# Gaussian Elimination


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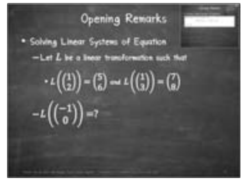
## Opening Remarks




- Solving Linear Systems of Equation

**Homework 2.4.1.2** Let $L$ be a linear transformation such that

$$L\left(\begin{pmatrix}1\\0\end{pmatrix}\right)=\begin{pmatrix}3\\5\end{pmatrix} \quad \text{and} \quad L\left(\begin{pmatrix}0\\1\end{pmatrix}\right)=\begin{pmatrix}2\\-1\end{pmatrix}.$$

Then $L\left(\begin{pmatrix}2\\3\end{pmatrix}\right)=$



## Opening Remarks

- Solving Linear Systems of Equation

$$\begin{pmatrix}2\\3\end{pmatrix} = ?\begin{pmatrix}1\\0\end{pmatrix} + ?\begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}2\\3\end{pmatrix} = \chi_0\begin{pmatrix}1\\0\end{pmatrix} + \chi_1\begin{pmatrix}0\\1\end{pmatrix}$$



## Opening Remarks

- Solving Linear Systems of Equation

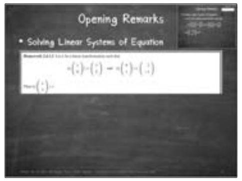
For the next three exercises, let $L$ be a linear transformation such that

$$L\left(\begin{pmatrix}1\\0\end{pmatrix}\right)=\begin{pmatrix}3\\5\end{pmatrix} \quad \text{and} \quad L\left(\begin{pmatrix}1\\1\end{pmatrix}\right)=\begin{pmatrix}5\\4\end{pmatrix}.$$

**Homework 2.4.1.5** $L\left(\begin{pmatrix}2\\3\end{pmatrix}\right)=$

$$\begin{pmatrix}2\\3\end{pmatrix} = \chi_0\begin{pmatrix}1\\0\end{pmatrix} + \chi_1\begin{pmatrix}1\\1\end{pmatrix}$$

# Opening Remarks



- Solving Linear Systems of Equation
  - Let $L$ be a linear transformation such that
    - $L\left(\begin{pmatrix}1\\2\end{pmatrix}\right)=\begin{pmatrix}5\\6\end{pmatrix}$ and $L\left(\begin{pmatrix}1\\3\end{pmatrix}\right)=\begin{pmatrix}7\\8\end{pmatrix}$
  - $L\left(\begin{pmatrix}-1\\0\end{pmatrix}\right)=?$



# Opening Remarks

- Solving Linear Systems of Equation

$$\begin{array}{rcrcr} \chi_0 & + & \chi_1 & = & -1 \\ 2\chi_0 & + & 3\chi_1 & = & 0 \end{array}$$

$$\begin{pmatrix}1 & 1\\2 & 3\end{pmatrix}\begin{pmatrix}\chi_0\\\chi_1\end{pmatrix}=\begin{pmatrix}-1\\0\end{pmatrix}$$

$$\left(\begin{array}{cc|c}1 & 1 & -1\\2 & 3 & 0\end{array}\right)$$



# Outline

- Solving system of linear equation
- Representing system of linear equation as appended matrices
- Reducing matrices to row echelon form
- LU factorization



# Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System
- Appended Matrices
- Gauss Transforms
- Computing Separately with the Matrix and Right-Hand Side (Forward Substitution)
- Towards an Algorithm

## Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System

$$\left(\begin{array}{cccc|c} \chi_{0,0} & \chi_{0,1} & \cdots & \chi_{0,m} & \alpha_0 \\ 0 & \chi_{1,1} & \cdots & \chi_{1,1} & \alpha_1 \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \chi_{m,m} & \alpha_m \end{array}\right)$$



## Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System
  - **Gaussian elimination (transform linear system of equations to an upper triangular system)**
    - Solving the above linear system relies on the fact that its solution does not change if
      - Equations are reordered (not used until next week);
      - An equation in the system is modified by subtracting a multiple of another equation in the system from it; and/or
      - Both sides of an equation in the system are scaled by a nonzero number.



## Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System

$$\begin{array}{ccccccc} 2x & + & 4y & + & 2z & = & -10 \\ 4x & - & 2y & + & 6z & = & 20 \\ 6x & - & 4y & + & 2z & = & 18 \end{array}$$

$$\begin{array}{ccccccc} 2\chi_0 & + & 4\chi_1 & + & 2\chi_2 & = & -10 \\ 4\chi_0 & - & 2\chi_1 & + & 6\chi_2 & = & 20 \\ 6\chi_0 & - & 4\chi_1 & + & 2\chi_2 & = & 18 \end{array}$$



## Gaussian Elimination

- Linear Equations

$$\begin{array}{ccccccc} 2\chi_0 & + & 4\chi_1 & + & 2\chi_2 & = & -10 \\ 4\chi_0 & - & 2\chi_1 & + & 6\chi_2 & = & 20 \\ 6\chi_0 & - & 4\chi_1 & + & 2\chi_2 & = & 18 \end{array}$$

- Gaussian elimination
  - Following Step 1
    - $row1 - \lambda_{1,0} row0$
    - $row2 - \lambda_{2,0} row0$
  - Following Step 2
    - $row2 - \lambda_{2,1} row0$
- Back substitution
- Check your answer

# Gaussian Elimination

- Practice with Gaussian Elimination
  - http://ulaff.s3.amazonaws.com/GaussianEliminationPractice/index.html

**Homework 6.2.1.2** Compute the solution of the linear system of equations given by

$$\begin{array}{rcrcrcr} -2\chi_0 & + & \chi_1 & + & 2\chi_2 & = & 0 \\ 4\chi_0 & - & \chi_1 & - & 5\chi_2 & = & 4 \\ 2\chi_0 & - & 3\chi_1 & - & \chi_2 & = & -6 \end{array}$$

- $\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$



# Gaussian Elimination

- Appended Matrices

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & -10 \\ 4 & 2 & 6 & 20 \\ 6 & 4 & 2 & 18 \end{array}\right)$$

Represent

$$\begin{array}{rcrcrcr} 2\chi_0 & + & 4\chi_1 & + & 2\chi_2 & = & -10 \\ 4\chi_0 & - & 2\chi_1 & + & 6\chi_2 & = & 20 \\ 6\chi_0 & - & 4\chi_1 & + & 2\chi_2 & = & 18 \end{array}$$



# Gaussian Elimination

- Appended Matrices

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & -10 \\ 4 & 2 & 6 & 20 \\ 6 & 4 & 2 & 18 \end{array}\right)$$

- Gaussian elimination
  - Following Step 1
    - $row1 - \lambda_{1,0} row0$ & $row2 - \lambda_{2,0} row0$
  - Following Step 2
    - $row2 - \lambda_{2,1} row0$
- Back substitution
- Check your answer



# Gaussian Elimination

- Appended Matrice

**Homework 6.2.2.2** Compute the solution of the linear system of equations expressed as an appended matrix given by

$$\left(\begin{array}{rrr|r} -1 & 2 & -3 & 2 \\ -2 & 2 & -8 & 10 \\ 2 & -6 & 6 & -2 \end{array}\right)$$

- $\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$

# Gaussian Elimination

- Gauss Transforms

**Theorem 6.1** *Let $\widehat{L}_j$ be a matrix that equals the identity, except that for $i > j$ the $(i, j)$ elements (the ones below the diagonal in the $j$th column) have been replaced with $-\lambda_{i,j}$:*

$$\widehat{L}_j = \begin{pmatrix} I_j & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+1,j} & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+2,j} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\lambda_{m-1,j} & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

*Then $\widehat{L}_j A$ equals the matrix $A$ except that for $i > j$ the $i$th row is modified by subtracting $\lambda_{i,j}$ times the $j$th row from it. Such a matrix $\widehat{L}_j$ is called a Gauss transform.*



