

Gaussian Elimination

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Opening Remarks

• Solving Linear Systems of Equation

Opening Remarks

- Solving Linear Systems of Equation
 - Let L be a linear transformation such that
 - $L\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$ and $L\left(\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$
 - $-L\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) = ?$

Robert van de Geijn and Maggie Myers, Linear Algebra - Foundations to Frontiers, <https://www.edx.org/>

Homework 2.4.1.2 Let L be a linear transformation such that

$$L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and} \quad L\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

$$\text{Then } L\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) =$$

Opening Remarks

- Solving Linear Systems of Equation

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = ? \begin{pmatrix} 1 \\ 0 \end{pmatrix} + ? \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \chi_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \chi_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Opening Remarks

• Solving Linear Systems of Equation

For the next three exercises, let L be a linear transformation such that

$$L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and} \quad L\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

Homework 2.4.1.5 $L\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) =$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \chi_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \chi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Opening Remarks

- Solving Linear Systems of Equation

— Let L be a linear transformation such that

$$\bullet L \begin{pmatrix} (1) \\ (2) \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \text{ and } L \begin{pmatrix} (1) \\ (3) \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

$$-L \begin{pmatrix} (-1) \\ 0 \end{pmatrix} = ?$$

Opening Remarks

- Solving Linear Systems of Equation

Homework 2.4.1.2 Let L be a linear transformation such that
 $L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and $L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$.
Then $L\begin{pmatrix} 2 \\ 3 \end{pmatrix} = ?$

Robert van de Geijn and Maggie Myers, Linear Algebra - Foundations to Frontiers, <https://www.edx.org/>

2

Opening Remarks

- Solving Linear Systems of Equation

$$\begin{array}{rcl} \chi_0 & + & \chi_1 = -1 \\ 2\chi_0 & + & 3\chi_1 = 0 \end{array}$$

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 1 & -1 \\ 2 & 3 & 0 \end{array} \right)$$

Outline

- Solving system of linear equation
- Representing system of linear equation as appended matrices
- Reducing matrices to row echelon form
- LU factorization

Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System
- Appended Matrices
- Gauss Transforms
- Computing Separately with the Matrix and Right-Hand Side (Forward Substitution)
- Towards an Algorithm

Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System

$$\left(\begin{array}{cccc|c} \chi_{0,0} & \chi_{0,1} & \dots & \chi_{0,m} & \alpha_0 \\ 0 & \chi_{1,1} & \dots & \chi_{1,1} & \alpha_1 \\ 0 & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \chi_{m,m} & \alpha_m \end{array} \right)$$

Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System
 - Gaussian elimination (transform linear system of equations to an upper triangular system)
 - Solving the above linear system relies on the fact that its solution does not change if
 - Equations are reordered (not used until next week);
 - An equation in the system is modified by subtracting a multiple of another equation in the system from it; and/or
 - Both sides of an equation in the system are scaled by a nonzero number.

Gaussian Elimination

- Reducing a System of Linear Equations to an Upper Triangular System

$$2x + 4y + 2z = -10$$

$$4x - 2y + 6z = 20$$

$$6x - 4y + 2z = 18$$

$$2\chi_0 + 4\chi_1 + 2\chi_2 = -10$$

$$4\chi_0 - 2\chi_1 + 6\chi_2 = 20$$

$$6\chi_0 - 4\chi_1 + 2\chi_2 = 18$$

Gaussian Elimination

- Linear Equations

$$\begin{array}{rclcl} 2\chi_0 & + & 4\chi_1 & + & 2\chi_2 = -10 \\ 4\chi_0 & - & 2\chi_1 & + & 6\chi_2 = 20 \\ 6\chi_0 & - & 4\chi_1 & + & 2\chi_2 = 18 \end{array}$$

- Gaussian elimination

- Following Step 1

- $\text{row1} - \lambda_{1,0}\text{row0}$
 - $\text{row2} - \lambda_{2,0}\text{row0}$

- Following Step 2

- $\text{row2} - \lambda_{2,1}\text{row0}$

- Back substitution

- Check your answer

Gaussian Elimination

- Practice with Gaussian Elimination
 - <http://ulaff.s3.amazonaws.com/GaussianEliminationPractice/index.html>

