ความน่าจะเป็นและสถิติ
(Probability and Statistics)

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เนื้อหาที่จะเรียน

- Introduction
- Descriptive Statistics
- Probability
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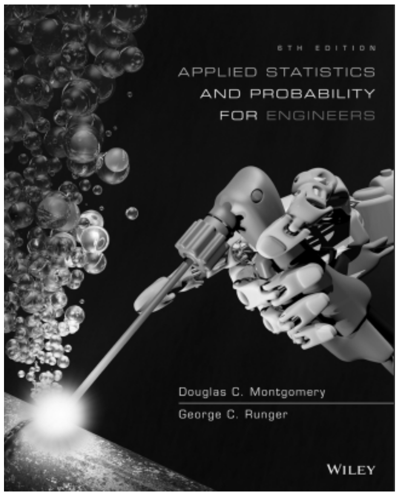
- Continuous Random Variables and Probability Distributions
- Point Estimation of Parameters and Sampling Distributions
- Statistical Intervals for a Single Sample
- Tests of Hypotheses for a Single Sample
- Simple Linear Regression and Correlation

ตำราเรียน

- หลัก :
  - Douglas C. Montgomery and George C. Runger , Applied Statistics and Probability for Engineers, 6th Edition, John Wiley & Sons, Inc, 2014
- รอง :
  - Prem S. Mann, Introductory Statistics (7th Edition), John Wiley & Sons, Inc, 2010



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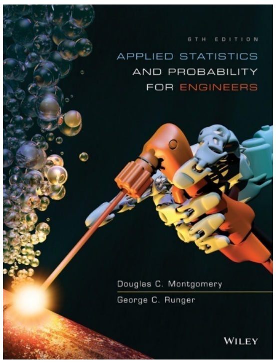


Applied Statistics and Probability for Engineers

Sixth Edition

Douglas C. Montgomery George C. Runger

Chapter 1

The Role of Statistics in Engineering

1 The Role of Statistics in Engineering

CHAPTER OUTLINE

Learning Objectives for Chapter 1

After careful study of this chapter, you should be able to do the following:

1. Identify the role of statistics in engineering problem-solving process.
2. How variability affects the data collected and used for engineering decisions.
3. Differentiate between enumerative and analytical studies.
4. Discuss the different methods that engineers use to collect data.
5. Identify the advantages that designed experiments have in comparison to the other methods of collecting engineering data.
6. Explain the differences between mechanistic models & empirical models.
7. Discuss how probability and probability models are used in engineering and science.

What Engineers Do?

An engineer is someone who solves problems of interest to society with the efficient application of scientific principles by:

- Refining existing products
- Designing new products or processes

The Creative Process

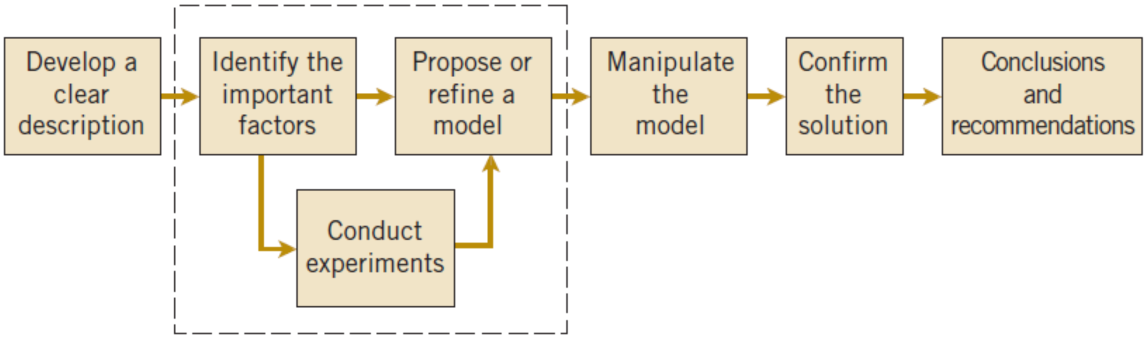


Figure 1-1 The engineering method

Statistics Supports The Creative Process

The field of statistics deals with the collection, presentation, analysis, and use of data to:

- Make decisions
- Solve problems
- Design products and processes

It is the science of data.

Variability

- Statistical techniques are useful to describe and understand variability.
- By variability, we mean successive observations of a system or phenomenon do *not* produce exactly the same result.
- Statistics gives us a framework for describing this variability and for learning about potential sources of variability.

An Engineering Example of Variability

Eight prototype units are produced and their pull-off forces are measured (in pounds): 12.6, 12.9, 13.4, 12.3, 13.6, 13.5, 12.6, 13.1.

All of the prototypes does not have the same pull-off force. We can see the variability in the above measurements as they exhibit variability.

The **dot diagram** is a very useful plot for displaying a small body of data - say up to about 20 observations.

This plot allows us to see easily two features of the data; the **location,** or the middle, and the **scatter** or **variability**.

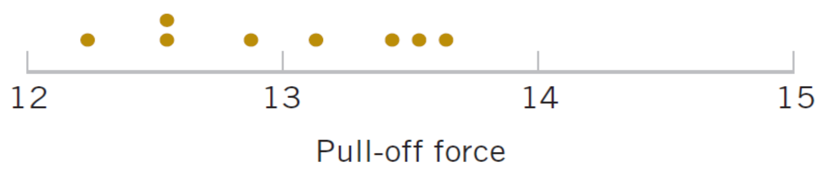


Figure 1-2 Dot diagram of the pull-off force data.

Basic Methods of Collecting Data

Three basic methods of collecting data:

- A **retrospective** study using historical data
  - Data collected in the past for other purposes.
- An **observational** study
  - Data, presently collected, by a passive observer.
- A **designed experiment**
  - Data collected in response to process input changes.

Hypothesis Tests

Hypothesis Test

- A statement about a process behavior value.
- Compared to a claim about another process value.
- Data is gathered to support or refuse the claim.

One-sample hypothesis test:

- Example: Ford avg mpg = 30

  vs

  Ford avg mpg < 30

Two-sample hypothesis test:

- Example: Ford avg mpg – Chevy avg mpg = 0

  vs

  Ford avg mpg – Chevy avg mpg > 0

Factorial Experiment & Example

An experiment design which uses every possible combination of the factor levels to form a basic experiment with “k” different settings for the process. This type of experiment is called a **factorial experiment.**

Example:

Consider a petroleum distillation column:

- Output is acetone concentration
- Inputs (factors) are:
  1. Reboil temperature
  2. Condensate temperature
  3. Reflux rate
- Output changes as the inputs are changed by experimenter.

Factorial Experiment Example

- Each factor is set at 2 reasonable levels (-1 and +1)
- 8 (2^3) runs are made, at every combination of factors, to observe acetone output.
- Resultant data is used to create a mathematical model of the process representing cause and effect.

Reboil Temp.	Condensate Temp.	Reflux Rate
−1	−1	−1
+1	−1	−1
−1	+1	−1
+1	+1	−1
−1	−1	+1
+1	−1	+1
−1	+1	+1
+1	+1	+1

Table 1-1 The Designed Experiment (Factorial Design) for the Distillation Column

Fractional Factorial Experiment

- Factor experiments can get too large. For example, 8 factors will require $2^8 = 256$ experimental runs of the distillation column.
- Certain combinations of factor levels can be deleted from the experiments without degrading the resultant model.
- The result is called a **fractional factorial experiment**.

Fractional Factorial Experiment - Example

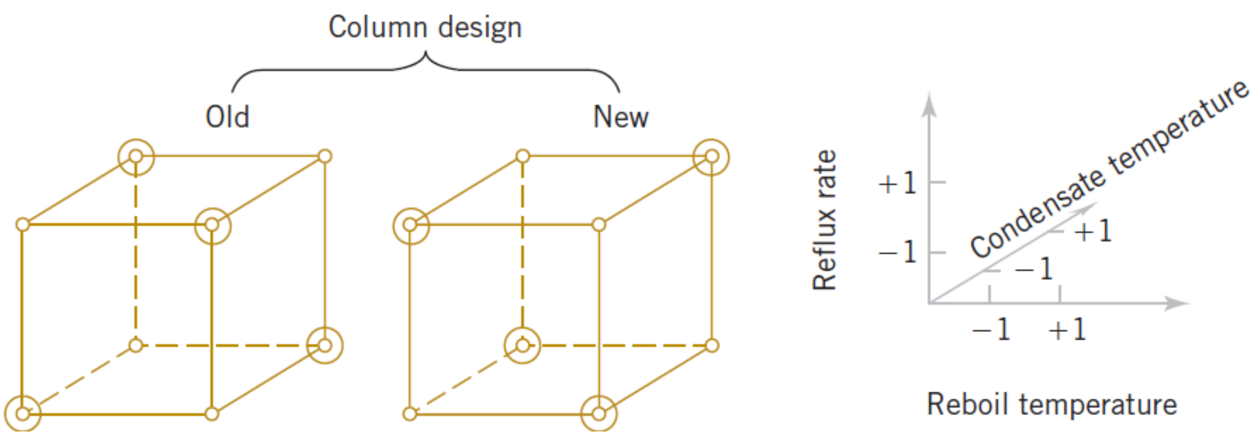


Figure 1-3 A fractional factorial experiment for the distillation column (one-half fraction) $2^4 / 2 = 8$ circled settings.

An Experiment in Variation

W. Edwards Deming, a famous industrial statistician & contributor to the Japanese quality revolution, conducted a illustrative experiment on process **over-control** or **tampering**.

Let's look at his apparatus and experimental procedure.

Deming's Experimental Set-up

Marbles were dropped through a funnel onto a target and the location where the marble struck the target was recorded.
Variation was caused by several factors:
Marble placement in funnel & release dynamics, vibration, air currents, measurement errors.

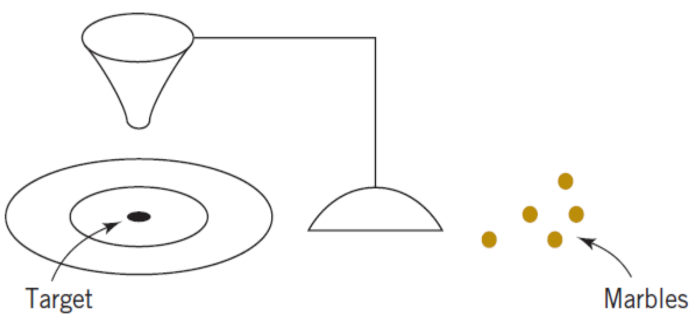


Figure 1-4 Deming's Funnel experiment

Deming's Experimental Procedure

- The funnel was aligned with the center of the target. Marbles were dropped. The distance from the strike point to the target center was measured and recorded.

- Strategy 1: The funnel was not moved. Then the process was repeated.

- Strategy 2: The funnel was moved an equal distance in the opposite direction to compensate for the error. He continued to make this type of adjustment after each marble was dropped. Then the process was repeated.

Deming's Experimental Procedure

- When both strategies were completed, he noticed the variability of the distance from the target for strategy 2 was approximately twice as large than for strategy 1.
- The deviations from the target is increased due to the adjustments to the funnel . Hence, adjustments to the funnel do not decrease future errors. Instead, they tend to move the funnel farther from the target.
- This experiment explains that the adjustments to a process based on random disturbances can actually increase the variation of the process. This is referred to as **overcontrol** or **tampering.**

Conclusions from the Deming Experiment

The lesson of the Deming experiment is that a process should not be adjusted in response to random variation, but only when a clear shift in the process value becomes apparent.

Then a process adjustment should be made to return the process outputs to their normal values.

To identify when the shift occurs, a control chart is used. Output values, plotted over time along with the outer limits of normal variation, pinpoint when the process leaves normal values and should be adjusted.

How Is the Change Detected?

- A control chart is used. Its characteristics are:
  - Time-oriented horizontal axis, e.g., hours.
  - Variable-of-interest vertical axis, e.g., % acetone.
- Long-term average is plotted as the center-line.
- Long-term usual variability is plotted as an upper and lower control limit around the long-term average.
- A sample of size n is taken and the averages are plotted over time. If the plotted points are between the control limits, then the process is normal; if not, it needs to be adjusted.

How Is the Change Detected Graphically?

The center line on the control chart is just the average of the concentration measurements for the first 20 samples

$$\bar{X} = 91.5 g / l$$

when the process is stable. The upper control limit and the lower control limit are located 3 standard deviations of the concentration values above and below the center line.

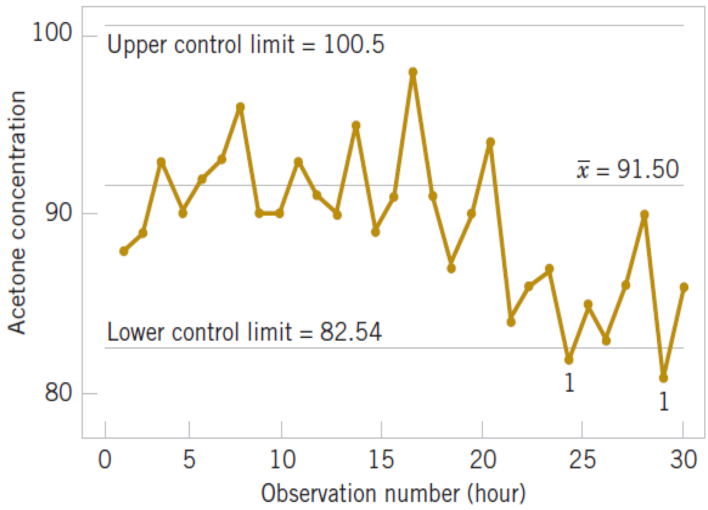


Figure 1-5 A control chart for the chemical process concentration data. Process steps out at hour 24 & 29. Shut down & adjust process.

Use of Control Charts

Deming contrasted two purposes of control charts:

1. **Enumerative studies**: Control chart of past production lots. Used for lot-by-lot acceptance sampling.

2. **Analytic studies**: Real-time control of a production process.

Mechanistic and Empirical Models

A **mechanistic model** is built from our underlying knowledge of the basic physical mechanism that relates several variables.

Example: Ohm's Law

Current = V/R

$$I = E/R$$

$$I = E/R + \varepsilon$$

where ε is a term added to the model to account for the fact that the observed values of current flow do not perfectly conform to the mechanistic model.

• The form of the function is known.

Mechanistic and Empirical Models

An **empirical model** is built from our engineering and scientific knowledge of the phenomenon, but is not directly developed from our theoretical or first-principles understanding of the underlying mechanism.

The form of the function is not known *a priori*.

An Example of an Empirical Model

- In a semiconductor manufacturing plant, the finished semiconductor is wire-bonded to a frame. In an observational study, the variables recorded were:
  - Pull strength to break the bond (y)
  - Wire length (x_1)
  - Die height (x_2)
- The data recorded are shown on the next slide.

An Example of an Empirical Model

Table 1-2 Wire Bond Pull Strength Data

Observation Number	Pull Strength y	Wire Length x_1	Die Height x_2
1	9.95	2	50
2	24.45	8	110
3	31.75	11	120
4	35.00	10	550
5	25.02	8	295
6	16.86	4	200
7	14.38	2	375
8	9.60	2	52
9	24.35	9	100
10	27.50	8	300
11	17.08	4	412
12	37.00	11	400
13	41.95	12	500
14	11.66	2	360
15	21.65	4	205
16	17.89	4	400
17	69.00	20	600
18	10.30	1	585
19	34.93	10	540
20	46.59	15	250
21	44.88	15	290
22	54.12	16	510
23	56.63	17	590
24	22.13	6	100
25	21.15	5	400

An Example of an Empirical Model

$$\widehat{\text{Pull strength}} = \beta_0 + \beta_1(\text{wire length}) + \beta_2(\text{die height}) + \epsilon$$

where the "hat," or circumflex, over pull strength indicates that this is an estimated or predicted quality.

In general, this type of empirical model is called a **regression model.**

The **estimated** regression relationship is given by:

$$\widehat{\text{Pull strength}} = 2.26 + 2.74(\text{wire length}) + 0.0125(\text{die height})$$

Visualizing the Data

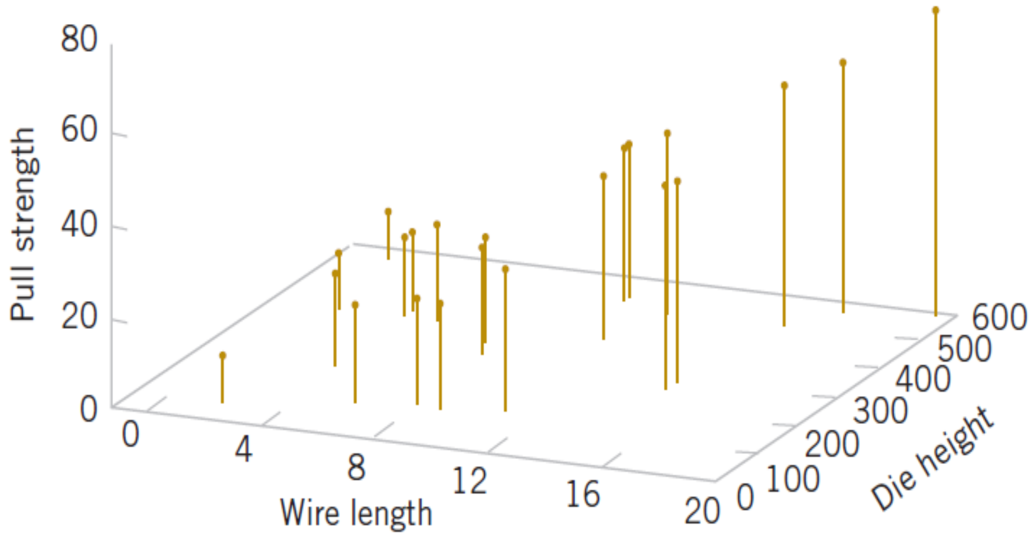


Figure 1-6 Three-dimensional plot of the pull strength (y), wire length (x_1) and die height (x_2) data.

Visualizing the Resultant Model Using Regression Analysis

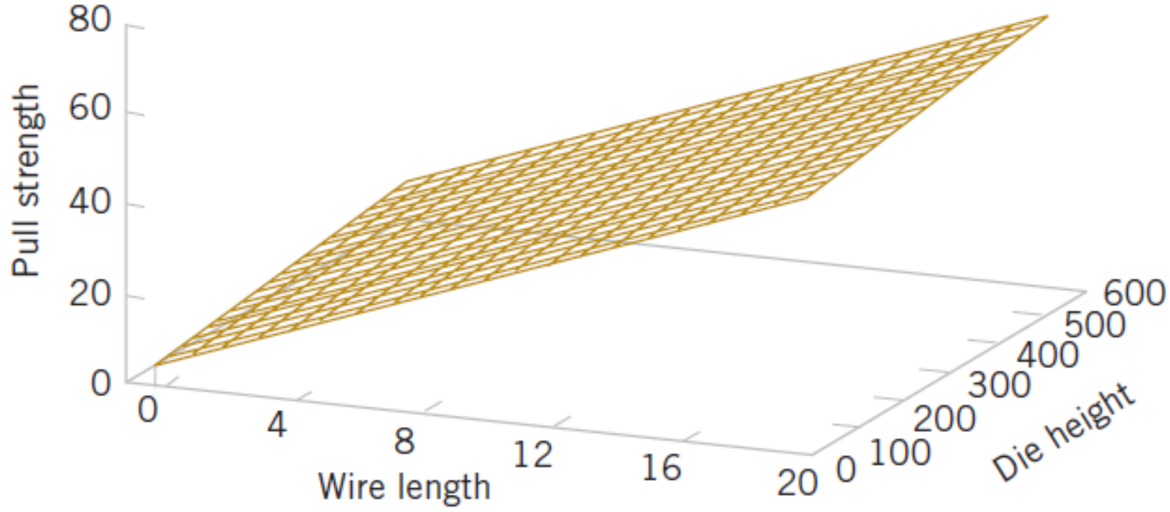


Figure 1-7 Plot of the predicted values (a plane) of pull strength from the empirical regression model.

Models Can Also Reflect Uncertainty

- **Probability models** help quantify the risks involved in statistical inference, that is, risks involved in decisions made every day.
- Probability provides the **framework** for the study and application of statistics.
- Probability concepts will be introduced in the next lecture.

6 Random Sampling and Data Description

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter you should be able to do the following:

1. Compute and interpret the sample mean, sample variance, sample standard deviation, sample median, and sample range
2. Explain the concepts of sample mean, sample variance, population mean, and population variance
3. Construct and interpret visual data displays, including the stem-and-leaf display, the histogram, and the box plot
4. Explain the concept of random sampling
5. Construct and interpret normal probability plots
6. Explain how to use box plots and other data displays to visually compare two or more samples of data
7. Know how to use simple time series plots to visually display the important features of time-oriented data.

6-1 Numerical Summaries

Definition: Sample Mean

If the n observations in a sample are denoted by $x_1, x_2, \ldots, x_n$, the **sample mean** is

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad (6\text{-}1)$$

6-1 Numerical Summaries

Example 6-1

Let's consider the eight observations collected from the prototype engine connectors from Chapter 1. The eight observations are $x_1 = 12.6$, $x_2 = 12.9$, $x_3 = 13.4$, $x_4 = 12.3$, $x_5 = 13.6$, $x_6 = 13.5$, $x_7 = 12.6$, and $x_8 = 13.1$. The sample mean is

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{8} x_i}{8} = \frac{12.6 + 12.9 + \cdots + 13.1}{8}$$

$$= \frac{104}{8} = 13.0 \text{ pounds}$$

A physical interpretation of the sample mean as a measure of location is shown in the dot diagram of the pull-off force data. See Figure 6-1. Notice that the sample mean $\bar{x} = 13.0$ can be thought of as a "balance point." That is, if each observation represents 1 pound of mass placed at the point on the x-axis, a fulcrum located at $\bar{x}$ would exactly balance this system of weights.

6-1 Numerical Summaries

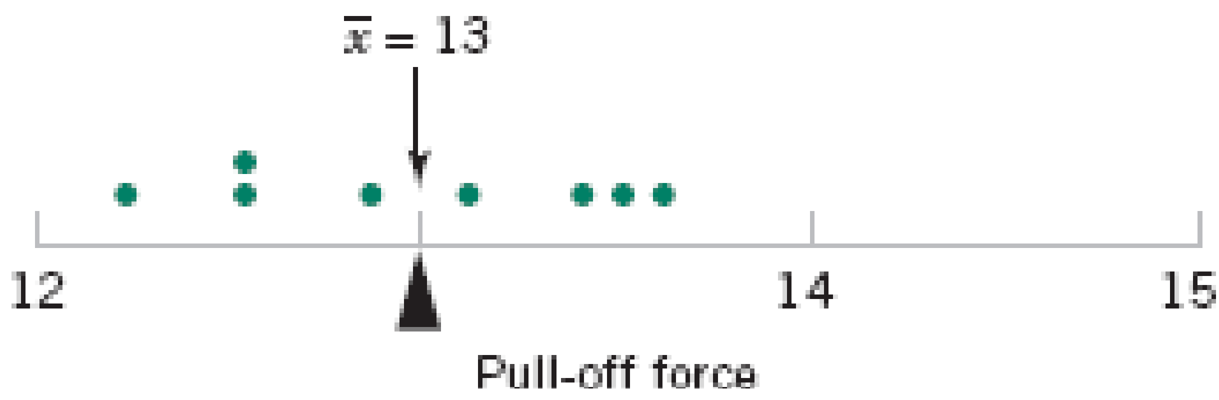


Figure 6-1 The sample mean as a balance point for a system of weights.

6-1 Numerical Summaries

Population Mean

For a finite population with N measurements, the mean is

$$\mu = \sum_{i=1}^{N} x_i f(x_i) = \frac{\sum_{i=1}^{N} x_i}{N} \qquad (6\text{-}2)$$

The sample mean is a reasonable estimate of the population mean.

6-1 Numerical Summaries

Definition: Sample Variance

If $x_1, x_2, \ldots, x_n$ is a sample of n observations, the **sample variance** is

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n - 1} \qquad (6\text{-}3)$$

The **sample standard deviation,** s, is the positive square root of the sample variance.

6-1 Numerical Summaries

How Does the Sample Variance Measure Variability?

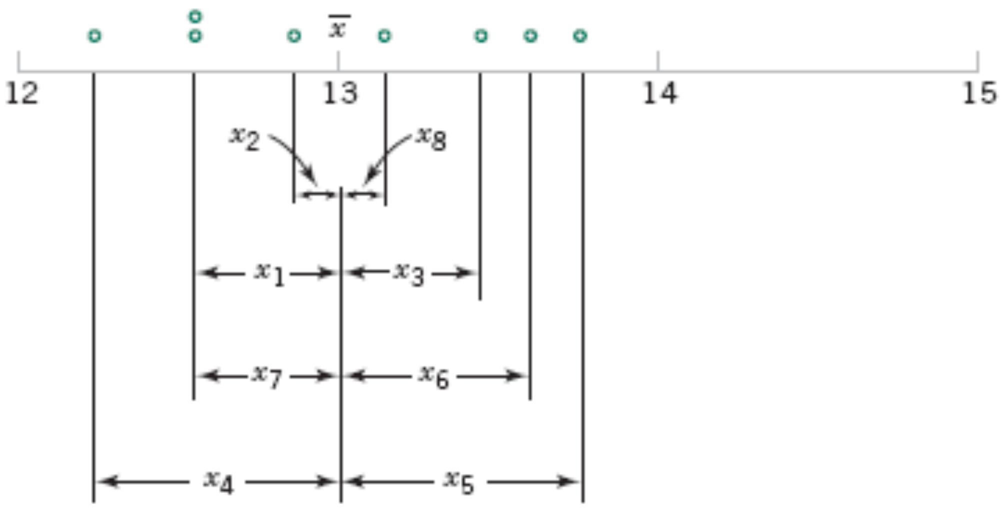


Figure 6-2 How the sample variance measures variability through the deviations $x_i - \bar{x}$.

6-1 Numerical Summaries

Example 6-2

Table 6-1 displays the quantities needed for calculating the sample variance and sample standard deviation for the pull-off force data. These data are plotted in Fig. 6-2. The numerator of s^2 is

$$\sum_{i=1}^{8} (x_i - \bar{x})^2 = 1.60$$

so the sample variance is

$$s^2 = \frac{1.60}{8 - 1} = \frac{1.60}{7} = 0.2286 \text{ (pounds)}^2$$

and the sample standard deviation is

$$s = \sqrt{0.2286} = 0.48 \text{ pounds}$$

6-1 Numerical Summaries

Table 6-1 Calculation of Terms for the Sample Variance and Sample Standard Deviation

i	x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
1	12.6	−0.4	0.16
2	12.9	−0.1	0.01
3	13.4	0.4	0.16
4	12.3	−0.7	0.49
5	13.6	0.6	0.36
6	13.5	0.5	0.25
7	12.6	−0.4	0.16
8	13.1	0.1	0.01
	104.0	0.0	1.60

6-1 Numerical Summaries

Computation of s^2

$$s^2 = \frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n-1} \qquad (6\text{-}4)$$

6-1 Numerical Summaries

Population Variance

When the population is finite and consists of N values, we may define the population variance as

$$\sigma^2 = \frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N} \qquad (6\text{-}5)$$

The sample variance is a reasonable estimate of the population variance.

6-1 Numerical Summaries

Definition

If the n observations in a sample are denoted by $x_1, x_2, \ldots, x_n$, the **sample range** is

$$r = \max(x_i) - \min(x_i) \qquad (6\text{-}6)$$

6-2 Stem-and-Leaf Diagrams

A **stem-and-leaf diagram** is a good way to obtain an informative visual display of a data set $x_1, x_2, \ldots, x_n$, where each number x_i consists of at least two digits. To construct a stem-and-leaf diagram, use the following steps.

Steps for Constructing a Stem-and-Leaf Diagram

(1) Divide each number x_i into two parts: a **stem**, consisting of one or more of the leading digits and a **leaf**, consisting of the remaining digit.

(2) List the stem values in a vertical column.

(3) Record the leaf for each observation beside its stem.

(4) Write the units for stems and leaves on the display.

6-2 Stem-and-Leaf Diagrams

Example 6-4

To illustrate the construction of a stem-and-leaf diagram, consider the alloy compressive strength data in Table 6-2. We will select as stem values the numbers 7, 8, 9, . . . , 24. The resulting stem-and-leaf diagram is presented in Fig. 6-4. The last column in the diagram is a frequency count of the number of leaves associated with each stem. Inspection of this display immediately reveals that most of the compressive strengths lie between 110 and 200 psi and that a central value is somewhere between 150 and 160 psi. Furthermore, the strengths are distributed approximately symmetrically about the central value. The stem-and-leaf diagram enables us to determine quickly some important features of the data that were not immediately obvious in the original display in Table 6-2.

6-2 Stem-and-Leaf Diagrams

Table 6-2 Compressive Strength (in psi) of 80 Aluminum-Lithium Alloy Specimens

105	221	183	186	121	181	180	143
97	154	153	174	120	168	167	141
245	228	174	199	181	158	176	110
163	131	154	115	160	208	158	133
207	180	190	193	194	133	156	123
134	178	76	167	184	135	229	146
218	157	101	171	165	172	158	169
199	151	142	163	145	171	148	158
160	175	149	87	160	237	150	135
196	201	200	176	150	170	118	149

6-2 Stem-and-Leaf Diagrams

Figure 6-4 Stem-and-leaf diagram for the compressive strength data in Table 6-2.

Stem	Leaf	Frequency
7	6	1
8	7	1
9	7	1
10	5 1	2
11	5 8 0	3
12	1 0 3	3
13	4 1 3 5 3 5	6
14	2 9 5 8 3 1 6 9	8
15	4 7 1 3 4 0 8 8 6 8 0 8	12
16	3 0 7 3 0 5 0 8 7 9	10
17	8 5 4 4 1 6 2 1 0 6	10
18	0 3 6 1 4 1 0	7
19	9 6 0 9 3 4	6
20	7 1 0 8	4
21	8	1
22	1 8 9	3
23	7	1
24	5	1

Stem : Tens and hundreds digits (psi); Leaf: Ones digits (psi)

6-2 Stem-and-Leaf Diagrams

Example 6-5

Figure 6-5 illustrates the stem-and-leaf diagram for 25 observations on batch yields from a chemical process. In Fig. 6-5(a) we have used 6, 7, 8, and 9 as the stems. This results in too few stems, and the stem-and-leaf diagram does not provide much information about the data. In Fig. 6-5(b) we have divided each stem into two parts, resulting in a display that more

6-2 Stem-and-Leaf Diagrams

Stem	Leaf
6	1 3 4 5 5 6
7	0 1 1 3 5 7 8 8 9
8	1 3 4 4 7 8 8
9	2 3 5

(a)

Stem	Leaf
6L	1 3 4
6U	5 5 6
7L	0 1 1 3
7U	5 7 8 8 9
8L	1 3 4 4
8U	7 8 8
9L	2 3
9U	5

(b)

Stem	Leaf
6z	1
6t	3
6f	4 5 5
6s	6
6e	
7z	0 1 1
7t	3
7f	5
7s	7
7e	8 8 9
8z	1
8t	3
8f	4 4
8s	7
8e	8 8
9z	
9t	2 3
9f	5
9s	
9e	

(c)

Figure 6-5 Stem-and-leaf displays for Example 6-5. Stem: Tens digits. Leaf: Ones digits.

6-2 Stem-and-Leaf Diagrams

Figure 6-6 Stem-and-leaf diagram from Minitab.

Character Stem-and-Leaf Display

Stem-and-leaf of Strength
N = 80 Leaf Unit = 1.0

1	7	6
2	8	7
3	9	7
5	10	1 5
8	11	0 5 8
11	12	0 1 3
17	13	1 3 3 4 5 5
25	14	1 2 3 5 6 8 9 9
37	15	0 0 1 3 4 4 6 7 8 8 8 8
(10)	16	0 0 0 3 3 5 7 7 8 9
33	17	0 1 1 2 4 4 5 6 6 8
23	18	0 0 1 1 3 4 6
16	19	0 3 4 6 9 9
10	20	0 1 7 8
6	21	8
5	22	1 8 9
2	23	7
1	24	5

6-2 Stem-and-Leaf Diagrams

Data Features

• The **median** is a measure of central tendency that divides the data into two equal parts, half below the median and half above. If the number of observations is even, the median is halfway between the two central values.

From Fig. 6-6, the 40th and 41st values of strength as 160 and 163, so the median is (160 + 163)/2 = 161.5. If the number of observations is odd, the median is the *central* value.

The **range** is a measure of variability that can be easily computed from the ordered stem-and-leaf display. It is the maximum minus the minimum measurement. From Fig.6-6 the range is 245 - 76 = 169.

6-2 Stem-and-Leaf Diagrams

Data Features

• When an **ordered** set of data is divided into four equal parts, the division points are called **quartiles.**

The **first** or **lower quartile**, q_1 , is a value that has approximately one-fourth (25%) of the observations below it and approximately 75% of the observations above.

The **second quartile**, q_2, has approximately one-half (50%) of the observations below its value. The second quartile is *exactly* equal to the **median**.

The **third** or **upper quartile**, q_3, has approximately three-fourths (75%) of the observations below its value. As in the case of the median, the quartiles may not be unique.

6-2 Stem-and-Leaf Diagrams

Data Features

• The compressive strength data in Figure 6-6 contains n = 80 observations. Minitab software calculates the first and third quartiles as the$(n + 1)/4$ and $3(n + 1)/4$ ordered observations and interpolates as needed.

For example, $(80 + 1)/4 = 20.25$ and $3(80 + 1)/4 = 60.75$.

Therefore, Minitab interpolates between the 20th and 21st ordered observation to obtain $q_1 = 143.50$ and between the 60th and 61st observation to obtain $q_3 = 181.00$.

6-2 Stem-and-Leaf Diagrams

Data Features

• The ***interquartile range*** is the difference between the upper and lower quartiles, and it is sometimes used as a measure of variability.

• In general, the $100k$th ***percentile*** is a data value such that approximately $100k\%$ of the observations are at or below this value and approximately $100(1 - k)\%$ of them are above it.

6-3 Frequency Distributions and Histograms

- A **frequency distribution** is a more compact summary of data than a stem-and-leaf diagram.

- To construct a frequency distribution, we must divide the range of the data into intervals, which are usually called **class intervals**, **cells**, or **bins**.

Constructing a Histogram (Equal Bin Widths):

(1) Label the bin (class interval) boundaries on a horizontal scale.
(2) Mark and label the vertical scale with the frequencies or the relative frequencies.
(3) Above each bin, draw a rectangle where height is equal to the frequency (or relative frequency) corresponding to that bin.

6-3 Frequency Distributions and Histograms

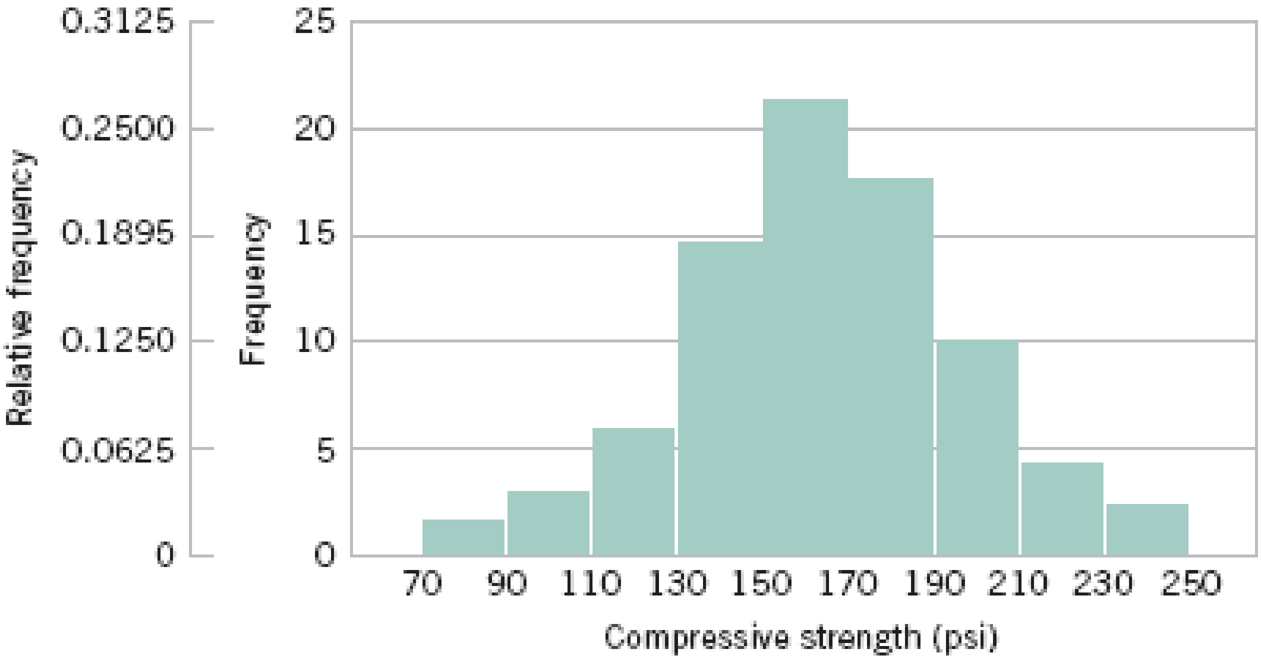


Figure 6-7 Histogram of compressive strength for 80 aluminum-lithium alloy specimens.

6-3 Frequency Distributions and Histograms

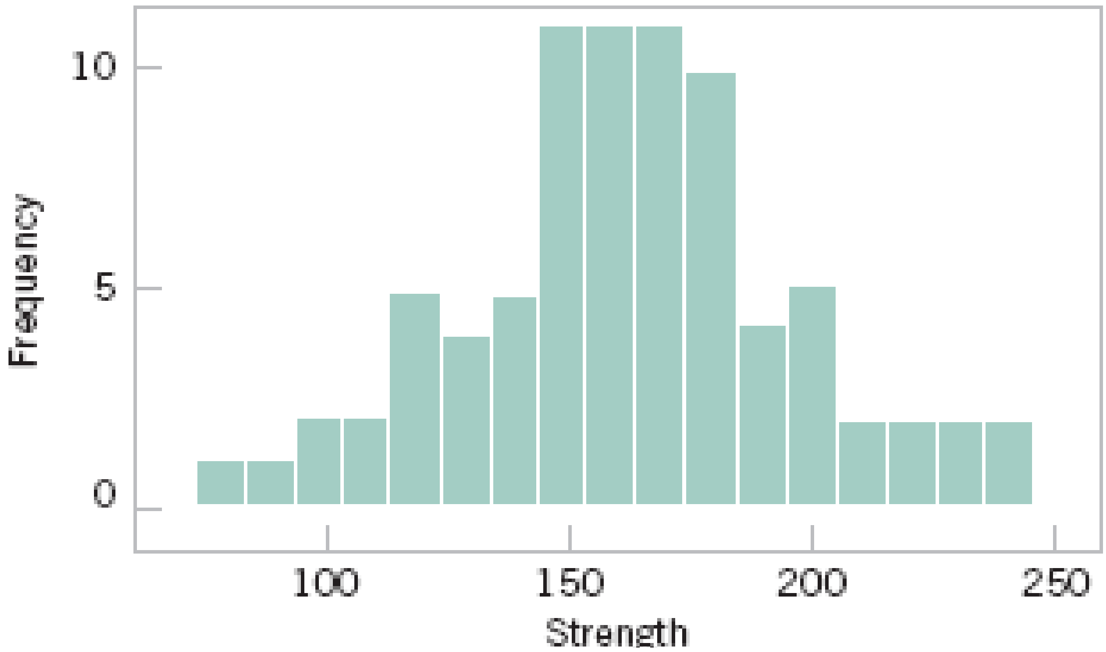


Figure 6-8 A histogram of the compressive strength data from Minitab with 17 bins.

6-3 Frequency Distributions and Histograms

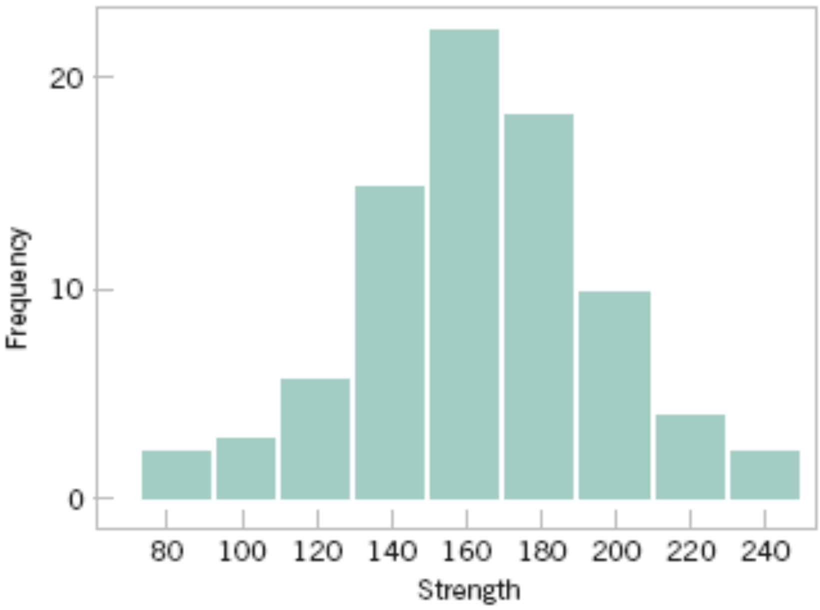


Figure 6-9 A histogram of the compressive strength data from Minitab with nine bins.

Figure 6-9 A histogram of the compressive strength data from Minitab with nine bins.

