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THE GENERALIZED PATCH TEST*

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Abstract. The paper establishes the basic convergence conditions for the method of nonconforming finite elements applied to a class of generalized elliptic boundary value problems with variable, not necessarily smooth coefficients. The main result is a new, generalized patch test. Approximability and success in this test is the necessary and sufficient condition for convergence of the nonconforming approximations. It is proved that nonconforming elements of Wilson, Adini, Crouzeit-Raviart, Morley, and de Veubeke pass the generalized patch test and thus yield convergent approximations of the boundary value problems.

Introduction. This paper proves the convergence of the method of nonconforming finite elements for a large class of generalized elliptic boundary value problems with variable, not necessarily smooth coefficients. The basic tool is a new generalized patch test. It is shown that nonconforming elements of Wilson, Adini on rectangles and of Crouzeit–Raviart, Morley, de Veubecke on triangles stand the generalized patch test and thus yield convergent approximations for the boundary value problems.

Success in the generalized patch test is proved by the technique that Ciarlet [6], Lascaux–Lesaint [12] have used for obtaining discretization error estimates and orders of convergence for nonconforming approximations. In contrast to the form of their local patch test, the form of the generalized patch test neither depends on the special element nor on the boundary value problems under consideration. As the approximability condition, the generalized patch relates to the underlying spaces only. Success in the generalized patch test is not only sufficient but also necessary for convergence when approximating sequences of spaces are used. This result has been stated already in [22] for a class of model examples.

In our paper [23], a counterexample to the patch test of Irons and Strang (see Irons–Razzaque [10], Strang [17], Strang–Fix [18, pp. 176, 301]) has been established by a simple nonconforming approximation that passes this test but does not yield approximation solutions converging to the solution of the given boundary value problem. At present it seems not to be known whether this test would yield a sufficient convergence condition for at least a sufficiently restricted class of nonconforming elements. Moreover it is not clear whether and how this test could be applied to problems with variable coefficients. Neither is the patch test of Irons and Strang, in general, a necessary condition for convergence (see [3], [16]). This may be seen, for example, by means of nonconforming elements satisfying the required continuity conditions only approximately or excepting sufficiently small subsets of nodes. Examples of this type will be analyzed in another context, as well as Zienkiewicz triangles [25, § 10.6] and the strange convergence behavior described in [11] that can be explained by the methods of our paper [23].

The generalized patch test is the concrete form of the so-called closedness condition. The approximability and the closedness condition together are basic preconditions for convergence. The forthcoming paper [24] establishes the fundamental compactness properties of nonconforming approximations. Using these properties, a very general criterion is proved concerning the validity of the solvability and stability assumption (V) in § 1.2. In addition, also the convergence of spectra and eigenspaces in nonconforming approximations of eigenvalue problems (see [21]) is ensured.

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1. Convergence conditions for the method of nonconforming finite elements. First a class of generalized boundary value problems and their nonconforming approximations is defined. The starting point for the following investigation of convergence properties is an analysis of the convergence concept for nonconforming finite elements. In this way, the natural embedding of the generalized boundary value problems or variational equations and their approximations into the big space $L^{m,2}(G)$ is motivated. A corresponding embedding has already been used successfully in our papers [19], [20], studying Sobolev spaces and boundary value problems under perturbation of domains.

The fundamental convergence theorem 1.1.(12) shows that two conditions together are necessary and sufficient for convergence of the approximations. The first is the well-known approximability condition. The second requires that the norms $\|d\|_{E_\iota'}$ of the error functional d on the subspaces E_ι converge to zero for $\iota \to \infty$. In the papers of de Arantes e Oliveira [3], Ciarlet [6], Lascaux–Lesaint [12], Nitsche [14], this is shown by deriving appropriate estimates for this sequence.

The present paper pursues another way. First it will be proved that the condition $\lim \|d\|_{E_\iota'} = 0$ can be characterized equivalently by the closedness of the sequence E_0, (E_ι) or V, (V_ι). This property requires that all limits of weakly convergent sequences of functions from V_ι belong to $V \subset H^m(G)$. Proving the validity of the closedness condition, in essence, consists in showing that the limits are m-times differentiable in the weak sense of Sobolev spaces H^m. Exactly this property is proved by our generalized patch test.

1.1. Nonconforming approximations of variational equations. We shall study approximations of *variational equations* of the form

$$(1) \qquad u_0 \in V; \qquad \sum_{|\sigma|,|\tau| \leq m} \int_G a_{\sigma\tau} D^\sigma \varphi \overline{D^\tau u_0}\, dx = \sum_{|\sigma| \leq m} \int_G D^\sigma \varphi \bar{f}_\sigma\, dx, \qquad \varphi \in V.$$

By G is meant a not necessarily bounded open polyhedral domain in $\mathbb{R}^n$ and by V a closed subspace of the Sobolev space $H^m(G)$. The coefficients $a_{\sigma\tau}$ are real-valued bounded measurable functions on G and f_σ functions in $L^2(G)$ for $|\sigma| \leq m$. A classical example is obtained by the differential equation

$$\sum_{|\sigma|,|\tau| \leq m} (-1)^{|\sigma|} D^\sigma (a_{\sigma\tau} D^\tau v) = f \quad \text{in } G$$

together with inhomogeneneous Dirichlet boundary conditions. This problem is understood in the generalized sense, for example, as

$$\sum_{|\sigma|,|\tau| \leq m} \int_G a_{\sigma\tau} D^\sigma \varphi \overline{D^\tau u_0}\, dx = \int_G \varphi \bar{f}\, dx - \sum_{|\sigma|,|\tau| \leq m} \int_G a_{\sigma\tau} D^\sigma \varphi \overline{D^\tau g}\, dx, \qquad \varphi \in H_0^m(G),$$

where $u_0 \in H_0^m(G)$ and the generalized solution v is specified by $v = u_0 + g$ and some function $g \in H^m(G)$.

For every $\iota = 1, 2, \cdots$ let $\mathcal{K}_\iota$ be a subdivision of $\bar{G}$ by bounded closed convex polyhedra $K \subset \bar{G}$ having nonvoid interiors $\underset{\sim}{K}$. These polyhedra constitute the *finite elements* of the subdivisions of $\bar{G}$. By $\delta_1(K)$ is meant the greatest diameter of the element K and by $\delta_0(K)$ the greatest diameter of all balls contained in K. We shall assume that the subdivisions $\mathcal{K}_\iota$ have the following properties:

(K1) $\mathcal{K}_\iota$ is a locally finite subdivision of $\bar{G}$ and

$$(2) \qquad \bar{G} = \bigcup_{K \in \mathcal{K}_\iota} K, \qquad \iota = 1, 2, \cdots;$$

(K2) For each $K_1, K_2 \in \mathcal{K}_\iota$, the intersection $K_1 \cap K_2$ is empty or a face of both K_1 and K_2;

(K3) There exists a constant ζ and a null sequence (h_ι) of positive numbers such that the inequalities

$$\text{(i)} \quad \delta_1(K) \leqq \zeta \delta_0(K), \qquad \text{(ii)} \quad \delta_1(K) \leqq h_\iota$$

hold uniformly for all elements $K \in \mathcal{K}_\iota$ and all ι.

The subdivisions $\mathcal{K}_\iota$ generate the open subsets

$$G_\iota = \bigcup_{K \in \mathcal{K}_\iota} \underline{K} \subset G, \qquad \iota = 1, 2, \cdots. \tag{3}$$

By assumption (K2), different elements cannot have common interior points, that is, $\underline{K}_1 \cap \underline{K}_2 = \emptyset$. The subsets G_ι define the Sobolev spaces $H^m(G_\iota)$. The method of nonconforming finite elements approximates the Sobolev space $H^m(G)$ or a subspace of $H^m(G)$ by a sequence of suitable closed subspaces $V_\iota \subset H^m(G_\iota)$ for $\iota = 1, 2, \cdots$. The variational equation (1) is thus approximated by the sequence

$$u_\iota \in V_\iota; \qquad \sum_{|\sigma|, |\tau| \leqq m} \sum_{K \in \mathcal{K}_\iota} \int_{\underline{K}} a_{\sigma\tau} D^\sigma \varphi \overline{D^\tau u_\iota}\, dx = \sum_{|\sigma| \leqq m} \sum_{K \in \mathcal{K}_\iota} \int_{\underline{K}} D^\sigma \varphi \bar{f}_\sigma \, dx, \qquad \varphi \in V_\iota, \tag{4}$$

for $\iota = 1, 2, \cdots$.

A sequence of functions $u_\iota \in H^m(G_\iota)$, $\iota \in \mathbb{N}' \subset \mathbb{N}$, is said to be *strongly* (*weakly*) *convergent* in the sense of the method of nonconforming finite elements iff the sequences of generalized partial derivatives $D^\sigma u_\iota$ are strongly (weakly) convergent in $L^2(G)$ for $\iota \to \infty$ and every multiindex σ for $|\sigma| \leqq m$. This definition makes sense because the open subsets G_ι and G differ only by sets of $(n-1)$-dimensional faces of the elements $K \in \mathcal{K}_\iota$ having the n-dimensional measure zero. Hence all partial derivatives $D^\sigma u_\iota$ may be viewed as functions of $L^2(G)$.

Consequently, the appropriate setting for the convergence study is the space $L^{m,2}(G)$ of all vector-valued functions $\mathbf{u} = (u^\sigma)$ with components $u^\sigma \in L^2(G)$ for $|\sigma| \leqq m$. This is a Hilbert space with scalar product and norm

$$(\mathbf{v}, \mathbf{w})_m = \sum_{|\sigma| \leqq m} \int_G v^\sigma \overline{w^\sigma}\, dx, \qquad \|\mathbf{v}\|_m = \left(\sum_{|\sigma| \leqq m} \int_G |v^\sigma|^2 \, dx \right)^{1/2}, \qquad \mathbf{v}, \mathbf{w} \in L^{m,2}(G). \tag{5}$$

The *natural embedding*

$$\mathbf{u}_\iota = (u_\iota^\sigma), \qquad u_\iota^\sigma(x) = D^\sigma u_\iota(x), \qquad x \in G_\iota, \qquad |\sigma| \leqq m, \tag{6}$$

assigns to each function $u_\iota \in H^m(G_\iota)$ a function $\mathbf{u}_\iota$ in $L^{m,2}(G)$. Evidently,

$$(\mathbf{v}_\iota, \mathbf{w}_\iota)_m = \sum_{K \in \mathcal{K}_\iota} (v_\iota, w_\iota)_{m,\underline{K}} = \sum_{K \in \mathcal{K}_\iota} \sum_{|\sigma| \leqq m} \int_{\underline{K}} D^\sigma v_\iota \overline{D^\sigma w_\iota}\, dx \tag{7}$$

for all $v_\iota, w_\iota \in H^m(G_\iota)$, $\iota = 0, 1, 2, \cdots$. The natural embedding maps the Sobolev spaces $H^m(G_\iota)$ and the subspaces $V_\iota \subset H^m(G_\iota)$ onto closed subspaces ${}_EH^m(G)$ and ${}_EV_\iota$ of $L^{m,2}(G)$ for $\iota = 0, 1, 2, \cdots$. For brevity, we write $G = G_0$ and $V = V_0$. The strong (weak) convergence of sequences $u_\iota \in H^m(G_\iota)$, $\iota \in \mathbb{N}' \subset \mathbb{N}$ is then identical to the strong (weak) convergence of the associated embedded sequences $\mathbf{u}_\iota$ in $L^{m,2}(G)$. In view of the representation (7), the strong convergence of a sequence of functions $u_\iota \in H^m(G_\iota)$, $\iota = 0, 1, 2, \cdots$, can now be expressed in the form

$$\|\mathbf{u}_\iota - \mathbf{u}_0\|_m^2 = \sum_{K \in \mathcal{K}_\iota} \sum_{|\sigma| \leqq m} \int_{\underline{K}} |D^\sigma u_\iota - D^\sigma u_0|^2 \, dx \to 0 \qquad (\iota \to \infty). \tag{8}$$