Homework 6.2.1.2 Compute the solution of the linear system of equations given by

$$\begin{aligned}-2\chi_0 + \chi_1 + 2\chi_2 &= 0 \\ 4\chi_0 - \chi_1 - 5\chi_2 &= 4 \\ 2\chi_0 - 3\chi_1 - \chi_2 &= -6\end{aligned}$$

$$\cdot \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$$

Gaussian Elimination

- Appended Matrices

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & -10 \\ 4 & 2 & 6 & 20 \\ 6 & 4 & 2 & 18 \end{array} \right)$$

Represent

$$\begin{aligned} 2\chi_0 + 4\chi_1 + 2\chi_2 &= -10 \\ 4\chi_0 - 2\chi_1 + 6\chi_2 &= 20 \\ 6\chi_0 - 4\chi_1 + 2\chi_2 &= 18 \end{aligned}$$

Gaussian Elimination

- Appended Matrices

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & -10 \\ 4 & 2 & 6 & 20 \\ 6 & 4 & 2 & 18 \end{array} \right)$$

- Gaussian elimination
 - Following Step 1
 - $\text{row1} - \lambda_{1,0}\text{row0}$ & $\text{row2} - \lambda_{2,0}\text{row0}$
 - Following Step 2
 - $\text{row2} - \lambda_{2,1}\text{row0}$
- Back substitution
- Check your answer

Gaussian Elimination

- Appended Matrice

Homework 6.2.2.2 Compute the solution of the linear system of equations expressed as an appended matrix given by

$$\left(\begin{array}{ccc|c} -1 & 2 & -3 & 2 \\ -2 & 2 & -8 & 10 \\ 2 & -6 & 6 & -2 \end{array} \right)$$

$$\cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \square \\ \square \\ \square \end{pmatrix}$$

Gaussian Elimination

• Gauss Transforms

Theorem 6.1 Let \hat{L}_j be a matrix that equals the identity, except that for $i > j$ the (i, j) elements (the ones below the diagonal in the j th column) have been replaced with $-\lambda_{i,j}$:

$$\hat{L}_j = \begin{pmatrix} I_j & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+1,j} & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+2,j} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\lambda_{m-1,j} & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then $\hat{L}_j A$ equals the matrix A except that for $i > j$ the i th row is modified by subtracting $\lambda_{i,j}$ times the j th row from it. Such a matrix \hat{L}_j is called a Gauss transform.

Gaussian Elimination

Proof: Let

$$\hat{L}_j = \begin{pmatrix} I_j & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+1,j} & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+2,j} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\lambda_{m-1,j} & 0 & 0 & \cdots & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ \check{a}_{j+1}^T \\ \check{a}_{j+2}^T \\ \vdots \\ \check{a}_{m-1}^T \end{pmatrix},$$

where I_k equals a $k \times k$ identity matrix, $A_{s:t,:}$ equals the matrix that consists of rows s through t from matrix A , and \check{a}_k^T equals the k th row of A . Then

$$\begin{aligned} \hat{L}_j A &= \begin{pmatrix} I_j & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+1,j} & 1 & 0 & \cdots & 0 \\ 0 & -\lambda_{j+2,j} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\lambda_{m-1,j} & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ \check{a}_{j+1}^T \\ \check{a}_{j+2}^T \\ \vdots \\ \check{a}_{m-1}^T \end{pmatrix} \\ &= \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ -\lambda_{j+1,j}\check{a}_j^T + \check{a}_{j+1}^T \\ -\lambda_{j+2,j}\check{a}_j^T + \check{a}_{j+2}^T \\ \vdots \\ -\lambda_{m-1,j}\check{a}_j^T + \check{a}_{m-1}^T \end{pmatrix} = \begin{pmatrix} A_{0:j-1,:} \\ \check{a}_j^T \\ \check{a}_{j+1}^T - \lambda_{j+1,j}\check{a}_j^T \\ \check{a}_{j+2}^T - \lambda_{j+2,j}\check{a}_j^T \\ \vdots \\ \check{a}_{m-1}^T - \lambda_{m-1,j}\check{a}_j^T \end{pmatrix}. \end{aligned}$$

