ECS 203 (ME2) - Part 3A
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CHAPTER 9

Sinusoids and Phasors

We now begins the analysis of circuits in which the voltage or current sources are time-varying. In this chapter, we are particularly interested in sinusoidally time-varying excitation, or simply, excitation by a sinusoid.

9.0.1. Some terminology:

(a) A **sinusoid** is a signal that has the form of the sine or cosine function.
   - Turn out that you can express them all under the same notation using only cosine (or only sine) function. We ill use cosine.

(b) A sinusoidal current is referred to as **alternating current (AC)**.

(c) Circuits driven by sinusoidal current or voltage sources are called **AC circuits**.

9.1. Sinusoids

9.1.1. Consider the sinusoidal signal (in cosine form)

$$x(t) = X_m \cos(\omega t + \phi) = X_m \cos(2\pi f t + \phi),$$

where

X_m: the amplitude of the sinusoid,
ω: the angular frequency in radians/s (or rad/s),
ϕ: the phase.

- First, we consider the case when $\phi = 0$:

- When $\phi \neq 0$, we shift the graph of $X_m \cos(\omega t)$ to the ***left*** "by ϕ".

9.1.2. The **period** (the time of one complete cycle) of the sinusoid is

$$T = \frac{2\pi}{\omega}.$$

The unit of the period is in second if the angular frequency unit is in radian per second.

The **frequency** f (the number of cycles per second or hertz (Hz)) is the reciprocal of this quantity, i.e.,

$$f = \frac{1}{T}.$$

9.1.3. ***Standard form*** for sinusoid: In this class, when you are asked to find the sinusoid representation of a signal, make sure that your answer is in the form

$$x(t) = X_m \cos(\omega t + \phi) = X_m \cos(2\pi f t + \phi),$$

where X_m is nonnegative and ϕ is between $-180°$ and $+180°$.

- When the signal is given in the sine form, it can be converted into its cosine form via the identity

  $$\sin(x) = \cos(x - 90°).$$

  In particular,

  $$X_m \sin(\omega t + \phi) = X_m \cos(\omega t + \phi - 90°).$$

- X_m is always non-negative. We can avoid having the negative sign by the following conversion:

$$-\cos(x) = \cos(x \pm 180^\circ).$$

In particular,

$$-A\cos(\omega t + \phi) = A\cos(2\pi f t + \phi \pm 180^\circ).$$

Note that usually you do not have the choice between $+180^\circ$ or -180°. The one that you need to use is the one that makes $\phi \pm 180^\circ$ falls somewhere between -180° and $+180^\circ$.

9.2. Phasors

Sinusoids are easily expressed in terms of phasors, which are more convenient to work with than sine and cosine functions. The tradoff is that phasors are complex-valued.

9.2.1. The idea of phasor representation is based on Euler's identity:

$$e^{j\phi} = \cos\phi + j\sin\phi,$$

From the identity, we may regard $\cos\phi$ and $\sin\phi$ as the real and imaginary parts of $e^{j\phi}$:

$$\cos\phi = \mathrm{Re}\left\{e^{j\phi}\right\}, \quad \sin\phi = \mathrm{Im}\left\{e^{j\phi}\right\},$$

where Re and Im stand for the real part of and the imaginary part of $e^{j\phi}$.

9.2.2. A phasor is a complex number that represents the amplitude and phase of a sinusoid. Given a sinusoid $v(t) = V_m\cos(\omega t + \phi)$, then

$$v(t) = V_m\cos(\omega t+\phi) = \mathrm{Re}\left\{V_m e^{j(\omega t+\phi)}\right\} = \mathrm{Re}\left\{V_m e^{j\phi}\cdot e^{j\omega t}\right\} = \mathrm{Re}\left\{\mathbf{V}e^{j\omega t}\right\},$$

where

$$\mathbf{V} = V_m e^{j\phi} = V_m\angle\phi.$$

$\mathbf{V}$ is called the ***phasor representation*** of the sinusoid $v(t)$. In other words, a phasor is a complex number that represents amplitude and phase of a sinusoid.

9.2.3. Remarks:

- Whenever a sinusoid is expressed as a phasor, the term $e^{j\omega t}$ is **implicitly** present. It is therefore important, when dealing with phasors, to keep in mind the (angular) frequency ω of the phasor.