# Gaussian Elimination

**Proof:** Let

$$\widehat{L}_j = \begin{pmatrix} I_j & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+1,j} & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+2,j} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\lambda_{m-1,j} & 0 & 0 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ \check{a}_{j+1}^T \\ \check{a}_{j+2}^T \\ \vdots \\ \check{a}_{m-1}^T \end{pmatrix},$$

where $I_k$ equals a $k \times k$ identity matrix, $A_{s:t,:}$ equals the matrix that consists of rows $s$ through $t$ from matrix $A$, and $\check{a}_k^T$ equals the $k$th row of $A$. Then

$$\begin{aligned} \widehat{L}_j A &= \begin{pmatrix} I_j & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+1,j} & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+2,j} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\lambda_{m-1,j} & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ \check{a}_{j+1}^T \\ \check{a}_{j+2}^T \\ \vdots \\ \check{a}_{m-1}^T \end{pmatrix} \\ &= \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ -\lambda_{j+1,j}\check{a}_j^T + \check{a}_{j+1}^T \\ -\lambda_{j+2,j}\check{a}_j^T + \check{a}_{j+2}^T \\ \vdots \\ -\lambda_{m-1,j}\check{a}_j^T + \check{a}_{m-1}^T \end{pmatrix} = \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ \check{a}_{j+1}^T - \lambda_{j+1,j}\check{a}_j^T \\ \check{a}_{j+2}^T - \lambda_{j+2,j}\check{a}_j^T \\ \vdots \\ \check{a}_{m-1}^T - \lambda_{m-1,j}\check{a}_j^T \end{pmatrix}. \end{aligned}$$



# Gaussian Elimination

- Gauss Transforms

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 4 & -2 & 6 \\ 6 & -4 & 2 \end{pmatrix} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \square & 1 & 0 \\ \square & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 2 & -2 & 6 \\ -4 & -4 & 2 \end{pmatrix} = \begin{pmatrix} \square & \square & \square \\ 0 & \square & \square \\ 0 & \square & \square \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \square & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & -10 & 10 \\ 0 & -16 & 8 \end{pmatrix} = \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & 0 & \square \end{pmatrix}.$$



# Gaussian Elimination

- Gauss Transforms

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & -10 \\ 4 & 2 & 6 & 20 \\ 6 & 4 & 2 & 18 \end{array}\right)$$

- Gaussian elimination

$$- \begin{pmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ -6/2 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 2 & 4 & 2 & -10 \\ 4 & 2 & 6 & 20 \\ 6 & 4 & 2 & 18 \end{array}\right)$$

$$- \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(-16/-10) & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 2 & 4 & 2 & -10 \\ 0 & -10 & 10 & 40 \\ 0 & -16 & 8 & 48 \end{array}\right)$$

- Back substitution
- Check your answer

# Gaussian Elimination

- **Computing Separately with the Matrix and Right-Hand Side (Forward Substitution)**

$$\begin{pmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ -6/2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 2 & -10 \\ 4 & 2 & 6 & 20 \\ 6 & 4 & 2 & 18 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 2 & -10 \\ 0 & -10 & 10 & 40 \\ 0 & -16 & 8 & 48 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ -6/2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 & 2 \\ 4 & 2 & 6 \\ 6 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 2 \\ 0 & -10 & 10 \\ 0 & -16 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ -4/2 & 0 & 1 \end{pmatrix}\begin{pmatrix} -10 \\ 20 \\ 18 \end{pmatrix} = \begin{pmatrix} -10 \\ 40 \\ 48 \end{pmatrix}$$



# Gaussian Elimination

$$\left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & \boxed{a_{21}} & \boxed{A_{22}} \end{array}\right)$$

- Towards an Algorithm

Before

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -2 & 1 & 0 \\ -3 & 0 & 1 \end{array}\right)\left(\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 4 & -2 & 6 \\ 6 & -4 & 2 \end{array}\right)$$

After

$$\left(\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 2 & -10 & 10 \\ 3 & -16 & 8 \end{array}\right)$$

Before

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -1.6 & 1 \end{array}\right)\left(\begin{array}{cc|c} 2 & 4 & -2 \\ \hline 2 & -10 & 10 \\ \hline 3 & -16 & 8 \end{array}\right)$$

