Vector Spaces

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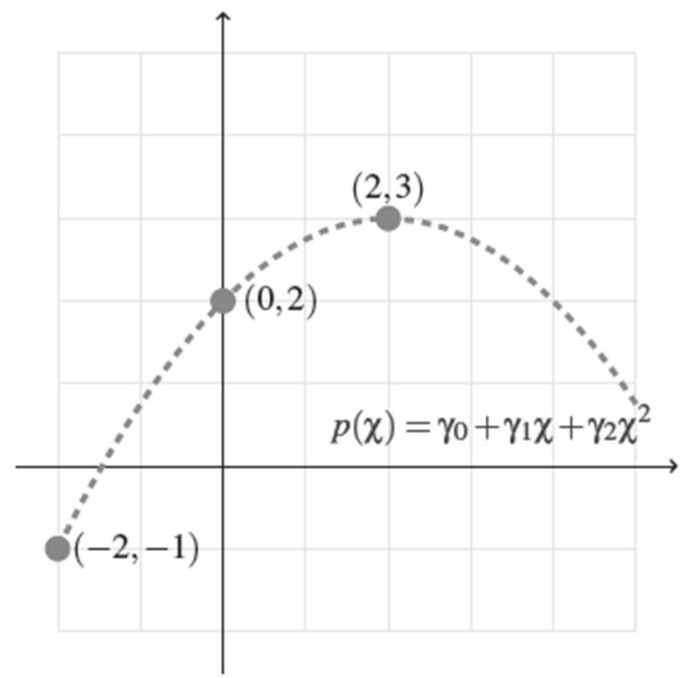
Opening Remarks

$$p(\chi) = \gamma_0 + \gamma_1 \chi + \gamma_2 \chi^2$$

$$\begin{array}{rcccccccl} p(-2) & = & \gamma_0 & + & \gamma_1(-2) & + & \gamma_2(-2)^2 & = & -1 \\ p(0) & = & \gamma_0 & + & \gamma_1(0) & + & \gamma_2(0)^2 & = & 2 \\ p(2) & = & \gamma_0 & + & \gamma_1(2) & + & \gamma_2(2)^2 & = & 3 \end{array}$$

$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$



Robert van de Geijn and Maggie Myers. Linear Algebra - Foundations to Frontiers. https://www.edx.org/

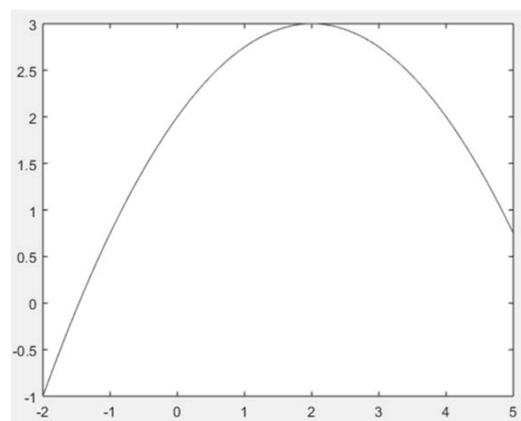
Opening Remarks

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & -1 \\ 1 & 0 & 0 & 2 \\ 1 & 2 & 4 & 3 \end{array}\right)$$

- Gaussian Elimination | Gauss-Jordan Elimination

$$\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -0.25 \end{pmatrix} \Rightarrow p(\chi) = 2 + \chi - \frac{1}{4}\chi^2$$

