SIAM J. NUMER. ANAL.
Vol. 29, No. 5, pp. 1261-1293, October 1992

ON LOCKING AND ROBUSTNESS IN THE FINITE ELEMENT METHOD*

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Abstract. A numerical scheme for the approximation of a parameter-dependent problem is said to exhibit *locking* if the accuracy of the approximations deteriorates as the parameter tends to a limiting value. A *robust* numerical scheme for the problem is one that is essentially uniformly convergent for all values of the parameter. Precise mathematical definitions for these terms are developed, their quantitative characterization is given, and some general theorems involving locking and robustness are proven. A model problem involving heat transfer is analyzed in detail using this mathematical framework, and various related computational results are described. Applications to some different problems involving locking are presented.

Key words. locking, robustness, finite element method, parameter-dependent, elliptic, partial differential equation

AMS(MOS) subject classifications. 65N30, 35B30, 65N15

1. Introduction. The mathematical formulation of various problems involves dependency on a crucial parameter t arising out of physical considerations. For example, plate and shell models involve the thickness d of the plate or shell, the analysis of elastic materials in general involves the Poisson ratio ν, heat transfer involves the ratio of conductivities μ in different directions, etc.

The numerical approximation of such parameter-dependent problems may suffer when t lies close to a limiting value t_0 (thickness $d \to 0$, Poisson ratio $\nu \to 0.5$, ratio μ of conductivities is very small ($\mu \to 0$), etc.). Most a priori error estimates yield optimal asymptotic convergence rates for $t > t_0$ fixed. Since these estimates are not uniform in t, a degeneration often occurs for values of t close to t_0. This is manifested in actual computations, where the error may not decrease at the predicted rate for t close to t_0 for most practical choices of the discretization parameter. We refer to this phenomenon as *locking*. Various examples of locking have been reported in the engineering and mathematical literature; see for example [1], [6]-[10], [14], and others.

Locking involves the "shifting" of the asymptotic range of the calculations and will eventually disappear when the level of discretization is increased enough, depending upon the strength of the degeneracy. Unfortunately, using brute force may lead to an infeasible level of discretization required before convergence can be observed. Moreover, the prediction of the discretization level for a required error tolerance is now quite complicated.

The most ideal remedy to locking is to employ a method that is *robust*, i.e., one which is more or less uniformly convergent for all t. Various robust methods (particularly mixed methods) have been proposed and analyzed in the context of locking; see for example [7] for the plate problem, [1], [9] for a beam problem, [6], [14] for Poisson ratio locking, among others.

Generally, in the above mentioned papers, locking was addressed in an ad hoc manner. In this paper our primary goal is to develop a systematic mathematical

* Received by the editors June 13, 1990; accepted for publication (in revised form) October 11, 1991.

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approach which will allow precise characterization of locking and robustness of a method as well as a quantitative measure of the locking and robustness strength. In the next section we present the definitions of these concepts.

Our definitions are quite general in nature, so that we can treat various different types of locking phenomena from the same point of view. The examples analyzed in §§ 2–5 (both theoretical and computational) illustrate the need for this flexibility.

In the next section, we also derive various general theorems for a special class of locking problems which are important in the context of applications and have been studied, for example, in [6]. These problems have a well-defined associated limit problem for $t \to t_0$, satisfying certain properties. An example is the case of elasticity, where we get the Stokes' problem in the limit as the Poisson ratio $\nu \to 0.5$. In [6], it was shown that a necessary condition for the absence of locking is the satisfaction of a suitable approximability condition for this limit problem. We show in Theorems 2.2 and 2.3 that this condition is sufficient as well, under certain conditions.

Section 3 contains detailed results on locking and robustness for the model problem introduced in § 2. Section 4 shows how the nonquasi-uniformity of meshes used can be interpreted in terms of locking. In § 5, we present a simplified analysis for the beam problem which was analyzed in [1], [9]. In § 6, we discuss a similar simplified analysis that is possible for the Stokes' problem, detailed results for which (using the theorems from § 2) are presented in [5].

2. A model problem and the definitions. To show the main features of locking, we first discuss an illustrative model elliptic problem which arises in the study of (for example) heat transfer through highly orthotropic materials with the conductivity being very different in two perpendicular directions. Let the conductivities in the x_1 and x_2 directions be $k_1 = 1$ and $k_2 = 1/t$, respectively. We will be particularly interested in the case where t is close to 0.

We assume homogeneous Dirichlet boundary data on Γ_D and Neumann data on Γ_N (with $\Gamma = \Gamma_D \cup \Gamma_N$). The corresponding partial differential equation we study is

$$-\frac{\partial^2 u}{\partial x_1^2} - \frac{1}{t}\frac{\partial^2 u}{\partial x_2^2} = f(x, y) \quad \text{in } \Omega, \tag{2.1}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{2.2a}$$

$$\frac{\partial u}{\partial n_c} = g \quad \text{on } \Gamma_N, \tag{2.2b}$$

where n_c is the usual co-normal.

Let $H_D^1(\Omega) = \{u \in H^1(\Omega) \mid u = 0 \text{ on } \Gamma_D\}$. Then this problem may be put in the variational form: Find $u \in H_D^1(\Omega)$ such that for all $v \in H_D^1(\Omega)$,

$$B_t^1(u, v) = F(v), \tag{2.3}$$

where

$$B_t^1(u, v) = \int_\Omega \left[\left(\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial v}{\partial x_1}\right) + \frac{1}{t}\left(\frac{\partial u}{\partial x_2}\right)\left(\frac{\partial v}{\partial x_2}\right)\right] dx_1\, dx_2 \tag{2.4}$$

and

$$F(v) = \int_\Omega fv\, dx_1\, dx_2 + \int_{\Gamma_N} gv\, ds.$$

Let $\Omega=(-1,1)^2$ for simplicity. We will be particularly interested in the case where $\Gamma_D=\emptyset$, $f=0$, and g is chosen so that the true solution (unique up to a constant) is given by

$$u_t^1=\sin x_1\, e^{-\sqrt{t}\,x_2}. \tag{2.5}$$

As $t\to 0$, it is seen that u_t^1 has a well-defined limit $\sin x_1$ which is constant in the x_2 direction.

We also consider (2.3) with B_t^1 replaced by its rotated form

$$B_t^2(u,v)=\int_\Omega\left[\left(\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_2}\right)\left(\frac{\partial v}{\partial x_1}+\frac{\partial v}{\partial x_2}\right)+\frac{1}{t}\left(\frac{\partial u}{\partial x_2}-\frac{\partial u}{\partial x_1}\right)\left(\frac{\partial v}{\partial x_2}-\frac{\partial v}{\partial x_1}\right)\right]dx_1\,dx_2, \tag{2.6}$$

which corresponds to the case where the directions of the orthotropy do not coincide with the x_1, x_2 axes. By choosing g appropriately, we obtain as the solution of our new variational problem the rotated version of (2.5):

$$u_t^2=\sin(x_1+x_2)\, e^{-\sqrt{t}(x_2-x_1)}, \tag{2.7}$$

which is once again unique up to a constant. As $t\to 0$, u_t^2 becomes constant in the $\eta=x_2-x_1$ direction.

We now approximate the above problems by the finite element method. First we consder the h-version on a uniform square mesh on Ω with 1, 4, and 16 square elements. In Figs. 2.1–2.3 we have plotted the percentage relative error versus the number of degrees of freedom in a log-log scale for $p=1, 2$, and 4. In each case, results are shown for both u_t^1 and u_t^2, for $t=10^{-1}$, and 10^{-6}. The dotted lines show the error in the H^1 norm, while the solid lines represent the error in the energy norm—their behavior is very similar (see Theorem 2.4).

It is observed that for $t=10^{-1}$, both cases lead to the error decreasing at the predicted asymptotic optimal rate. However, when $t=10^{-6}$, only the unrotated case shows the same behavior. The error for the rotated case hardly decreases—this is described as a *locking effect.*

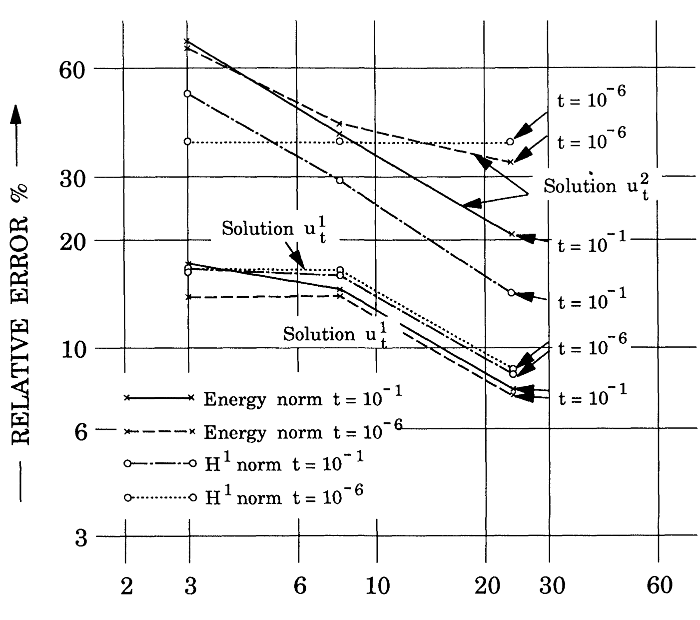

FIG. 2.1. *Relative error in u_t^1 and u_t^2 for the h-version, $p=1$.*

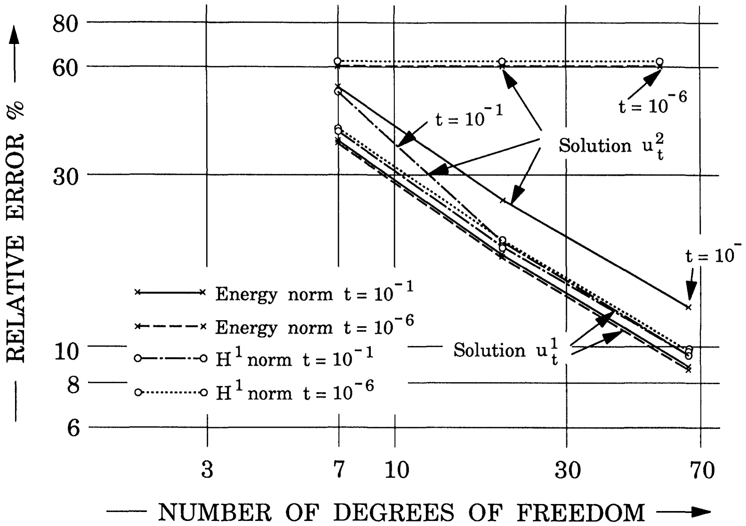

FIG. 2.2. *Relative error in u_t^1 and u_t^2 for the h-version, $p=2$.*

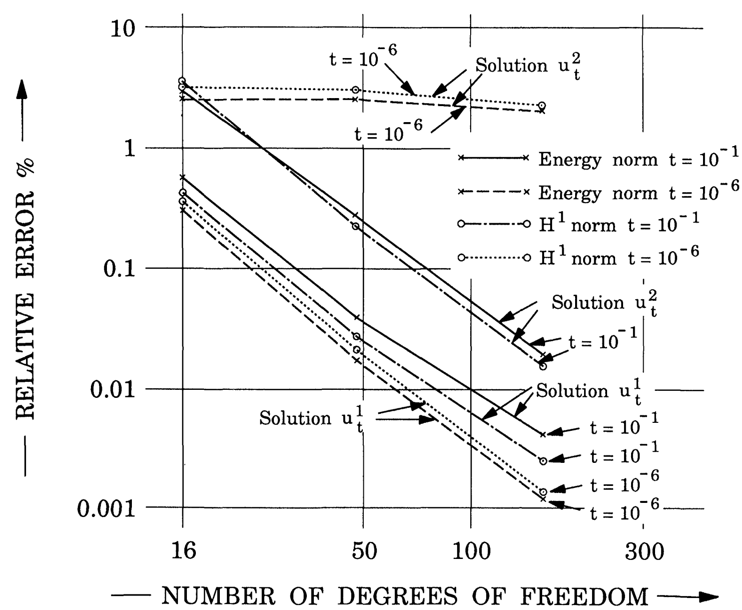

FIG. 2.3. *Relative error in u_t^1 and u_t^2 for the h-version, $p=4$.*

In Figs. 2.4–2.5 the p-version has been used, with $h=2, 1$ (i.e., one and four square elements). Figure 2.4 shows that the p-version with one square is free of locking for both u_t^1 and u_t^2 since the value of t has no appreciable effect on the observed rate of convergence. In Fig. 2.5, it is seen that the error curve for u_t^2 with $t=10^{-6}$ is shifted upwards compared to the other curves, although the slope of this curve remains the same. This suggests that the *order* of convergence remains the same, but the constant in the error estimate is larger for t small. (This "shift" will, of course, occur for the h-version as well, but will only be visible when h is of the same order as t. Hence, in practice it is not observed.)