1.2. Convergence theorem and discretization error estimate. The space $L^{m,2}(G)$ is viewed in the following as a Hilbert space with an arbitrary but fixed scalar product $(\cdot,\cdot)$ and an associated norm $\|\cdot\|$ equivalent to the $\|\cdot\|_m$-norm 1.1.(5). The coefficient matrix $(a_{\sigma\tau})$ specifies the sesquilinear form

$$(1)\qquad a(\mathbf{v},\mathbf{w})=\sum_{|\sigma|,|\tau|\leqq m}\int_G a_{\sigma\tau}v^\sigma\overline{w^\tau}\,dx,\qquad \mathbf{v},\mathbf{w}\in L^{m,2}(G).$$

For the sake of simplicity, we shall use the notation

$$(2)\qquad E_0={}_EV,\qquad E_\iota={}_EV_\iota,\qquad \iota=1,2,\cdots,$$

for the embedded subspaces V, V_ι in $L^{m,2}(G)$. The given variational equation, 1.1.(1), with the solution $u_0\in V$ may then be written in the form

$$(3)\qquad \mathbf{u}_0\in E_0;\qquad a(\boldsymbol{\varphi},\mathbf{u}_0)=l(\boldsymbol{\varphi}),\qquad \boldsymbol{\varphi}\in E_0;$$

and the sequence of variational equations 1.1.(4) in the form

$$(4)\qquad \mathbf{u}_\iota\in E_\iota;\qquad a(\boldsymbol{\varphi},\mathbf{u}_\iota)=l(\boldsymbol{\varphi}),\qquad \boldsymbol{\varphi}\in E_\iota,\quad \iota=1,2,\cdots;$$

where the right sides of these equations are specified by

$$(5)\qquad l(\boldsymbol{\varphi})=\sum_{|\sigma|\leqq m}\int_G \varphi^\sigma\bar f_\sigma\,dx,\qquad \boldsymbol{\varphi}=(\varphi^\sigma)\in L^{m,2}(G).$$

The following *assumption*, concerning the sesquilinear form a and the subspaces V_ι or E_ι, will be made in this paper:

(V) The variational equations (3), (4) are uniquely solvable for each inhomogeneous term l of the form (5) and there exist positive constants α_0, α_1 such that the bistability condition

$$(6)\qquad \alpha_0\|\mathbf{v}_\iota\|\leqq\sup_{0\neq\boldsymbol{\varphi}\in E_\iota}\frac{|a(\boldsymbol{\varphi},\mathbf{v}_\iota)|}{\|\boldsymbol{\varphi}\|}\leqq\alpha_1\|\mathbf{v}_\iota\|$$

holds uniformly for all $\mathbf{v}_\iota\in E_\iota$ and all $\iota=1,2,\cdots$.

This assumption is used in an equivalent form, for example, by Babuška and Aziz [5, § 6.2]. If the coefficients $a_{\sigma\tau}$ belong to $L^\infty(G)$, obviously, there exists a constant α_1 such that the inequality

$$(7)\qquad |a(\mathbf{v},\mathbf{w})|\leqq\alpha_1\|\mathbf{v}\|\,\|\mathbf{w}\|,\qquad \mathbf{v},\mathbf{w}\in L^{m,2}(G),$$

is valid. In particular, this implies the second inequality in (6). A well-known sufficient condition for the validity of the above assumption (V) then reads

$$(8)\qquad \alpha_0\|\mathbf{v}_\iota\|^2\leqq|a(\mathbf{v}_\iota,\mathbf{v}_\iota)|,\qquad \mathbf{v}_\iota\in E_\iota,$$

uniformly for all $\iota=0,1,2,\cdots$. For, the first inequality in (6) follows immediately from (8) and the theorem of Lax–Milgram guarantees the unique solvability of the variational equations (3), (4). Let us remark in passing, that our paper [24], using compactness properties of nonconforming finite elements, will state much weaker conditions for the validity of assumption (V).

Example. For $n=2$, $m=1$ let

$$a^1(\mathbf{v},\mathbf{w})=\int_G(a_{10}v^{10}\overline{w^{10}}+a_{01}v^{01}\overline{w^{01}}+a_{00}v^{00}\overline{w^{00}})\,dx$$

and for $n=2$, $m=2$ let

$$a^2(\mathbf{v},\mathbf{w})=\int_G (a_{20}v^{20}\overline{w^{20}}+a_{11}v^{11}\overline{w^{11}}+a_{02}v^{02}\overline{w^{02}})\,dx+a^1(\mathbf{v},\mathbf{w}).$$

When the real coefficients a_σ satisfy the inequalities

$$0<\alpha_0\leqq a_\sigma(x)\leqq\alpha_1,\qquad x\in G,\qquad \sigma=(\sigma_1,\sigma_2),$$

one has

$$\alpha_0\|\mathbf{v}\|_m^2\leqq a^m(\mathbf{v},\mathbf{v})\leqq\alpha_1\|\mathbf{v}\|_m^2,\qquad \mathbf{v}\in L^{m,2}(G),$$

for $m=1,2$. Hence a^m is a scalar product for $L^{m,2}(G)$ and the associated norm is equivalent to the $\|\cdot\|_m$-norm on $L^{m,2}(G)$. The conditions (7), (8) are fulfilled and thus the assumption (V) holds.

We will now formulate necessary and sufficient conditions for the convergence of the solution $\mathbf{u}_\iota$ of (4). Let Q_ι be the orthogonal projection of $L^{m,2}(G)$ onto the subspace E_ι for $\iota=0,1,2,\cdots$. For each $\mathbf{u}\in L^{m,2}(G)$ the *shortest distance* from $\mathbf{u}$ to E_ι has the form

(9) $$|\mathbf{u},E_\iota|=\min_{\boldsymbol{\varphi}_\iota\in E_\iota}\|\mathbf{u}-\boldsymbol{\varphi}_\iota\|=\|\mathbf{u}-Q_\iota\mathbf{u}\|.$$

In addition, the *error functional*

(10) $$d(\boldsymbol{\varphi})=l(\boldsymbol{\varphi})-a(\boldsymbol{\varphi},\mathbf{u}_0),\qquad \boldsymbol{\varphi}\in L^{m,2}(G).$$

is needed with the solution $\mathbf{u}_0$ of (3) and the linear form l in (5). Evidently, the error functional is a continuous linear form on $L^{m,2}(G)$ possessing the characteristic property

(11) $$d\in E_0^\perp\Leftrightarrow d(\boldsymbol{\varphi})=0,\qquad \boldsymbol{\varphi}\in E_0.$$

The norms of the error functional on the subspaces E_ι are given by

$$\|d\|_{E_\iota'}=\sup_{0\neq\boldsymbol{\varphi}\in E_\iota}\frac{|d(\boldsymbol{\varphi})|}{\|\boldsymbol{\varphi}\|}.$$

Using these concepts, the following convergence theorem is true. The two-sided error estimate has been stated already in our paper [20] for a more general class of approximation methods.

(12) *The solutions* $\mathbf{u}_\iota$ *of the variational equations* (3), (4) *satisfy the two-sided discretization error estimate*

(i) $$|\mathbf{u}_0,E_\iota|^2+\frac{1}{\alpha_1^2}\|e_\iota\|_{E_\iota'}^2\leqq\|\mathbf{u}_0-\mathbf{u}_\iota\|^2\leqq|\mathbf{u}_0,E_\iota|^2+\frac{1}{\alpha_0^2}\|e_\iota\|_{E_\iota'}^2,$$

using the abbreviation

(ii) $$e_\iota(\boldsymbol{\varphi})=a(\boldsymbol{\varphi},\mathbf{u}_0-Q_\iota\mathbf{u}_0)+d(\boldsymbol{\varphi}),\qquad \boldsymbol{\varphi}\in E_\iota.$$

The sequence $(\mathbf{u}_\iota)$ *converges to* $\mathbf{u}_0$ *if and only if the two conditions*

(iii) $$\lim_{\iota\to\infty}|\mathbf{u}_0,E_\iota|=0,\qquad \lim_{\iota\to\infty}\|d\|_{E_\iota'}=0$$

are valid.

Proof. Pythagoras' theorem implies the identity

$$\|\mathbf{u}_0-\mathbf{u}_\iota\|^2=|\mathbf{u}_0,E_\iota|^2+\|\mathbf{u}_\iota-Q_\iota\mathbf{u}_0\|^2.$$

Using the bistability condition (6) for $\mathbf{v}_\iota = \mathbf{u}_\iota - Q_\iota \mathbf{u}_0$ and the variational equation (4) it follows that the error estimate (12i) holds. In view of (7), the first term on the right side of the defining equations (12ii) for e_ι has the upper bound $\alpha_1 \|\varphi\| \, |\mathbf{u}_0, E_\iota|$. Thus

$$(13) \qquad \|e_\iota\|_{E_\iota'} \leqq \alpha_1 |\mathbf{u}_0, E_\iota| + \|d\|_{E_\iota'}, \qquad \|d\|_{E_\iota'} \leqq \alpha_1 |\mathbf{u}_0, E_\iota| + \|e_\iota\|_{E_\iota'}.$$

The convergence condition (12iii) is hence equivalent to

$$\lim_{\iota\to\infty} |\mathbf{u}_0, E_\iota| = 0, \qquad \lim_{\iota\to\infty} \|e_\iota\|_{E_\iota'} = 0,$$

and, by (12i), further equivalent to the convergence of $\mathbf{u}_\iota$ to $\mathbf{u}_0$ in $L^{m,2}(G)$ for $\iota \to \infty$. □

To illustrate the above convergence conditions, let us consider the special case that the sesquilinear form a is a scalar product and the associated norm equivalent to the $\|\cdot\|_m$-norm on $L^{m,2}(G)$. These conditions are true for the sesquilinear forms a^1 on $L^{1,2}(G)$ and a^2 on $L^{2,2}(G)$ in the above example. The results of this section may then be applied to $L^{m,2}(G)$ with the "energy inner product" and the associated "energy norm"

$$(14) \qquad (\mathbf{u}, \mathbf{v})_a = a(\mathbf{u}, \mathbf{v}), \qquad \|\mathbf{u}\|_a = a(\mathbf{u}, \mathbf{u})^{1/2}, \qquad \mathbf{u}, \mathbf{v} \in L^{m,2}(G).$$

The representation theorem of Fréchet–Riesz ensures the existence of a unique solution $\mathbf{w} \in L^{m,2}(G)$ of the variational equation

$$(15) \qquad a(\varphi, \mathbf{w}) = l(\varphi), \qquad \varphi \in L^{m,2}(G),$$

for each continuous linear form l on $L^{m,2}(G)$. The bistability condition holds trivially with $\alpha_0 = \alpha_1 = 1$. Further $\mathbf{u}_0 - Q_\iota \mathbf{u}_0$ is a-orthogonal to E_ι and thus $e_\iota(\varphi) = d(\varphi)$ for $\varphi \in E_\iota$. The error estimate (12i) takes on the form

$$(16) \qquad \|\mathbf{u}_0 - \mathbf{u}_\iota\|_a^2 = |\mathbf{u}_0, E_\iota|_a^2 + \|d\|_{E_\iota'}^2, \qquad \iota = 1, 2, \cdots.$$

