were employed and solution time was 144 sec on an ICL 1904S computer. Figure 3(b) shows the contours of maximum shear stress in the fillet region and for this case the stress concentration factor was calculated to be 1·36. It should be noted that in this analysis only the effect of tangential acceleration was considered and the centripetal forces due to the angular velocity, $\mathring{\theta}$, were not included. These would be obtained by a normal axi-symmetric stress analysis.

CONCLUSIONS

It has been shown how the finite element method can be easily employed to determine the stress and displacement fields of rotating axisymmetric bodies. The results agree well with those of experimental techniques for the static problem considered. When angular acceleration effects are included the numerical approach appears to provide the only feasible method of analysis since, for such a case, the electrical analogy no longer holds and the application of the appropriate body forces in photoelasticity presents insurmountable difficulties.

For the accelerating shaft problem it is seen that stresses in excess of 30,000 $\mathrm{lb/in^2}$ are set up in the fillet region for the acceleration assumed. In a real shaft-turbine system the acceleration is highly non-linear and short-term accelerations of two or three times the quoted value may exist, producing stresses above the initial plastic yield value. Whilst the solution of this problem could be approximated to without the use of body forces, in other situations (short shafts, projectiles, etc.) inclusion of body force acceleration terms is essential.

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THE STATIC CONDENSATION ALGORITHM

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SUMMARY

The application of the static condensation method is discussed in general. The technique is presented as an extension of the Gauss elimination algorithm. An efficient FORTRAN subroutine is given which simultaneously reduces both the stiffness matrix and the stress–displacement transformation matrix.

INTRODUCTION

The static condensation method was initially used to eliminate the internal degrees of freedom in a quadrilateral finite element constructed from four triangles.[1] The method is more general than the application at the element stiffness level and can be used to reduce the number of degrees of freedom of the complete structural system. In many cases, it is similar to the substructure technique, frontal solution method or matrix partitioning. However, all of these methods appear to an application of the basic Gaussian elimination procedure.

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The elimination of internal degrees of freedom at the element stiffness level reduces the overall size and bandwidth of the resulting set of equations. In the case of dynamic analysis the elimination of the massless degrees of freedom reduce the size of the resulting eigenvalue problems. The terminology 'static' condensation was coined in the application of the method to dynamic analysis.[2]

The purpose of this paper is to present the static condensation procedure as an efficient numerical algorithm. In addition, the procedure is extended to the condensation of the stress–displacement transformation matrix; therefore, the calculation of the eliminated degrees of freedom is not required in the subsequent evaluation of the element stresses.

MATRIX FORMULATION OF THE STATIC CONDENSATION METHOD

The basic static condensation procedure can be illustrated by the use of matrix notation. The equilibrium equations can be written in matrix partitional forms as

$$\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ab} \\ \mathbf{K}_{ba} & \mathbf{K}_{bb} \end{bmatrix} \begin{bmatrix} \mathbf{u}_a \\ \mathbf{u}_b \end{bmatrix} \quad \begin{bmatrix} \mathbf{P}_a \\ \mathbf{P}_b \end{bmatrix} \tag{1}$$

where $\mathbf{u}_a$ indicates the degrees of freedom to be eliminated and $\mathbf{u}_b$ indicates the degrees of freedom which are associated with the reduced stiffness matrix. The solution of the first submatrix equation for $\mathbf{u}_a$ yields

$$\mathbf{u}_a = \mathbf{C} - T\mathbf{u}_b \tag{2}$$

in which

$$\mathbf{C} = \mathbf{K}_{aa}^{-1}\mathbf{P}_a \tag{3}$$

$$\mathbf{T} = \mathbf{K}_{aa}^{-1}\mathbf{K}_{ab} \tag{4}$$

Substitution of equation (2) into the second submatrix equation results in a set of equilibrium equations with respect to the 'b' degrees of freedom

$$\mathbf{K}^*\mathbf{u}_b = \mathbf{P}^* \tag{5}$$

where

$$\mathbf{K}^* = \mathbf{K}_{bb} - \mathbf{K}_{ba}T \tag{6}$$

$$\mathbf{P}^* = \mathbf{P} - \mathbf{K}_{ba}\mathbf{C} \tag{7}$$

Physically, the term $\mathbf{K}_{ba}\mathbf{T}$ indicates the stiffness modification due to the release of the 'a' degrees of freedom and $\mathbf{K}_{ba}C$ represents the forces carried over from the 'a' to the 'b' degrees of freedom.

