More on Matrix Inversion

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Opening Remarks

- The following statements are equivalent statements about $A \in \mathbb{R}^{n\times n}$:
  - A is nonsingular.
  - A is invertible.
  - A^{-1} exists.
  - $AA^{-1} = A^{-1}A = I$.
  - A represents a linear transformation that is a bijection.
  - $Ax = b$ has a unique solution for all $\mathbf{b} \in \mathbb{R}^n$.
  - $Ax = 0$ implies that $x = 0$.
  - $Ax = e_j$ has a solution for all $j \in \{0, \ldots, n-1\}$.

Opening Remarks

Homework 8.1.1.1 Assume that $A, B, C \in \mathbb{R}^{n\times n}$, let $BA = C$, and B be nonsingular.

A is nonsingular if and only if C is nonsingular.

True/False

- Proof
  - ($\Rightarrow$) Assume A is nonsingular.
  - ($\Leftarrow$) Assume C is nonsingular.

Opening Remarks

- Gauss transforms & Permutation

$$A = \begin{pmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\times & 1 & 0 & 0 & 0 \\ -\times & 0 & 1 & 0 & 0 \\ -\times & 0 & 0 & 1 & 0 \\ -\times & 0 & 0 & 0 & 1 \end{pmatrix} P_0 \begin{pmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{pmatrix} = \begin{pmatrix} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\times & 1 & 0 & 0 \\ 0 & -\times & 0 & 1 & 0 \\ 0 & -\times & 0 & 0 & 1 \end{pmatrix} P_1 \begin{pmatrix} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \\ 0 & x & x & x & x \end{pmatrix} = \begin{pmatrix} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & x & x & x \end{pmatrix}$$

When LU Factorization with Row Pivoting Fails?

Opening Remarks

- When LU Factorization with Row Pivoting Fails

$$\overbrace{\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&-\times&1&0&0\\0&-\times&0&1&0\\0&-\times&0&0&1\end{pmatrix}}^{\widetilde{L}_1} P_1 \overbrace{\begin{pmatrix}\times&\times&\times&\times&\times\\0&\times&\times&\times&\times\\0&\times&\times&\times&\times\\0&\times&\times&\times&\times\\0&\times&\times&\times&\times\end{pmatrix}}^{\widetilde{L}_0 P_0 A} = \overbrace{\left(\begin{array}{cc|c|cc}\times&\times&\times&\times&\times\\0&\times&\times&\times&\times\\ \hline 0&0&0&\times&\times\\0&0&0&\times&\times\\0&0&0&\times&\times\end{array}\right)}^{\left(\begin{array}{c|c|c}U_{00}&u_{01}&U_{02}\\ \hline 0&\alpha_{11}&a_{12}^T\\ \hline 0&a_{21}&A_{22}\end{array}\right)}.$$

$$\left(\begin{array}{c|c|c}U_{00}&u_{01}&U_{02}\\ \hline 0&0&a_{12}^T\\ \hline 0&0&A_{22}\end{array}\right) \overbrace{\left(\begin{array}{c}-U_{00}^{-1}u_{01}\\ \hline 1\\ \hline 0\end{array}\right)}^{x} = \left(\begin{array}{c}-U_{00}U_{00}^{-1}u_{01}+u_{01}\\ \hline 0\\ \hline 0\end{array}\right) = \left(\begin{array}{c}0\\ \hline 0\\ \hline 0\end{array}\right)$$

Opening Remarks

Algorithm: $[A,p] := \text{LU\_PIV}(A,p)$

Partition $A \rightarrow \left(\begin{array}{c|c}A_{TL}&A_{TR}\\ \hline A_{BL}&A_{BR}\end{array}\right)$, $p \rightarrow \left(\begin{array}{c}p_T\\ \hline p_B\end{array}\right)$

where A_{TL} is 0×0 and p_T has 0 components

while $m(A_{TL}) < m(A)$ do

Repartition

$$\left(\begin{array}{c|c}A_{TL}&A_{TR}\\ \hline A_{BL}&A_{BR}\end{array}\right) \rightarrow \left(\begin{array}{c|c|c}A_{00}&a_{01}&A_{02}\\ \hline a_{10}^T&\alpha_{11}&a_{12}^T\\ \hline A_{20}&a_{21}&A_{22}\end{array}\right), \left(\begin{array}{c}p_T\\ \hline p_B\end{array}\right) \rightarrow \left(\begin{array}{c}p_0\\ \hline \pi_1\\ \hline p_2\end{array}\right)$$

$$\pi_1 = \text{PIVOT}\left(\left(\begin{array}{c}\alpha_{11}\\ \hline a_{21}\end{array}\right)\right)$$

$$\left(\begin{array}{c|c|c}a_{10}^T&\alpha_{11}&a_{12}^T\\ \hline A_{20}&a_{21}&A_{22}\end{array}\right) := P(\pi_1)\left(\begin{array}{c|c|c}a_{10}^T&\alpha_{11}&a_{12}^T\\ \hline A_{20}&a_{21}&A_{22}\end{array}\right)$$

$a_{21} := a_{21}/\alpha_{11}$ (a_{21} now contains l_{21})

$$\left(\begin{array}{c}a_{12}^T\\ \hline A_{22}\end{array}\right) = \left(\begin{array}{c}a_{12}^T\\ \hline A_{22}-a_{21}a_{12}^T\end{array}\right)$$

Continue with

$$\left(\begin{array}{c|c}A_{TL}&A_{TR}\\ \hline A_{BL}&A_{BR}\end{array}\right) \leftarrow \left(\begin{array}{c|c|c}A_{00}&a_{01}&A_{02}\\ \hline a_{10}^T&\alpha_{11}&a_{12}^T\\ \hline A_{20}&a_{21}&A_{22}\end{array}\right), \left(\begin{array}{c}p_T\\ \hline p_B\end{array}\right) \leftarrow \left(\begin{array}{c}p_0\\ \hline \pi_1\\ \hline p_2\end{array}\right)$$

endwhile

$$\left(\begin{array}{c|c|c}U_{00}&u_{01}&U_{02}\\ \hline 0&\alpha_{11}&a_{12}^T\\ \hline 0&a_{21}&A_{22}\end{array}\right)$$

$$\left(\begin{array}{c|c|c}U_{00}&u_{01}&U_{02}\\ \hline 0&0&a_{12}^T\\ \hline 0&0&A_{22}\end{array}\right)$$

