{1}{2} \rho_2 \frac{d\tilde{\nu}}{d\beta_2} (\beta_2) (\beta_2')^2 = \rho_2 \eta - \frac{\partial \tilde{\sigma}}{\partial \beta_2} + \text{div} \left( \frac{\partial \tilde{\sigma}}{\partial \text{grad } \beta_2} \right) \quad (18)$$

where, in the new context,  $\tilde{\nu}(\beta_2)$  and  $\tilde{\sigma}(\rho_1, F_2, \beta, \text{grad } \beta, \beta_2, \text{grad } \beta_2)$  are new appropriate inertial and potential functions, respectively. Thus, we recover the dynamic part of the theory of Passman and Batra for an elastic material with unsaturated voids (see equations (2.3)<sub>{1-4}</sub> of [8]).

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