The stresses within the elements may be expressed by an equation of the form

$$\sigma = [\mathbf{A}_a \mathbf{A}_b] \begin{bmatrix} \mathbf{u}_a \\ \mathbf{u}_b \end{bmatrix} + \tau \tag{8}$$

where τ are the initial stresses before the system is subjected to the displacements $\mathbf{u}_a$ and $\mathbf{u}_b$. The displacement $\mathbf{u}_b$ may be eliminated from equation (8) by the substitution of equation (2). Or

$$\boldsymbol{\sigma} = \mathbf{A}^*\mathbf{u}_a + \boldsymbol{\tau}^* \tag{9}$$

where

$$\mathbf{A}^* = \mathbf{A}_b - \mathbf{A}_a \mathbf{T} \tag{10}$$

$$\boldsymbol{\tau}^* = \boldsymbol{\tau} - \mathbf{A}_a \mathbf{C} \tag{11}$$

As an example of the application of this procedure, consider a quadrilateral element formed by the combination of four six-node triangles as shown in Figure 1. In the case of two-dimensional stress problems this element has 13 nodes or 26 degrees of freedom; however, the 10 degrees of freedom associated with the interior points may be eliminated before the element stiffness is

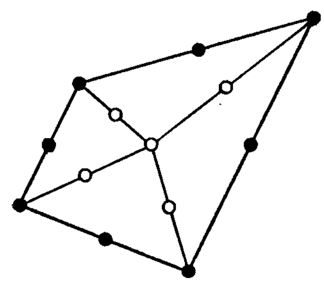

Figure 1. Quadrilateral element

added to the total stiffness of the structure. If the stress matrices $\mathbf{A}^*$ and $\boldsymbol{\tau}^*$ are evaluated at the same time the reduced stiffness matrix $\mathbf{K}^*$ is formed, the internal displacements $\mathbf{u}_a$ need not be calculated. Of course, this will require that the matrices $\mathbf{A}^*$ and $\boldsymbol{\tau}^*$ be saved on a low speed storage unit until after the displacements $\mathbf{u}_b$ are evaluated.

THE STATIC CONDENSATION ALGORITHM

The matrix notation used in the previous section serves to illustrate the basic concept of the technique; however, the matrix operations of multiplication and inversion are not efficient within a computer program. Equation (1) can be written as

$$\mathbf{K}\mathbf{u} = \mathbf{P} \tag{12}$$

If this set of equations is for the element shown in Figure 1 and if the first ten unknowns are associated with the internal points, we can apply the Gauss elimination procedure directly to the system to eliminate these degrees of freedom. Also, if the stress–displacement equation is written as

$$\boldsymbol{\sigma} = \mathbf{A}\mathbf{u} + \boldsymbol{\tau} \tag{13}$$

the appropriate unknowns can be eliminated at the same time. If we consider three components of stress, the application of the Gauss algorithm to these equations can be summarized as follows:

For $n = 1, 10$

1. Solve for u_n

$$u_n = C_n - \sum_{j=n+1}^{26} T_{nj} u_j$$

where

$$C_n = P_n^* / K_{nn}^*$$

$$T_{nj} = K_{nj}^* / K_{nn}^*$$

2. Substitute into the remaining equations will result in the modification of the coefficients

$$\left.\begin{aligned} K_{ij}^{**} &= K_{ij}^{*} - K_{in}^{*} T_{nj} \\ P_i^{**} &= P_i^{*} - K_{in}^{*} C_n \end{aligned}\right\} \begin{aligned} i &= n+1, 26 \\ j &= n+1, 26 \end{aligned}$$

3. Substitute into the stress–displacement equations

$$\left.\begin{aligned} A_{ij}^{**} &= A_{ij}^{*} - A_{in}^{*} T_{nj} \\ \tau_i^{**} &= \tau_i^{*} - A_{in}^{*} C_n \end{aligned}\right\} \begin{aligned} i &= 1, 3 \\ j &= n+1, 26 \end{aligned}$$

The * indicates the repeated modification of the coefficients for each value of n Within the computer program these coefficients are modified and stored at the same storage location.

The FORTRAN statements which apply to the elimination of these ten degrees of freedom are

```
      DO  500  N =1,10
      NN =N +1
      C =P(N)/S(N,N)
      DO  300  J =NN,26
      P(J) =P(J) -S(J,N)*C
      T =S(N,J)/S(N,N)
      DO  200  I =NN,26
200   S(I,J) =S(I,J) -S(I,N)*T
      DO  300  I =1,3
300   A(I,J) =A(I,J) -A(I,N)*T
      DO  400  I =1,3
400   TAU(I) =TAU(I) -A(I,N)*C
500   CONTINUE
```

where the stiffness term K_{ij} is indicated in FORTRAN as S(I, J). In this case, it is apparent that the FORTRAN statements are a very efficient summary of the algorithm. The results of the reduction are stored in the same area as the original matrices, with the first ten columns of the S and A array and the first ten rows of the S array eliminated.