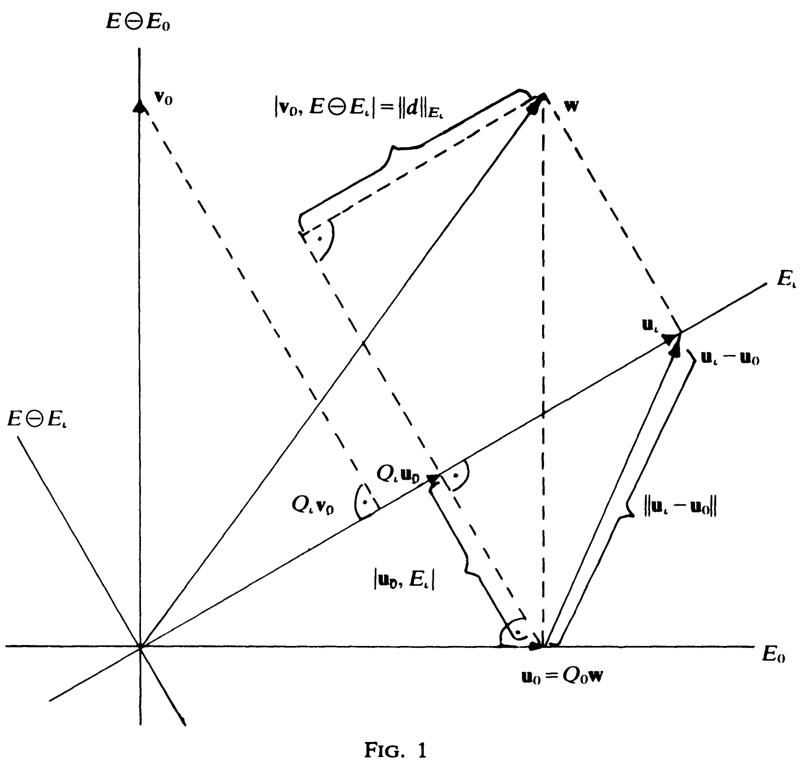


FIG. 1

The solutions $\mathbf{u}_\iota$ of (3), (4) are the a-orthogonal projections $Q_\iota\mathbf{w}$ of $\mathbf{w}$ onto E_ι for $\iota = 0, 1, 2, \cdots$. The error functional d now becomes

$$d(\varphi) = a(\varphi, \mathbf{v}_0), \qquad \varphi \in L^{m,2}(G),$$

where $\mathbf{v}_0 = \mathbf{w} - Q_0\mathbf{w}$. The norms of the error functional d on E_ι therefore permit the representation

(17) $$\|d\|_{E_\iota'} = \|Q_\iota \mathbf{v}_0\|_a = |\mathbf{v}_0, L^{m,2}(G) \ominus E_\iota|_a, \qquad \iota = 1, 2, \cdots,$$

where $L^{m,2}(G) \ominus E_\iota$ denotes the a-orthogonal complement of E_ι in $L^{m,2}(G)$. Figure 1 gives a geometrical interpretation of the error equations (16), (17).

1.3. Convergence, approximability and closedness. A sequence of subspaces (E_ι) *approximates* the subspace E_0 of $L^{m,2}(G)$ iff for each function $\varphi \in E_0$ there exists a sequence of functions $\varphi_\iota \in E_\iota$, $\iota \in \mathbb{N}$, converging to φ. In this sense, the sequence of subspaces $V_\iota \subset H^m(G_\iota)$, $\iota \in \mathbb{N}$, approximates the subspace $V \subset H^m(G)$ iff the embedded spaces $E_0 = {}_EV$, $E_\iota = {}_EV_\iota$ in $L^{m,2}(G)$ have this property. The *approximability condition* permits the equivalent formulation

(1) $$\forall \varphi \in E_0 \colon \lim_{\iota\to\infty} |\varphi, E_\iota| = 0.$$

Let $\mathcal{P}(G_\iota)$ be the space of all piecewise polynomial functions φ_ι on G_ι such that $\varphi_\iota|\underline{K} \in \mathcal{P}(\underline{K})$ for all elements $K \in \mathcal{K}_\iota$ where $\mathcal{P}(\underline{K})$ denotes the space of all polynomials on $\underline{K}$. Typical subspaces V_ι in the method of nonconforming finite elements have the following property: There exists a natural number $r > m$ and a constant γ such that $V_\iota \subset \mathcal{P}(G_\iota) \cap H^m(G_\iota)$ and to each $u \in V \subset H^r(G)$ there exist functions $\varphi_\iota \in V_\iota$ satisfying the error estimate

(2) $$\|u - \varphi_\iota\|_{m,K} \leqq \gamma h_\iota^{r-m} |u|_{r,K}$$

uniformly for all elements $K \in \mathcal{K}_\iota$ and all $\iota = 1, 2, \cdots$ where h_ι is defined by assumption (K3). By virtue of 1.1.(7), it follows that

(3) $$|\mathbf{u}, E_\iota|_m \leqq \Big(\sum_{K \in \mathcal{K}_\iota} \|u - \varphi_\iota\|_{m,K}^2 \Big)^{1/2} \leqq \gamma h_\iota^{r-m} |u|_r.$$

Hence the sequence (V_ι) approximates the subspace $V \cap H^r(G)$. When this subspace is dense in V, condition (3) entails the approximability of V by the sequence (V_ι).

The second convergence condition in theorem 1.2.(12) requires that the norms of the error functional d on E_ι constitute a null sequence. The corresponding terms are denoted in Nitsche's paper [14] by Γ_h and in Strang [17], Strang–Fix [18] by Δ. The error functional d is the same as the functional E_h in the papers of Ciarlet [6], Lascaux–Lesaint [12]. Note also the discretization error estimate of de Arantes e Oliveira [3], although it deals with a different class of boundary value problems. Our convergence theorem generalizes corresponding theorems of these papers to the case of variable coefficients and, additionally, states not only sufficient but also necessary convergence conditions and an error estimate from below.

In the method of conforming finite elements, $V_\iota \subset V$ and $E_\iota \subset E_0$, thus $d|E_\iota = 0$ for all ι. In this case $\|d\|_{E_\iota'} = 0$ and 1.2.(12i) becomes the discretization error estimate for approximations by conforming finite elements (see [4], [5]).

For nonconforming elements, however, the functional d on E_ι is, in general, different from zero. The approximations 1.1.(4) of the variational equation 1.1.(1) have the error functional

(4) $$d(\boldsymbol{\varphi})=\sum_{K\in\mathcal{K}_\iota}\left\{\sum_{|\sigma|\leqq m}\int_K \varphi^\sigma \bar{f}_\sigma\,dx-\sum_{|\sigma|,|\tau|\leqq m}\int_K a_{\sigma\tau}\varphi^\sigma\overline{D^\tau u_0}\,dx\right\},\qquad \boldsymbol{\varphi}=(\varphi^\sigma)\in L^{m,2}(G),$$

with the solution u_0 of 1.1.(1). For nonsmooth coefficients this solution generally possesses no additional regularity so that partial integrations of the second integral in (4) on the subspaces $E_\iota={}_EV_\iota$ are impossible. The method of Ciarlet [6], Lascaux–Lesaint [12] to derive estimates for $\|d\|_{E_\iota'}$ or the approach of Patterson [15], therefore, cannot be applied in this case. Consequently, one has to go another way to prove the validity of the convergence condition $\lim \|d\|_{E_\iota'}=0$.

The stepping stones on this way are the two closedness criteria. A sequence E_0, (E_ι) is said to be *closed* iff the limits of all weakly convergent subsequences of functions $\mathbf{v}_\iota\in E_\iota$, $\iota\in\mathbb{N}'\subset\mathbb{N}$, belong to the subspace E_0. Similarly, the sequence of subspaces V, (V_ι) is said to be closed iff the sequence of embedded spaces $E_0={}_EV$, $E_\iota={}_EV_\iota$, $\iota\in\mathbb{N}$, is closed in $L^{m,2}(G)$. As stated in § 1.1, a sequence of functions $v_\iota\in V_\iota\subset H^m(G_\iota)$, $\iota\in\mathbb{N}'$, is said to be weakly convergent in the sense of the method of nonconforming finite elements if the partial derivatives $D^\sigma v_\iota$ are weakly convergent in $L^2(G)$ for $\iota\to\infty$ and all $|\sigma|\leqq m$. To each of these derivatives there exists a limit v^σ in $L^2(G)$. Thus the closedness condition guarantees that there exists an associated function $v\in V\subset H^m(G)$ with the property $v^\sigma=D^\sigma v$ for $|\sigma|\leqq m$.

Having made these preparations, we can now state the *abstract closedness criterion* where by $E_0^\perp$ we mean the space of all continuous linear forms d on $L^{m,2}(G)$ orthogonal to E_0, that is, $d(\boldsymbol{\varphi})=0$ for $\boldsymbol{\varphi}\in E_0$.

(5) *The sequence E_0, (E_ι) is closed if and only if the following convergence statement is true*

(i) $$\forall d\in E_0^\perp:\ \lim_{\iota\to\infty}\|d\|_{E_\iota'}=0.$$

Proof. (i) (Only if) Let us first assume that the closedness condition holds and let d be an arbitrary continuous linear form on $L^{m,2}(G)$ orthogonal E_0. Let $\mathbb{N}'$ be a subsequence of the sequence of natural numbers such that

$$\limsup_{\iota\to\infty}\|d\|_{E_\iota'}=\lim_{\iota\in\mathbb{N}'}\|d\|_{E_\iota'}.$$

By the representation theorem of Fréchet–Riesz, there exist unique solutions $\mathbf{v}_\iota\in E_\iota$ of the variational equations $(\boldsymbol{\varphi},\mathbf{v}_\iota)_m=d(\boldsymbol{\varphi})$ for all $\boldsymbol{\varphi}\in E_\iota$ and all $\iota=1,2,\cdots$, such that $\|d\|^2_{E_\iota'}=\|\mathbf{v}_\iota\|^2_m=d(\mathbf{v}_\iota)$. The sequence $(\mathbf{v}_\iota)$ is bounded in $L^{m,2}(G)$ and consequently weakly compact. Thus there exist a subsequence $\mathbb{N}''\subset\mathbb{N}'$ and a function $\mathbf{v}$ such that $\mathbf{v}_\iota\rightharpoonup\mathbf{v}$ in $L^{m,2}(G)$ for $\iota\to\infty$, $\iota\in\mathbb{N}''$. By virtue of the closedness condition, the limit $\mathbf{v}$ belongs to E_0 and so

$$\limsup_{\iota\to\infty}\|d\|^2_{E_\iota'}=\lim_{\iota\in\mathbb{N}''}\|d\|^2_{E_\iota'}=\lim_{\iota\in\mathbb{N}''}d(\mathbf{v}_\iota)=d(\mathbf{v})=0.$$

(ii) (If) Conversely, let now an arbitrary weakly convergent sequence $\mathbf{v}_\iota\in E_\iota$, $\iota\in\mathbb{N}'\subset\mathbb{N}$, be given. Under the condition (5i),

$$|d(\mathbf{v}_\iota)|\leqq\|d\|_{E_\iota'}\|\mathbf{v}_\iota\|_m\to 0\qquad(\iota\to\infty).$$

The limit $\mathbf{v}$ of this sequence then has the property

$$\forall d\in E_0^\perp:\ d(\mathbf{v})=\lim_{\iota\in\mathbb{N}'}d(\mathbf{v}_\iota)=0.$$

Therefore, all limits $\mathbf{v}$ of weakly convergent sequences belong to $E_0^{\perp\perp} = E_0$ such that the sequence E_0, (E_ι) is closed. □

In these terms the necessary and sufficient conditions for convergence can be formulated as follows:

(6) *The solutions u_ι of the variational equations* 1.1.(4) *converge to the solution u_0 of* 1.1.(1) *for all $f_\sigma \in L^2(G)$, $|\sigma| \leqq m$, if and only if (V_ι) approximates the subspace V and the sequence V, (V_ι) is closed.*

Proof. (i) From theorem 1.2.(12) it is readily seen that approximability and closedness imply the convergence of u_ι to u_0 for $\iota \to \infty$ in the sense of the method of nonconforming finite elements.