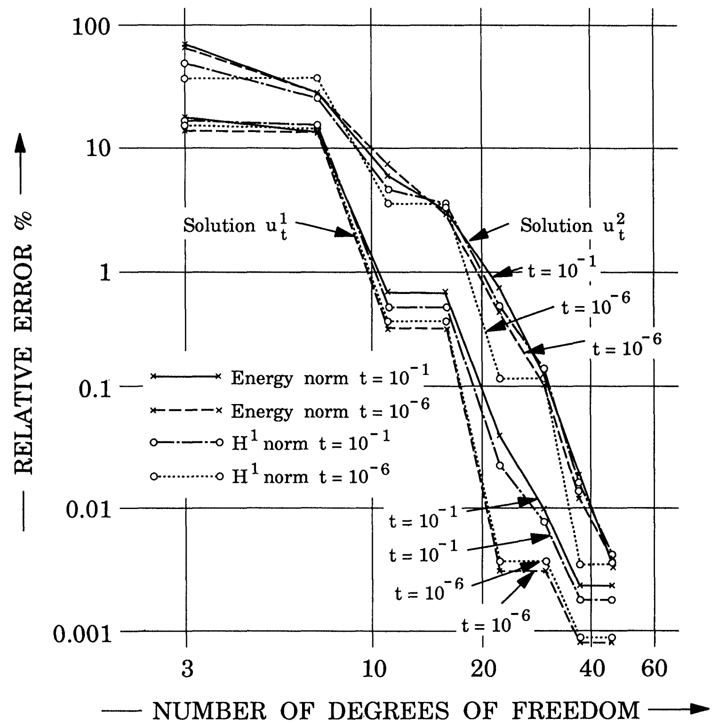

FIG. 2.4. *Relative error in u_t^1 and u_t^2 for the p-version using one square element.*

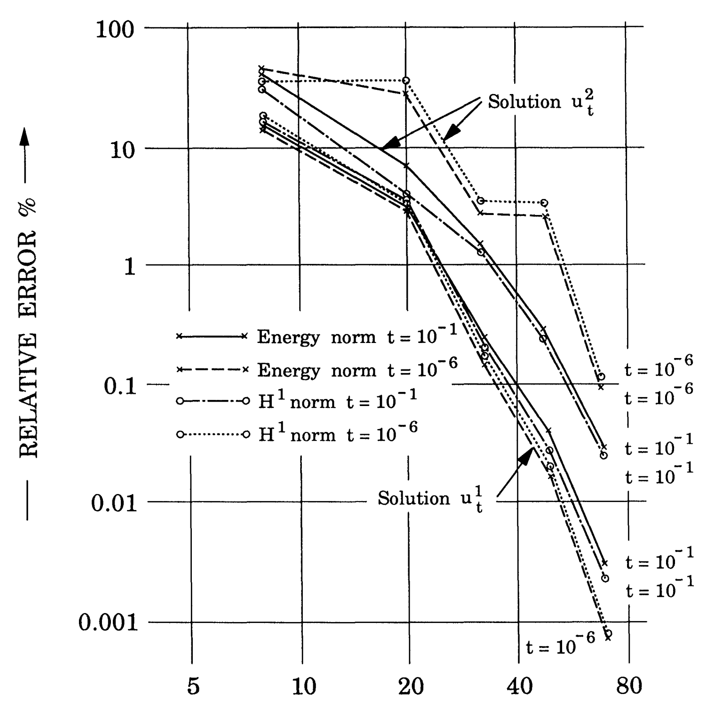

FIG. 2.5. *Relative error in u_t^1 and u_t^2 for the p-version using four elements.*

Figure 2.6 compares the h-version ($p=1,2$) and p-version (one square) for u_t^2 in terms of the $H^1(\Omega)$ norm error. This clearly illustrates the difference in terms of locking. It shows that the p-version is more *robust*.

In § 3, we will prove various results to explain these phenomena. It should be noted that the above numerical results are only valid for the practical range of h and p with which we have worked. If refinement is continued, the asymptotic convergence rate would eventually be observed for any fixed t.

We now develop mathematically precise definitions of locking and robustness.

Assume first that the parameter t is an element of a set S which is taken to be a semiopen interval of the form $(a, b]$ (with the limiting value of t being a). For each t, let $B_t(\cdot,\cdot)$ be a bounded symmetric bilinear form defined over a Hilbert space V such that the energy norm

$$\|u\|_{E,t}=(B_t(u,u))^{1/2} \tag{2.8}$$

satisfies

$$C_1(t)\|u\|_V \leqq \|u\|_{E,t} \leqq C_2(t)\|u\|_V \tag{2.9}$$

where $0<C_1,\ C_2<\infty$, though $C_1(t)/C_2(t)$ could $\to 0$ as $t\to a$. (Here, $\|\cdot\|_V$ represents the usual norm of the quotient space if, for example, we have a pure Neumann problem.)

Let $\mathscr{H}=\{H_t\}$ be a family of linear spaces, $H_t\subset H\subset V$, where H is a compactly imbedded subspace of V. For each H_t, let $\|\cdot\|_t$ denote an associated norm and let $H_t^B=\{u\in H_t,\ \|u\|_t\leqq K\}$ be the ball of radius K, where $\|u\|_H\leqq C\|u\|_t$ (C independent of t). Here, the radius K may represent different values, but it will always be $\leqq K_0$, a

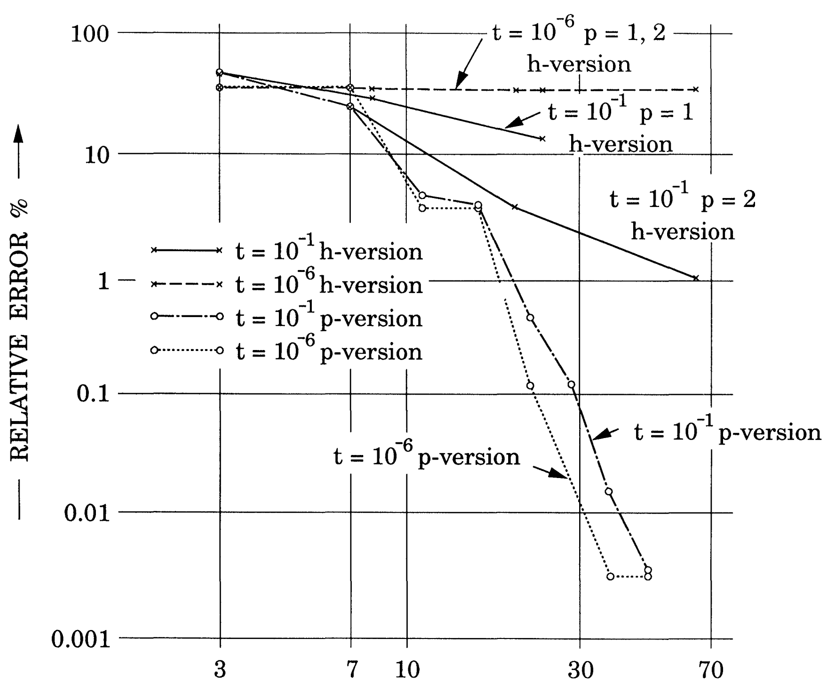

FIG. 2.6. *Comparison of the h- and p-version accuracies for u_t^2.*

constant independent of t. In our examples, H_t may often be identical to H. (Note that by $\mathcal{H}$, we actually mean both $\{H_t\}$ and H.)

We are interested in the approximation of "exact solutions" that lie in H_t. To this end, for each t, let $\mathcal{F}_t = \{V_t^N\}$ be a family of finite-dimensional subspaces of V with the dimension N being independent of t. (We assume that a V_t^N is given for each $N \in \mathcal{N}$, with $\mathcal{N}$ being unbounded.) For $u_t \in H_t$, we consider the sequence of approximations $u_t^N \in V_t^N$ given by

$$B_t(u_t^N, v) = B_t(u_t, v) \quad \forall v \in V_t^N, \tag{2.10}$$

i.e., a sequence of approximations to the *problem* $\mathcal{P}_t$ associated with bilinear form B_t and solution set H_t. The family $\{\mathcal{F}_t\}$ can be identified with an *extension procedure* $\mathcal{F}$, i.e., a rule which (for each t) gives a method regarding how the dimension N of the approximate subspace is increased. It is seen that (2.10) defines a projection of the space H_t into V_t^N. However, the exact form of (2.10) may not allow us to consider certain problems, for example, those involving inhomogeneous Dirichlet boundary conditions. A more general form of (2.10) is therefore

$$u_t^N = B_t^N u_t \tag{2.11}$$

where $B_t^N : H_t \to V_t^N$ is a projection operator. The definitions that follow are equally valid for (2.11) as well.

In case we are using (2.11), we will assume the existence of an energy norm corresponding to (2.8) such that

$$\|u_t - u_t^N\|_{E,t} \leqq C \inf_{v \in V_t^N} \|u_t - v\|_{E,t} \tag{2.12}$$

for all N, t, and C independent of t. Note that otherwise (2.12) follows immediately from (2.10).

Remark. We introduced the set H_t^B with the help of the norm $\|\cdot\|_t$. This characterization is a very special one. We can, instead, define H_t^B to be any *reasonable* set. We will do this in Theorem 3.5.

We denote $R_0 = S \times \mathcal{N}$ and let $R \subset R_0$ be a subset of the pairs (t, N) under consideration.

Our final component is a family $\mathcal{E} = \{E_t\}$ of measures for the error $E_t : V \to \mathbb{R}$. We are interested in the errors $E_t(u_t - u_t^N)$ for problems $\mathcal{P}_t$. Let us mention that some of B_t, H_t, $\mathcal{F}_t$, E_t may be independent of t (and usually are).

For any R, we now define the *locking ratio* $L(t, N) = L(t, N, R, \mathcal{H}, \mathcal{F}, \mathcal{E}, \eta)$ $(0 < \eta \leqq b - a)$ for the problem $\{\mathcal{P}_t\}$, $a < t \leqq b$, by

$$L(t, N) = \frac{\sup_{u_t \in H_t^B} E_t(u_t - u_t^N)}{\inf_{t}{}_{(t,N) \in R_\eta} \sup_{u_t \in H_t^B} E_t(u_t - u_t^N)} \tag{2.13}$$

where $R_\eta = \{(t, N),\ t \geqq a + \eta\} \cap R$ and where we assume that $L(t, N) < \infty$.

Remark. The locking ratio compares the performance of the method at parameter value t to the best possible performance for *reasonable* values of t, which are characterized by $t \geqq a + \eta$. In most applications, the infimum in the denominator over R_η is the same as that over R, so that η can be taken arbitrarily close to zero (and appears in the definition essentially as a technicality). This is because, typically, the error in u_t^N is smallest when the problem is least singular. However, in general, L depends on η.

Let us illustrate this ratio by considering the bilinear forms (2.4), (2.6) with $R = \{10^{-1}, 10^{-2}, 10^{-4}, 10^{-6}\} \times \mathcal{N}$. Let H_t^i $(i = 1, 2)$ be spanned by the single element u_t^i, $i = 1, 2$ (see (2.5) and (2.7)), and let $E_t(u) = \|u\|_{E,t}$. (Here, $\|\cdot\|_t$ is selected so that $\|u_t^i\|_t = 1$.)

We take $\mathcal{F}$ to be the h extension procedure (and let $\eta = 10^{-6}$). Then $L^1(t, N)$ and $L^2(t, N)$ are shown for various p in Tables 2.1, 2.2 using 1, 4, 9, and 16 square elements.

It is observed that for $p=1$, $L^1(t, N)$ remains close to unity. Similar results are observed for higher p for L^1 (not reproduced here). On the other hand, it is seen that for all p, $L^2(t, N)$ seems to be unbounded as $t \to 0$. (For t fixed, the locking ratio is, of course, bounded, but can be very large.)

In Table 2.3 we show $L^2(t, N)$ for the p-version for one and 16 square elements. We see that for one element, L^2 stays bounded while for 16 elements it becomes unbounded.

Let us now present the definition of locking.

TABLE 2.1.
Locking ratio $L^1(t, N)$ and $L^2(t, N)$ for the h-version, $p=1$ ($\eta = 10^{-6}$), and the energy norm.

	t \ N	3	8	15	24
$L^1(t, N)$	10^{-1}	1.21	1.06	1.05	1.05
	10^{-2}	1.02	1.01	1.01	1.01
	10^{-4}	1.00	1.00	1.00	1.00
	10^{-6}	1.00	1.00	1.00	1.00
$L^2(t, N)$	10^{-1}	1.02	1.00	1.00	1.00
	10^{-2}	1.00	1.07	1.23	1.41
	10^{-4}	1.00	1.08	1.29	1.53
	10^{-6}	1.00	1.08	1.29	1.53

TABLE 2.2.
Locking ratio $L^2(t, N)$ for the h-version, $p=2, 3, 4$ ($\eta = 10^{-6}$), and the energy norm.

	t \ N	7	20	39	64
$p=2$	10^{-1}	1.00	1.00	1.00	1.00
	10^{-2}	1.64	2.67	2.92	2.97
	10^{-4}	1.77	4.81	10.06	15.90
	10^{-6}	1.77	4.87	10.67	18.76
	t \ N	11	32	63	104
$p=3$	10^{-1}	1.00	1.00	1.00	1.00
	10^{-2}	1.04	1.46	2.17	2.40
	10^{-4}	1.05	1.79	4.98	10.43
	10^{-6}	1.05	1.79	5.11	11.74
	t \ N	16	48	96	160
$p=4$	10^{-1}	1.15	1.00	1.00	1.00
	10^{-2}	1.01	1.73	1.83	1.50
	10^{-4}	1.00	7.83	15.17	15.50
	10^{-6}	1.00	9.01	43.33	101.56

TABLE 2.3.
Locking ratio $L^2(t, N)$ for the p-version ($\eta = 10^{-6}$) and the energy norm.

# of elem.	t \ N	3	7	11	16	22
1	10^{-1}	1.02	1.00	1.00	1.15	1.35
	10^{-2}	1.00	1.64	1.04	1.01	1.00
	10^{-4}	1.00	1.77	1.05	1.00	1.00
	10^{-6}	1.00	1.77	1.05	1.00	1.00
# of elem.	**t \ N**	**24**	**64**	**104**	**160**	
16	10^{-1}	1.00	1.00	1.00	1.00	
	10^{-2}	1.41	2.97	2.39	1.52	
	10^{-4}	1.53	15.91	10.43	15.56	
	10^{-6}	1.53	18.76	11.74	101.53	

DEFINITION 2.1. For any $0 < \eta \leqq b - a$, the extension procedure $\mathcal{F}$ is *free from locking* over the region R for the family of problems $\{\mathcal{P}_t, (t, N) \in R\}$ with respect to η, the solution sets $\mathcal{H}$, and measures $\mathcal{E}$ if and only if

$$\limsup_{N \to \infty} \left(\sup_{\substack{t \\ (t,N) \in R}} L(t, N, R, \mathcal{H}, \mathcal{F}, \mathcal{E}, \eta) \right) = M < \infty. \tag{2.14}$$

$\mathcal{F}$ shows *locking of order* $f(N)$ (where $\lim_{N \to \infty} f(N) = \infty$) with respect to R, $\mathcal{H}$, $\mathcal{F}$, $\mathcal{E}$, η if and only if for some η,

$$0 < \limsup_{N \to \infty} \left(\sup_{\substack{t \\ (t, N) \in R}} L(t, N, R, \mathcal{H}, \mathcal{F}, \mathcal{E}, \eta) \frac{1}{f(N)} \right) = C < \infty. \tag{2.15}$$

For the case where $C < \infty$ (respectively, $C > 0$), we say that the *order of locking* is *at most* (respectively, *at least*) $f(N)$.

Related to the above definition, we may also define *robustness* as follows.