Gauss-Jordan Elimination

- Solving $Ax = b$ via Gauss-Jordan Elimination
- Solving $Ax = b$ via Gauss-Jordan Elimination: Gauss Transforms
- Solving $Ax = b$ via Gauss-Jordan Elimination: Multiple Right-Hand Sides
- Computing A^{-1} via Gauss-Jordan Elimination
- Computing A^{-1} via Gauss-Jordan Elimination, Alternative
- Pivoting
- Cost of Matrix Inversion

Gauss-Jordan Elimination

- Solving $Ax = b$ via Gauss-Jordan Elimination

Homework 8.2.1.1 Perform the following steps

- To transform the system on the left to the one on the right:

$$\begin{array}{rcrcrcr}-2\chi_0&+&2\chi_1&-&5\chi_2&=&-7\\ 2\chi_0&-&3\chi_1&+&7\chi_2&=&11\\ -4\chi_0&+&3\chi_1&-&7\chi_2&=&-9\end{array} \longrightarrow \begin{array}{rcrcrcr}-2\chi_0&+&2\chi_1&-&5\chi_2&=&-7\\ &&-\chi_1&+&2\chi_2&=&4\\ &&-\chi_1&+&3\chi_2&=&5\end{array}$$

one must subtract $\lambda_{1,0} = \square$ times the first row from the second row and subtract $\lambda_{2,0} = \square$ times the first row from the third row.

- To transform the system on the left to the one on the right:

$$\begin{array}{rcrcrcr}-2\chi_0&+&2\chi_1&-&5\chi_2&=&-7\\ &&-\chi_1&+&2\chi_2&=&4\\ &&-\chi_1&+&3\chi_2&=&5\end{array} \longrightarrow \begin{array}{rcrcrcr}-2\chi_0&&&-&\chi_2&=&1\\ &&-\chi_1&+&2\chi_2&=&4\\ &&&&\chi_2&=&1\end{array}$$

one must subtract $\upsilon_{0,1} = \square$ times the second row from the first row and subtract $\lambda_{2,1} = \square$ times the

Gauss-Jordan Elimination

- Solving $Ax = b$ via Gauss-Jordan Elimination

- To transform the system on the left to the one on the right:

$$\begin{array}{rcrcrcr} -2\chi_0 & & & - & \chi_2 & = & 1 \\ & & -\chi_1 & + & 2\chi_2 & = & 4 \\ \hline & & & & \chi_2 & = & 1 \\ \hline \end{array} \longrightarrow \begin{array}{rcrcrcr} -2\chi_0 & & & & & = & 2 \\ & & -\chi_1 & & & = & 2 \\ \hline & & & & \chi_2 & = & 1 \\ \hline \end{array}$$

one must subtract $\upsilon_{0,2} = \Box$ times the third row from the first row and subtract $\upsilon_{1,2} = \Box$ times the third row from the first row.

- To transform the system on the left to the one on the right:

$$\begin{array}{rrrcr} -2\chi_0 & & & = & 2 \\ & -\chi_1 & & = & 2 \\ & & \chi_2 & = & 1 \end{array} \longrightarrow \begin{array}{rrrcr} \chi_0 & & & = & -1 \\ & \chi_1 & & = & -2 \\ & & \chi_2 & = & 1 \end{array}$$

one must multiply the first row by $\delta_{0,0} = \Box$, the second row by $\delta_{1,1} = \Box$, and the third row by $\delta_{2,2} = \Box$.

- Use the above exercises to compute the vector x that solves

$$\begin{array}{rcrcrcr} -2\chi_0 & + & 2\chi_1 & - & 5\chi_2 & = & -7 \\ 2\chi_0 & - & 3\chi_1 & + & 7\chi_2 & = & 11 \\ -4\chi_0 & + & 3\chi_1 & - & 7\chi_2 & = & -9 \end{array}$$

Gauss-Jordan Elimination

- Solving $Ax = b$ via Gauss-Jordan Elimination

Homework 8.2.1.2 Perform the process illustrated in the last exercise to solve the systems of linear equations

- $\begin{pmatrix} 3 & 2 & 10 \\ -3 & -3 & -14 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} -7 \\ 9 \\ -5 \end{pmatrix}$
- $\begin{pmatrix} 2 & -3 & 4 \\ 2 & -2 & 3 \\ 6 & -7 & 9 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} -8 \\ -5 \\ -17 \end{pmatrix}$

Gauss-Jordan Elimination

- Solving $Ax = b$ via Gauss-Jordan Elimination: Gauss Transforms

$$\begin{array}{rcrcrcr} -2\chi_0 & + & 2\chi_1 & - & 5\chi_2 & = & -7 \\ 2\chi_0 & - & 3\chi_1 & + & 7\chi_2 & = & 11 \\ -4\chi_0 & + & 3\chi_1 & - & 7\chi_2 & = & -9 \end{array}$$

$$\left(\begin{array}{rrr||r} -2 & 2 & -5 & -7 \\ 2 & -3 & 7 & 11 \\ -4 & 3 & -7 & -9 \end{array}\right)$$

Gauss-Jordan Elimination

- Solving $Ax = b$ via Gauss-Jordan Elimination: Gauss Transforms

$$\left(\begin{array}{cc|c|cc} 1 & 0 & -\times & 0 & 0 \\ 0 & 1 & -\times & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -\times & 1 & 0 \\ 0 & 0 & -\times & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c|cc||c} x & 0 & x & x & x & x \\ 0 & x & x & x & x & x \\ \hline 0 & 0 & x & x & x & x \\ \hline 0 & 0 & x & x & x & x \\ 0 & 0 & x & x & x & x \end{array}\right) = \left(\begin{array}{cc|c|cc||c} x & 0 & 0 & x & x & x \\ 0 & x & 0 & x & x & x \\ \hline 0 & 0 & x & x & x & x \\ \hline 0 & 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x & x \end{array}\right)$$

$$\left(\begin{array}{c|c|c} I & -u_{01} & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -l_{21} & I \end{array}\right) \left(\begin{array}{c|c|c||c} D_{00} & a_{01} & A_{02} & b_0 \\ \hline 0 & a_{11} & a_{12}^T & \beta_1 \\ \hline 0 & a_{21} & A_{22} & b_2 \end{array}\right) = ?$$

$$\left(\begin{array}{c|c|c} I & -u_{01} & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -l_{21} & I \end{array}\right) \left(\begin{array}{c} A_0 \\ \hline a_1^T \\ \hline A_2 \end{array}\right) = \left(\begin{array}{c} A_0 - u_{01}\, a_1^T \\ \hline a_1^T \\ \hline A_2 - l_{21}\, a_1^T \end{array}\right)$$