(ii) Conversely, every $u_0 \in V$ defines a continuous linear form by $l(\boldsymbol{\varphi}) = a(\boldsymbol{\varphi}, \mathbf{u}_0)$, $\boldsymbol{\varphi} \in L^{m,2}(G)$. Consider the variational equations 1.2.(3), (4) using this linear form as inhomogeneous term. Then 1.2.(3) has the solution $\mathbf{u}_0$ and the convergence of the solutions $\mathbf{u}_\iota$ of 1.2.(4) to $\mathbf{u}_0$ entails the convergence of the shortest distances $|\mathbf{u}_0, E_\iota| \to 0$ for $\iota \to \infty$. Next choose any $d \in E_0^\perp$. Every continuous linear form on $L^{m,2}(G)$ has the form 1.2.(5). Hence one can set $l = d$ on the right sides of 1.2.(3), (4). The associated solution $\mathbf{u}_0$ then vanishes. From the convergence of the solutions $\mathbf{u}_\iota$ to $\mathbf{u}_0 = 0$, using the convergence theorem 1.2.(12), it follows that $\|d\|_{E_\iota'} \to 0$ for $\iota \to \infty$. By Theorem (5), this proves the closedness of the sequence E_0, (E_ι) and so too V, (V_ι). □

1.4. The test. The question now arises under which conditions sequences of function spaces $V_\iota \subset H^m(G_\iota)$, $\iota = 0, 1, 2, \cdots$, might be closed. The answer is given by the generalized patch test. This test proves the differentiability, in the generalized sense of Sobolev spaces, of the limits of weakly convergent sequences $v_\iota \in V_\iota$, $\iota \in \mathbb{N}' \subset \mathbb{N}$. It is well-known that this differentiability is a local property. Correspondingly, the generalized patch test also proves a local property of the sequence of function spaces $V_\iota \subset H^m(G_\iota)$. Without difficulty, both the closedness of the sequence $H^m(G)$, (V_ι) and of the sequence $H_0^m(G)$, (V_ι) can be treated together. For, the only difference is that in the first case the generalized patch test is applied in the open domain G and in the second case in the closure $\bar{G}$.

A function $\mathbf{v} = (v^\mu) \in L^{m,2}(G)$ belongs to the Sobolev space $H^m(G)$ if and only if the relation

(1) $$\int_G D_l \psi v^\mu \, dx = -\int_G \psi v^{\mu + e_l} \, dx, \quad l = 1, \cdots, n, \quad |\mu| \leqq m - 1,$$

holds for all test functions $\psi \in C_0^\infty(G)$ where e_l denotes the unit vectors $(\delta_{1l}, \cdots, \delta_{nl})$ in $\mathbb{R}^n$. For brevity, we write

(2) $$T_{l,\mu}(\psi, \mathbf{v}) = \int_G (D_l \psi v^\mu + \psi v^{\mu + e_l}) \, dx.$$

Hereby bounded bilinear forms on $H^1(G) x L^{m,2}(G)$ are defined satisfying the inequality

(3) $$|T_{l,\mu}(\psi, \mathbf{v})| \leqq \|\psi\|_1 \|\mathbf{v}\|_m, \qquad \psi \in H^1(G), \quad \mathbf{v} \in L^{m,2}(G).$$

In view of (1), a function $\mathbf{v} \in L^{m,2}(G)$ is in ${}_E H^m(G)$ if and only if the condition

(4) $$T_{l,\mu}(\psi, \mathbf{v}) = 0, \qquad l = 1, \cdots, n, \quad |\mu| \leqq m - 1,$$

is valid for all $\psi \in C_0^\infty(G)$. Similarly, the following statement is true.

(5) *A function $\mathbf{v} \in L^{m,2}(G)$ belongs to ${}_E H_0^m(G)$ if and only if* (4) *holds for all test functions $\psi \in C_0^\infty(\mathbb{R}^n)$.*

Proof. (i) (If) Every function $v \in H_0^m(G)$ can be extended to a function $\hat{v}$ on $\mathbb{R}^n$ by $\hat{v} = v$ on G, $\hat{v} = 0$ on $\mathbb{R}^n - G$. As one easily sees, this extension belongs to $H^m(\mathbb{R}^n)$; thus

$$\int_{\mathbb{R}^n} D_l\psi D^\mu \hat{v}\,dx = -\int_{\mathbb{R}^n} \psi D^{\mu+e_l}\hat{v}\,dx, \qquad \psi \in C_0^\infty(\mathbb{R}^n), \quad l = 1, \cdots, n, \quad |\mu| \leqq m-1.$$

The partial derivatives $D^\mu \hat{v}$ have their support in $\bar{G}$ and the boundary of the polyhedral domain has the measure zero. The functions $v^\mu = D^\mu v = D^\mu \hat{v}$ on G hence satisfy the condition (4) for all $\psi \in C_0^\infty(\mathbb{R}^n)$.

(ii) (Only if) Let $\mathbf{v}$ now be a function in $L^{m,2}(G)$ with this property and let $\hat{\mathbf{v}} \in L^{m,2}(\mathbb{R}^n)$ be the extension of $\mathbf{v}$ by $\hat{\mathbf{v}} = 0$ on $\mathbb{R}^n - G$. Then

$$\int_{\mathbb{R}^n} D_l\psi \hat{v}^\mu\,dx = -\int_{\mathbb{R}^n} \psi \hat{v}^{\mu+e_l}\,dx, \qquad \psi \in C_0^\infty(\mathbb{R}^n), \qquad l = 1, \cdots, n, \quad |\mu| \leqq m-1.$$

This shows that the function $\hat{v} = \hat{v}^{(0,\cdots,0)}$ is in $H^m(\mathbb{R}^n)$ and $\hat{v}^\mu = D^\mu \hat{v}$ for $|\mu| \leqq m$. The support of $\hat{v}$ is contained in $\bar{G}$. The polyhedral domain G possesses the segment property because by assumption (K1) the subdivisions $\mathcal{K}_\iota$ of $\bar{G}$ are locally finite and each element $K \in \mathcal{K}_\iota$ has the segment property. These conditions imply $v = \hat{v}|G \in H_0^m(G)$ and so $\mathbf{v} = \hat{\mathbf{v}}|G \in {}_EH_0^m(G)$ (see [19, pp. 25–26]). □

In view of the representation 1.1.(2) of $\bar{G}$, integrals over $\bar{G}$ or G may be written as a sum of corresponding integrals over the elements $K \in \mathcal{K}_\iota$. Functions $\mathbf{v}_\iota \in {}_EH^m(G_\iota)$ have the components $v_\iota^\mu = D^\mu v_\iota$ on the interior $\underline{K}$ of each element $K \in \mathcal{K}_\iota$ where $v_\iota \in H^m(G_\iota)$. For the sake of simplicity, we shall also write $T_{l,\mu}(\psi, v_\iota)$ instead of $T_{l,\mu}(\psi, \mathbf{v}_\iota)$ in this case. The integrand in (2) for $\mathbf{v} = \mathbf{v}_\iota$ permits the representation $D_l(\psi D^\mu v_\iota)$ on $\underline{K}$. Using the Gaussian theorem, we thus obtain

$$(6) \qquad T_{l,\mu}(\psi, v_\iota) = \sum_{K \in \mathcal{K}_\iota} \int_{\partial K} \psi D^\mu v_\iota^K N_l\,ds,$$

N_l for $l = 1, \cdots, n$ being the components of the unit vector in outward normal direction on the surface of the element K. The notation $D^\mu v_\iota^K$ reminds us that the trace of the function $D^\mu v_\iota|\underline{K}$ on the boundary ∂K has to be used in the surface integral.

The *generalized patch test in the subset* $S \subset \bar{G}$ for sequences of functions $v_\iota \in H^m(G_\iota)$, $\iota \in \mathbb{N}'$, is defined by the property:

(g.p.t.) *For every point $x \in S$ there exists an open neighborhood O in $\mathbb{R}^n$ such that*

$$(7) \qquad \lim_{\iota \in \mathbb{N}'} \sum_{K \in \mathcal{K}_\iota} \int_{\partial K} \chi D^\mu v_\iota^K N_l\,ds = 0$$

for all test functions $\chi \in C_0^\infty(O)$ and all indices $l = 1, \cdots, n$, $|\mu| \leqq m-1$.

This formulation emphasizes the fact that the generalized patch test deals with a local property. The set S could, for instance, consist of a single point. In the examples of § 2, the generalized patch test is proved for $S = G$ with the neighborhood $O = G$ and, in the case of generalized Dirichlet boundary conditions, for $S = \bar{G}$ with the neighborhood $O = \mathbb{R}^n$. That is, condition (7) is verified for all test functions $\chi \in C_0^\infty(G)$ or $\chi \in C_0^\infty(\mathbb{R}^n)$. In these two cases, the meaning of the generalized patch is seen from the following theorem.

(8) *Let any weakly convergent sequence of functions $v_\iota \in H^m(G_\iota)$, $\iota \in \mathbb{N}' \subset \mathbb{N}$, be given. The limit* w-lim $\mathbf{v}_\iota = \mathbf{v}$ *is a member of ${}_EH^m(G)$ (${}_EH_0^m(G)$) if and only if the sequence passes the generalized patch test in G (in $\bar{G}$).*

Proof. (i) (Only if) If the limit $\mathbf{v}$ belongs to ${}_EH^m(G)({}_EH_0^m(G))$, condition (4) holds for all $\chi=\psi\in C_0^\infty(G)(C_0^\infty(\mathbb{R}^n))$. From the representation (2) it is seen that $T_{l,\mu}(\chi,\cdot)$ is a continuous linear form on $L^{m,2}(G)$ for all l,μ,χ as indicated above. The weakly convergent sequence $(\mathbf{v}_\iota)$ hence satisfies the condition

$$\lim_{\iota\in\mathbb{N}'} T_{l,\mu}(\chi,v_\iota)=T_{l,\mu}(\psi,\mathbf{v})=0.$$

Since $\mathbf{v}_\iota\in{}_EH^m(G_\iota)$ and the representation (6) holds, the generalized patch test is passed in $S=G$ with the neighborhood $O=G$ (in $S=\bar{G}$ with $O=\mathbb{R}^n$) for all $x\in S$.