6-3 Frequency Distributions and Histograms

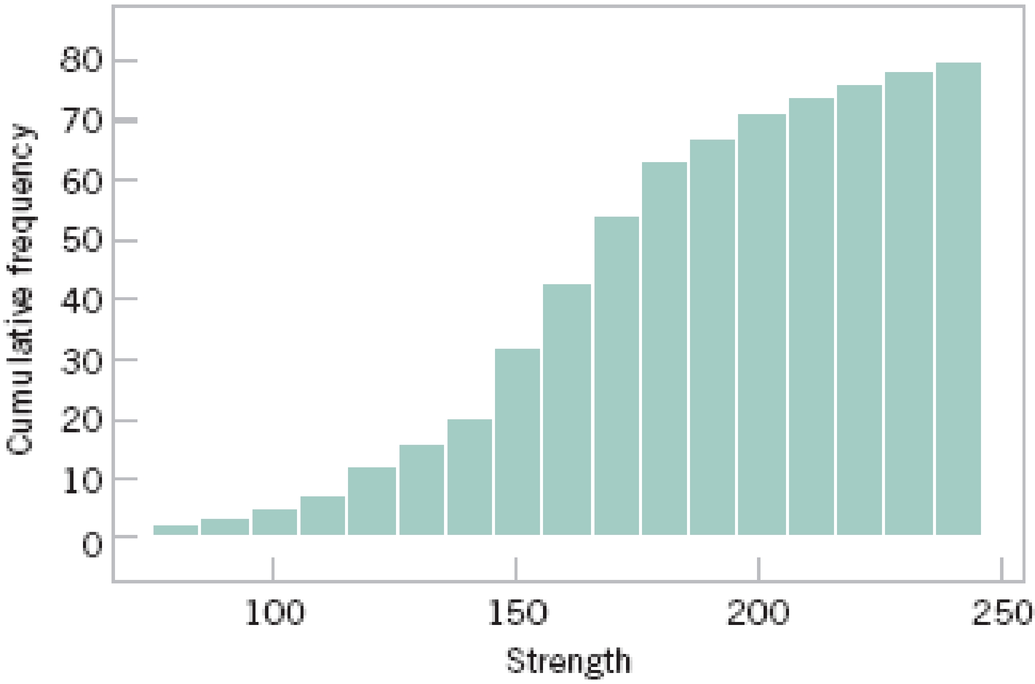


Figure 6-10 A cumulative distribution plot of the compressive strength data from Minitab.

6-3 Frequency Distributions and Histograms

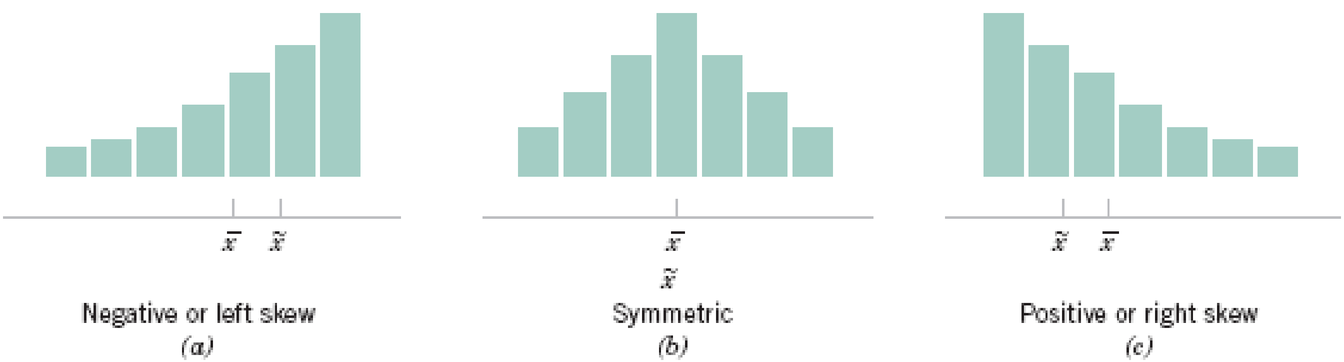


Figure 6-11 Histograms for symmetric and skewed distributions.

6-4 Box Plots

• The **box plot** is a graphical display that simultaneously describes several important features of a data set, such as center, spread, departure from symmetry, and identification of observations that lie unusually far from the bulk of the data.

• **Whisker**
• **Outlier**
• **Extreme outlier**

6-4 Box Plots

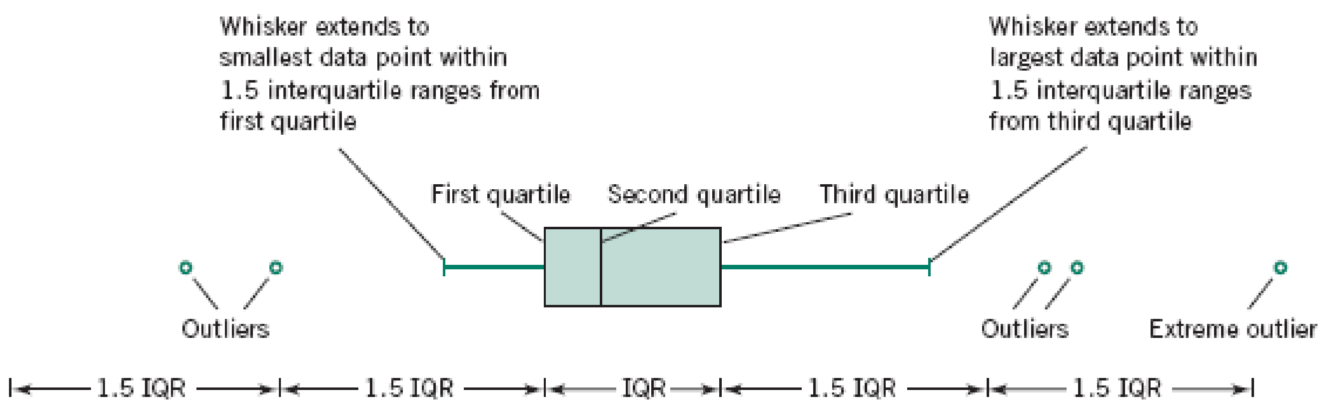


Figure 6-13 Description of a box plot.

6-4 Box Plots

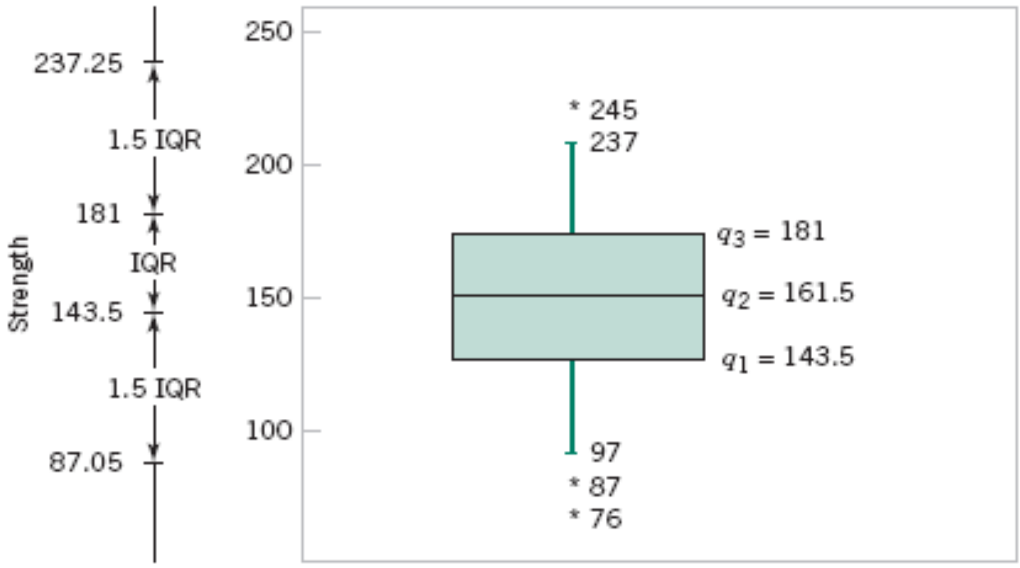
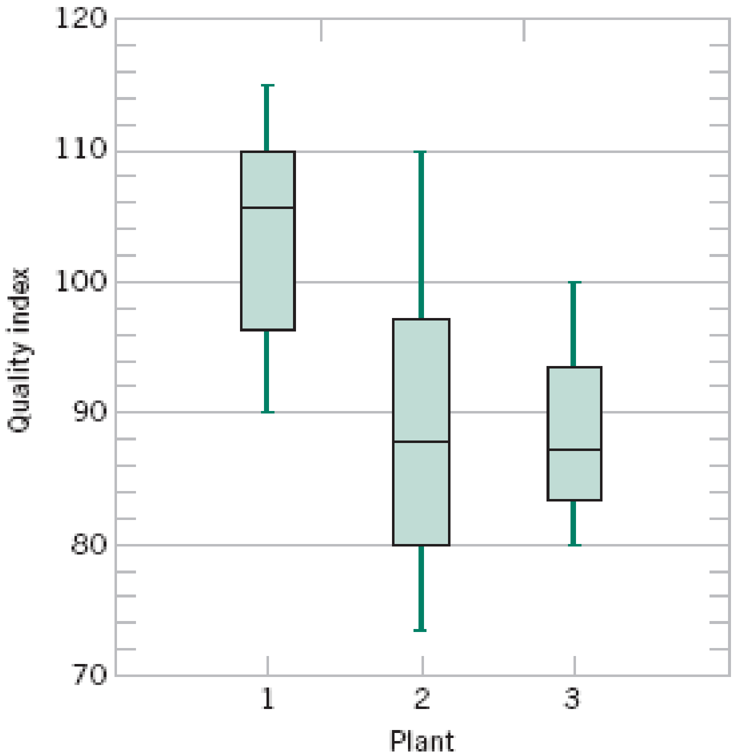


Figure 6-14 Box plot for compressive strength data in Table 6-2.

6-4 Box Plots

Figure 6-15
Comparative box plots of a quality index at three plants.



6-5 Time Sequence Plots

- A **time series** or **time sequence** is a data set in which the observations are recorded in the order in which they occur.
- A **time series plot** is a graph in which the vertical axis denotes the observed value of the variable (say x) and the horizontal axis denotes the time (which could be minutes, days, years, etc.).
- When measurements are plotted as a time series, we often see
  - **trends,**
  - **cycles, or**
  - **other broad features of the data**

6-5 Time Sequence Plots

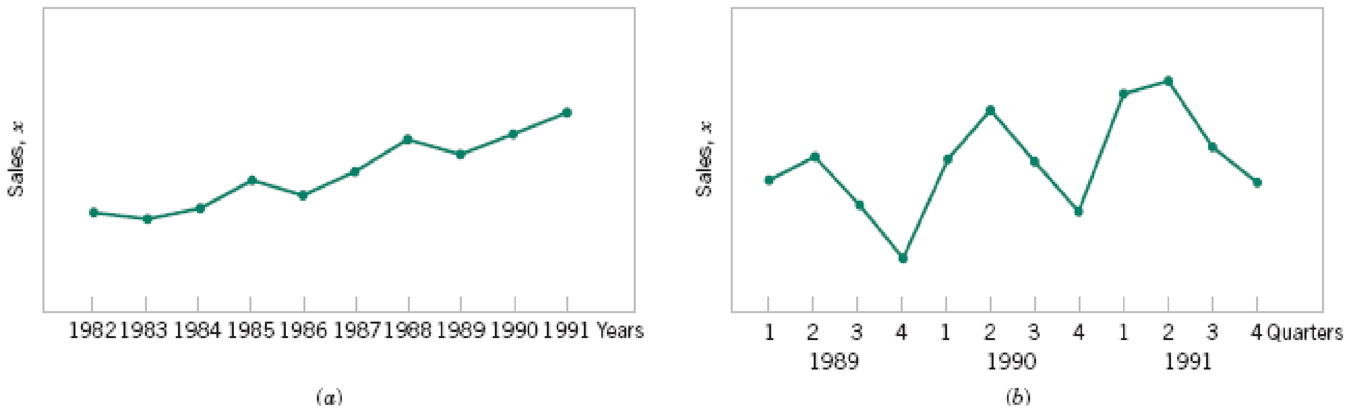


Figure 6-16 Company sales by year (a) and by quarter (b).

6-5 Time Sequence Plots

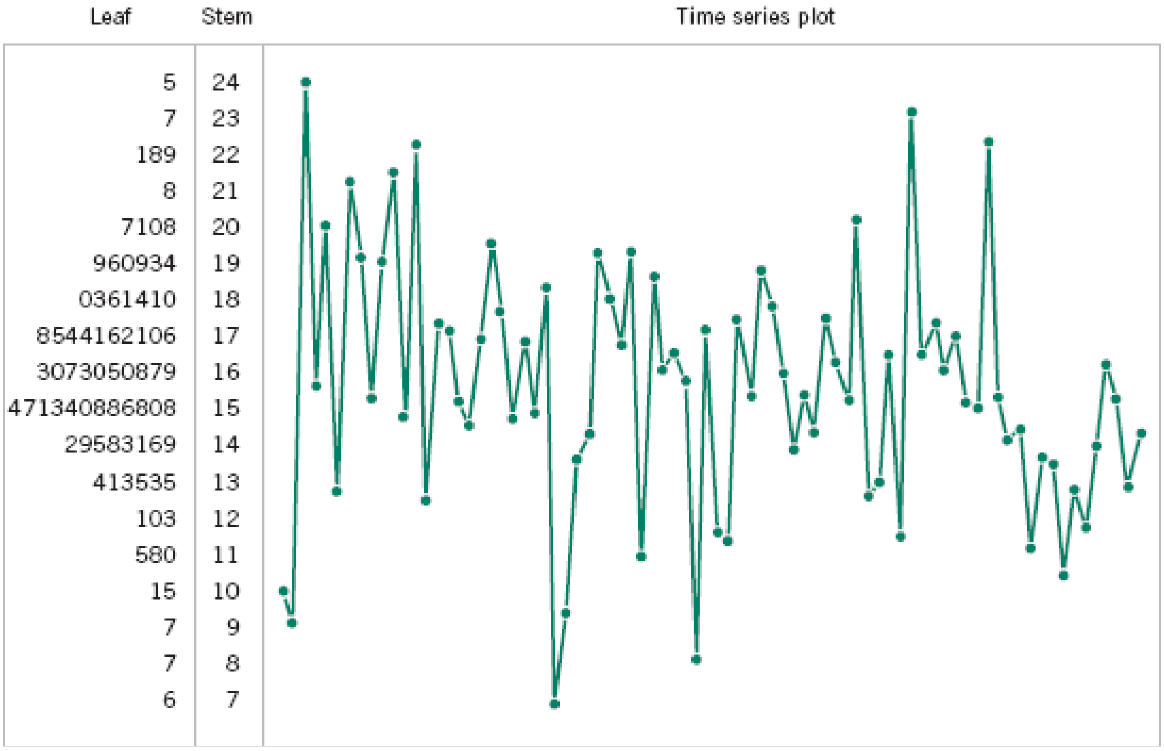


Figure 6-17 A digidot plot of the compressive strength data in Table 6-2.

6-5 Time Sequence Plots

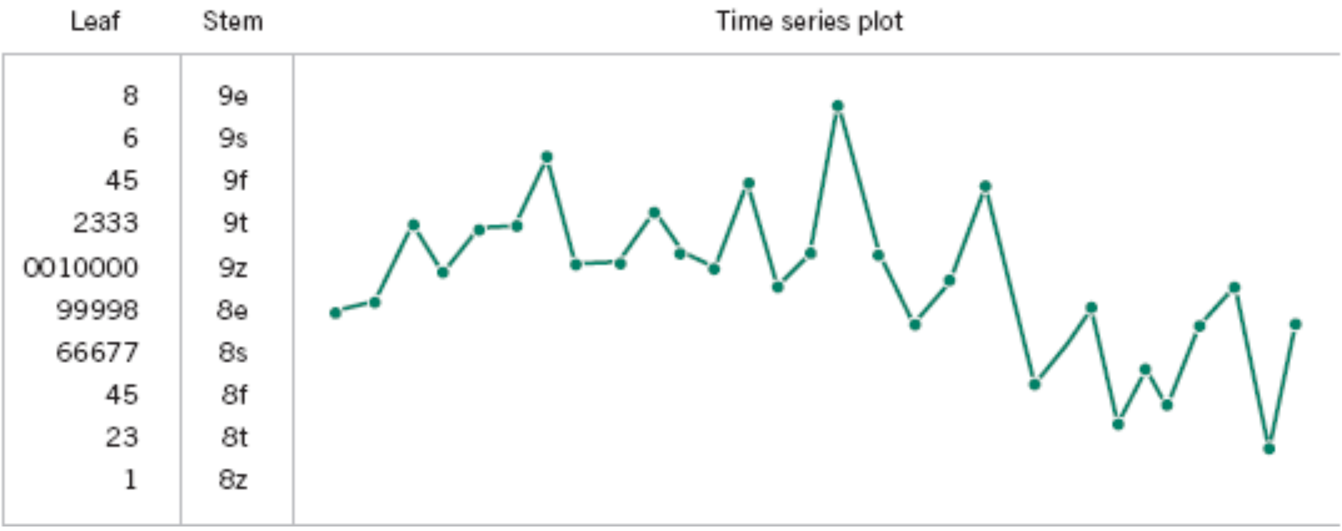


Figure 6-18 A digidot plot of chemical process concentration readings, observed hourly.

6-6 Probability Plots

- **Probability plotting** is a graphical method for determining whether sample data conform to a hypothesized distribution based on a subjective visual examination of the data.

- Probability plotting typically uses special graph paper, known as **probability paper,** that has been designed for the hypothesized distribution. Probability paper is widely available for the normal, lognormal, Weibull, and various chi-square and gamma distributions.

6-6 Probability Plots

Example 6-7

Ten observations on the effective service life in minutes of batteries used in a portable personal computer are as follows: 176, 191, 214, 220, 205, 192, 201, 190, 183, 185. We hypothesize that battery life is adequately modeled by a normal distribution. To use probability plotting to investigate this hypothesis, first arrange the observations in ascending order and calculate their cumulative frequencies $(j - 0.5)/10$ as shown in Table 6-6.

Table 6-6 Calculation for Constructing a Normal Probability Plot

j	$x_{(j)}$	$(j - 0.5)/10$	z_j
1	176	0.05	−1.64
2	183	0.15	−1.04
3	185	0.25	−0.67
4	190	0.35	−0.39
5	191	0.45	−0.13
6	192	0.55	0.13
7	201	0.65	0.39
8	205	0.75	0.67
9	214	0.85	1.04
10	220	0.95	1.64

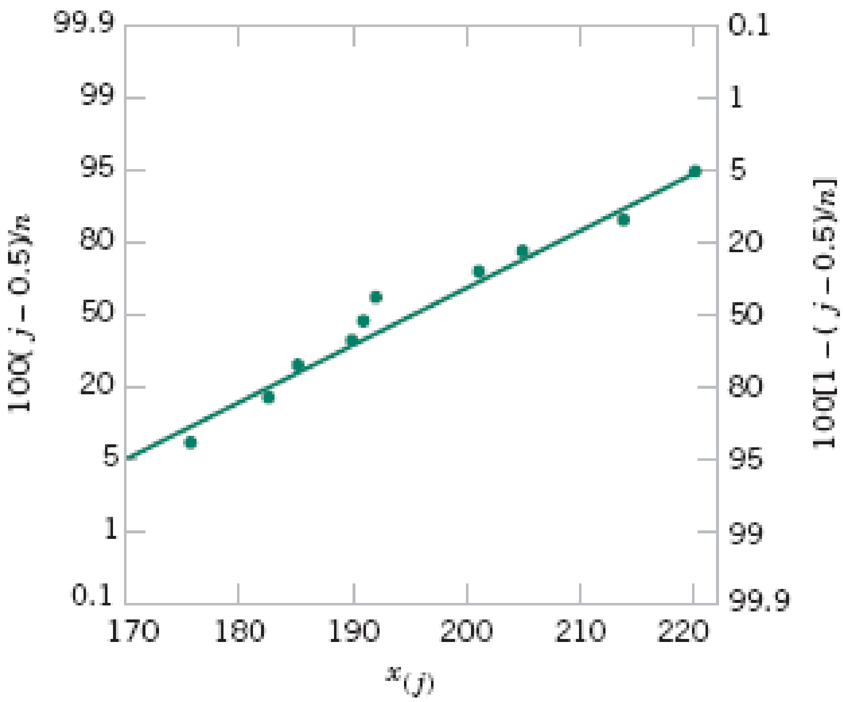
6-6 Probability Plots

Example 6-7 (continued)

The pairs of values $x_{(j)}$ and $(j - 0.5)/10$ are now plotted on normal probability paper. This plot is shown in Fig. 6-19. Most normal probability paper plots $100(j - 0.5)/n$ on the left vertical scale and $100[1 - (j - 0.5)/n]$ on the right vertical scale, with the variable value plotted on the horizontal scale. A straight line, chosen subjectively, has been drawn through the plotted points. In drawing the straight line, you should be influenced more by the points near the middle of the plot than by the extreme points. A good rule of thumb is to draw the line approximately between the 25th and 75th percentile points. This is how the line in Fig. 6-19 was determined. In assessing the "closeness" of the points to the straight line, imagine a "fat pencil" lying along the line. If all the points are covered by this imaginary pencil, a normal distribution adequately describes the data. Since the points in Fig. 6-19 would pass the "fat pencil" test, we conclude that the normal distribution is an appropriate model.

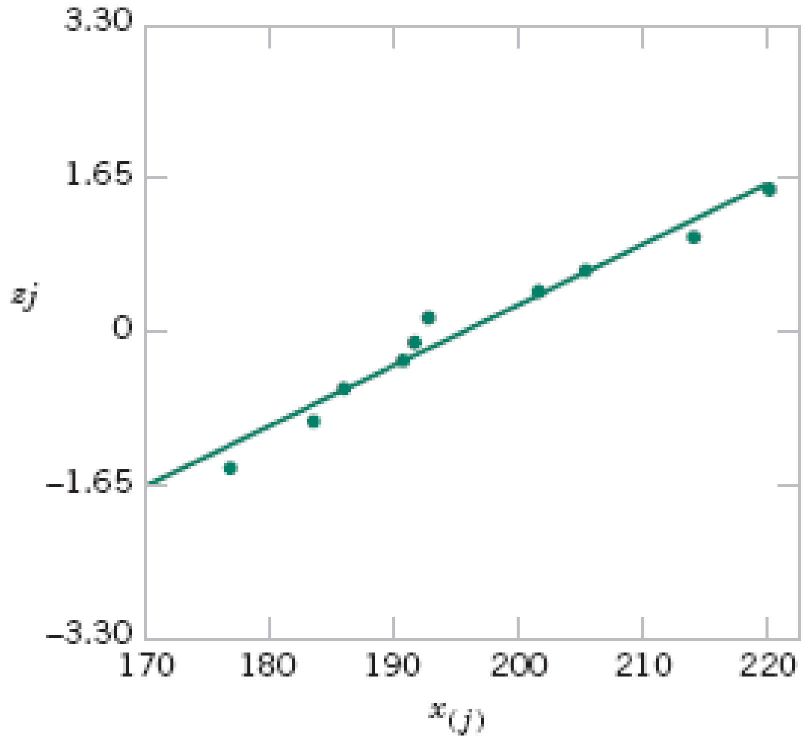
6-6 Probability Plots

Figure 6-19 Normal probability plot for battery life.



6-6 Probability Plots

Figure 6-20 Normal probability plot obtained from standardized normal scores.



6-6 Probability Plots

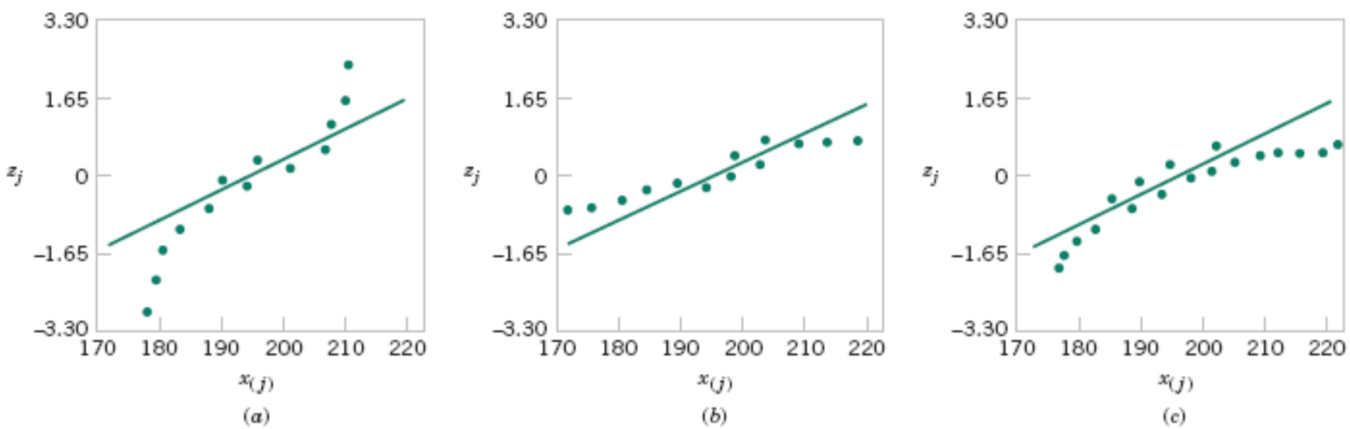


Figure 6-21 Normal probability plots indicating a nonnormal distribution. (a) Light-tailed distribution. (b) Heavy-tailed distribution. (c) A distribution with positive (or right) skew.

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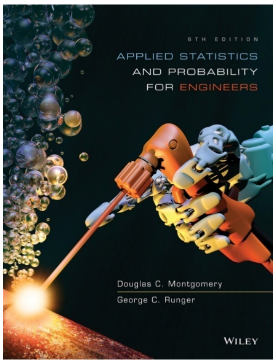


Applied Statistics and Probability for Engineers

Sixth Edition

Douglas C. Montgomery   George C. Runger

Chapter 2
Probability

2 Probability

CHAPTER OUTLINE

Learning Objectives for Chapter 2

After careful study of this chapter, you should be able to do the following:

1. Understand and describe sample spaces and events
2. Interpret probabilities and calculate probabilities of events
3. Use permutations and combinations to count outcomes
4. Calculate the probabilities of joint events
5. Interpret and calculate conditional probabilities
6. Determine independence and use independence to calculate probabilities
7. Understand Bayes' theorem and when to use it
8. Understand random variables

Random Experiment

- An experiment is a procedure that is
  - carried out under controlled conditions, and
  - executed to discover an unknown result.
- An experiment that results in different outcomes even when repeated in the same manner every time is a random experiment.

Sample Spaces

- The set of all possible outcomes of a random experiment is called the sample space, S.
- S is discrete if it consists of a finite or countable infinite set of outcomes.
- S is continuous if it contains an interval of real numbers.

Example 2-1: Defining Sample Spaces

- Randomly select a camera and record the recycle time of a flash. $S = R^+ = \{x \mid x > 0\}$, the positive real numbers.
- Suppose it is known that all recycle times are between 1.5 and 5 seconds. Then

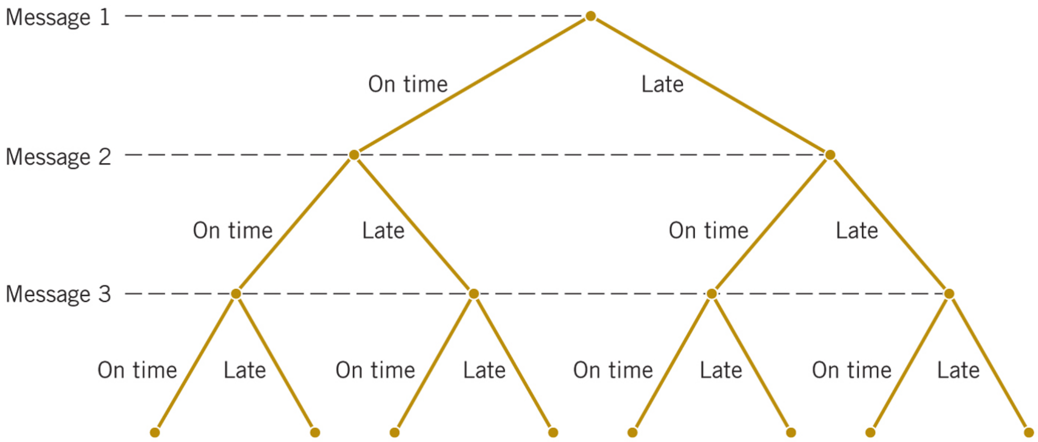
  $S = \{x \mid 1.5 < x < 5\}$ is continuous.
- It is known that the recycle time has only three values(low, medium or high). Then $S = \{$*low, medium, high*$\}$ is discrete.
- Does the camera conform to minimum recycle time specifications?

  $S = \{$*yes, no*$\}$ is discrete.

Sample Space Defined By A Tree Diagram

Example 2-2: Messages are classified as on-time(o) or late(l). Classify the next 3 messages.

S = {ooo, ool, olo, oll, loo, lol, llo, lll}



Events are Sets of Outcomes

- An event (E) is a subset of the sample space of a random experiment.
- Event combinations
  - The Union of two events consists of all outcomes that are contained in one event or the other, denoted as $E_1 \cup E_2$.
  - The Intersection of two events consists of all outcomes that are contained in one event and the other, denoted as $E_1 \wedge E_2$.
  - The Complement of an event is the set of outcomes in the sample space that are not contained in the event, denoted as E'.

Example 2-3 Discrete Events

Suppose that the recycle times of two cameras are recorded. Consider only whether or not the cameras conform to the manufacturing specifications. We abbreviate *yes* and *no* as *y* and *n*. The sample space is $S = \{yy, yn, ny, nn\}$.

Suppose, E_1 denotes an event that at least one camera conforms to specifications, then $E_1 = \{yy, yn, ny\}$

Suppose, E_2 denotes an event that no camera conforms to specifications, then $E_2 = \{nn\}$

Suppose, E_3 denotes an event that at least one camera does not conform.

then $E_3 = \{yn, ny, nn\}$,

- Then E_1 **U** $E_3 = S$
- Then E_1 **^** $E_3 = \{yn, ny\}$
- Then $E_1' = \{nn\}$

Example 2-4 Continuous Events

Measurements of the thickness of a part are modeled with the sample space: $S = R^+$.

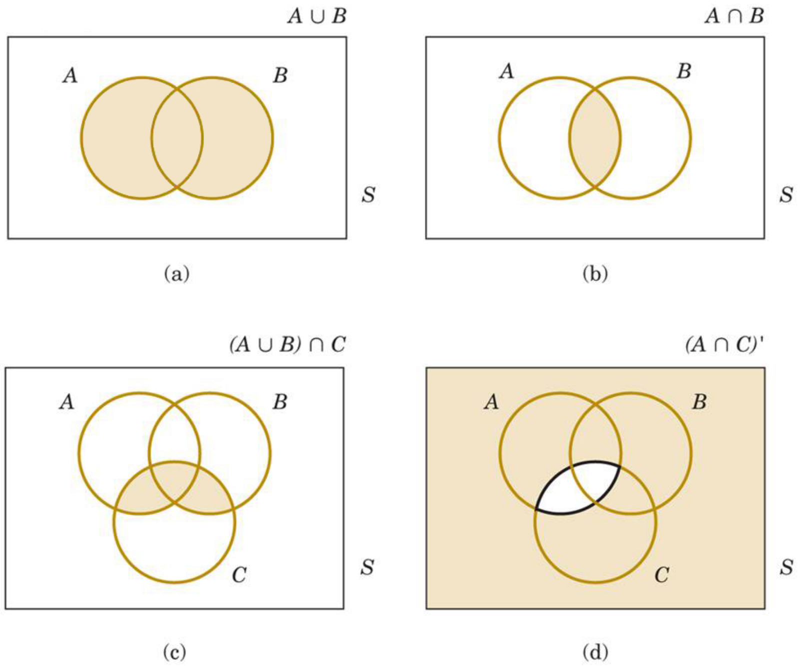
Let $E_1 = \{x \mid 10 \le x < 12\}$,

Let $E_2 = \{x \mid 11 < x < 15\}$

- Then E_1 **U** $E_2 = \{x \mid 10 \le x < 15\}$
- Then E_1 **^** $E_2 = \{x \mid 11 < x < 12\}$
- Then $E_1' = \{x \mid 0 < x < 10 \text{ or } x \ge 12\}$
- Then E_1' **^** $E_2 = \{x \mid 12 \le x < 15\}$

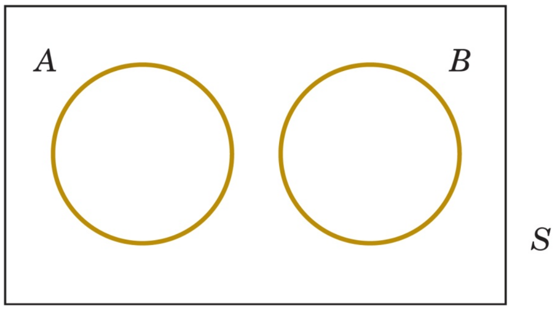
Venn Diagrams

Events *A* & *B* contain their respective outcomes. The shaded regions indicate the event relation of each diagram.



Mutually Exclusive Events

- Events *A* and *B* are mutually exclusive because they share no common outcomes.
- The occurrence of one event precludes the occurrence of the other.
- Symbolically, $A \wedge B = \varnothing$



Mutually Exclusive Events - Laws

- Commutative law (event order is unimportant):
  - $A \cup B = B \cup A$ and $A \cap B = B \cap A$
- Distributive law (like in algebra):
  - $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
  - $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- Associative law (like in algebra):
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$

Mutually Exclusive Events - Laws

- DeMorgan's law:
  - $(A \cup B)' = A' \cap B'$ The complement of the union is the intersection of the complements.
  - $(A \cap B)' = A' \cup B'$ The complement of the intersection is the union of the complements.
- Complement law:

$$(A')' = A.$$

Counting Techniques

- There are three special rules, or counting techniques, used to determine the number of outcomes in events.
- They are :
  1. Multiplication rule
  2. Permutation rule
  3. Combination rule
- Each has its special purpose that must be applied properly – the right tool for the right job.

Counting – Multiplication Rule

- Multiplication rule:
  – Let an operation consist of *k* steps and there are
    - n_1 ways of completing step 1,
    - n_2 ways of completing step 2, … and
    - n_k ways of completing step *k*.

  – Then, the total number of ways to perform *k* steps is:
    - $n_1 \cdot n_2 \cdot \ldots \cdot n_k$

Example 2-5 - Web Site Design

- In the design for a website, we can choose to use among:
  - 4 colors,
  - 3 fonts, and
  - 3 positions for an image.

  How many designs are possible?
- Answer via the multiplication rule: 4x3x3 = 36

Counting – Permutation Rule

- A permutation is a unique sequence of distinct items.
- If $S = \{a, b, c\}$, then there are 6 permutations
  - Namely: *abc*, *acb*, *bac*, *bca*, *cab*, *cba* (order matters)
- Number of permutations for a set of n items is $n!$
- $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$
- $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$ = FACT(7) in Excel
- By definition: $0! = 1$

Counting–Subset Permutations and an example

- For a sequence of *r* items from a set of *n* items:

$$P_r^n = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

- **Example 2-6**: Printed Circuit Board
- A printed circuit board has eight different locations in which a component can be placed. If four different components are to be placed on the board, how many designs are possible?
- Answer: Order is important, so use the permutation formula with $n = 8$, $r = 4$.

$$P_4^8 = \frac{8!}{(8-4)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 8 \cdot 7 \cdot 6 \cdot 5 = 1{,}680$$

Counting - Similar Item Permutations

- Used for counting the sequences when some items are identical.
- The number of permutations of:

  $n = n_1 + n_2 + \ldots + n_r$ items of which

  $n_1, n_2, \ldots, n_r$ are identical.

  is calculated as:

$$\frac{n!}{n_1!\,n_2!\,\ldots\,n_r!}$$

Example 2-7: Hospital Schedule

- In a hospital, a operating room needs to schedule three knee surgeries and two hip surgeries in a day. The knee surgery is denoted as *k* and the hip as *h*.
  - How many sequences are there?

    Since there are 2 identical hip surgeries and 3 identical knee surgeries, then

    $$\frac{5!}{2! \cdot 3!} = \frac{5 \cdot 4 \cdot 3!}{2 \cdot 1 \cdot 3!} = 10$$

  - What is the set of sequences?

    {*kkkhh*, *kkhkh*, *kkhhk*, *khkkh*, *khkhk*, *khhkk*, *hkkkh*, *hkkhk*, *hkhkk*, *hhkkk*}

Counting – Combination Rule

- A combination is a selection of *r* items from a set of *n* where order does not matter.
- If $S = \{a, b, c\}$, $n = 3$, then
  - If $r = 3$, there is 1 combination, namely: *abc*
  - If $r = 2$, there are 3 combinations, namely *ab*, *ac*, and *bc*
- # of permutations ≥ # of combinations
- Since order does not matter with combinations, we are dividing the # of permutations by *r*!, where *r*! is the # of arrangements of *r* elements.

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 2-8: Sampling w/o Replacement-1

- A bin of 50 parts contains 3 defectives and 47 non-defective parts. A sample of 6 parts is selected from the 50 without replacement. How many samples of size 6 contain 2 defective parts?
- First, how many ways are there for selecting 2 parts from the 3 defective parts?

$$C_2^3 = \frac{3!}{2! \cdot 1!} = 3 \text{ different ways}$$

- In Excel: | 3 | = COMBIN(3,2) |

Example 2-8: Sampling w/o Replacement-2

- Now, how many ways are there for selecting 4 parts from the 47 non-defective parts?

$$C_4^{47} = \frac{47!}{4! \cdot 43!} = \frac{47 \cdot 46 \cdot 45 \cdot 44 \cdot 43!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 43!} = 178{,}365 \text{ different ways}$$

- In Excel: | 178,365 | = COMBIN(47,4) |

Example 2-8: Sampling w/o Replacement-3

- Now, how many ways are there to obtain:
  - 2 from 3 defectives, and
  - 4 from 47 non-defectives?

$$C_2^3 C_4^{47} = 3 \cdot 178{,}365 = 535{,}095 \text{ different ways}$$

- In Excel:

535,095	=COMBIN(3,2)*COMBIN(47,4)

Probability

- Probability is the likelihood or chance that a particular outcome or event from a random experiment will occur.
- In this chapter, we consider only discrete (finite or countably infinite) sample spaces.
- Probability is a number in the [0,1] interval.
- A probability of:
  - 1 means certainty
  - 0 means impossibility

Types of Probability

- Subjective probability is a "degree of belief."

  Example: "There is a 50% chance that I'll study tonight."

- Relative frequency probability is based on how often an event occurs over a very large sample space.

  Example:

$$\lim_{n \to \infty} \frac{n(A)}{n}$$

Probability Based on Equally-Likely Outcomes

- Whenever a sample space consists of *N* possible outcomes that are equally likely, the probability of each outcome is 1/*N*.
- Example: In a batch of 100 diodes, 1 is laser diode. A diode is randomly selected from the batch. Random means each diode has an equal chance of being selected. The probability of choosing the laser diode is 1/100 or 0.01, because each outcome in the sample space is equally likely.

Probability of an Event

- For a discrete sample space, the *probability of an event E*, denoted by $P(E)$, equals the sum of the probabilities of the outcomes in E.

- The discrete sample space may be:
  - A finite set of outcomes
  - A countably infinite set of outcomes.

Example 2-9: Probabilities of Events

- A random experiment has a sample space $\{a,b,c,d\}$. These outcomes are not equally-likely; their probabilities are: 0.1, 0.3, 0.5, 0.1.
- Let Event $A = \{a,b\}$, $B = \{b,c,d\}$, and $C = \{d\}$
  - $P(A) = 0.1 + 0.3 = 0.4$
  - $P(B) = 0.3 + 0.5 + 0.1 = 0.9$
  - $P(C) = 0.1$
  - $P(A') = 0.6$ and $P(B') = 0.1$ and $P(C') = 0.9$
  - Since event $A \wedge B = \{b\}$, then $P(A \wedge B) = 0.3$
  - Since event $A \cup B = \{a,b,c,d\}$, then $P(A \cup B) = 1.0$
  - Since event $A \wedge C = \{\text{null}\}$, then $P(A \wedge C) = 0$

Axioms of Probability

- Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

  If S is the sample space and E is any event in the random experiment,
  1. $P(S) = 1$
  2. $0 \leq P(E) \leq 1$
  3. For any two events E_1 and E_2 with $E_1 \wedge E_2 = \varnothing$,
  $$P(E_1 \mathbf{U} E_2) = P(E_1) + P(E_2)$$
- The axioms imply that:
  - $P(\varnothing) = 0$ and $P(E') = 1 - P(E)$
  - If E_1 is contained in E_2, then $P(E_1) \leq P(E_2)$.

Addition Rules

- Joint events are generated by applying basic set operations to individual events, specifically:
  - Unions of events, A **U** B
  - Intersections of events, $A \wedge B$
  - Complements of events, A'
- Probabilities of joint events can often be determined from the probabilities of the individual events that comprise them.

Example 2-10: Semiconductor Wafers

A wafer is randomly selected from a batch that is classified by contamination and location.

– Let *H* be the event of high concentrations of contaminants. Then $P(H) = 358/940$.
– Let *C* be the event of the wafer being located at the center of a sputtering tool. Then $P(C) = 626/940$.
– $P(H \cap C) = 112/940$

Contamination	Location of Tool		Total
	Center	Edge	
Low	514	68	582
High	112	246	358
Total	626	314	940

– $P(H \cup C) = P(H) + P(C) - P(H \cap C)$
$= (358 + 626 - 112)/940$

This is the addition rule.

Probability of a Union

- For any two events *A* and *B*, the probability of union is given by:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If events *A* and *B* are mutually exclusive, then

$$P(A \cap B) = \varphi,$$

and therefore:

$$P(A \cup B) = P(A) + P(B)$$

Addition Rule: 3 or More Events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B)$$
$$- P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Note the alternating signs.

If a collection of events E_i are pairwise mutually exclusive;
that is $E_i \cap E_j = \phi$, for all i, j

$$\text{Then}: P(E_1 \cup E_2 \cup \ldots \cup E_k) = \sum_{i=1}^{k} P(E_i)$$

Conditional Probability

- $P(B \mid A)$ is the probability of event B occurring, given that event A has already occurred.
- A communications channel has an error rate of 1 per 1000 bits transmitted. Errors are rare, but do tend to occur in bursts. If a bit is in error, the probability that the next bit is also in error is greater than 1/1000.