Gauss-Jordan Elimination

- **Solving $Ax = b$ via Gauss-Jordan Elimination: Gauss Transforms**

Algorithm: $[A, b] :=$ GaussJordan_Part2(A, b)

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where A_{TL} is 0×0, b_T has 0 rows

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

$\beta_1 := \beta_1 / \alpha_{11}$

$\alpha_{11} := 1$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

endwhile

Algorithm: $[A, b] :=$ GaussJordan_Part1(A, b)

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where A_{TL} is 0×0, b_T has 0 rows

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

$a_{01} := a_{01} / \alpha_{11}$ $\quad (= u_{01})$

$a_{21} := a_{21} / \alpha_{11}$ $\quad (= l_{21})$

$A_{02} := A_{02} - a_{01} a_{12}^T$ $\quad (= A_{02} - u_{01} a_{12}^T)$

$A_{22} := A_{22} - a_{21} a_{12}^T$ $\quad (= A_{22} - l_{21} a_{12}^T)$

$b_0 := b_0 - \beta_1 a_{01}$ $\quad (= b_2 - \beta_1 u_{01})$

$b_2 := b_2 - \beta_1 a_{21}$ $\quad (= b_2 - \beta_1 l_{21})$

$a_{01} := 0$ (zero vector)

$a_{21} := 0$ (zero vector)

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

endwhile

Gauss-Jordan Elimination

- **Solving $Ax = b$ via Gauss-Jordan Elimination: Multiple Right-Hand Sides**

$$\begin{array}{rcrcrcr} -2\chi_0 & + & 2\chi_1 & - & 5\chi_2 & = & -7 \\ 2\chi_0 & - & 3\chi_1 & + & 7\chi_2 & = & 11 \\ -4\chi_0 & + & 3\chi_1 & - & 7\chi_2 & = & -9 \end{array}$$

and

$$\begin{array}{rcrcrcr} -2\chi_0 & + & 2\chi_1 & - & 5\chi_2 & = & 8 \\ 2\chi_0 & - & 3\chi_1 & + & 7\chi_2 & = & -13 \\ -4\chi_0 & + & 3\chi_1 & - & 7\chi_2 & = & 9 \end{array}$$

$$\left(\begin{array}{rrr||rr} -2 & 2 & -5 & -7 & 8 \\ 2 & -3 & 7 & 11 & -13 \\ -4 & 3 & -7 & -9 & 9 \end{array}\right)$$

Gauss-Jordan Elimination

- Solving $Ax = b$ via Gauss-Jordan Elimination: Multiple Right-Hand Sides

Homework 8.2.3.3 Evaluate

$$\begin{array}{rcrcrcr} 3\chi_{0,0} & + & 2\chi_{1,0} & + & 10\chi_{2,0} & = & -7 \\ -3\chi_{0,0} & - & 3\chi_{1,0} & - & 14\chi_{2,0} & = & 9 \\ 3\chi_{0,0} & + & 1\chi_{1,0} & + & 4\chi_{2,0} & = & -5 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcr} 3\chi_{0,0} & + & 2\chi_{1,0} & + & 10\chi_{2,0} & = & 16 \\ -3\chi_{0,0} & - & 3\chi_{1,0} & - & 14\chi_{2,0} & = & -25 \\ 3\chi_{0,0} & + & 1\chi_{1,0} & + & 4\chi_{2,0} & = & 3 \end{array}$$

Gauss-Jordan Elimination

- **Solving $Ax = b$ via Gauss-Jordan Elimination: Multiple Right-Hand Sides**

$$\left(\begin{array}{cc|c|cc} 1 & 0 & -\times & 0 & 0 \\ 0 & 1 & -\times & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -\times & 1 & 0 \\ 0 & 0 & -\times & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c|cc||cc} x & 0 & x & x & x & x & x \\ 0 & x & x & x & x & x & x \\ \hline 0 & 0 & x & x & x & x & x \\ \hline 0 & 0 & x & x & x & x & x \\ 0 & 0 & x & x & x & x & x \end{array}\right)$$

$$= \left(\begin{array}{cc|c|cc||cc} x & 0 & 0 & x & x & x & x \\ 0 & x & 0 & x & x & x & x \\ \hline 0 & 0 & x & x & x & x & x \\ \hline 0 & 0 & 0 & x & x & x & x \\ 0 & 0 & 0 & x & x & x & x \end{array}\right)$$

$$\left(\begin{array}{c|c|c} I & -u_{01} & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -l_{21} & I \end{array}\right) \left(\begin{array}{c|c|c||c} D_{00} & a_{01} & A_{02} & B_0 \\ \hline 0 & a_{11} & a_{12}^T & b_1^T \\ \hline 0 & a_{21} & A_{22} & B_2 \end{array}\right) = ?$$

Gauss-Jordan Elimination

- Solving $Ax = b$ via Gauss-Jordan Elimination: Multiple Right-Hand Sides

Algorithm: $[A,B] := \text{GAUSSJORDAN\_MRHS\_PART1}(A,B)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $B \rightarrow \left(\begin{array}{c} B_T \\ \hline B_B \end{array}\right)$
  where A_{TL} is 0×0, B_T has 0 rows
while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} B_T \\ \hline B_B \end{array}\right) \rightarrow \left(\begin{array}{c} B_0 \\ \hline b_1^T \\ \hline B_2 \end{array}\right)$$

