Linear Transformations and Matrices

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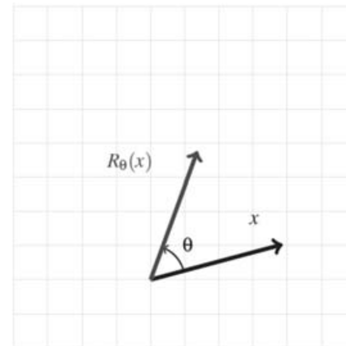
Rotating in 2D

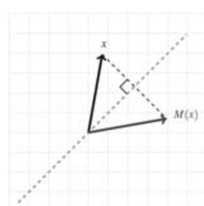
- Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates an input vector through an angle θ



- $R_\theta(\alpha x)$
- $\alpha R_\theta(x)$
- $R_\theta(x + y)$
- $R_\theta(x) + R_\theta(y)$

Example

- A reflection with respect to a 45 degree line is illustrated by
  - Think of the dashed green line as a mirror and $M : \mathbb{R}^2 \to \mathbb{R}^2$ as the vector function that maps a vector to its mirror image.
  - If $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, then $\mathrm{M}(\alpha x) = \alpha \mathrm{M}(x)$ and $\mathrm{M}(x + y) = \mathrm{M}(x) + \mathrm{M}(y)$ (in other words, M is a linear transformation).
    - True
    - False



Linear Transformations

- What Makes Linear Transformations so Special?
  - Many problems in science and engineering involve vector functions such as: $f : \mathbb{R}^n \to \mathbb{R}^m$. Given such a function, one often wishes to do the following:
    - Given vector $x \in \mathbb{R}^n$, evaluate $\mathrm{f}(x)$; or
    - Given vector $y \in \mathbb{R}^m$, find x such that $\mathrm{f}(x) = y$; or
    - Find scalar λ and vector x such that $\mathrm{f}(x) = \lambda x$ (only if $m = n$).

Linear Transformations

- What is a Linear Transformation?
  - Definition 2.1
    - A vector function $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is said to be a linear transformation, if for all $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$
      - Transforming a scaled vector is the same as scaling the transformed vector:

        $$L(\alpha x) = \alpha L(x)$$
      - Transforming the sum of two vectors is the same as summing the two transformed vectors:

        $$L(x + y) = L(x) + L(y)$$

Example

- The transformation $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix}\right) = \begin{pmatrix} \chi_0 + \chi_1 \\ \chi_0 \end{pmatrix}$ is a linear transformation?

Example

- The transformation $f\left(\begin{pmatrix} \chi \\ \psi \end{pmatrix}\right) = \begin{pmatrix} \chi + \psi \\ \chi + 1 \end{pmatrix}$ is a linear transformation?

Exercises

- The vector function $f\left(\begin{pmatrix} \chi \\ \psi \end{pmatrix}\right) = \begin{pmatrix} \chi\psi \\ \chi \end{pmatrix}$ is a linear transformation.
  - TRUE / FALSE
- The vector function $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} \chi_0 + 1 \\ \chi_1 + 2 \\ \chi_2 + 3 \end{pmatrix}$ is a linear transformation.
  - TRUE / FALSE
- The vector function $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}\right) = \begin{pmatrix} \chi_0 \\ \chi_0 + \chi_1 \\ \chi_0 + \chi_1 + \chi_2 \end{pmatrix}$ is a linear transformation.
  - TRUE / FALSE

Exercises

- If $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $L(0) = 0$. (Recall that 0 equals a vector with zero components of appropriate size.)
  - Always / Sometime / Never
- If $L: \mathbb{R}^n \to \mathbb{R}^m$ and $f(0) \neq 0$, Then f is not a linear transformation.
  - True/False
- If $L: \mathbb{R}^n \to \mathbb{R}^m$ and $f(0) = 0$, Then f is a linear transformation.
  - Always / Sometime / Never

Exercises

- Find an example of a function f such that $f(\alpha x) = \alpha f(x)$, but for some x, y it is the case that $f(x+y) \neq f(x) + f(y)$. (This is pretty tricky!)
- The vector function $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix}\right) = \begin{pmatrix} \chi_1 \\ \chi_0 \end{pmatrix}$ is a linear transformation.
  - TRUE / FALSE

Linear Transformations

- Of Linear Transformations and Linear Combinations
  - Lemma 2.4
    - $L: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if (iff) for all $u, v \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$
      $$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v)$$
  - Lemma 2.5
    - Let $v_0, v_1, \ldots, v_{k-1} \in \mathbb{R}^n$ and let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then
      $$L(v_0 + v_1 + \cdots + v_{k-1}) = L(v_0) + L(v_1) + \cdots + L(v_{k-1})$$

Mathematical Induction

- What is the Principle of Mathematical Induction?
  - The Principle of Mathematical Induction (weak induction) says that if one can show that
    - (Base case) a property holds for $n = k_b$; and
    - (Inductive step) if it holds for $n = K$, where $K \geq k_b$, then it is also holds for $n = K + 1$,
  - then one can conclude that the property holds for all integers $n \geq k_b$.
  - Often $k_b = 0$ or $k_b = 1$.