Gaussian Elimination

- Gauss Transforms

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 4 & -2 & 6 \\ 6 & -4 & 2 \end{pmatrix} = \begin{pmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \boxed{} & 1 & 0 \\ \hline 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 2 & -2 & 6 \\ -4 & -4 & 2 \end{pmatrix} = \begin{pmatrix} \boxed{} & \boxed{} & \boxed{} \\ 0 & \boxed{} & \boxed{} \\ 0 & \boxed{} & \boxed{} \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{} & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -2 \\ 0 & -10 & 10 \\ 0 & -16 & 8 \end{pmatrix} = \begin{pmatrix} \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} \\ 0 & \boxed{} & \boxed{} \end{pmatrix}.$$

Gaussian Elimination

- Gauss Transforms

$$\left(\begin{array}{ccc|c} 2 & 4 & 2 & -10 \\ 4 & 2 & 6 & 20 \\ 6 & 4 & 2 & 18 \end{array} \right)$$

- Gaussian elimination

$$- \left(\begin{array}{ccc} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ -6/2 & 0 & 1 \end{array} \right) \left(\begin{array}{ccc|c} 2 & 4 & 2 & -10 \\ 4 & 2 & 6 & 20 \\ 6 & 4 & 2 & 18 \end{array} \right)$$

$$- \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(-16/-10) & 1 \end{array} \right) \left(\begin{array}{ccc|c} 2 & 4 & 2 & -10 \\ 0 & -10 & 10 & 40 \\ 0 & -16 & 8 & 48 \end{array} \right)$$

- Back substitution
- Check your answer

Gaussian Elimination

- Computing Separately with the Matrix and Right-Hand Side (Forward Substitution)

$$\begin{pmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ -6/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 & -10 \\ 4 & 2 & 6 & 20 \\ 6 & 4 & 2 & 18 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 2 & -10 \\ 0 & -10 & 10 & 40 \\ 0 & -16 & 8 & 48 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ -6/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 2 \\ 4 & 2 & 6 \\ 6 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 2 \\ 0 & -10 & 10 \\ 0 & -16 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -4/2 & 1 & 0 \\ -4/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -10 \\ 20 \\ 18 \end{pmatrix} = \begin{pmatrix} -10 \\ 40 \\ 48 \end{pmatrix}$$

Gaussian Elimination

$$\left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

- Towards an Algorithm

Before

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -2 & 1 & 0 \\ -3 & 0 & 1 \end{array} \right) \left(\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 4 & -2 & 6 \\ 6 & -4 & 2 \end{array} \right)$$

After

$$\left(\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 2 & -10 & 10 \\ 3 & -16 & 8 \end{array} \right)$$

Before

$$\left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & -1.6 & 1 \end{array} \right) \left(\begin{array}{c|cc} 2 & 4 & -2 \\ \hline 2 & -10 & 10 \\ 3 & -16 & 8 \end{array} \right)$$

After

$$\left(\begin{array}{ccc} 2 & 4 & -2 \\ 2 & -10 & 10 \\ 3 & 1.6 & -8 \end{array} \right)$$