After

$$\begin{pmatrix} 2 & 4 & -2 \\ 2 & -10 & 10 \\ 3 & 1.6 & -8 \end{pmatrix}$$



# Gaussian Elimination

$$\left(\begin{array}{ccc|c|ccc} 1 & & & 0 & & 0 & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \hline 0 & & & 1 & & 0 & \\ \hline & & & \times & 1 & & \\ & 0 & & \vdots & & \ddots & \\ & & & \times & & & 1 \end{array}\right)\left(\begin{array}{ccc|c|ccc} \times & \cdots & \times & \times & \times & \cdots & \times \\ & \ddots & \times & \vdots & \times & \ddots & \times \\ & & \times & \times & \times & \cdots & \times \\ \hline & & & \times & \times & \cdots & \times \\ \hline & & & \times & \times & \cdots & \times \\ & & & \vdots & \times & \ddots & \times \\ & & & \times & \times & \cdots & \times \end{array}\right)$$

$$= \left(\begin{array}{ccc|c|ccc} \times & \cdots & \times & \times & \times & \cdots & \times \\ & \ddots & \times & \vdots & \times & \ddots & \times \\ & & \times & \times & \times & \cdots & \times \\ \hline & & & \times & \times & \cdots & \times \\ \hline & & & 0 & \times & \cdots & \times \\ & & & \vdots & \times & \ddots & \times \\ & & & 0 & \times & \cdots & \times \end{array}\right)$$

$$\underbrace{\left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -l_{21} & I \end{array}\right)\left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{22} \\ \hline 0 & \alpha_{11} & a_{12}^T \\ \hline 0 & a_{21} & A_{22} \end{array}\right)} = \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{22} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & A_{22}^{\text{new}} \end{array}\right)$$

$$\underbrace{\left(\begin{array}{cc|c} U_{00} & u_{01} & U_{22} \\ \hline 0 & \multicolumn{2}{c}{\left(\begin{array}{c|c} 1 & 0 \\ \hline -l_{21} & I \end{array}\right)\left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array}\right)} \end{array}\right)}$$

$$\left(\begin{array}{cc|c} U_{00} & u_{01} & U_{22} \\ \hline 0 & \multicolumn{2}{c}{\left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline -l_{21}\alpha_{11} + a_{21} & -l_{21}a_{12}^T + A_{22} \end{array}\right)} \end{array}\right)$$



# Gaussian Elimination

Before

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -2 & 1 & 0 \\ -3 & 0 & 1 \end{array}\right)\left(\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 4 & -2 & 6 \\ 6 & -4 & 2 \end{array}\right)$$

Before

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -1.6 & 1 \end{array}\right)\left(\begin{array}{cc|c} 2 & 4 & -2 \\ \hline 2 & -10 & 10 \\ \hline 3 & -16 & 8 \end{array}\right)$$

**Algorithm:** $A :=$ GAUSSIAN_ELIMINATION $(A)$

**Partition** $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$

where $A_{TL}$ is $0 \times 0$

**while** $m(A_{TL}) < m(A)$ **do**

**Repartition**

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & \boxed{a_{21}} & \boxed{A_{22}} \end{array}\right)$$

$\boxed{a_{21}} := a_{21}/\alpha_{11} \quad (= l_{21})$

$\boxed{A_{22}} := A_{22} - a_{21}a_{12}^T \quad (= A_{22} - l_{21}a_{12}^T)$

**Continue with**

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

**endwhile**

# Gaussian Elimination

$$\underbrace{\left(\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -l_{21} & I \end{array}\right)\left(\begin{array}{c} y_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)} = \left(\begin{array}{c} y_0 \\ \hline \psi_1 \\ \hline b_2^{\mathrm{new}} \end{array}\right)$$

$$\underbrace{\left(\begin{array}{c} y_0 \\ \hline \left(\begin{array}{c|c} 1 & 0 \\ \hline -l_{21} & I \end{array}\right)\left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right) \end{array}\right)}$$

$$\left(\begin{array}{c} y_0 \\ \hline \left(\begin{array}{c} \beta_1 \\ \hline -l_{21}\beta_{11} + b_2 \end{array}\right) \end{array}\right)$$