DEFINITION 2.2. The extension procedure $\mathcal{F}$ is *robust* over the region $R \subset R_0$ for the family of problems $\{\mathcal{P}_t, (t, N) \in R\}$ with respect to solution sets $\mathcal{H}$ and measures $\mathcal{E}$ if and only if

$$\lim_{N \to \infty} \sup_{\substack{t \\ (t,N) \in R}} \sup_{u_t \in H_t^B} E_t(u_t - u_t^N) = 0.$$

It is *robust with uniform order* $g(N)$ if and only if

$$\limsup_{N \to \infty} \left(\left(\sup_{\substack{t \\ (t,N) \in R}} \sup_{u_t \in H_t^B} E_t(u_t - u_t^N) \right) \frac{1}{g(N)} \right) = C < \infty \tag{2.16}$$

where $g(N) \to 0$ as $N \to \infty$.

Definitions 2.1 and 2.2 allow great flexibility in the way various components may depend on t. This is because, as illustrated by the examples in this and the succeeding sections, different formulations require different t dependencies.

The measure E_t in our definition can take various forms, for example, the maximum stresses, stress at a point, etc. The two main choices discussed here will be the V norm and the energy norm.

As $t \to a$, the approximation may deteriorate due to reasons other than locking. For example, the exact solution of the Reissner–Mindlin plate model with uniform load has a boundary layer of increasing strength as $t \to 0$. Hence the deterioration of the approximation is due to the loss in regularity of the solution as well as to locking. The numerical resolution of these two effects is based on different strategies. Our definition permits us to effectively isolate the locking effects by sufficiently restricting the set $\mathcal{H}$ to exclude unsmooth solutions.

By locking, we typically understand the behavior of the extension procedure when t is much smaller than N^{-1} (for $a=0$). Hence, instead of considering the region $R_0=(0,b]\times\mathcal{N}$ (Fig. 2.7(a)), we may be interested in a region of the type $R=\{(t,N), 0<t\leqq b_N\}$ shown in Fig. 2.7(b). This is why we have R explicitly in our definition. For the examples discussed in this paper, however, we are able to prove all our results for the entire region $R=R_0$.

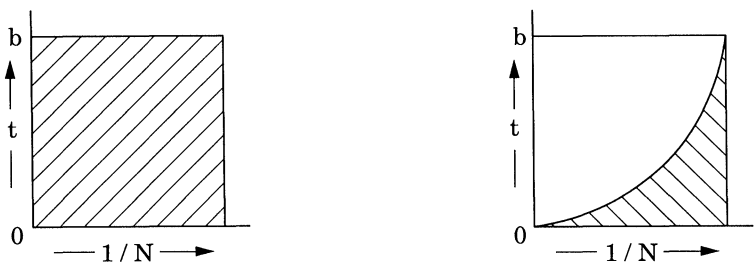

FIG. 2.7(a). *The region* R_0.

FIG. 2.7(b). *The region* R.

We will often need to bound below the denominator in (2.13). In this connection, the following definition of the n-width will be useful. (Here H^B is the ball of radius K in H.)

The n-width of the set H in V with respect to the measure E is defined as

$$d_N(V,H,E)=K^{-1}\inf_M\left\{\sup_{u\in H^B}\inf_{v\in M}E(u-v)\,|\,M\subset V, \dim M=N\right\} \tag{2.17}$$

with M being a linear manifold. When $E(u)$ is given by $\|u\|_V$, we denote this by $d_N(V,H)$.

The choice of the measures $\|\cdot\|_t$ and E_t are not independent if we want to study problems which make sense mathematically. We will restrict our attention to choices of $\mathcal{H}$, $\mathcal{E}$, R such that for any $(t,N)\in R$,

$$0<d_N(V,H_t,E_t)\leqq CF_A(N) \tag{2.18}$$

where $F_A(N)\to 0$ as $N\to\infty$, F_A independent of t, C independent of N and t. (We take H_t^B instead of H^B in (2.17) to define $d_N(V,H_t,E_t)$.)

Condition (2.18) ensures that there are extension procedures for which it is at least *possible* that the robustness can be achieved. Related to (2.18) is the following condition, which ensures that our choice of extension procedure has a chance to avoid locking: For any $(t,N)\in R$,

$$C_1F_B(N)\leqq \sup_{u\in H_t^B}\inf_{v\in V_t^N}E_t(u-v)\leqq \sup_{u\in H^B}\inf_{v\in V_t^N}E_t(u-v)\leqq C_2F_B(N) \tag{2.19}$$

where $F_B(N)\to 0$ as $N\to\infty$, F_B independent of t, C independent of N and t. We say that $\{\mathcal{H},\mathcal{F},\mathcal{E},R\}$ is F_B-admissible if (2.19) is satisfied. Note that (2.19) can be an unduly broad condition to assume; for example, if $E_t = \|\cdot\|_{E,t}$, then (2.19) already implies robustness with uniform order F_B, as well as absence of locking.

We now formulate the following assumption.

Assumption A. There exists $0 < \eta \leqq b-a$ such that for each $(t, N) \in R_\eta$,

$$C_1 F_0(N) \leqq \sup_{u_t \in H_t^B} E_t(u_t - u_t^N) \leqq C_2 F_0(N) \tag{2.20}$$

where $F_0(N) \to 0$ as $N \to \infty$, F_0 independent of t; $C_1, C_2 > 0$ independent of t, N, but in general depending on η (u_t^N is the finite element solution).

Assumption A says that there exists an R_η on which the extension procedure is robust with uniform order $F_0(N)$. Under this assumption, the locking ratio L with respect to η takes a simpler form and Definition 2.1 is equivalent to saying that $\mathscr{F}$ is free from locking with respect to η if and only if

$$\limsup_{N\to\infty} \left(\sup_{\substack{t \\ (t,N)\in R}} \sup_{u_t \in H_t^B} E_t(u_t - u_t^N)(F_0(N))^{-1} \right) = CM < \infty \tag{2.21}$$

(a similar reformulation holds for (2.15)).

In the sequel, we will not specify η from Definition 2.1 explicitly; whenever Assumption A holds, it will be assumed that the statements about locking that follow pertain to *any* η for which Assumption A is satisfied.

The following theorem relates the concepts of locking and robustness and follows immediately from their definitions.

THEOREM 2.1. *Let S, $R_0 = S \times \mathscr{N}$, $R \subset R_0$, $\mathscr{H}$, $\mathscr{E}$, $\mathscr{F}$ be given and let Assumption* A *hold. Then the extension procedure $\mathscr{F}$ is free from locking over the region R if and only if it is robust over the region R with uniform order $F_0(N)$. Moreover, if $f(N)$ is such that*

$$f(N)F_0(N) = g(N) \to 0 \quad \text{as } N \to \infty,$$

then $\mathscr{F}$ shows locking of order $f(N)$ if and only if it is robust with maximum uniform order $g(N)$.

Note that $\mathscr{F}$ is nonrobust if and only if it shows locking of order $(F_0(N))^{-1}$. In this case we say that $\mathscr{F}$ shows *complete locking.*

Let us now discuss a special case of the bilinear form $B_t(\cdot,\cdot)$ in terms of which many locking problems can be formulated. Let V be imbedded in the Hilbert space W, with norm $\|\cdot\|_W$ and inner product $(\cdot,\cdot)$. We assume that, for $t \in S = (0, b]$

$$B_t(u, v) = a(u, v) + \frac{1}{t}(Cu, Cv) \tag{2.22}$$

is a symmetric bilinear form defined on $V \times V$, where $a(\cdot,\cdot)$ is a symmetric bilinear form satisfying

$$C_1\|u\|_V \leqq (a(u,u))^{1/2} \leqq C_2\|u\|_V \tag{2.23}$$

and $C: V \to W$ is a linear form satisfying

$$\|Cv\|_W \leqq C_3\|v\|_V. \tag{2.24}$$

Here C_1, C_2, C_3 are constants independent of t ($a(\cdot,\cdot)$ and C could be made t-dependent as long as (2.23)–(2.24) were satisfied). We have chosen $(a, b] = (0, b]$ for convenience.

Equations (2.22), (2.23), and (2.24) show immediately that for any $t \in (0, b]$, the following version of (2.9) holds:

$$\tilde{C}_1\|u\|_V \leqq \|u\|_{E,t} \leqq \tilde{C}_2 t^{-1/2}\|u\|_V. \tag{2.25}$$

We will give examples of problems that satisfy the form (2.22) in §§ 3, 5, and 6. For a number of such problems and the set $\mathscr{H}=\{H_t\}$, $H_t \subset H \subset V$, it is useful to define $\lim_{t\to 0} H_t = H_0 \subset H \subset V$. We define it as the set of all $u_0 \in H$ such that there exists a sequence $\{u_t\}$, $u_t \in H_t$, $t \in S$, and a constant C independent of t such that

$$\|u_t\|_{E,t} \leqq C\|u_0\|_H, \qquad \|u_t\|_t \leqq C\|u_0\|_H \tag{2.26}$$

$$\|u_t - u_0\|_V \to 0 \quad \text{as } t \to 0. \tag{2.27}$$

Assume now that $u_0 \in H_0$, then by definition there are $u_t \in H_t$ such that (2.26) and (2.27) hold. Hence

$$\frac{1}{t}\|Cu_t\|_W^2 \leqq C$$

so that

$$\|Cu_t\|_W = O(t^{1/2}) \tag{2.28}$$

from which it follows that for any $u_0 \in H_0$ we have

$$Cu_0 = 0. \tag{2.29}$$

Define

$$H_0^B = H_0 \cap H^B. \tag{2.30}$$

In many applications, the following condition holds.

CONDITION (α). *For any $u_t \in H_t^B$, there is a $u_0 \in H_0^B$ (u_0 depending on u_t) such that*

$$\|u_t - u_0\|_H \leqq Ct^{1/2} \tag{2.31}$$

with C independent of t and u_t.

As we will see, Condition (α) is important in the context of various theorems.

We now prove some general theorems for the form (2.22) that hold when the spaces V_t^N are assumed independent of t.

THEOREM 2.2. (A) *Let $\mathscr{H}=\{H_t\}$, $H_t \subset H \subset V$, $\mathscr{F}=\{\mathscr{F}_t\}$, $\mathscr{F}_t=\{V_t^N\}$, $V_t^N = V^N$, $\mathscr{E}=\{E_t\}$, $E_t(u)=\|u\|_V$, $S=(0,b]$, $R=\{(t,N)\,|\,0<t\leqq b_N\}\subset R_0$, and let $\lim_{t\to\infty} H_t = H_0 \neq \emptyset$. Assume further that Assumption* A *holds. Then the extension process $\mathscr{F}$ is free from locking over R only if*

$$g(N) = \sup_{u\in H_0^B} \inf_{\substack{w \in V^N\\ Cw=0}} \|u-w\|_V \leqq CF_0(N). \tag{2.32}$$

More generally, it shows locking of order at most $f(N)$ only if

$$g(N) \leqq CF_0(N)f(N). \tag{2.33}$$

Moreover, if

$$g(N) \geqq CF_0(N)f(N) \tag{2.34}$$

then the extension process shows locking of at least order $f(N)$.

(B) *Suppose, in addition, that Condition (α) holds and that $\{\mathscr{H}, \mathscr{F}, \mathscr{E}, R\}$ is F_0-admissible. Then $\mathscr{F}$ is free from locking over R if and only if* (2.32) *holds. It shows locking of order $f(N)$ if and only if*

$$C_1F_0(N)f(N) \leqq g(N) \leqq C_2F_0(N)f(N). \tag{2.35}$$

Proof. Suppose there is no locking (i.e., $\mathcal{F}$ is free of locking) over R. Let N be fixed. Then, using Assumption A, by (2.21), for any (t, N) we have for $u_t \in H_t^B$,

$$\|u_t - u_t^N\|_V \leqq CF_0(N).$$

Now let $u_0 \in H_0^B$ and $u_t \in H_t$ be such that (2.26) and (2.27) hold. Then $u_t \in H_t^B$ by (2.26). Because V^N is finite dimensional, there exists a subsequence of $\{u_t^N\}$ such that $u_{t_i}^N \to u_0^N \in V^N$ in V and for t small enough,

$$\|u_0 - u_0^N\|_V \leqq \|u_0 - u_t\|_V + \|u_t - u_t^N\|_V + \|u_t^N - u_0^N\|_V \leqq CF_0(N)$$

with $Cu_0 = Cu_0^N = 0$. ($Cu_0^N = 0$ because by (2.10), u_t^N also satisfies (2.28).) Hence (2.32) holds. Equation (2.33) follows similarly by replacing $F_0(N)$ by $F_0(N)f(N)$.

Next suppose that the locking is of order $o(f(N))$. Then by (2.33),

$$g(N) = o(F_0(N)f(N)),$$

which contradicts (2.34). Hence the locking is of at least order $f(N)$.

Suppose now that Condition (α) holds. We have to show in addition that if (2.32) holds there is no locking. So for any $u_t \in H_t^B$, let $u_0 \in H_0^B$ be such that (2.31) holds. Then we may write

$$u_t = u_0 + (u_t - u_0) = u_0 + \chi_t, \tag{2.36}$$

where $\chi_t \in H$ satisfies

$$\|\chi_t\|_H \leqq Ct^{1/2}. \tag{2.37}$$

Hence we have

$$\begin{aligned}
\|u_t - u_t^N\|_{E,t} &\leqq C \inf_{v \in V^N} \|u_t - v\|_{E,t} \\
&\leqq C \inf_{v \in V^N} \|u_0 + \chi_t - v\|_{E,t} \\
&\leqq C \left\{ \inf_{\substack{v_1 \in V^N \\ Cv_1 = 0}} \|u_0 - v_1\|_{E,t} + \inf_{v_2 \in V^N} \|\chi_t - v_2\|_{E,t} \right\} \\
&\leqq C \left\{ \inf_{\substack{v_1 \in V^N \\ Cv_1 = 0}} \|u_0 - v_1\|_V + \inf_{v_2 \in V^N} \|\chi_t - v_2\|_{E,t} \right\}
\end{aligned}$$

using the fact that $u_0 \in H_0^B$ and hence $Cu_0 = 0$. This gives, by (2.32) and (2.25),

$$\leqq C\{F_0(N) + t^{-1/2} \inf_{v_2 \in V^N} \|\chi_t - v_2\|_V\}.$$

Since χ_t satisfies (2.37), we use the F_0-admissibility (2.19) to get

$$\leqq C\{F_0(N) + t^{-1/2} t^{1/2} F_0(N)\} \leqq CF_0(N).$$

Then we see that

$$\|u_t - u_t^N\|_V \leqq C\|u_t - u_t^N\|_{E,t} \leqq CF_0(N)$$

so that on R, $L(t, N)$ remains bounded, i.e., there is no locking in the V norm.