Gaussian Elimination

$$\begin{array}{c}
 \left(\begin{array}{ccc|cc} 1 & & & & \\ & \ddots & & & \\ & & 0 & 0 & \\ \hline 1 & & & & \\ \hline 0 & 1 & 0 & & \\ \hline \end{array} \right) \left(\begin{array}{ccc|cc} \times & \cdots & \times & \times & \cdots & \times \\ & \ddots & \times & \vdots & & \times \\ & & \times & \times & \cdots & \times \\ \hline & & & \times & \times & \cdots & \times \\ & & & & \times & \cdots & \times \\ & & & & & \vdots & \ddots & \times \\ & & & & & \times & \cdots & \times \end{array} \right) \\
 = \left(\begin{array}{ccc|cc} \times & \cdots & \times & \times & \cdots & \times \\ & \ddots & \times & \vdots & & \times \\ & & \times & \times & \cdots & \times \\ \hline & & & \times & \times & \cdots & \times \\ & & & & \times & \cdots & \times \\ & & & & & \vdots & \ddots & \times \\ & & & & & \times & \cdots & \times \end{array} \right) \underbrace{\left(\begin{array}{ccc|c} I & 0 & 0 & \\ 0 & 1 & 0 & \\ 0 & -l_{21} & I & \\ \hline \end{array} \right) \left(\begin{array}{ccc|c} U_{00} & u_{01} & & U_{22} \\ 0 & \alpha_{11} & a_{12}^T & \\ 0 & a_{21} & A_{22} & \\ \hline \end{array} \right)}_{= \left(\begin{array}{ccc|c} U_{00} & u_{01} & & U_{22} \\ 0 & v_{11} & u_{12}^T & \\ 0 & 0 & A_{22}^{\text{new}} & \\ \hline \end{array} \right)} \\
 \underbrace{\left(\begin{array}{ccc|c} U_{00} & u_{01} & & U_{22} \\ 0 & \left(\begin{array}{cc} 1 & 0 \\ -l_{21} & I \end{array} \right) & \left(\begin{array}{cc} \alpha_{11} & a_{12}^T \\ a_{21} & A_{22} \end{array} \right) & \\ \hline \end{array} \right)}_{\left(\begin{array}{ccc|c} U_{00} & u_{01} & & U_{22} \\ 0 & \left(\begin{array}{cc} \alpha_{11} & a_{12}^T \\ -l_{21}\alpha_{11} + a_{21} & -l_{21}a_{12}^T + A_{22} \end{array} \right) & & \\ \hline \end{array} \right)}
 \end{array}$$

Gaussian Elimination

Before

$$\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -2 & 1 & 0 \\ -3 & 0 & 1 \end{array} \right) \quad \left(\begin{array}{c|ccc} 2 & 4 & -2 \\ \hline 4 & -2 & 6 \\ 6 & -4 & 2 \end{array} \right)$$

Before

$$\left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & -1.6 & 1 & 1 \end{array} \right) \quad \left(\begin{array}{c|ccc} 2 & 4 & -2 & \\ \hline 2 & -10 & 10 & \\ 3 & -16 & 8 & \end{array} \right)$$

Algorithm: $A := \text{GAUSSIAN_ELIMINATION}(A)$

$$\text{Partition } A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$$

where A_{TL} is 0×0

while $m(A_{TL}) < m(A)$ do

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c|cc|c} A_{00} & a_{01} & A_{02} & \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T & \\ \hline A_{20} & a_{21} & A_{22} & \end{array} \right)$$

$$a_{21} := a_{21}/\alpha_{11} \quad (= l_{21})$$

$$A_{22} := A_{22} - a_{21}a_{12}^T \quad (= A_{22} - l_{21}a_{12}^T)$$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c|cc|c} A_{00} & a_{01} & A_{02} & \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T & \\ \hline A_{20} & a_{21} & A_{22} & \end{array} \right)$$

endwhile

Gaussian Elimination

$$\underbrace{\left(\begin{array}{c|cc|c} I & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & -l_{21} & I \end{array} \right)}_{\text{Augmented matrix}} \left(\begin{array}{c} y_0 \\ \beta_1 \\ b_2 \end{array} \right) = \left(\begin{array}{c} y_0 \\ \psi_1 \\ b_2^{\text{new}} \end{array} \right)$$

$$\underbrace{\left(\begin{array}{c} y_0 \\ \hline \left(\begin{array}{c|c} 1 & 0 \\ \hline -l_{21} & I \end{array} \right) \left(\begin{array}{c} \beta_1 \\ b_2 \end{array} \right) \end{array} \right)}$$

$$\left(\begin{array}{c} y_0 \\ \hline \left(\begin{array}{c} \beta_1 \\ -l_{21}\beta_{11} + b_2 \end{array} \right) \end{array} \right)$$