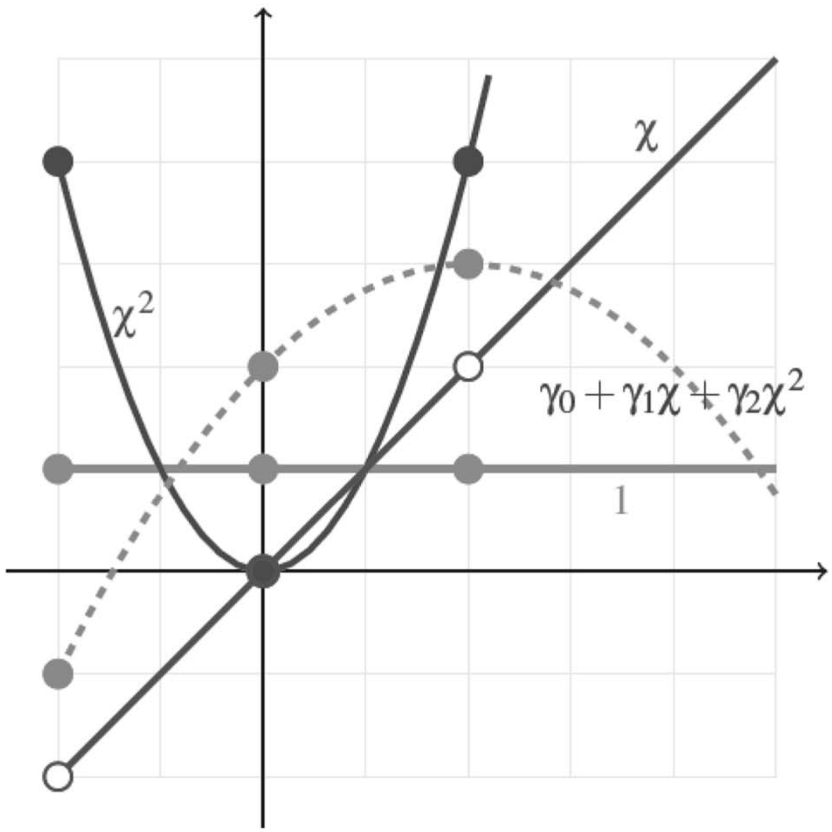
- Matlab Script
  - x = [-2:0.1:5]; p = 2 + x -0.25*x.^2;
  - plot(x,p)



Opening Remarks

$$p(\chi) = 2 + \chi - \frac{1}{4}\chi^2.$$

Now, let's look at this problem a little differently. $p(\chi)$ is a linear combination (a word you now understand well) of the polynomials $p_0(\chi) = 1$, $p_1(\chi) = \chi$, and $p_2(\chi) = \chi^2$. These basic polynomials are called "parent functions".



$$\begin{aligned}
\begin{pmatrix} p(-2) \\ p(0) \\ p(2) \end{pmatrix} &= \begin{pmatrix} \gamma_0 & + & \gamma_1(-2) & + & \gamma_2(-2)^2 \\ \gamma_0 & + & \gamma_1(0) & + & \gamma_2(0)^2 \\ \gamma_0 & + & \gamma_1(2) & + & \gamma_2(2)^2 \end{pmatrix} \\
&= \gamma_0 \begin{pmatrix} p_0(-2) \\ p_0(0) \\ p_0(2) \end{pmatrix} + \gamma_1 \begin{pmatrix} p_1(-2) \\ p_1(0) \\ p_1(2) \end{pmatrix} + \gamma_2 \begin{pmatrix} p_2(-2) \\ p_2(0) \\ p_2(2) \end{pmatrix} \\
&= \gamma_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \gamma_1 \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + \gamma_2 \begin{pmatrix} (-2)^2 \\ 0^2 \\ 2^2 \end{pmatrix} \\
&= \gamma_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \gamma_1 \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + \gamma_2 \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}.
\end{aligned}$$

Opening Remarks

What we notice is that this last vector must equal a linear combination of the first three vectors:

