Tutorial on Proof-theoretic Semantics

Bogdan Dicher[1,*]

[1]*LanCog Group, Centre of Philosophy of the University of Lisbon, Lisbon, Portugal*

[*]Email: `bdicher@me.com`

Proof-theoretic semantics (PTS) is an inferentialist theory of meaning which originates in the work of Gentzen in the 1930s and was subsequently developed by Prawitz, Martin-Löf, Dummett, and more recently by Schroeder-Heister, who also baptised the theory, and many others. It is an alternative to Tarskian model-theoretic semantics, aiming to explain the meaning of the logical constants in terms of the rules of inference that govern their behaviour in proofs.

The *orthodox* version of PTS, developed against the background of natural deduction, can be described as an extended attempt to develop Gentzen's suggestion that 'the introduction [rules] represent, as it were, the 'definitions' of the [logical constants], and the eliminations are no more [...] than the consequences of these definitions'. At its core lies the notion of *harmony*: a kind of balance between the relative strength of the introductions and eliminations of a logical constant that testifies to their successfully defining a logical constant. The quest for a formal property that accurately captures the intuitive notion of harmony has dominated much of PTS. Said quest is the source of *less orthodox* versions of PTS. By and large, these retain the focus on harmony, while taking revisionary stances with respect to other aspects of orthodox PTS, such as the priority of the standard assertionist setting of PTS, or of natural deduction.

The first part of the tutorial will critically discuss orthodox PTS, focusing on the development of different conceptions of harmony and their connection with formal properties of Gentzen-style calculi, such as reducibility, invertibility and normalizability or cut-elimination.

The second part of the tutorial is devoted to less orthodox stances in PTS. We will first look at bilateralist versions of the programme, which put denial on a par with assertion and thus introduce a new dimension of harmony, between the conditions for asserting and, respectively, denying a sentence. Finally, we will look at versions of PTS that take the sequent calculus as the framework of choice for specifying definitional rules for the logical connectives. We will explore the motivation(s) for going down this path and some results obtained within this framework.

A detailed list of the topics of the tutorial, including recommended readings, is available at `bdicher.me` *in the section* PSAST.

TUTORIAL ON PROOF-THEORETIC SEMANTICS

BOGDAN DICHER

LanCog Group, Centre of Philosophy of the University of Lisbon, Lisbon, Portugal

1. Topics

(1) Orthodox PTS
  (a) The Gentzenian roots of PTS
    (i) Gentzen style calculi
    (ii) Introductions as definitions; philosophical grounding and harmony
    (iii) Reducibility and the justification of the eliminations; proof-theoretic validity
  (b) Further developments
    (i) Prior's *tonk* and a different measure of harmony
    (ii) Conservativeness, normalisability and cut-elimination; the subformula property
    (iii) Local and global harmony; definitional rules revisited
    (iv) Weak and strong disharmony; stability
(2) Less orthodox PTS
  (a) Bilateralism
    (i) Assertion and denial on all fours
    (ii) Bilateralism in natural deduction; which rules?
    (iii) Bilateralism in the sequent calculus
  (b) The sequent calculus as a basis for PTS
    (i) The metatheoretic interpretations of the sequent calculus; sequents as relata of consequence
    (ii) Harmony and invertibility
    (iii) Non-transitive and non-reflexive logics; invertibility revisited

Literature guide

The classics. The fundamental texts of PTS are [12, 17] and [9]. Another important historical reference is [16], although this tutorial will not engage much with this strand of PTS.

Overviews. For a fast paced yet comprehensive survey of many important topics in PTS, see [30]. [11] is the only extant modern monograph on the topic, reporting a wealth of new results mainly in the orthodox tradition, with interesting glimpses into bilateralism as well.

The texts mentioned in the previous two paragraph just about cover topic (I), particularly if the list of references in [30] is used as a resource. An extensive analysis of the extant conceptions of proof-theoretic validity can be found in [25].

Prior's *tonk* is introduced in [19]; PTS (in particular [9]) draws heavily on the solution to it proposed in [1]. For a wider perspective on Belnap's approach to definitional success, see [22]. The relation between normalisation and cut-elimination is

E-mail address: bdicher@me.com.

explored at length in [33]. The mysteriously titled topic 'definitional rules revisited' has to do with the topics broached in [18]; see also [2], [27]; see also [7].

For topics b ii an iii (local and global harmony) as well as b iv, one can start from [31] and [5]. [20] is important for reviving Gentzen's idea that the elimination rules are a 'function' of the introductions. For the last topic (stability), see also [15].

Proof-theoretic bilateralism is by now a well-established field of inquiry. Against the background of natural deduction, it has resurged with particular vigour in [23]; see also [11]. For a historically informed wider analysis of bilaterlaism, see [14]. Rumfitt's proposal is problematic, as shown by [10] and his solution ([24]) less than satisfactory. See also [3]. For sequent-calculus approaches to bilateralism, see [21] and also [6]. For a different take on the same issue, see [26].