Now suppose that (2.35) holds. Then we may replace $F_0(N)$ in the above by $F_0(N)f(N)$ so that

$$\|u_t - u_t^N\|_V \leqq C\|u_t - u_t^N\|_{E,t} \leqq CF_0(N)f(N)$$

which shows locking of order *at most* $f(N)$. Also, since (2.34) is satisfied, the locking is of order *at least* $f(N)$, i.e., of order $f(N)$.

Conversely, if the locking is of order $f(N)$, then (2.33) holds. If $g(N)=o(F_0(N)f(N))$, then, just like before, we establish that

$$\|u_t-u_t^N\|_V=o(F_0(N)f(N))$$

so that the locking is of order at most $o(f(N))$, a contradiction. □

Remark. Note that in the above, if $\{\mathscr{H},\mathscr{F},\mathscr{E},R\}$ is F_0-admissible, then Assumption A is satisfied with the same F_0 because for $t\geqq\eta$, the energy norm $\|\cdot\|_{E,t}$ is equivalent to the norm $\|\cdot\|_V$.

We have actually proved a stronger result, namely, the following theorem.

THEOREM 2.3. (A) *Let the conditions of Theorem* 2.2 (A) *hold. Then* $\mathscr{F}$ *is free from locking with respect to the energy norm over R only if* (2.32) *holds. It shows locking of order* $f(N)$ *only if* (2.33) *holds. It shows locking of order at least* $f(N)$ *if* (2.34) *holds.*

(B) *Let the conditions of Theorem* 2.2 (B) *hold as well. Then* $\mathscr{F}$ *is free from locking over R if and only if* (2.32) *holds. It shows locking of order* $f(N)$ *if and only if* (2.35) *holds.*

Proof. Assuming that Assumption A holds with $E_t(u)=\|u\|_V$ implies that it is true with $E_t(u)=\|u\|_{E,t}$ as well. Hence, in both cases we can use the same $F_0(N)$. The rest of the proof is the same. □

Theorems 2.2 and 2.3 show that under certain circumstances, we need only check condition (2.32) (or (2.35)) and admissibility to determine whether or not locking exists.

The energy norm and the V norm are two important error measures. The relation between them in terms of locking is expressed in the following theorem.

THEOREM 2.4. (A) *Let the assumptions of Theorem* 2.2(A) *hold. Then* $\mathscr{F}$ *is free from locking over the region* $R\subset R_0$ *with respect to the V norm if it is free from locking with respect to the energy norm. It shows locking of order* $f(N)$ *in the V norm only if it shows locking of at least order* $f(N)$ *in the energy norm.*

(B) *Moreover, if, in addition, the conditions of Theorem* 2.2 (B) *hold, then* $\mathscr{F}$ *is free from locking with respect to the V norm if and only if it is free with respect to the energy norm. It shows locking of order* $f(N)$ *in the V norm if and only if it shows locking of order* $f(N)$ *in the energy norm.*

Proof. Part (A) follows immediately using the fact that (2.20) holds with the same $F_0(N)$ for both $\|\cdot\|_V$ and $\|\cdot\|_{E,t}$. Part (B) follows by using Theorems 2.2(B) and 2.3(B). □

3. Locking and robustness results for the model problem. In this section we state and prove some theoretical results for the model problem that explain the computational observations presented in the previous section.

We first note that we may write for $t\in(0,\beta]$ $(\beta<1)$

$$B_t^1(u,v)=\int_\Omega\left[\left(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1}+\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right)+\left(\frac{1}{t}-1\right)\left(\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\right)\right]dx_1\,dx_2, \tag{3.1}$$

which is of the form (2.22) with

$$a(u,v)=\int_\Omega\nabla u\cdot\nabla v\,dx_1\,dx_2,\qquad Cu=\frac{\partial u}{\partial x_2}.$$

Replacing $(t/1-t)$ by t, we have

$$t\in(0,\beta/1-\beta]=(0,b].$$

Clearly (2.23)-(2.24) are satisfied. (Here, $V=H^1(\Omega)$ and $W=L_2(\Omega)$.)

Instead of analyzing the forms B_t^1 and B_t^2 on the same meshes as in § 2, we will now deal with the analogous problem of considering only B_t^1 on unrotated and rotated meshes (Fig. 3.1). This will simplify some notation. Although the problems analyzed now are not totally identical with the ones addressed in § 2, they show the same features and are essentially equivalent.

Let $S \subseteq \mathbb{R}^2$ be a triangle or parallelogram and let $\mathcal{P}_p(S)$ denote the set of all polynomials of total degree (degree in each variable) $\leqq p$ if S is a triangle (parallelogram). We will talk about spaces V^N of continuous piecewise polynomials of degree $\leqq p$ on the meshes in Fig. 3.1 satisfying $u|_S \in \mathcal{P}_p(S)$ for any S in the mesh, for any $u \in V^N$.

We will also find it convenient to work with a suitably scaled version of (3.1), given by the change of variables

$$\hat{x}_1 = x_1, \quad \hat{x}_2 = x_2\sqrt{t}, \quad \hat{u}(\hat{x}_1, \hat{x}_2) = u(x_1, x_2). \tag{3.2}$$

Under the transformation (3.2) we see that (3.1) may be written as

$$B_t(u, v) = \frac{1}{\sqrt{t}} \int_{\Omega_t} \nabla \hat{u} \cdot \nabla \hat{v} \, d\hat{x}_1 \, d\hat{x}_2$$

with $\Omega_t = \{(\hat{x}_1, \hat{x}_2) | \; |\hat{x}_1| \leqq 1, |\hat{x}_2| \leqq \sqrt{t}\}$ so that we just get Laplace's equation over a very thin domain. The corresponding meshes from Fig. 3.1 now appear as shown in Fig. 3.2.

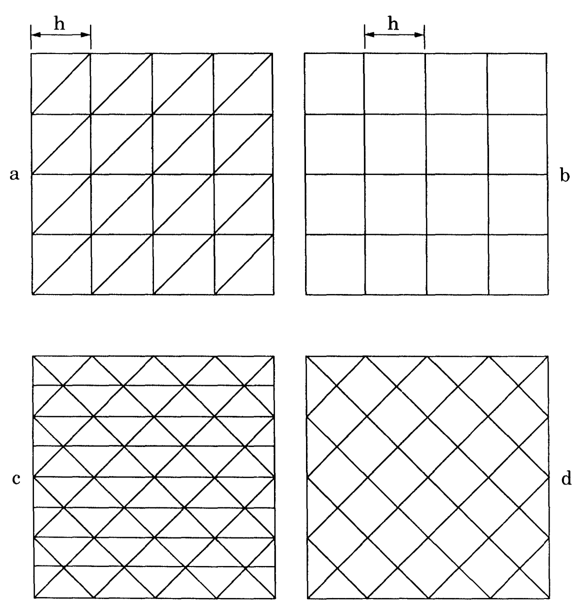

FIG. 3.1. *Unrotated and rotated meshes.* (a) *Unrotated triangular mesh.* (b) *Unrotated rectangular mesh.* (c) *Rotated triangular mesh.* (d) *Rotated rectangular mesh.*

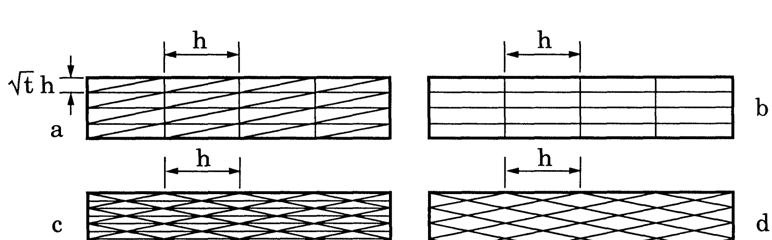

FIG. 3.2. *Scaled unrotated and rotated meshes.* (a) *Scaled unrotated triangular mesh.* (b) *Scaled unrotated rectangular mesh.* (c) *Scaled rotated triangular mesh.* (d) *Scaled rotated rectangular mesh.*

The above families of uniform meshes are all examples of more general *quasi-uniform* meshes. We will refer to triangular meshes satisfying the *minimum* (respectively, *maximum*) angle condition if the minimal angle of any triangle in the mesh is larger than ε (respectively, maximal angle is smaller than $\pi-\varepsilon$), where ε is fixed for the family of meshes (see [2]). Analogously, we will say that a parallelogram mesh satisfies the *maximum angle condition* if the maximum angle of any parallelogram in the mesh is smaller than $\pi-\varepsilon$.

Let us now look at the scaled triangular meshes. We observe that as $t\to 0$, the minimum angle condition is violated for both (a) and (c). However, we observe a clear difference in terms of the *maximum* angle condition. Mesh (a) satisfies the maximum angle condition *uniformly* with respect to t as $h\to 0$, while mesh (c) violates this condition as $t\to 0$. Similarly, the maximum angle condition for the rectangular mesh (b) holds uniformly as $t\to 0$ but not for the rotated (parallelogram) mesh (d). The reason that meshes (c) and (d) lead to locking is essentially due to the violation of the maximum angle condition as $t\to 0$.

Let us define the following weighted norms over Ω in terms of the corresponding norms over Ω_t

$$\|u\|_{H_t^k(\Omega)}=\frac{1}{t^{1/4}}\|\hat{u}\|_{H^k(\Omega_t)}. \tag{3.3}$$

Note that $|u|_{H_t^1(\Omega)}=\|u\|_{E,t}$, the energy norm.

We will need the following lemma about n-widths.

LEMMA 3.1. *Let $\Omega\subset\mathbb{R}^i$, $i=1,2$, be a polygonal domain (for $i=1$, Ω is an interval). Then*

$$d_N(H^1(\Omega),H^{k+1}(\Omega))\approx N^{-k/i},\qquad k\geqq 1, \tag{3.4}$$

where the equivalency constants depend on k and Ω but are independent of N.

Proof. For the proof, see [12].

LEMMA 3.2. *Let $\{V^N\}\subset H^1(\Omega)$ be subspaces of piecewise polynomials of fixed degree p on quasi-uniform triangular or rectangular meshes which satisfy the uniform maximum angle condition. Then*

$$\sup_{\substack{u\in H^{k+1}(\Omega)\\ \|u\|_{H^{k+1}(\Omega)}\leqq 1}}\ \inf_{v\in V^N}\|u-v\|_{H^1(\Omega)}\geqq CN^{-\min(p,k)/2} \tag{3.5}$$

where C is independent of N, but depends on p, k, Ω and the characterization of the mesh (quasi-uniformity constant and maximum angle condition).

Proof. For $k\leqq p$ (3.5) follows from Lemma 3.1. Let $k>p$. In (3.5) we assumed that $V^N\subset H^1(\Omega)$, i.e., V^N is the space of conforming piecewise polynomials of degree p. Let us denote $\tilde{V}^{\tilde{N}}\subset L_2(\Omega)$ to be the space of piecewise polynomials of degree p without continuity conditions between elements. Then $C\tilde{N}\leqq N\leqq\tilde{N}$ and

$$\Phi(\tilde{N})=\sup_{\substack{u\in H^{k+1}(\Omega)\\ \|u\|_{H^{k+1}(\Omega)}\leqq 1}}\ \inf_{v\in\tilde{V}^{\tilde{N}}}\left(\sum_{\mathcal{T}}\|u-v\|^2_{H^1(\mathcal{T})}\right)^{1/2}\leqq\sup_{\substack{u\in H^{k+1}(\Omega)\\ \|u\|_{H^{k+1}(\Omega)}\leqq 1}}\ \inf_{v\in V^N}\|u-v\|_{H^1(\Omega)}$$

where the sum above is over all single elements $\mathcal{T}$ of the mesh.

Hence it is sufficient to prove

$$\Phi(\tilde{N})\geqq CN^{-p/2}.$$

Obviously also, $N=N(h)\approx h^{-2}\approx\tilde{N}$. Now consider a polynomial u of degree $p+1$ (total (triangles) or separate degrees (rectangles)). Then $u\in H^{k+1}(\Omega)$. Hence if (3.5) is not true then

$$\Phi(\tilde{N}) \leqq f(h)h^p \tag{3.6}$$

where $f(h) \to 0$ as $h \to 0$. Since the mesh is quasi-uniform with uniform angle condition, the following inverse inequality holds on every element $\mathcal{T}$ for any polynomial of degree at most $p+1$

$$\|v\|_{H^s(\mathcal{T})} \leqq Ch^{t-s}\|v\|_{H^t(\mathcal{T})}, \qquad 0 \leqq t \leqq s$$

where C is independent of $\mathcal{T}$, t, s, h. Hence if (3.6) is satisfied then we have, for $v_h \in \tilde{V}^{\tilde{N}}$,

$$\left(\sum_{\mathcal{T}} \|u - v_h\|^2_{H^{p+1}(\mathcal{T})}\right)^{1/2} \leqq Cf(h)$$

assuming that $\|u\|_{H^{p+1}(\Omega)} = 1$. But since v_h is a polynomial of degree p, $\partial^{p+1}v_h/\partial x_i^{p+1} = 0$, $i = 1, 2$ and hence

$$\left\|\frac{\partial^{p+1}u}{\partial x_i^{p+1}}\right\|_{L_2(\Omega)} \to 0 \quad \text{as } h \to 0,$$

which is a contradiction. □

Remark. Note that for any fixed t, $0 < t \leqq b$, equation (3.5) will hold with $H^{k+1}(\Omega)$ replaced by $H_t^{k+1}(\Omega)$ as well.

We now consider the case of the h-version on unrotated meshes followed by the analysis of the rotated meshes. For simplicity we will assume that either $\Gamma_N = \emptyset$ or $\Gamma_D = \emptyset$. We have not explicitly mentioned the value of η from Definition 2.1 in the theorems in this section. They will hold for any fixed $0 < \eta \leqq b$.