A more convenient form for computer programming is to store the coefficients to be eliminated in sequence after the coefficients which are to be modified; then, the subscripts for the reduced stiffness start with one and rearrangement of the reduced coefficients is not required. A general FORTRAN IV subroutine is given in the Appendix which is based on this method of storing the coefficients. The only modification of the above procedure is that the Gauss elimination is applied to the matrices in reverse order—the first term to be eliminated is the last unknown in the displacement vector.

FINAL REMARKS

The static condensation algorithm as presented in this paper is an efficient method to eliminate element degrees of freedom. However, in the case where a large number of stresses are required, it may be more efficient not to form the stress displacement A and to re-calculate the internal displacements U_a from the equivalent of equation (2). This requires that certain element information be retained in low speed storage until the evaluation of element stresses.

The static condensation algorithm has definite advantages if there are many elements in the system with the same geometry and material properties. For this case, the reduced element stiffness matrix and the reduced stress–displacement matrix need be formed and stored only once.

In the case of dynamic analysis by the mode superposition method the stress–displacement matrix can be multiplied by the mode shapes and a transformation between element stresses and generalized modal response can be developed. This will allow the direct evaluation of the desired element stresses without the need to calculate all the time-dependent joint displacements of the system.

APPENDIX

General subroutine for static condensation

The subroutine is called by the following statement:

CALL STATIC(ND, NR, NS, NL, S, P, A, TAU)

where

ND = The size of original matrix—the unknowns to be eliminated are stored last.

NR = The size of the reduced matrix—(ND–NR) are the number of degrees of freedom to be eliminated.

NS = The number of stress components to be evaluated.

NL = The number of load conditions considered.

S = The original ND × ND stiffness matrix, **K**.

P = The original ND × NL load matrix, **P**.

A = The original NS × ND stress–displacement matrix, **A**.

TAU = The original NS × NL initial stress matrix, $\boldsymbol{\tau}$.

After the return from the subroutine the arrays are modified as follows:

S = The reduced NR × NR stiffness matrix, **K***.

P = The reduced NR × NL load matrix, **P***.

A = The reduced NS × NR stress matrix, **A**.

TAU = The reduced NS × NL initial stress matrix, $\boldsymbol{\tau}^*$.

```
      SUBROUTINE  STATIC(ND,NR,NS,NL,S,P,A,TAU)
      DIMENSION  S(24,24),P(24,4),A(12,24),TAU(12,4)
C
      NE =ND -NR
C
      DO  500  M =1,NE
C
      MAX =ND -M
      N =MAX +1
C
      DO  300  J =1,MAX
      IF(S(N,J).EQ.O.O)  GO  TO  300
      T =S(N,J)/S(N,N)
C
      DO  200  I =J,MAX
      S(I,J) =S(I,J) -S(I,N)*T
  200 S(J,I) =S(I,J)
```

```
C
          DO  250  I=1,NS
    250   A(I,J)=A(I,J)-A(I,N)*T
    300   CONTINUE
C
          DO  400  L=1,NL
          C=P(N,L)/S(N,N)
          DO  350  I=1,MAX
    350   P(I,L)=P(I,L)-S(I,N)*C
          DO  400  I=1,NS
    400   TAU(I,L)=TAU(I,L)-A(I,N)*C
C
    500   CONTINUE
C
          RETURN
          END
```

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A TECHNIQUE FOR DEGENERATING BRICK-TYPE ISOPARAMETRIC ELEMENTS USING HIERARCHICAL MIDSIDE NODES

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INTRODUCTION

This technical note was presented verbally at the IUTAM conference at Liège in 1970 but was not published. Indeed, publication would have been premature because the results of the computations were not understood. Three years later, they are still not understood, and this original note is now published so that other workers may perhaps provide the explanations.

An isoparametric brick element with corner nodes 1, 2, 3,... 8 may be progressively distorted until say vertices 5 and 8 eventually coalesce and edge 58 disappears. After several such modifications the original brick may be unrecognizable, as in Figure 1. We might expect that a good

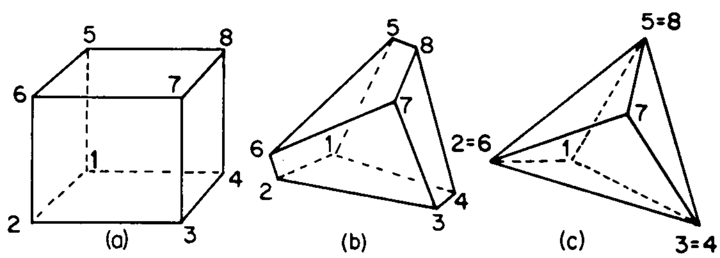

Figure 1. Stages in the degeneration towards the di-tetrahedron

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