$a_{01} := a_{01}/\alpha_{11}$ $\quad (= u_{01})$
$a_{21} := a_{21}/\alpha_{11}$ $\quad (= l_{21})$
$A_{02} := A_{02} - a_{01}a_{12}^T$ $\quad (= A_{02} - u_{01}a_{12}^T)$
$A_{22} := A_{22} - a_{21}a_{12}^T$ $\quad (= A_{22} - l_{21}a_{12}^T)$
$B_0 := B_0 - a_{01}b_1^T$ $\quad (= B_0 - u_{01}b_1^T)$
$B_2 := B_2 - a_{21}b_1^T$ $\quad (= B_2 - l_{21}b_1^T)$
$a_{01} := 0$ (zero vector)
$a_{21} := 0$ (zero vector)

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} B_T \\ \hline B_B \end{array}\right) \leftarrow \left(\begin{array}{c} B_0 \\ \hline b_1^T \\ \hline B_2 \end{array}\right)$$

endwhile

Algorithm: $[A,B] := \text{GAUSSJORDAN\_MRHS\_PART2}(A,B)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $B \rightarrow \left(\begin{array}{c} B_T \\ \hline B_B \end{array}\right)$
  where A_{TL} is 0×0, B_T has 0 rows
while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} B_T \\ \hline B_B \end{array}\right) \rightarrow \left(\begin{array}{c} B_0 \\ \hline b_1^T \\ \hline B_2 \end{array}\right)$$

$b_1^T := (1/\alpha_{11})b_1^T$
$\alpha_{11} := 1$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} B_T \\ \hline B_B \end{array}\right) \leftarrow \left(\begin{array}{c} B_0 \\ \hline b_1^T \\ \hline B_2 \end{array}\right)$$

endwhile

Gauss-Jordan Elimination

- Computing A^{-1} via Gauss-Jordan Elimination
  - Recall the following observation about the inverse of matrix A. If we let X equal the inverse of A, then

$$AX = I$$

$$A\left(x_0 \mid x_1 \mid \cdots \mid x_{n-1} \right) = \left(e_0 \mid e_1 \mid \cdots \mid e_{n-1} \right),$$

  - so that $Ax_j = e_j$. In other words, the j th column of $X = A^{-1}$ can be computed by solving $Ax_j = e_j$. Clearly, we can use the routine that performs Gauss-Jordan with the appended system $(A \parallel B)$ to compute A^{-1} by feeding it $B = I$!

Gauss-Jordan Elimination

- Computing A^{-1} via Gauss-Jordan Elimination

$$\begin{array}{rcrcrcr} -2\chi_0 & + & 2\chi_1 & - & 5\chi_2 & = & 1 \\ 2\chi_0 & - & 3\chi_1 & + & 7\chi_2 & = & 0 \\ -4\chi_0 & + & 3\chi_1 & - & 7\chi_2 & = & 0 \end{array}$$

and

$$\begin{array}{rcrcrcr} -2\chi_0 & + & 2\chi_1 & - & 5\chi_2 & = & 0 \\ 2\chi_0 & - & 3\chi_1 & + & 7\chi_2 & = & 1 \\ -4\chi_0 & + & 3\chi_1 & - & 7\chi_2 & = & 0 \end{array}$$

and

$$\begin{array}{rcrcrcr} -2\chi_0 & + & 2\chi_1 & - & 5\chi_2 & = & 0 \\ 2\chi_0 & - & 3\chi_1 & + & 7\chi_2 & = & 0 \\ -4\chi_0 & + & 3\chi_1 & - & 7\chi_2 & = & 1 \end{array}$$

Then

$$\left(\begin{array}{rrr||rrr} -2 & 2 & -5 & 1 & 0 & 0 \\ 2 & -3 & 7 & 0 & 1 & 0 \\ -4 & 3 & -7 & 0 & 0 & 1 \end{array}\right)$$

Gauss-Jordan Elimination

- Computing A^{-1} via Gauss-Jordan Elimination

Homework 8.2.4.1 Evaluate

- $$\left(\begin{array}{r|rr} 1 & 0 & 0 \\ \hline 1 & 1 & 0 \\ -2 & 0 & 1 \end{array}\right) \left(\begin{array}{r|rr||r|rr} -2 & 2 & -5 & 1 & 0 & 0 \\ \hline 2 & -3 & 7 & 0 & 1 & 0 \\ -4 & 3 & -7 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{r|rr||c|c|c} -2 & 2 & -5 & \square & \square & \square \\ \hline 0 & -1 & 2 & \square & \square & \square \\ 0 & -1 & 3 & \square & \square & \square \end{array}\right)$$
- $$\left(\begin{array}{r|r|r} 1 & 2 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -1 & 1 \end{array}\right) \left(\begin{array}{r|r|r||r|r|r} -2 & 2 & -5 & 1 & 0 & 0 \\ \hline 0 & -1 & 2 & 1 & 1 & 0 \\ \hline 0 & -1 & 3 & -2 & 0 & 1 \end{array}\right) = \left(\begin{array}{r|r|r||c|c|c} -2 & 0 & -1 & \square & \square & \square \\ \hline 0 & -1 & 2 & \square & \square & \square \\ \hline 0 & 0 & 1 & \square & \square & \square \end{array}\right)$$
- $$\left(\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -2 \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{rr|r||rr|r} -2 & 0 & -1 & 3 & 2 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right) = \left(\begin{array}{rr|r||cc|c} -2 & 0 & 0 & \square & \square & \square \\ 0 & -1 & 0 & \square & \square & \square \\ \hline 0 & 0 & 1 & \square & \square & \square \end{array}\right)$$
- $$\left(\begin{array}{rrr} -\frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right) \left(\begin{array}{rrr||rrr} -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 7 & 3 & -2 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right) = \left(\begin{array}{rrr||ccc} 1 & 0 & 0 & \square & \square & \square \\ 0 & 1 & 0 & \square & \square & \square \\ 0 & 0 & 1 & \square & \square & \square \end{array}\right)$$
- $$\left(\begin{array}{rrr} -2 & 2 & -5 \\ 2 & -3 & 7 \\ -4 & 3 & -7 \end{array}\right) \left(\begin{array}{rrr} 0 & -\frac{1}{2} & -\frac{1}{2} \\ -7 & -3 & 2 \\ -3 & -1 & 1 \end{array}\right) =$$