- To obtain the sinusoid corresponding to a given phasor $\mathbf{V}$, multiply the phasor by the time factor $e^{j\omega t}$ and take the real part.

  Equivalently, given a phasor, we obtain the time-domain representation as the cosine function with the same magnitude as the phasor and the argument as ωt plus the phase of the phasor.
- Any complex number z (including any phasor) can be equivalently represented in three forms.
  (a) Rectangular form: $z = x + jy$.
  (b) Polar form: $z = r\angle\phi$.
  (c) Exponential form: $z = re^{j\phi}$

  where the relations between them are

  $$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\frac{y}{x} \pm 180^\circ.$$

  $$x = r\cos\phi, \quad y = r\sin\phi.$$

  Note that for ϕ, the choice of using $+180^\circ$ or -180° in the formula is determined by the actual quadrant in which the complex number lies.

- As a complex quantity, a phasor may be expressed in rectangular form, polar form, or exponential form. In this class, we focus on polar form.

9.2.4. ***Summary***: By suppressing the time factor, we transform the sinusoid from the time domain to the phasor domain. This transformation is summarized as follows:

$$v(t) = V_m\cos(\omega t + \phi) \Leftrightarrow \mathbf{V} = V_m\angle\phi.$$

Time domain representation $\Leftrightarrow$ Phasor domain representation

9.2.5. ***Standard form*** for phasor: In this class, when you are asked to find the phasor representation of a signal, make sure that your answer is a complex number in polar form, i.e. $r\angle\phi$ where r is nonnegative and ϕ is between $-180°$ and $+180°$.

EXAMPLE 9.2.6. Transform these sinusoids to phasors:
(a) $i = 6\cos(50t - 40°)$ A

(b) $v = -4\sin(30t + 50°)$ V

EXAMPLE 9.2.7. Find the sinusoids represented by these phasors:
(a) $\mathbf{I} = -3 + j4$ A

(b) $\mathbf{V} = j8e^{-j20°}$ V

9.2.8. The differences between $v(t)$ and $\mathbf{V}$ should be emphasized:
(a) $v(t)$ is the instantaneous or time-domain representation, while V is the frequency or phasor-domain representation.
(b) $v(t)$ is time dependent, while $\mathbf{V}$ is not.
(c) $v(t)$ is always real with no complex term, while $\mathbf{V}$ is generally complex.

9.2.9. Adding sinusoids of the *same frequency* is equivalent to adding their corresponding phasors. To see this,

$$\begin{aligned} A_1\cos(\omega t + \phi_1) + A_2\cos(\omega t + \phi_2) &= \mathrm{Re}\left\{\mathbf{A}_1 e^{j\omega t}\right\} + \mathrm{Re}\left\{\mathbf{A}_2 e^{j\omega t}\right\} \\ &= \mathrm{Re}\left\{(\mathbf{A}_1 + \mathbf{A}_2)\, e^{j\omega t}\right\}. \end{aligned}$$

9.2.10. Properties involving differentiation and integration:

(a) **Differentiating** a sinusoid is equivalent to multiplying its corresponding phasor by $j\omega$. In other words,

$$\frac{dv(t)}{dt} \Leftrightarrow j\omega\mathbf{V}.$$

To see this, suppose $v(t) = V_m \cos(\omega t + \phi)$. Then,

$$\begin{aligned}\frac{dv}{dt}(t) &= -\omega V_m \sin(\omega t + \phi) = \omega V_m \cos(\omega t + \phi - 90^\circ + 180^\circ) \\ &= \operatorname{Re}\left\{\omega V_m e^{j\phi} e^{j90^\circ} \cdot e^{j\omega t}\right\} = \operatorname{Re}\left\{j\omega \mathbf{V} e^{j\omega t}\right\}\end{aligned}$$

Alternatively, express $v(t)$ as

$$v(t) = \operatorname{Re}\left\{V_m e^{j(\omega t + \phi)}\right\}.$$

Then,

$$\frac{d}{dt}v(t) = \operatorname{Re}\left\{V_m j\omega e^{j(\omega t+\phi)}\right\}.$$

(b) **Integrating** a sinusoid is equivalent to dividing its corresponding phasor by $j\omega$. In other words,

$$\int v(t)dt \Leftrightarrow \frac{\mathbf{V}}{j\omega}.$$

EXAMPLE 9.2.11. Find the voltage $v(t)$ in a circuit described by the intergrodifferential equation

$$2\frac{dv}{dt} + 5v + 10\int v dt = 20\cos(5t - 30^\circ)$$

using the phasor approach.