THEOREM 3.1. *Consider the problem* (2.10) *with B_t given by* (3.1). *Let V_t^N be the set of piecewise polynomials of degree $\leqq p$ ($p \geqq 1$) on an unrotated mesh of the form shown in Fig.* 3.1(a), (b), *with mesh spacing $h = h(N)$. Let H_t be the space of functions given by*

$$H_t = \{u \mid \|u\|_{H_t^{k+1}(\Omega)} < \infty\}$$

with $k \geqq p$ and let $H = H^{k+1}(\Omega) \subset V = H^1(\Omega)$. Let the error measure $E_t(u) = \|u\|_{E,t} = \|u\|_{H_t^1(\Omega)}$. Then the extension procedure $\mathcal{F}$ is free from locking over the region *$R_0 = (0, b] \times \mathcal{N}$ and is robust with uniform order $g(N) = O(N^{-p/2})$ over this region.*

Proof. Let $u \in H_t$. Then $\hat{u} \in H^{k+1}(\Omega_t)$ with $k \geqq p$. Consider first the case where piecewise linear functions are used on a triangular mesh. Then the proof of the sufficiency of the maximum angle condition [2] easily gives

$$\inf_{\hat{v} \in \hat{V}_t^N} \|\hat{u} - \hat{v}\|_{H^1(\Omega_t)} \leqq Ch\|\hat{u}\|_{H^2(\Omega_t)} \tag{3.7}$$

where $\hat{V}_t^N$ is the image of V_t^N and (due to the mesh satisfying the maximum angle condition) C is independent of h and t.

Translating back to Ω, we obtain

$$\inf_{v \in V_t^N} \|u - v\|_{E,t} \leqq Ch\|u\|_{H_t^2(\Omega)}.$$

Now by (2.12) we have (since $h^{-2} \approx N$)

$$\|u - u^N\|_{E,t} \leqq Ch\|u\|_{H_t^2(\Omega)} \leqq CN^{-1/2}\|u\|_{H_t^k(\Omega)} \tag{3.8}$$

for all t. We now take a fixed $0 < \eta \leqq b$ in (2.13) which gives, using Lemma 3.2,

$$L(t, N) = \sup_{\substack{u \in H_t^2 \\ \|u\|_{H_t^2} \leqq 1}} \|u - u^N\|_{E,t}N^{1/2} \leqq C.$$

Hence, Definition 2.1 shows that $\mathcal{F}$ is free from locking. The robustness follows easily from (3.8).

For the case of piecewise polynomials of degree $p>1$, the sufficiency of the maximum angle condition may once again be established using the arguments in [2]. This leads to an analog of (3.8) of the form

$$\|u-u^N\|_{E,t} \leqq Ch^p \|u\|_{H_t^{p+1}(\Omega)} \leqq CN^{-p/2}\|u\|_{H_t^{k+1}(\Omega)}.$$

Using Lemma 3.2 again, we establish the required results. The rectangular case follows similarly. □

Remark. The above theorem will hold for more general quasi-uniform meshes as well. It explains the results in Figs. 2.1–2.3 for the unrotated case.

Note that Theorem 2.3 shows immediately that with the same choice of H_t, there is no locking (and we obtain uniform robustness) in the V (i.e., H^1) norm as well.

Another interesting question is to examine locking in the V norm when, instead of $H_t^{k+1}(\Omega)$, H_t is taken to be an unweighted space. For this case, we expect once again that there will be no locking. We prove this assertion for the case $p=1$, using the equivalence between finite difference and finite element schemes.

THEOREM 3.2. *Consider the problem* (2.10) *with B_t given by* (3.1) *to be the approximate problem corresponding to* (2.1)–(2.2) *with $\Gamma_N=0$. Assume that $\mathscr{F}=\{\mathscr{F}_t\}$ where $\mathscr{F}_t=\{V^N\}$ is the space of continuous, piecewise linear functions on the triangular mesh shown in Fig.* 3.1(a). *Let $H_t=C^4(\bar{\Omega})=H$ (independently of t) and $V=H^1(\Omega)$. Let the error measure be $E_t(u)=\|u\|_{H^1(\Omega)}$. Then $\mathscr{F}$ is free of locking over $R_0=(0,b]\times\mathcal{N}$ and is robust with uniform order $O(N^{-1/2})$ there.*

Proof. Consider the mesh shown in Fig. 3.1(a), with Ω being given by $(-1,1)^2$ and the nodes being labeled A_{ij}, $i=0,1,\cdots,M$, $j=0,1,\cdots,M$, where $M=2/h$ (A_{00} corresponding to the node with coordinates $(-1,-1)$).

Let u_t, the exact solution, and u_t^N, the finite element solution at A_{ij}, be denoted by u_{ij}, u_{ij}^N, respectively. Then by the usual relation between finite element and finite difference solutions for a regular triangular mesh, we have for i,j such that $A_{ij}\in\Omega$ with $u_{ij}^N=0$ when $A_{ij}\in\Gamma$,

$$-\Delta_t^{ij}u_t^N=t(-u_{i-1,j}^N+2u_{i,j}^N-u_{i+1,j})+(-u_{i,j-1}^N+2u_{i,j}^N-u_{i,j+1}^N)=th^2\bar{f}_{ij}, \tag{3.9}$$

where

$$\bar{f}_{ij}=\frac{1}{h^2}\int_\Omega f\phi_{ij}\,dx_1\,dx_2.$$

Here, ϕ_{ij} is the standard "hat function" at A_{ij}. We are given $u\in C^{(4)}(\bar{\Omega})$, i.e.,

$$\|u\|_{C^{(4)}(\bar{\Omega})}\leqq K.$$

Using this, we may show

$$\bar{f}_{ij}=f_{ij}+B_{ij}h^2$$

where $f_{ij}=f(A_{ij})$ and B_{ij} satisfies $|B_{ij}|\leqq CK$ with C independent of u, h, and t. Moreover, for the exact solution we have

$$-\Delta_t^{ij}u_t=h^2(tf_{ij}+D_{ij}h^2) \tag{3.10}$$

where again $|D_{ij}|\leqq CK$. Using (3.9)–(3.10), we see that with $\chi=u_t^N-u_t$, we have $\chi_{ij}=0$ on Γ and

$$-\Delta_t^{i,j}\chi=h^4H_{ij} \tag{3.11}$$

with $|H_{ij}|\leqq CK$. Let

$$\eta(x,y)=1-y^2$$

so that

$$\eta_{ij} = (jh)(2-jh) \geqq 0.$$

Then $\eta_{ij} = 0$ for $j = 0, M$, and for A_{ij} in the interior of Ω,

$$-\Delta_t^{ij}\eta = 2h^2.$$

Let $\mathcal{K} = Ch^2K\eta - \chi$. Then

$$-\Delta_t^{ij}\mathcal{K} = 2Ch^4K - h^4H_{ij} \geqq Ch^4K > 0. \tag{3.12}$$

Also, $\mathcal{K}_{ij} \geqq 0$ on the boundary Γ of Ω. Suppose $\mathcal{K}_{ij} < 0$ in Ω for some $A_{ij} \notin \Gamma$. Then there exists a minimum value of $\mathcal{K}_{ij} < 0$ in Ω. At such a value, for any t,

$$-\Delta_t^{ij}\mathcal{K} \leqq 0,$$

contradicting (3.12). Hence, $\mathcal{K}_{ij} \geqq 0$ on $\bar{\Omega}$, i.e.,

$$\chi_{ij} \leqq CKh^2\eta_{ij}.$$

Similarly, taking $\xi = Ch^2K\eta + \chi$, we see that

$$\chi_{ij} \geqq -CKh^2\eta_{ij}.$$

Noting that $|\eta_{ij}| \leqq 1$, we have

$$|\chi_{ij}| \leqq CKh^2.$$

This shows that the difference between the interpolant of u_t and the finite element solution at the nodal points is $O(h^2)$, from which

$$\|u_t - u_t^N\|_{H^1(\Omega)} \leqq Ch = O(N^{-1/2}). \tag{3.13}$$

Using Lemma 3.2 we see that we have the same rate as (3.5) with $p = 1$. Since (3.13) holds for all t, the theorem follows. $\square$

Remark. In Theorem 3.2 we have used some assumptions only for the sake of simplicity. For example, we assumed $\Gamma = \Gamma_D$, which can be relaxed. Further, we assumed $H_t = C^{(4)}(\bar{\Omega})$ and triangular meshes, which can also be relaxed.

We could of course consider the case where $H_t = H^k(\Omega) = H$ and $E_t(u) = \|u\|_{E,t}$. To see why this combination is not appropriate for analysis, consider functions on Ω that are functions of the variable x_2 alone, i.e., $\Phi \in H^k(I)$. For such a function, by Lemma 3.1,

$$d_N(H^1(I), H^k(I)) \approx N^{-(k-1)}.$$

Now considering these functions as functions of both x_1 and x_2, we have $\Phi \in H^k(\Omega)$ so that we get from the above

$$d_N(H^1(\Omega), H^k(\Omega), E_t) \geqq \frac{1}{\sqrt{t}} CN^{-(k-1)}.$$

Hence, since $d_N(H^1(\Omega), H^k(\Omega), E_t)$ does not converge to zero uniformly in t as $N \to \infty$, condition (2.18) is violated. Therefore, locking will exist (albeit in a trivial way) for any choice of approximation.

Let us now consider the effect of using rotated meshes. Figures 2.1–2.3 indicate that there is locking both in the V norm and the energy norm. To explain this, let us first prove the following lemma.

LEMMA 3.3. *Let the extension process $\mathscr{F}$ be based on the h-version using a continuous piecewise polynomial of fixed degree $\leqq p$ on a rotated mesh* (*Figs.* 3.1(c), (d)). *Let u be a function defined on Ω such that $Cu = \partial u/\partial x_2 = 0$ and u is not a polynomial on Ω. Then there exists a constant $C > 0$, independent of N, such that*

$$\inf_{\substack{w \in V^N \\ Cw=0}} \|u-w\|_V = \inf_{\substack{w \in \mathscr{P}_p(\Omega) \\ Cw=0}} \|u-w\|_V \geqq C \quad \textit{for all } N. \tag{3.14}$$

Proof. Consider a triangular mesh of the form shown in Fig. 3.1(c). Let V^N be the space of piecewise polynomials of total degree $\leqq p$ on this mesh and let $w \in V^N$ satisfy $Cw = 0$. Then over any element $\mathscr{T}_j$, w must be of the form

$$w = \sum_{i=0}^{p} a_{ij} x_1^i.$$

For any two triangles $\mathscr{T}_j$, $\mathscr{T}_k$ with a common side Γ_{jk}, we see that the variable x_1 is never fixed along Γ_{jk}. Hence the continuity of w implies that

$$a_{ij} = a_{ik}.$$

Repeating this argument, we see that $w \in \mathscr{P}_p(\Omega)$. Since $u \notin \mathscr{P}_p(\Omega)$, we obtain (3.14). The same argument works for rectangular meshes. □

We now obtain the following theorem.

THEOREM 3.3. *Consider the problems* (2.10), (3.1). *Let V_t^N be the set of piecewise polynomials of fixed degree $\leqq p$ $(p \geqq 1)$ on a rotated mesh of the form shown in Figs.* 3.1(c) *or* 3.1(d), *with mesh spacing $h = h(N)$. Let the error measure be $\mathscr{E}_1 = \{E_t^1\}$, $E_t^1(u) = \|u\|_{H^1(\Omega)}$ or $\mathscr{E}_2 = \{E_t^2\}$, $E_t^2(u) = \|u\|_{E,t}$, and let $R = \{(t, N), 0 < t \leqq b_N\} \subset R_0$. Let $\mathscr{H}$ be as in Theorem* 3.1. *Then the extension procedure $\mathscr{F}$ shows complete locking and is not robust with any uniform order over R.*

Proof. For $t \geqq \eta > 0$, we see that the spaces V_t^N will satisfy the maximum angle (or ratio) condition. Hence, by a standard approximability result and Lemma 3.2, we see that Assumption A will hold, with $F_0(N) = N^{-\min(p,k)/2}$. We now apply Theorem 2.2(A) or Theorem 2.3(A). Lemma 3.3 shows that the condition (2.32) is not satisfied. Hence $\mathscr{F}$ cannot be free from locking over R. The lack of robustness follows as well from Lemma 3.3. □

Suppose we now choose H_t to be a space of the form $H^k(\Omega)$ as in Theorem 3.2. Then, since the space of admissible functions in Theorem 3.3 is more restrictive, we again see that there will be locking in the $H^1(\Omega)$ norm. Let us mention here that an alternate proof of Theorem 3.3 may be derived by considering the necessity of the maximum angle condition (see [2]).

Remark. In the proof of Theorem 3.3, we used the necessity of condition (2.32) to prove that there was locking. We mention here that for the spaces H_t in Theorems 3.1 and 3.2, Condition (α) does not hold, so that (2.32) is not sufficient to show the lack of locking. Of course, we could restrict $\mathscr{H}$ in such a way that we only look at a class of problems for which (α) holds. For example, taking $H_t = \operatorname{span}\{u_t^1\}$ gives

$$u_t^1 \to u_0 = \sin x_1$$

for which, noting that $H_t \subset H^k(\Omega)$,

$$\|u_t^1 - u_0\|_{H^k(\Omega)} \leqq C\sqrt{t}$$

so that (α) holds.

Let us now examine what happens when we use the p-version instead of the h-version. We have the following.