(ii) (If) Conversely, let now the generalized patch test be satisfied and let ψ be any test function in $C_0^\infty(G)$ (in $C_0^\infty(\mathbb{R}^n)$). The function ψ has compact support M in G (in $\mathbb{R}^n$). The open neighborhoods O, associated to each $x\in M$ $(x\in M\cap\bar{G})$, constitute a covering of M (of $M\cap\bar{G}$). By virtue of the Heine–Borel theorem, there exists a finite subcovering $\{O_1,\cdots,O_t\}$ of M (of $M\cap\bar{G}$). Using an associated partition of unity

$$1=\sum_{k=1}^{t}\zeta_k(x),\quad x\in M\ (x\in M\cap\bar{G});\qquad \zeta_k\in C_0^\infty(O_k),\quad k=1,\cdots,t,$$

one has

$$\psi(x)=\sum_{k=1}^{t}\chi_k(x),\quad x\in M(x\in M\cap\bar{G});\qquad \chi_k=\zeta_k\psi\in C_0^\infty(O_k),\quad k=1,\cdots,t.$$

The generalized patch test then implies

$$T_{l,\mu}(\psi,\mathbf{v})=\sum_{k=1}^{t}T_{l,\mu}(\chi_k,\mathbf{v})=\sum_{k=1}^{t}\lim_{\iota\in\mathbb{N}'}T_{l,\mu}(\chi_k,v_\iota)=0.$$

Thus condition (4) is true for all $\psi\in C_0^\infty(G)$ $(C_0^\infty(\mathbb{R}^n))$ and hence $\mathbf{v}\in{}_EH^m(G)({}_EH_0^m(G))$. $\square$

The generalized patch test now yields the fundamental *concrete closedness criterion.*

(9) *The sequence $H^m(G)$, (V_ι) (or $H_0^m(G)$, (V_ι)) is closed if and only if every bounded sequence $v_\iota\in V_\iota$, $\iota\in\mathbb{N}'\subset\mathbb{N}$, satisfies the generalized patch test in G (in $\bar{G}$).*

Proof. (i) (If) If bounded sequences satisfy the generalized patch test, a fortiori also weakly convergent sequences $(\mathbf{v}_\iota)$ in $L^{m,2}(G)$ do. From theorem (8) it follows that the limit w-$\lim\mathbf{v}_\iota=\mathbf{v}$ belongs to ${}_EH^m(G)$ (to ${}_EH_0^m(G)$). Consequently, the sequence $H^m(G)$, (V_ι) (or $H_0^m(G)$, (V_ι)) is closed.

(ii) (Only if) To prove the converse, let now the closedness condition be valid and let $v_\iota\in V_\iota$, $\iota\in\mathbb{N}'$, be any bounded sequence. Then there exists for each $l=1,\cdots,n$, $|\mu|\leqq m-1$, and $\psi\in C_0^\infty(G)$ $(C_0^\infty(\mathbb{R}^n))$ a subsequence $\mathbb{N}''\subset\mathbb{N}'$ such that the representation

$$\limsup_{\iota\in\mathbb{N}'}|T_{l,\mu}(\psi,v_\iota)|=\lim_{\iota\in\mathbb{N}''}|T_{l,\mu}(\psi,v_\iota)|$$

is valid. The sequence $(\mathbf{v}_\iota)$ is bounded in $L^{m,2}(G)$ and hence weakly compact in this space. That is there exist a subsequence $\mathbb{N}'''\subset\mathbb{N}''$ and an element $\mathbf{v}$ such that $\mathbf{v}_\iota\rightharpoonup\mathbf{v}$ in $L^{m,2}(G)$ for $\iota\to\infty$, $\iota\in\mathbb{N}''$. The closedness condition implies $\mathbf{v}\in{}_EH^m(G)$ $(\mathbf{v}\in{}_EH_0^m(G))$. By (4), (5), one obtains

$$\limsup_{\iota\in\mathbb{N}'}|T_{l,\mu}(\psi,v_\iota)|=\lim_{\iota\in\mathbb{N}'''}|T_{l,\mu}(\psi,v_\iota)|=|T_{l,\mu}(\psi,\mathbf{v})|=0.$$

Consequently, $T_{l,\mu}(\psi, v_\iota)$ converges to zero for $\iota \to \infty$, $\iota \in \mathbb{N}'$, and by (6) the generalized patch test is satisfied in $S = G$ with $O = G$ (in $S = \bar{G}$ with $O = \mathbb{R}^n$) for each $x \in S$. $\square$

In view of the above theorem, the sequence of *subspaces* $V_\iota \subset H^m(G_\iota)$, $\iota \in \mathbb{N}$, is said to *pass the generalized patch test* in G (in $\bar{G}$) iff each bounded sequence $\mathbf{v}_\iota \in E_\iota = {}_E V_\iota$, $\iota \in \mathbb{N}' \subset \mathbb{N}$, passes the test in G (in $\bar{G}$). Using theorem 1.3.(6), we thus obtain the *main convergence theorem.*

(10) *The solutions u_ι of the variational equations* 1.1.(4) *converge to the solution $u_0 \in V = H^m(G)$ ($V = \mathring{H}^m_0(G)$) of* 1.1.(1) *for all $f_\sigma \in L^2(G)$, $|\sigma| \leqq m$, if and only if the sequence (V_ι) both approximates the subspace V and passes the generalized patch test in G (in $\bar{G}$).*

2. Examples of nonconforming finite elements. It will be proved in this section that a series of well-known nonconforming elements pass the generalized test. It is well known that these elements also possess the required approximability properties. The main convergence theorem 1.4.(10) then ensures the convergence of the associated nonconforming approximations of the variational equations described in §§ 1.1, 1.2. In this way, the convergence of the method of nonconforming finite elements is proved for a large class of generalized boundary value problems with not necessarily smooth coefficients. In contrast to this result, Ciarlet [6], Lascaux–Lesaint [12] limit their study to a special plate problem with constant coefficients. These papers develop, however, a technique for estimating the error functional and establish the order of convergence of the nonconforming approximations. Using this technique we shall be able to prove that the nonconforming elements under consideration pass the generalized patch test.

2.1. General properties. The approximating spaces V_ι in the following examples consist of piecewise polynomial functions in $\mathcal{P}(G_\iota)$ having some continuity properties on interelement boundaries. Additionally, they satisfy certain boundary conditions in the case that Dirichlet boundary conditions have to be approximated. For every function $v_\iota \in V_\iota$ and every element $K \in \mathcal{K}_\iota$, v_ι^K denotes the uniquely determined polynomial in $\mathcal{P}(\mathbb{R}^n)$ with the property $v_\iota^K(x) = v_\iota(x)$ for $x \in \underline{K}$. Similarly V_ι^K is the space of all these polynomials v_ι^K for $v_\iota \in V_\iota$. The restriction $v_\iota|\underline{K}$ can be extended to the boundary ∂K by $v_\iota^K|\partial K$. We shall always assume in the following that the functions v_ι are extended correctly, with respect to the given context, onto the element boundary ∂K. For notational simplicity, we shall write T_l, $T_{l,k}$ instead of $T_{l,0}$, $T_{l,\mu}$ for $|\mu| = 1$ and $\mu = e_k = (\delta_{1k}, \cdots, \delta_{nk})$, $k = 1, \cdots, n$.

Special nonconforming elements are characterized by the following data: (L) degrees of freedom or nodal parameters, (M) subspaces V_ι of $\mathcal{P}(G_\iota)$. The data (M) are grouped into: (M1) space of polynomials V_ι^K for $K \in \mathcal{K}_\iota$, (M2) continuity conditions, (M3) boundary conditions if required.

We shall consider subdivisions $\mathcal{K}_\iota$ of polyhedral subdomains $\bar{G}$ of the plane $\mathbb{R}^2$ into rectangles and triangles. The boundary of the elements $K \in \mathcal{K}_\iota$ is the union of the four sides of the rectangle or the three sides of the triangle. Under the general assumptions (K1), (K2) the following *alternative*, concerning these sides F, is true: either F is contained in the boundary ∂G or F is a common side of two distinct elements $K, K' \in \mathcal{K}_\iota$. Those sides F, which are not common sides of two elements $K, K' \in \mathcal{K}_\iota$, are said to be *free.* Using these terms, the following statement is true.

(1) *For every subdivision $\mathcal{K}_\iota$ of $\bar{G}$, the boundary ∂G is the union of all free sides of elements $K \in \mathcal{K}_\iota$.*

The following property of the bilinear forms $T_{l,\mu}$ plays an important role in applications of the generalized patch test to special nonconforming elements.

(2) *For every continuous function w on $\bar{G}$ and every test function $\psi \in C_0^\infty(G)$,*

$$\text{(i)} \qquad \sum_{K \in \mathcal{K}_\iota} \int_{\partial K} \psi w N_l \, ds = \sum_{K \in \mathcal{K}_\iota} \sum_{F \subset \partial K} \int_F \psi w N_l^K \, ds = 0, \qquad l = 1, 2.$$

In the case that $w = 0$ on ∂G, this condition holds moreover for all $\psi \in C_0^\infty(\mathbb{R}^n)$.

Proof. The sides $F \subset \partial K$ are either contained in ∂G or common sides of two elements K, K'. For $F \subset \partial G$ one has $\psi w = 0$ on F because $\psi \in C_0^\infty(G)$ or $w = 0$ on ∂G. Hence all integrals over free sides $F \subset \partial G$ vanish in the above sum (2i). If F is a common side of two elements K, K', the associated normal vectors are constant on this side and $N^K = -N^{K'}$. Therefore, (2i) is a telescoping sum in which terms pairwise cancel. □

Success in the generalized patch test will be proved by deriving appropriate estimates of the bilinear forms $T_{l,\mu}$. To this purpose, one needs some well-known inequalities from the theory of finite elements. Under the assumption (K3), there exists some constant α such that

$$\text{(3)} \qquad \int_{\partial K} |w|^2 \, ds \leqq \frac{\alpha^2}{h_K} \|w\|_{1,K}^2, \qquad w \in H^1(\underline{K}), \quad h_K = \delta_1(K),$$

uniformly for all $K \in \mathcal{K}_\iota$, $\iota = 1, 2, \cdots$. Analogously, the *inverse property* reads

$$\text{(4)} \qquad |w|_{t+1,K} \leqq \frac{\beta}{h_K} |w|_{t,K}, \qquad w \in \mathcal{Q}_r, \quad t = 0, 1, 2, \cdots,$$

with seminorms of the form (6). $\mathcal{P}_r$ denotes the space of all polynomials of at most rth degree, and $\mathcal{Q}_r$ the space of all polynomials of highest degree r in each of the independent variables x_i.