Gauss-Jordan Elimination

- Computing A^{-1} via Gauss-Jordan Elimination

$$\left(\begin{array}{cc|c|cc} 1 & 0 & -\times & 0 & 0 \\ 0 & 1 & -\times & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -\times & 1 & 0 \\ 0 & 0 & -\times & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c|cc||cc|c|cc} x & 0 & x & x & x & x & x & 0 & 0 & 0 \\ 0 & x & x & x & x & x & x & 0 & 0 & 0 \\ \hline 0 & 0 & x & x & x & x & x & 1 & 0 & 0 \\ \hline 0 & 0 & x & x & x & x & x & 0 & 1 & 0 \\ 0 & 0 & x & x & x & x & x & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{cc|c|cc||cc|c|cc} x & 0 & 0 & x & x & x & x & x & 0 & 0 \\ 0 & x & 0 & x & x & x & x & x & 0 & 0 \\ \hline 0 & 0 & x & x & x & x & x & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & x & x & x & x & x & 1 & 0 \\ 0 & 0 & 0 & x & x & x & x & x & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{c|c|c} I & -u_{01} & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -l_{21} & I \end{array}\right) \left(\begin{array}{c|c|c||c|c|c} D_{00} & a_{01} & A_{02} & B_{00} & 0 & 0 \\ \hline 0 & \alpha_{11} & a_{12}^T & b_{10}^T & 1 & 0 \\ \hline 0 & a_{21} & A_{22} & B_{20} & 0 & I \end{array}\right) = ?$$

Gauss-Jordan Elimination

- Computing A^{-1} via Gauss-Jordan Elimination

$$\left(\begin{array}{c|c|c} I & -u_{01} & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -l_{21} & I \end{array}\right) \left(\begin{array}{c|c|c||c|c|c} D_{00} & a_{01} & A_{02} & B_{00} & 0 & 0 \\ \hline 0 & \alpha_{11} & a_{12}^T & b_{10}^T & 1 & 0 \\ \hline 0 & a_{21} & A_{22} & B_{20} & 0 & I \end{array}\right)$$

$$= \left(\begin{array}{c|c|c||c|c|c} D_{00} & a_{01} - \alpha_{11} u_{01} & A_{02} - u_{01} a_{12}^T & B_{00} - u_{01} b_{10}^T & -u_{01} & 0 \\ \hline 0 & \alpha_{11} & a_{12}^T & b_{10}^T & 1 & 0 \\ \hline 0 & a_{21} - \alpha_{11} l_{21} & A_{22} - l_{21} a_{12}^T & B_{20} - l_{21} b_{10}^T & -l_{21} & I \end{array}\right)$$

$$= \left(\begin{array}{c|c|c||c|c|c} D_{00} & 0 & A_{02} - u_{01} a_{12}^T & B_{00} - u_{01} b_{10}^T & -u_{01} & 0 \\ \hline 0 & \alpha_{11} & a_{12}^T & b_{10}^T & 1 & 0 \\ \hline 0 & 0 & A_{22} - l_{21} a_{12}^T & B_{20} - l_{21} b_{10}^T & -l_{21} & I \end{array}\right)$$

Gauss-Jordan Elimination

- Computing A^{-1} via Gauss-Jordan Elimination

Algorithm: $[A, B] := \text{GJ\_INVERSE\_PART1}(A, B)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $B \rightarrow \left(\begin{array}{c|c} B_{TL} & B_{TR} \\ \hline B_{BL} & B_{BR} \end{array}\right)$

where A_{TL} is 0×0, B_{TL} is 0×0

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} B_{TL} & B_{TR} \\ \hline B_{BL} & B_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} B_{00} & b_{01} & B_{02} \\ \hline b_{10}^T & \beta_{11} & b_{12}^T \\ \hline B_{20} & b_{21} & B_{22} \end{array}\right)$$

where α_{11} is 1×1, β_{11} is 1×1

$a_{01} := a_{01}/\alpha_{11}$	$A_{02} := A_{02} - a_{01} a_{12}^T$	$B_{00} := B_{00} - a_{01} b_{10}^T$	$b_{01} := -a_{01}$
$a_{21} := a_{21}/\alpha_{11}$	$A_{22} := A_{22} - a_{21} a_{12}^T$	$B_{20} := B_{20} - a_{21} b_{10}^T$	$b_{21} := -a_{21}$

(Note: a_{01} and a_{21} on the left need to be updated first.)

$a_{01} := 0$ (zero vector)

$a_{21} := 0$ (zero vector)

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} B_{TL} & B_{TR} \\ \hline B_{BL} & B_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} B_{00} & b_{01} & B_{02} \\ \hline b_{10}^T & \beta_{11} & b_{12}^T \\ \hline B_{20} & b_{21} & B_{22} \end{array}\right)$$

endwhile

Gauss-Jordan Elimination

- Computing A^{-1} via Gauss-Jordan Elimination

Algorithm: $[A, B] := \text{GJ\_INVERSE\_PART2}(A, B)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $B \rightarrow \left(\begin{array}{c|c} B_{TL} & B_{TR} \\ \hline B_{BL} & B_{BR} \end{array}\right)$

where A_{TL} is 0×0, B_{TL} is 0×0

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} B_{TL} & B_{TR} \\ \hline B_{BL} & B_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} B_{00} & b_{01} & B_{02} \\ \hline b_{10}^T & \beta_{11} & b_{12}^T \\ \hline B_{20} & b_{21} & B_{22} \end{array}\right)$$

where α_{11} is 1×1, β_{11} is 1×1

$b_{10}^T := b_{10}^T/\alpha_{11}$

$\beta_{11} := \beta_{11}/\alpha_{11}$

$b_{12}^T := b_{12}^T/\alpha_{11}$

$\alpha_{11} := 1$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} B_{TL} & B_{TR} \\ \hline B_{BL} & B_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} B_{00} & b_{01} & B_{02} \\ \hline b_{10}^T & \beta_{11} & b_{12}^T \\ \hline B_{20} & b_{21} & B_{22} \end{array}\right)$$