THEOREM 3.4. *Consider the problems* (2.10), (3.1), *with* $\Gamma_D = 0$. *Let* V_t^N *be the set of piecewise polynomials of degree* $\leqq p = p(N)$ *on any of the meshes shown in Fig.* 3.1 (*i.e., rotated or unrotated*), *with fixed mesh size h independent of N. Let* H_t^B *be the space of functions given by*

$$H_t^B = \{u \mid \|u\|_{H_t^{k+1}(\Omega)} \leqq 1\}$$

with $k > 0$ *and let* $H = H^{k+1}(\Omega)$. *Let the error measure be* $E_t(u) = \|u\|_{E,t}$. *Then the p-version extension procedure* $\mathcal{F}$ *is free from locking over the region* $R_0 = (0, b] \times \mathcal{N}$ *and is robust with uniform order* $f(N) = O(N^{-k/2})$ *over this region.*

Proof. Let $u \in H_t^{k+1}(\Omega)$. Denote $\tilde{\Omega} = \{|\hat{x}_1| < \frac{5}{4}, |\hat{x}_2| < 1\}$, $\hat{\Omega} = \{|\hat{x}_1| < \frac{5}{4}, |\hat{x}_2| < \frac{5}{4}\}$. By the extension approach of Nikolsky and Babich (see [11, Thm. 3.9]) we extend u to $\tilde{u}$ on $\tilde{\Omega}$: for $-\frac{5}{4} < \hat{x}_1 < -1$,

$$\tilde{u}(\hat{x}_1, \hat{x}_2) = \sum_{l=1}^{k+1} \lambda_l u\left(-1 - (\hat{x}_1 + 1)\frac{l}{k+1}, \hat{x}_2\right)$$

with the exact form of the coefficients λ_l being described in [11]. $\tilde{u}$ is analogously extended on $1 < \hat{x} < \frac{5}{4}$ and then on $\hat{\Omega}$ to get $\hat{u}(\hat{x}_1, \hat{x}_2)$, which preserves the norm, i.e.,

$$\|\hat{u}\|_{H_t^{k+1}(\hat{\Omega})} \leqq C\|u\|_{H_t^{k+1}(\Omega)}.$$

Now we use the same approach as in [3, Lem. 3.1]. We transform $\hat{\Omega}$ onto

$$Q = \left\{|\xi_1| < \frac{\pi}{2}, |\xi_2| < \frac{\pi}{2}\right\}$$

with the transformation

$$\hat{x}_1 = \tfrac{5}{4}\sin \xi_1, \qquad \hat{x}_2 = \tfrac{5}{4}\sin \xi_2$$

and let $v(\xi_1, \xi_2) = \hat{u}(\hat{x}_1, \hat{x}_2)$. Then obviously

$$\|v\|_{H_t^{k+1}(Q)} \leqq C\|\hat{u}\|_{H_t^{k+1}(\hat{\Omega})} \leqq C\|u\|_{H_t^{k+1}(\Omega)}.$$

Now function v can be extended periodically on $\tilde{R} = \{|\xi_1| < \pi, |\xi_2| < \pi\}$ and it is symmetric with respect to the lines $\xi_1 = \pi/2$, $\xi_2 = \pi/2$. Writing

$$v = Re \sum_{j=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_{j\ell}\, e^{i(j\xi_1 + \ell\xi_2)}$$

we denote

$$v_p = Re \sum_{j=-p}^{p} \sum_{\ell=-p}^{p} a_{j\ell}\, e^{i(j\xi_1 + \ell\xi_2)}$$

and get

$$\|v - v_p\|_{H_t^1(Q)} \leqq Cp^{-k}\|u\|_{H_t^{k+1}(\Omega)}.$$

Transforming v_p back into variables $\hat{x}_1$ and $\hat{x}_2$ and denoting it by V we see that V is a polynomial of degree p in x_1 and x_2. We get on Ω

$$\|u - V\|_{H_t^1(\Omega)} \leqq Cp^{-k}\|u\|_{H_t^{k+1}(\Omega)},$$

and because $|u|_{H_t^1(\Omega)} = \|u\|_{E,t}$ we have

$$\inf \|u - w\|_{E,t} \leqq CN^{-k/2}\|u\|_{H_t^{k+1}(\Omega)} \tag{3.15}$$

where the infimum is taken over all polynomials of degree $2p$ and where C depends on k but is independent of N, t, and u.

For $t=\eta>0$, $\|\cdot\|_{E,t}$ and $\|\cdot\|_{H_t^{k+1}(\Omega)}$ are equivalent to $\|\cdot\|_{H^1(\Omega)}$ and $\|u\|_{H^{k+1}(\Omega)}$, respectively, with the equivalency constants dependent on η. Using Lemma 3.1 we get

$$d_N(H^1(\Omega), H^{k+1}(\Omega)) \geqq CN^{-k/2} \tag{3.16}$$

and the statement follows from (3.16) and (3.15). □

Remark. The statement and proof of Theorem 3.4 above carries over identically if, instead of (3.3), we use the following weaker definition of the norm $\|\cdot\|_{H_t^k(\Omega)}$:

$$\|u\|^2_{H_t^k(\Omega)} = \sum_{0\leqq \alpha_1 \leqq k} \left\| \frac{\partial^{\alpha_1} u}{\partial x_1^{\alpha_1}} \right\|_0^2 + t^{-1} \sum_{\substack{0\leqq \alpha_1+\alpha_2\leqq k \\ \alpha_2 \geqq 1}} \left\| \frac{\partial^{\alpha_1+\alpha_2} u}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}} \right\|_0^2. \tag{3.17}$$

In fact, Theorem 3.1 also holds with the above weaker definition; instead of using the maximum angle condition, we can establish it directly by considering interpolation on the region Ω. Hence the results in this section are valid if (3.3) is replaced by (3.17).

For the case of just one square, the above theorem guarantees that the rate of convergence is the same, independent of t, for both the rotated and unrotated case. This is clearly observed in Fig. 2.4. Note that our function there was $u_t^1 = \sin x_1\, e^{-\sqrt{t}\, x_2}$ which satisfies

$$u_t^1 \in H_t^k(\Omega) \quad \forall k.$$

Since this function is very well behaved, we can actually assert a stronger robustness result for it (see Theorem 3.5 below).

For the case of more than one square, Theorem 3.4 asserts that the *rate* of convergence is independent of t both with rotated and unrotated meshes. However, in actual practice, while the unrotated case will be insensitive to t, the rotated case will be sensitive. This is because it is known from [4] that the error for the finite element method using piecewise polynomials of degree $\leqq p$ on a quasi-uniform mesh with mesh size h is of the form

$$\|u-u^N\|_{N^1(\Omega)} \leqq \frac{Ch^{\min(k,p)}}{p^k} \|u\|_{H^{k+1}(\Omega)}. \tag{3.18}$$

When the p-version is used, the factor $Ch^{\min(k,p)} = Ch^k$ (for p large) appears as a constant. In terms of locking, when t is bounded away from 0, we observe the full effect of this constant. However, for t close to 0, Lemma 3.3 shows that this constant is much larger. This explains the data in Table 2.3. For a rigorous demonstration of this effect, we begin by proving some lemmas on the approximability of smooth solutions, such as (2.5).

LEMMA 3.4. *Let $f(x) \in C^\infty(I)$, $I=(-1,1)$ be such that for any $k \geqq 0$,*

$$\max |f^{(k)}(x)| \leqq Ad^k \tag{3.19}$$

with $0<d\leqq 1$ and A independent of k and d. Then

$$\inf_{v\in \mathcal{P}_p(I)} \|f-v\|_0 \leqq CAd^{p+1}\rho(p) \tag{3.20}$$

where

$$\rho(p) = \frac{1}{2^{(p+1)}(p+1)!}.$$

Moreover, suppose that there is a $B>0$ such that

$$\min |f^{(p+1)}(x)| \geqq Bd^{p+1}. \tag{3.21}$$

Then

$$\inf_{v \in \mathcal{P}_p(I)} \|f-v\|_0 \geqq CBd^{p+1}\rho(p), \tag{3.22}$$

where C is independent of p. (*Here* $\|\cdot\|_0 = \|\cdot\|_{L_2(I)}$.)

Proof. We write

$$f(x) = \sum_{k=0}^{\infty} a_k L_k(x) \tag{3.23}$$

where $L_k(x)$ is the kth Legendre polynomial. Then we have

$$L_k(x) = \frac{(-1)^k}{2^k k!} \frac{d^k}{dx^k}\{(1-x^2)^k\}$$

and coefficients a_k in (3.23) satisfy

$$a_k = \frac{2k+1}{2} \int_{-1}^{1} f(x) L_k(x)\, dx.$$

Hence we have

$$|a_k| = \frac{2k+1}{2^{k+1}k!} \left| \int_{-1}^{1} (1-x^2)^k f^{(k)}(x)\, dx \right| \leqq Ad^k \frac{2k+1}{2^{k+1}k!} \int_{-1}^{1} (1-x^2)^k\, dx. \tag{3.24}$$

Denoting

$$\mathcal{H}(k) = \int_{-1}^{1} (1-x^2)^k\, dx = \frac{(k!)^2}{(2k+1)!} 2^{k+1}$$

we get

$$C_1 k^{-1/2} \leqq \mathcal{H}(k) \leqq C_2 k^{-1/2}$$

so that

$$|a_k| \leqq \frac{Ck^{1/2}Ad^k}{2^k k!}.$$

Now

$$\begin{aligned} E_p^2 = \inf_{v \in \mathcal{P}_p(I)} \|f-v\|_0^2 = \sum_{k=p+1}^{\infty} a_k^2 \frac{2}{k+1} &\leqq CA^2 \sum_{p+1}^{\infty} d^{2k} \frac{1}{(2^k k!)^2} \\ &\leqq \frac{CA^2}{((p+1)!)^2} \sum_{p+1}^{\infty} \left(\frac{d}{2}\right)^{2k} \leqq \frac{CA^2}{((p+1)!)^2} \left(\frac{d}{2}\right)^{2(p+1)}, \end{aligned}$$

since $(d/2) \leqq \frac{1}{2}$ and the sum is a geometric series. This gives (3.20). On the other hand,

$$E_p^2 \geqq \max_{k \geqq p+1} a_k^2 \frac{2}{2k+1}.$$

Hence using (3.21) and (3.24), we see that (3.22) holds. $\square$

For any $0 < d \leqq 1$, let us denote by $\mathcal{L}_d$ the set of all functions $f(x) \in C^\infty(I)$ such that for each $p \geqq 1$ there exist at most M intervals $I_i^{(p)}$, $i = 1, \cdots, M$ (M independent of p), with $J^p = I - \bigcup_{j=1}^{M} I_j^{(p)}$ having measure $\geqq \tau > 0$ (τ independent of p) and

$$\min_{x \in J^p} |f^{(p)}(x)| \geqq Bd^p,$$

and

$$\max_{x\in I}|f^{(p)}(x)|\leqq Ad^p$$

with A, B independent of p. Let us norm this space by

$$\|f\|_{\mathscr{L}_d}=\sup_p\left\{d^{-p}\max_{x\in I}|f^{(p)}(x)|\right\}.$$

Then we have the following lemma.

LEMMA 3.5. *Let $\{\mathscr{T}^h\}$ be a sequence of quasi-uniform meshes on $I=(-1,1)$ and let $I=\bigcup_{j=1}^{N(h)} I_j^h$, $I_j^h=(x_j^h, x_{j+1}^h)$, $h_j=x_{j+1}^h-x_j^h$, $\tilde{C}_1h\leqq h_j\leqq\tilde{C}_2h$. Let $\{V_h^p\}$ denote a sequence of subspaces of $H^1(I)$ of piecewise polynomials of degree $\leqq p$ on the mesh $\mathscr{T}^h$. Let $f\in\mathscr{L}_d$. Then*

$$C_1d^ph^p\rho(p-1)\leqq\inf_{v\in V_h^p}|f-v|_{H^1(I)}\leqq C_2d^ph^p\rho(p-1) \tag{3.25}$$

where C_1, C_2 depend on τ, A, B, M, but are independent of d, h, p and where the first inequality holds provided h is sufficiently small.

Proof. Let $g(x)=f'(x)$ and let

$$e_{p-1,j}^h=\inf_{w\in\mathscr{P}_{p-1}(I_j^h)}\|g-w\|_{L_2(I_j^h)}. \tag{3.26}$$

Transforming I_j^h to the standard interval I we get, by applying Lemma 3.4,

$$e_{p-1,j}^h\leqq Cd^ph^{p+(1/2)}\rho(p-1). \tag{3.27}$$

If $I_j^h\in J^p$, then also

$$e_{p-1,j}^h\geqq Cd^ph^{p+(1/2)}\rho(p-1). \tag{3.28}$$

Now let $\bar{w}(x)$ be the polynomial of degree $p-1$ which attains the infimum (3.26). Then on I_j^h let

$$v(x)=f(x_j)+\int_{x_j}^x\bar{w}(x)\,dx.$$

Because of the construction of $v(x)$, by expanding $f'(x)$ into Legendre polynomials we have

$$v(x_{j+1})=f(x_{j+1}).$$

Hence in this manner we construct $v(x)$ on the entire I with $v(x)\in H^1(I)$. From (3.27) we have, by squaring and summing over j,

$$\|f'(x)-v'(x)\|_{L_2(I)}\leqq Cd^ph^p\rho(p-1).$$

For h sufficiently small, we have C/h intervals contained in J^p with C independent of p. Hence we also get, using (3.22),

$$\|v'(x)-f'(x)\|_0\geqq Cd^ph^p\rho(p-1). \qquad\square$$

Let us note that the function $\sin x_1\in\mathscr{L}_1$ and $e^{-\sqrt{t}x_2}\in\mathscr{L}_{\sqrt{t}}$. Hence the function u_t^1 given by (2.5) satisfies

$$u_t^1\in H_t=\mathscr{L}_1\otimes\mathscr{L}_{\sqrt{t}}. \tag{3.29}$$

Note that for $\sin x_1$, $M=1$ in the definition of $\mathscr{L}_1$ and (3.25) will hold for $h \leqq 1$. Also, if $h>1$, then (3.25) holds for p even. Let us define for $u=u_1(x_1)u_2(x_2) \in H_t$,

$$\|u\|_t = \|u_1\|_{\mathscr{L}_1} + \|u_2\|_{\mathscr{L}_{\sqrt{t}}}.$$

Obviously, taking $H=H^{k+1}(\Omega)$, we have $H_t \subset H$ for any k.

We now prove the following result for the case when the solution set $\mathscr{H}=\{H_t\}$ where H_t is given by (3.29). (Here H_t is in fact a countably normed space and H_t^B is characterized by t, A, B, d, τ, M.)