Conditional Probability Rule

- The conditional probability of an event *B* given an event *A*, denoted as *P*(*B* | *A*), is:

  P(*B* | *A*) = *P*(*A*^*B*) / *P*(*A*) for *P*(*A*) > 0.
- From a relative frequency perspective of *n* equally likely outcomes:
  - *P*(*A*) = (number of outcomes in *A*) / *n*
  - *P*(*A*^*B*) = (number of outcomes in *A*^*B*) / *n*
  - *P*(*B* | *A*) = number of outcomes in *A*^*B* / number of outcomes in *A*

Example 2-11

There are 4 probabilities conditioned on flaws in the below table.

Parts Classified			
Defective	Surface Flaws		Total
	Yes (F)	No (F')	
Yes (D)	10	18	28
No (D')	30	342	372
Total	40	360	400

$$P(F) = 40/400 \text{ and } P(D) = 28/400$$

$$P(D|F) = P(D \cap F)/P(F) = \tfrac{10}{400}\Big/\tfrac{40}{400} = \tfrac{10}{40}$$

$$P(D'|F) = P(D' \cap F)/P(F) = \tfrac{30}{400}\Big/\tfrac{40}{400} = \tfrac{30}{40}$$

$$P(D|F') = P(D \cap F')/P(F') = \tfrac{18}{400}\Big/\tfrac{360}{400} = \tfrac{18}{360}$$

$$P(D'|F') = P(D' \cap F')/P(F') = \tfrac{342}{400}\Big/\tfrac{360}{400} = \tfrac{342}{360}$$

Random Samples

- Random means each item is equally likely to be chosen. If more than one item is sampled, random means that every sampling outcome is equally likely.
  - 2 items are taken from $S = \{a,b,c\}$ without replacement.
  - Ordered sample space: $S = \{ab,ac,bc,ba,ca,cb\}$
  - Unordered sample space: $S = \{ab,ac,bc\}$

Example 2-12 : Sampling Without Enumeration

- A batch of 50 parts contains 10 made by Tool 1 and 40 made by Tool 2. If 2 parts are selected randomly*,

  a) What is the probability that the 2nd part came from Tool 2, given that the 1st part came from Tool 1?

  - $P(E_1) = P$(1st part came from Tool 1) = 10/50
  - $P(E_2 \mid E_1) = P$(2nd part came from Tool 2 given that 1st part came from Tool 1) = 40/49

  b) What is the probability that the 1st part came from Tool 1 and the 2nd part came from Tool 2?

  - $P(E_1 \cap E_2) = P$(1st part came from Tool 1 and 2nd part came from Tool 2) = (10/50)·(40/49) = 8/49

*Selected randomly implies that at each step of the sample, the items remain in the batch are equally likely to be selected.

Multiplication Rule

- The conditional probability can be rewritten to generalize a multiplication rule.

$$P(A \wedge B) = P(B \mid A) \cdot P(A) = P(A \mid B) \cdot P(B)$$

- The last expression is obtained by exchanging the roles of *A* and *B*.

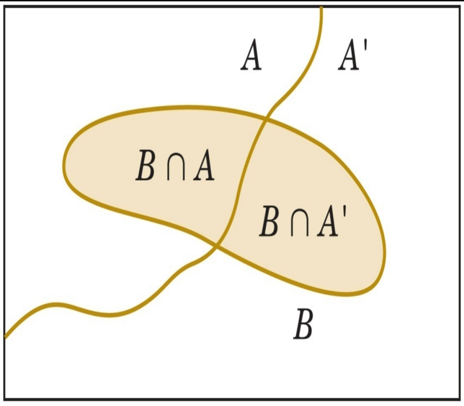
Example 2-13: Machining Stages

The probability that a part made in the 1st stage of a machining operation meets specifications is 0.90. The probability that it meets specifications in the 2nd stage, given that met specifications in the first stage is 0.95. What is the probability that both stages meet specifications?

- Let *A* and *B* denote the events that the part has met 1st and 2nd stage specifications, respectively.
- $P(A \wedge B) = P(B \mid A) \cdot P(A) = 0.95 \cdot 0.90 = 0.855$

Two Mutually Exclusive Subsets

- A and A' are mutually exclusive.
- A^B and A'^B are mutually exclusive
- $B = (A$^$B)$ **U** $(A'$^$B)$



<u>Total Probability Rule</u>

For any two events A and B

$$P(B) = P(B \cap A) + P(B \cap A')$$
$$= P(B \mid A) \bullet P(A) + P(B \mid A') \bullet P(A')$$

Example 2-14: Semiconductor Contamination

Information about product failure based on chip manufacturing process contamination is given below. Find the probability of failure.

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not High	0.8

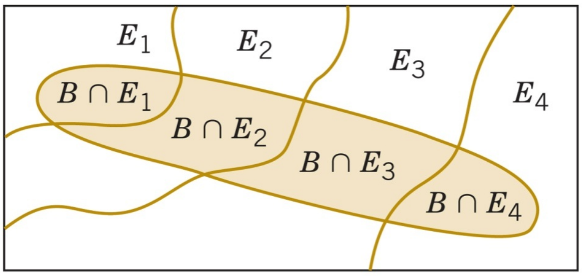
Let F denote the event that the product fails.
Let H denote the event that the chip is exposed to high contamination during manufacture. Then

- $P(F \mid H) = 0.100$ and $P(H) = 0.2$, so $P(F \quad H) = 0.02$
- $P(F \mid H') = 0.005$ and $P(H') = 0.8$, so $P(F \quad H') = 0.004$
- $P(F) = P(F \quad H) + P(F \quad H')$ (Using Total Probability rule)
  $= 0.020 + 0.004 = 0.024$

Total Probability Rule (Multiple Events)

- A collection of sets E_1, E_2, … E_k such that
  E_1 E_2 …… $E_k = S$ is said to be **exhaustive**.
- Assume E_1, E_2, … E_k are k mutually exclusive and exhaustive. Then

$$P(B) = P(B \cap E_1) + P(B \cap E_2) + \ldots + P(B \cap E_k)$$
$$= P(B \mid E_1) \cdot P(E_1) + P(B \mid E_2) \cdot P(E_2) + \ldots + P(B \mid E_k) \cdot P(E_k)$$

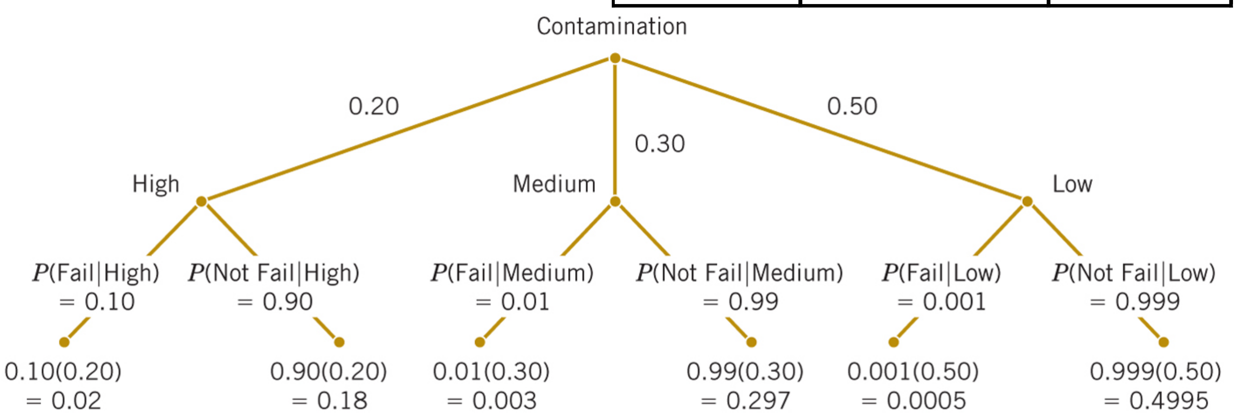


$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

Example 2-15: Semiconductor Failures-1

Continuing the discussion of contamination during chip manufacture, find the probability of failure.

Probability of Failure	Level of Contamination	Probability of Level
0.100	High	0.2
0.010	Medium	0.3
0.001	Low	0.5



$P(\text{Fail}) = 0.02 + 0.003 + 0.0005 = 0.0235$

Example 2-15: Semiconductor Failures-2

- Let F denote the event that a chip fails
- Let H denote the event that a chip is exposed to high levels of contamination
- Let M denote the event that a chip is exposed to medium levels of contamination
- Let L denote the event that a chip is exposed to low levels of contamination.
- Using Total Probability Rule,

$$\begin{aligned} P(F) &= P(F|H)P(H) + P(F|M)P(M) + P(F|L)P(L) \\ &= (0.1)(0.2) + (0.01)(0.3) + (0.001)(0.5) \\ &= 0.0235 \end{aligned}$$

Event Independence

- Two events are independent if any one of the following equivalent statements is true:
  1. $P(A \mid B) = P(A)$
  2. $P(B \mid A) = P(B)$
  3. $P(A \wedge B) = P(A) \cdot P(B)$
- This means that occurrence of one event has no impact on the probability of occurrence of the other event.

Example 2-16: Flaws and Functions

Table 1 provides an example of 400 parts classified by surface flaws and as (functionally) defective. Suppose that the situation is different and follows Table 2. Let *F* denote the event that the part has surface flaws. Let *D* denote the event that the part is defective.

The data shows whether the events are independent.

TABLE 1 Parts Classified				TABLE 2 Parts Classified (data chg'd)			
	Surface Flaws				Surface Flaws		
Defective	Yes (F)	No (F')	Total	Defective	Yes (F)	No (F')	Total
Yes (D)	10	18	28	Yes (D)	2	18	20
No (D')	30	342	372	No (D')	38	342	380
Total	40	360	400	Total	40	360	400
	$P(D \mid F) =$	10/40 =	0.25		$P(D \mid F) =$	2/40 =	0.05
	$P(D) =$	28/400 =	0.10		$P(D) =$	20/400 =	0.05
			not same				same
	Events D & F are **dependent**				Events D & F are **independent**		

Independence with Multiple Events

The events $E_1, E_2, \ldots, E_k$ are independent if and only if, for any subset of these events:

$$P(E_{i1} \quad E_{i2} \quad \ldots, \quad E_{ik}) = P(E_{i1}) \cdot P(E_{i2}) \cdot \ldots \cdot P(E_{ik})$$

Example 2-17: Semiconductor Wafers

Assume the probability that a wafer contains a large particle of contamination is 0.01 and that the wafers are independent; that is, the probability that a wafer contains a large particle does not depend on the characteristics of any of the other wafers. If 15 wafers are analyzed, what is the probability that no large particles are found?

Solution:

Let E_i denote the event that the i^{th} wafer contains no large particles, i = 1, 2, …,15.

Then , $P(E_i) = 0.99$.
The required probability is $P(E_1 \quad E_2 \quad \ldots \quad E_{15})$.
From the assumption of independence,

$$P(E_1 \quad E_2 \quad \ldots \quad E_{15}) = P(E_1) \cdot P(E_2) \cdot \ldots \cdot P(E_{15})$$
$$= (0.99)^{15}$$
$$= 0.86.$$

Bayes' Theorem

- Thomas Bayes (1702-1761) was an English mathematician and Presbyterian minister.
- His idea was that we observe conditional probabilities through prior information.
- Bayes' theorem states that,

$$P(A \mid B) = \frac{P(B \mid A) \bullet P(A)}{P(B)} \quad \text{for } P(B) > 0$$

Example 2-18

The conditional probability that a high level of contamination was present when a failure occurred is to be determined. The information from Example 2-14 is summarized here.

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not High	0.8

Solution:

Let *F* denote the event that the product fails, and let *H* denote the event that the chip is exposed to high levels of contamination. The requested probability is *P*(*F*).

$$P(H \mid F) = \frac{P(F \mid H) \cdot P(H)}{P(F)} = \frac{0.10 \cdot 0.20}{0.024} = 0.83$$

$$P(F) = P(F \mid H) \cdot P(H) + P(F \mid H') \cdot P(H')$$
$$= 0.1 \cdot 0.2 + 0.005 \cdot 0.8 = 0.024$$

Bayes Theorem with Total Probability

If E_1, E_2, … E_k are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1 \mid B) = \frac{P(B \mid E_1)P(E_1)}{P(B \mid E_1)P(E_1) + P(B \mid E_2)P(E_2) + \ldots + P(B \mid E_k)P(E_k)}$$

where $P(B) > 0$

Note : Numerator expression is always one of the terms in the sum of the denominator.

Example 2-19: Bayesian Network

A printer manufacturer obtained the following three types of printer failure probabilities. Hardware $P(H) = 0.3$, software $P(S) = 0.6$, and other $P(O) = 0.1$. Also, $P(F \mid H) = 0.9$, $P(F \mid S) = 0.2$, and $P(F \mid O) = 0.5$.

If a failure occurs, determine if it's most likely due to hardware, software, or other.

$$P(F) = P(F \mid H)P(H) + P(F \mid S)P(S) + P(F \mid O)P(O)$$
$$= 0.9(0.1) + 0.2(0.6) + 0.5(0.3) = 0.36$$
$$P(H \mid F) = \frac{P(F \mid H) \cdot P(H)}{P(F)} = \frac{0.9 \cdot 0.1}{0.36} = 0.250$$
$$P(S \mid F) = \frac{P(F \mid S) \cdot P(S)}{P(F)} = \frac{0.2 \cdot 0.6}{0.36} = 0.333$$
$$P(O \mid F) = \frac{P(F \mid O) \cdot P(O)}{P(F)} = \frac{0.5 \cdot 0.3}{0.36} = 0.417$$

Note that the conditionals given failure add to 1. Because $P(O \mid F)$ is largest, the most likely cause of the problem is in the *other* category.

Random Variable and its Notation

- A variable that associates a number with the outcome of a random experiment is called a random variable.
- A random variable is a function that assigns a real number to each outcome in the sample space of a random experiment.
- A random variable is denoted by an uppercase letter such as X. After the experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as $x = 70$ milliamperes. X and x are shown in italics, e.g., $P(X = x)$.

Discrete & Continuous Random Variables

- A discrete random variable is a random variable with a finite or countably infinite range. Its values are obtained by counting.
- A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range. Its values are obtained by measuring.

Examples of Discrete & Continuous Random Variables

- Discrete random variables:
  - Number of scratches on a surface.
  - Proportion of defective parts among 100 tested.
  - Number of transmitted bits received in error.
  - Number of common stock shares traded per day.
- Continuous random variables:
  - Electrical current and voltage.
  - Physical measurements, e.g., length, weight, time, temperature, pressure.

3 Discrete Random Variables and Probability Distributions

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter you should be able to do the following:

1. Determine probabilities from probability mass functions and the reverse
2. Determine probabilities from cumulative distribution functions and cumulative distribution functions from probability mass functions, and the reverse
3. Calculate means and variances for discrete random variables
4. Understand the assumptions for each of the discrete probability distributions presented
5. Select an appropriate discrete probability distribution to calculate probabilities in specific applications
6. Calculate probabilities, determine means and variances for each of the discrete probability distributions presented

3-1 Discrete Random Variables

Many physical systems can be modeled by the same or similar random experiments and random variables. The distribution of the random variables involved in each of these common systems can be analyzed, and the results of that analysis can be used in different applications and examples. In this chapter, we present the analysis of several random experiments and **discrete random variables** that frequently arise in applications. We often omit a discussion of the underlying sample space of the random experiment and directly describe the distribution of a particular random variable.

3-1 Discrete Random Variables

Example 3-1

A voice communication system for a business contains 48 external lines. At a particular time, the system is observed, and some of the lines are being used. Let the random variable X denote the number of lines in use. Then, X can assume any of the integer values 0 through 48. When the system is observed, if 10 lines are in use, $x = 10$.

3-2 Probability Distributions and Probability Mass Functions

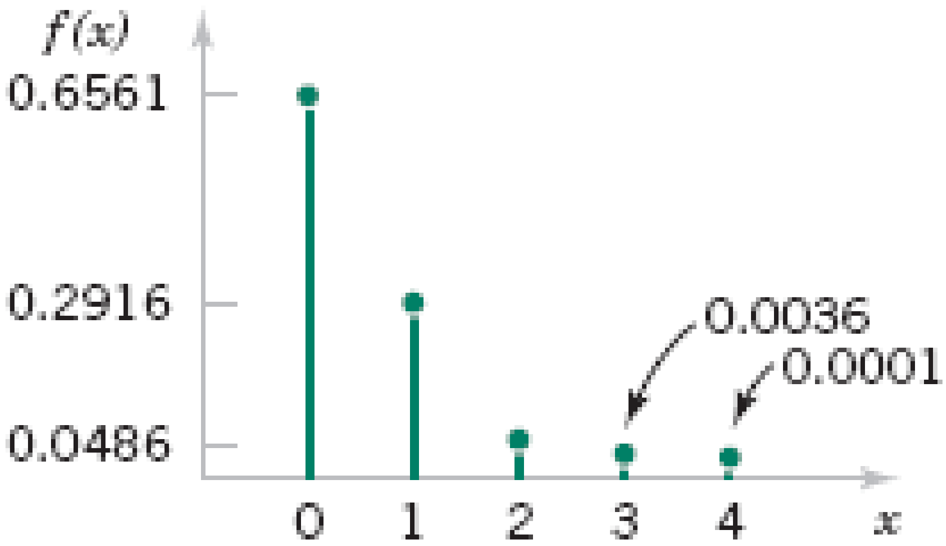


Figure 3-1 Probability distribution for bits in error.

3-2 Probability Distributions and Probability Mass Functions

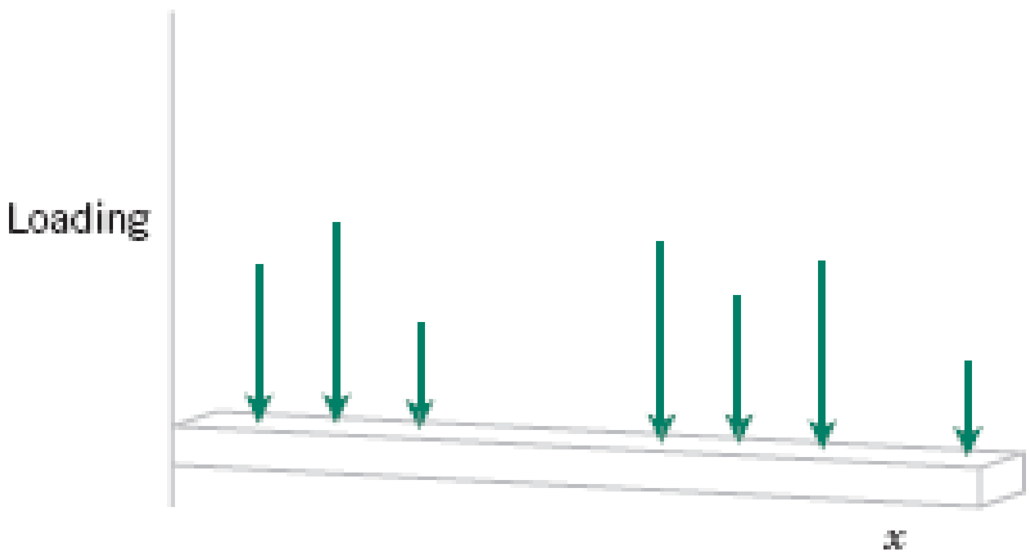


Figure 3-2 Loadings at discrete points on a long, thin beam.

3-2 Probability Distributions and Probability Mass Functions

Definition

For a discrete random variable X with possible values $x_1, x_2, \ldots, x_n$, a **probability mass function** is a function such that

(1) $f(x_i) \geq 0$

(2) $\sum_{i=1}^{n} f(x_i) = 1$

(3) $f(x_i) = P(X = x_i)$ (3-1)

Example 3-5

Let the random variable X denote the number of semiconductor wafers that need to be analyzed in order to detect a large particle of contamination. Assume that the probability that a wafer contains a large particle is 0.01 and that the wafers are independent. Determine the probability distribution of X.

Let p denote a wafer in which a large particle is present, and let a denote a wafer in which it is absent. The sample space of the experiment is infinite, and it can be represented as all possible sequences that start with a string of a's and end with p. That is,

$$s = \{p, ap, aap, aaap, aaaap, aaaaap, \text{and so forth}\}$$

Consider a few special cases. We have $P(X = 1) = P(p) = 0.01$. Also, using the independence assumption

$$P(X = 2) = P(ap) = 0.99(0.01) = 0.0099$$

Example 3-5 (continued)

A general formula is

$$P(X = x) = \underbrace{P(aa \ldots ap)}_{(x-1)a\text{'s}} = 0.99^{x-1}(0.01), \qquad \text{for } x = 1, 2, 3, \ldots$$

Describing the probabilities associated with X in terms of this formula is the simplest method of describing the distribution of X in this example. Clearly $f(x) \geq 0$. The fact that the sum of the probabilities is 1 is left as an exercise. This is an example of a geometric random variable, and details are provided later in this chapter.

3-3 Cumulative Distribution Functions

Definition

The **cumulative distribution function** of a discrete random variable X, denoted as $F(x)$, is

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$$

For a discrete random variable X, $F(x)$ satisfies the following properties.

(1) $F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i)$

(2) $0 \leq F(x) \leq 1$

(3) If $x \leq y$, then $F(x) \leq F(y)$ (3-2)

Example 3-8

Suppose that a day's production of 850 manufactured parts contains 50 parts that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch. Let the random variable X equal the number of nonconforming parts in the sample. What is the cumulative distribution function of X?

The question can be answered by first finding the probability mass function of X.

$$P(X = 0) = \frac{800}{850} \cdot \frac{799}{849} = 0.886$$

$$P(X = 1) = 2 \cdot \frac{800}{850} \cdot \frac{50}{849} = 0.111$$

$$P(X = 2) = \frac{50}{850} \cdot \frac{49}{849} = 0.003$$

Therefore,

$$F(0) = P(X \le 0) = 0.886$$

$$F(1) = P(X \le 1) = 0.886 + 0.111 = 0.997$$

$$F(2) = P(X \le 2) = 1$$

The cumulative distribution function for this example is graphed in Fig. 3-4. Note that $F(x)$ is defined for all x from $-\infty < x < \infty$ and not only for 0, 1, and 2.

Example 3-8

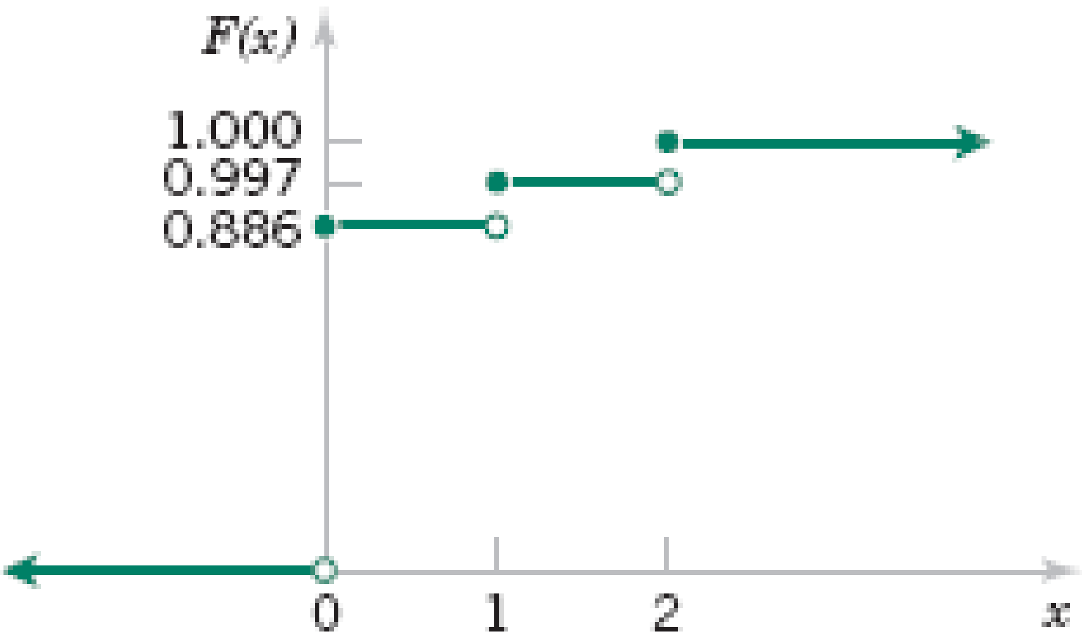


Figure 3-4 Cumulative distribution function for Example 3-8.

3-4 Mean and Variance of a Discrete Random Variable

Definition

The **mean** or **expected value** of the discrete random variable X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \sum_x x f(x) \qquad (3\text{-}3)$$

The **variance** of X, denoted as σ^2 or $V(X)$, is

$$\sigma^2 = V(X) = E(X-\mu)^2 = \sum_x (x-\mu)^2 f(x) = \sum_x x^2 f(x) - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

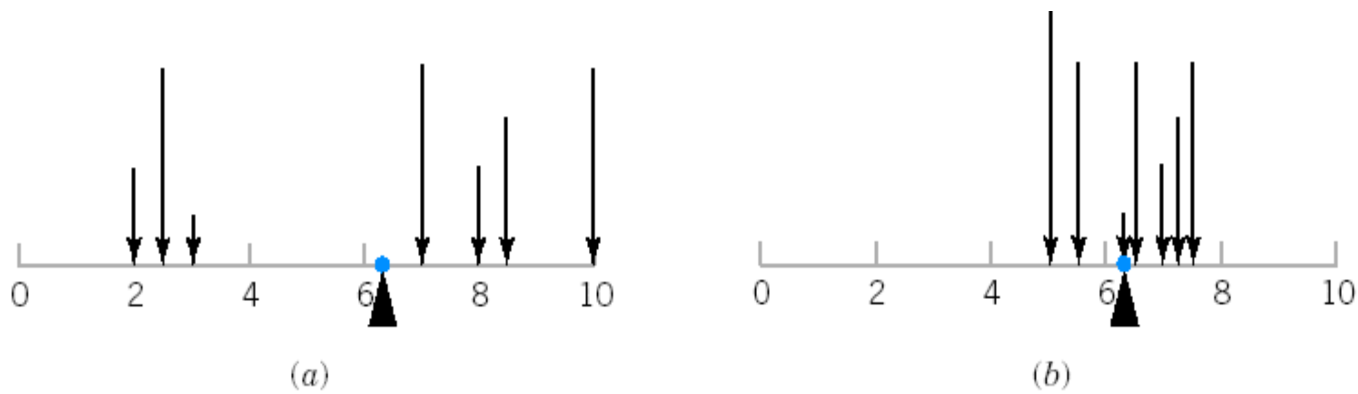


Figure 3-5 A probability distribution can be viewed as a loading with the mean equal to the balance point. Parts (a) and (b) illustrate equal means, but Part (a) illustrates a larger variance.

3-4 Mean and Variance of a Discrete Random Variable

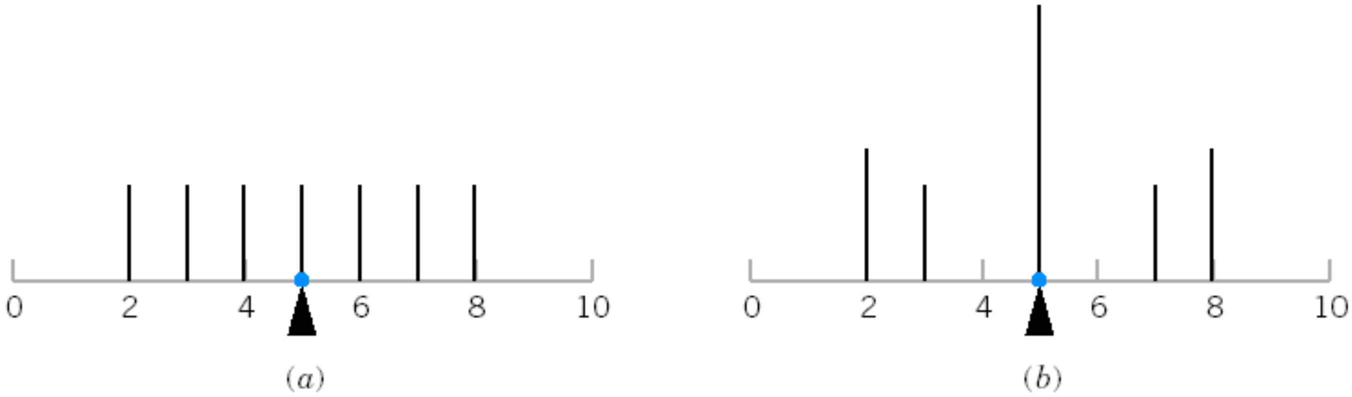


Figure 3-6 The probability distribution illustrated in Parts (a) and (b) differ even though they have equal means and equal variances.

Example 3-11

The number of messages sent per hour over a computer network has the following distribution:

x = number of messages	10	11	12	13	14	15
$f(x)$	0.08	0.15	0.30	0.20	0.20	0.07

Determine the mean and standard deviation of the number of messages sent per hour.

$$E(X) = 10(0.08) + 11(0.15) + \cdots + 15(0.07) = 12.5$$

$$V(X) = 10^2(0.08) + 11^2(0.15) + \cdots + 15^2(0.07) - 12.5^2 = 1.85$$

$$\sigma = \sqrt{V(X)} = \sqrt{1.85} = 1.36$$

3-4 Mean and Variance of a Discrete Random Variable

Expected Value of a Function of a Discrete Random Variable

If X is a discrete random variable with probability mass function $f(x)$,

$$E[h(X)] = \sum_x h(x)f(x) \qquad (3\text{-}4)$$

3-5 Discrete Uniform Distribution

Definition

A random variable X has a **discrete uniform distribution** if each of the n values in its range, say, $x_1, x_2, \ldots, x_n$, has equal probability. Then,

$$f(x_i) = 1/n \qquad (3\text{-}5)$$

3-5 Discrete Uniform Distribution

Example 3-13

The first digit of a part's serial number is equally likely to be any one of the digits 0 through 9. If one part is selected from a large batch and X is the first digit of the serial number, X has a discrete uniform distribution with probability 0.1 for each value in $R = \{0, 1, 2, \ldots, 9\}$. That is,

$$f(x) = 0.1$$

for each value in R. The probability mass function of X is shown in Fig. 3-7.

3-5 Discrete Uniform Distribution

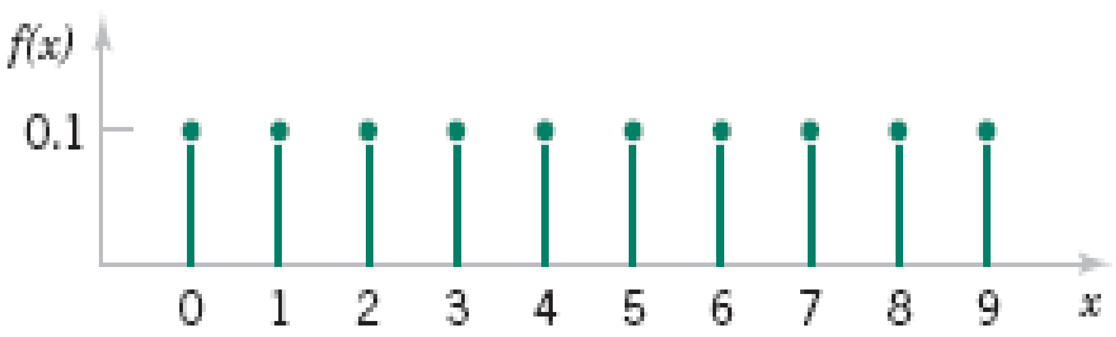


Figure 3-7 Probability mass function for a discrete uniform random variable.

3-5 Discrete Uniform Distribution

Mean and Variance

Suppose X is a discrete uniform random variable on the consecutive integers $a, a+1, a+2, \ldots, b$, for $a \leq b$. The mean of X is

$$\mu = E(X) = \frac{b+a}{2}$$

The variance of X is

$$\sigma^2 = \frac{(b-a+1)^2 - 1}{12} \qquad (3\text{-}6)$$

3-6 Binomial Distribution

Random experiments and random variables

1. Flip a coin 10 times. Let X = number of heads obtained.
2. A worn machine tool produces 1% defective parts. Let X = number of defective parts in the next 25 parts produced.
3. Each sample of air has a 10% chance of containing a particular rare molecule. Let X = the number of air samples that contain the rare molecule in the next 18 samples analyzed.
4. Of all bits transmitted through a digital transmission channel, 10% are received in error. Let X = the number of bits in error in the next five bits transmitted.

3-6 Binomial Distribution

Random experiments and random variables

5. A multiple choice test contains 10 questions, each with four choices, and you guess at each question. Let X = the number of questions answered correctly.
6. In the next 20 births at a hospital, let X = the number of female births.
7. Of all patients suffering a particular illness, 35% experience improvement from a particular medication. In the next 100 patients administered the medication, let X = the number of patients who experience improvement.

3-6 Binomial Distribution

Definition

A random experiment consists of n Bernoulli trials such that

(1) The trials are independent

(2) Each trial results in only two possible outcomes, labeled as "success" and "failure"

(3) The probability of a success in each trial, denoted as p, remains constant

The random variable X that equals the number of trials that result in a success has a **binomial random variable** with parameters $0 < p < 1$ and $n = 1, 2, \ldots$. The probability mass function of X is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots, n \tag{3-7}$$

3-6 Binomial Distribution

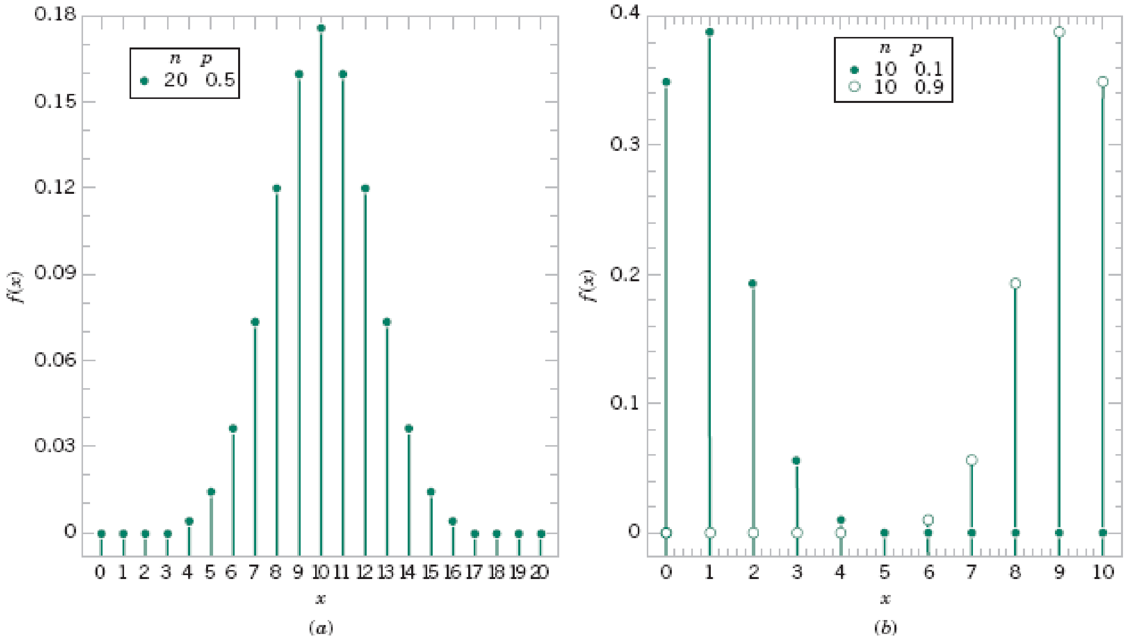


Figure 3-8 Binomial distributions for selected values of *n* and *p*.

3-6 Binomial Distribution

Example 3-18

Each sample of water has a 10% chance of containing a particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant. Find the probability that in the next 18 samples, exactly 2 contain the pollutant.

Let X = the number of samples that contain the pollutant in the next 18 samples analyzed. Then X is a binomial random variable with $p = 0.1$ and $n = 18$. Therefore,

$$P(X = 2) = \binom{18}{2}(0.1)^2(0.9)^{16}$$

Now $\binom{18}{2} = 18!/[2!\ 16!] = 18(17)/2 = 153$. Therefore,

$$P(X = 2) = 153(0.1)^2(0.9)^{16} = 0.284$$

3-6 Binomial Distribution

Example 3-18

Determine the probability that at least four samples contain the pollutant. The requested probability is

$$P(X \geq 4) = \sum_{x=4}^{18} \binom{18}{x} (0.1)^x (0.9)^{18-x}$$

However, it is easier to use the complementary event,

$$P(X \geq 4) = 1 - P(X < 4) = 1 - \sum_{x=0}^{3} \binom{18}{x} (0.1)^x (0.9)^{18-x}$$
$$= 1 - [0.150 + 0.300 + 0.284 + 0.168] = 0.098$$

Determine the probability that $3 \leq X < 7$. Now

$$P(3 \leq X < 7) = \sum_{x=3}^{6} \binom{18}{x} (0.1)^x (0.9)^{18-x}$$
$$= 0.168 + 0.070 + 0.022 + 0.005$$
$$= 0.265$$

3-6 Binomial Distribution

Mean and Variance

If X is a binomial random variable with parameters p and n,

$$\mu = E(X) = np \quad \text{and} \quad \sigma^2 = V(X) = np(1-p) \tag{3-8}$$

3-6 Binomial Distribution

Example 3-19

For the number of transmitted bits received in error in Example 3-16, $n = 4$ and $p = 0.1$, so

$$E(X) = 4(0.1) = 0.4 \qquad \text{and} \qquad V(X) = 4(0.1)(0.9) = 0.36$$

and these results match those obtained from a direct calculation in Example 3-9.

3-7 Geometric and Negative Binomial Distributions

Example 3-20

The probability that a bit transmitted through a digital transmission channel is received in error is 0.1. Assume the transmissions are independent events, and let the random variable X denote the number of bits transmitted *until* the first error.

Then, $P(X = 5)$ is the probability that the first four bits are transmitted correctly and the fifth bit is in error. This event can be denoted as $\{OOOOE\}$, where O denotes an okay bit. Because the trials are independent and the probability of a correct transmission is 0.9,

$$P(X = 5) = P(OOOOE) = 0.9^4 0.1 = 0.066$$

Note that there is some probability that X will equal any integer value. Also, if the first trial is a success, $X = 1$. Therefore, the range of X is $\{1, 2, 3, \ldots\}$, that is, all positive integers.

3-7 Geometric and Negative Binomial Distributions

Definition

In a series of Bernoulli trials (independent trials with constant probability p of a success), let the random variable X denote the number of trials until the first success. Then X is a **geometric random variable** with parameter $0 < p < 1$ and

$$f(x) = (1-p)^{x-1}p \qquad x = 1, 2, \ldots \tag{3-9}$$

3-7 Geometric and Negative Binomial Distributions

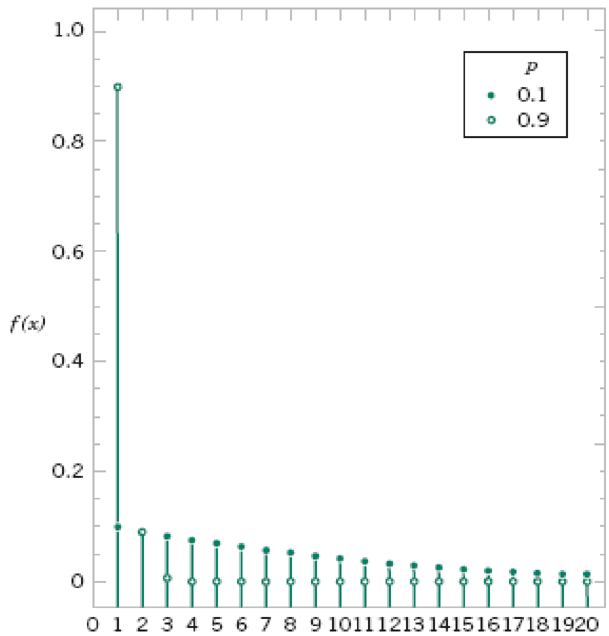


Figure 3-9. Geometric distributions for selected values of the parameter p.

3-7 Geometric and Negative Binomial Distributions

3-7.1 Geometric Distribution

Example 3-21

The probability that a wafer contains a large particle of contamination is 0.01. If it is assumed that the wafers are independent, what is the probability that exactly 125 wafers need to be analyzed before a large particle is detected?

Let X denote the number of samples analyzed until a large particle is detected. Then X is a geometric random variable with $p = 0.01$. The requested probability is

$$P(X = 125) = (0.99)^{124} 0.01 = 0.0029$$

3-7 Geometric and Negative Binomial Distributions

Definition

If X is a geometric random variable with parameter p,

$$\mu = E(X) = 1/p \qquad \text{and} \qquad \sigma^2 = V(X) = (1 - p)/p^2 \qquad \text{(3-10)}$$

3-7 Geometric and Negative Binomial Distributions

Lack of Memory Property

A geometric random variable has been defined as the number of trials until the first success. However, because the trials are independent, the count of the number of trials until the next success can be started at any trial without changing the probability distribution of the random variable. For example, in the transmission of bits, if 100 bits are transmitted, the probability that the first error, after bit 100, occurs on bit 106 is the probability that the next six outcomes are *OOOOOE*. This probability is $(0.9)^5(0.1) = 0.059$, which is identical to the probability that the initial error occurs on bit 6.

The implication of using a geometric model is that the system presumably will not wear out. The probability of an error remains constant for all transmissions. In this sense, the geometric distribution is said to lack any memory. The **lack of memory property** will be discussed again in the context of an exponential random variable in Chapter 4.