endwhile

Gauss-Jordan Elimination

- Computing A^{-1} via Gauss-Jordan Elimination, Alternative

Homework 8.2.5.1

- Determine $\delta_{0,0}, \lambda_{1,0}, \lambda_{2,0}$ so that

$$\left(\begin{array}{c|cc} \delta_{0,0} & 0 & 0 \\ \hline \lambda_{1,0} & 1 & 0 \\ \lambda_{2,0} & 0 & 1 \end{array}\right)\left(\begin{array}{ccc|ccc} -1 & -4 & -2 & 1 & 0 & 0 \\ 2 & 6 & 2 & 0 & 1 & 0 \\ -1 & 0 & 3 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 4 & 2 & -1 & 0 & 0 \\ 0 & -2 & -2 & 2 & 1 & 0 \\ 0 & 4 & 5 & -1 & 0 & 1 \end{array}\right)$$

- Determine $\upsilon_{0,1}, \delta_{1,1}$, and $\lambda_{2,1}$ so that

$$\left(\begin{array}{c|c|c} 1 & \upsilon_{0,1} & 0 \\ \hline 0 & \delta_{1,1} & 0 \\ \hline 0 & \lambda_{2,1} & 1 \end{array}\right)\left(\begin{array}{c|c|c||c|c|c} 1 & 4 & 2 & -1 & 0 & 0 \\ \hline 0 & -2 & -2 & 2 & 1 & 0 \\ \hline 0 & 4 & 5 & -1 & 0 & 1 \end{array}\right) = \left(\begin{array}{c|c|c||c|c|c} 1 & 0 & -2 & 3 & 2 & 0 \\ \hline 0 & 1 & 1 & -1 & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 1 & 3 & 2 & 1 \end{array}\right)$$

- Determine $\upsilon_{0,2}, \upsilon_{0,2}$, and $\delta_{2,2}$ so that

$$\left(\begin{array}{cc|c} 1 & 0 & \upsilon_{0,2} \\ 0 & 1 & \upsilon_{1,2} \\ \hline 0 & 0 & \delta_{2,2} \end{array}\right)\left(\begin{array}{cc|c||cc|c} 1 & 0 & -2 & 3 & 2 & 0 \\ 0 & 1 & 1 & -1 & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 1 & 3 & 2 & 1 \end{array}\right) = \left(\begin{array}{cc|c||cc|c} 1 & 0 & 0 & 9 & 6 & 2 \\ 0 & 1 & 0 & -4 & -\frac{5}{2} & -1 \\ \hline 0 & 0 & 1 & 3 & 2 & 1 \end{array}\right)$$

- Evaluate

$$\begin{pmatrix} -1 & -4 & -2 \\ 2 & 6 & 2 \\ -1 & 0 & 3 \end{pmatrix}\begin{pmatrix} 9 & 6 & 2 \\ -4 & -\frac{5}{2} & -1 \\ 3 & 2 & 1 \end{pmatrix} =$$

Gauss-Jordan Elimination

- Computing A^{-1} via Gauss-Jordan Elimination, Alternative

$$\left(\begin{array}{c|c|c} I & -u_{01} & 0 \\ \hline 0 & \delta_{11} & 0 \\ \hline 0 & -l_{21} & I \end{array}\right)\left(\begin{array}{c|c|c||c|c|c} I & a_{01} & A_{02} & B_{00} & 0 & 0 \\ \hline 0 & \alpha_{11} & a_{12}^T & b_{10}^T & 1 & 0 \\ \hline 0 & a_{21} & A_{22} & B_{20} & 0 & I \end{array}\right)$$

$$= \left(\begin{array}{c|c|c||c|c|c} I & a_{01}-\alpha_{11}u_{01} & A_{02}-u_{01}a_{12}^T & B_{00}-u_{01}b_{10}^T & -u_{01} & 0 \\ \hline 0 & \delta_{11}\alpha_{11} & \delta_{11}a_{12}^T & \delta_{11}b_{10}^T & \delta_{11} & 0 \\ \hline 0 & a_{21}-\alpha_{11}l_{21} & A_{22}-l_{21}a_{12}^T & B_{20}-l_{21}b_{10}^T & -l_{21} & I \end{array}\right)$$

$$= \left(\begin{array}{c|c|c||c|c|c} I & 0 & A_{02}-u_{01}a_{12}^T & B_{00}-u_{01}b_{10}^T & -u_{01} & 0 \\ \hline 0 & 1 & a_{12}^T/\alpha_{11} & b_{10}^T/\alpha_{11} & 1/\alpha_{11} & 0 \\ \hline 0 & 0 & A_{22}-l_{21}a_{12}^T & B_{20}-l_{21}b_{10}^T & -l_{21} & I \end{array}\right)$$

Gauss-Jordan Elimination

- Computing A^{-1} via Gauss-Jordan Elimination, Alternative

Algorithm: $[B] := \text{GJ\_INVERSE\_ALT}(A, B)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $B \rightarrow \left(\begin{array}{c|c} B_{TL} & B_{TR} \\ \hline B_{BL} & B_{BR} \end{array}\right)$

where A_{TL} is 0×0, B_{TL} is 0×0

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} B_{TL} & B_{TR} \\ \hline B_{BL} & B_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} B_{00} & b_{01} & B_{02} \\ \hline b_{10}^T & \beta_{11} & b_{12}^T \\ \hline B_{20} & b_{21} & B_{22} \end{array}\right)$$

where α_{11} is 1×1, β_{11} is 1×1

$$\left(\begin{array}{c|c|c} & a_{01} := a_{01}/\alpha_{11} & A_{02} := A_{02} - a_{01}a_{12}^T \\ \hline & & \\ \hline & a_{21} := a_{21}/\alpha_{11} & A_{22} := A_{22} - a_{21}a_{12}^T \end{array}\right) \quad \left(\begin{array}{c|c|c} B_{00} := B_{00} - a_{01}b_{10}^T & b_{01} := -a_{01} & \\ \hline & & \\ \hline B_{20} := B_{20} - a_{21}b_{10}^T & b_{21} := -a_{21} & \end{array}\right)$$

(Note: above a_{01} and a_{21} must be updated before the operations to their right.)

$$\left(\begin{array}{c|c|c} & a_{01} := 0 & \\ \hline & \alpha_{11} := 1 & a_{12}^T := a_{12}^T/\alpha_{11} \\ \hline & a_{21} := 0 & \end{array}\right) \quad \left(\begin{array}{c|c|c} & & \\ \hline b_{10}^T := b_{10}^T/\alpha_{11} & \beta_{11} = 1/\alpha_{11} & \\ \hline & & \end{array}\right)$$

(Note: above α_{11} must be updated last.)