Mathematical Induction

- Examples

$$\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}; n \geq 1$$

  - Base case : $n = 1$
  - Inductive step : Inductive Hypothesis (IH)
    - Assume that the result is true for $n = k$ where $k \geq 1$
    - Show that the result is then also true for $n = k + 1$
  - Find : $2\sum_{i=0}^{n-1} i$

Exercises

- Let $n \geq 1$. Then $\sum_{i=1}^{n} i = n(n+1)/2$.
  - Always / Sometimes / Never
- Let $n \geq 1$. Then $\sum_{i=0}^{n-1} 1 = n$.
  - Always / Sometimes / Never
- Let $n \geq 1$ and $x \in \mathbb{R}^m$. Then $\sum_{i=0}^{n-1} x = x + x + \cdots + x = nx$.
  - Always / Sometimes / Never
- Let $n \geq 1$. $\sum_{i=0}^{n-1} i^2 = (n-1)n(2n-1)/6$.
  - Always / Sometimes / Never

Representing Linear Transformations as Matrices

- From Linear Transformation to Matrix-Vector Multiplication
  - Theorem 2.6
    - Let $v_0, v_1, \ldots, v_{n-1} \in \mathbb{R}^n$, $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$, and let $L: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation.
    - Then

$$L(\alpha_0 v_0 + \alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1}) \\ = \alpha_0 L(v_0) + \alpha_1 L(v_1) + \cdots + \alpha_{n-1} L(v_{n-1})$$

Exercises

Homework 2.4.1.2 Let L be a linear transformation such that

$$L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and} \quad L\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Then $L\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) =$

For the next three exercises, let L be a linear transformation such that

$$L\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \quad \text{and} \quad L\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

Homework 2.4.1.3 $L\left(\begin{pmatrix} 3 \\ 3 \end{pmatrix}\right) =$

Homework 2.4.1.4 $L\left(\begin{pmatrix} -1 \\ 0 \end{pmatrix}\right) =$

Homework 2.4.1.5 $L\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right) =$

Representing Linear Transformations as Matrices

Now we are ready to link linear transformations to matrices and matrix-vector multiplication. Recall that any vector $x \in \mathbb{R}^n$ can be written as

$$x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \chi_0 \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e_0} + \chi_1 \underbrace{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{e_1} + \cdots + \chi_{n-1} \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}_{e_{n-1}} = \sum_{j=0}^{n-1} \chi_j e_j.$$

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Given $x \in \mathbb{R}^n$, the result of $y = L(x)$ is a vector in $\mathbb{R}^m$. But then

$$y = L(x) = L\left(\sum_{j=0}^{n-1} \chi_j e_j.\right) = \sum_{j=0}^{n-1} \chi_j L(e_j) = \sum_{j=0}^{n-1} \chi_j a_j,$$

where we let $a_j = L(e_j)$.

> **The Big Idea.** The linear transformation L is completely described by the vectors
>
> $$a_0, a_1, \ldots, a_{n-1}, \quad \text{where } a_j = L(e_j)$$
>
> because for any vector x, $L(x) = \sum_{j=0}^{n-1} \chi_j a_j$.

By arranging these vectors as the columns of a two-dimensional array, which we call the matrix A, we arrive at the observation that the matrix is simply a representation of the corresponding linear transformation L.

Representing Linear Transformations as Matrices

– Definition 2.7 ($\mathbb{R}^{m \times n}$)

- The set of all $m \times n$ real valued matrices is denoted by $\mathbb{R}^{m \times n}$.
- Thus, $A \in \mathbb{R}^{m \times n}$ means that A is a real valued matrix of size $m \times n$.