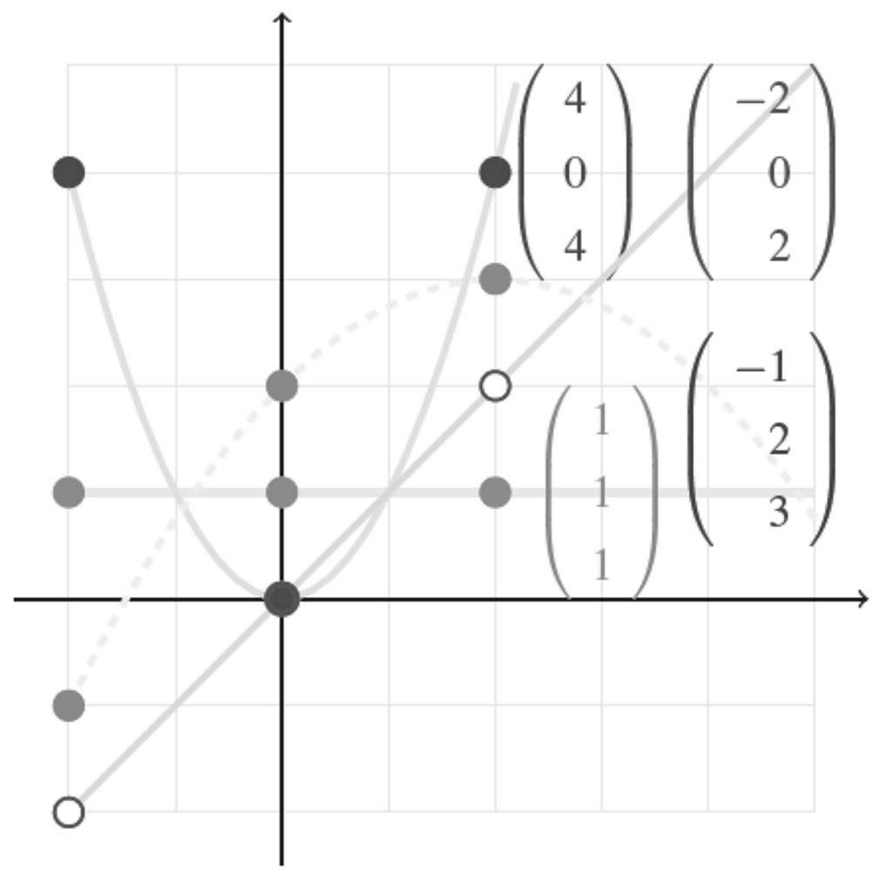
$$\gamma_0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \gamma_1 \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} + \gamma_2 \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

Again, this gives rise to the matrix equation

$$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

with the solution

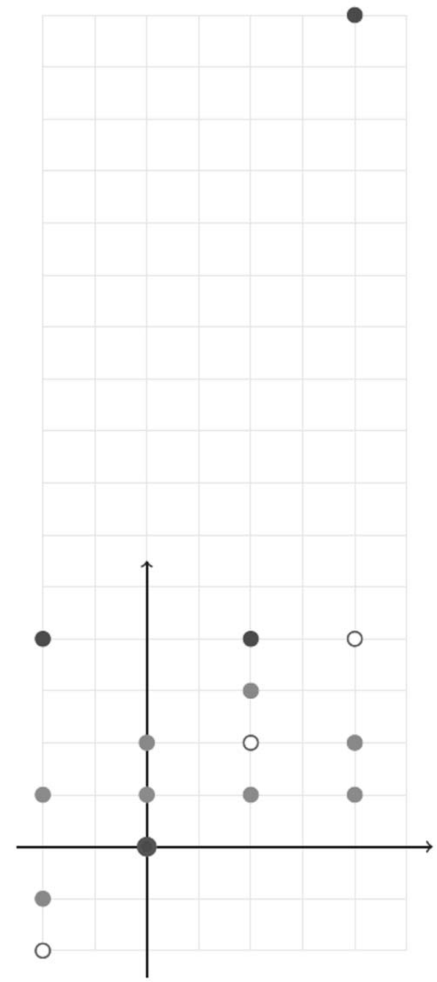
$$\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -0.25 \end{pmatrix}.$$