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} B_{TL} & B_{TR} \\ \hline B_{BL} & B_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} B_{00} & b_{01} & B_{02} \\ \hline b_{10}^T & \beta_{11} & b_{12}^T \\ \hline B_{20} & b_{21} & B_{22} \end{array}\right)$$

endwhile

Gauss-Jordan Elimination

- Pivoting

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c|c} B_{TL} & B_{TR} \\ \hline B_{BL} & B_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} B_{00} & b_{01} & B_{02} \\ \hline b_{10}^T & \beta_{11} & b_{12}^T \\ \hline B_{20} & b_{21} & B_{22} \end{array}\right)$$

where α_{11} is 1×1, β_{11} is 1×1

$$\left(\begin{array}{c|c|c} & a_{01} := a_{01}/\alpha_{11} & A_{02} := A_{02} - a_{01}a_{12}^T \\ \hline & & \\ \hline & a_{21} := a_{21}/\alpha_{11} & A_{22} := A_{22} - a_{21}a_{12}^T \end{array}\right) \quad \left(\begin{array}{c|c|c} B_{00} := B_{00} - a_{01}b_{10}^T & b_{01} := -a_{01} & \\ \hline & & \\ \hline B_{20} := B_{20} - a_{21}b_{10}^T & b_{21} := -a_{21} & \end{array}\right)$$

(Note: above a_{01} and a_{21} must be updated before the operations to their right.)

$$\left(\begin{array}{c|c|c} & a_{01} := 0 & \\ \hline & \alpha_{11} := 1 & a_{12}^T := a_{12}^T/\alpha_{11} \\ \hline & a_{21} := 0 & \end{array}\right) \quad \left(\begin{array}{c|c|c} & & \\ \hline b_{10}^T := b_{10}^T/\alpha_{11} & \beta_{11} = 1/\alpha_{11} & \\ \hline & & \end{array}\right)$$

(Note: above α_{11} must be updated last.)

Gauss-Jordan Elimination

- Cost of Matrix Inversion
  - LU factorization
    - Factor $A = LU$ $\frac{2}{3}n^3$ flops
    - Solve $Lz = e_j$ n^2 flops
    - Solve $Ux_j = z$ n^2 flops
  - Gauss-Jordan matrix inversion
    - Invert A $2n^3$ flops
    - Multiply $\mathrm{b} = A^{-1}x$ $2n^2$ flops

(Very Important) Enrichment

- Symmetric Positive Denite Matrices
  - Symmetric positive definite (SPD) matrices are an important class of matrices that occur naturally as part of applications. We will see SPD matrices come up later in this course, when we discuss how to solve overdetermined systems of equations:

$$Bx = y \ where \in \mathbb{R}^{n\times m} \ and \ m > n:$$

  - In other words, when there are more equations than there are unknowns in our linear system of equations. When B has "linearly independent columns," a term with which you will become very familiar later in the course, the best solution to $Bx = y$ satisfies $B^T Bx = B^T y$. If we set $\mathrm{A} = B^T B$ and $\mathrm{b} = B^T y$, then we need to solve $Ax = b$, and now A is square and nonsingular (which we will prove later in the course). Now, we could solve $Ax = b$ via any of the methods we have discussed so far. However, these methods ignore the fact that A is symmetric. So, the question becomes how to take advantage of symmetry.

(Very Important) Enrichment

- Symmetric Positive Denite Matrices

Definition 8.1 *Let $A \in \mathbb{R}^{n\times n}$. Matrix A is said to be symmetric positive definite (SPD) if*

- *A is symmetric; and*
- *$x^T Ax > 0$ for all nonzero vectors $x \in \mathbb{R}^n$.*

A nonsymmetric matrix can also be positive definite and there are the notions of a matrix being negative definite or indefinite. We won't concern ourselves with these in this course.

Here is a way to relate what a positive definite matrix is to something you may have seen before. Consider the quadratic polynomial

$$p(\chi) = \alpha\chi^2 + \beta\chi + \gamma = \chi\alpha\chi + \beta\chi + \gamma.$$

The graph of this function is a parabola that is "concaved up" if $\alpha > 0$. In that case, it attains a minimum at a unique value χ.

Now consider the vector function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = x^T Ax + b^T x + \gamma$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$ are all given. If A is a SPD matrix, then this equation is minimized for a unique vector x. If $n = 2$, plotting this function when A is SPD yields a paraboloid that is concaved up:

(Very Important) Enrichment

- Solving $Ax = b$ when A is Symmetric Positive Denite

Cholesky factorization theorem

Theorem 8.2 *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Then there exists a lower triangular matrix $L \in \mathbb{R}^{n\times n}$ such that $A = LL^T$. If the diagonal elements of L are chosen to be positive, this factorization is unique.*

We will not prove this theorem.

Unblocked Cholesky factorization

We are going to closely mimic the derivation of the LU factorization algorithm from Unit 6.3.1.

Partition

$$A \to \left(\begin{array}{c|c} \alpha_{11} & \star \\ \hline a_{21} & A_{22} \end{array}\right), \quad \text{and} \quad L \to \left(\begin{array}{c|c} \lambda_{11} & 0 \\ \hline l_{21} & L_{22} \end{array}\right).$$

Here we use $\star$ to indicate that we are not concerned with that part of the matrix because A is symmetric and hence we should be able to just work with the lower triangular part of it.