THEOREM 3.5. *Consider the problem in Theorem 3.4 with $H_t = \mathscr{L}_t = \mathscr{L}_1 \otimes \mathscr{L}_{\sqrt{t}}$ and $H = H^{k+1}(\Omega)$, $k>0$. Then the $h-p$ extension procedure $\mathscr{F}$, using piecewise polynomials of degree $\leqq p_N$, on unrotated meshes of mesh size h_N, is free from locking over the region $R_0 = (0, b] \times \mathscr{N}$ and is robust with uniform order $f(N) = O(h_N^{p_N}\rho(p_N - 1))$ there.*

Proof. We consider for simplicity only the case $\Gamma_D = \emptyset$. Let $u_t \in H_t^B$ so that

$$u_t(x_1, x_2) = \xi(x_1)\chi(x_2)$$

with $\xi \in \mathscr{L}_1$, $\chi \in \mathscr{L}_{\sqrt{t}}$. Then for any $y = v(x_1)w(x_2) \in V^N$,

$$\begin{aligned} \|u_t - y\|_{E,t}^2 &= \|\xi'\chi - v'w\|_0^2 + \frac{1}{t}\|\xi\chi' - vw'\|_0^2 \\ &\leqq \|\xi'\chi - v'\chi\|_0^2 + \|v'\chi - v'w\|_0^2 + \frac{1}{t}\|\xi\chi' - \xi w'\|_0^2 + \frac{1}{t}\|\xi w' - vw'\|_0^2 \\ &= \|\xi' - v'\|_0^2\|\chi\|_0^2 + \|\chi - w\|_0^2\|v'\|_0^2 \\ &\quad + \frac{1}{t}\{\|\xi\|_0^2\|\chi' - w'\|_0^2 + \|\xi - v\|_0^2\|w'\|_0^2\}. \end{aligned} \tag{3.30}$$

Suppose $v, w \in V_h^p$ are the best approximations to ξ, χ, respectively, in the sense of Lemma 3.5, i.e.,

$$\|\xi' - v'\|_0 \leqq Ch^p\rho(p-1) \qquad \|\chi' - w'\|_0 \leqq Ct^{p/2}h^p\rho(p-1).$$

Then (3.24) gives for $p \geqq 1$,

$$\|u_t - y\|_{E,t} \leqq Ch^p\rho(p-1) \tag{3.31}$$

uniformly in t. This proves the robustness result. To show that there is no locking, we must verify that (3.31) is the best rate possible. Now it may be easily shown that

$$\begin{aligned} \inf_{y \in V^N} \|u_t - y\|_{E,t}^2 &= \inf_{v,w \in V_h^p} \|\xi\chi - vw\|_{E,t}^2 \\ &\geqq \inf_{v \in V_h^p} \|\xi'\chi - v'\chi\|_0^2 = \inf_{v \in V_h^p} \|\xi' - v'\|_0^2\|\chi\|_0^2. \end{aligned}$$

Using Lemma 3.5 then shows that (3.31) is the best rate possible. Then by Definition 2.1, there is no locking over R_0. $\square$

Remark. Note that the above theorem also shows that there is no locking for the p-version.

We now look at the case where rotated meshes are used. We will distinguish between two cases. For the case where no refinement is present, i.e., only one element is used, we essentially get the unrotated case with one element. (We could understand this case also by considering the rotated solution u_t^2 as in § 2 (as mentioned in the beginning of this section, this is equivalent).) We will observe no locking in this case. When refinement is present, Lemma 3.3 tells us that, for the limit problem, the mesh behaves essentially like one with a single element. This will show locking for both the p-version and $h-p$ version.

THEOREM 3.6. *Consider the problems* (2.10), (3.1) ($\Gamma_D = \emptyset$). *Let* V_t^N *be the set of piecewise polynomials of degree* $p = p_N$ *on a family of rotated meshes of the form shown in Fig.* 3.1(d), *with mesh spacing* $h = h_N$. *Let* $H_t = \mathcal{L}_1 \otimes \mathcal{L}_{\sqrt{t}}$, $H = H^{k+1}(\Omega)$, $k > 0$, *and* $E_t(u) = \|u\|_{E,t}$ *or* $\|u\|_{H^1(\Omega)}$. *Let* $\mathcal{F}$ *consist of the p-version* (h_N *fixed*) *or the h-p version. For the p-version on a single element,* $\mathcal{F}$ *is free from locking and uniformly robust with order* $O(\rho(p_N - 1))$. *For a refined mesh with mesh size* $h_N \leqq 1$, *the extension process* $\mathcal{F}$ *shows locking of at least order* $(2/h_N)^{p_N}$ *but is uniformly robust with order* $\rho(p_N - 1)$.

Proof. For a single element, we have $h = 2$. We may either consider the pair u_t^1, $B_t^1(\cdot, \cdot)$ or u_t^2, $B_t^2(\cdot, \cdot)$. In either case, Lemma 3.5 may be used, as in the proof of Theorem 3.5, to show that there is no locking, and we get uniform robustness of order $f(N) = O(\rho(p_N - 1))$.

When refinement is present with $h \leqq 1$, we see that for the limit problem, by Lemma 3.3, the mesh behaves like a single element, i.e., one with mesh size two. However, for t bounded away from zero, we get the same convergence rate as in Theorem 3.5. This implies that in Assumption A,

$$F_0(N) = \inf_{w \in V^N} \|u - w\|_V = O(h_N^{p_N} \rho(p_N - 1)),$$

but for $Cu = 0$,

$$\inf_{\substack{w \in V^N \\ Cw = 0}} \|u - w\|_V (F_0(N))^{-1} \approx O\left(\left(\frac{2}{h_N}\right)^{p_N}\right).$$

Hence by (2.34) we get locking of at least order $(2/h_N)^{p_N}$. The method $\mathcal{F}$ is still uniformly robust with the rate using one square, i.e., $O(\rho(p_N - 1))$. $\square$

Let us summarize the various locking phenomena we have dealt with in this section in Table 3.1. ($U \equiv$ Unrotated, $R \equiv$ Rotated).

TABLE 3.1

Extension Process	H_t	E_t	Locking	Robustness (uniform order)
h-version, degree p, U	$H_t^{k+1}(\Omega)$, $k \geqq p$	H_t^1, H^1	No	$O(N^{-p/2})$
h-version, degree 1, U	$C^{(4)}(\bar{\Omega})$	H^1	No	$O(N^{-1/2})$
h-version, degree p, R	$H_t^{k+1}(\Omega)$, $k > 0$	H_t^1, H^1	Yes	None
	$H^{k+1}(\Omega)$, $k > 0$	H^1	Yes	None
p-version, U	$H_t^{k+1}(\Omega)$, $k > 0$	H_t^1, H^1	No	$O(N^{-k/2})$
p-version, R	$H_t^{k+1}(\Omega)$, $k > 0$	H_t^1, H^1	No	$O(N^{-k/2})$
p or h-p version, U	$\mathcal{L}_1 \times \mathcal{L}_{\sqrt{t}}$	H_t^1	No	$O(h_N^{p_N} \rho(p_N - 1))$
p-version, R, 1 square	$\mathcal{L}_1 \times \mathcal{L}_{\sqrt{t}}$	H_t^1	No	$O(\rho(p_N - 1))$
p or h-p version, R refined	$\mathcal{L}_1 \times \mathcal{L}_{\sqrt{t}}$	H_t^1	Yes	$O(\rho(p_N - 1))$

We see from the above table the effect of changing various choices of the extension procedure, H_t and E_t.

Let us remark that our results will be valid for *any* rotated mesh. We considered the case where the rotation is 45° for convenience, since locking will be observed most readily in this case. However, Lemma 3.3 is true for other angles of rotation as well.

4. Locking and nonquasi-uniform meshes. Our definition for locking in § 2 allows considerable flexibility in terms of the various components $\mathcal{H}$, $\mathcal{E}$, $\mathcal{F}$, etc., which may depend on the parameter t. We now exploit this flexibility to analyze an example where locking is caused by the failure of an inverse estimate to hold when the meshes are nonquasi-uniform.

Let $I=[0,1]$ and $t\in(0,\frac{1}{3}]$. For each N, let $h=N^{-1}$, $x_j=jh$, $j=0,\cdots,N$. Define $\delta=h(1-2t)$, $\varepsilon=ht$, and let $\{\mathcal{T}_t^h\}$ be the sequence of meshes shown in Fig. 4.1 consisting of nodal points x_i^ℓ, x_i, x_i^r. Let $V_t^N=V_t^h\subset H_0^1(I)$ be the space of piecewise linear functions on $\mathcal{T}_t^h$. This defines the family $\{\mathcal{F}_t\}=\mathcal{F}$. Given $u\in H_0^1(I)$, for $t>0$, we define $u_t^h=u_t^N\in V_t^N=V_t^h$ by (2.10) with

$$B_t(u,v)=B(u,v)=\int_I uv\,dx,$$

i.e., u_t^h is the L_2 projection of u on V_t^h. This defines our extension procedure $\mathcal{F}$. We choose $H_t=H^2(I)\cap H_0^1(I)$ and $E_t(u)=\|u\|_1$ and ask whether locking occurs as $t\to 0$. Note that the only component depending on t is $\mathcal{F}=\{\mathcal{F}_t\}$ (due to the mesh). We let $R_0=\{(t,N),0<t\leqq\frac{1}{3}\}$.

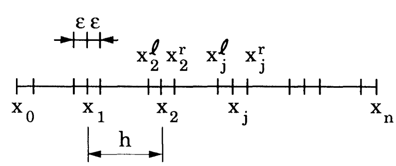

FIG. 4.1. *The mesh* $\mathcal{T}_t^h$.

THEOREM 4.1. *Let* $\{P_t\}$, $\mathcal{H}$ *and* $\mathcal{E}$ *be as above. Then for* $0<\eta\leqq 1$, $\mathcal{F}$ *shows locking of* $O(N)$ *for any region* $R=\{(t,N),0<t\leqq b_N\}\subset R_0$, *and is not robust with any uniform order.*

Proof. First it is easy to see that the quadruple $(\mathcal{H},\mathcal{F},\mathcal{E},R)$ is F_0-admissible with $F_0(N)\approx CN^{-1}$ due to standard results about the $H^1(I)$ projection on V_t^N. For any $\eta\geqq\eta_0>0$, the mesh is quasi-uniform on R_η so that the space V_t^N satisfies the inverse assumption. This leads to Assumption A being satisfied so that Theorem 2.1 holds with $F_0(N)$ as above.

Let us keep h fixed and let $t\to 0$. Then it may be easily verified that in the L_2 norm, $u_t^h\to u_0^h\in V_0^h$ where V_0^h is a set of discontinuous linear functions on $[(j-1)h,jh]$. Also u_0^h satisfies the limit problem

$$\int_{(j-1)h}^{jh} u_0^h v\,dx=\int_{(j-1)h}^{jh} uv\,dx \tag{4.1}$$

with v being a linear function on $[(j-1)h,jh]$, $j=1,\cdots,N$.

Suppose $\mathcal{F}$ is robust with some uniform order. Then for $t>0$, h fixed,

$$\|u_t^h\|_1\leqq\|u\|_1+\|u-u_t^h\|_1\leqq C$$

with C independent of t. Hence there exists $\{u_{t_k}^h\}$, a convergent subsequence such that $u_{t_k}^h$ converges to $\tilde{u}_0^h$ strongly in $L_2(I)$ and weakly in $H^1(I)$. Hence the limit $\tilde{u}_0^h\in V_0^h$. Since $u_t^h\to u_0^h$ in L_2, we must have

$$\tilde{u}_0^h=u_0^h.$$

But this is a contradiction, since $u_0^h\in V_0^h$, satisfying (4.1) will not be continuous in general, i.e., $u_0^h\notin H_0^1(I)$. Hence $\mathcal{F}$ is not robust with any uniform order. Also, because $F_0(N)\approx N^{-1}$, $\mathcal{F}$ shows locking of order N. □

Remark 4.1. Note that if we choose $E(u)=\|u\|_0$, then $\mathcal{F}$ will be free from locking.

5. The clamped loaded beam. In this section we look at the robustness of some discretization schemes for a one-dimensional problem involving the Timoshenko beam.

This problem (along with various extension processes for it) has been analyzed in the context of locking in [1], [9]. Here, we present a simplified alternate analysis, using our theory, to prove some of the results in these references. We do this by showing that Condition (α) is satisfied so that Theorems 2.2(A) and (B) hold. We may therefore restrict our attention to the limiting problem which is essentially a one-dimensional biharmonic problem.

The problem we look at is

$$-\phi_t'' + \frac{1}{t}(\phi_t - w_t') = f \quad \text{on } I = (0, 1), \tag{5.1}$$

$$\frac{1}{t}(\phi_t - w_t')' = g \quad \text{on } I, \tag{5.2}$$

$$\phi_t(0) = \phi_t(1) = w_t(0) = w_t(1) = 0. \tag{5.3}$$

Here, w_t and ϕ_t represent the vertical displacement and rotation, respectively, of the vertical fibers of the beam, which is subject to a vertical body force $-tg(x)$ (f can be related in applications to dislocations or moments). The thickness $\sqrt{t}$ is assumed to lie in the interval $(0, 1]$.

We may cast (5.1)–(5.3) into the following variational form. Let $V = H_0^1(I) \times H_0^1(I)$. Then we wish to find $u_t = (\phi_t, w_t) \in V$ satisfying

$$B_t(u_t, v) = b(u_t, v) + \frac{1}{t}(Cu_t, Cv) = F(v) \tag{5.4}$$

for all $v = (\psi, z) \in V$, with $b(u_t, v) = (\phi_t', \psi')$, $Cu_t = \phi_t - w_t'$. Here $(\cdot, \cdot)$ represents the $L_2(I)$ inner product and F is given by

$$F(v) = \langle f, \psi\rangle + \langle g, z\rangle,$$

where $\langle \cdot, \cdot\rangle$ is the duality between $H^1(I)$ and $H^{-1}(I)$.

It is easy to see [1] that

$$\frac{1}{3}\|u_t\|_V^2 \leqq \|u_t\|_{E,t}^2 \leqq \left(1 + \frac{2}{t}\right)\|u_t\|_V^2 \tag{5.5}$$

with the energy norm being defined as usual. The bilinear form can be put in the form (2.22) if, as in (3.1), we take $t/(1-t)$ instead of t, giving for $u = (\phi, w)$

$$a(u, u) = b(u, u) + (Cu, Cu) \geqq \tfrac{1}{3}\|u\|_V^2$$

so that (2.23), (2.24) are satisfied.

In [1] it has been shown that for $f, g \in H^{-1}(I)$ and $0 < t \leqq 1$, the following a priori estimate holds for $k = 0, 1, \cdots$, for the problem (5.4) (or equivalently, (5.1)–(5.3))

$$\|u_t\|_{k+1} + \frac{1}{t}\|Cu_t\|_k \leqq C(\|f\|_{k-1} + \|g\|_{k-1}), \tag{5.6}$$

where C is a constant depending only on k and

$$\|u_t\|_{k+1} = \|\phi_t\|_{k+1} + \|w_t\|_{k+1}.$$

Hence a natural choice for the solution set $H_t = H_{t,k+1}$ is to define $H_{t,k+1} \subset H = (H^{k+1}(I) \cup H_0^1(I))^2$ by

$$H_{t,k+1} = \{u \mid \|u\|_{t,k+1} < \infty\} \tag{5.7a}$$

with the norm $\|\cdot\|_t = \|\cdot\|_{t,k+1}$ given by

$$\|u\|_{t,k+1} = \|u\|_{k+1} + \frac{1}{t}\|Cu\|_k. \tag{5.7b}$$

Then $H_{t,k+1} \subset H = H^{k+1}(I)$.