Opening Remarks

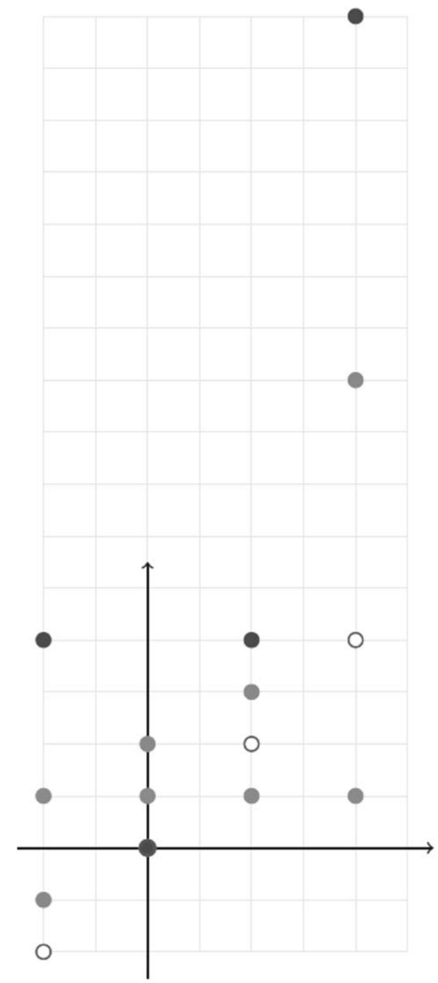
- Solvable or not solvable, that's the question

$$
-\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 2 \end{pmatrix}
$$



Opening Remarks

- Solvable or not solvable, that's the question
  - $$\begin{pmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 9 \end{pmatrix}$$



When Systems Don't Have a Unique Solution

- **When Solutions Are Not Unique**

Example 9.1 Consider

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

Does $Ax = b_0$ have a solution? The answer is yes:

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}. \quad \checkmark$$

But this is not the only solution:

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ 0 \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \quad \checkmark$$

and

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}. \quad \checkmark$$

Indeed, later we will see there are an infinite number of solutions!

When Systems Don't Have a Unique Solution

- **When Solutions Are Not Unique**

Homework 9.2.1.1 Evaluate

1. $\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} =$

2. $\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} =$

3. $\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} =$

Does the system $\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$ have multiple solutions? Yes/No

When Systems Don't Have a Unique Solution

- When Linear Systems Have No Solutions

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$$

$$\left(\begin{array}{ccc||c} 2 & 2 & -2 & 0 \\ -2 & -3 & 4 & 3 \\ 4 & 3 & -2 & 4 \end{array}\right)$$

When Systems Don't Have a Unique Solution

- **When Linear Systems Have Many Solutions**

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 2 & 2 & -2 & 0 \\ -2 & -3 & 4 & 3 \\ 4 & 3 & -2 & 3 \end{array}\right)$$

When Systems Don't Have a Unique Solution

- What is Going On?
  - Consider $Ax = b$ and assume that we have
    - One solution to the system $Ax = b$, the specific solution we denote by x_s so that $Ax_s = b$.
    - One solution to the system $Ax = 0$ that we denote by x_n so that $Ax_n = 0$.
  - Then

$$A(x_s + x_n) = Ax_s + Ax_n = b + 0 = b$$

  - So, $x_s + x_n$ is also a solution.

$$A(x_s + \beta x_n) = Ax_s + A(\beta x_n) = b + 0 = b$$

  - So $A(x_s + \beta x_n)$ is a solution for every $\beta \in \mathbb{R}$.

When Systems Don't Have a Unique Solution

- Toward a Systematic Approach to Finding All Solutions

Let's focus on finding nontrivial solutions to $Ax = 0$, for the same example as in Unit 9.2.3. (The trivial solution to $Ax = 0$ is $x = 0$.)