3-7 Geometric and Negative Binomial Distributions

3-7.2 Negative Binomial Distribution

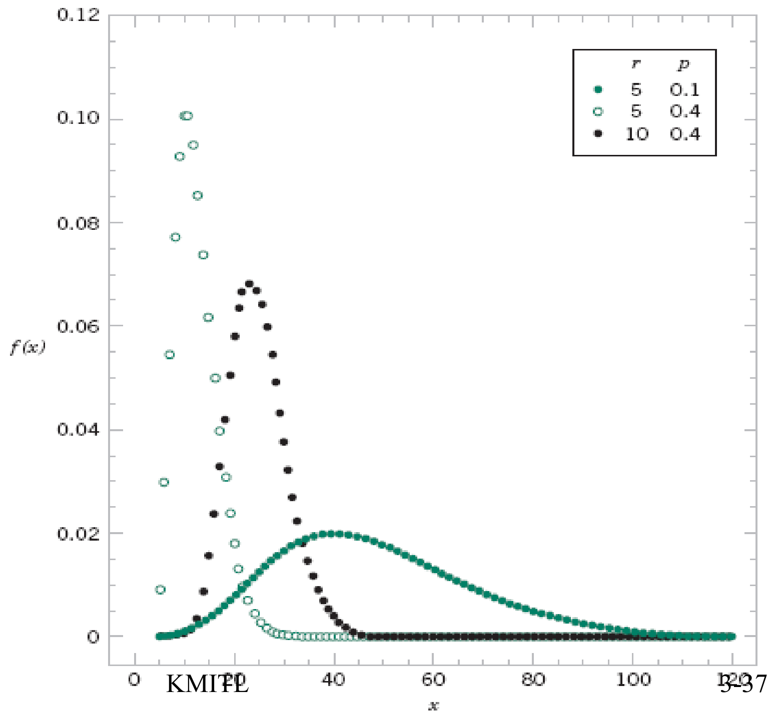
A generalization of a geometric distribution in which the random variable is the number of Bernoulli trials required to obtain r successes results in the **negative binomial distribution**.

In a series of Bernoulli trials (independent trials with constant probability p of a success), let the random variable X denote the number of trials until r successes occur. Then X is a **negative binomial random variable** with parameters $0 < p < 1$ and $r = 1, 2, 3, \ldots$, and

$$f(x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r \qquad x = r, r+1, r+2, \ldots \qquad (3\text{-}11)$$

3-7 Geometric and Negative Binomial Distributions

Figure 3-10. Negative binomial distributions for selected values of the parameters *r* and *p*.



3-7 Geometric and Negative Binomial Distributions

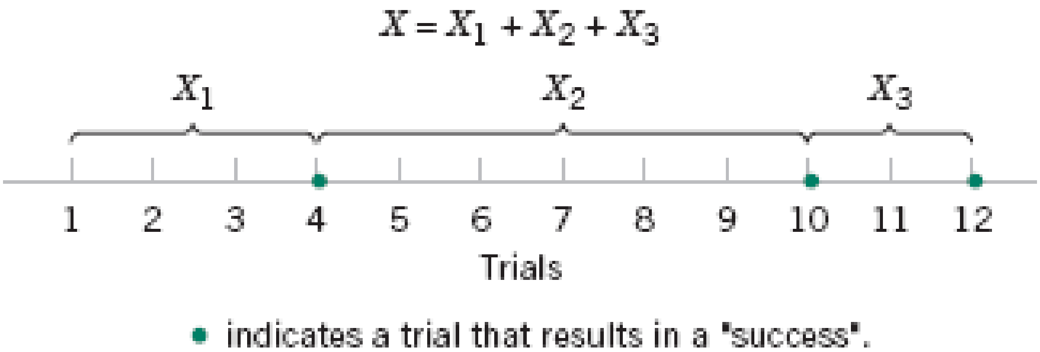


Figure 3-11. Negative binomial random variable represented as a sum of geometric random variables.

3-7 Geometric and Negative Binomial Distributions

3-7.2 Negative Binomial Distribution

If X is a negative binomial random variable with parameters p and r,

$$\mu = E(X) = r/p \qquad \text{and} \qquad \sigma^2 = V(X) = r(1-p)/p^2 \qquad (3\text{-}12)$$

3-7 Geometric and Negative Binomial Distributions

Example 3-25

A Web site contains three identical computer servers. Only one is used to operate the site, and the other two are spares that can be activated in case the primary system fails. The probability of a failure in the primary computer (or any activated spare system) from a request for service is 0.0005. Assuming that each request represents an independent trial, what is the mean number of requests until failure of all three servers?

Let X denote the number of requests until all three servers fail, and let X_1, X_2, and X_3 denote the number of requests before a failure of the first, second, and third servers used, respectively. Now, $X = X_1 + X_2 + X_3$. Also, the requests are assumed to comprise independent trials with constant probability of failure $p = 0.0005$. Furthermore, a spare server is not affected by the number of requests before it is activated. Therefore, X has a negative binomial distribution with $p = 0.0005$ and $r = 3$. Consequently,

$$E(X) = 3/0.0005 = 6000 \text{ requests}$$

3-7 Geometric and Negative Binomial Distributions

Example 3-25

What is the probability that all three servers fail within five requests? The probability is $P(X \le 5)$ and

$$\begin{aligned} P(X \le 5) &= P(X = 3) + P(X = 4) + P(X = 5) \\ &= 0.0005^3 + \binom{3}{2} 0.0005^3(0.9995) + \binom{4}{2} 0.0005^3(0.9995)^2 \\ &= 1.25 \times 10^{-10} + 3.75 \times 10^{-10} + 7.49 \times 10^{-10} \\ &= 1.249 \times 10^{-9} \end{aligned}$$

3-8 Hypergeometric Distribution

Definition

A set of N objects contains

K objects classified as successes

$N - K$ objects classified as failures

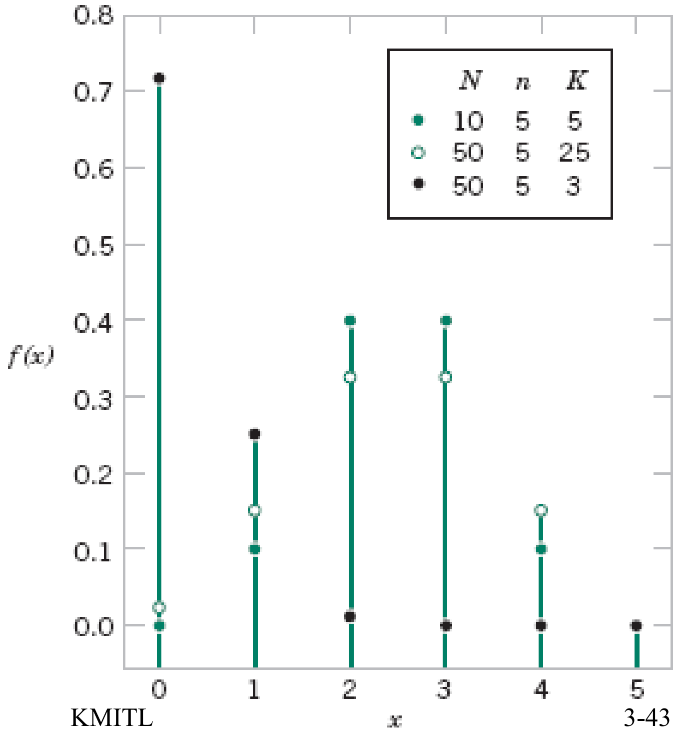
A sample of size n objects is selected randomly (without replacement) from the N objects, where $K \le N$ and $n \le N$.

Let the random variable X denote the number of successes in the sample. Then X is a **hypergeometric random variable** and

$$f(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} \qquad x = \max\{0, n + K - N\} \text{ to } \min\{K, n\} \tag{3-13}$$

3-8 Hypergeometric Distribution

Figure 3-12. Hypergeometric distributions for selected values of parameters N, K, and n.



3-8 Hypergeometric Distribution

Example 3-27

A batch of parts contains 100 parts from a local supplier of tubing and 200 parts from a supplier of tubing in the next state. If four parts are selected randomly and without replacement, what is the probability they are all from the local supplier?

Let X equal the number of parts in the sample from the local supplier. Then, X has a hypergeometric distribution and the requested probability is $P(X = 4)$. Consequently,

$$P(X = 4) = \frac{\binom{100}{4}\binom{200}{0}}{\binom{300}{4}} = 0.0119$$

3-8 Hypergeometric Distribution

Example 3-27

What is the probability that two or more parts in the sample are from the local supplier?

$$P(X \geq 2) = \frac{\binom{100}{2}\binom{200}{2}}{\binom{300}{4}} + \frac{\binom{100}{3}\binom{200}{1}}{\binom{300}{4}} + \frac{\binom{100}{4}\binom{200}{0}}{\binom{300}{4}}$$

$$= 0.298 + 0.098 + 0.0119 = 0.408$$

What is the probability that at least one part in the sample is from the local supplier?

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{\binom{100}{0}\binom{200}{4}}{\binom{300}{4}} = 0.804$$

3-8 Hypergeometric Distribution

Mean and Variance

If X is a hypergeometric random variable with parameters N, K, and n, then

$$\mu = E(X) = np \quad \text{and} \quad \sigma^2 = V(X) = np(1-p)\left(\frac{N-n}{N-1}\right) \qquad (3\text{-}14)$$

where $p = K/N$.

Here p is interpreted as the proportion of successes in the set of N objects.

3-8 Hypergeometric Distribution

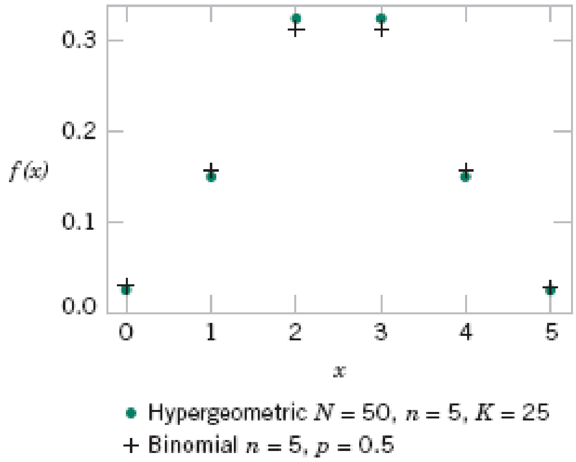
Finite Population Correction Factor

The term in the variance of a hypergeometric random variable

$$\frac{N-n}{N-1} \tag{3-15}$$

is called the finite population correction factor.

3-8 Hypergeometric Distribution



	0	1	2	3	4	5
Hypergeometric probability	0.025	0.149	0.326	0.326	0.149	0.025
Binomial probability	0.031	0.156	0.321	0.312	0.156	0.031

Figure 3-13. Comparison of hypergeometric and binomial distributions.

3-9 Poisson Distribution

Example 3-30

Consider the transmission of n bits over a digital communication channel. Let the random variable X equal the number of bits in error. When the probability that a bit is in error is constant and the transmissions are independent, X has a binomial distribution. Let p denote the probability that a bit is in error. Let $\lambda = pn$. Then, $E(x) = pn = \lambda$ and

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}$$

Now, suppose that the number of bits transmitted increases and the probability of an error decreases exactly enough that pn remains equal to a constant. That is, n increases and p decreases accordingly, such that $E(X) = \lambda$ remains constant. Then, with some work, it can be shown that

$$\binom{n}{x}\left(\frac{1}{n}\right)^x \to 1 \qquad \left(1 - \frac{\lambda}{n}\right)^{-x} \to 1 \qquad \left(1 - \frac{\lambda}{n}\right)^{n} \to e^{-\lambda}$$

so that

$$\lim_{n\to\infty} P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots$$

Also, because the number of bits transmitted tends to infinity, the number of errors can equal any nonnegative integer. Therefore, the range of X is the integers from zero to infinity.

3-9 Poisson Distribution

Definition

Given an interval of real numbers, assume events occur at random throughout the interval. If the interval can be partitioned into subintervals of small enough length such that

(1) the probability of more than one event in a subinterval is zero,

(2) the probability of one event in a subinterval is the same for all subintervals and proportional to the length of the subinterval, and

(3) the event in each subinterval is independent of other subintervals, the random experiment is called a **Poisson process.**

The random variable X that equals the number of events in the interval is a **Poisson random variable** with parameter $0 < \lambda$, and the probability mass function of X is

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, 2, \ldots \qquad (3\text{-}16)$$

3-9 Poisson Distribution

Consistent Units

It is important to **use consistent units** in the calculation of probabilities, means, and variances involving Poisson random variables. The following example illustrates unit conversions. For example, if the

average number of flaws per millimeter of wire is 3.4, then the
average number of flaws in 10 millimeters of wire is 34, and the
average number of flaws in 100 millimeters of wire is 340.

3-9 Poisson Distribution

Example 3-33

Contamination is a problem in the manufacture of optical storage disks. The number of particles of contamination that occur on an optical disk has a Poisson distribution, and the average number of particles per centimeter squared of media surface is 0.1. The area of a disk under study is 100 squared centimeters. Find the probability that 12 particles occur in the area of a disk under study.

Let X denote the number of particles in the area of a disk under study. Because the mean number of particles is 0.1 particles per cm^2

$$E(X) = 100 \text{ cm}^2 \times 0.1 \text{ particles/cm}^2 = 10 \text{ particles}$$

Therefore,

$$P(X = 12) = \frac{e^{-10}10^{12}}{12!} = 0.095$$

3-9 Poisson Distribution

Example 3-33

The probability that zero particles occur in the area of the disk under study is

$$P(X = 0) = e^{-10} = 4.54 \times 10^{-5}$$

Determine the probability that 12 or fewer particles occur in the area of the disk under study. The probability is

$$P(X \leq 12) = P(X = 0) + P(X = 1) + \cdots + P(X = 12) = \sum_{i=0}^{12} \frac{e^{-10}10^i}{i!}$$

3-9 Poisson Distribution

Mean and Variance

If X is a Poisson random variable with parameter λ, then

$$\mu = E(X) = \lambda \qquad \text{and} \qquad \sigma^2 = V(X) = \lambda \qquad (3\text{-}17)$$

Continuous Random Variables and Probability Distributions

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter you should be able to do the following:

1. Determine probabilities from probability density functions.
2. Determine probabilities from cumulative distribution functions and cumulative distribution functions from probability density functions, and the reverse.
3. Calculate means and variances for continuous random variables.
4. Understand the assumptions for each of the continuous probability distributions presented.
5. Select an appropriate continuous probability distribution to calculate probabilities in specific applications.
6. Calculate probabilities, determine means and variances for each of the continuous probability distributions presented.
7. Standardize normal random variables.
8. Use the table for the cumulative distribution function of a standard normal distribution to calculate probabilities.
9. Approximate probabilities for some binomial and Poisson distributions.

4-1 Continuous Random Variables

Previously, we discussed the measurement of the current in a thin copper wire. We noted that the results might differ slightly in day-to-day replications because of small variations in variables that are not controlled in our experiment—changes in ambient temperatures, small impurities in the chemical composition of the wire, current source drifts, and so forth.

Another example is the selection of one part from a day's production and very accurately measuring a dimensional length. In practice, there can be small variations in the actual measured lengths due to many causes, such as vibrations, temperature fluctuations, operator differences, calibrations, cutting tool wear, bearing wear, and raw material changes. Even the measurement procedure can produce variations in the final results.

4-2 Probability Distributions and Probability Density Functions

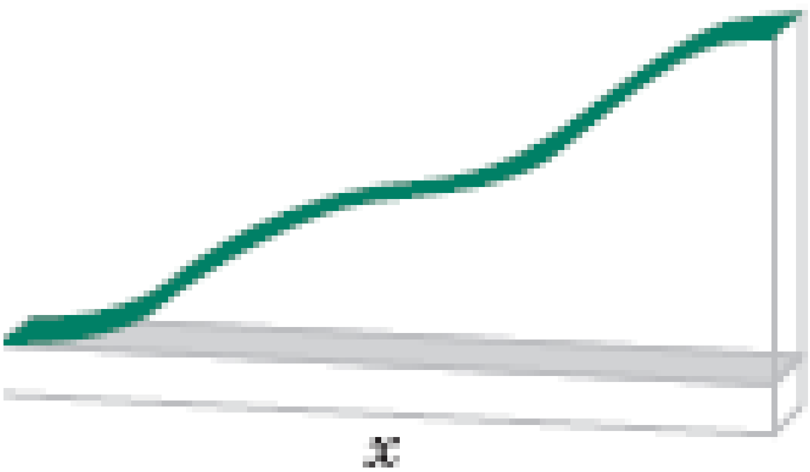


Figure 4-1 Density function of a loading on a long, thin beam.

4-2 Probability Distributions and Probability Density Functions

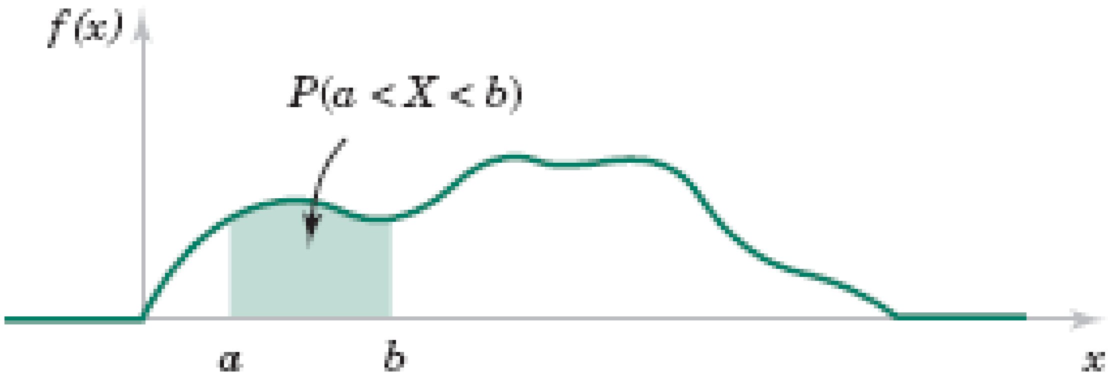


Figure 4-2 Probability determined from the area under $f(x)$.

4-2 Probability Distributions and Probability Density Functions

Definition

For a continuous random variable X, a **probability density function** is a function such that

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(3) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\,dx =$ area under $f(x)$ from a to b

for any a and b (4-1)

4-2 Probability Distributions and Probability Density Functions

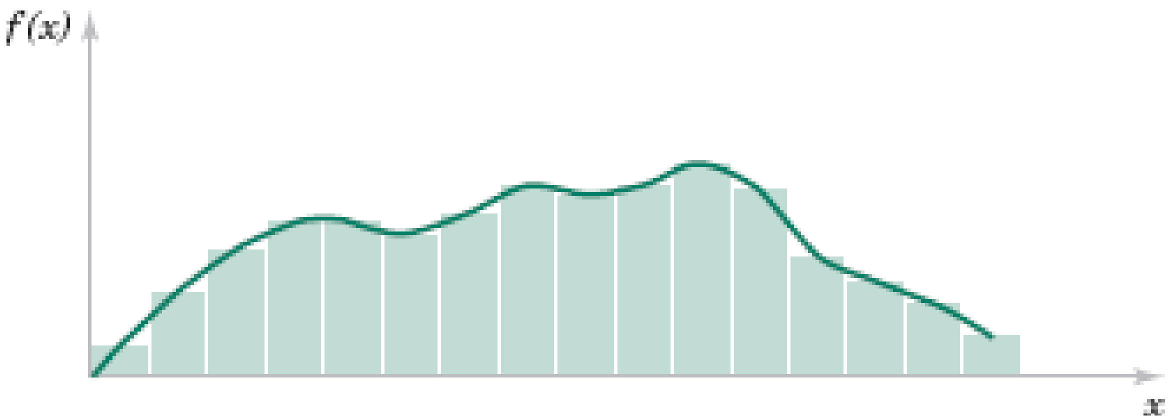


Figure 4-3 Histogram approximates a probability density function.

4-2 Probability Distributions and Probability Density Functions

If X is a **continuous random variable,** for any x_1 and x_2,

$$P(x_1 \le X \le x_2) = P(x_1 < X \le x_2) = P(x_1 \le X < x_2) = P(x_1 < X < x_2) \quad (4\text{-}2)$$

4-2 Probability Distributions and Probability Density Functions

Example 4-2

Let the continuous random variable X denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of X can be modeled by a probability density function $f(x) = 20e^{-20(x-12.5)}$, $x \geq 12.5$.

If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped? The density function and the requested probability are shown in Fig. 4-5. A part is scrapped if $X > 12.60$. Now,

$$P(X > 12.60) = \int_{12.6}^{\infty} f(x)\, dx = \int_{12.6}^{\infty} 20e^{-20(x-12.5)}\, dx = -e^{-20(x-12.5)}\Big|_{12.6}^{\infty} = 0.135$$

4-2 Probability Distributions and Probability Density Functions

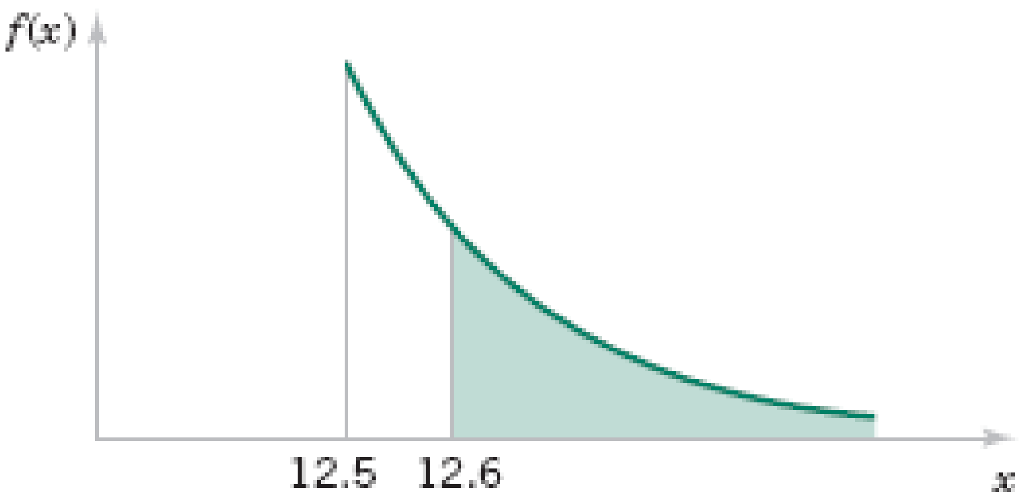


Figure 4-5 Probability density function for Example 4-2.

4-2 Probability Distributions and Probability Density Functions

Example 4-2 (continued)

What proportion of parts is between 12.5 and 12.6 millimeters? Now,

$$P(12.5 < X < 12.6) = \int_{12.5}^{12.6} f(x)\,dx = -e^{-20(x-12.5)}\Big|_{12.5}^{12.6} = 0.865$$

Because the total area under $f(x)$ equals 1, we can also calculate $P(12.5 < X < 12.6) = 1 - P(X > 12.6) = 1 - 0.135 = 0.865$.

4-3 Cumulative Distribution Functions

Definition

The **cumulative distribution function** of a continuous random variable X is

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(u)\,du \qquad (4\text{-}3)$$

for $-\infty < x < \infty$.

4-3 Cumulative Distribution Functions

Example 4-4

For the drilling operation in Example 4-2, $F(x)$ consists of two expressions.

$$F(x) = 0 \quad \text{for} \quad x < 12.5$$

and for $12.5 \le x$

$$F(x) = \int_{12.5}^{x} 20e^{-20(u-12.5)}\,du$$
$$= 1 - e^{-20(x-12.5)}$$

Therefore,

$$F(x) = \begin{cases} 0 & x < 12.5 \\ 1 - e^{-20(x-12.5)} & 12.5 \le x \end{cases}$$

Figure 4-7 displays a graph of $F(x)$.

4-3 Cumulative Distribution Functions

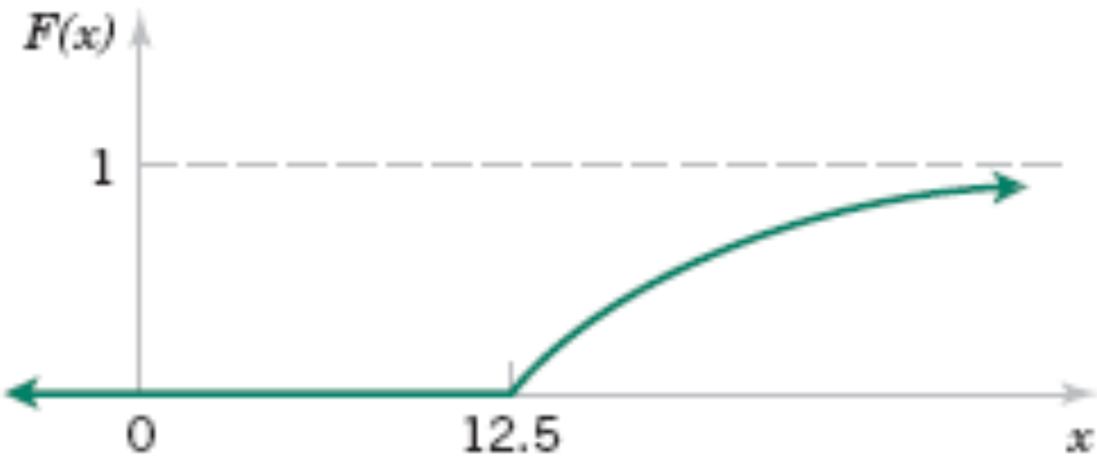


Figure 4-7 Cumulative distribution function for Example 4-4.

4-4 Mean and Variance of a Continuous Random Variable

Definition

Suppose X is a continuous random variable with probability density function $f(x)$. The **mean** or **expected value** of X, denoted as μ or $E(X)$, is

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx \qquad (4\text{-}4)$$

The **variance** of X, denoted as $V(X)$ or σ^2, is

$$\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

The **standard deviation** of X is $\sigma = \sqrt{\sigma^2}$.

4-4 Mean and Variance of a Continuous Random Variable

Example 4-6

For the copper current measurement in Example 4-1, the mean of X is

$$E(X) = \int_{0}^{20} x f(x)\, dx = 0.05x^2/2 \Big|_{0}^{20} = 10$$

The variance of X is

$$V(X) = \int_{0}^{20} (x-10)^2 f(x)\, dx = 0.05(x-10)^3/3 \Big|_{0}^{20} = 33.33$$

4-4 Mean and Variance of a Continuous Random Variable

Expected Value of a Function of a Continuous Random Variable

If X is a continuous random variable with probability density function $f(x)$,

$$E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)\,dx \tag{4-5}$$

4-4 Mean and Variance of a Continuous Random Variable

For the drilling operation in Example 4-2, the mean of X is

Example 4-8

$$E(X) = \int_{12.5}^{\infty} xf(x)\,dx = \int_{12.5}^{\infty} x\,20e^{-20(x-12.5)}\,dx$$

Integration by parts can be used to show that

$$E(X) = -xe^{-20(x-12.5)} - \frac{e^{-20(x-12.5)}}{20}\bigg|_{12.5}^{\infty} = 12.5 + 0.05 = 12.55$$

The variance of X is

$$V(X) = \int_{12.5}^{\infty} (x - 12.55)^2 f(x)\,dx$$

Although more difficult, integration by parts can be used two times to show that $V(X) = 0.0025$.

4-5 Continuous Uniform Random Variable

Definition

A continuous random variable X with probability density function

$$f(x) = 1/(b - a), \qquad a \le x \le b \tag{4-6}$$

is a **continuous uniform random variable.**

4-5 Continuous Uniform Random Variable

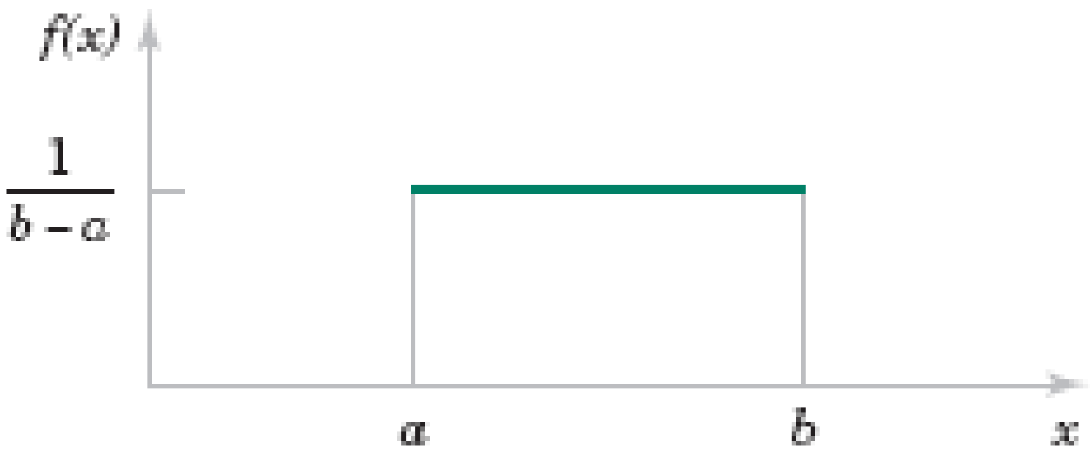


Figure 4-8 Continuous uniform probability density function.

4-5 Continuous Uniform Random Variable

Mean and Variance

If X is a continuous uniform random variable over $a \le x \le b$,

$$\mu = E(X) = \frac{(a+b)}{2} \quad \text{and} \quad \sigma^2 = V(X) = \frac{(b-a)^2}{12} \qquad (4\text{-}7)$$

4-5 Continuous Uniform Random Variable

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0, 20 mA], and assume that the probability density function of X is $f(x) = 0.05,\ 0 \le x \le 20$.

What is the probability that a measurement of current is between 5 and 10 milliamperes? The requested probability is shown as the shaded area in Fig. 4-9.

Example 4-9

$$P(5 < X < 10) = \int_{5}^{10} f(x)\,dx$$
$$= 5(0.05) = 0.25$$

The mean and variance formulas can be applied with $a = 0$ and $b = 20$. Therefore,

$$E(X) = 10 \text{ mA} \quad \text{and} \quad V(X) = 20^2/12 = 33.33 \text{ mA}^2$$

Consequently, the standard deviation of X is 5.77 mA.

4-5 Continuous Uniform Random Variable

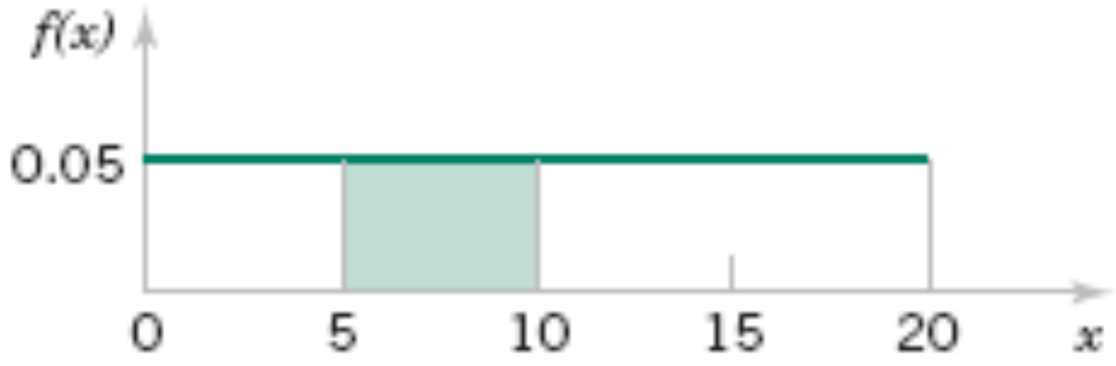


Figure 4-9 Probability for Example 4-9.

4-5 Continuous Uniform Random Variable

The cumulative distribution function of a continuous uniform random variable is obtained by integration. If $a < x < b$,

$$F(x) = \int_a^x 1/(b-a)\,du = x/(b-a) - a/(b-a)$$

Therefore, the complete description of the cumulative distribution function of a continuous uniform random variable is

$$F(x) = \begin{cases} 0 & x < a \\ (x-a)/(b-a) & a \le x < b \\ 1 & b \le x \end{cases}$$

4-6 Normal Distribution

Definition

A random variable X with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty \tag{4-8}$$

is a **normal random variable** with parameters μ, where $-\infty < \mu < \infty$, and $\sigma > 0$. Also,

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2 \tag{4-9}$$

and the notation $N(\mu, \sigma^2)$ is used to denote the distribution. The mean and variance of X are shown to equal μ and σ^2, respectively, at the end of this Section 5-6.

4-6 Normal Distribution

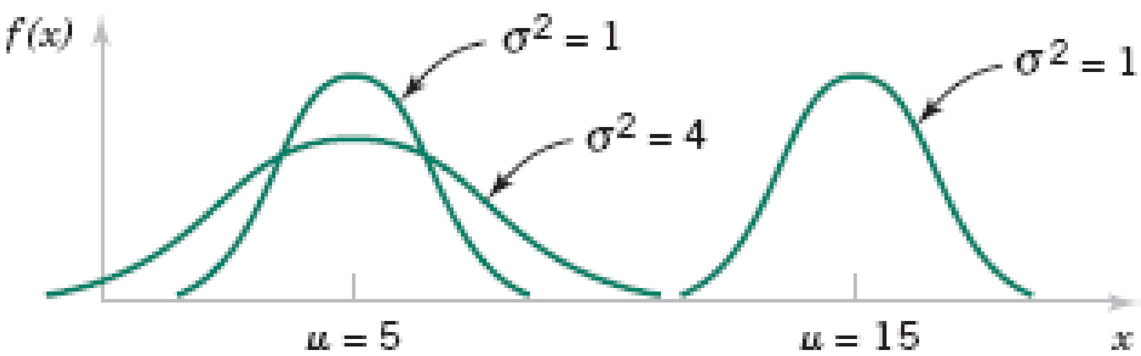


Figure 4-10 Normal probability density functions for selected values of the parameters μ and σ^2.

4-6 Normal Distribution

Some useful results concerning the normal distribution

For any normal random variable,

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6827$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

4-6 Normal Distribution

Definition : Standard Normal

A normal random variable with

$$\mu = 0 \quad \text{and} \quad \sigma^2 = 1$$

is called a **standard normal random variable** and is denoted as Z.

The cumulative distribution function of a standard normal random variable is denoted as

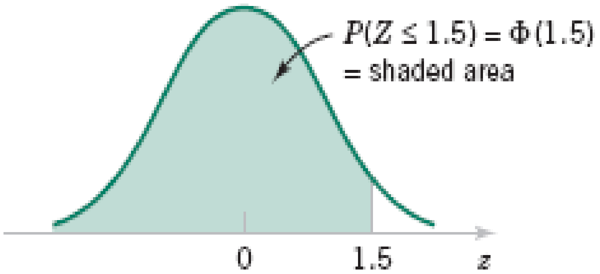
$$\Phi(z) = P(Z \leq z)$$

4-6 Normal Distribution

Example 4-11

Assume Z is a standard normal random variable. Appendix Table II provides probabilities of the form $P(Z \leq z)$. The use of Table II to find $P(Z \leq 1.5)$ is illustrated in Fig. 4-13. Read down the z column to the row that equals 1.5. The probability is read from the adjacent column, labeled 0.00, to be 0.93319.

The column headings refer to the hundredth's digit of the value of z in $P(Z \leq z)$. For example, $P(Z \leq 1.53)$ is found by reading down the z column to the row 1.5 and then selecting the probability from the column labeled 0.03 to be 0.93699.



z	0.00	0.01	0.02	0.03
0	0.50000	0.50399	0.50398	0.51197
⋮		⋮		
1.5	0.93319	0.93448	0.93574	0.93699

Figure 4-13 Standard normal probability density function.

4-6 Normal Distribution

Standardizing

If X is a normal random variable with $E(X) = \mu$ and $V(X) = \sigma^2$, the random variable

$$Z = \frac{X - \mu}{\sigma} \tag{4-10}$$

is a normal random variable with $E(Z) = 0$ and $V(Z) = 1$. That is, Z is a standard normal random variable.

4-6 Normal Distribution

Example 4-13

Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)^2. What is the probability that a measurement will exceed 13 milliamperes?

Let X denote the current in milliamperes. The requested probability can be represented as $P(X > 13)$. Let $Z = (X - 10)/2$. The relationship between the several values of X and the transformed values of Z are shown in Fig. 4-15. We note that $X > 13$ corresponds to $Z > 1.5$. Therefore, from Appendix Table II,

$$P(X > 13) = P(Z > 1.5) = 1 - P(Z \leq 1.5) = 1 - 0.93319 = 0.06681$$

Rather than using Fig. 4-15, the probability can be found from the inequality $X > 13$. That is,

$$P(X > 13) = P\left(\frac{(X - 10)}{2} > \frac{(13 - 10)}{2}\right) = P(Z > 1.5) = 0.06681$$

4-6 Normal Distribution

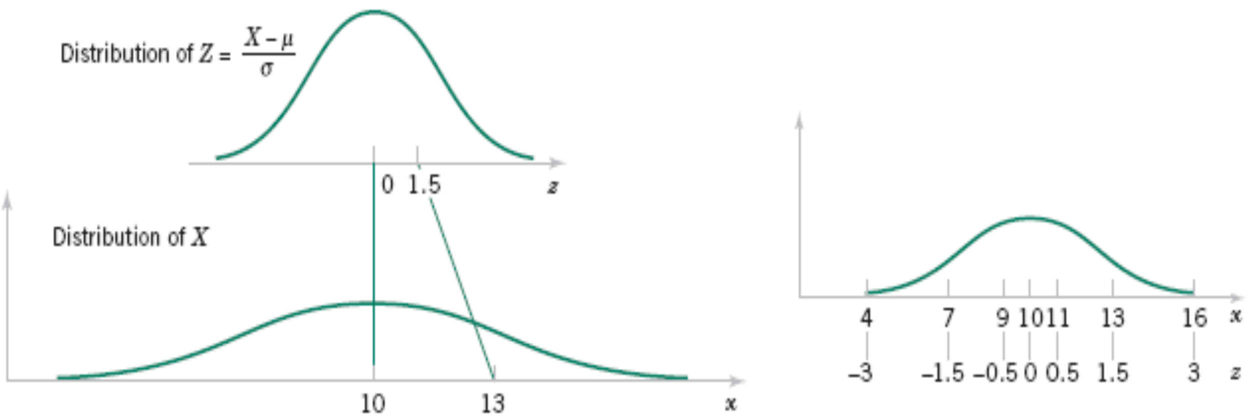


Figure 4-15 Standardizing a normal random variable.

4-6 Normal Distribution

To Calculate Probability

Suppose X is a normal random variable with mean μ and variance σ^2. Then,

$$P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = P(Z \leq z) \tag{4-11}$$

where Z is a **standard normal random variable,** and $z = \dfrac{(x-\mu)}{\sigma}$ is the **z-value** obtained by **standardizing** X.

The probability is obtained by entering Appendix Table II with $z = (x - \mu)/\sigma$.

4-6 Normal Distribution

Example 4-14

Continuing the previous example, what is the probability that a current measurement is between 9 and 11 milliamperes? From Fig. 4-15, or by proceeding algebraically, we have

$$\begin{aligned} P(9 < X < 11) &= P((9-10)/2 < (X-10)/2 < (11-10)/2) \\ &= P(-0.5 < Z < 0.5) = P(Z < 0.5) - P(Z < -0.5) \\ &= 0.69146 - 0.30854 = 0.38292 \end{aligned}$$

4-6 Normal Distribution

Example 4-14 (continued)

Determine the value for which the probability that a current measurement is below this value is 0.98. The requested value is shown graphically in Fig. 4-16. We need the value of x such that $P(X < x) = 0.98$. By standardizing, this probability expression can be written as

$$\begin{aligned} P(X < x) &= P((X - 10)/2 < (x - 10)/2) \\ &= P(Z < (x - 10)/2) \\ &= 0.98 \end{aligned}$$

Appendix Table II is used to find the z-value such that $P(Z < z) = 0.98$. The nearest probability from Table II results in

$$P(Z < 2.05) = 0.97982$$

Therefore, $(x - 10)/2 = 2.05$, and the standardizing transformation is used in reverse to solve for x. The result is

$$x = 2(2.05) + 10 = 14.1 \text{ milliamperes}$$

4-6 Normal Distribution

Example 4-14 (continued)

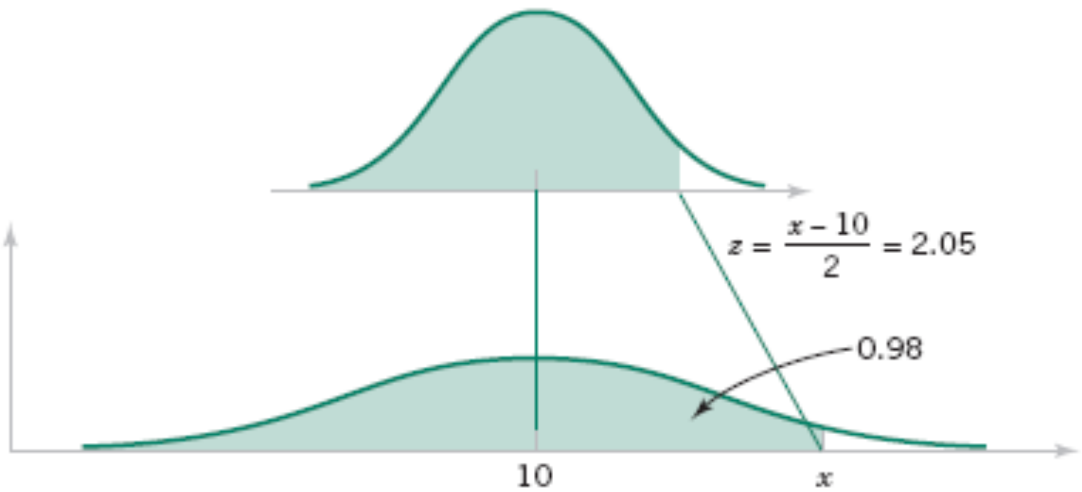


Figure 4-16 Determining the value of x to meet a specified probability.

4-7 Normal Approximation to the Binomial and Poisson Distributions

• Under certain conditions, the normal distribution can be used to approximate the binomial distribution and the Poisson distribution.