**Algorithm:** $b :=$ FORWARD_SUBSTITUTION$(A, b)$

**Partition** $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where $A_{TL}$ is $0 \times 0$, $b_T$ has 0 rows

**while** $m(A_{TL}) < m(A)$ **do**

**Repartition**

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

$b_2 := b_2 - \beta_1 a_{21} \quad (= b_2 - \beta_1 l_{21})$

**Continue with**

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

**endwhile**



# Solving Ax = b via LU Factorization

- LU factorization (Gaussian elimination)
- Solving Lz = b (Forward substitution)
- Solving Ux = b (Back substitution)
- Putting it all together to solve Ax = b
- Cost



# Solving Ax = b via LU Factorization

- LU factorization (Gaussian elimination)
  - Idea
    - A matrix $A \in \mathbb{R}^{n \times n}$ can be factored into the product of two matrices $\mathbf{L}, \mathbf{U} \in \mathbb{R}^{n \times n}$

      $$A = LU$$

      - where $L$ is unit lower triangular and $U$ is upper triangular.



# Solving Ax = b via LU Factorization

- LU factorization (Gaussian elimination)

Assume $A \in \mathbb{R}^{n \times n}$ is given and that $L$ and $U$ are to be computed such that $A = LU$, where $L \in \mathbb{R}^{n \times n}$ is unit lower triangular and $U \in \mathbb{R}^{n \times n}$ is upper triangular. We derive an algorithm for computing this operation by partitioning

$$A \rightarrow \left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array}\right), \quad L \rightarrow \left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right), \quad \text{and} \quad U \rightarrow \left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right).$$

Now, $A = LU$ implies (using what we learned about multiplying matrices that have been partitioned into submatrices)

$$\begin{aligned} \overbrace{\left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array}\right)}^{A} &= \overbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right)}^{L} \overbrace{\left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right)}^{U} \\ &= \overbrace{\left(\begin{array}{c|c} 1 \times \upsilon_{11} + 0 \times 0 & 1 \times u_{12}^T + 0 \times U_{22} \\ \hline l_{21}\upsilon_{11} + L_{22} \times 0 & l_{21}u_{12}^T + L_{22}U_{22} \end{array}\right)}^{LU}. \\ &= \overbrace{\left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline l_{21}\upsilon_{11} & l_{21}u_{12}^T + L_{22}U_{22} \end{array}\right)}^{LU}. \end{aligned}$$

# Solving Ax = b via LU Factorization

- **LU factorization (Gaussian elimination)**

For two matrices to be equal, their elements must be equal, and therefore, if they are partitioned conformally, their submatrices must be equal:

$$\begin{array}{c|c} \alpha_{11} = \upsilon_{11} & a_{12}^T = u_{12}^T \\ \hline a_{21} = l_{21}\upsilon_{11} & A_{22} = l_{21}u_{12}^T + L_{22}U_{22} \end{array}$$

or, rearranging,

$$\begin{array}{c|c} \upsilon_{11} = \alpha_{11} & u_{12}^T = a_{12}^T \\ \hline l_{21} = a_{21}/\upsilon_{11} & L_{22}U_{22} = A_{22} - l_{21}u_{12}^T \end{array}.$$

This suggests the following steps for **overwriting** a matrix $A$ with its LU factorization:

- Partition

$$A \rightarrow \left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array}\right).$$

- Update $a_{21} = a_{21}/\alpha_{11} (= l_{21})$. (Scale $a_{21}$ by $1/\alpha_{11}$!)