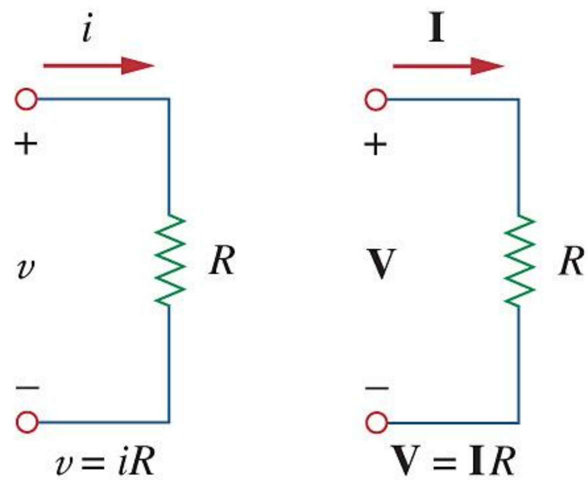
9.3. Phasor relationships for circuit elements

9.3.1. Resistor R: If the current through a resistor R is

$$i(t) = I_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{I} = I_m \angle \phi,$$

the voltage across it is given by

$$v(t) = i(t)R = RI_m \cos(\omega t + \phi).$$



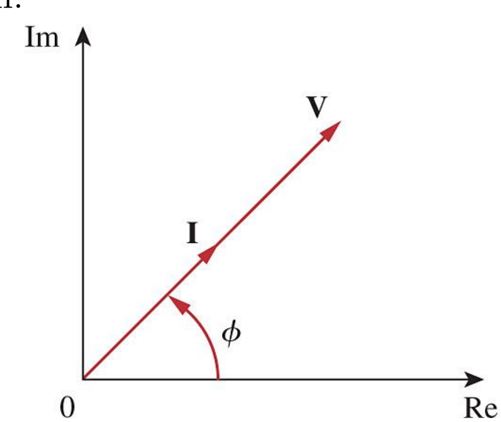
The phasor of the voltage is

$$\mathbf{V} = RI_m \angle \phi.$$

Hence,

$$\mathbf{V} = \mathbf{I}R.$$

We note that voltage and current are **in phase** and that the voltage-current relation for the resistor in the phasor domain continues to be Ohms law, as in the time domain.

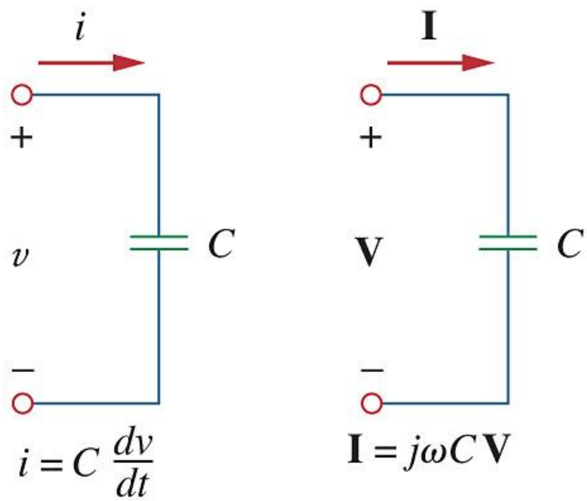


9.3.2. Capacitor C: If the voltage across a capacitor C is

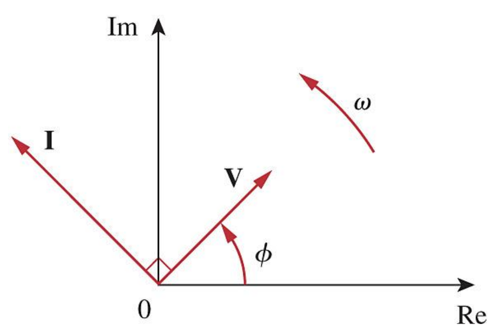
$$v(t) = V_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{V} = V_m \angle \phi,$$

the current through it is given by

$$i(t) = C\frac{dv(t)}{dt} \Leftrightarrow \mathbf{I} = j\omega C\mathbf{V} = \omega C V_m \angle(\phi + 90^\circ).$$



The voltage and current are 90° out of phase. Specifically, the current leads the voltage by 90°.