4-7 Normal Approximation to the Binomial and Poisson Distributions

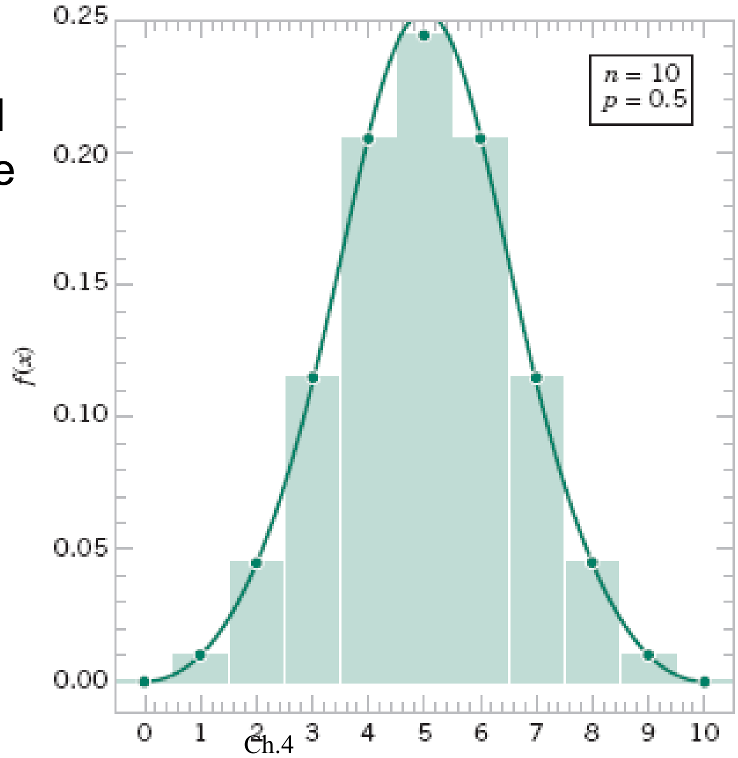


Figure 4-19 Normal approximation to the binomial.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Example 4-17

In a digital communication channel, assume that the number of bits received in error can be modeled by a binomial random variable, and assume that the probability that a bit is received in error is 1×10^{-5}. If 16 million bits are transmitted, what is the probability that more than 150 errors occur?

Let the random variable X denote the number of errors. Then X is a binomial random variable and

$$P(X > 150) = 1 - P(x \le 150) = 1 - \sum_{x=0}^{150} \binom{16{,}000{,}000}{x} (10^{-5})^x (1 - 10^{-5})^{16{,}000{,}000-x}$$

Clearly, the probability in Example 4-17 is difficult to compute. Fortunately, the normal distribution can be used to provide an excellent approximation in this example.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Binomial Distribution

If X is a binomial random variable, with parameters n and p

$$Z = \frac{X - np}{\sqrt{np(1-p)}} \tag{4-12}$$

is approximately a standard normal random variable. To approximate a binomial probability with a normal distribution a **continuity correction** is applied as follows

$$P(X \le x) = P(X \le x + 0.5) \cong P\left(Z \le \frac{x + 0.5 - np}{\sqrt{np(1-p)}}\right)$$

and

$$P(x \le X) = P(x - 0.5 \le X) \cong P\left(\frac{x - 0.5 - np}{\sqrt{np(1-p)}} \le Z\right)$$

The approximation is good for $np > 5$ and $n(1 - p) > 5$.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Example 4-18

The digital communication problem in the previous example is solved as follows:

$$P(X > 150) = P\left(\frac{X - 160}{\sqrt{160(1 - 10^{-5})}} > \frac{150 - 160}{\sqrt{160(1 - 10^{-5})}}\right)$$
$$= P(Z > -0.79) = P(Z < 0.79) = 0.785$$

Because $np = (16 \times 10^6)(1 \times 10^{-5}) = 160$ and $n(1 - p)$ is much larger, the approximation is expected to work well in this case.

4-7 Normal Approximation to the Binomial and Poisson Distributions

hypergeometric distribution	$\approx$ $\frac{n}{N} < 0.1$	binomial distribution	$\approx$ $np > 5$ $n(1-p) > 5$	normal distribution

Figure 4-21 Conditions for approximating hypergeometric and binomial probabilities.

4-7 Normal Approximation to the Binomial and Poisson Distributions

Normal Approximation to the Poisson Distribution

If X is a Poisson random variable with $E(X) = \lambda$ and $V(X) = \lambda$,

$$Z = \frac{X - \lambda}{\sqrt{\lambda}} \tag{4-13}$$

is approximately a standard normal random variable. The approximation is good for

$$\lambda > 5$$

4-7 Normal Approximation to the Binomial and Poisson Distributions

Example 4-20

Assume that the number of asbestos particles in a squared meter of dust on a surface follows a Poisson distribution with a mean of 1000. If a squared meter of dust is analyzed, what is the probability that less than 950 particles are found?

This probability can be expressed exactly as

$$P(X \leq 950) = \sum_{x=0}^{950} \frac{e^{-1000} x^{1000}}{x!}$$

The computational difficulty is clear. The probability can be approximated as

$$P(X \leq x) = P\left(Z \leq \frac{950 - 1000}{\sqrt{1000}}\right) = P(Z \leq -1.58) = 0.057$$

4-8 Exponential Distribution

Definition

The random variable X that equals the distance between successive events of a Poisson process with mean $\lambda > 0$ is an **exponential random variable** with parameter λ. The probability density function of X is

$$f(x) = \lambda e^{-\lambda x} \quad \text{for} \quad 0 \le x < \infty \qquad (4\text{-}14)$$

4-8 Exponential Distribution

Mean and Variance

If the random variable X has an exponential distribution with parameter λ,

$$\mu = E(X) = \frac{1}{\lambda} \quad \text{and} \quad \sigma^2 = V(X) = \frac{1}{\lambda^2} \qquad (4\text{-}15)$$

It is important to **use consistent units** in the calculation of probabilities, means, and variances involving exponential random variables. The following example illustrates unit conversions.

4-8 Exponential Distribution

Example 4-21

In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 25 log-ons per hour. What is the probability that there are no log-ons in an interval of 6 minutes?

Let X denote the time in hours from the start of the interval until the first log-on. Then, X has an exponential distribution with $\lambda = 25$ log-ons per hour. We are interested in the probability that X exceeds 6 minutes. Because λ is given in log-ons per hour, we express all time units in hours. That is, 6 minutes = 0.1 hour. The probability requested is shown as the shaded area under the probability density function in Fig. 4-23. Therefore,

$$P(X > 0.1) = \int_{0.1}^{\infty} 25e^{-25x}\,dx = e^{-25(0.1)} = 0.082$$

4-8 Exponential Distribution

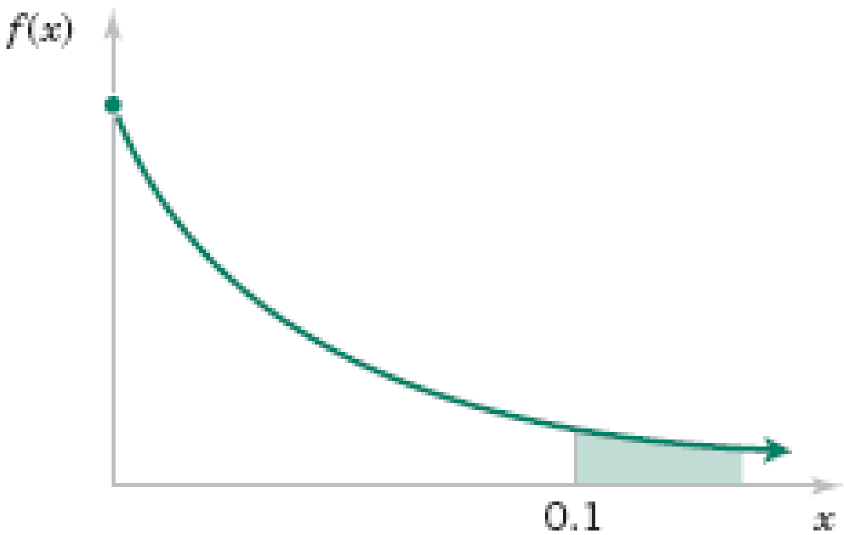


Figure 4-23 Probability for the exponential distribution in Example 4-21.

4-8 Exponential Distribution

Example 4-21 (continued)

Also, the cumulative distribution function can be used to obtain the same result as follows:

$$P(X > 0.1) = 1 - F(0.1) = e^{-25(0.1)}$$

An identical answer is obtained by expressing the mean number of log-ons as 0.417 log-ons per minute and computing the probability that the time until the next log-on exceeds 6 minutes. Try it.

What is the probability that the time until the next log-on is between 2 and 3 minutes? Upon converting all units to hours,

$$P(0.033 < X < 0.05) = \int_{0.033}^{0.05} 25e^{-25x}\,dx = -e^{-25x}\Big|_{0.033}^{0.05} = 0.152$$

4-8 Exponential Distribution

Example 4-21 (continued)

An alternative solution is

$$P(0.033 < X < 0.05) = F(0.05) - F(0.033) = 0.152$$

Determine the interval of time such that the probability that no log-on occurs in the i val is 0.90. The question asks for the length of time x such that $P(X > x) = 0.90$. Now,

$$P(X > x) = e^{-25x} = 0.90$$

Take the (natural) log of both sides to obtain $-25x = \ln(0.90) = -0.1054$. Therefore,

$$x = 0.00421 \text{ hour} = 0.25 \text{ minute}$$

4-8 Exponential Distribution

Example 4-21 (continued)

Furthermore, the mean time until the next log-on is

$$\mu = 1/25 = 0.04 \text{ hour} = 2.4 \text{ minutes}$$

The standard deviation of the time until the next log-on is

$$\sigma = 1/25 \text{ hours} = 2.4 \text{ minutes}$$

4-8 Exponential Distribution

Our starting point for observing the system does not matter.

•An even more interesting property of an exponential random variable is the **lack of memory property**.

In Example 4-21, suppose that there are no log-ons from 12:00 to 12:15; the probability that there are no log-ons from 12:15 to 12:21 is still 0.082. Because we have already been waiting for 15 minutes, we feel that we are "due." That is, the probability of a log-on in the next 6 minutes should be greater than 0.082. ***However, for an exponential distribution this is not true.***

4-8 Exponential Distribution

Example 4-22

Let X denote the time between detections of a particle with a geiger counter and assume that X has an exponential distribution with $\lambda = 1.4$ minutes. The probability that we detect a particle within 30 seconds of starting the counter is

$$P(X < 0.5 \text{ minute}) = F(0.5) = 1 - e^{-0.5/1.4} = 0.30$$

In this calculation, all units are converted to minutes. Now, suppose we turn on the geiger counter and wait 3 minutes without detecting a particle. What is the probability that a particle is detected in the next 30 seconds?

4-8 Exponential Distribution

Example 4-22 (continued)

Because we have already been waiting for 3 minutes, we feel that we are "due." ' is, the probability of a detection in the next 30 seconds should be greater than 0.3. Howe for an exponential distribution, this is not true. The requested probability can be expre as the conditional probability that $P(X < 3.5 \mid X > 3)$. From the definition of conditi probability,

$$P(X < 3.5 \mid X > 3) = P(3 < X < 3.5)/P(X > 3)$$

where

$$P(3 < X < 3.5) = F(3.5) - F(3) = [1 - e^{-3.5/1.4}] - [1 - e^{-3/1.4}] = 0.0035$$

and

$$P(X > 3) = 1 - F(3) = e^{-3/1.4} = 0.117$$

4-8 Exponential Distribution

Example 4-22 (continued)

Therefore,

$$P(X < 3.5 \mid X > 3) = 0.035/0.117 = 0.30$$

After waiting for 3 minutes without a detection, the probability of a detection in the next 30 seconds is the same as the probability of a detection in the 30 seconds immediately after starting the counter. The fact that you have waited 3 minutes without a detection does not change the probability of a detection in the next 30 seconds.

4-8 Exponential Distribution

Example 4-22 illustrates the **lack of memory property** of an exponential random variable and a general statement of the property follows. In fact, the exponential distribution is the only continuous distribution with this property.

Lack of Memory Property

For an exponential random variable X,

$$P(X < t_1 + t_2 \mid X > t_1) = P(X < t_2) \qquad (4\text{-}16)$$

4-8 Exponential Distribution

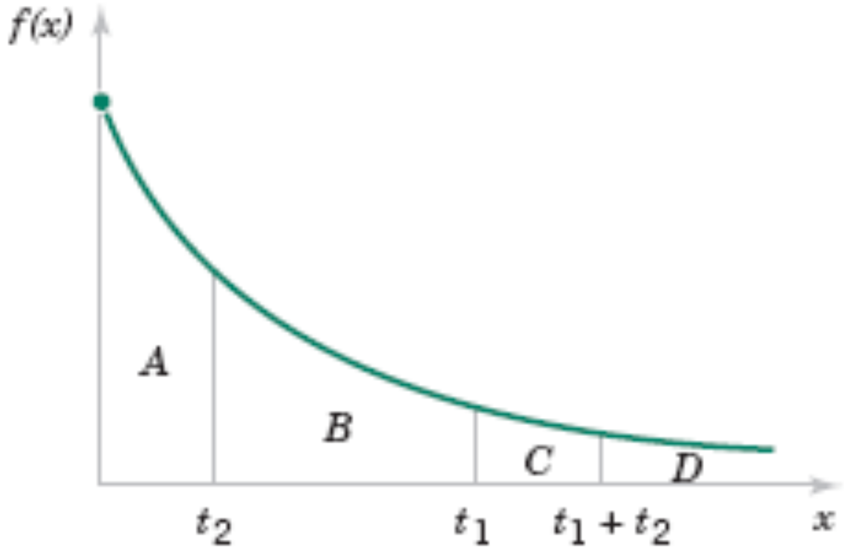


Figure 4-24 Lack of memory property of an Exponential distribution.

IMPORTANT TERMS AND CONCEPTS

- Chi-squared distribution
- Continuous uniform distribution
- Continuity correction
- Cumulative probability distribution function-continuous random variable
- Erlang distribution
- Exponential distribution
- Gamma distribution
- Lack of memory property-continuous random variable
- Lognormal distribution
- Mean-continuous random variable
- Mean-function of a continuous random variable
- Normal approximation to binomial and Poisson probabilities
- Normal distribution
- Probability density function
- Probability distribution-continuous random variable
- Standard deviation-continuous random variable
- Standard normal distribution
- Standardizing
- Variance-continuous random variable
- Weibull distribution

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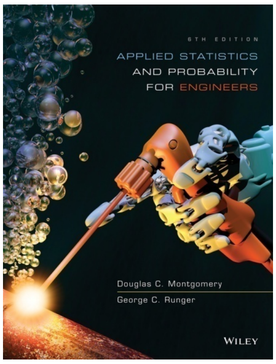

Applied Statistics and Probability for Engineers

Sixth Edition

Douglas C. Montgomery   George C. Runger

Chapter 7

Point Estimation of Parameters and Sampling Distributions

7 Point Estimation of Parameters and Sampling Distributions

CHAPTER OUTLINE

7-1 Point Estimation
7-2 Sampling Distributions and the Central Limit Theorem
7-3 General Concepts of Point Estimation
  7-3.1 Unbiased Estimators
  7-3.2 Variance of a Point Estimator
  7-3.3 Standard Error: Reporting a Point Estimate
  7-3.4 Mean Squared Error of an Estimator
7-4 Methods of Point Estimation
  7-4.1 Method of Moments
  7-4.2 Method of Maximum Likelihood
  7-4.3 Bayesian Estimation of Parameters

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Learning Objectives for Chapter 7

After careful study of this chapter, you should be able to do the following:

1. General concepts of estimating the parameters of a population or a probability distribution.
2. Important role of the normal distribution as a sampling distribution.
3. The central limit theorem.
4. Important properties of point estimators, including bias, variances, and mean square error.
5. Constructing point estimators using the method of moments, and the method of maximum likelihood.
6. Compute and explain the precision with which a parameter is estimated.
7. Constructing a point estimator using the Bayesian approach.

Point Estimation

- A point estimate is a reasonable value of a population parameter.
- $X_1, X_2, \ldots, X_n$ are random variables.
- Functions of these random variables, x-bar and s^2, are also random variables called statistics.
- Statistics have their unique distributions which are called sampling distributions.

Point Estimator

A point estimate of some population parameter θ is a single numerical value $\hat{\theta}$ of a statistic $\hat{\Theta}$.
The statistic $\hat{\Theta}$ is called the point estimator.

As an example, suppose the random variable X is normally distributed with an unknown mean μ. The sample mean is a point estimator of the unknown population mean μ. That is, $\hat{\mu} = \bar{X}$. After the sample has been selected, the numerical value $\bar{x}$ is the point estimate of μ.
Thus if $x_1 = 25, x_2 = 30, x_3 = 29$, and $x_4 = 31$, the point estimate of μ is

$$\bar{x} = \frac{25+30+29+31}{4} = 28.75$$

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Some Parameters & Their Statistics

Parameter	Measure	Statistic
μ	Mean of a single population	x-bar
σ^2	Variance of a single population	s^2
σ	Standard deviation of a single population	s
p	Proportion of a single population	p-hat
$\mu_1 - \mu_2$	Difference in means of two populations	x bar$_1$ - x bar$_2$
$p_1 - p_2$	Difference in proportions of two populations	p hat$_1$ - p hat$_2$

- There could be choices for the point estimator of a parameter.
- To estimate the mean of a population, we could choose the:
  - Sample mean.
  - Sample median.
  - Average of the largest & smallest observations in the sample.

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Some Definitions

- The random variables X_1, X_2,…,X_n are a random sample of size n if:
  a) The X_i 's are independent random variables.
  b) Every X_i has the same probability distribution.
- A statistic is any function of the observations in a random sample.
- The probability distribution of a statistic is called a sampling distribution.

Central Limit Theorem

If $X_1, X_2, \ldots, X_n$ is a random sample of size n is taken from a population (either finite or infinite) with mean μ and finite variance σ^2, and if $\bar{X}$ is the sample mean, then the limiting form of the distribution of

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

as $n \to \infty$, is the standard normal distribution.

Example 7-2: Central Limit Theorem

Suppose that a random variable X has a continuous uniform distribution:

$$f(x)=\begin{cases}1/2, & 4\le x\le 6\\ 0, & \text{otherwise}\end{cases}$$

Find the distribution of the sample mean of a random sample of size $n = 40$.

By the CLT the distribution $\bar{X}$ is normal .

$$\mu=\frac{b+a}{2}=\frac{6+4}{2}=5$$

$$\sigma^2=\frac{(b-a)^2}{12}=\frac{(6-4)^2}{12}1/3$$

$$\sigma_{\bar{X}}^2=\frac{\sigma^2}{n}=\frac{1/3}{40}=\frac{1}{120}$$

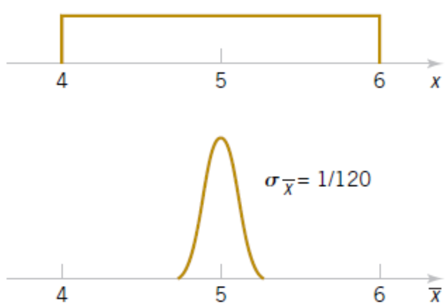


Figure 7-5 The distribution of X and $\bar{X}$ for Example 7-2.

Sampling Distribution of a Difference in Sample Means

- If we have two independent populations with means μ_1 and μ_2, and variances σ_1^2 and σ_2^2, and
- If X-bar$_1$ and X-bar$_2$ are the sample means of two independent random samples of sizes n_1 and n_2 from these populations:
- Then the sampling distribution of:

$$Z=\frac{(\bar{X}_1-\bar{X}_2)-(\mu_1-\mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}}$$

is approximately standard normal, if the conditions of the central limit theorem apply.

- If the two populations are normal, then the sampling distribution of Z is exactly standard normal.

Example 7-3: Aircraft Engine Life

The effective life of a component used in jet-turbine aircraft engine is a random variable with mean 5000 and SD 40 hours and is close to a normal distribution. The engine manufacturer introduces an improvement into the Manufacturing process for this component that changes the parameters to 5050 and 30. Random samples of size 16 and 25 are selected.

What is the probability that the difference in the two sample means is at least 25 hours?

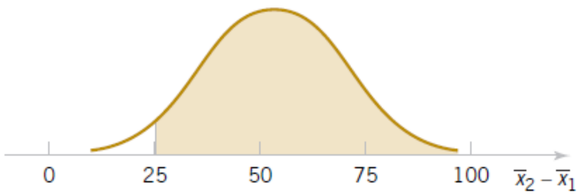


Figure 7-6 The sampling distribution of $\overline{X}_2 - \overline{X}_1$ in Example 7-3.

	Process		
	Old (1)	New (2)	Diff (2-1)
x-bar =	5,000	5,050	50
s =	40	30	
n =	16	25	
Calculations			
$s / \sqrt{n}$ =	10	6	11.7
		z =	-2.14
P(xbar$_2$-xbar$_1$ > 25) = P(Z > z) =			0.9838
		= 1 - NORMSDIST(z)	

Unbiased Estimators Defined

The point estimator $\hat{\Theta}$ is an unbiased estimator for the parameter θ if:

$$E\left(\hat{\Theta}\right) = \theta$$

If the estimator is not unbiased, then the difference:

$$E\left(\hat{\Theta}\right) - \theta$$

is called the bias of the estimator $\hat{\Theta}$.

The mean of the sampling distribution of $\hat{\Theta}$ is equal to θ.

Example 7-4: Sample Mean & Variance Are Unbiased-1

- X is a random variable with mean μ and variance σ^2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n.
- Show that the sample mean (X-bar) is an unbiased estimator of μ.

$$E(\bar{X}) = E\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right)$$

$$= \frac{1}{n}\left[E(X_1) + E(X_2) + \ldots + E(X_n)\right]$$

$$= \frac{1}{n}[\mu + \mu + \ldots + \mu] = \frac{n\mu}{n} = \mu$$

Example 7-4: Sample Mean & Variance Are Unbiased-2

Show that the sample variance (S^2) is a unbiased estimator of σ^2.

$$E(S^2) = E\left(\frac{\sum_{i=1}^{n}(X - \bar{X})^2}{n-1}\right) = \frac{1}{n-1}E\left[\sum_{i=1}^{n}\left(X_i^2 + \bar{X}^2 - 2\bar{X}X_i\right)\right]$$

$$= \frac{1}{n-1}\left[E\left(\sum_{i=1}^{n}X_i^2 - n\bar{X}^2\right)\right] = \frac{1}{n-1}\left[\sum_{i=1}^{n}E\left(X_i^2\right) - nE\left(\bar{X}^2\right)\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}\left(\mu^2 + \sigma^2\right) - n\left(\mu^2 + \sigma^2/n\right)\right]$$

$$= \frac{1}{n-1}\left[n\mu^2 + n\sigma^2 - n\mu^2 - \sigma^2\right] = \frac{1}{n-1}\left[(n-1)\sigma^2\right] = \sigma^2$$

Minimum Variance Unbiased Estimators

- If we consider all unbiased estimators of θ, the one with the smallest variance is called the minimum variance unbiased estimator (MVUE).
- If X_1, X_2,…, X_n is a random sample of size n from a normal distribution with mean μ and variance σ^2, then the sample X-bar is the MVUE for μ.

Standard Error of an Estimator

The standard error of an estimator $\hat{\Theta}$ is its standard deviation, given by

$$\sigma_{\hat{\Theta}} = \sqrt{V\left(\hat{\Theta}\right)}.$$

If the standard error involves unknown parameters that can be estimated,

substitution of these values into $\sigma_{\hat{\Theta}}$

produces an estimated standard error, denoted by $\hat{\sigma}_{\hat{\Theta}}$.

Equivalent notation: $\hat{\sigma}_{\hat{\Theta}} = s_{\hat{\Theta}} = se\left(\hat{\Theta}\right)$

If the X_i are $\sim N\left(\mu, \sigma^2\right)$, then standard error of $\bar{X}$ is

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}}.$$

If σ is not known, then $\hat{\sigma}_{\bar{X}} = \dfrac{s}{\sqrt{n}}$.

Example 7-5: Thermal Conductivity

- These observations are 10 measurements of thermal conductivity of Armco iron.
- Since σ is not known, we use s to calculate the standard error.
- Since the standard error is 0.2% of the mean, the mean estimate is fairly precise. We can be very confident that the true population mean is 41.924 ± 2(0.0898) or between 41.744 and 42.104.

x_i	
41.60	
41.48	
42.34	
41.95	
41.86	
42.18	
41.72	
42.26	
41.81	
42.04	
41.924	= Mean
0.284	= Std dev (s)
0.0898	= Std error

Mean Squared Error

The mean squared error of an estimator $\hat{\Theta}$ of the parameter θ is defined as:

$$\text{MSE}\left(\hat{\Theta}\right) = E\left(\hat{\Theta} - \theta\right)^2$$

$$\text{Can be rewritten as} = E\left[\hat{\Theta} - E\left(\hat{\Theta}\right)\right]^2 + \left[\theta - E\left(\hat{\Theta}\right)\right]^2$$

$$= V\left(\hat{\Theta}\right) + \left(\text{bias}\right)^2$$

Conclusion: The mean squared error (MSE) of the estimator is equal to the variance of the estimator plus the bias squared.

8 Statistical Intervals for a Single Sample

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter, you should be able to do the following:

1. Construct confidence intervals on the mean of a normal distribution, using either the normal distribution or the t distribution method
2. Construct confidence intervals on the variance and standard deviation of a normal distribution
3. Construct confidence intervals on a population proportion
4. Use a general method for constructing an approximate confidence interval on a parameter
5. Construct prediction intervals for a future observation
6. Construct a tolerance interval for a normal population
7. Explain the three types of interval estimates: confidence intervals, prediction intervals, and tolerance intervals

8-1 Introduction

- In the previous chapter we illustrated how a parameter can be estimated from sample data. However, it is important to understand how good is the estimate obtained.

- Bounds that represent an interval of plausible values for a parameter are an example of an interval estimate.

- Three types of intervals will be presented:
  - Confidence intervals
  - Prediction intervals
  - Tolerance intervals

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.1 Development of the Confidence Interval and its Basic Properties

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with unknown mean μ and known variance σ^2. From the results of Chapter 5 we know that the sample mean $\overline{X}$ is normally distributed with mean μ and variance σ^2/n. We may **standardize** $\overline{X}$ by subtracting the mean and dividing by the standard deviation, which results in the variable

$$Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \tag{8-3}$$

Now Z has a standard normal distribution.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.1 Development of the Confidence Interval and its Basic Properties

A **confidence interval** estimate for μ is an interval of the form $l \leq \mu \leq u$, where the end-points l and u are computed from the sample data. Because different samples will produce different values of l and u, these end-points are values of random variables L and U, respectively. Suppose that we can determine values of L and U such that the following probability statement is true:

$$P\{L \leq \mu \leq U\} = 1 - \alpha \tag{8-4}$$

where $0 \leq \alpha \leq 1$. There is a probability of $1 - \alpha$ of selecting a sample for which the CI will contain the true value of μ. Once we have selected the sample, so that $X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$, and computed l and u, the resulting **confidence interval** for μ is

$$l \leq \mu \leq u \tag{8-5}$$

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.1 Development of the Confidence Interval and its Basic Properties

- The endpoints or bounds l and u are called lower- and upper-confidence limits, respectively.
- Since Z follows a standard normal distribution, we can write:

$$P\left\{-z_{\alpha/2} \leq \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2}\right\} = 1 - \alpha$$

Now manipulate the quantities inside the brackets by (1) multiplying through by $\sigma/\sqrt{n}$, (2) subtracting $\overline{X}$ from each term, and (3) multiplying through by -1. This results in

$$P\left\{\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right\} = 1 - \alpha \tag{8-6}$$

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.1 Development of the Confidence Interval and its Basic Properties

Definition

If $\overline{x}$ is the sample mean of a random sample of size n from a normal population with known variance σ^2, a $100(1-\alpha)\%$ CI on μ is given by

$$\overline{x} - z_{\alpha/2}\sigma/\sqrt{n} \leq \mu \leq \overline{x} + z_{\alpha/2}\sigma/\sqrt{n} \qquad (8\text{-}7)$$

where $z_{\alpha/2}$ is the upper $100\alpha/2$ percentage point of the standard normal distribution.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Example 8-1

ASTM Standard E23 defines standard test methods for notched bar impact testing of metallic materials. The Charpy V-notch (CVN) technique measures impact energy and is often used to determine whether or not a material experiences a ductile-to-brittle transition with decreasing temperature. Ten measurements of impact energy (J) on specimens of A238 steel cut at 60°C are as follows: 64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2, and 64.3. Assume that impact energy is normally distributed with $\sigma = 1J$. We want to find a 95% CI for μ, the mean impact energy. The required quantities are $z_{\alpha/2} = z_{0.025} = 1.96$, $n = 10$, $\sigma = 1$, and $\overline{x} = 64.46$. The resulting 95% CI is found from Equation 8-7 as follows:

$$\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$64.46 - 1.96\frac{1}{\sqrt{10}} \leq \mu \leq 64.46 + 1.96\frac{1}{\sqrt{10}}$$

$$63.84 \leq \mu \leq 65.08$$

That is, based on the sample data, a range of highly plausible vaules for mean impact energy for A238 steel at 60°C is $63.84J \leq \mu \leq 65.08J$.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Interpreting a Confidence Interval

- The confidence interval is a random interval
- The appropriate interpretation of a confidence interval (for example on μ) is: The observed interval $[l, u]$ brackets the true value of μ, with confidence $100(1-\alpha)$.
- Examine Figure 8-1 on the next slide.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

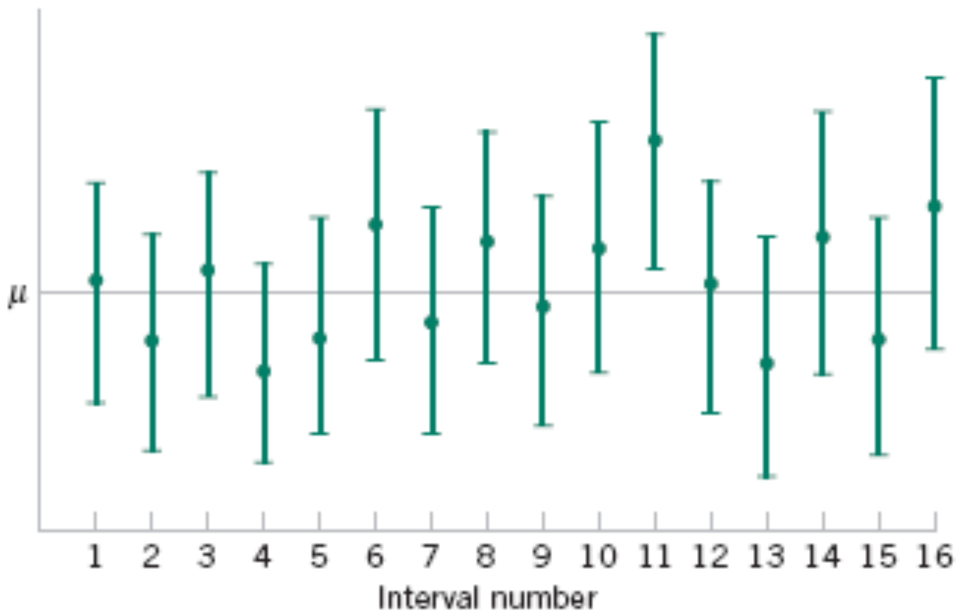


Figure 8-1 Repeated construction of a confidence interval for μ.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Confidence Level and Precision of Error

The length of a confidence interval is a measure of the precision of estimation.

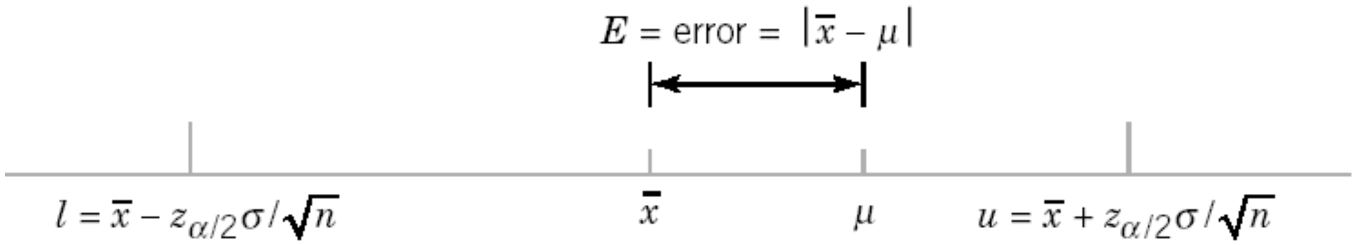


Figure 8-2 Error in estimating μ with $\bar{x}$.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.2 Choice of Sample Size

Definition

If $\bar{x}$ is used as an estimate of μ, we can be $100(1 - \alpha)\%$ confident that the error $|\bar{x} - \mu|$ will not exceed a specified amount E when the sample size is

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 \qquad (8\text{-}8)$$

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Example 8-2

To illustrate the use of this procedure, consider the CVN test described in Example 8-1, and suppose that we wanted to determine how many specimens must be tested to ensure that the 95% CI on μ for A238 steel cut at 60°C has a length of at most $1.0J$. Since the bound on error in estimation E is one-half of the length of the CI, to determine n we use Equation 8-8 with $E = 0.5$, $\sigma = 1$, and $z_{\alpha/2} = 0.025$. The required sample size is 16

$$n = \left(\frac{z_{\alpha/2}\sigma}{E}\right)^2 = \left[\frac{(1.96)1}{0.5}\right]^2 = 15.37$$

and because n must be an integer, the required sample size is $n = 16$.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.3 One-Sided Confidence Bounds

Definition

A $100(1 - \alpha)\%$ **upper-confidence bound** for μ is

$$\mu \leq u = \bar{x} + z_\alpha \sigma/\sqrt{n} \tag{8-9}$$

and a $100(1 - \alpha)\%$ **lower-confidence bound** for μ is

$$\bar{x} - z_\alpha \sigma/\sqrt{n} = l \leq \mu \tag{8-10}$$

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.4 General Method to Derive a Confidence Interval

It is easy to give a general method for finding a confidence interval for an unknown parameter θ. Let $X_1, X_2, \ldots, X_n$ be a random sample of n observations. Suppose we can find a statistic $g(X_1, X_2, \ldots, X_n; \theta)$ with the following properties:

1. $g(X_1, X_2, \ldots, X_n; \theta)$ depends on both the sample and θ.
2. The probability distribution of $g(X_1, X_2, \ldots, X_n; \theta)$ does not depend on θ or any other unknown parameter.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.4 General Method to Derive a Confidence Interval

In the case considered in this section, the parameter $\theta = \mu$. The random variable $g(X_1, X_2, \ldots, X_n; \mu) = (\overline{X} - \mu)/(\sigma/\sqrt{n})$ and satisfies both conditions above; it depends on the sample and on μ, and it has a standard normal distribution since σ is known. Now one must find constants C_L and C_U so that

$$P[C_L \le g(X_1, X_2, \ldots, X_n; \theta) \le C_U] = 1 - \alpha \qquad (8\text{-}11)$$

Because of property 2, C_L and C_U do not depend on θ. In our example, $C_L = -z_{\alpha/2}$ and $C_U = z_{\alpha/2}$. Finally, you must manipulate the inequalities in the probability statement so that

$$P[L(X_1, X_2, \ldots, X_n) \le \theta \le U(X_1, X_2, \ldots, X_n)] = 1 - \alpha \qquad (8\text{-}12)$$

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.4 General Method to Derive a Confidence Interval

This gives $L(X_1, X_2, \ldots, X_n)$ and $U(X_1, X_2, \ldots, X_n)$ as the lower and upper confidence limits defining the $100(1-\alpha)\%$ confidence interval for θ. The quantity $g(X_1, X_2, \ldots, X_n; \theta)$ is often called a "pivotal quantity" because we pivot on this quantity in Equation 8-11 to produce Equation 8-12. In our example, we manipulated the pivotal quantity $(\overline{X}-\mu)/(\sigma/\sqrt{n})$ to obtain $L(X_1, X_2, \ldots, X_n) = \overline{X} - z_{\alpha/2}\sigma/\sqrt{n}$ and $U(X_1, X_2, \ldots, X_n) = \overline{X} + z_{\alpha/2}\sigma/\sqrt{n}$.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

8-2.5 A Large-Sample Confidence Interval for μ

Definition

When n is large, the quantity

$$\frac{\overline{X}-\mu}{S/\sqrt{n}}$$

has an approximate standard normal distribution. Consequently,

$$\overline{x} - z_{\alpha/2}\frac{s}{\sqrt{n}} \leq \mu \leq \overline{x} + z_{\alpha/2}\frac{s}{\sqrt{n}} \tag{8-13}$$

is a **large sample confidence interval** for μ, with confidence level of approximately $100(1-\alpha)\%$.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Example 8-4

An article in the 1993 volume of the *Transactions of the American Fisheries Society* reports the results of a study to investigate the mercury contamination in largemouth bass. A sample of fish was selected from 53 Florida lakes and mercury concentration in the muscle tissue was measured (ppm). The mercury concentration values are

1.230	0.490	0.490	1.080	0.590	0.280	0.180	0.100	0.940
1.330	0.190	1.160	0.980	0.340	0.340	0.190	0.210	0.400
0.040	0.830	0.050	0.630	0.340	0.750	0.040	0.860	0.430
0.044	0.810	0.150	0.560	0.840	0.870	0.490	0.520	0.250
1.200	0.710	0.190	0.410	0.500	0.560	1.100	0.650	0.270
0.270	0.500	0.770	0.730	0.340	0.170	0.160	0.270	

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Example 8-4 (continued)

The summary statistics from Minitab are displayed below:

Descriptive Statistics: Concentration

Variable	N	Mean	Median	TrMean	StDev	SE Mean
Concentration	53	0.5250	0.4900	0.5094	0.3486	0.0479

Variable	Minimum	Maximum	Q1	Q3
Concentration	0.0400	1.3300	0.2300	0.7900

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Example 8-4 (continued)

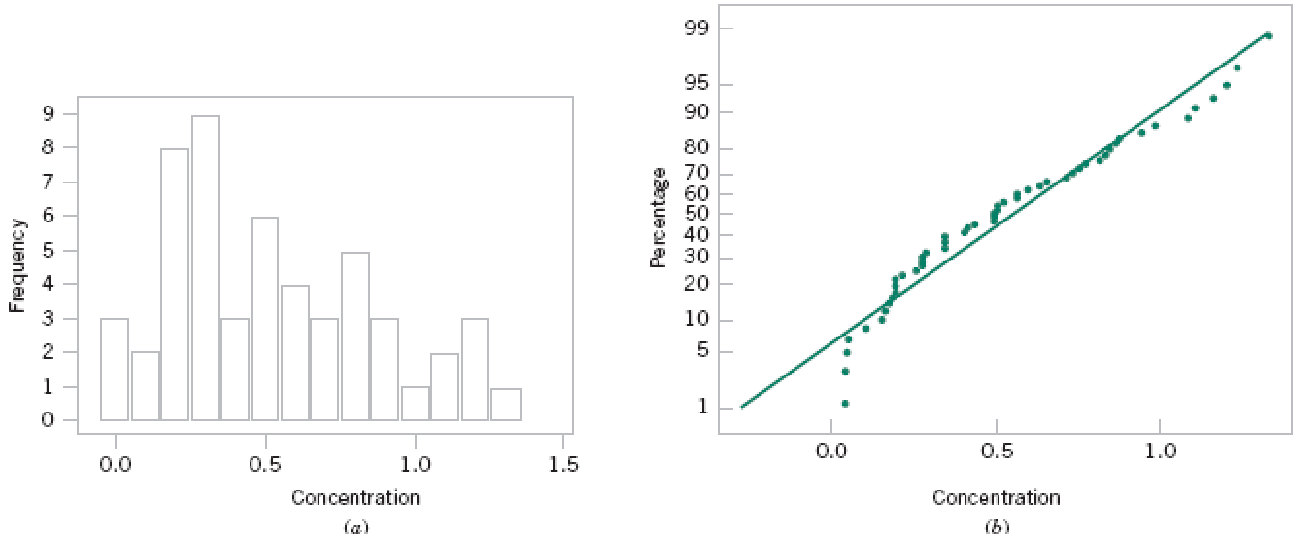


Figure 8-3 Mercury concentration in largemouth bass (a) Histogram. (b) Normal probability plot

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

Example 8-4 (continued)

Figure 8-3(a) and (b) presents the histogram and normal probability plot of the mercury concentration data. Both plots indicate that the distribution of mercury concentration is not normal and is positively skewed. We want to find an approximate 95% CI on μ. Because $n > 40$, the assumption of normality is not necessary to use Equation 8-13. The required quantities are $n = 53$, $\overline{x} = 0.5250$, $s = 0.3486$, and $z_{0.025} = 1.96$. The approximate 95% CI on μ is

$$\overline{x} - z_{0.025}\frac{s}{\sqrt{n}} \leq \mu \leq \overline{x} + z_{0.025}\frac{s}{\sqrt{n}}$$

$$0.5250 - 1.96\frac{0.3486}{\sqrt{53}} \leq \mu \leq 0.5250 + 1.96\frac{0.3486}{\sqrt{53}}$$

$$0.4311 \leq \mu \leq 0.6189$$

This interval is fairly wide because there is a lot of variability in the mercury concentration measurements.