# Solving Ax = b via LU Factorization

- **LU factorization (Gaussian elimination)**

**Algorithm:** $A := \text{LU\_UNB\_VAR5}(A)$

**Partition** $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right)$

where $A_{TL}$ is $0 \times 0$

**while** $m(A_{TL}) < m(A)$ **do**

**Repartition**

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

where $\alpha_{11}$ is $1 \times 1$

$a_{21} := a_{21}/\alpha_{11}$ $\quad (= l_{21})$

$A_{22} := A_{22} - a_{21}a_{12}^T$ $\quad (= A_{22} - l_{21}a_{12}^T)$

**Continue with**

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

**endwhile**

| Step | $\left(\begin{array}{c\|c\|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$ | $a_{21}/\alpha_{11}$ | $A_{22} - a_{21}a_{12}^T$ |
|---|---|---|---|
| 1-2 | $\begin{pmatrix} -2 & -1 & 1 \\ 2 & -2 & -3 \\ -4 & 4 & 7 \end{pmatrix}$ | $\begin{pmatrix} 2 \\ -4 \end{pmatrix} / (-2) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ | $\begin{pmatrix} -2 & -3 \\ 4 & 7 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 6 & 5 \end{pmatrix}$ |
| 3 | $\begin{pmatrix} -2 & -1 & 1 \\ -1 & -3 & -2 \\ 2 & 6 & 5 \end{pmatrix}$ | $(6)/(-3) = (-2)$ | $(5) - (-2)\ (-2) = (1)$ |
| | $\begin{pmatrix} -2 & -1 & 1 \\ -1 & -3 & -2 \\ 2 & -2 & 1 \end{pmatrix}$ | | |

| Step | Current system | Multiplier | Operation |
|---|---|---|---|
| 1 | $\left(\begin{array}{ccc\|c} -2 & -1 & 1 & 6 \\ 2 & -2 & -3 & 3 \\ -4 & 4 & 7 & -3 \end{array}\right)$ | $\frac{2}{-2} = -1$ | $\begin{array}{rrrrr} & 2 & -2 & -3 & 3 \\ -1\times( & -2 & -1 & 1 & 6) \\ \hline & 0 & -3 & -2 & 9 \end{array}$ |
| 2 | $\left(\begin{array}{ccc\|c} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ -4 & 4 & 7 & -3 \end{array}\right)$ | $\frac{-4}{-2} = 2$ | $\begin{array}{rrrrr} & -4 & 4 & 7 & -3 \\ -(2)\times( & -2 & -1 & 1 & 6) \\ \hline & 0 & 6 & 5 & -15 \end{array}$ |
| 3 | $\left(\begin{array}{ccc\|c} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ 0 & 6 & 5 & -15 \end{array}\right)$ | $\frac{6}{-3} = -2$ | $\begin{array}{rrrrr} & 0 & 6 & 5 & -15 \\ -(-2)\times( & 0 & -3 & -2 & 9) \\ \hline & 0 & 0 & 1 & 3 \end{array}$ |
| 4 | $\left(\begin{array}{ccc\|c} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ 0 & 0 & 1 & 3 \end{array}\right)$ | | |



# Solving Ax = b via LU Factorization

- **Solving Lz = b (Forward substitution)**

Given a unit lower triangular matrix $L \in \mathbb{R}^{n\times n}$ and vectors $z, b \in \mathbb{R}^n$, consider the equation $Lz = b$ where $L$ and $b$ are known and $z$ is to be computed. Partition

$$L \rightarrow \left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right), \quad z \rightarrow \left(\begin{array}{c} \zeta_1 \\ \hline z_2 \end{array}\right), \quad \text{and} \quad b \rightarrow \left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right).$$

(Recall: the horizontal line here partitions the result. It is *not* a division.) Now, $Lz = b$ implies that

$$\overbrace{\left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right)}^{b} = \overbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array}\right)}^{L} \overbrace{\left(\begin{array}{c} \zeta_1 \\ \hline z_2 \end{array}\right)}^{z}$$

$$= \overbrace{\left(\begin{array}{c} 1\times\zeta_1 + 0\times z_2 \\ \hline l_{21}\zeta_1 + L_{22}z_2 \end{array}\right)}^{Lz} = \overbrace{\left(\begin{array}{c} \zeta_1 \\ \hline l_{21}\zeta_1 + L_{22}z_2 \end{array}\right)}^{Lz}$$

so that

$$\begin{array}{c} \beta_1 = \zeta_1 \\ \hline b_2 = l_{21}\zeta_1 + L_{22}z_2 \end{array} \quad \text{or, equivalently,} \quad \begin{array}{c} \zeta_1 = \beta_1 \\ \hline L_{22}z_2 = b_2 - \zeta_1 l_{21} \end{array}.$$