- Mnemonic: CIVIL

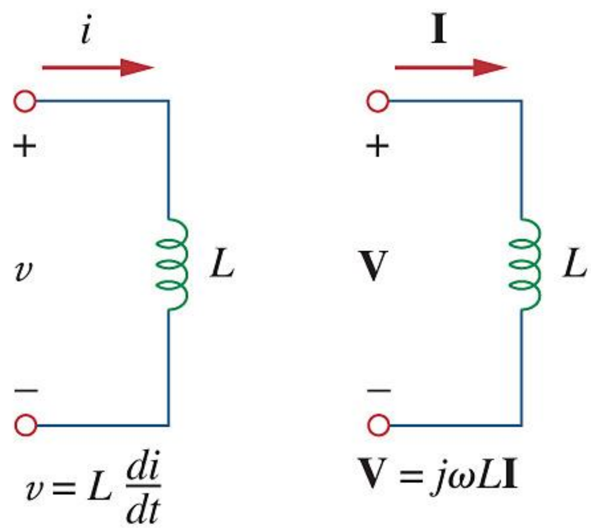
  In a Capacitive (C) circuit, I leads V. In an inductive (L) circuit, V leads V.

9.3.3. Inductor L: If the current through an inductor L is

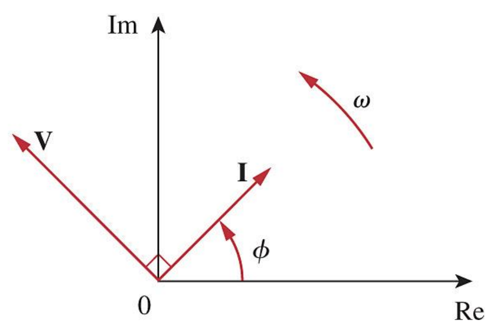
$$i(t) = I_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{I} = I_m \angle \phi,$$

the voltage across it is given by

$$v(t) = L\frac{di(t)}{dt} \Leftrightarrow \mathbf{V} = j\omega L\mathbf{I} = \omega L I_m \angle(\phi + 90^\circ).$$



The voltage and current are 90° out of phase. Specifically, the current lags the voltage by 90°.



Element	Time domain	Frequency domain
R	$v = Ri$	$\mathbf{V} = R\mathbf{I}$
L	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$
C	$i = C\dfrac{dv}{dt}$	$\mathbf{V} = \dfrac{\mathbf{I}}{j\omega C}$

9.4. Impedance and Admittance

Thus, we obtained the voltage current relations for the three passive elements as

$$\mathbf{V} = \mathbf{I}R, \quad \mathbf{V} = j\omega L\mathbf{I}, \quad \mathbf{I} = j\omega C\mathbf{V}.$$

These equations may be written in terms of the ratio of the phasor voltage to the phasor of current as

$$\frac{\mathbf{V}}{\mathbf{I}} = R, \quad \frac{\mathbf{V}}{\mathbf{I}} = j\omega L, \quad \frac{\mathbf{V}}{\mathbf{I}} = \frac{1}{j\omega C}.$$

From these equations, we obtain Ohm's law in phasor form for any type of element as

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{or} \quad \mathbf{V} = \mathbf{I}\mathbf{Z}.$$

Definition 9.4.1. The impedance $\mathbf{Z}$ of a circuit is the ratio of the phasor voltage $\mathbf{V}$ to the phasor current $\mathbf{I}$, measured in ohms (Ω).

As a complex quantity, the impedance may be expressed in rectangular form as

$$\mathbf{Z} = R + jX = |\mathbf{Z}|\angle\theta,$$

with

$$|\mathbf{Z}| = \sqrt{R^2 + X^2}, \quad \theta = \tan^{-1}\frac{X}{R}, \quad R = |\mathbf{Z}|\cos\theta, \quad X = |\mathbf{Z}|\sin\theta.$$

$R = \operatorname{Re}\{\mathbf{Z}\}$ is called the **resistance** and $X = \operatorname{Im}\{\mathbf{Z}\}$ is called the **reactance**.