In addition, one needs error estimates for piecewise polynomial approximations of functions by (generalized) interpolation. These estimates can be obtained by well-known techniques (see Ciarlet–Raviart [7]). For every subdivision $\mathcal{K}_\iota$, the associated interpolation operators are continuous linear projections P of $H^r(G_\iota)$ onto a subspace M of piecewise polynomial functions. Under the assumptions (K1), (K2), (K3), there exist constants γ such that the remainder terms $Rw = w - Pw$ satisfy the error estimate

$$\text{(5)} \qquad \int_{\partial K} |Rw|^2 \, ds \leqq \gamma^2 h_K^{2r-1} |w|_{r,K}^2, \qquad w \in H^r(G_\iota), \quad h_K = \delta_1(K),$$

uniformly for all elements $K \in \mathcal{K}_\iota$ and $\iota = 1, 2, \cdots$. For triangulations $\mathcal{K}_\iota$ the space M consists of functions whose restrictions to any open triangle $\underline{K} \in \mathcal{K}_\iota$ are polynomials in $\mathcal{P}_{r-1}(\underline{K})$. The associated seminorm in (5) reads

$$\text{(6)} \qquad |w|_{r,K} = \left(\sum_{|\sigma| = r} \int_{\underline{K}} |D^\sigma w|^2 \, dx \right)^{1/2}.$$

The error estimate (5) is obtained by the well-known generalized Poincaré inequality (see Nečas [13, p. 117]) and affine transformations onto a reference triangle.

On rectangular subdivisions $\mathcal{K}_\iota$, the spaces M consist of functions whose restrictions to each open rectangle $\underline{K} \in \mathcal{K}_\iota$ are polynomials in $\mathcal{Q}_{r-1}(\underline{K})$. In these cases the associated seminorm is

$$\text{(7)} \qquad |w|_{r,K} = \left(\sum_{i=1}^n \int_{\underline{K}} |D_i^r w|^2 \, dx \right)^{1/2}.$$

The error estimate (5) is obtained by means of inequalities of Aronszajn–Smith (see Agmon [1, p. 167]) and affine transformations onto a reference rectangle. Obviously, the seminorm (7) is bounded by (6) so that in both cases the error estimate (5) holds with the stronger seminorm (6).

Finally two special examples of Dirichlet boundary value problems are stated in which the associated error functional 1.2.(10) can be represented directly by the bilinear forms $T_{l,\mu}$. The estimates for $T_{l,\mu}$ in the following sections are continuously extendable to $H^1(G)\times E_\iota$ or $H^r(G)\times E_\iota$. In this way, they immediately yield estimates of the norms $\|d\|_{E_\iota'}$ for sufficiently regular solutions u_0.

First let $m=1$ and let $u_0\in \mathring{H}^1_0(G)\times H^2(G)$ be a solution of the differential equation

$$-\Delta u_0+u_0=f \quad \text{in } G.$$

Then u_0 is a solution of the variational equation

$$\int_G (\nabla\varphi\cdot\overline{\nabla u_0}+\varphi\overline{u_0})\,dx=\int_G \varphi\bar f\,dx,\qquad \varphi\in \mathring{H}^1_0(G). \tag{8}$$

By partial integration, the associated error functional becomes

$$\begin{aligned} d(\varphi)&=\sum_{K\in\mathcal{K}_\iota}\int_{K}\{\varphi\bar f-(\nabla\varphi\cdot\overline{\nabla u_0}+\varphi\overline{u_0})\}\,dx\\ &=-\sum_{K\in\mathcal{K}_\iota}\int_{\partial K}\frac{\overline{\partial u_0}}{\partial N}\varphi^K\,ds,\qquad \varphi\in H^1(G_\iota).\end{aligned}$$

Thus d may be written as

$$d(\varphi)=-\sum_{k=1}^{n} T_k(\overline{D_k u_0},\varphi),\qquad \varphi\in E_\iota={}_EV_\iota. \tag{9}$$

An example in the case $m=2$ is obtained by the boundary value problem

$$\Delta^2u_0+u_0=f \quad \text{in } G$$

and the associated variational equation

$$\int_G (\Delta\varphi\,\overline{\Delta u_0}+\varphi\overline{u_0})\,dx=\int_G \varphi\bar f\,dx,\qquad \varphi\in \mathring{H}^2_0(G), \tag{10}$$

for a solution $u_0\in \mathring{H}^2_0(G)\cap H^4(G)$. In this case, partially integrating the error functional d gives

$$\begin{aligned} d(\varphi)&=\sum_{K\in\mathcal{K}_\iota}\int_{K}\{\varphi\bar f-(\Delta\varphi\,\overline{\Delta u_0}+\varphi\overline{u_0})\}\,dx\\ &=\sum_{K\in\mathcal{K}_\iota}\int_{\partial K}\left\{\frac{\overline{\partial\Delta u_0}}{\partial N}\varphi^K-\overline{\Delta u_0}\frac{\partial\varphi^K}{\partial N}\right\}ds,\qquad \varphi\in H^2(G_\iota).\end{aligned}$$

Consequently one has the representation

$$d(\varphi)=\sum_{k=1}^{n}\{T_k(\overline{D_k\,\Delta u_0},\varphi)-T_{k,k}(\overline{\Delta u_0},\varphi)\},\qquad \varphi\in E_\iota={}_EV_\iota. \tag{11}$$

2.2. Wilson's element. This element is defined in the plane $\mathbb{R}^2$ on rectangles whose sides are parallel to the coordinate axes x and y. Due to assumption (K1), the domain $\bar G$ must in this case be a rectangle, a T- or L-shaped domain or the like. The boundary of each element K is the union of the four sides $F_1,\cdots,F_4$. These are line segments parallel to an interval of the x- or y-axis (see Fig. 2).

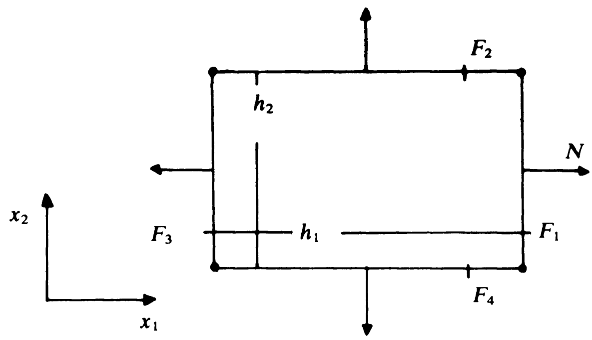


FIG. 2

Degrees of freedom and polynomial spaces of Wilson's elements are specified as follows:

(L) Nodal parameters are the function values at the vertices of the rectangles K and the values of the second derivatives $\partial^2 v_\iota/\partial x_1^2$, $\partial^2 v_\iota/\partial x_2^2$ on $\underline{K}$ for each $K \in \mathcal{K}_\iota$.

(M1) $V_\iota^K = \mathcal{P}_2(\mathbb{R}^2)$ for all $K \in \mathcal{K}_\iota$;

(M2) functions $v_\iota \in V_\iota$ are continuous at the vertices of the rectangles $K \in \mathcal{K}_\iota$;

(M3) in the case of Dirichlet boundary conditions, $v_\iota = 0$ at vertices on the boundary ∂G.

In order to apply the generalized patch test, one needs special interpolation operators. Let P_0 be the linear operator assigning to each integrable function w on G the following *step function*

(1) $$(P_0 w)(x) = w(K) = \frac{1}{|K|} \int_{\partial K} w \, dx, \qquad x \in \underline{K}, \quad K \in \mathcal{K}_\iota.$$

By $w(K)$ is meant the *mean value* of w over the element K. Next let P_1 be the operator of *piecewise bilinear interpolation* of functions v on $\bar{G}$, specified by $P_1 v|\underline{K}$ is bilinear and $P_1 v = v$ at the four vertices of K for each $K \in \mathcal{K}_\iota$. By virtue of the continuity condition (M2), $P_1 v$ is a continuous function on $\bar{G}$. Under the assumption (M3), additionally, $P_1 v = 0$ on the boundary ∂G of the domain G. The remainder terms of these two interpolations satisfy the error estimate 2.1.(5) with the seminorm 2.1.(6) for $r = 1$ and $r = 2$ respectively.

(2) *The estimate*

$$|T_l(\psi, v_\iota)| \leqq \beta \gamma^2 h_\iota |\psi|_1 |v_\iota|_{1,G_\iota}, \qquad l = 1, 2,$$

holds uniformly for all $v_\iota \in V_\iota$, $\iota = 1, 2, \cdots$, *and all test functions* $\psi \in C_0^\infty(G)$ ($\psi \in C_0^\infty(\mathbb{R}^2)$, *in case* (M3) *is valid*). *Thus the sequence* (V_ι) *passes the generalized patch test with* $m = 1$ *in* G (*in* $\bar{G}$).

Proof. (i) The piecewise bilinear interpolation $w = P_1 v$ for $v \in V_\iota$ satisfies the assumptions of theorem 2.1.(2) so that $T_l(\psi, P_1 v) = 0$ and for $R_1 v = v - P_1 v$ then

$$T_l(\psi, v) = T_l(\psi, R_1 v), \qquad v \in V_\iota,$$

for all $\psi \in C_0^\infty(G)$ ($C_0^\infty(\mathbb{R}^2)$). Next one has the relation

$$\int_{\partial K} R_1 v N_l \, ds = \sum_{i=1}^{4} \int_{F_i} R_1 v N_l \, ds = 0$$

for all $K \in \mathcal{K}_\iota$. For, each side F_i is a line segment, the normal vector $N = (N_1, N_2)$ is constant on F_i and $R_1 v$ on F_i a polynomial of second degree in one variable vanishing at the two endpoints of the line segment. Applying the trapezoidal rule yields

$$\int_{F_i} R_1 v N_l \, ds = -\int_{F_{i+2}} R_1 v N_l \, ds = -\frac{h_{i+1}^3}{12} \frac{\partial^2 v^K}{\partial x_{i+1}^2} \delta_{il}, \qquad i, l = 1, 2,$$

where $x_3 = x_1$ and $h_3 = h_1$. The partial derivatives of second order are constant on $\underline{K}$ so that in the above sum integrals over opposite sides cancel.

(ii) From the result of the first part of this proof it is seen that

$$\int_{\partial K} \psi R_1 v N_l \, ds = \int_{\partial K} R_0 \psi R_1 v N_l \, ds$$

where $P_0\psi$ is the piecewise constant approximation of ψ and $R_0\psi = \psi - P_0\psi$ the associated remainder term. Application of Schwarz inequality and of the estimate 2.1.(5) yields

$$\left| \int_{\partial K} \psi R_1 v N_l \, ds \right| \leqq \gamma^2 \sqrt{h_K} |\psi|_{1,K} \sqrt{h_K^3} |v|_{2,K}.$$

Using the inverse property 2.1.(4), Schwarz's inequality for sums and the representation 1.1.(7) for the norms, one finally obtains

$$\begin{aligned} |T_l(\psi, v)| &\leqq \sum_{K \in \mathcal{K}_\iota} \left| \int_{\partial K} \psi R_1 v N_l \, ds \right| \\ &\leqq \beta \gamma^2 h_\iota \sum_{K \in \mathcal{K}_\iota} |\psi|_{1,K} |v|_{1,K} \leqq \beta \gamma^2 h_\iota |\psi|_1 |v|_{1,G_\iota}. \end{aligned}$$

Hence $T_l(\psi, v_\iota)$ converges to zero for $\iota \to \infty$, for every bounded sequence of functions $v_\iota \in V_\iota \in H^1(G_\iota)$, $\iota \in \mathbb{N}'$, and every $\psi \in C_0^\infty(G)$ ($C_0^\infty(\mathbb{R}^2)$). The sequence (V_ι) thus passes the generalized patch test in G (in $\bar{G}$). □

2.3. Adini's element. Adini's nonconforming element is defined on rectangles in the plane. The rectangles and the polyhedral domain $\bar{G}$ have the same shape as described in § 2.2. Degrees of freedom and piecewise polynomial subspaces are specified by the following:

(L) Nodal parameters are the function values and the values of the two first partial derivatives with respect to x_1, x_2 at the vertices of the rectangles $K \in \mathcal{K}_\iota$.