# Solving Ax = b via LU Factorization

- **Solving Lz = b (Forward substitution)**

**Algorithm:** $[b] := \text{LTRSV\_UNB\_VAR1}(L, b)$

**Partition** $L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where $L_{TL}$ is $0 \times 0$, $b_T$ has 0 rows

**while** $m(L_{TL}) < m(L)$ **do**

**Repartition**

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

where $\lambda_{11}$ is $1 \times 1$, $\beta_1$ has 1 row

$b_2 := b_2 - \beta_1 l_{21}$

**Continue with**

$$\left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

**endwhile**

| Step | $\left(\begin{array}{c\|c\|c} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{array}\right)$ | $\left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$ | $b_2 - l_{21}\beta_1$ |
|---|---|---|---|
| 1-2 | $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$ | $\begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix}$ | $\begin{pmatrix} 3 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} (6) = \begin{pmatrix} 9 \\ -15 \end{pmatrix}$ |
| 3 | $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$ | $\begin{pmatrix} 6 \\ 9 \\ -15 \end{pmatrix}$ | $(-15) - (-2)(9) = (3)$ |
| | $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$ | $\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}$ | |

# Solving Ax = b via LU Factorization

- Solving Ux = b (Back substitution)

Given upper triangular matrix $U \in \mathbb{R}^{n\times n}$ and vectors $x, b \in \mathbb{R}^n$, consider the equation $Ux = b$ where $U$ and $b$ are known and $x$ is to be computed. Partition

$$U \rightarrow \left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right), \quad x \rightarrow \left(\begin{array}{c} \chi_1 \\ \hline x_2 \end{array}\right) \quad \text{and} \quad b \rightarrow \left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right).$$

Now, $Ux = b$ implies

$$\overbrace{\left(\begin{array}{c} \beta_1 \\ \hline b_2 \end{array}\right)}^{b} = \overbrace{\left(\begin{array}{c|c} \upsilon_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array}\right)}^{U} \overbrace{\left(\begin{array}{c} \chi_1 \\ \hline x_2 \end{array}\right)}^{x}$$

$$= \overbrace{\left(\begin{array}{c} \upsilon_{11}\chi_1 + u_{12}^T x_2 \\ \hline 0 \times \chi_1 + U_{22} x_2 \end{array}\right)}^{Ux} = \overbrace{\left(\begin{array}{c} \upsilon_{11}\chi_1 + u_{12}^T x_2 \\ \hline U_{22} x_2 \end{array}\right)}^{Ux}$$

so that

$$\begin{array}{c} \beta_1 = \upsilon_{11}\chi_1 + u_{12}^T x_2 \\ \hline b_2 = U_{22} x_2 \end{array} \quad \text{or, equivalently,} \quad \begin{array}{c} \chi_1 = (\beta_1 - u_{12}^T x_2)/\upsilon_{11} \\ \hline U_{22} x_2 = b_2 \end{array}.$$



# Solving Ax = b via LU Factorization

- Solving Ux = b (Back substitution)

**Algorithm:** $[b] :=$ UTRSV_UNB_VAR1$(U, b)$

**Partition** $U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array}\right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right)$

where $U_{BR}$ is $0 \times 0$, $b_B$ has 0 rows

**while** $m(U_{BR}) < m(U)$ **do**

**Repartition**

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \rightarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

$\beta_1 := \beta_1 - u_{12}^T b_2$

$\beta_1 := \beta_1 / \upsilon_{11}$

**Continue with**

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & \upsilon_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array}\right), \left(\begin{array}{c} b_T \\ \hline b_B \end{array}\right) \leftarrow \left(\begin{array}{c} b_0 \\ \hline \beta_1 \\ \hline b_2 \end{array}\right)$$

**endwhile**



# Solving Ax = b via LU Factorization

- Putting it all together to solve Ax = b

| | |
|---|---|
| Want to solve: | $Ax = b.$ |
| We can now find triangular $L$ and $U$ so that | $A = LU.$ |
| Substitute: | $(LU)x = b.$ |
| Matrix multiplication is associative: | $L(Ux) = b.$ |
| We don't know $x$ but we can create a dummy vector $y = Ux$. | $L \underbrace{y}_{Ux} = b.$ |
| Solve<br>Solving a (lower) triangular system is easy! | $Ly = b$ for $y$ |
| Solve<br>Solving an (upper) triangular system is easy! | $Ux = y$ for $x$ |



# Questions and Answers