Algorithm: $b := \text{FORWARD_SUBSTITUTION}(A, b)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$, $b \rightarrow \left(\begin{array}{c} b_T \\ b_B \end{array} \right)$
where A_{TL} is 0×0 , b_T has 0 rows

while $m(A_{TL}) < m(A)$ **do**

Repartition

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right), \left(\begin{array}{c} b_T \\ b_B \end{array} \right) \rightarrow \left(\begin{array}{c} b_0 \\ \beta_1 \\ b_2 \end{array} \right)$$

$$b_2 := b_2 - \beta_1 a_{21} \quad (= b_2 - \beta_1 l_{21})$$

Continue with

$$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right), \left(\begin{array}{c} b_T \\ b_B \end{array} \right) \leftarrow \left(\begin{array}{c} b_0 \\ \beta_1 \\ b_2 \end{array} \right)$$

endwhile

Solving $Ax = b$ via LU Factorization

- LU factorization (Gaussian elimination)
- Solving $Lz = b$ (Forward substitution)
- Solving $Ux = b$ (Back substitution)
- Putting it all together to solve $Ax = b$
- Cost

Solving $Ax = b$ via LU Factorization

- LU factorization (Gaussian elimination)
 - Idea
 - A matrix $A \in \mathbb{R}^{n \times n}$ can be factored into the product of two matrices $L, U \in \mathbb{R}^{n \times n}$
 - $$A = LU$$
 - where L is unit lower triangular and U is upper triangular.

Solving $Ax = b$ via LU Factorization

- LU factorization (Gaussian elimination)

Assume $A \in \mathbb{R}^{n \times n}$ is given and that L and U are to be computed such that $A = LU$, where $L \in \mathbb{R}^{n \times n}$ is unit lower triangular and $U \in \mathbb{R}^{n \times n}$ is upper triangular. We derive an algorithm for computing this operation by partitioning

$$A \rightarrow \left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array} \right), \quad L \rightarrow \left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array} \right), \quad \text{and} \quad U \rightarrow \left(\begin{array}{c|c} v_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array} \right).$$

Now, $A = LU$ implies (using what we learned about multiplying matrices that have been partitioned into submatrices)

$$\begin{aligned} \overbrace{\left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array} \right)}^A &= \overbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline l_{21} & L_{22} \end{array} \right)}^L \overbrace{\left(\begin{array}{c|c} v_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array} \right)}^U \\ &= \overbrace{\left(\begin{array}{c|c} 1 \times v_{11} + 0 \times 0 & 1 \times u_{12}^T + 0 \times U_{22} \\ \hline l_{21}v_{11} + L_{22} \times 0 & l_{21}u_{12}^T + L_{22}U_{22} \end{array} \right)}^{LU} \\ &= \overbrace{\left(\begin{array}{c|c} v_{11} & u_{12}^T \\ \hline l_{21}v_{11} & l_{21}u_{12}^T + L_{22}U_{22} \end{array} \right)}^{LU}. \end{aligned}$$

Solving $Ax = b$ via LU Factorization

- LU factorization (Gaussian elimination)

For two matrices to be equal, their elements must be equal, and therefore, if they are partitioned conformally, their submatrices must be equal:

$$\begin{array}{c|c} \alpha_{11} = v_{11} & a_{12}^T = u_{12}^T \\ \hline a_{21} = l_{21}v_{11} & A_{22} = l_{21}u_{12}^T + L_{22}U_{22} \end{array}$$

or, rearranging,

$$\begin{array}{c|c} v_{11} = \alpha_{11} & u_{12}^T = a_{12}^T \\ \hline l_{21} = a_{21}/v_{11} & L_{22}U_{22} = A_{22} - l_{21}u_{12}^T \end{array}.$$

This suggests the following steps for **overwriting** a matrix A with its LU factorization:

- Partition

$$A \rightarrow \left(\begin{array}{c|c} \alpha_{11} & a_{12}^T \\ \hline a_{21} & A_{22} \end{array} \right).$$

- Update $a_{21} = a_{21}/\alpha_{11} (= l_{21})$. (Scale a_{21} by $1/\alpha_{11}$!)