The reactance X may be positive or negative. We say that the impedance is **inductive** when X is positive or **capacitive** when X is negative.

Definition 9.4.2. The **admittance** ($\mathbf{Y}$) is the reciprocal of impedance, measured in Siemens (S). The admittance of an element(or a circuit) is the ratio of the phasor current through it to phasor voltage across it, or

$$\mathbf{Y} = \frac{\mathbf{1}}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}}.$$

9.4.3. **Kirchhoff's laws (KCL and KVL) hold in the phasor form**.

To see this, suppose $v_1, v_2, \dots, v_n$ are the voltages around a closed loop, then

$$v_1 + v_2 + \cdots + v_n = 0.$$

If each voltage v_i is a sinusoid, i.e.

$$v_i = V_{mi}\cos(\omega t + \phi_i) = \operatorname{Re}\left\{\mathbf{V}_i e^{j\omega t}\right\}$$

with phasor $\mathbf{V}_i = V_{mi}\angle\phi_i = V_{mi}e^{j\phi_i}$, then

$$\operatorname{Re}\left\{\left(\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n\right)e^{j\omega t}\right\} = 0,$$

which must be true for all time t. To satisfy this, we need

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0.$$

Hence, KVL holds for phasors.

Similarly, we can show that KCL holds in the frequency domain, i.e., if the currents $i_1, i_2, \dots, i_n$ be the currents entering or leaving a closed surface at time t, then

$$i_1 + i_2 + \cdots + i_n = 0.$$

If the currents are sinusoids and $\mathbf{I}_1, \mathbf{I}_2, \dots, \mathbf{I}_n$ are their phasor forms, then

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0.$$

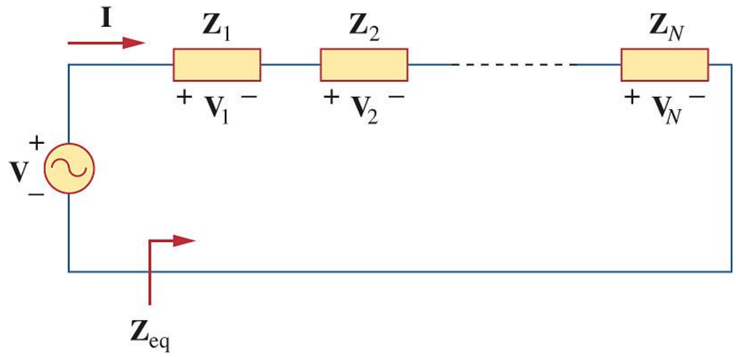
Major Implication: Since Ohm's Law and Kirchoff's Laws hold in phasor domain, **all resistance combination, analysis methods** (nodal and mesh analysis) **and circuit theorems** (linearity, superposition, source transformation, and Thevenin's and Norton's equivalent circuits) that we have previously studied for dc circuits **apply to ac circuits**!!!

Just think of impedance as a complex-valued resistance!!

In addition, our ac circuits can now effortlessly include capacitors and inductors which can be considered as impedances whose values depend on the frequency ω of the ac sources!!

9.5. Impedance Combinations

Consider N series-connected impedances as shown below.



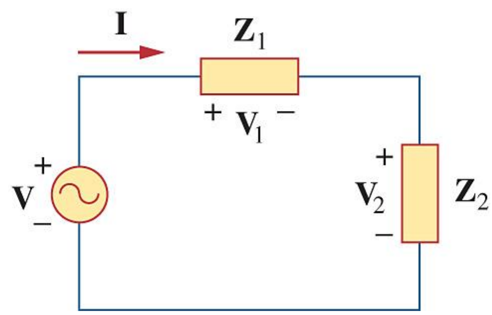
The same current $\mathbf{I}$ flows through the impedances. Applying KVL around the loop gives

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_N = \mathbf{I}(\mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N)$$

The equivalent impedance at the input terminals is

$$\mathbf{Z}_{eq} = \frac{\mathbf{V}}{\mathbf{I}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_N.$$