8-2 Confidence Interval on the Mean of a Normal Distribution, Variance Known

A General Large Sample Confidence Interval

$$\hat{\theta} - z_{\alpha/2}\sigma_{\hat{\Theta}} \le \theta \le \hat{\theta} + z_{\alpha/2}\sigma_{\hat{\Theta}} \qquad (8\text{-}14)$$

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

8-3.1 The *t* distribution

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with unknown mean μ and unknown variance σ^2. The random variable

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} \qquad (8\text{-}15)$$

has a t distribution with $n - 1$ degrees of freedom.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

8-3.1 The *t* distribution

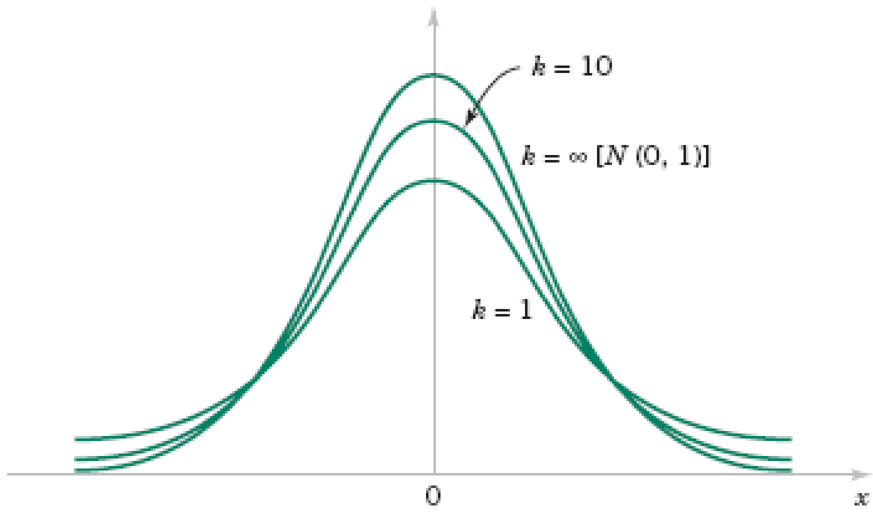


Figure 8-4 Probability density functions of several *t* distributions.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

8-3.1 The *t* distribution

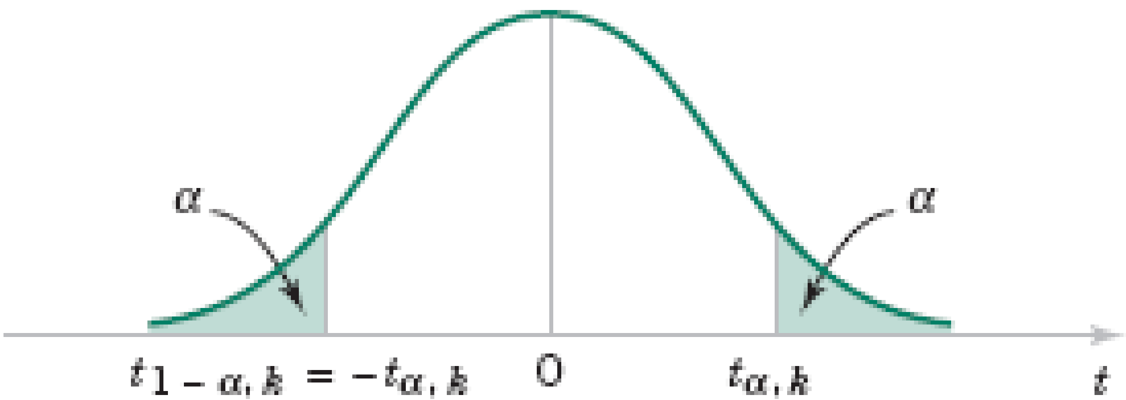


Figure 8-5 Percentage points of the *t* distribution.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

8-3.2 The *t* Confidence Interval on μ

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal distribution with unknown variance σ^2, a **100(1 − α) percent confidence interval on μ** is given by

$$\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n} \le \mu \le \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n} \quad (8\text{-}18)$$

where $t_{\alpha/2,n-1}$ is the upper $100\alpha/2$ percentage point of the t distribution with $n-1$ degrees of freedom.

One-sided confidence bounds on the mean are found by replacing $t_{\alpha/2,n-1}$ in Equation 8-18 with $t_{\alpha,n-1}$.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

Example 8-5

An article in the journal *Materials Engineering* (1989, Vol. II, No. 4, pp. 275–281) describes the results of tensile adhesion tests on 22 U-700 alloy specimens. The load at specimen failure is as follows (in megapascals):

19.8	10.1	14.9	7.5	15.4	15.4
15.4	18.5	7.9	12.7	11.9	11.4
11.4	14.1	17.6	16.7	15.8	
19.5	8.8	13.6	11.9	11.4	

The sample mean is $\bar{x} = 13.71$, and the sample standard deviation is $s = 3.55$. Figures 8-6 and 8-7 show a box plot and a normal probability plot of the tensile adhesion test data, respectively. These displays provide good support for the assumption that the population is normally distributed. We want to find a 95% CI on μ. Since $n = 22$, we have $n - 1 = 21$ degrees of freedom for t, so $t_{0.025,21} = 2.080$. The resulting CI is

$$\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n} \le \mu \le \bar{x} + t_{\alpha/2,n-1}s/\sqrt{n}$$

$$13.71 - 2.080(3.55)/\sqrt{22} \le \mu \le 13.71 + 2.080(3.55)/\sqrt{22}$$

$$13.71 - 1.57 \le \mu \le 13.71 + 1.57$$

$$12.14 \le \mu \le 15.28$$

The CI is fairly wide because there is a lot of variability in the tensile adhesion test measurements.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

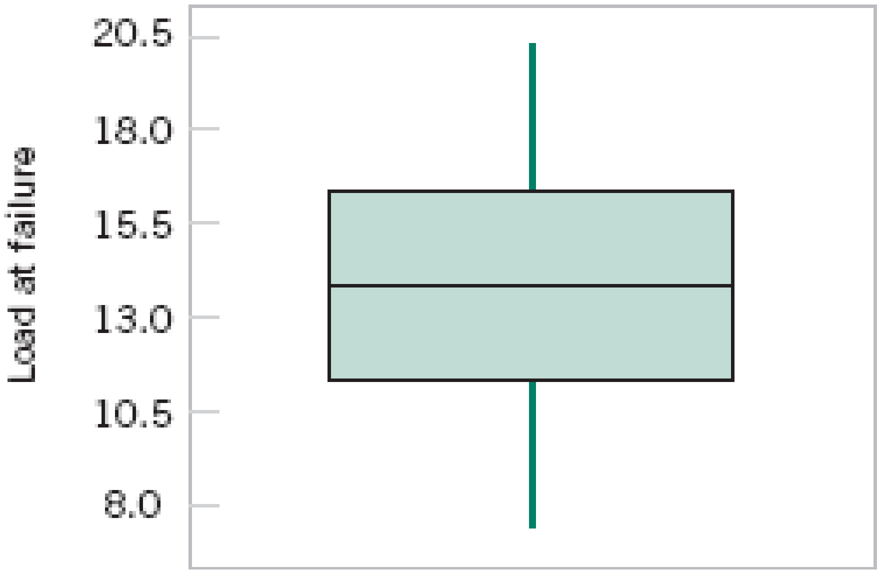


Figure 8-6 Box and Whisker plot for the load at failure data in Example 8-5.

8-3 Confidence Interval on the Mean of a Normal Distribution, Variance Unknown

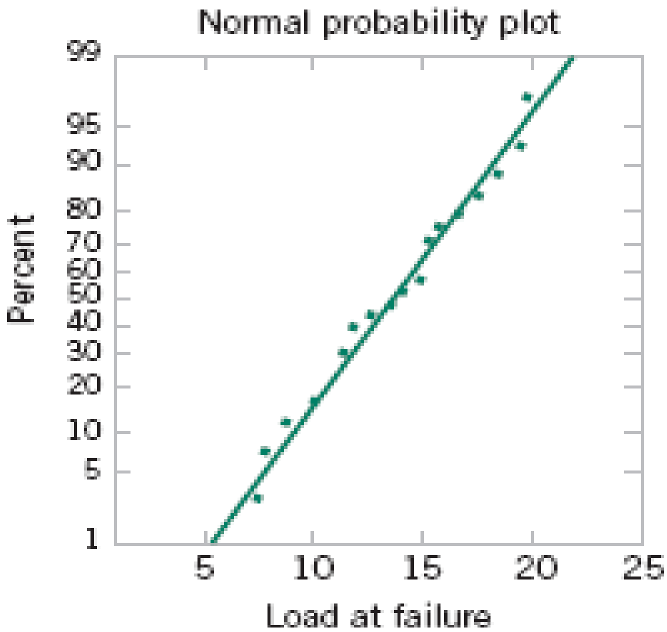


Figure 8-7 Normal probability plot of the load at failure data in Example 8-5.

8-4 Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

Definition

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with mean μ and variance σ^2, and let S^2 be the sample variance. Then the random variable

$$X^2 = \frac{(n-1)\,S^2}{\sigma^2} \tag{8-19}$$

has a chi-square (χ^2) distribution with $n - 1$ degrees of freedom.

8-4 Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

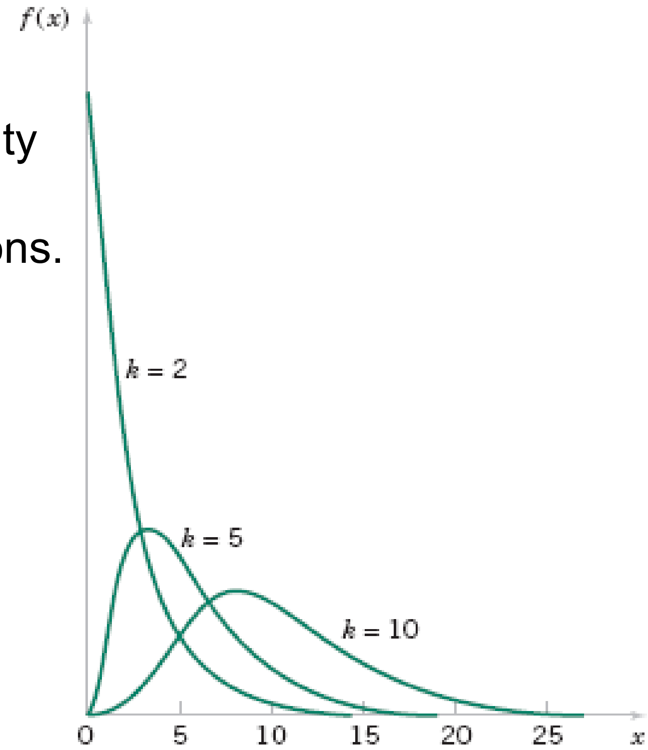


Figure 8-8 Probability density functions of several χ^2 distributions.

8-4 Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

Definition

If s^2 is the sample variance from a random sample of n observations from a normal distribution with unknown variance σ^2, then a **100(1 − α)% confidence interval on σ^2** is

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \leq \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}} \qquad (8\text{-}21)$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the upper and lower $100\alpha/2$ percentage points of the chi-square distribution with $n-1$ degrees of freedom, respectively. A **confidence interval for σ** has lower and upper limits that are the square roots of the corresponding limits in Equation 8-21.

8-4 Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

One-Sided Confidence Bounds

The 100(1 − α)% lower and upper confidence bounds on σ^2 are

$$\frac{(n-1)s^2}{\chi^2_{\alpha,n-1}} \leq \sigma^2 \quad \text{and} \quad \sigma^2 \leq \frac{(n-1)s^2}{\chi^2_{1-\alpha,n-1}} \qquad (8\text{-}22)$$

respectively.

8-4 Confidence Interval on the Variance and Standard Deviation of a Normal Distribution

An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s^2 = 0.0153$ (fluid ounces)^2. If the variance of fill volume is too large, an unacceptable proportion of bottles will be under- or overfilled. We will assume that the fill volume is approximately normally distributed. A 95% upper-confidence interval is found from Equation 8-22 as follows:

Example 8-6

$$\sigma^2 \le \frac{(n-1)s^2}{\chi^2_{0.95,19}}$$

or

$$\sigma^2 \le \frac{(19)0.0153}{10.117} = 0.0287 \text{ (fluid ounce)}^2$$

This last expression may be converted into a confidence interval on the standard deviation σ by taking the square root of both sides, resulting in

$$\sigma \le 0.17$$

Therefore, at the 95% level of confidence, the data indicate that the process standard deviation could be as large as 0.17 fluid ounce.

8-5 A Large-Sample Confidence Interval For a Population Proportion

Normal Approximation for Binomial Proportion

If n is large, the distribution of

$$Z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{\hat{P} - p}{\sqrt{\dfrac{p(1-p)}{n}}}$$

is approximately standard normal.

The quantity $\sqrt{p(1-p)/n}$ is called the standard error of the point estimator $\hat{P}$.

8-5 A Large-Sample Confidence Interval For a Population Proportion

If $\hat{p}$ is the proportion of observations in a random sample of size n that belongs to a class of interest, an approximate $100(1-\alpha)\%$ confidence interval on the proportion p of the population that belongs to this class is

$$\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \qquad (8\text{-}25)$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution.

8-5 A Large-Sample Confidence Interval For a Population Proportion

Example 8-7

In a random sample of 85 automobile engine crankshaft bearings, 10 have a surface finish that is rougher than the specifications allow. Therefore, a point estimate of the proportion of bearings in the population that exceeds the roughness specification is $\hat{p} = x/n = 10/85 = 0.12$. A 95% two-sided confidence interval for p is computed from Equation 8-25 as

$$\hat{p} - z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \le \hat{p} + z_{0.025}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

or

$$0.12 - 1.96\sqrt{\frac{0.12(0.88)}{85}} \le p \le 0.12 + 1.96\sqrt{\frac{0.12(0.88)}{85}}$$

which simplifies to

$$0.05 \le p \le 0.19$$

8-5 A Large-Sample Confidence Interval For a Population Proportion

Choice of Sample Size

The sample size for a specified value E is given by

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 p(1 - p) \qquad (8\text{-}26)$$

An upper bound on n is given by

$$n = \left(\frac{z_{\alpha/2}}{E}\right)^2 (0.25) \qquad (8\text{-}27)$$

8-5 A Large-Sample Confidence Interval For a Population Proportion

Example 8-8

Consider the situation in Example 8-7. How large a sample is required if we want to be 95% confident that the error in using $\hat{p}$ to estimate p is less than 0.05? Using $\hat{p} = 0.12$ as an initial estimate of p, we find from Equation 8-26 that the required sample size is

$$n = \left(\frac{z_{0.025}}{E}\right)^2 \hat{p}(1 - \hat{p}) = \left(\frac{1.96}{0.05}\right)^2 0.12(0.88) \cong 163$$

If we wanted to be *at least* 95% confident that our estimate $\hat{p}$ of the true proportion p was within 0.05 regardless of the value of p, we would use Equation 8-27 to find the sample size

$$n = \left(\frac{z_{0.025}}{E}\right)^2 (0.25) = \left(\frac{1.96}{0.05}\right)^2 (0.25) \cong 385$$

Notice that if we have information concerning the value of p, either from a preliminary sample or from past experience, we could use a smaller sample while maintaining both the desired precision of estimation and the level of confidence.

8-5 A Large-Sample Confidence Interval For a Population Proportion

One-Sided Confidence Bounds

The approximate $100(1-\alpha)\%$ lower and upper confidence bounds are

$$\hat{p} - z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \le p \quad \text{and} \quad p \le \hat{p} + z_\alpha \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \tag{8-28}$$

respectively.

8-6 Guidelines for Constructing Confidence Intervals

The most difficult step in constructing a confidence interval is often the match of the appropriate calculation to the objective of the study. Common cases are listed in Table 8-1 along with the reference to the section that covers the appropriate calculation for a confidence interval test. Table 8-1 provides a simple road map to help select the appropriate analysis. Two primary comments can help identify the analysis:

1. Determine the parameter (and the distribution of the data) that will be bounded by the confidence interval or tested by the hypothesis.
2. Check if other parameters are known or need to be estimated.

9 Tests of Hypotheses for a Single Sample

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter, you should be able to do the following:

1. Structure engineering decision-making problems as hypothesis tests
2. Test hypotheses on the mean of a normal distribution using either a Z-test or a t-test procedure
3. Test hypotheses on the variance or standard deviation of a normal distribution
4. Test hypotheses on a population proportion
5. Use the P-value approach for making decisions in hypotheses tests
6. Compute power, type II error probability, and make sample size selection decisions for tests on means, variances, and proportions
7. Explain and use the relationship between confidence intervals and hypothesis tests
8. Use the chi-square goodness of fit test to check distributional assumptions
9. Use contingency table tests

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

Statistical hypothesis testing and confidence interval estimation of parameters are the fundamental methods used at the data analysis stage of a **comparative experiment,** in which the engineer is interested, for example, in comparing the mean of a population to a specified value.

Definition

A **statistical hypothesis** is a statement about the parameters of one or more populations.

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

For example, suppose that we are interested in the burning rate of a solid propellant used to power aircrew escape systems.

- Now burning rate is a random variable that can be described by a probability distribution.
- Suppose that our interest focuses on the **mean** burning rate (a parameter of this distribution).
- Specifically, we are interested in deciding whether or not the mean burning rate is 50 centimeters per second.

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

Two-sided Alternative Hypothesis

H_0: μ = 50 centimeters per second null hypothesis

H_1: $\mu \neq$ 50 centimeters per second alternative hypothesis

One-sided Alternative Hypotheses

H_0: μ = 50 centimeters per second H_0: μ = 50 centimeters per second

or

H_1: $\mu <$ 50 centimeters per second H_1: $\mu >$ 50 centimeters per second

9-1 Hypothesis Testing

9-1.1 Statistical Hypotheses

Test of a Hypothesis

• A procedure leading to a decision about a particular hypothesis

• Hypothesis-testing procedures rely on using the information in a **random sample from the population of interest**.

• If this information is *consistent* with the hypothesis, then we will conclude that the hypothesis is **true**; if this information is *inconsistent* with the hypothesis, we will conclude that the hypothesis is **false**.

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

$$H_0\text{: } \mu = 50 \text{ centimeters per second}$$
$$H_1\text{: } \mu \neq 50 \text{ centimeters per second}$$

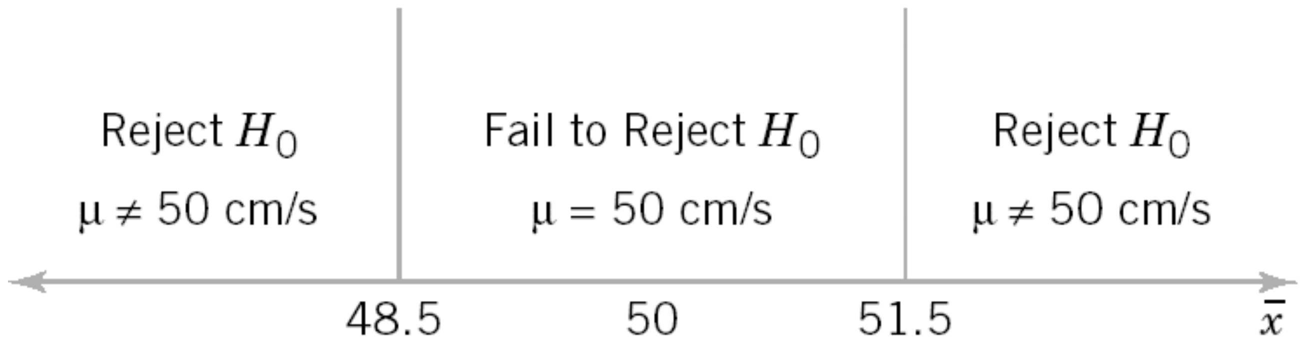


Figure 9-1 Decision criteria for testing H_0:μ = 50 centimeters per second versus H_1:$\mu \neq$ 50 centimeters per second.

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

Definitions

Rejecting the null hypothesis H_0 when it is true is defined as a **type I error.**

Failing to reject the null hypothesis when it is false is defined as a **type II error.**

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

Table 9-1 Decisions in Hypothesis Testing

Decision	H_0 Is True	H_0 Is False
Fail to reject H_0	no error	type II error
Reject H_0	type I error	no error

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true})$$

Sometimes the type I error probability is called the **significance level**, or the **α-error**, or the **size** of the test.

9-1 Hypothesis Testing

9-1.2 Tests of Statistical Hypotheses

$$\alpha = P(\overline{X} < 48.5 \text{ when } \mu = 50) + P(\overline{X} > 51.5 \text{ when } \mu = 50)$$

The z-values that correspond to the critical values 48.5 and 51.5 are

$$z_1 = \frac{48.5 - 50}{0.79} = -1.90 \quad \text{and} \quad z_2 = \frac{51.5 - 50}{0.79} = 1.90$$

Therefore

$$\alpha = P(Z < -1.90) + P(Z > 1.90) = 0.028717 + 0.028717 = 0.057434$$

9-1 Hypothesis Testing

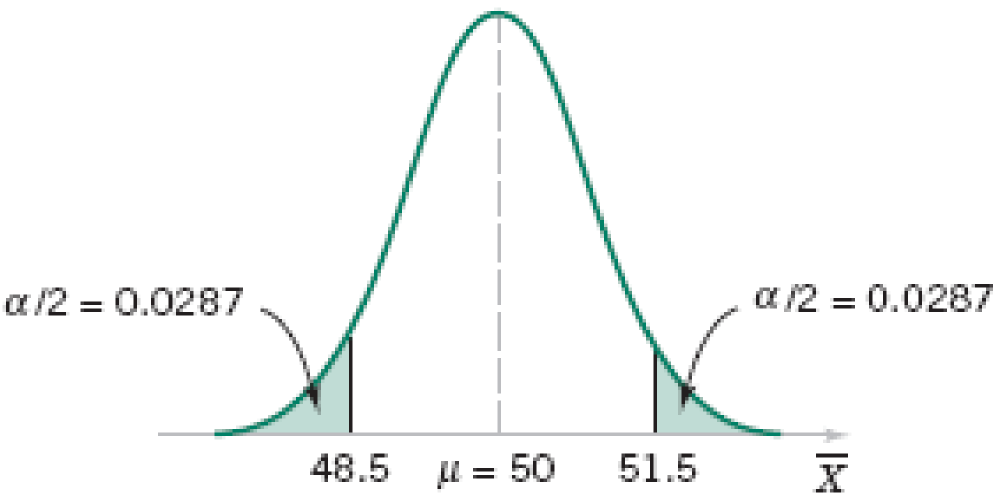


Figure 9-2 The critical region for H_0: $\mu = 50$ versus H_1: $\mu \neq 50$ and $n = 10$.

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \text{ when } H_0 \text{ is true}) \qquad (9\text{-}3)$$

9-1 Hypothesis Testing

$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \text{ when } H_0 \text{ is false}) \qquad (9\text{-}4)$$

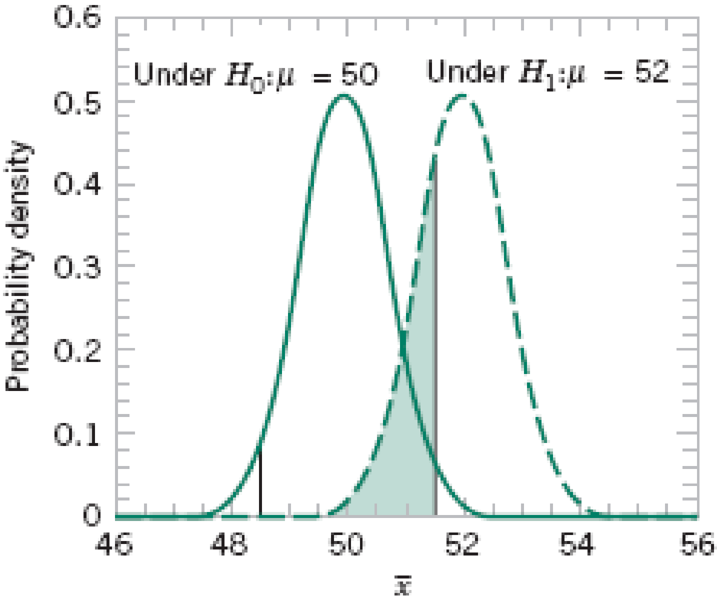


Figure 9-3 The probability of type II error when μ = 52 and n = 10.

9-1 Hypothesis Testing

$$\beta = P(48.5 \le \overline{X} \le 51.5 \text{ when } \mu = 52)$$

The z-values corresponding to 48.5 and 51.5 when $\mu = 52$ are

$$z_1 = \frac{48.5 - 52}{0.79} = -4.43 \quad \text{and} \quad z_2 = \frac{51.5 - 52}{0.79} = -0.63$$

Therefore

$$\begin{aligned}\beta &= P(-4.43 \le Z \le -0.63) = P(Z \le -0.63) - P(Z \le -4.43)\\ &= 0.2643 - 0.0000 = 0.2643\end{aligned}$$

9-1 Hypothesis Testing

$$\beta = P(48.5 \le \overline{X} \le 51.5 \text{ when } \mu = 50.5)$$

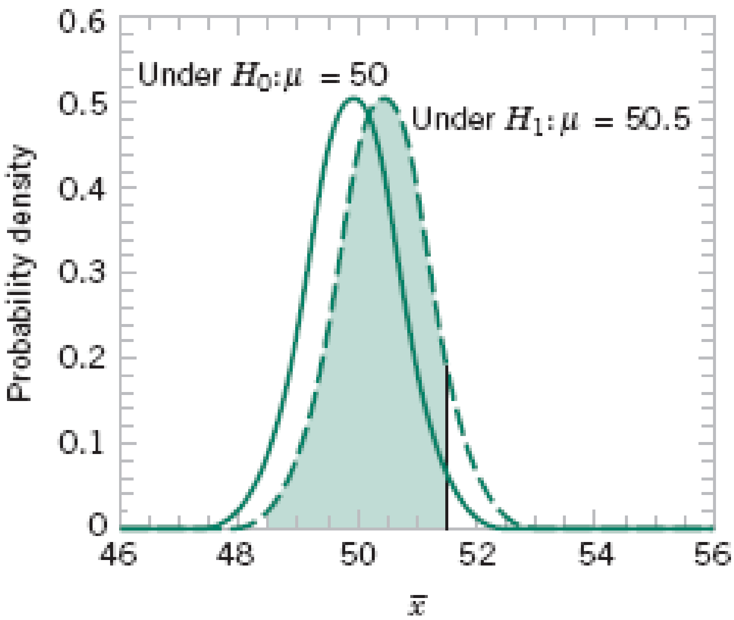


Figure 9-4 The probability of type II error when μ = 50.5 and n = 10.

9-1 Hypothesis Testing

$$\beta = P(48.5 \le \overline{X} \le 51.5 \text{ when } \mu = 50.5)$$

As shown in Fig. 9-4, the z-values corresponding to 48.5 and 51.5 when $\mu = 50.5$ are

$$z_1 = \frac{48.5 - 50.5}{0.79} = -2.53 \qquad \text{and} \qquad z_2 = \frac{51.5 - 50.5}{0.79} = 1.27$$

Therefore

$$\begin{aligned}\beta &= P(-2.53 \le Z \le 1.27) = P(Z \le 1.27) - P(Z \le -2.53)\\ &= 0.8980 - 0.0057 = 0.8923\end{aligned}$$

9-1 Hypothesis Testing

$$\beta = P(48.5 \le \bar{X} \le 51.5 \text{ when } \mu = 52)$$

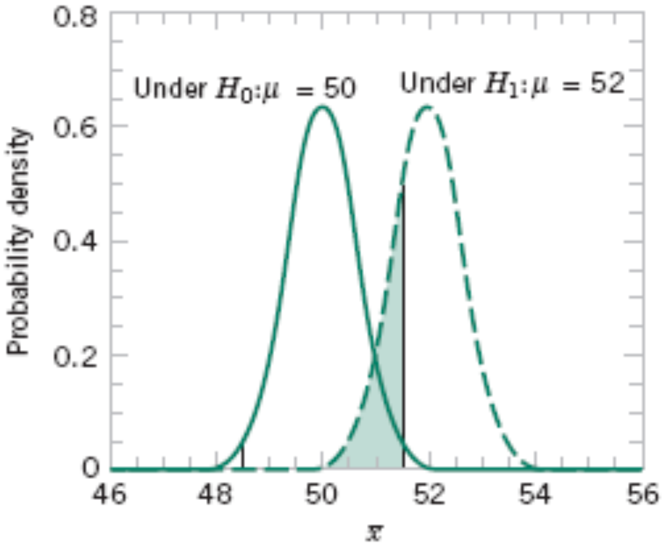


Figure 9-5 The probability of type II error when μ = 2 and n = 16.

9-1 Hypothesis Testing

$$\beta = P(48.5 \le \bar{X} \le 51.5 \text{ when } \mu = 52)$$

When $n = 16$, the standard deviation of $\bar{X}$ is $\sigma/\sqrt{n} = 2.5/\sqrt{16} = 0.625$, and the z-values corresponding to 48.5 and 51.5 when $\mu = 52$ are

$$z_1 = \frac{48.5 - 52}{0.625} = -5.60 \quad \text{and} \quad z_2 = \frac{51.5 - 52}{0.625} = -0.80$$

Therefore

$$\begin{aligned}\beta &= P(-5.60 \le Z \le -0.80) = P(Z \le -0.80) - P(Z \le -5.60) \\ &= 0.2119 - 0.0000 = 0.2119\end{aligned}$$

9-1 Hypothesis Testing

Acceptance Region	Sample Size	α	β at $\mu = 52$	β at $\mu = 50.5$
$48.5 < \bar{x} < 51.5$	10	0.0576	0.2643	0.8923
$48 < \bar{x} < 52$	10	0.0114	$\boxed{0.5000}$	$\boxed{0.9705}$
$48.5 < \bar{x} < 51.5$	16	0.0164	0.2119	$\boxed{0.9445}$
$48 < \bar{x} < 52$	16	$\boxed{0.0014}$	$\boxed{0.5000}$	$\boxed{0.9918}$

9-1 Hypothesis Testing

Definition

The **power** of a statistical test is the probability of rejecting the null hypothesis H_0 when the alternative hypothesis is true.

- The power is computed as 1 - β, and power can be interpreted as *the probability of correctly rejecting a false null hypothesis*. We often compare statistical tests by comparing their power properties.
- For example, consider the propellant burning rate problem when we are testing $H_0 : \mu = 50$ centimeters per second against $H_1 : \mu$ not equal 50 centimeters per second . Suppose that the true value of the mean is $\mu = 52$. When $n = 10$, we found that $\beta = 0.2643$, so the power of this test is $1 - \beta = 1 - 0.2643 = 0.7357$ when $\mu = 52$.

9-1 Hypothesis Testing

9-1.3 One-Sided and Two-Sided Hypotheses

Two-Sided Test:

$$H_0\colon \mu = \mu_0$$
$$H_1\colon \mu \neq \mu_0$$

One-Sided Tests:

$$H_0\colon \mu = \mu_0 \qquad\qquad H_0\colon \mu = \mu_0$$
$$\text{or}$$
$$H_1\colon \mu > \mu_0 \qquad\qquad H_1\colon \mu < \mu_0$$

9-1 Hypothesis Testing

Example 9-1

Consider the propellant burning rate problem. Suppose that if the burning rate is less than 50 centimeters per second, we wish to show this with a strong conclusion. The hypotheses should be stated as

$$H_0\colon \mu = 50 \text{ centimeters per second}$$
$$H_1\colon \mu < 50 \text{ centimeters per second}$$

Here the critical region lies in the lower tail of the distribution of $\overline{X}$. Since the rejection of H_0 is always a strong conclusion, this statement of the hypotheses will produce the desired outcome if H_0 is rejected. Notice that, although the null hypothesis is stated with an equal sign, it is understood to include any value of μ not specified by the alternative hypothesis. Therefore, failing to reject H_0 does not mean that $\mu = 50$ centimeters per second exactly, but only that we do not have strong evidence in support of H_1.

9-1 Hypothesis Testing

The bottler wants to be sure that the bottles meet the specification on mean internal pressure or bursting strength, which for 10-ounce bottles is a minimum strength of 200 psi. The bottler has decided to formulate the decision procedure for a specific lot of bottles as a hypothesis testing problem. There are two possible formulations for this problem, either

$$H_0: \mu = 200 \text{ psi} \qquad \text{or} \qquad H_0: \mu = 200 \text{ psi}$$
$$H_1: \mu > 200 \text{ psi} \qquad\qquad H_1: \mu < 200 \text{ psi}$$

9-1 Hypothesis Testing

9-1.4 P-Values in Hypothesis Tests

Definition

The ***P*-value** is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.

9-1 Hypothesis Testing

9-1.4 P-Values in Hypothesis Tests

Consider the two-sided hypothesis test for burning rate

$$H_0: \mu = 50 \quad H_1: \mu \neq 50$$

with $n = 16$ and $\sigma = 2.5$. Suppose that the observed sample mean is $\bar{x} = 51.3$ centimeters per second. Figure 9-6 shows a critical region for this test with critical values at 51.3 and the symmetric value 48.7. The P-value of the test is the α associated with this critical region. Any smaller value for α expands the critical region and the test fails to reject the null hypothesis when $\bar{x} = 51.3$. The P-value is easy to compute after the test statistic is observed. In this example

$$\begin{aligned} P\text{-value} &= 1 - P(48.7 < \bar{X} < 51.3) \\ &= 1 - P\left(\frac{48.7 - 50}{2.5/\sqrt{16}} < Z < \frac{51.3 - 50}{2.5/\sqrt{16}}\right) \\ &= 1 - P(-2.08 < Z < 2.08) \\ &= 1 - 0.962 = 0.038 \end{aligned}$$

9-1 Hypothesis Testing

9-1.4 P-Values in Hypothesis Tests

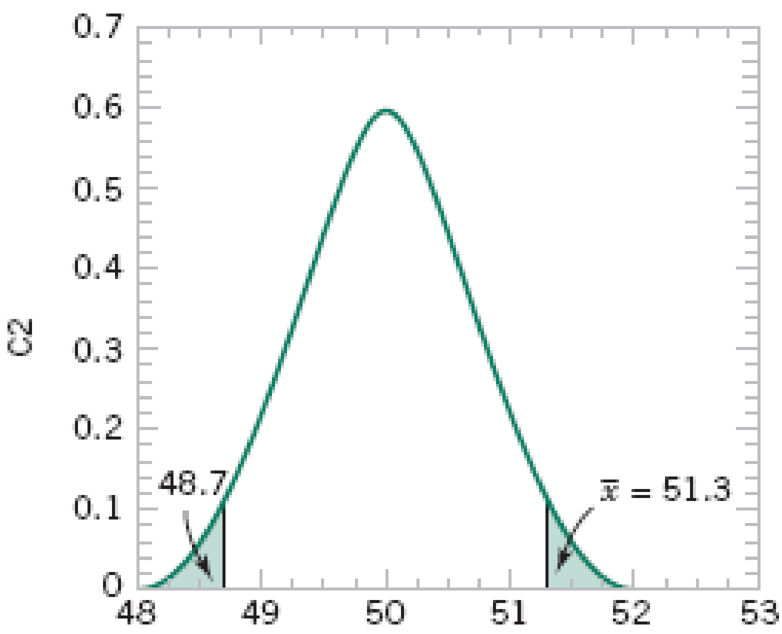


Figure 9-6 P-value is area of shaded region when $\bar{x} = 51.3$.

9-1 Hypothesis Testing

9-1.5 Connection between Hypothesis Tests and Confidence Intervals

There is a close relationship between the test of a hypothesis about any parameter, say θ, and the confidence interval for θ. If $[l, u]$ is a $100(1-\alpha)\%$ confidence interval for the parameter θ, the test of size α of the hypothesis

$$H_0\colon \theta = \theta_0$$
$$H_1\colon \theta \neq \theta_0$$

will lead to rejection of H_0 if and only if θ_0 is **not** in the $100(1-\alpha)\%$ CI $[l, u]$. As an illustration, consider the escape system propellant problem with $\bar{x} = 51.3$, $\sigma = 2.5$, and $n = 16$. The null hypothesis H_0: $\mu = 50$ was rejected, using $\alpha = 0.05$. The 95% two-sided CI on μ can be calculated using Equation 8-7. This CI is $51.3 \pm 1.96(2.5/\sqrt{16})$ and this is $50.075 \leq \mu \leq 52.525$. Because the value $\mu_0 = 50$ is not included in this interval, the null hypothesis H_0: $\mu = 50$ is rejected.

9-1 Hypothesis Testing

9-1.6 General Procedure for Hypothesis Tests

1. From the problem context, identify the parameter of interest.
2. State the null hypothesis, H_0 .
3. Specify an appropriate alternative hypothesis, H_1.
4. Choose a significance level, α.
5. Determine an appropriate tst statistic.
6. State the rejection region for the statistic.
7. Compute any necessary sample quantities, substitute these into the equation for the test statistic, and compute that value.
8. Decide whether or not H_0 should be rejected and report that in the problem context.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean

We wish to test:

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

The **test statistic** is:

$$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \qquad (9\text{-}8)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean

Reject H_0 if the observed value of the test statistic z_0 is either:

$$z_0 > z_{\alpha/2} \text{ or } z_0 < -z_{\alpha/2}$$

Fail to reject H_0 if

$$-z_{\alpha/2} < z_0 < z_{\alpha/2}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

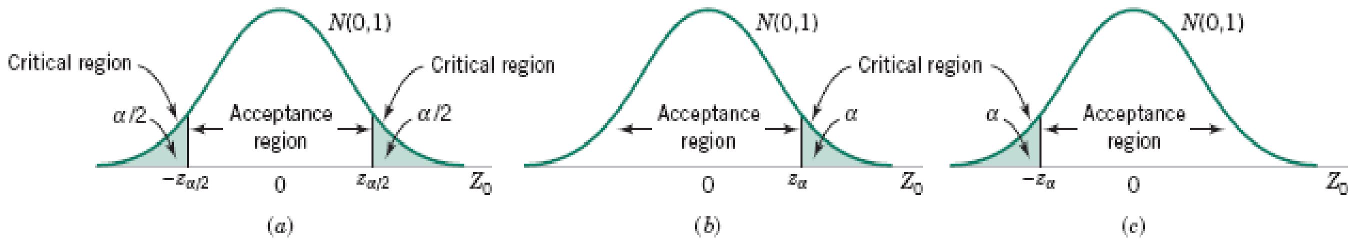


Figure 9-7 The distribution of Z_0 when H_0: $\mu = \mu_0$ is true, with critical region for (a) the two-sided alternative $H_1: \mu \neq \mu_0$, (b) the one-sided alternative $H_1: \mu > \mu_0$, and (c) the one-sided alternative $H_1: \mu < \mu_0$.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-2

Aircrew escape systems are powered by a solid propellant. The burning rate of this propellant is an important product characteristic. Specifications require that the mean burning rate must be 50 centimeters per second. We know that the standard deviation of burning rate is $\sigma = 2$ centimeters per second. The experimenter decides to specify a type I error probability or significance level of $\alpha = 0.05$ and selects a random sample of $n = 25$ and obtains a sample average burning rate of $\overline{x} = 51.3$ centimeters per second. What conclusions should be drawn?