Solving $Ax = b$ via LU Factorization

- LU factorization (Gaussian elimination)

Algorithm: $A := \text{LU_UNB_VAR5}(A)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$
 where A_{TL} is 0×0

while $m(A_{TL}) < m(A)$ do

Repartition

$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$
 where α_{11} is 1×1

$a_{21} := a_{21}/\alpha_{11}$ ($= l_{21}$)
 $A_{22} := A_{22} - a_{21}a_{12}^T$ ($= A_{22} - l_{21}a_{12}^T$)

Continue with

$\left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$

endwhile

Step	$\left(\begin{array}{c cc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$	a_{21}/α_{11}	$A_{22} - a_{21}a_{12}^T$																				
1-2	$\left(\begin{array}{c cc} -2 & -1 & 1 \\ \hline 2 & -2 & -3 \\ -4 & 4 & 7 \end{array} \right)$	$\begin{pmatrix} 2 \\ -4 \end{pmatrix} / (-2) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} -2 & -3 \\ 4 & 7 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix} \\ = \begin{pmatrix} -3 & -2 \\ 6 & 5 \end{pmatrix}$																				
3	$\left(\begin{array}{c cc} -2 & -1 & 1 \\ \hline -1 & -3 & -2 \\ 2 & 6 & 5 \end{array} \right)$	$(6) / (-3) = (-2)$	$\begin{pmatrix} 5 \\ -2 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$																				
	$\left(\begin{array}{ccc c} -2 & -1 & 1 & 6 \\ -1 & -3 & -2 & 3 \\ 2 & -2 & -3 & -3 \\ \hline 2 & -2 & 1 & 1 \end{array} \right)$	<table border="1"> <thead> <tr> <th>Step</th> <th>Current system</th> <th>Multiplier</th> <th>Operation</th> </tr> </thead> <tbody> <tr> <td>1</td> <td>$\begin{pmatrix} -2 & -1 & 1 & 6 \\ 2 & -2 & -3 & 3 \\ -4 & 4 & 7 & -3 \end{pmatrix}$</td> <td>$\frac{2}{-2} = -1$</td> <td>$\begin{array}{cccc} 2 & -2 & -3 & 3 \\ -1 \times (-2 & -1 & 1 & 6) \\ \hline 0 & -3 & -2 & 9 \end{array}$</td> </tr> <tr> <td>2</td> <td>$\begin{pmatrix} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ -4 & 4 & 7 & -3 \end{pmatrix}$</td> <td>$\frac{-4}{-2} = 2$</td> <td>$\begin{array}{cccc} -4 & 4 & 7 & -3 \\ -(2) \times (-2 & -1 & 1 & 6) \\ \hline 0 & 6 & 5 & -15 \end{array}$</td> </tr> <tr> <td>3</td> <td>$\begin{pmatrix} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ 0 & 6 & 5 & -15 \end{pmatrix}$</td> <td>$\frac{6}{-3} = -2$</td> <td>$\begin{array}{cccc} 0 & 6 & 5 & -15 \\ -(-2) \times (0 & -3 & -2 & 9) \\ \hline 0 & 0 & 1 & 3 \end{array}$</td> </tr> <tr> <td>4</td> <td>$\begin{pmatrix} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ 0 & 0 & 1 & 3 \end{pmatrix}$</td> <td></td> <td></td> </tr> </tbody> </table>		Step	Current system	Multiplier	Operation	1	$\begin{pmatrix} -2 & -1 & 1 & 6 \\ 2 & -2 & -3 & 3 \\ -4 & 4 & 7 & -3 \end{pmatrix}$	$\frac{2}{-2} = -1$	$\begin{array}{cccc} 2 & -2 & -3 & 3 \\ -1 \times (-2 & -1 & 1 & 6) \\ \hline 0 & -3 & -2 & 9 \end{array}$	2	$\begin{pmatrix} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ -4 & 4 & 7 & -3 \end{pmatrix}$	$\frac{-4}{-2} = 2$	$\begin{array}{cccc} -4 & 4 & 7 & -3 \\ -(2) \times (-2 & -1 & 1 & 6) \\ \hline 0 & 6 & 5 & -15 \end{array}$	3	$\begin{pmatrix} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ 0 & 6 & 5 & -15 \end{pmatrix}$	$\frac{6}{-3} = -2$	$\begin{array}{cccc} 0 & 6 & 5 & -15 \\ -(-2) \times (0 & -3 & -2 & 9) \\ \hline 0 & 0 & 1 & 3 \end{array}$	4	$\begin{pmatrix} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ 0 & 0 & 1 & 3 \end{pmatrix}$		
Step	Current system	Multiplier	Operation																				
1	$\begin{pmatrix} -2 & -1 & 1 & 6 \\ 2 & -2 & -3 & 3 \\ -4 & 4 & 7 & -3 \end{pmatrix}$	$\frac{2}{-2} = -1$	$\begin{array}{cccc} 2 & -2 & -3 & 3 \\ -1 \times (-2 & -1 & 1 & 6) \\ \hline 0 & -3 & -2 & 9 \end{array}$																				
2	$\begin{pmatrix} -2 & -1 & 1 & 6 \\ 0 & -3 & -2 & 9 \\ -4 & 4 & 7 & -3 \end{pmatrix}$	$\frac{-4}{-2} = 2$	$\begin{array}{cccc} -4 & 4 & 7 & -3 \\ -(2) \times (-2 & -1 & 1 & 6) \\ \hline 0 & 6 & 5 & -15 \end{array}$																				
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Solving $Ax = b$ via LU Factorization