(M1) $V_\iota^K = \mathcal{P}_3(\mathbb{R}^2) + [x_1^3 x_2, x_1 x_2^3]$ for all $K \in \mathcal{K}_\iota$,

(M2) functions $v_\iota \in V_\iota$ together with their partial derivatives of first order are continuous at vertices of the rectangles $K \in \mathcal{K}_\iota$;

(M3) in the case of Dirichlet boundary conditions, additionally, v_ι, $\partial v_\iota / \partial x_1$, $\partial v_\iota / \partial x_2$ vanish at vertices on ∂G.

(1) *The equations*

(i) $$T_l(\psi, v_\iota) = 0, \qquad T_{l,k}(\psi, v_\iota) = 0, \qquad l \neq k,$$

and the estimate

(ii) $$|T_{k,k}(\psi, v_\iota)| \leqq \beta \gamma^2 h_\iota |\psi|_1 |v_\iota|_{2,G_\iota}, \qquad k = 1, 2,$$

hold uniformly for all $v_\iota \in V_\iota$, $\iota = 1, 2, \cdots$, and all $\psi \in C_0^\infty(G)$ ($\psi \in C_0^\infty(\mathbb{R}^2)$, provided that (M3) *is true). Consequently, the sequence (V_ι) stands the generalized patch test with $m = 2$ in G (in $\bar{G}$).*

Proof. (i) It will first be verified that functions $v \in V_\iota$ are continuous on $\bar{G}$. Let F be a common side of two rectangles $K, K' \in \mathcal{K}_\iota$. Then v^K, $v^{K'}$ are polynomials of third degree in one variable on the line segment F. The two polynomials have the same function values and first derivatives at the endpoints of F. It is well known that under these conditions the two polynomials must be identical, that is $v^K = v^{K'}$ on F. In addition, $v = 0$ on ∂G when the boundary conditions (M3) hold. This is obtained from the fact that a polynomial of third degree vanishing together with its first derivatives at the endpoints of a line segment F, must be identically zero on $F \subset \partial G$. The boundary ∂G is the union of all free sides F on rectangles $K \in \mathcal{K}_\iota$ and so $v = 0$ on ∂G. Theorem 2.1.(2) finally implies $T_l(\psi, v) = 0$ for all $\psi \in C_0^\infty(G)$ $(C_0^\infty(\mathbb{R}^2))$.

(ii) The analysis of the expressions $T_{l,k}(\psi, v)$ proceeds in two steps. Consider first the case $l \neq k$. The normal vector $N = (N_1, N_2)$ on the sides F_i of the rectangle K has the components

$$N_1 = \pm 1, N_2 = 0 \quad \text{on } F_1, F_3; \qquad N_1 = 0,\ N_2 = \pm 1 \quad \text{on } F_2, F_4.$$

Hence, in the sum $T_{1,2}(\psi, v)$ the integrals over F_2, F_4 are equal to zero. On the sides F_1, F_3, $D_2 v$ is the derivative of v in tangential direction. As v is continuous on interelement boundaries, $D_2 v$ is continuous on F_1, F_3. Consequently, in the sum

$$T_{1,2}(\psi, v) = \sum_{K \in \mathcal{K}_\iota} \sum_{F \subset \partial K} \int_F \psi D_2 v^K N_1^K \, ds \tag{2}$$

all integrals over common sides F_1, F_3 cancel pairwise. In the case of Dirichlet boundary conditions (M3), $v = 0$ on ∂G and so the tangential derivatives $D_2 v$ vanish on $F_1, F_3 \subset \partial G$. Analogous considerations are valid for $T_{2,1}(\psi, v)$. In this way, the second equation in (1i) has been proved.

(ii) Again let P_1 be the operator of piecewise bilinear interpolation of functions on $\bar{G}$ and let R_1 be the associated remainder term operator. The two partial derivatives $D_1 v, D_2 v$ are continuous at vertices of the rectangles K and, in the case of the boundary condition (M3), vanish at vertices on ∂G. Thus $P_1 D_k v$ is continuous on $\bar{G}$ (and zero on ∂G) so that, by theorem 2.1.(2),

$$T_{k,k}(\psi, v) = \sum_{K \in \mathcal{K}_\iota} \int_{\partial K} \psi R_1 D_k v N_k \, ds, \qquad k = 1, 2,$$

for all $\psi \in C_0^\infty(G)$ $(\psi \in C_0^\infty(\mathbb{R}^2))$. Further, the relation

$$\int_{\partial K} R_1 D_k v N_k \, ds = \left(\int_{F_k} - \int_{F_{k+2}} \right) R_1 D_k v \, ds = 0$$

holds. For, $w = R_1 D_k v$ is a polynomial of third degree in one variable on each side F, vanishing at the endpoints of the line segment. Simpson's rule is exact and yields

$$\int_{F_k} R_1 w \, ds = \tfrac{2}{3} h_{k+1} w(c_k), \qquad h_{k+2} = h_k, \quad k = 1, \cdots, 4,$$

where c_k is the midpoint of the line segment F_k. Every polynomial p of third degree in one variable on an interval $[a, b]$ with midpoint $c = \frac{1}{2}(a + b)$ has the property

$$p(c) = \tfrac{1}{2}(p(a) + p(b)) - \tfrac{1}{8}(b - a)^2 p''(c).$$

By applying this relation to $p=R_1D_kv$ on the four sides of K, it follows that

$$\begin{aligned}\int_{\partial K} R_1D_kvN_k\,ds &= \tfrac{2}{3}h_{k+1}(w(c_k)-w(c_{k+2}))\\ &= -\frac{h_{k+1}^3}{12}\left\{\frac{\partial^3 v^K}{\partial x_k\,\partial x_{k+1}^2}(c_k)-\frac{\partial^3 v^K}{\partial x_k\,\partial x_{k+1}^2}(c_{k+2})\right\}\\ &= -h_k\frac{h_{k+1}^3}{12}\frac{\partial^4 v^K}{\partial x_k^2\,\partial x_{k+1}^2}=0, \qquad x_{k+2}=x_k, \quad k=1,2.\end{aligned}$$

The vanishing of the fourth order derivative is readily seen from the definition of V_ι^K in (M1). By use of the operator P_0 of piecewise constant interpolation and the associated operator $R_0=1-P_0$, the above results entail

$$\int_{\partial K}\psi R_1D_kvN_k\,ds=\int_{\partial K}(R_0\psi)(R_1D_kv)N_k\,ds$$

and, by 2.1.(5), finally the estimate

$$\left|\int_{\partial K}\psi R_1D_kvN_k\,ds\right|\leqq\gamma^2\sqrt{h_K}|\psi|_{1,K}\sqrt{h_K^3}|D_kv|_{2,K}.$$

This implies the inequality (1ii) in the same way as in the proof of theorem 2.2.(2). □

2.4. The elements of Crouzeit–Raviart. We consider this class of nonconforming elements on triangulations $\mathcal{K}_\iota$ of $\bar{G}\subset\mathbb{R}^2$. Each side F of a triangle K is a compact interval $[a, b]$ of a real line. By *Gaussian points* of rth order are meant the zeros of the Legendre polynomials of rth order on the interval $F=[a, b]$. These polynomials are, save for a constant factor, uniquely defined by their orthogonality to the subspace $\mathcal{P}_{r-1}(F)$ in $L^2(F)$. The defining properties of the class of nonconforming elements of Crouzeit–Raviart (see [8]) read:

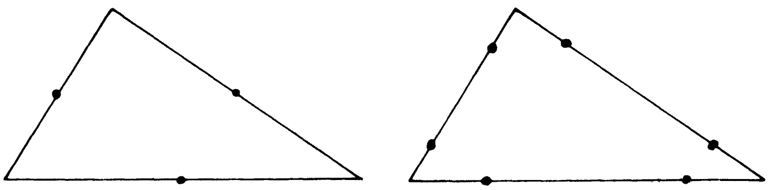
(L) The function values at Gaussian points of rth order on every side F of all triangles $K\in\mathcal{K}_\iota$ are nodal parameters.

(M1) $V_\iota^K\subset\mathcal{P}_t(\mathbb{R}^2)$ and $V_\iota^K|F\subset\mathcal{P}_r(F)$ for each side F of K and each triangle $K\in\mathcal{K}_\iota$;

(M2) functions $v_\iota\in V_\iota$ are continuous at Gaussian points of rth order on inter-element boundaries;

(M3) the function values are zero at Gaussian points on free sides $F\subset\partial G$ in the case of Dirichlet boundary conditions.

The simplest example is obtained for $r=1$ (see Fig. 3a). In this case, the mid-side node is a Gaussian point of first order. The spaces V_ι consist of piecewise linear functions, that is, $V_\iota^K=\mathcal{P}_1(\mathbb{R}^2)$ for all $K\subset\mathcal{K}_\iota$, which are continuous at mid-side nodes. Crouzeit–Raviart [8] study a further example of order $r=3$.



(a) $r=1$ (b) $r=2$

FIG. 3

In the following, $P^F_{r-1}w$ denotes the Lagrange interpolation of continuous functions w on the side F of a triangle $K \in \mathcal{K}_\iota$ at Gaussian points of rth order by a polynomial of $(r-1)$th degree in one variable. The associated remainder term is $R^F_{r-1}w = w - P^F_{r-1}w$.

(1) *In these terms, the inequality*

$$\left|\int_F \psi R^F_{r-1} v\, ds\right| \leqq \gamma^2 h_K^{2r-1} |\psi|_{r,K} |v|_{r,K}, \qquad h_K = \delta_1(K),$$

is true uniformly for all sides F of triangles $K \in \mathcal{K}_\iota$, $\iota = 1, 2, \cdots$, all $\psi \in C_0^\infty(\mathbb{R}^2)$ and all polynomials v with the property $v|F \in \mathcal{P}_r(F)$.