For the topics under 2(b): the so-called metatheoretic interpretation of the sequent calculus is pervasive, albeit mostly understated, throughout the history of PTS. Clear statements of the position can be found in [13, 32]. A forceful criticism of the view appears in [29]; see also [28]. The view that sequent are the proper consequence carrier if further developed in [8]. A tentative theory of harmony appropriate for this new setting is provided in [4].

References

[1] Nuel D. Belnap. Tonk, plonk and plink. *Analysis*, 22:130–134, 1962.

[2] Ruy de Queiroz. On reduction rules, meaning-as-use, and proof-theoretic semantics. *Studia Logica*, 2008.

[3] Pedro del Valle-Inclan and Julian J Schlöder. Coordination and Harmony in Bilateral Logic. *Mind*, 07 2022. fzac012.

[4] B. Dicher and F. Paoli. On harmony and metasequents (or: Harmony without identity and cut). Manuscript in progress.

[5] Bogdan Dicher. Weak disharmony: some lessons for proof-theoretic semantics. *Review of Symbolic Logic*, 9(3):583–602, 2016.

[6] Bogdan Dicher. Ask not what bilateralist intuitionists can do for Cut, but what Cut can do for bilateralist intuitionism. *Analysis*, 80(1):30–40, 06 2019.

[7] Bogdan Dicher. Hopeful monsters: A note on multiple conclusions. *Erkenntnis*, 85:77–98, 2020.

[8] Bogdan Dicher and Francesco Paoli. The original sin of proof-theoretic semantics. *Synthese*, 198:615–640, 2021.

[9] Michael Dummett. *The Logical Basis of Metaphysics*. Duckworth, London, 1991.

[10] Fernando Ferreira. The Co-ordination Principles: A Problem for Bilateralism. *Mind*, 117(468):1051–1057, 10 2008.

[11] Nissim Francez. *Proof-theoretic semantics*. College Publications, 2015.

[12] Gerhard Gentzen. Untersuchungen über das logische schließen. I,II. *Mathematische Zeitschrift*, 39(1):176–210; 405–431, 1935.

[13] Ian Hacking. What is logic? *The Journal of Philosophy*, 76:285–319, 1979.

[14] Lloyd Humberstone. The revival of rejective negation. 29(4):331–381, 2000.

[15] Bruno Jacinto and Stephen Read. General elimination stability. *Studia Logica*, 105:361–405, 2017.

[16] Per Martin-Löf. On the meaning of the logical constants and the justification of the logial laws. *Nordic Journal of Philosophical Logic*, 1:11–69, 1996.

[17] Dag Prawitz. *Natural Deduction: A Proof Theoretical Study*. Almqvist and Wiksell, Stockholm, 1965.

[18] Dag Prawitz. Pragmatist and verificationist theories of meaning. In R.E. Auxier and L.E. Hahn, editors, *The Philosophy of Michael Dummett*, pages 455–481. Open Court, 2007.

[19] Arthur N. Prior. The runabout inference-ticket. *Analysis*, 21(2):38–39, 1960.

[20] Stephen Read. General-elimination harmony and the meaning of the logical constants. *Journal of Philosophical Logic*, 39:557–576, 2010.

[21] Greg Restall. Multiple conclusions. In Petr Hájek, Luis Valdés-Villanueva, and Dag Westerståhl, editors, *Logic, Methodology and Philosophy of Science: Proceedings of the Twelfth International Congress*, pages 189–205. KCL Publications, 2005.

[22] Greg Restall. Proof theory and meaning: On the context of deducibility. In *Logic Colloquium 2007*, pages 204–219. Cambridge University Press, 2010.

[23] Ian Rumfitt. "Yes" and "No". *Mind*, 109(436):781–823, 2000.
[24] Ian Rumfitt. Knowledge by deduction. *Grazer Philosophische Studien*, 77(61–84), 2008.
[25] Peter Schroeder-Heister. Validity concepts in proof-theoretic semantics. *Synthese*, 148(3):525–571, February 2006.
[26] Peter Schroeder-Heister. Definitional reasoning in proof-theoretic semantics and the square of opposition. In Jean-Yves Béziau and Gillman Payette, editors, *The Square of Opposition: A General Framework for Cognition*, pages 323–349. Peter Lang, 2007.
[27] Peter Schroeder-Heister. Generalized definitional reflection and the inversion principle. *Logica Universalis*, 1:355–376, 2007.
[28] Peter Schroeder-Heister. Sequent calculi and bidirectional natural deduction: On the proper basis of proof-theoretic semantics. *The logica yearbook*, pages 237–251, 2008.
[29] Peter Schroeder-Heister. The categorical and the hypothetical: a critique of some fundamental assumptions of standard semantics. *Synthese*, 187(3):925–942, March 2011.
[30] Peter Schroeder-Heister. Proof-theoretic semantics. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Spring 2018 edition, 2018.
[31] Florian Steinberger. What harmony could and could not be. *Australasian Journal of Philosophy*, 89:617–639, 2011.
[32] Neil Tennant. *Core Logic*. Oxford University Press, 2017.
[33] A. M. Ungar. *Normalization, cut-elimination, and the theory of proofs*. CSLI Publications, 1992.