- Solving $Lz = b$ (Forward substitution)

Given a unit lower triangular matrix $L \in \mathbb{R}^{n \times n}$ and vectors $z, b \in \mathbb{R}^n$, consider the equation $Lz = b$ where L and b are known and z is to be computed. Partition

$$L \rightarrow \left(\begin{array}{c|c} 1 & 0 \\ l_{21} & L_{22} \end{array} \right), \quad z \rightarrow \left(\begin{array}{c} \zeta_1 \\ z_2 \end{array} \right), \quad \text{and} \quad b \rightarrow \left(\begin{array}{c} \beta_1 \\ b_2 \end{array} \right).$$

(Recall: the horizontal line here partitions the result. It is *not* a division.) Now, $Lz = b$ implies that

$$\begin{aligned} \overbrace{\left(\begin{array}{c} \beta_1 \\ b_2 \end{array} \right)}^b &= \overbrace{\left(\begin{array}{c|c} 1 & 0 \\ l_{21} & L_{22} \end{array} \right)}^L \overbrace{\left(\begin{array}{c} \zeta_1 \\ z_2 \end{array} \right)}^z \\ &= \overbrace{\left(\begin{array}{c} 1 \times \zeta_1 + 0 \times z_2 \\ l_{21}\zeta_1 + L_{22}z_2 \end{array} \right)}^{Lz} = \overbrace{\left(\begin{array}{c} \zeta_1 \\ l_{21}\zeta_1 + L_{22}z_2 \end{array} \right)}^{Lz} \end{aligned}$$

so that

$$\frac{\beta_1 = \zeta_1}{b_2 = l_{21}\zeta_1 + L_{22}z_2} \quad \text{or, equivalently,} \quad \frac{\zeta_1 = \beta_1}{L_{22}z_2 = b_2 - \zeta_1 l_{21}}.$$

Solving $Ax = b$ via LU Factorization

- Solving $Lz = b$ (Forward substitution)

Algorithm: $[b] := \text{LTSV_UNB_VAR1}(L, b)$

Partition $L \rightarrow \begin{pmatrix} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{pmatrix}$, $b \rightarrow \begin{pmatrix} b_T \\ b_B \end{pmatrix}$

where L_{TL} is 0×0 , b_T has 0 rows

while $m(L_{TL}) < m(L)$ do

Repartition

$$\begin{pmatrix} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{pmatrix}, \begin{pmatrix} b_T \\ b_B \end{pmatrix} \rightarrow \begin{pmatrix} b_0 \\ \beta_1 \\ b_2 \end{pmatrix}$$

where λ_{11} is 1×1 , β_1 has 1 row

$$b_2 := b_2 - \beta_1 l_{21}$$

Continue with

$$\begin{pmatrix} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{pmatrix} \leftarrow \begin{pmatrix} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{pmatrix}, \begin{pmatrix} b_T \\ b_B \end{pmatrix} \leftarrow \begin{pmatrix} b_0 \\ \beta_1 \\ b_2 \end{pmatrix}$$