In particular, if $N = 2$, the current through the impedance is



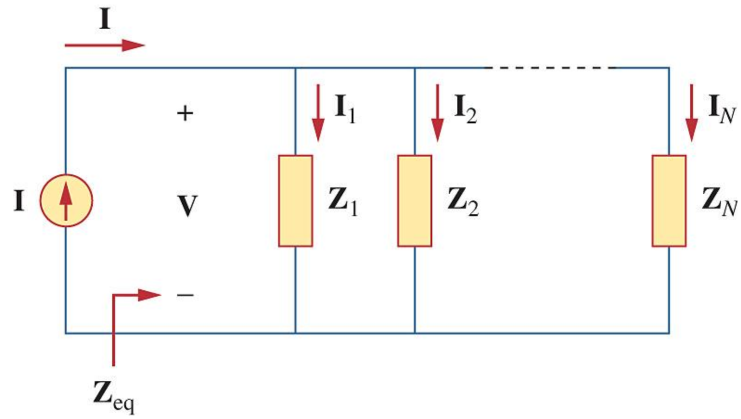
$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_1 + \mathbf{Z}_2}.$$

Because $\mathbf{V}_1 = \mathbf{Z}_1\mathbf{I}$ and $\mathbf{V}_2 = \mathbf{Z}_2\mathbf{I}$,

$$\mathbf{V_1} = \frac{\mathbf{Z_1}}{\mathbf{Z_1} + \mathbf{Z_2}}\mathbf{V}, \quad \mathbf{V_2} = \frac{\mathbf{Z_2}}{\mathbf{Z_1} + \mathbf{Z_2}}\mathbf{V}$$

which is the **voltage-division** relationship.

Now, consider N parallel-connected impedances as shown below.



The voltage across each impedance is the same. Applying KCL at the top node gives

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_N = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}\right).$$

The equivalent impedance $\mathbf{Z}_{eq}$ can be found from

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \cdots + \frac{1}{\mathbf{Z}_N}.$$

When $N = 2$,

$$\mathbf{Z}_{eq} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}.$$

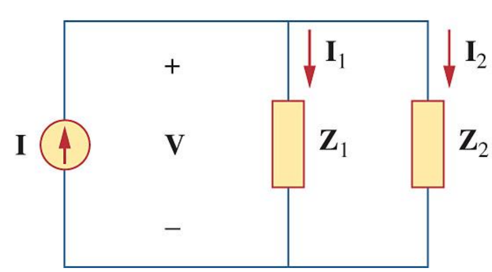
Because

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}_1\mathbf{Z}_1 = \mathbf{I}_2\mathbf{Z}_2,$$

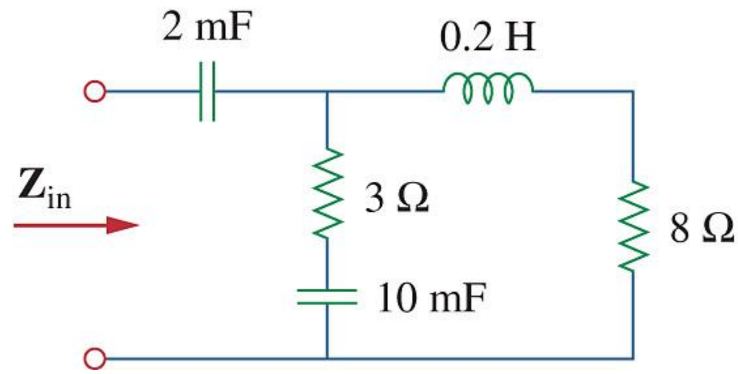
we have

$$\mathbf{I_1} = \frac{\mathbf{Z_2}}{\mathbf{Z_1} + \mathbf{Z_2}}\mathbf{I}, \quad \mathbf{I_2} = \frac{\mathbf{Z_1}}{\mathbf{Z_1} + \mathbf{Z_2}}\mathbf{I}$$

which is the **current-division** principle.



EXAMPLE 9.5.1. Find the input impedance of the circuit below. Assume that the circuit operates at $\omega = 50$ rad/s.



EXAMPLE 9.5.2. Determine $v_o(t)$ in the circuit below.