(Very Important) Enrichment

- Solving $Ax = b$ when A is Symmetric Positive Denite

We want L to satisfy $A = LL^T$. Hence

$$
\begin{aligned}
\overbrace{\left(\begin{array}{c|c} \alpha_{11} & \star \\ \hline a_{21} & A_{22} \end{array}\right)}^{A}
&= \overbrace{\left(\begin{array}{c|c} \lambda_{11} & 0 \\ \hline l_{21} & L_{22} \end{array}\right)}^{L}
\overbrace{\left(\begin{array}{c|c} \lambda_{11} & 0 \\ \hline l_{21} & L_{22} \end{array}\right)^T}^{L^T} \\
&= \overbrace{\left(\begin{array}{c|c} \lambda & 0 \\ \hline l_{21} & L_{22} \end{array}\right)}^{L}
\overbrace{\left(\begin{array}{c|c} \lambda_{11} & l_{21}^T \\ \hline 0 & L_{22}^T \end{array}\right)}^{L^T} \\
&= \overbrace{\left(\begin{array}{c|c} \lambda_{11}^2 + 0 \times 0 & \star \\ \hline l_{21}\lambda_{11} + L_{22} \times 0 & l_{21}l_{21}^T + L_{22}L_{22}^T \end{array}\right)}^{LL^T}. \\
&= \overbrace{\left(\begin{array}{c|c} \lambda_{11}^2 & \star \\ \hline l_{21}\lambda_{11} & l_{21}l_{21}^T + L_{22}L_{22}^T \end{array}\right)}^{LL^T}.
\end{aligned}
$$

where, again, the $\star$ refers to part of the matrix in which we are not concerned because of symmetry.

(Very Important) Enrichment

- Solving $Ax = b$ when A is Symmetric Positive Denite

For two matrices to be equal, their elements must be equal, and therefore, if they are partitioned conformally, their submatrices must be equal:

$$
\begin{array}{c|c} \alpha_{11} = \lambda_{11}^2 & \star \\ \hline a_{21} = l_{21}\lambda_{11} & A_{22} = l_{21}l_{21}^T + L_{22}L_{22}^T \end{array}
$$

or, rearranging,

$$
\begin{array}{c|c} \lambda_{11} = \sqrt{\alpha_{11}} & \star \\ \hline l_{21} = a_{21}/\lambda_{11} & L_{22}L_{22}^T = A_{22} - l_{21}l_{21}^T \end{array}.
$$

(Very Important) Enrichment

- Solving $Ax = b$ when A is Symmetric Positive Denite

Algorithm: $[A] :=$ CHOL_UNB_VAR3(A)

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$
  where A_{TL} is 0×0

while $m(A_{TL}) < m(A)$ **do**

  Repartition

$$
\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)
$$

  $\alpha_{11} := \sqrt{\alpha_{11}}$
  $a_{21} := a_{21}/\alpha_{11}$
  $A_{22} := A_{22} - a_{21}a_{21}^T$
    (updating only the lower triangular part)

  Continue with

$$
\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)
$$

endwhile

(Very Important) Enrichment

- Solving $Ax = b$ when A is Symmetric Positive Denite

LU factorization	Cholesky factorization
$A \rightarrow \left(\begin{array}{c\|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array}\right), L \rightarrow \left(\begin{array}{c\|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right), U \rightarrow \left(\begin{array}{c\|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right)$	$A \rightarrow \left(\begin{array}{c\|c} \alpha_{11} & \star \\ \hline a_{21} & A_{22} \end{array}\right), L \rightarrow \left(\begin{array}{c\|c} \lambda_{11} & 0 \\ \hline l_{21} & L_{22} \end{array}\right).$
$\left(\begin{array}{c\|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array}\right) = \underbrace{\left(\begin{array}{c\|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right)\left(\begin{array}{c\|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right)}_{\left(\begin{array}{c\|c} \upsilon_{11} & u_{12}^T \\ \hline l_{21}\upsilon_{11} & l_{21}u_{12}^T + L_{22}U_{22} \end{array}\right)}$	$\left(\begin{array}{c\|c} \alpha_{11} & \star \\ \hline a_{21} & A_{22} \end{array}\right) = \underbrace{\left(\begin{array}{c\|c} \lambda_{11} & 0 \\ \hline l_{21} & L_{22} \end{array}\right)\left(\begin{array}{c\|c} \lambda_{11} & 0 \\ \hline l_{21} & L_{22} \end{array}\right)^T}_{\left(\begin{array}{c\|c} \lambda_{11}^2 & \star \\ \hline l_{21}\lambda_{11} & l_{21}l_{12}^T + L_{22}L_{22}^T \end{array}\right)}.$
$\begin{array}{c\|c} \alpha_{11} = \upsilon_{11} & a_{12}^T = u_{12}^T \\ \hline a_{21} = l_{21}\upsilon_{11} & A_{22} = l_{21}u_{12}^T + L_{22}U_{22} \end{array}$	$\begin{array}{c\|c} \alpha_{11} = \lambda_{11}^2 & \star \\ \hline a_{21} = l_{21}\lambda_{11} & A_{22} = l_{21}l_{12}^T + L_{22}L_{22}^2 \end{array}$
$\alpha_{11} := \alpha_{11}$ $a_{12}^T := a_{12}^T$ $a_{21} := a_{21}/\alpha_{11}$ $A_{22} := A_{22} - a_{21}a_{12}^T$	$\alpha_{11} := \sqrt{\alpha_{11}}$ $a_{21} := a_{21}/\alpha_{11}$ $A_{22} := A_{22} - a_{21}a_{12}^T$ (update only lower triangular part)

(Very Important) Enrichment

- **Other Factorizations**

We have now encountered the LU factorization,

$$A = LU,$$

the LU factorization with row pivoting,

$$PA = LU,$$

and the Cholesky factorization,

$$A = LL^T.$$

Later in this course you will be introduced to the QR factorization,

$$A = QR,$$

where Q has the special property that $Q^T Q = I$ and R is an upper triangular matrix.

When a matrix is *indefinite symmetric*, there is a factorization called the LDL^T (pronounce as L D L transpose) factorization,

$$A = LDL^T,$$

where L is unit lower triangular and D is diagonal. You may want to see if you can modify the derivation of the Cholesky factorization to yield an algorithm for the LDL^T factorization.

Questions and Answers