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-2

We may solve this problem by following the eight-step procedure outlined in Section 9-1.4. This results in

1. The parameter of interest is μ, the mean burning rate.
2. H_0: $\mu = 50$ centimeters per second
3. H_1: $\mu \neq 50$ centimeters per second
4. $\alpha = 0.05$
5. The test statistic is

$$z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-2

6. Reject H_0 if $z_0 > 1.96$ or if $z_0 < -1.96$. Note that this results from step 4, where we specified $\alpha = 0.05$, and so the boundaries of the critical region are at $z_{0.025} = 1.96$ and $-z_{0.025} = -1.96$.
7. Computations: Since $\bar{x} = 51.3$ and $\sigma = 2$,

$$z_0 = \frac{51.3 - 50}{2/\sqrt{25}} = 3.25$$

8. Conclusion: Since $z_0 = 3.25 > 1.96$, we reject H_0: $\mu = 50$ at the 0.05 level of significance. Stated more completely, we conclude that the mean burning rate differs from 50 centimeters per second, based on a sample of 25 measurements. In fact, there is strong evidence that the mean burning rate exceeds 50 centimeters per second.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean

We may also develop procedures for testing hypotheses on the mean μ where the alternative hypothesis is one-sided. Suppose that we specify the hypotheses as

$$H_0\colon \mu = \mu_0 \qquad\qquad (9\text{-}11)$$
$$H_1\colon \mu > \mu_0$$

In defining the critical region for this test, we observe that a negative value of the test statistic Z_0 would never lead us to conclude that H_0: $\mu = \mu_0$ is false. Therefore, we would place the critical region in the **upper tail** of the standard normal distribution and reject H_0 if the computed value of z_0 is too large. That is, we would reject H_0 if

$$z_0 > z_\alpha \qquad\qquad (9\text{-}12)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean (Continued)

as shown in Figure 9-7(b). Similarly, to test

$$H_0\colon \mu = \mu_0$$
$$H_1\colon \mu < \mu_0 \qquad\qquad (9\text{-}13)$$

we would calculate the test statistic Z_0 and reject H_0 if the value of z_0 is too small. That is, the critical region is in the **lower tail** of the standard normal distribution as shown in Figure 9-7(c), and we reject H_0 if

$$z_0 < -z_\alpha \qquad\qquad (9\text{-}14)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.1 Hypothesis Tests on the Mean (Continued)

Null hypothesis: $H_0: \mu = \mu_0$

Test statistic: $$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}}$$

Alternative hypothesis	Rejection criteria
$H_1: \mu \neq \mu_0$	$z_0 > z_{\alpha/2,n-1}$ or $z_0 < -z_{\alpha/2,n-1}$
$H_1: \mu > \mu_0$	$z_0 > z_{\alpha,n-1}$
$H_1: \mu < \mu_0$	$z_0 < -z_{\alpha,n-1}$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

P-Values in Hypothesis Tests

The ***P*-value** is the smallest level of significance that would lead to rejection of the null hypothesis H_0 with the given data.

$$P = \begin{cases} 2[1 - \Phi(|z_0|)] & \text{for a two-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu \neq \mu_0 \\ 1 - \Phi(z_0) & \text{for a upper-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu > \mu_0 \\ \Phi(z_0) & \text{for a lower-tailed test: } H_0: \mu = \mu_0 \quad H_1: \mu < \mu_0 \end{cases} \tag{9-15}$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Finding the Probability of Type II Error β

Consider the two-sided hypothesis

$$H_0: \mu = \mu_0$$
$$H_1: \mu \neq \mu_0$$

Suppose that the null hypothesis is false and that the true value of the mean is $\mu = \mu_0 + \delta$, say, where $\delta > 0$. The test statistic Z_0 is

$$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} = \frac{\overline{X} - (\mu_0 + \delta)}{\sigma/\sqrt{n}} + \frac{\delta\sqrt{n}}{\sigma}$$

Therefore, the distribution of Z_0 when H_1 is true is

$$Z_0 \sim N\left(\frac{\delta\sqrt{n}}{\sigma}, 1\right) \qquad (9\text{-}16)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Finding the Probability of Type II Error β

$$\beta = \Phi\left(z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) - \Phi\left(-z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right) \qquad (9\text{-}17)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Finding the Probability of Type II Error β

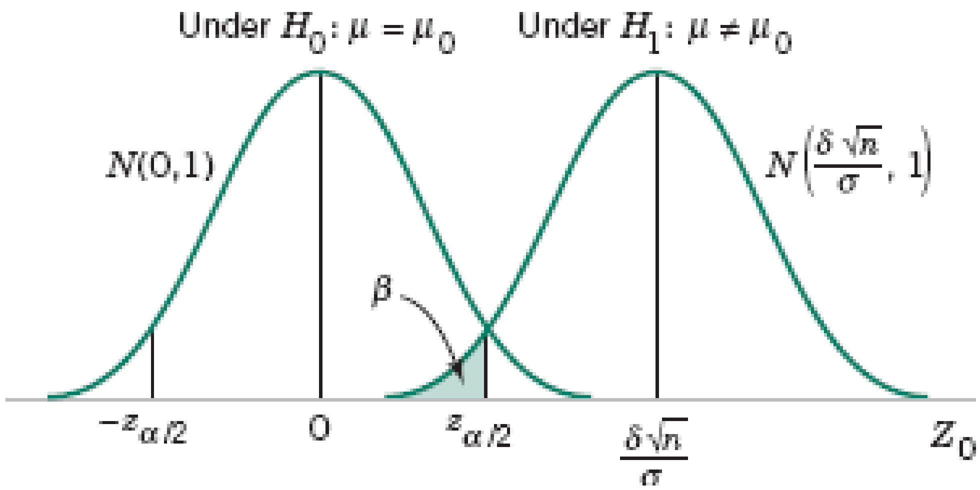


Figure 9-7 The distribution of Z_0 under H_0 and H_1

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Sample Size Formulas

For a two-sided alternative hypothesis:

$$n \simeq \frac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{\delta^2} \quad \text{where} \quad \delta = \mu - \mu_0 \qquad (9\text{-}19)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Sample Size Formulas

For a one-sided alternative hypothesis:

$$n = \frac{(z_\alpha + z_\beta)^2 \sigma^2}{\delta^2} \quad \text{where} \quad \delta = \mu - \mu_0 \qquad (9\text{-}20)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-3

Consider the rocket propellant problem of Example 9-2. Suppose that the analyst wishes to design the test so that if the true mean burning rate differs from 50 centimeters per second by as much as 1 centimeter per second, the test will detect this (i.e., reject H_0: $\mu = 50$) with a high probability, say 0.90. Now, we note that $\sigma = 2$, $\delta = 51 - 50 = 1$, $\alpha = 0.05$, and $\beta = 0.10$. Since $z_{\alpha/2} = z_{0.025} = 1.96$ and $z_\beta = z_{0.10} = 1.28$, the sample size required to detect this departure from H_0: $\mu = 50$ is found by Equation 9-19 as

$$n \simeq \frac{(z_{\alpha/2} + z_\beta)^2 \sigma^2}{\delta^2} = \frac{(1.96 + 1.28)^2 2^2}{(1)^2} \simeq 42$$

The approximation is good here, since $\Phi(-z_{\alpha/2} - \delta\sqrt{n}/\sigma) = \Phi(-1.96 - (1)\sqrt{42}/2) = \Phi(-5.20) \simeq 0$, which is small relative to β.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Using Operating Characteristic Curves

When performing sample size or type II error calculations, it is sometimes more convenient to use the **operating characteristic (OC) curves** in Appendix Charts VI*a* and VI*b*. These curves plot β as calculated from Equation 9-17 against a parameter d for various sample sizes n. Curves are provided for both $\alpha = 0.05$ and $\alpha = 0.01$. The parameter d is defined as

$$d = \frac{|\mu - \mu_0|}{\sigma} = \frac{|\delta|}{\sigma} \qquad (9\text{-}21)$$

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.2 Type II Error and Choice of Sample Size

Using Operating Characteristic Curves

so one set of operating characteristic curves can be used for all problems regardless of the values of μ_0 and σ. From examining the operating characteristic curves or Equation 9-17 and Fig. 9-7, we note that

1. The further the true value of the mean μ is from μ_0, the smaller the probability of type II error β for a given n and α. That is, we see that for a specified sample size and α, large differences in the mean are easier to detect than small ones.
2. For a given δ and α, the probability of type II error β decreases as n increases. That is, to detect a specified difference δ in the mean, we may make the test more powerful by increasing the sample size.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

Example 9-4

Consider the propellant problem in Example 9-2. Suppose that the analyst is concerned about the probability of type II error if the true mean burning rate is $\mu = 51$ centimeters per second. We may use the operating characteristic curves to find β. Note that $\delta = 51 - 50 = 1$, $n = 25$, $\sigma = 2$, and $\alpha = 0.05$. Then using Equation 9-21 gives

$$d = \frac{|\mu - \mu_0|}{\sigma} = \frac{|\delta|}{\sigma} = \frac{1}{2}$$

and from Appendix Chart VII*a*, with $n = 25$, we find that $\beta = 0.30$. That is, if the true mean burning rate is $\mu = 51$ centimeters per second, there is approximately a 30% chance that this will not be detected by the test with $n = 25$.

9-2 Tests on the Mean of a Normal Distribution, Variance Known

9-2.3 Large Sample Test

We have developed the test procedure for the null hypothesis H_0: $\mu = \mu_0$ assuming that the population is normally distributed and that σ^2 is known. In many if not most practical situations σ^2 will be unknown. Furthermore, we may not be certain that the population is well modeled by a normal distribution. In these situations if n is large (say $n > 40$) the sample standard deviation s can be substituted for σ in the test procedures with little effect. Thus, while we have given a test for the mean of a normal distribution with known σ^2, it can be easily converted into a **large-sample test procedure for unknown σ^2** that is valid regardless of the form of the distribution of the population. This large-sample test relies on the central limit theorem just as the large-sample confidence interval on μ that was presented in the previous chapter did. Exact treatment of the case where the population is normal, σ^2 is unknown, and n is small involves use of the t distribution and will be deferred until Section 9-3.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

9-3.1 Hypothesis Tests on the Mean

One-Sample *t*-Test

Null hypothesis: $H_0\colon \mu = \mu_0$

Test statistic: $$T_0 = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$

Alternative hypothesis	Rejection criteria
$H_1\colon \mu \neq \mu_0$	$t_0 > t_{\alpha/2,n-1}$ or $t_0 < -t_{\alpha/2,n-1}$
$H_1\colon \mu > \mu_0$	$t_0 > t_{\alpha,n-1}$
$H_1\colon \mu < \mu_0$	$t_0 < -t_{\alpha,n-1}$

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

9-3.1 Hypothesis Tests on the Mean

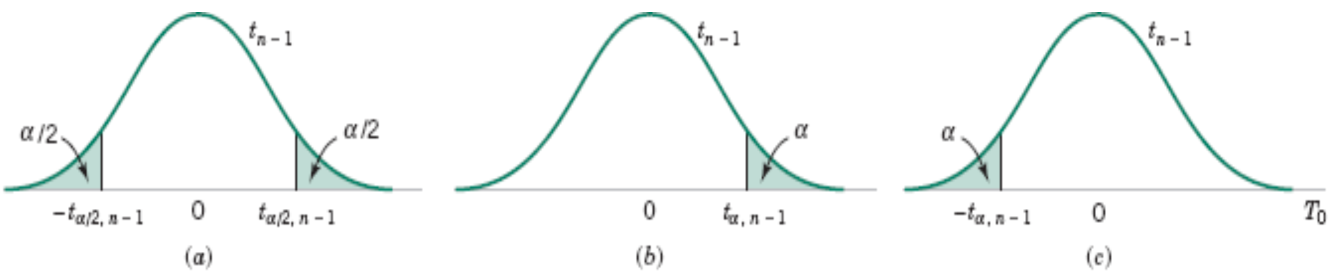


Figure 9-9 The reference distribution for $H_0\colon \mu = \mu_0$ with critical region for (a) $H_1\colon \mu \neq \mu_0$, (b) $H_1\colon \mu > \mu_0$, and (c) $H_1\colon \mu < \mu_0$.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

The increased availability of light materials with high strength has revolutionized the design and manufacture of golf clubs, particularly drivers. Clubs with hollow heads and very thin faces can result in much longer tee shots, especially for players of modest skills. This is due partly to the "spring-like effect" that the thin face imparts to the ball. Firing a golf ball at the head of the club and measuring the ratio of the outgoing velocity of the ball to the incoming velocity can quantify this spring-like effect. The ratio of velocities is called the coefficient of restitution of the club. An experiment was performed in which 15 drivers produced by a particular club maker were selected at random and their coefficients of restitution measured. In the experiment the golf balls were fired from an air cannon so that the incoming velocity and spin rate of the ball could be precisely controlled. It is of interest to determine if there is evidence (with $\alpha = 0.05$) to support a claim that the mean coefficient of restitution exceeds 0.82. The observations follow:

0.8411	0.8191	0.8182	0.8125	0.8750
0.8580	0.8532	0.8483	0.8276	0.7983
0.8042	0.8730	0.8282	0.8359	0.8660

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

The sample mean and sample standard deviation are $\bar{x} = 0.83725$ and $s = 0.02456$. The normal probability plot of the data in Fig. 9-9 supports the assumption that the coefficient of restitution is normally distributed. Since the objective of the experimenter is to demonstrate that the mean coefficient of restitution exceeds 0.82, a one-sided alternative hypothesis is appropriate.

The solution using the eight-step procedure for hypothesis testing is as follows:

1. The parameter of interest is the mean coefficient of restitution, μ.
2. H_0: $\mu = 0.82$
3. H_1: $\mu > 0.82$. We want to reject H_0 if the mean coefficient of restitution exceeds 0.82.
4. $\alpha = 0.05$
5. The test statistic is

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

6. Reject H_0 if $t_0 > t_{0.05,14} = 1.761$

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

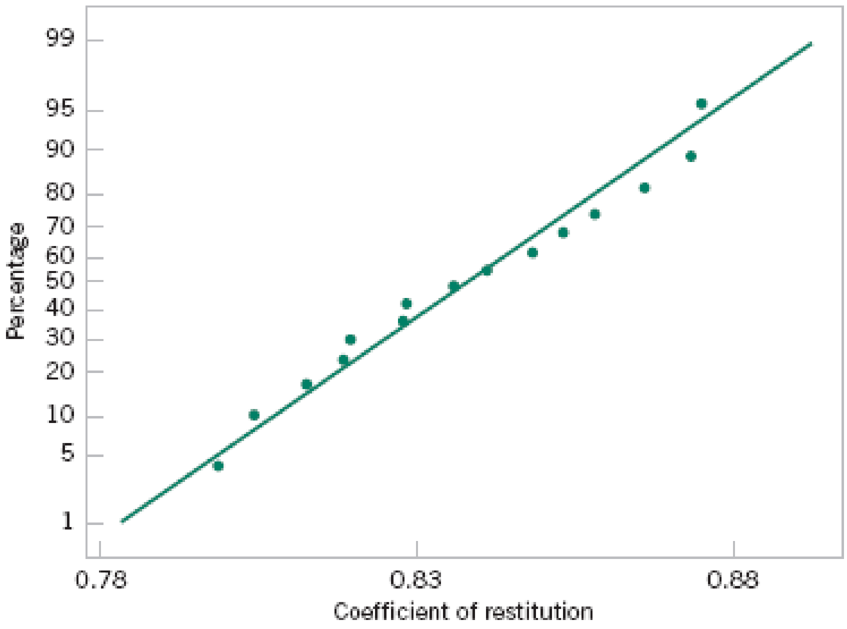


Figure 9-10 Normal probability plot of the coefficient of restitution data from Example 9-6.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-6

7. Computations: Since $\bar{x} = 0.83725$, $s = 0.02456$, $\mu_0 = 0.82$, and $n = 15$, we have

$$t_0 = \frac{0.83725 - 0.82}{0.02456/\sqrt{15}} = 2.72$$

8. Conclusions: Since $t_0 = 2.72 > 1.761$, we reject H_0 and conclude at the 0.05 level of significance that the mean coefficient of restitution exceeds 0.82.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

9-3.2 *P*-value for a *t*-Test

The *P*-value for a *t*-test is just the smallest level of significance at which the null hypothesis would be rejected.

To illustrate, consider the *t*-test based on 14 degrees of freedom in Example 9-6. The relevant critical values from Appendix Table IV are as follows:

Critical Value:	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
Tail Area:	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005

Notice that $t_0 = 2.72$ in Example 9-6, and that this is between two tabulated values, 2.624 and 2.977. Therefore, the *P*-value must be between 0.01 and 0.005. These are effectively the upper and lower bounds on the *P*-value.

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

9-3.3 Type II Error and Choice of Sample Size

The type II error of the two-sided alternative (for example) would be

$$\beta = P\{-t_{\alpha/2,n-1} \le T_0 \le t_{\alpha/2,n-1} \mid \delta \ne 0\}$$
$$= P\{-t_{\alpha/2,n-1} \le T_0' \le t_{\alpha/2,n-1}\}$$

9-3 Tests on the Mean of a Normal Distribution, Variance Unknown

Example 9-7

Consider the golf club testing problem from Example 9-6. If the mean coefficient of restitution exceeds 0.82 by as much as 0.02, is the sample size $n = 15$ adequate to ensure that H_0: $\mu = 0.82$ will be rejected with probability at least 0.8?

To solve this problem, we will use the sample standard deviation $s = 0.02456$ to estimate σ. Then $d = |\delta|/\sigma = 0.02/0.02456 = 0.81$. By referring to the operating characteristic curves in Appendix Chart VIIg (for $\alpha = 0.05$) with $d = 0.81$ and $n = 15$, we find that $\beta = 0.10$, approximately. Thus, the probability of rejecting H_0: $\mu = 0.82$ if the true mean exceeds this by 0.02 is approximately $1 - \beta = 1 - 0.10 = 0.90$, and we conclude that a sample size of $n = 15$ is adequate to provide the desired sensitivity.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

9-4.1 Hypothesis Test on the Variance

Suppose that we wish to test the hypothesis that the variance of a normal population σ^2 equals a specified value, say σ_0^2, or equivalently, that the standard deviation σ is equal to σ_0. Let X_1, $X_2, \ldots, X_n$ be a random sample of n observations from this population. To test

$$H_0: \sigma^2 = \sigma_0^2$$
$$H_1: \sigma^2 \neq \sigma_0^2 \qquad (9\text{-}26)$$

we will use the test statistic:

$$X_0^2 = \frac{(n-1)S^2}{\sigma_0^2} \qquad (9\text{-}27)$$

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

9-4.1 Hypothesis Test on the Variance

If the null hypothesis H_0: $\sigma^2 = \sigma_0^2$ is true, the test statistic X_0^2 defined in Equation 9-27 follows the chi-square distribution with $n - 1$ degrees of freedom. This is the reference distribution for this test procedure. Therefore, we calculate χ_0^2, the value of the test statistic X_0^2, and the null hypothesis H_0: $\sigma^2 = \sigma_0^2$ would be rejected if

$$\chi_0^2 > \chi^2_{\alpha/2,n-1} \qquad \text{or if} \quad \chi_0^2 < \chi^2_{1-\alpha/2,n-1}$$

where $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the upper and lower $100\alpha/2$ percentage points of the chi-square distribution with $n - 1$ degrees of freedom, respectively. Figure 9-10(a) shows the critical region.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

9-4.1 Hypothesis Test on the Variance

The same test statistic is used for one-sided alternative hypotheses. For the one-sided hypothesis

$$H_0\colon \sigma^2 = \sigma_0^2$$
$$H_1\colon \sigma^2 > \sigma_0^2 \tag{9-28}$$

we would reject H_0 if $\chi_0^2 > \chi^2_{\alpha,n-1}$, whereas for the other one-sided hypothesis

$$H_0\colon \sigma^2 = \sigma_0^2$$
$$H_1\colon \sigma^2 < \sigma_0^2 \tag{9-29}$$

we would reject H_0 if $\chi_0^2 < \chi^2_{1-\alpha,n-1}$. The one-sided critical regions are shown in Figure 9-10(b) and (c).

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

9-4.1 Hypothesis Test on the Variance

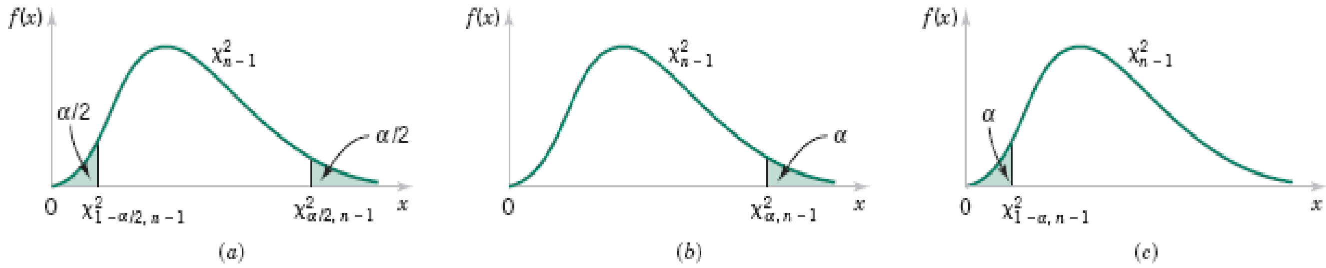


Figure 9-11 Reference distribution for the test of H_0: $\sigma^2 = \sigma_0^2$ with critical region values for (a) H_1: $\sigma^2 \neq \sigma_0^2$, (b) H_1: $\sigma^2 > \sigma_0^2$, and (c) H_1: $\sigma^2 < \sigma_0^2$.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

Example 9-8

An automatic filling machine is used to fill bottles with liquid detergent. A random sample of 20 bottles results in a sample variance of fill volume of $s^2 = 0.0153$ (fluid ounces)^2. If the variance of fill volume exceeds 0.01 (fluid ounces)^2, an unacceptable proportion of bottles will be underfilled or overfilled. Is there evidence in the sample data to suggest that the manufacturer has a problem with underfilled or overfilled bottles? Use $\alpha = 0.05$, and assume that fill volume has a normal distribution.

Using the eight-step procedure results in the following:

1. The parameter of interest is the population variance σ^2.
2. H_0: $\sigma^2 = 0.01$
3. H_1: $\sigma^2 > 0.01$
4. $\alpha = 0.05$
5. The test statistic is

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2}$$

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

Example 9-8

6. Reject H_0 if $\chi_0^2 > \chi_{0.05,19}^2 = 30.14$.
7. Computations:

$$\chi_0^2 = \frac{19(0.0153)}{0.01} = 29.07$$

8. Conclusions: Since $\chi_0^2 = 29.07 < \chi_{0.05,19}^2 = 30.14$, we conclude that there is no strong evidence that the variance of fill volume exceeds 0.01 (fluid ounces)^2.

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

9-4.2 Type II Error and Choice of Sample Size

For the two-sided alternative hypothesis:

$$\lambda = \frac{\sigma}{\sigma_0}$$

Operating characteristic curves are provided in Charts VI*i* and VI*j*:

9-4 Hypothesis Tests on the Variance and Standard Deviation of a Normal Distribution

Example 9-9

Consider the bottle-filling problem from Example 9-8. If the variance of the filling process exceeds 0.01 (fluid ounces)^2, too many bottles will be underfilled. Thus, the hypothesized value of the standard deviation is $\sigma_0 = 0.10$. Suppose that if the true standard deviation of the filling process exceeds this value by 25%, we would like to detect this with probability at least 0.8. Is the sample size of $n = 20$ adequate?

To solve this problem, note that we require

$$\lambda = \frac{\sigma}{\sigma_0} = \frac{0.125}{0.10} = 1.25$$

This is the abscissa parameter for Chart VII*k*. From this chart, with $n = 20$ and $\lambda = 1.25$, we find that $\beta \simeq 0.6$. Therefore, there is only about a 40% chance that the null hypothesis will be rejected if the true standard deviation is really as large as $\sigma = 0.125$ fluid ounce.

To reduce the β-error, a larger sample size must be used. From the operating characteristic curve with $\beta = 0.20$ and $\lambda = 1.25$, we find that $n = 75$, approximately. Thus, if we want the test to perform as required above, the sample size must be at least 75 bottles.

9-5 Tests on a Population Proportion

9-5.1 Large-Sample Tests on a Proportion

Many engineering decision problems include hypothesis testing about *p*.

$$H_0: p = p_0$$
$$H_1: p \neq p_0$$

An appropriate **test statistic** is

$$Z_0 = \frac{X - np_0}{\sqrt{np_0(1 - p_0)}} \qquad (9\text{-}32)$$

and reject $H_0: p = p_0$ if

$$z_0 > z_{\alpha/2} \quad \text{or} \quad z_0 < -z_{\alpha/2}$$

9-5 Tests on a Population Proportion

Example 9-10

A semiconductor manufacturer produces controllers used in automobile engine applications. The customer requires that the process fallout or fraction defective at a critical manufacturing step not exceed 0.05 and that the manufacturer demonstrate process capability at this level of quality using $\alpha = 0.05$. The semiconductor manufacturer takes a random sample of 200 devices and finds that four of them are defective. Can the manufacturer demonstrate process capability for the customer?

We may solve this problem using the eight-step hypothesis-testing procedure as follows:

1. The parameter of interest is the process fraction defective p.
2. H_0: $p = 0.05$
3. H_1: $p < 0.05$
   This formulation of the problem will allow the manufacturer to make a strong claim about process capability if the null hypothesis H_0: $p = 0.05$ is rejected.
4. $\alpha = 0.05$

9-5 Tests on a Population Proportion

Example 9-10

5. The test statistic is (from Equation 9-32)

$$z_0 = \frac{x - np_0}{\sqrt{np_0(1 - p_0)}}$$

   where $x = 4$, $n = 200$, and $p_0 = 0.05$.
6. Reject H_0: $p = 0.05$ if $z_0 < -z_{0.05} = -1.645$
7. Computations: The test statistic is

$$z_0 = \frac{4 - 200(0.05)}{\sqrt{200(0.05)(0.95)}} = -1.95$$

8. Conclusions: Since $z_0 = -1.95 < -z_{0.05} = -1.645$, we reject H_0 and conclude that the process fraction defective p is less than 0.05. The P-value for this value of the test statistic z_0 is $P = 0.0256$, which is less than $\alpha = 0.05$. We conclude that the process is capable.

9-5 Tests on a Population Proportion

Another form of the test statistic Z_0 is

$$Z_0 = \frac{X/n - p_0}{\sqrt{p_0(1 - p_0)/n}} \quad \text{or} \quad Z_0 = \frac{\hat{P} - p_0}{\sqrt{p_0(1 - p_0)/n}}$$

9-5 Tests on a Population Proportion

9-5.2 Type II Error and Choice of Sample Size

For a two-sided alternative

$$\beta = \Phi\left(\frac{p_0 - p + z_{\alpha/2}\sqrt{p_0(1 - p_0)/n}}{\sqrt{p(1 - p)/n}}\right) - \Phi\left(\frac{p_0 - p - z_{\alpha/2}\sqrt{p_0(1 - p_0)/n}}{\sqrt{p(1 - p)/n}}\right) \quad (9\text{-}34)$$

If the alternative is $p < p_0$

$$\beta = 1 - \Phi\left(\frac{p_0 - p - z_{\alpha}\sqrt{p_0(1 - p_0)/n}}{\sqrt{p(1 - p)/n}}\right) \quad (9\text{-}35)$$

If the alternative is $p > p_0$

$$\beta = \Phi\left(\frac{p_0 - p + z_{\alpha}\sqrt{p_0(1 - p_0)/n}}{\sqrt{p(1 - p)/n}}\right) \quad (9\text{-}36)$$

9-5 Tests on a Population Proportion

9-5.3 Type II Error and Choice of Sample Size

For a two-sided alternative

$$n = \left[\frac{z_{\alpha/2}\sqrt{p_0(1-p_0)} + z_\beta \sqrt{p(1-p)}}{p - p_0} \right]^2 \quad (9\text{-}37)$$

For a one-sided alternative

$$n = \left[\frac{z_{\alpha}\sqrt{p_0(1-p_0)} + z_\beta \sqrt{p(1-p)}}{p - p_0} \right]^2 \quad (9\text{-}38)$$

9-5 Tests on a Population Proportion

Example 9-11

Consider the semiconductor manufacturer from Example 9-10. Suppose that its process fallout is really $p = 0.03$. What is the β-error for a test of process capability that uses $n = 200$ and $\alpha = 0.05$?

The β-error can be computed using Equation 9-35 as follows:

$$\beta = 1 - \Phi\left[\frac{0.05 - 0.03 - (1.645)\sqrt{0.05(0.95)/200}}{\sqrt{0.03(1-0.03)/200}} \right] = 1 - \Phi(-0.44) = 0.67$$

Thus, the probability is about 0.7 that the semiconductor manufacturer will fail to conclude that the process is capable if the true process fraction defective is $p = 0.03$ (3%). That is, the power of the test against this particular alternative is only about 0.3. This appears to be a large β-error (or small power), but the difference between $p = 0.05$ and $p = 0.03$ is fairly small, and the sample size $n = 200$ is not particularly large.

9-5 Tests on a Population Proportion

Example 9-11

Suppose that the semiconductor manufacturer was willing to accept a β-error as large as 0.10 if the true value of the process fraction defective was $p = 0.03$. If the manufacturer continues to use $\alpha = 0.05$, what sample size would be required?

The required sample size can be computed from Equation 9-38 as follows:

$$n = \left[\frac{1.645\sqrt{0.05(0.95)} + 1.28\sqrt{0.03(0.97)}}{0.03 - 0.05}\right]^2$$
$$\simeq 832$$

where we have used $p = 0.03$ in Equation 9-38. Note that $n = 832$ is a very large sample size. However, we are trying to detect a fairly small deviation from the null value $p_0 = 0.05$.

IMPORTANT TERMS AND CONCEPTS

- α and β
- Connection between hypothesis tests and confidence intervals
- Critical region for a test statistic
- Hypothesis test
- Inference
- Null distribution
- Null hypothesis
- One- and two-sided alternative hypotheses
- Operating characteristic (OC) curves
- Power of a test
- *P*-value
- Reference distribution for a test statistic
- Sample size determination for hypothesis tests
- Significance level of a test
- Statistical hypotheses
- Statistical versus practical significance
- Test for goodness of fit
- Test for homogeneity
- Test for independence
- Test statistic
- Type I and type II errors

CHAPTER 13

SIMPLE LINEAR REGRESSION

13.1 Simple Linear Regression

- Simple Regression
- Linear Regression

Definition

A regression model is a mathematical equation that describes the relationship between two or more variables. A ***simple regression*** model includes only two variables: one independent and one dependent. The dependent variable is the one being explained, and the independent variable is the one used to explain the variation in the dependent variable.

Linear Regression

Definition

A (simple) regression model that gives a straight-line relationship between two variables is called a ***linear regression*** model.

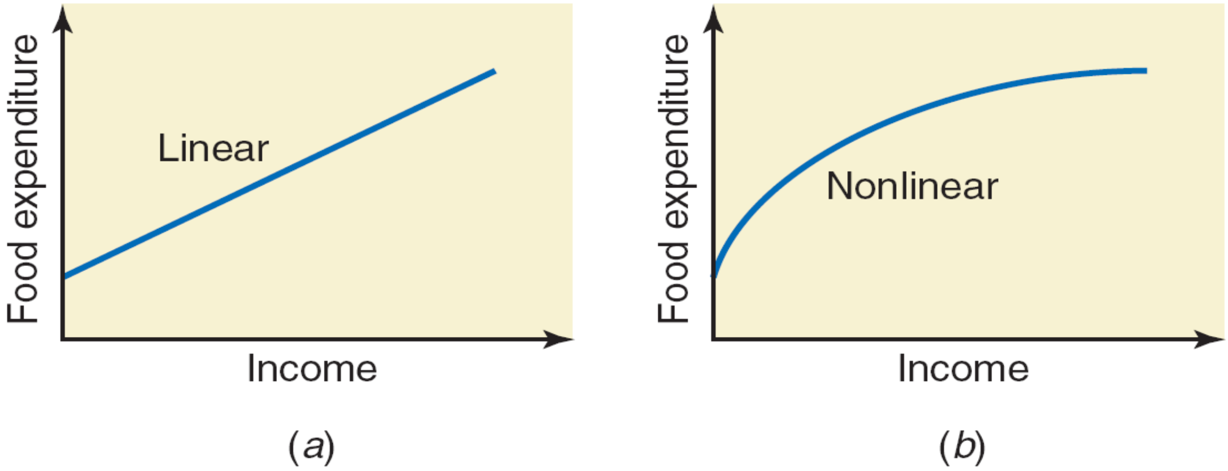


Figure 13.2 Plotting a linear equation.

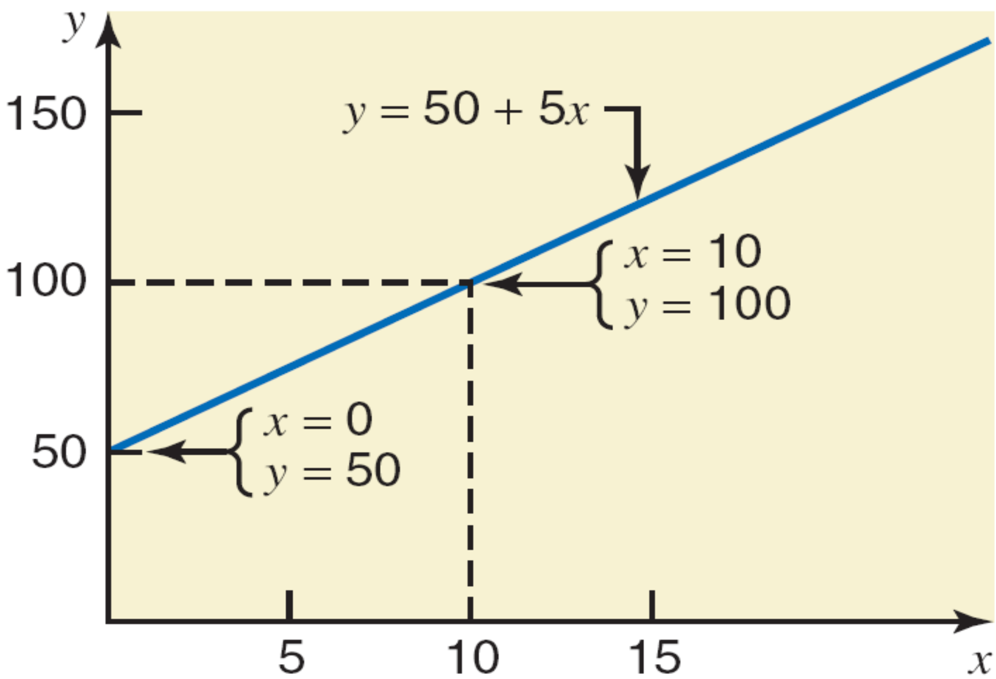
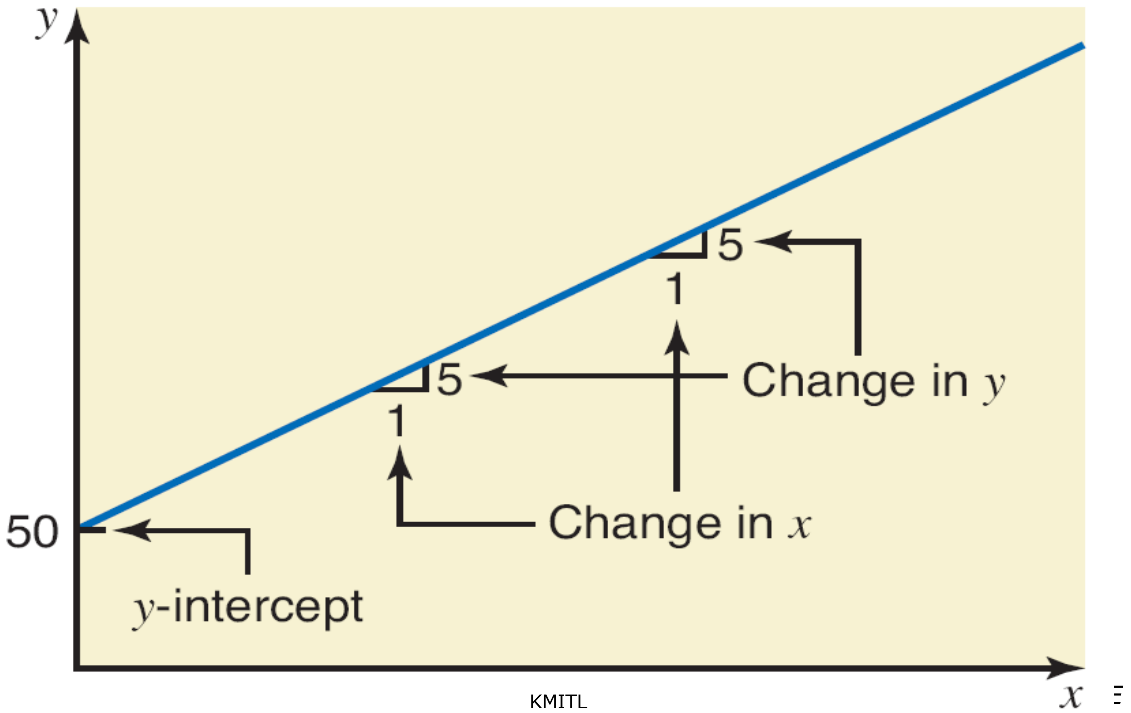


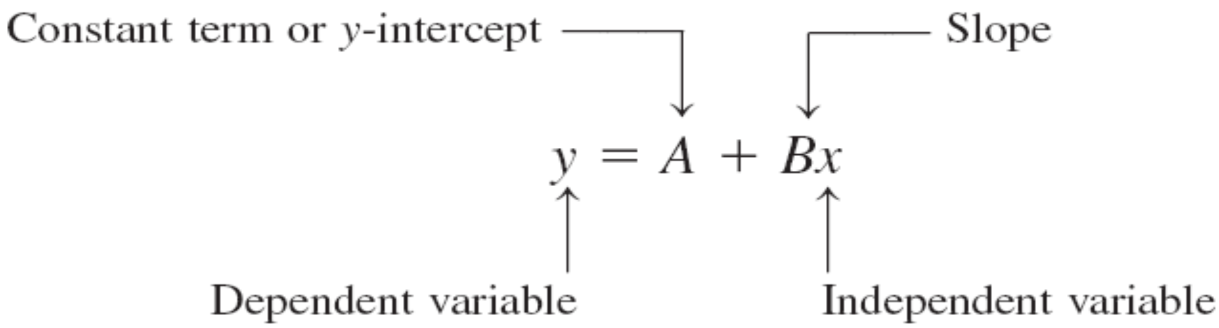
Figure 13.3 y-intercept and slope of a line.



SIMPLE LINEAR REGRESSION ANALYSIS

Definition

In the ***regression model*** $y = A + Bx + \varepsilon$, A is called the y-intercept or constant term, B is the slope, and ε is the random error term. The dependent and independent variables are y and x, respectively.



SIMPLE LINEAR REGRESSION ANALYSIS

Definition

In the model $\hat{y} = a + bx$, *a* and *b*, which are calculated using sample data, are called the ***estimates of A and B***, respectively.

Table 13.1 Incomes (in hundreds of dollars) and Food Expenditures of Seven Households

Income	Food Expenditure
55	14
83	24
38	13
61	16
33	9
49	15
67	17

Scatter Diagram

Definition

A plot of paired observations is called a ***scatter diagram***.

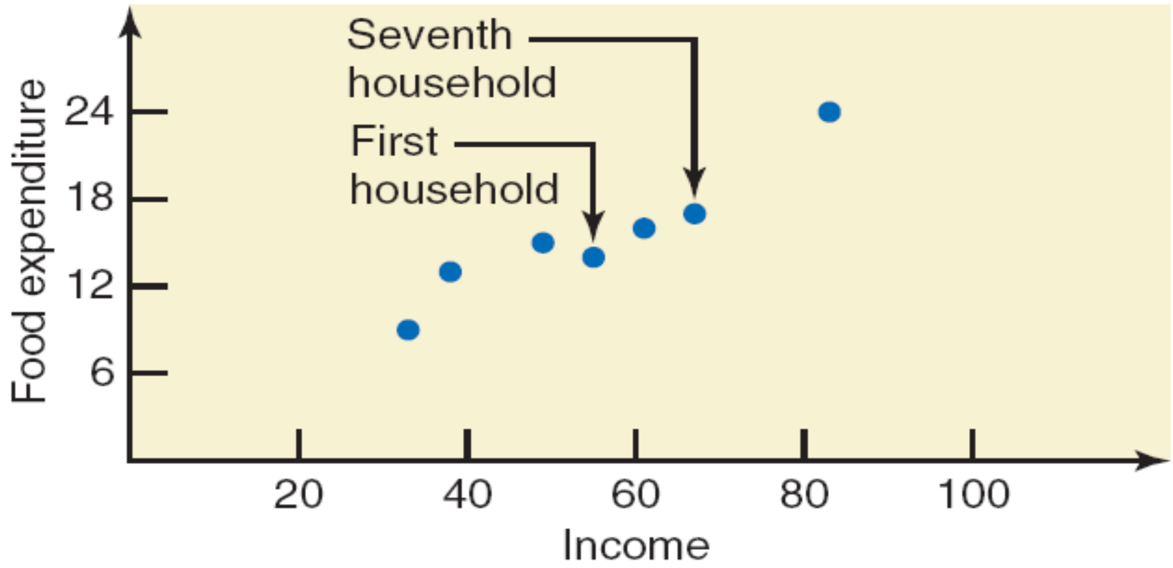


Figure 13.5 Scatter diagram and straight lines.

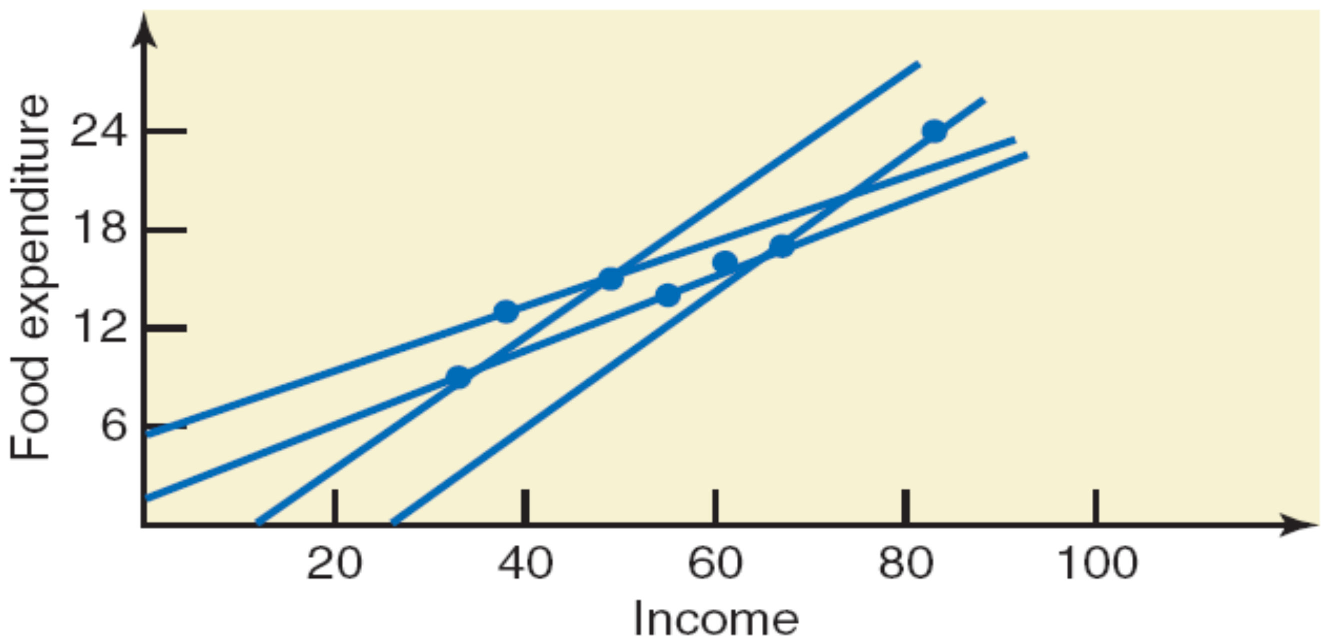
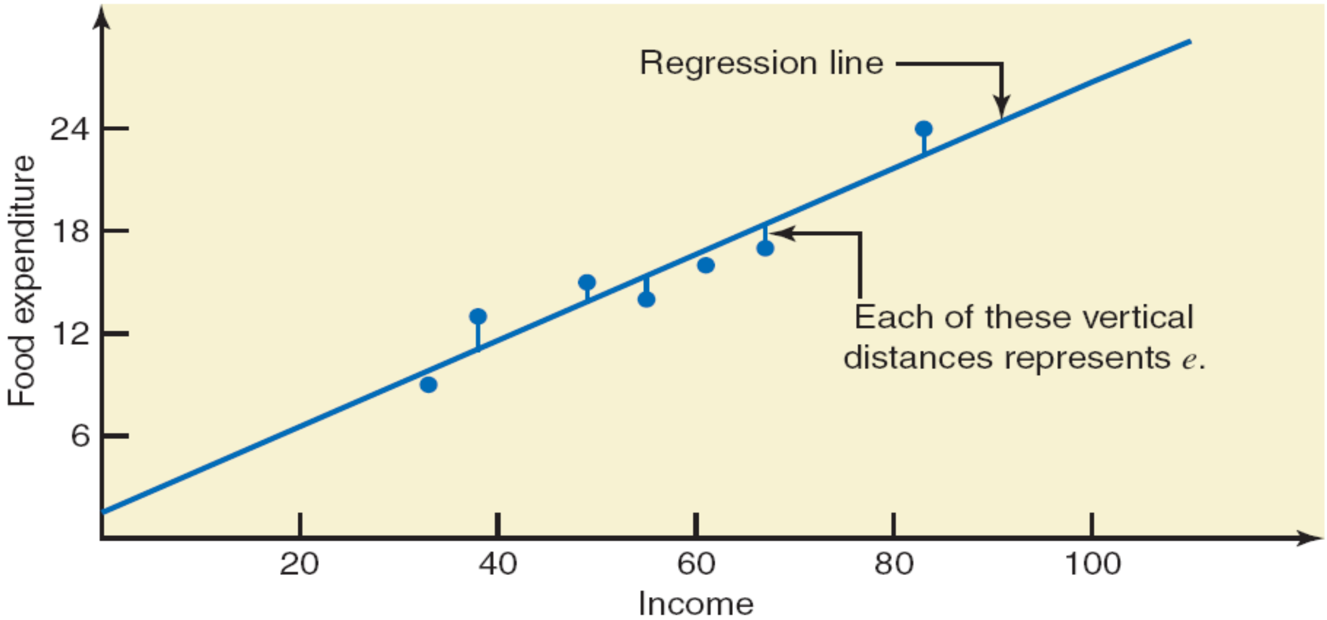


Figure 13.6 Regression Line and random errors.