Using (5.1), (5.2), we also have for $k = 0, 1, \cdots$,

$$\|f\|_{k-1} + \|g\|_{k-1} \leqq C\|u_t\|_{t,k+1}. \tag{5.8}$$

From (5.4), we see that as $t \to 0$, the Kirchhoff hypothesis

$$Cu = 0 \tag{5.9}$$

gets imposed so that any u_0 in the limiting set H_0^B must satisfy (5.9). Moreover, using (5.6) with $k = 0$, we see that if f, g are held fixed in (5.4) and $t \to 0$, then

$$\|Cu_t\|_0 = O(t).$$

Suppose now we are given a $u_t \in H_{t,k+1}^B$. We show that Condition (α) holds. Define f, g by equations (5.1)–(5.3) so that by (5.8), $\|f\|_{k-1}$, $\|g\|_{k-1} \leqq C$, and denote by $(u_0, \zeta_0) = (\phi_0, w_0, \zeta_0)$ the solution of

$$\begin{aligned} -\phi_0'' + \zeta_0 &= f \quad \text{on } I, \\ \zeta_0' &= g \quad \text{on } I, \\ \phi_0 - w_0' &= 0 \quad \text{on } I, \\ \phi_0(0) = \phi_0(1) &= w_0(0) = w_0(1). \end{aligned} \tag{5.10}$$

Equation (5.10) has the equivalent variational form; find $(u_0, \zeta_0) \in V \times S$ satisfying

$$\begin{aligned} b(u_0, v) + (\zeta_0, Cv) &= F(v) \quad \text{for all } v \in V, \\ (Cu_0, \eta) &= 0 \quad \text{for all } \eta \in S, \end{aligned} \tag{5.11}$$

with $S = L_2(I)$. Then (5.11) has a unique solution which satisfies

$$\|u_0\|_{k+1} + \|\zeta_0\|_k \leqq C(\|f\|_{k-1} + \|g\|_{k-1}). \tag{5.12}$$

Equation (5.12) is proven in [1] for $k = 0$ and can be obtained for higher k from (5.10) by differentiation. Equation (5.12) shows that $u_0 \in H_0 \cap H_{t,k+1}$. Using (5.8),

$$\|u_0\|_{k+1} + \|\zeta_0\|_k \leqq C\|u_t\|_{t,k+1}. \tag{5.13}$$

Now by (5.1) and (5.10), we have

$$-(\phi_t - \phi_0)'' + \frac{1}{t}C(u_t - u_0) = \zeta_0, \tag{5.14}$$

which corresponds to (using (5.4) and (5.11))

$$b((u_t - u_0), v) + \frac{1}{t}(C(u_t - u_0), Cv) = (\zeta_0, Cv) \tag{5.15}$$

from which, putting $v = u_t - u_0$, we get

$$\|u_t - u_0\|_{E,t}^2 \leqq \frac{t}{2}\|\zeta_0\|_0^2 + \frac{1}{2t}\|C(u_t - u_0)\|_0^2 \leqq \frac{t}{2}\|\zeta_0\|_0^2 + \frac{1}{2}\|u_t - u_0\|_{E,t}^2. \tag{5.16}$$

Hence

$$\|u_t - u_0\|_{E,t}^2 \leqq t\|\zeta_0\|_0^2,$$

i.e.,

$$(5.17) \qquad \|u_t - u_0\|_{E,t} \leqq t^{1/2}\|\zeta_0\|_0.$$

Next, differentiating (5.14) and using (5.2), (5.10) we see

$$(\phi_t - \phi_0)''' = 0$$

so that

$$(\phi_t - \phi_0)'' = a_t, \qquad (\phi_t - \phi_0)' = a_t x + b_t$$

where a_t, b_t are constants depending on t. Using (5.17),

$$|a_t|, |b_t| \leqq t^{1/2}\|\zeta_0\|_0$$

from which (using (5.1)-(5.3), (5.10), and (5.14)), the following estimate may be established:

$$\|u_t - u_0\|_{k+1} \leqq t^{1/2}\|\zeta_0\|_k.$$

Using (5.13), this gives

$$\|u_t - u_0\|_{k+1} \leqq t^{1/2}\|u_t\|_{t,k+1}$$

so that (α) is satisfied. Hence we may take $H^B_{0,k+1}$ to be the set of all u_0's constructed from $u_t \in H^B_{t,k}$ through the equations (5.10).

We will be interested in locking in the $\|\cdot\|_V$ norm (with η in Definition 2.1 chosen to be a fixed value >0). Then by Theorem 2.2, a necessary and sufficient condition for the absence of locking is that for all $u \in H^B_{0,k+1}$,

$$(5.18) \qquad \inf_{\substack{w \in V^N \\ Cw=0}} \|u - w\|_V \leqq CF_0(N) = \inf_{w \in V^N} \|u - w\|_V.$$

We will denote by $\{S_h^p\}$ a sequence of C^0 subspaces consisting of piecewise polynomials of degree $\leqq p$ on a quasi-uniform family of meshes on I.

THEOREM 5.1. *Consider the problem* (5.4). *Let* $\{V_t^N = (S_h^p)^2\} = \mathcal{F}_t$, *where* $p \geqq 1$ *is kept fixed and the h extension is used* $(h = h_N)$. *Let* $\mathcal{H} = \{H_t\}$, $H_t = H_{t,k+1}$ *be given by* (5.7), *with* $k \geqq p$. *Let* $H = H^{k+1}(\Omega)$ *and the error measure* $E_t(u) = \|u\|_V$. *Then the h-version extension procedure* $\mathcal{F}$ *shows locking of order* h_N^{-1}. *It is uniformly robust with order* h_N^{p-1} *for* $p > 1$.

Proof. We must estimate the two infimums in (5.18). First we have, for $u = (\phi, w) \in H^B_{0,k+1}$,

$$(5.19) \qquad F_0(N) = \inf_{\psi, z \in S_h^p} \{\|\phi - \psi\|_1 + \|w - z\|_1\} \approx O(h_N^p)$$

by standard results, since $\phi, w \in H^{k+1}(I)$, $k \geqq p$. Next, we estimate for $u = (\phi, w) \in H^B_{t,k+1}$ and $\phi - w' = 0$,

$$A_N = \inf_{\substack{\psi, z \in S_h^p \\ \psi - z' = 0}} \{\|\phi - \psi\|_1 + \|w - z\|_1\}.$$

Obviously,

$$(5.20) \qquad A_N = \inf_{z \in S_h^p} \{\|w' - z'\|_1 + \|w - z\|_1\} \approx \inf_{z \in S_h^p} \|w - z\|_2.$$

For $p=1$, $A_N=O(1)$. For $p\geqq 2$, standard spline theory [13] shows that for $k\geqq p$,

$$A_N=O(h_N^{p-1}). \tag{5.21}$$

Equations (5.19), (5.21), together with Theorem 2.2, prove the result. □

Let us now consider the p-version.

THEOREM 5.2. *Consider the problem* (5.4). *Let* $\{V_t^N=(S_h^p)^2\}=\mathcal{F}_t$, *where h is fixed and the p-extension* $(p=p_N)$ *is used. Let* $H_t=H_{t,k+1}$, $k>0$, $H=H^{k+1}(\Omega)$ *and* $E_t(u)=\|u\|_V$. *Then the p-version extension procedure* $\mathcal{F}$ *is free from locking and uniformly robust with order* p_N^{-k}.

Proof. By standard p-version approximation theory we have, for $u=(\phi,w)\in H_{0,k+1}^B$,

$$F_0(N)=\inf_{\psi,z\in S_h^p}\{\|\phi-\psi\|_1+\|w-z\|_1\}\approx O(p_N^{-k})$$

since $\phi, w\in H^{k+1}(I)$. Next, for $u\in H_{0,k+1}^B$, $Cu=0$ so that $w'=\phi\in H^{k+1}(I)$, i.e., $w\in H^{k+2}(I)$. Hence

$$A_N=\inf_{z\in S_h^p}\|w-z\|_2=O(p_N^{-(k+2-2)})=O(p_N^{-k}).$$

This proves the theorem. □

The results in [1], [9] also indicate what happens when the measure E_t is changed. We have summarized these results in Table 3.2. The set $H_t=H_{t,k+1}$ in each case (with $k\geqq p$ for the h-version).

Remark 5.1. Additional results in [1] deal with the h-version of a mized formulation for which locking is eliminated. The reference [9] also analyzes an h-p extension procedure which is free from locking.

6. Nearly incompressible materials. The procedure outlined in the previous section can be used to analyze "Poisson locking" as well, which occurs in the case of the

TABLE 3.2.
Locking and robustness for different choices of $\mathcal{F}$ *and* E_t.

Extension Process $\mathcal{F}$	E_t	Locking	Robustness (Uniform order)
h-version	$\|u\|_V$	Yes	None
$p=1$	$\|\phi\|_1$	Yes	None
[1]	$\|w\|_1$	Yes	None
	$\|\phi\|_0$	Yes	None
	$\|w\|_0$	Yes	None
h-version	$\|u\|_V$	Yes	$O(h_N^{p-1})$
$p\geqq 2$	$\|\phi\|_1$	Yes	$O(h_N^{p-1})$
[1]	$\|w\|_1$	No	$O(h_N^p)$
	$\|\phi\|_0$	Yes	$O(h_N^p)$
	$\|w\|_0$	Yes, experimentally	$O(h_N^p)$
p-version	$\|u\|_V$	No	$O(p_N^{-k})$
[9]	$\|\phi\|_1$	No	$O(p_N^{-k})$
	$\|w\|_1$	No	$O(p_N^{-k})$
	$\|\phi\|_0$	No	$O(p_N^{-(k+1)})$
	$\|w\|_0$	No	$O(p_N^{-(k+1)})$

elasticity equations when the Poisson ratio ν is close to 0.5. More specifically, consider the following problem:

$$-\Delta \mathbf{u}_\nu - \frac{1}{1-2\nu} \operatorname{grad} \operatorname{div} \mathbf{u}_\nu = \mathbf{f} \quad \text{in } \Omega, \tag{6.1}$$

$$\sum_{j=1}^{2} \left(\varepsilon_{ij}(\mathbf{u}_\nu) + \delta_{ij} \frac{\nu}{1-2\nu} \operatorname{div} \mathbf{u}_\nu \right) n_j = g_i, \qquad 1 \leqq i \leqq 2 \quad \text{on } \Gamma, \tag{6.2}$$

where $\{n_j\}_{j=1}^2$ is the unit outward normal to Γ and the strain tensor $\{\varepsilon_{ij}\}$ is given by

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial}{\partial x_i} u_j + \frac{\partial}{\partial x_j} u_i \right).$$

It is assumed that

$$\int_\Omega \mathbf{f} \cdot \mathbf{R} \, d\mathrm{x} + \int_\Gamma \mathbf{g} \cdot \mathbf{R} \, ds = 0$$

for any rigid motion R. Let us denote $t = (1-2\nu/\nu)$ and $\mathbf{u}_t = \mathbf{u}_\nu$. As $t \to 0$, the constraint

$$C\mathbf{u} = \operatorname{div} \mathbf{u} = 0 \tag{6.3}$$

gets enforced. $\mathbf{u}_t$ converges to a limit $\mathbf{u}_0$ which satisfies

$$\begin{gathered} -\Delta \mathbf{u}_0 - \operatorname{grad} \zeta_0 = \mathbf{f} \quad \text{in } \Omega, \quad \operatorname{div} \mathbf{u}_0 = 0 \quad \text{in } \Omega, \\ \sum_{j=1}^{2} \varepsilon_{ij}(\mathbf{u}_0) n_j + \nu \zeta_0 n_i = g_i, \qquad 1 \leqq i \leqq 2 \quad \text{on } \Gamma \end{gathered} \tag{6.4}$$

which is the Stokes equation.

The corresponding variational forms for (6.1) and (6.4) are, respectively, given by (5.4) and (5.11) with $V = \mathbf{H}_0^1(\Omega)$, $S = \{\eta \in L^2(\Omega), \int_\Omega \eta \, dx = 0\}$,

$$b(\mathbf{u}, \mathbf{v}) = \iint_\Omega \sum_{i,j=1}^{2} \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v}) \, dx, \qquad F(\mathbf{v}) = \int_\Gamma \mathbf{g} \cdot \mathbf{v} \, ds + \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx.$$

Let us define the spaces $H_{t,k+1}$ by (5.7) as before and take $H = H^{k+1}(\Omega)$ again. (Here, Cu is given by (6.3).) If the domain is smooth, we obtain all the usual shift theorems (as in the case of the example in § 5). Also the argument of equations (5.15), (5.16) carries over identically so that (5.17) will again hold. In fact, as we have shown in [5], Condition (α) will hold for this problem, even for cases when the domain is not smooth.

As a result, Theorem 2.2 applies again and we have the following result.

THEOREM 6.1. Let $\mathcal{H} = \{H_t\}$, $H_t = H_{t,k+1}$ as in (5.7) and $H = H^{k+1}(\Omega)$. Then the extension procedure $\mathcal{F}$ is *robust* with respect to error measure $E_t(\mathbf{u}) = \|\mathbf{u}\|_1$, for the variational form of (6.1)-(6.2) with uniform order $g(N)$ given by

$$g(N) = \sup_{\substack{\mathbf{u} \in H^B \\ \operatorname{div} \mathbf{u} = 0}} \inf_{\substack{\mathbf{w} \in V^N \\ \operatorname{div} \mathbf{w} = 0}} \|\mathbf{u} - \mathbf{w}\|_1. \tag{6.5}$$

It is free from locking if and only if

$$g(N) \leqq C F_0(N) = C \sup_{\mathbf{u} \in H^B} \inf_{\mathbf{w} \in V^N} \|\mathbf{u} - \mathbf{w}\|_1 \tag{6.6}$$

and shows locking of order $f(N)$ if and only if

$$g(N) \approx C F_0(N) f(N).$$

Note that in [6], (6.5) was stated only as a necessary condition, whereas we show that it is sufficient as well, which simplifies the subsequent analysis.

In [5], we use Theorem 6.1 and some results developed in § 2 to investigate the robustness of various schemes for nearly incompressible materials.

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