endwhile

Step	$\begin{pmatrix} L_{00} & 0 & 0 \\ \hline l_{10}^T & \lambda_{11} & 0 \\ \hline L_{20} & l_{21} & L_{22} \end{pmatrix}$	$\begin{pmatrix} b_0 \\ \beta_1 \\ b_2 \end{pmatrix}$	$b_2 - l_{21}\beta_1$
1-2	$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 3 \\ -3 \end{pmatrix}$	$\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} (6) = \begin{pmatrix} 9 \\ -3 \\ 12 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 9 \\ -15 \end{pmatrix}$	$(-15) - (-2)(9) = (3)$
	$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{pmatrix}$	$\begin{pmatrix} 6 \\ 9 \\ 3 \end{pmatrix}$	

Solving $Ax = b$ via LU Factorization

- Solving $Ux = b$ (Back substitution)

Given upper triangular matrix $U \in \mathbb{R}^{n \times n}$ and vectors $x, b \in \mathbb{R}^n$, consider the equation $Ux = b$ where U and b are known and x is to be computed. Partition

$$U \rightarrow \left(\begin{array}{c|c} v_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array} \right), \quad x \rightarrow \begin{pmatrix} \chi_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad b \rightarrow \begin{pmatrix} \beta_1 \\ b_2 \end{pmatrix}.$$

Now, $Ux = b$ implies

$$\begin{aligned} \overbrace{\begin{pmatrix} \beta_1 \\ b_2 \end{pmatrix}}^b &= \overbrace{\left(\begin{array}{c|c} v_{11} & u_{12}^T \\ \hline 0 & U_{22} \end{array} \right)}^U \overbrace{\begin{pmatrix} \chi_1 \\ x_2 \end{pmatrix}}^x \\ &= \overbrace{\begin{pmatrix} v_{11}\chi_1 + u_{12}^T x_2 \\ 0 \times \chi_1 + U_{22} x_2 \end{pmatrix}}^{Ux} = \overbrace{\begin{pmatrix} \beta_1 \\ U_{22} x_2 \end{pmatrix}}^{Ux} \end{aligned}$$

so that

$$\frac{\beta_1 = v_{11}\chi_1 + u_{12}^T x_2}{b_2 = U_{22} x_2} \quad \text{or, equivalently,} \quad \frac{\chi_1 = (\beta_1 - u_{12}^T x_2) / v_{11}}{U_{22} x_2 = b_2}.$$

Solving $Ax = b$ via LU Factorization

- Solving $Ux = b$ (Back substitution)

Algorithm: $[b] := \text{UTRSV_UNB_VAR1}(U, b)$

Partition $U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array} \right), b \rightarrow \left(\begin{array}{c} b_T \\ b_B \end{array} \right)$

where U_{BR} is 0×0 , b_B has 0 rows

while $m(U_{BR}) < m(U)$ do

Repartition

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \rightarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & v_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right), \left(\begin{array}{c} b_T \\ b_B \end{array} \right) \rightarrow \left(\begin{array}{c} b_0 \\ \beta_1 \\ b_2 \end{array} \right)$$

$$\beta_1 := \beta_1 - u_{12}^T b_2$$

$$\beta_1 := \beta_1 / v_{11}$$

Continue with

$$\left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline 0 & U_{BR} \end{array} \right) \leftarrow \left(\begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline 0 & v_{11} & u_{12}^T \\ \hline 0 & 0 & U_{22} \end{array} \right), \left(\begin{array}{c} b_T \\ b_B \end{array} \right) \leftarrow \left(\begin{array}{c} b_0 \\ \beta_1 \\ b_2 \end{array} \right)$$

endwhile

Solving $Ax = b$ via LU Factorization

- Putting it all together to solve $Ax = b$

Want to solve:	$Ax = b.$
We can now find triangular L and U so that	$A = LU.$
Substitute:	$(LU)x = b.$
Matrix multiplication is associative:	$L(Ux) = b.$
We don't know x but we can create a dummy vector $y = Ux.$	$L \underbrace{y}_{Ux} = b.$
Solve Solving a (lower) triangular system is easy!	$Ly = b$ for y
Solve Solving an (upper) triangular system is easy!	$Ux = y$ for x

Questions and Answers