Error Sum of Squares (SSE)

The ***error sum of squares***, denoted SSE, is

$$\mathbf{SSE} = \sum \mathbf{e}^2 = \sum (y - \hat{y})^2$$

The values of *a* and *b* that give the minimum SSE are called the ***least square estimates*** of A and B, and the regression line obtained with these estimates is called the ***least squares line***.

The Least Squares Line

For the least squares regression line $\hat{y} = a + bx$,

$$b = \frac{SS_{xy}}{SS_{xx}} \quad \text{and} \quad a = \bar{y} - b\bar{x}$$

where

$$SS_{xy} = \sum xy - \frac{(\sum x)(\sum y)}{n} \quad \text{and} \quad SS_{xx} = \sum x^2 - \frac{(\sum x)^2}{n}$$

and SS stands for "sum of squares." The least squares regression line $\hat{y} = a + bx$ is also called the regression of y on x.

Example 13-1

Find the least squares regression line for the data on incomes and food expenditure on the seven households given in the Table 13.1. Use income as an independent variable and food expenditure as a dependent variable.

Table 13.2

Income x	Food Expenditure y	xy	x^2
55	14	770	3025
83	24	1992	6889
38	13	494	1444
61	16	976	3721
33	9	297	1089
49	15	735	2401
67	17	1139	4489
$\Sigma x = 386$	$\Sigma y = 108$	$\Sigma xy = 6403$	$\Sigma x^2 = 23{,}058$

Example 13-1: Solution

$$\sum x = 386 \qquad \sum y = 108$$

$$\bar{x} = \sum x / n = 386/7 = 55.1429$$

$$\bar{y} = \sum y / n = 108/7 = 15.4286$$

$$SS_{xy} = \sum xy - \frac{\left(\sum x\right)\left(\sum y\right)}{n} = 6403 - \frac{(386)(108)}{7} = 447.5714$$

$$SS_{xx} = \sum x^2 - \frac{\left(\sum x\right)^2}{n} = 23{,}058 - \frac{(386)^2}{7} = 1772.8571$$

Example 13-1: Solution

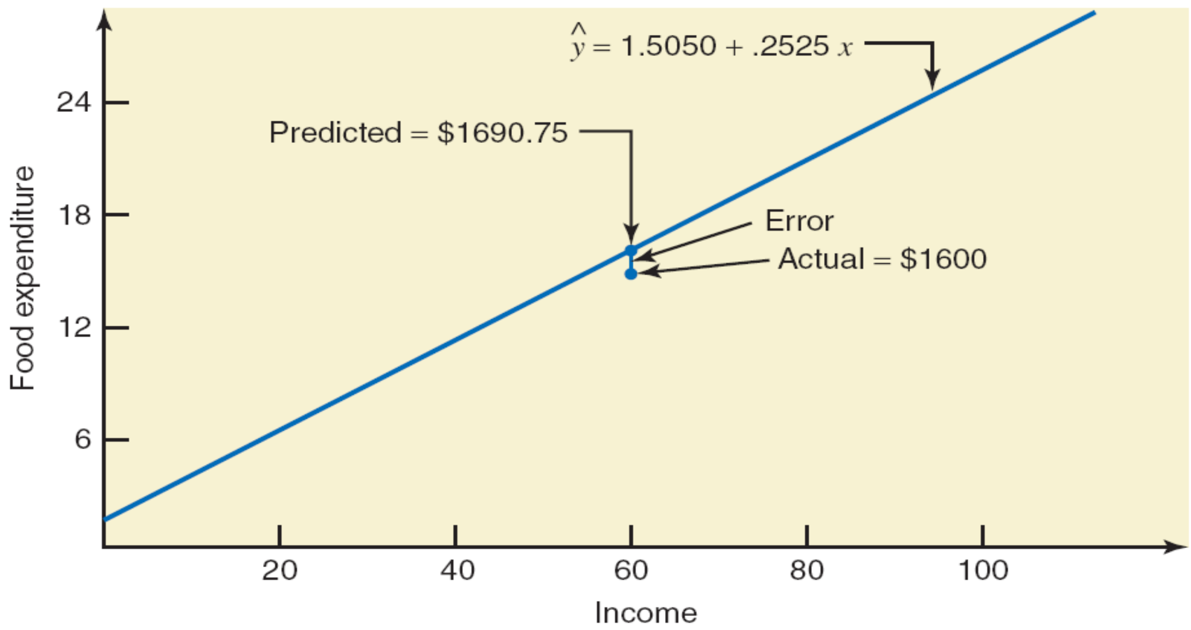
$$b = \frac{SS_{xy}}{SS_{xx}} = \frac{447.5714}{1772.8571} = .2525$$

$$a = \bar{y} - b\bar{x} = 15.4286 - (.2525)(55.1429) = 1.5050$$

Thus, our estimated regression model is

$$\hat{y} = 1.5050 + .2525\,x$$

Figure 13.7 Error of prediction.



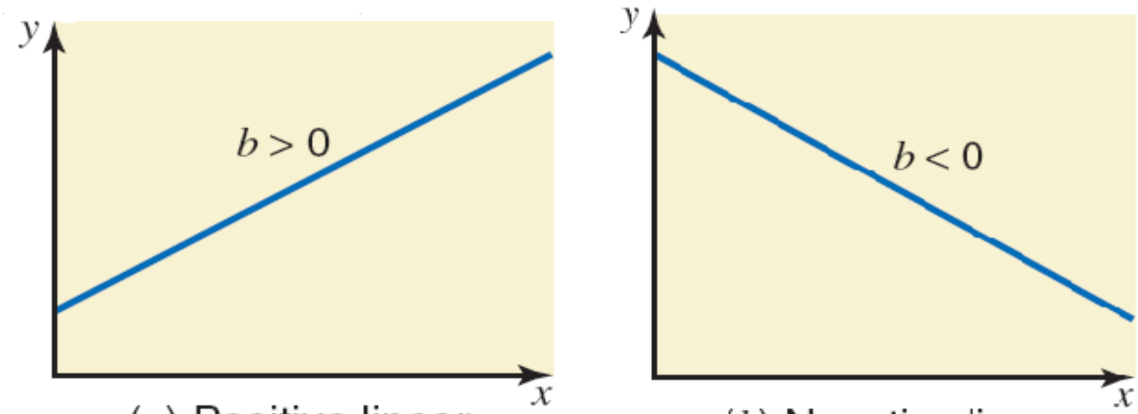
Interpretation of *a*

- Consider a household with zero income. Using the estimated regression line obtained in Example 13-1,
  - $\hat{y}$ = 1.5050 + .2525(0) = \$1.5050 hundred.
- Thus, we can state that a household with no income is expected to spend \$150.50 per month on food.
- The regression line is valid only for the values of x between 33 and 83.

Interpretation of *b*

- The value of b in the regression model gives the change in y (dependent variable) due to a change of one unit in x (independent variable).
- We can state that, on average, a \$100 (or \$1) increase in income of a household will increase the food expenditure by \$25.25 (or \$.2525).

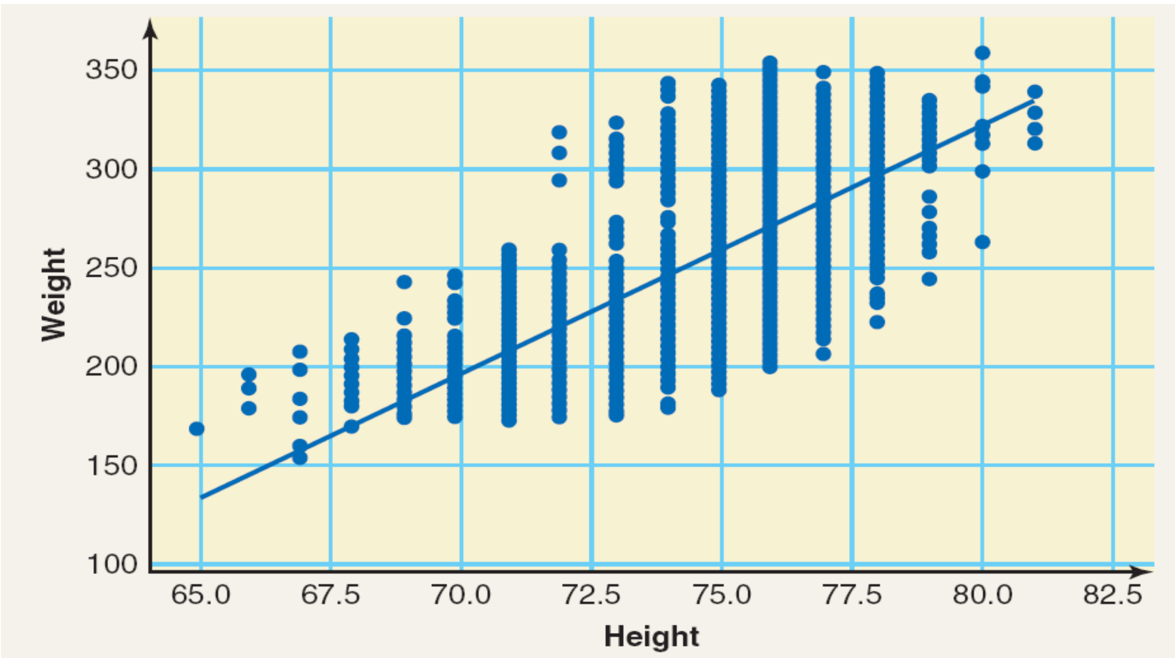
Figure 13.8 Positive and negative linear relationships between x and y.



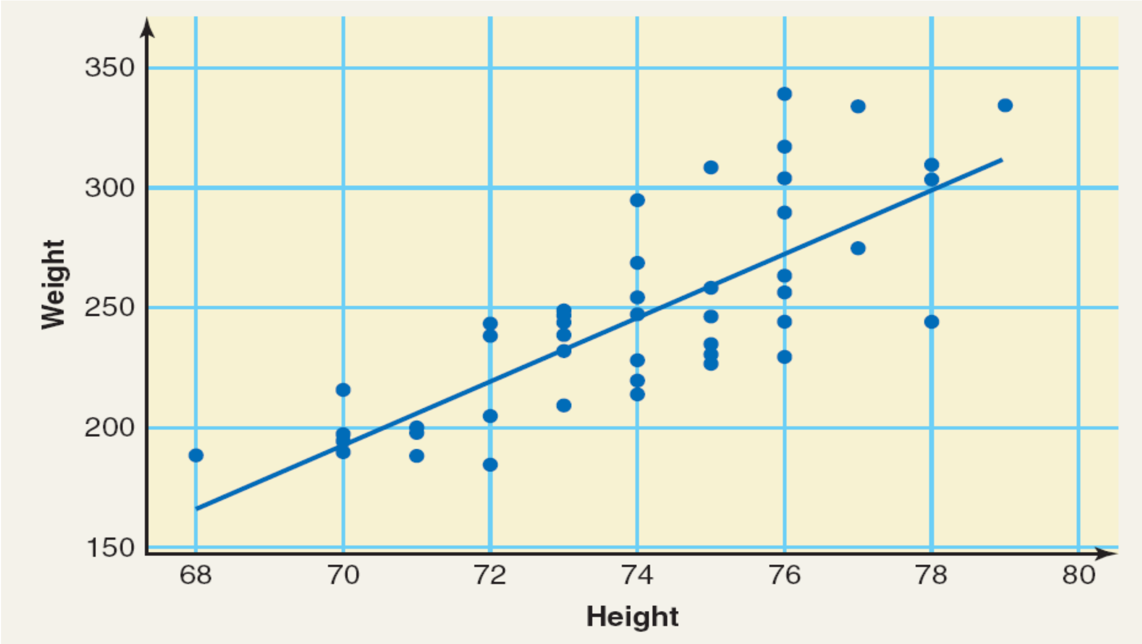
(*a*) Positive linear relationship

(*b*) Negative linear relationship

Case Study 13-1 Regression of Weights on Heights for NFL Players



Case Study 13-1 Regression of Weights on Heights for NFL Players



Assumptions of the Regression Model

Assumption 1: The random error term Є has a mean equal to zero for each x

Assumption 2: The errors associated with different observations are independent

Assumption 3: For any given x, the distribution of errors is normal

Assumption 4: The distribution of population errors for each x has the same (constant) standard deviation, which is denoted σ_ϵ

Figure 13.11 (a) Errors for households with an income of $4000 per month.

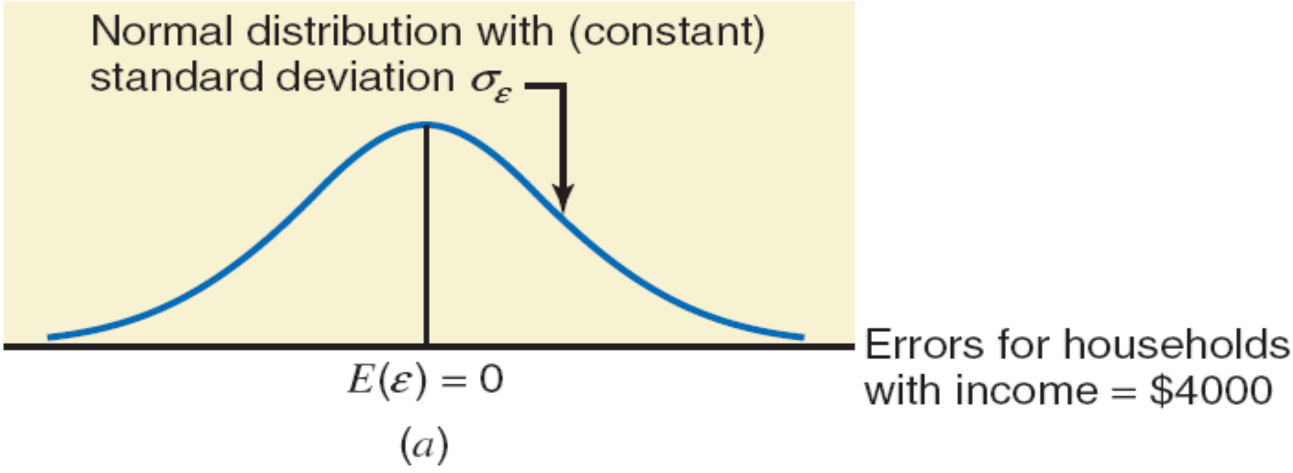


Figure 13.11 (b) Errors for households with an income of $ 7500 per month.

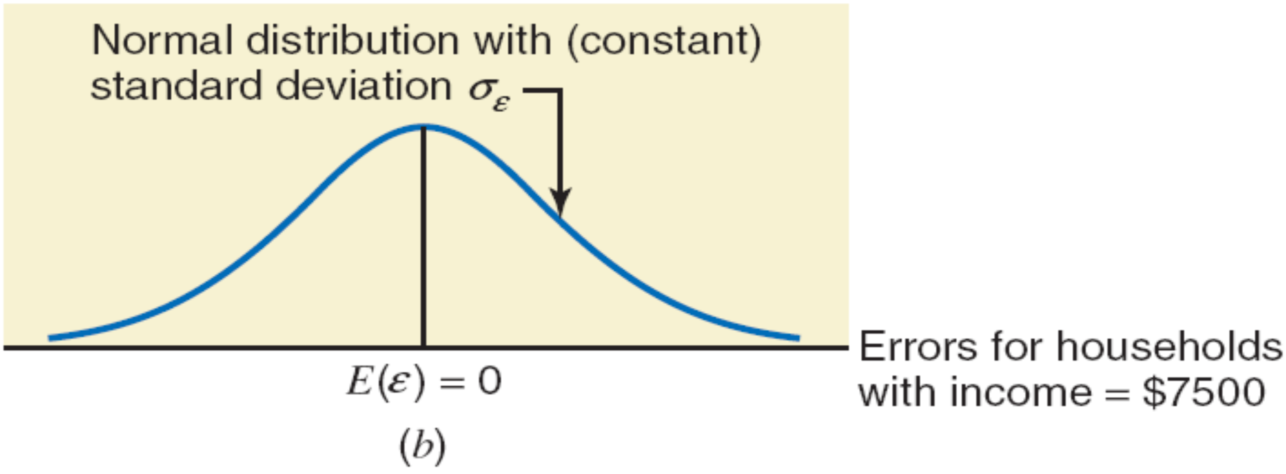


Figure 13.12 Distribution of errors around the population regression line.

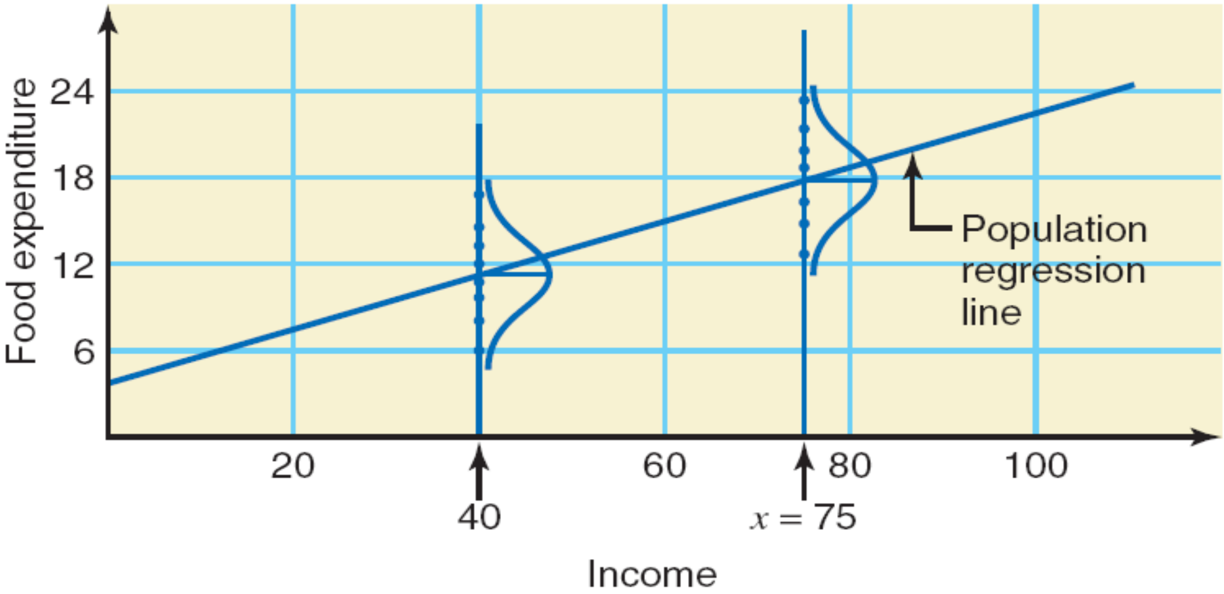
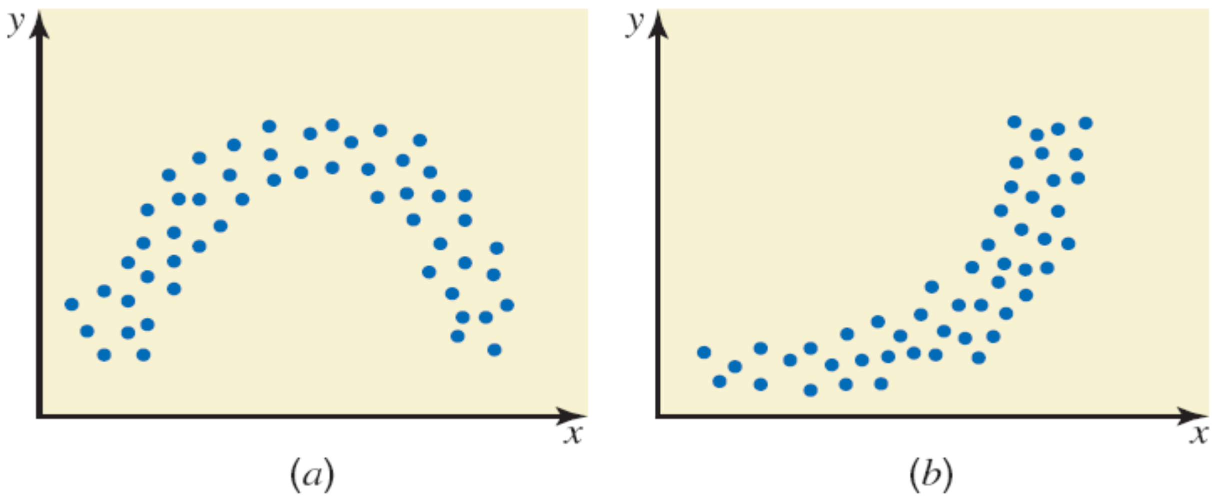
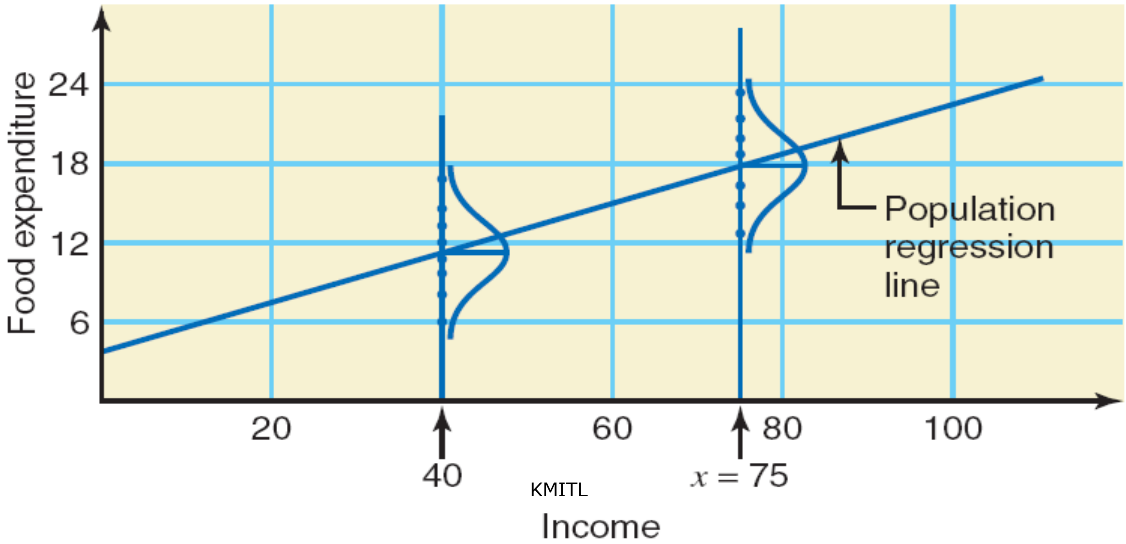


Figure 13.13 Nonlinear relations between x and y.



13.2 Standard Deviation of Errors and Coefficient of Determination

Degrees of Freedom for a Simple Linear Regression Model The *degrees of freedom* for a simple linear regression model are $df = n - 2$



STANDARD DEVIATION OF ERRORS AND COEFFICIENT OF DETERMINATION

The *standard deviation of errors* is calculated as

$$s_e = \sqrt{\frac{SS_{yy} - bSS_{xy}}{n-2}}$$

where

$$SS_{yy} = \sum y^2 - \frac{(\sum y)^2}{n}$$

Example 13-2

Compute the standard deviation of errors s_e for the data on monthly incomes and food expenditures of the seven households given in Table 13.1.

Income x	Food Expenditure y	y^2
55	14	196
83	24	576
38	13	169
61	16	256
33	9	81
49	15	225
67	17	289
$\Sigma x = 386$	$\Sigma y = 108$	$\Sigma y^2 = 1792$

Example 13-2: Solution

$$SS_{yy} = \sum y^2 - \frac{\left(\sum y\right)^2}{n} = 1792 - \frac{(108)^2}{7} = 125.7143$$

$$s_e = \sqrt{\frac{SS_{yy} - bSS_{xy}}{n-2}} \sqrt{\frac{125.7143 - .2525(447.5714)}{7-2}} = 1.5939$$

COEFFICIENT OF DETERMINATION

Total Sum of Squares (SST)

The ***total sum of squares***, denoted by **SST**, is calculated as

$$SST = \sum y^2 - \frac{\left(\sum y\right)^2}{n}$$

Note that this is the same formula that we used to calculate SS_{yy}.

Figure 13.15 Total errors.

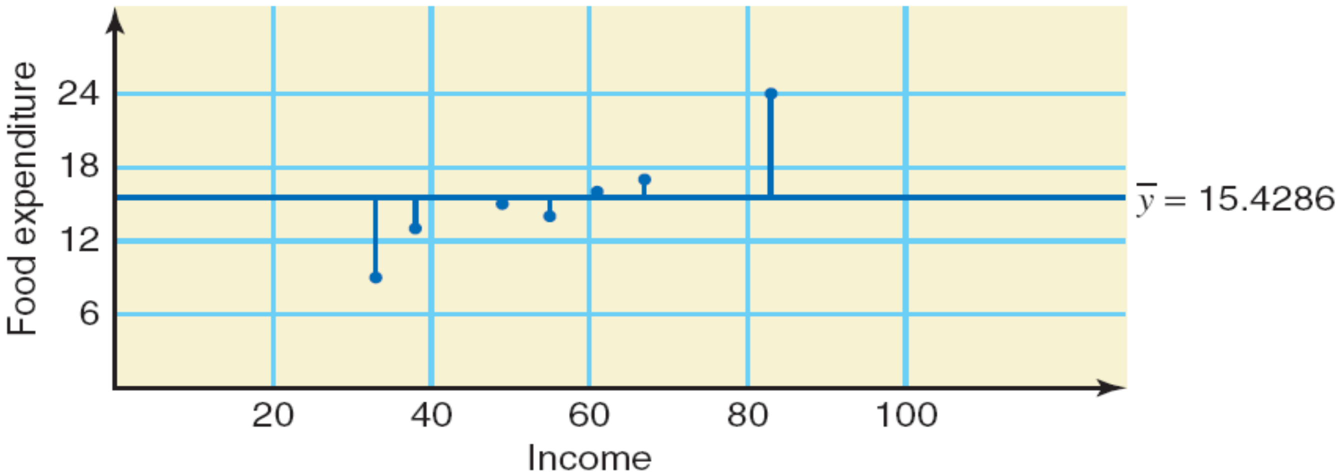
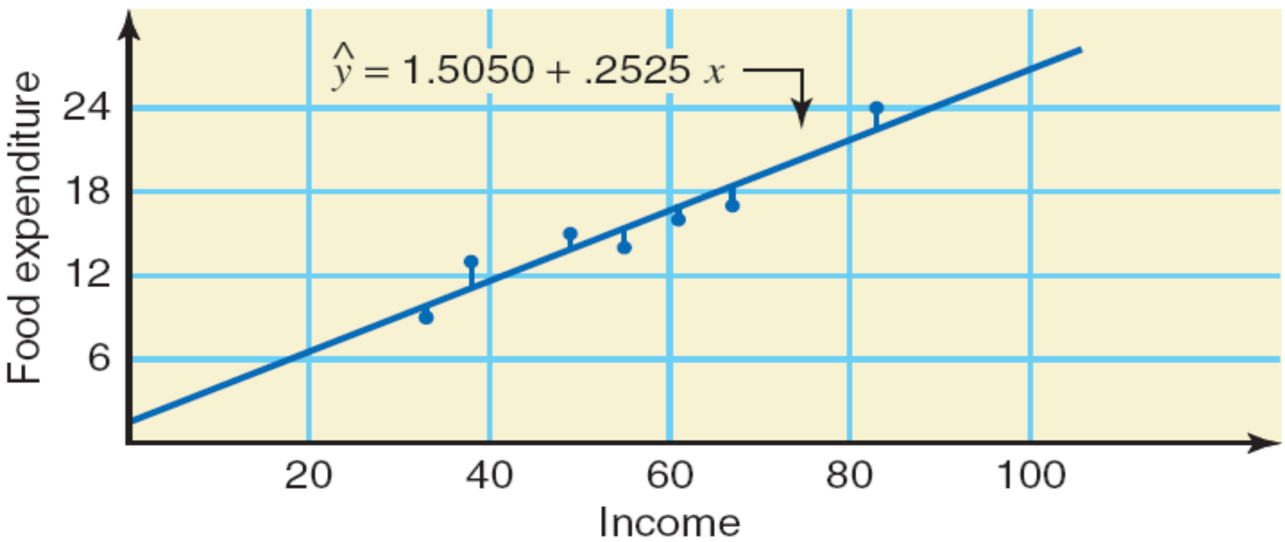


Table 13.4

x	y	$\hat{y} = 1.5050 + .2525x$	$e = y - \hat{y}$	$e^2 = (y - \hat{y})^2$
55	14	15.3925	−1.3925	1.9391
83	24	22.4625	1.5375	2.3639
38	13	11.1000	1.9000	3.6100
61	16	16.9075	−.9075	.8236
33	9	9.8375	−.8375	.7014
49	15	13.8775	1.1225	1.2600
67	17	18.4225	−1.4225	2.0235
				$\Sigma e^2 = \Sigma(y - \hat{y})^2 = 12.7215$

Figure 13.16 Errors of prediction when regression model is used.



COEFFICIENT OF DETERMINATION

Regression Sum of Squares (SSR)

The ***regression sum of squares*** , denoted by **SSR**, is

$$SSR = SST - SSE$$

COEFFICIENT OF DETERMINATION

Coefficient of Determination

The ***coefficient of determination***, denoted by r^2, represents the proportion of SST that is explained by the use of the regression model. The computational formula for r^2 is

$$r^2 = \frac{b\ SS_{xy}}{SS_{yy}}$$

and $0 \leq r^2 \leq 1$

Example 13-3

For the data of Table 13.1 on monthly incomes and food expenditures of seven households, calculate the coefficient of determination.

- From earlier calculations made in Examples 13-1 and 13-2,
- **b = .2525, SS_{xx} = 447.5714, SS_{yy} = 125.7143**

$$r^2 = \frac{b\,SS_{xy}}{SS_{yy}} = \frac{(.2525)(447.5714)}{125.7143} = .90$$

13.4 Linear Correlation

- Linear Correlation Coefficient
- Hypothesis Testing About the Linear Correlation Coefficient

Value of the Correlation Coefficient

The ***value of the correlation coefficient*** always lies in the range of –1 to 1; that is,

$$-1 \leq \rho \leq 1 \quad \text{and} \quad -1 \leq r \leq 1$$

Figure 13.18 Linear correlation between two variables.

(a) Perfect positive linear correlation, r = 1

(b) (b) Perfect negative linear correlation, r = -1

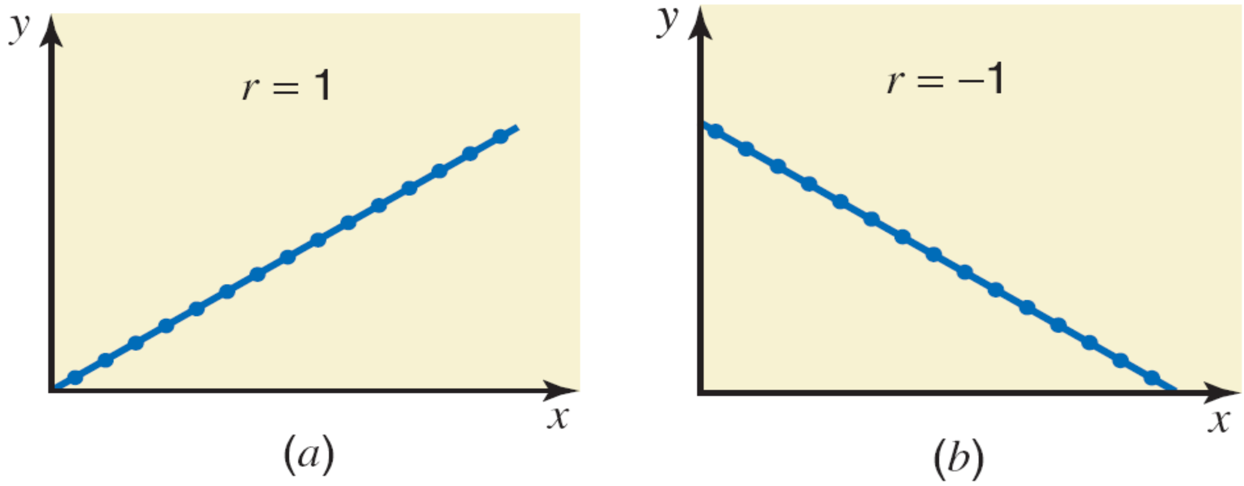


Figure 13.18 Linear correlation between two variables.

(c) No linear correlation, , $r \approx$ 0

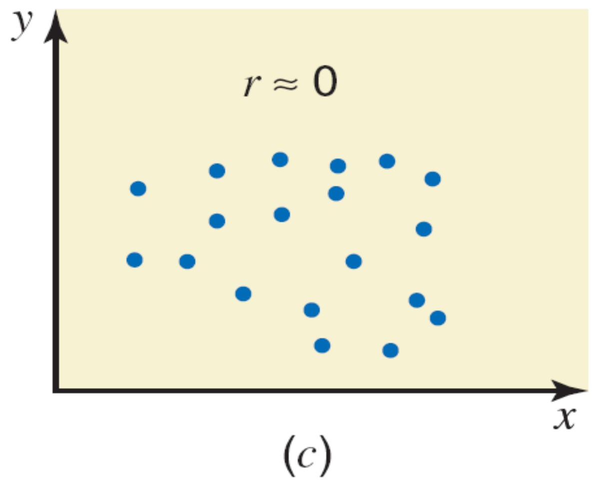
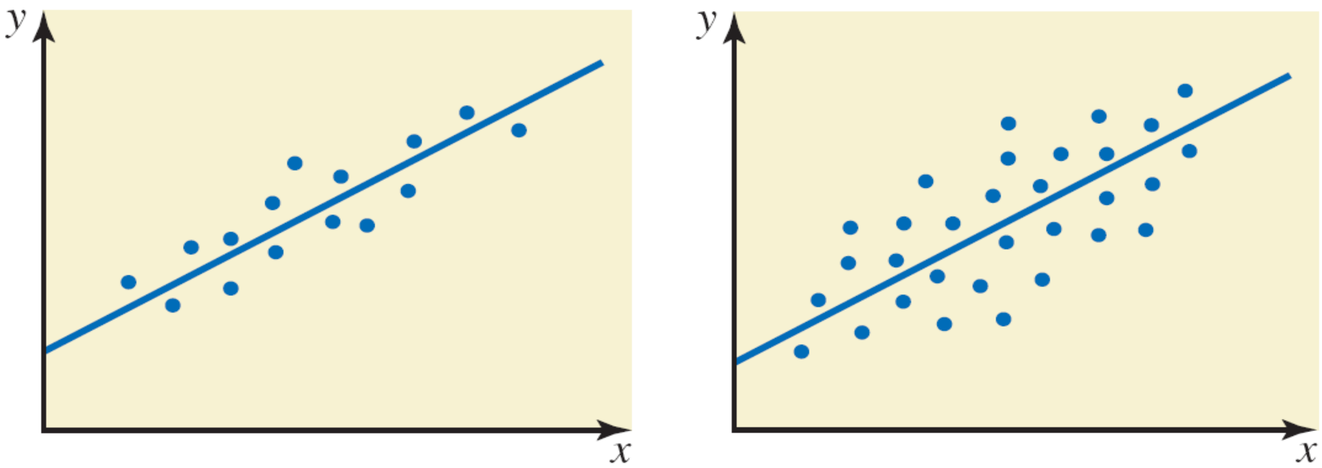


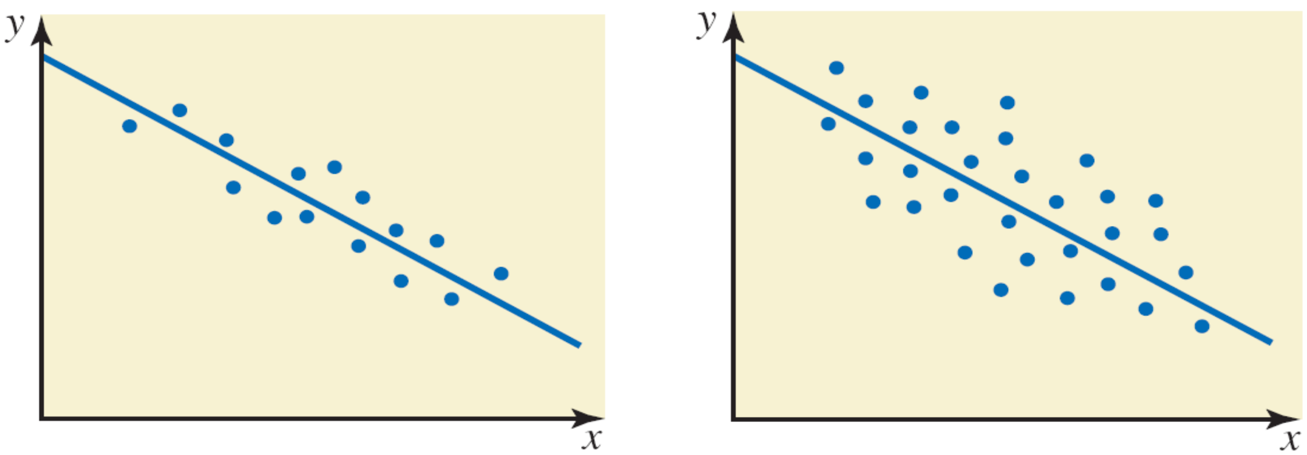
Figure 13.19 Linear correlation between variables.



(*a*) Strong positive linear correlatio (*r* is close to 1)

(*b*) Weak positive linear correlation (*r* is positive but close to zero)

Figure 13.19 Linear correlation between variables.



(*c*) Strong negative linear correlation (*r* is close to –1)

(*d*) Weak negative linear correlation (*r* is negative and close to zero)

Linear Correlation Coefficient

Linear Correlation Coefficient

The ***simple linear correlation coefficient***, denoted by *r*, measures the strength of the linear relationship between two variables for a sample and is calculated as

$$r = \frac{SS_{xy}}{\sqrt{SS_{xx}SS_{yy}}}$$

Example 13-6

Calculate the correlation coefficient for the example on incomes and food expenditures of seven households.

$$r = \frac{SS_{xy}}{\sqrt{SS_{xx}SS_{yy}}}$$

$$= \frac{447.5714}{\sqrt{(1772.8571)(125.7143)}} = .95$$

Hypothesis Testing About the Linear Correlation Coefficient

Test Statistic for r

If both variables are normally distributed and the null hypothesis is H_0: $\rho = 0$, then the value of the test statistic t is calculated as

$$t = r\sqrt{\frac{n-2}{1-r^2}}$$

Here n – 2 are the degrees of freedom.

Example 13-7

Using the 1% level of significance and the data from Example 13-1, test whether the linear correlation coefficient between incomes and food expenditures is positive. Assume that the populations of both variables are normally distributed.

Example 13-7: Solution

- **Step 1:**

  H_0: ρ = 0 (The linear correlation coefficient is zero)
  H_1: ρ > 0 (The linear correlation coefficient is positive)

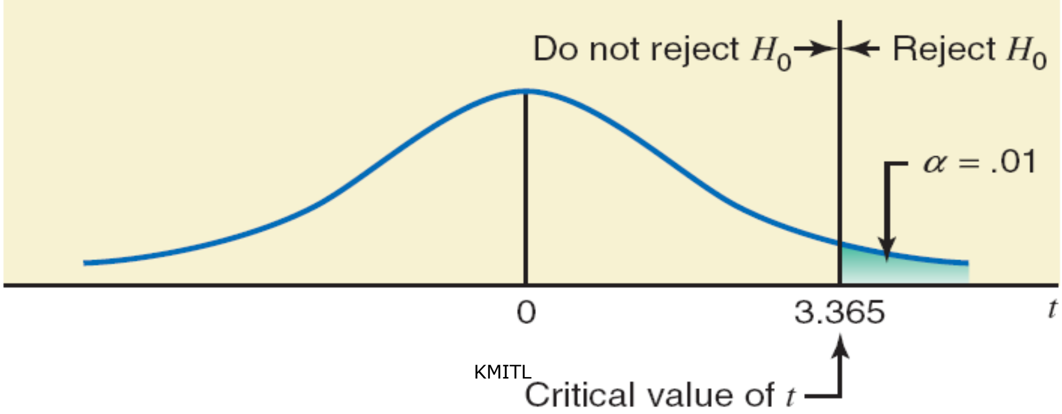
- **Step 2:**

  The population distributions for both variables are normally distributed. Hence, we can use the *t* distribution to perform this test about the linear correlation coefficient.

Example 13-7: Solution

- **Step 3:**

  Area in the right tail = .01
  $df = n - 2 = 7 - 2 = 5$
  The critical value of t = 3.365



Example 13-7: Solution

- **Step 4:**

$$t = r\sqrt{\frac{n-2}{1-r^2}}$$

$$t = 0.9481\sqrt{\frac{7-2}{1-(0.9481)^2}}$$

$$= 6.667$$

Example 13-7: Solution

- **Step 5:**

  The value of the test statistic t = 6.667
  - It is greater than the critical value of $t=3.365$
  - It falls in the rejection region

  Hence, we reject the null hypothesis.

  We conclude that there is a positive relationship between incomes and food expenditures.