Proof. (i) The quadrature formula, having the Gaussian points of rth order as nodal points at the side F of an element K, is exact for all polynomials of $(2r-1)$th degree. By the above assumption, the restriction of the polynomials v to the side F yields polynomials in $\mathcal{P}_r(F)$. This implies the orthogonality relation

$$\int_F p R^F_{r-1} v\, ds = 0, \qquad p \in \mathcal{P}_{r-1}(F),$$

because the remainder term $R^F_{r-1}v$ vanishes at the Gaussian points of rth order. Thus $P^F_{r-1}v$ is the orthogonal projection of $v|F$ into the subspace $\mathcal{P}_{r-1}(F)$ of $L^2(F)$. Therefore, the relations

$$\|R^F_{r-1}v\|_F = |v, \mathcal{P}_{r-1}(F)|_F, \qquad \int_F \psi R^F_{r-1} v\, ds = \int_F (\psi - p) R^F_{r-1} v\, ds$$

are valid for all polynomials $p \in \mathcal{P}_{r-1}(F)$. In this way, one obtains the estimate

(2) $$\left|\int_F \psi R^F_{r-1} v\, ds\right| \leqq |\psi, \mathcal{P}_{r-1}(F)|_F |v, \mathcal{P}_{r-1}(F)|_F.$$

(ii) Let P^K_{r-1} be the orthogonal projection of $H^r(\underline{K})$ onto the subspace $\mathcal{P}_{r-1}(\underline{K})$. For each $w \in H^r(\underline{K})$ the restriction of $P^K_{r-1}w$ to F is a polynomial in $\mathcal{P}_{r-1}(F)$ so that

$$|w, \mathcal{P}_{r-1}(F)|_F \leqq \|w - P^K_{r-1}w\|_F.$$

By 2.1.(5), one further has the estimate

$$\|w - P^K_{r-1}w\|_F \leqq \gamma h_K^{r-1/2} |w|_{r,K}, \qquad w \in H^r(\underline{K}),$$

uniformly for all $F \subset \partial K$, $K \in \mathcal{K}_\iota$, $\iota = 1, 2, \cdots$. The asserted inequality finally follows from (2) and the above estimates. □

By virtue of lemma (1), we can now prove the theorem:

(3) $$|T_l(\psi, v_\iota)| \leqq 3\beta^{r-1}\gamma^2 h_\iota^r |\psi|_r |v_\iota|_{1,G_\iota}, \qquad l = 1, 2,$$

uniformly for all $v_\iota \in V_\iota$, $\iota = 1, 2, \cdots$, and all $\psi \in C_0^\infty(G)$ ($\psi \in C_0^\infty(\mathbb{R}^2)$ in the case that (M3) *holds). Hence the sequence (V_ι) passes the generalized patch test with $m = 1$ in G (in $\bar{G}$).*

Proof. Let F be a common side of two triangles K, K'. Then $N^K = -N^{K'}$ and $\psi P^F_{r-1}v$ is continuous on F by assumption (M2) and definition of the interpolation $P^F_{r-1}v$ for functions $v \in V_\iota$. When the boundary condition (M3) is assumed, additionally, $\psi P^F_{r-1}v = 0$ on all sides $F \subset \partial G$. The bilinear forms T_l permit the representation

$$T_l(\psi, v) = \sum_{K \in \mathcal{K}_\iota} \sum_{F \subset \partial K} \int_F \psi R^F_{r-1} v N_l\, ds$$

for all $\psi \in C_0^\infty(G)$ ($\psi \in C_0^\infty(\mathbb{R}^2)$) because integrals having integrands $\psi P_{r-1}^F w N_l$ over interelement sides $F = K \cap K'$ cancel pairwise and intergrals over sides $F \subset \partial G$ vanish. From lemma (1) and the inverse property 2.1.(4) it is seen that

$$\left| \int_F \psi R_{r-1}^F v N_l \, ds \right| \leqq \beta^{r-1} \gamma^2 h_K^r |\psi|_{r,K} |v|_{1,K}.$$

This immediately implies the above estimate (3). □

2.5. Morley's element. Morley's element is defined on triangulations of the polyhedral domain $\bar{G} \subset \mathbb{R}^2$ by the following specifications:

(L) Nodal parameters are the function values at the vertices of the triangles and the values of the first derivatives in normal direction at mid-side nodes (see Fig. 4a).

(M1) $V_\iota^K = \mathscr{P}_2(\mathbb{R}^2)$ for every triangle $K \in \mathscr{K}_\iota$;

(M2) functions $v_\iota \in V_\iota$ are continuous at the vertices of the triangles $K \in \mathscr{K}_\iota$ and their first derivatives in normal direction at mid-side nodes are continuous;

(M3) nodal parameters are zero on the boundary ∂G in the case of Dirichlet boundary conditions.

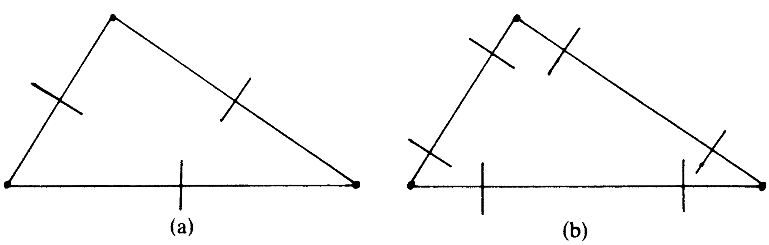


FIG. 4

Under these conditions, one has the statement:

(1) *The inequalities*

$$|T_{l,\mu}(\psi, v_\iota)| \leqq (\alpha\gamma + 3\gamma^2) h_\iota \|\psi\|_1 |v_\iota|_{2,G_\iota}, \qquad l = 1, 2, \quad |\mu| \leqq 1,$$

hold uniformly for all $v_\iota \in V_\iota$, $\iota = 1, 2, \cdots$, and all $\psi \in C_0^\infty(G)$ ($\psi \in C_0^\infty(\mathbb{R}^2)$ in case (M3) *is valid). Hence the sequence (V_ι) passes the generalized patch test with $m = 2$ in G (in $\bar{G}$).*

Proof. (i) For every function $v \in V_\iota$, the piecewise linear interpolation $P_1 v$ at the vertices of the triangles is a continuous function on $\bar{G}$ because the function values of v are continuous at the vertices. In addition, $P_1 v = 0$ on ∂G under the assumption (M3). From theorem 2.1.(2) it is thus seen that

$$T_l(\psi, v) = T_l(\psi, R_1 v)$$

for all $\psi \in C_0^\infty(G)$ ($\psi \in C_0^\infty(\mathbb{R}^2)$). Consequently, using the inequalities 2.1.(3), (5), one obtains the estimates

$$\left| \int_{\partial K} \psi R_1 v N_l \, ds \right|^2 \leqq \int_{\partial K} |\psi|^2 \, ds \int_{\partial K} |R_1 v|^2 \, ds \leqq \frac{\alpha^2}{h_K} \|\psi\|_{1,K}^2 \gamma^2 h_K^3 |v|_{2,K}^2,$$

and so

$$|T_l(\psi, v)| \leqq \sum_{K \in \mathscr{K}_\iota} \left| \int_{\partial K} \psi R_1 v N_l \, ds \right| \leqq \alpha\gamma h_\iota \|\psi\|_1 |v|_{2,G_\iota}.$$

(ii) By assumption (M2), the first derivative $D_N v$ in normal direction is continuous at midside nodes for all $v \in V_\iota$. This implies that the mean values

$$(D_N v)(F) = \frac{1}{|F|} \int_F D_N v \, ds, \qquad |F| = \int_F 1 \, ds,$$

are continuous on common sides $F = K \cap K'$ of two triangles. For, $D_N v^K$ and $D_N v^{K'}$ are polynomials of first degree on F and the midpoint rule is exact. Hence $(D_N v)(F) = (D_N v)(c)$ at the midside node c of F and $(D_N v^K)(c) = (D_N v^{K'})(c)$. Condition (M3) implies analogously that $(D_N v)(F) = 0$ for all sides $F \subset \partial G$. In addition, the mean values $(D_T v)(F)$ over the first derivatives in tangential direction are continuous at interelement sides F. For,

$$(D_T v)(F) = \frac{1}{|F|} \int_F D_T v \, ds = \frac{1}{|F|}(v(b) - v(a)),$$

a, b being the endpoints of the side F. The continuity of v at vertices of the triangles $K \in \mathcal{K}_\iota$ thus entails the continuity of the mean values $(D_T v)(F)$. Assuming (M3), obviously, $(D_T v)(F) = 0$ for all sides $F \subset \partial G$. These properties guarantee further that the mean values over the gradient $\nabla v = (D_1 v, D_2 v)$ are continuous at common sides F of triangles $K \in \mathcal{K}_\iota$ and zero at free sides $F \subset \partial G$.

(iii) For every function $w \in L^2(F)$ let

$$P_0^F w = w(F), \qquad R_0^F w = w - w(F).$$

The continuity (and the vanishing on the boundary) of the mean values of $D_k v$ yields the representation

$$T_{l,k}(\psi, v) = \sum_{K \in \mathcal{K}_\iota} \sum_{F \subset \partial K} \int_F \psi R_0^F (D_k v) N_l \, ds.$$

Obviously, $P_0^F w$ is the orthogonal projection of w onto $\mathcal{P}_0(F)$ in $L^1(F)$ and so

$$\|R_0^F w\|_F = |w, \mathcal{P}_0(F)|_F \leqq \|w - P_0^K w\|_F$$

where $P_0^K w = w(K)$ is the mean value of w over K. From 2.1.(5) for $r = 1$ it follows that

$$\int_F |R_0^F w|^2 \, ds \leqq \gamma^2 h_\iota |w|_{1,K}^2, \qquad w \in H^1(\underline{K}).$$

Finally

$$\int_F \psi R_0^F w \, ds = \int_F R_0^F \psi R_0^F w \, ds$$

and hence

$$\begin{aligned} |T_{l,k}(\psi, v)| &\leqq \gamma^2 h_\iota \sum_{K \in \mathcal{K}_\iota} \sum_{F \subset \partial K} |\psi|_{1,K} |D_k v|_{1,K} \\ &\leqq 3\gamma^2 h_\iota |\psi|_1 |v|_{2,G_\iota}. \end{aligned}$$

□

2.6. De Veubeke's element. We consider the type of element on triangulations which is defined by the following degrees of freedom and piecewise polynomial spaces:

(L) Nodel parameters are the function values at the vertices of the triangles $K \in \mathcal{K}_\iota$ and at the center, and the values of the first derivatives in normal direction at the Gaussian points of second order on each side (see Fig. 4b).

(M1) $V_\iota^K = \mathscr{P}_3(\mathbb{R}^2)$ for every triangle $K \in \mathscr{K}_\iota$;

(M2) the values of nodal parameters of the functions in V_ι are continuous on the common sides of the triangles $K \in \mathscr{K}_\iota$;

(M3) the values of nodal parameters on the boundary ∂G are zero when Dirichlet boundary conditions are imposed.

For these elements we have the following theorem.

(1) $$|T_{l,\mu}(\psi, v_\iota)| \leqq (\alpha\gamma + 3\gamma^2) h_\iota \|\psi\|_1 |v_\iota|_{2,G_\iota}, \qquad l = 1, 2, \quad |\mu| \leqq 1,$$

uniformly for all $v_\iota \in V_\iota$, $\iota = 1, 2, \cdots$, and all test functions $\psi \in C_0^\infty(G)$ ($\psi \in C_0^\infty(\mathbb{R}^2)$ in the case that (M3) *holds). Thus the sequence (V_ι) passes the generalized patch test with $m = 2$ in G (in $\bar{G}$).*

Proof. (i) As in the case of Morley's elements one obtains the estimate

$$|T_l(\psi, v)| \leqq \alpha\gamma h_\iota \|\psi\|_1 |v|_{2,G_\iota}$$

for all $v \in V_\iota$ and $\psi \in C_0^\infty(G)$ ($\psi \in C_0^\infty(\mathbb{R}^2)$).

(ii) In the same way as in the proof of theorem 2.5.(1), one sees next that the mean values of the tangential derivatives $D_T v$ over interelement sides F are continuous. The difference $D_N v^K - D_N v^{K'}$ of the normal derivatives is a polynomial of second degree on $F = K \cap K'$ having its zeros at the Gaussian points of second order. This polynomial is proportional to the second order Legendre polynomial on the interval $F = [a, b]$ so that its mean value over F vanishes. Consequently, the mean values of both $D_T v$ and $D_N v$ and hence of the gradients $\nabla v = (D_1 v, D_2 v)$ are continuous on interelement sides F. On free sides $F \subset \partial G$ the mean values of the gradients vanish when condition (M3) is assumed. As in part (iii) of the proof of theorem 2.5.(1), these properties imply the asserted inequality and hence the success of this sequence (V_ι) in the generalized patch test. □

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