Representing Linear Transformations as Matrices

– Definition 2.8 (Matrix-vector multiplication or product)

- Let $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$ with

$$A = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,0} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,0} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,0} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} \quad \text{and} \quad x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$$

- then

$$\begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,n-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,n-1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0}\chi_0 + & \alpha_{0,1}\chi_1 + & \cdots + & \alpha_{0,n-1}\chi_{n-1} \\ \alpha_{1,0}\chi_0 + & \alpha_{1,1}\chi_1 + & \cdots + & \alpha_{1,n-1}\chi_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{m-1,0}\chi_0 + & \alpha_{m-1,1}\chi_1 + & \cdots + & \alpha_{m-1,n-1}\chi_{n-1} \end{pmatrix}$$

Representing Linear Transformations as Matrices

- Practice with Matrix-Vector Multiplication

Homework 2.4.2.1 Compute Ax when $A = \begin{pmatrix} -1 & 0 & 2 \\ -3 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Homework 2.4.2.2 Compute Ax when $A = \begin{pmatrix} -1 & 0 & 2 \\ -3 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix}$ and $x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Homework 2.4.2.3 If A is a matrix and e_j is a unit basis vector of appropriate length, then $Ae_j = a_j$, where a_j is the jth column of matrix A.

Always/Sometimes/Never

Homework 2.4.2.4 If x is a vector and e_i is a unit basis vector of appropriate size, then their dot product, $e_i^T x$, equals the ith entry in x, χ_i.

Always/Sometimes/Never

Homework 2.4.2.5 Compute

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \left(\begin{pmatrix} -1 & 0 & 2 \\ -3 & 1 & -1 \\ -2 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = \_\_\_\_\_$$

Representing Linear Transformations as Matrices

- It Goes Both Ways
  - Theorem 2.9
    - Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be defined by $L(x) = Ax$ where $A \in \mathbb{R}^{m \times n}$. Then L is a linear transformation.
    - A function $f: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it can be written as a matrix-vector multiplication.

Exercises

Homework 2.4.3.1 Give the linear transformation that corresponds to the matrix

$$\begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Homework 2.4.3.2 Give the linear transformation that corresponds to the matrix

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Representing Linear Transformations as Matrices

- Example 2.10 (from 2.2)
  - The transformation $f\left(\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix}\right) = \begin{pmatrix} \chi_0 + \chi_1 \\ \chi_0 \end{pmatrix}$ is a linear transformation.
- Example 2.11 (from 2.3)
  - The transformation $f\left(\begin{pmatrix} \chi \\ \psi \end{pmatrix}\right) = \begin{pmatrix} \chi + \psi \\ \chi + 1 \end{pmatrix}$ is not a linear transformation.

Exercises

Homework 2.4.3.3 Let f be a vector function such that $f(\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix}) = \begin{pmatrix} \chi_0^2 \\ \chi_1 \end{pmatrix}$ Then

- (a) f is a linear transformation.
- (b) f is not a linear transformation.
- (c) Not enough information is given to determine whether f is a linear transformation.

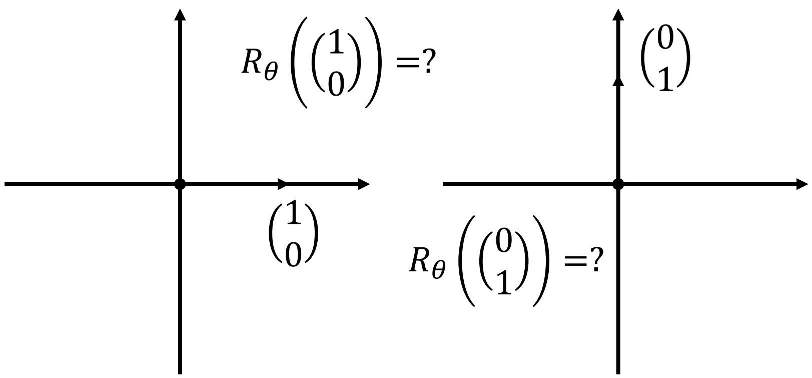
How do you know?

Homework 2.4.3.4 For each of the following, determine whether it is a linear transformation or not:

- $f(\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}) = \begin{pmatrix} \chi_0 \\ 0 \\ \chi_2 \end{pmatrix}$.
- $f(\begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix}) = \begin{pmatrix} \chi_0^2 \\ 0 \end{pmatrix}$.

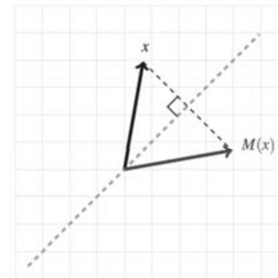
Representing Linear Transformations as Matrices

- **Rotations and Reflections, Revisited**



Representing Linear Transformations as Matrices

Homework 2.4.4.2 A reflection with respect to a 45 degree line is illustrated by



Again, think of the dashed green line as a mirror and let $M : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector function that maps a vector to its mirror image. Compute the matrix that represents M (by examining the picture).

Questions and Answers