Recall the example

$$\begin{pmatrix} 2 & 2 & -2 \\ -2 & -3 & 4 \\ 4 & 3 & -2 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$$

$$-A(x_s + \beta x_n)$$

which had the general solution

$$\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

When Systems Don't Have a Unique Solution

- Toward a Systematic Approach to Finding All Solutions

$$\left(\begin{array}{ccc|cc} 2 & 2 & -2 & 0 & 0 \\ -2 & -3 & 4 & 3 & 0 \\ 4 & 3 & -2 & 3 & 0 \end{array}\right)$$

  - Gaussian elimination

$$\left(\begin{array}{ccc|cc} 1 & 0 & 1 & 3 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

When Systems Don't Have a Unique Solution

Continuing on again

- Observe that there was no need to perform all the transformations with the appended system on the right. One could have simply applied them only to the appended system on the left. Then, to obtain the results on the right we simply set the right-hand side (the appended vector) equal to the zero vector.

So, let's translate the left appended system back into a system of linear systems:

$$\begin{array}{rcrcrcr} \chi_0 & & & + & \chi_2 & = & 3 \\ & & \chi_1 & - & 2\chi_2 & = & -3 \\ & & & & 0 & = & 0 \end{array}$$

As before, we have two equations and three unknowns, plus an equation that says that "$0 = 0$", which is *true*, but doesn't help much! We are going to find *one solution* (a specific solution), by choosing the free variable $\chi_2 = 0$. We can set it to equal anything, but zero is an easy value with which to compute. Substituting $\chi_2 = 0$ into the first two equations yields

$$\begin{array}{rcrcrcr} \chi_0 & & & + & 0 & = & 3 \\ & & \chi_1 & - & 2(0) & = & -3 \\ & & & & 0 & = & 0 \end{array}$$

We conclude that a specific solution is given by

$$x_s = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}.$$

When Systems Don't Have a Unique Solution

Next, let's look for *one* non-trivial solution to $Ax = 0$ by translating the right appended system back into a system of linear equations:

$$\begin{array}{rcrcr} \chi_0 & & + \ \chi_2 & = & 0 \\ & \chi_1 & - \ 2\chi_2 & = & 0 \end{array}$$

Now, if we choose the free variable $\chi_2 = 0$, then it is easy to see that $\chi_0 = \chi_1 = 0$, and we end up with the trivial solution, $x = 0$. So, instead choose $\chi_2 = 1$. (We, again, can choose any value, but it is easy to compute with 1.) Substituting this into the first two equations yields

$$\begin{array}{rcrcr} \chi_0 & & + \ 1 & = & 0 \\ & \chi_1 & - \ 2(1) & = & 0 \end{array}$$

Solving for χ_0 and χ_1 gives us the following non-trivial solution to $Ax = 0$:

$$x_n = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

But if $Ax_n = 0$, then $A(\beta x_n) = 0$. This means that all vectors

$$x_s + \beta x_n = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

solve the linear system. This is the general solution that we saw before.

In this particular example, it was not necessary to exchange (pivot) rows.

When Systems Don't Have a Unique Solution

- Toward a Systematic Approach to Finding All Solutions

Homework 9.2.5.1 Find the general solution (an expression for all solutions) for

$$\begin{pmatrix} 2 & -2 & -4 \\ -2 & 1 & 4 \\ 2 & 0 & -4 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}.$$

Homework 9.2.5.2 Find the general solution (an expression for all solutions) for

$$\begin{pmatrix} 2 & -4 & -2 \\ -2 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}.$$

Review of Sets

- **Definition and Notation**

We very quickly discuss what a set is and some properties of sets. As part of discussing vector spaces, we will see lots of examples of sets and hence we keep examples down to a minimum.

Definition 9.3 *In mathematics, a set is defined as a collection of distinct objects.*

The objects that are members of a set are said to be its elements. If S is used to denote a given set and x is a member of that set, then we will use the notation $x \in S$ which is pronounced x is an element of S.

If x, y, and z are distinct objects that together are the collection that form a set, then we will often use the notation $\{x,y,z\}$ to describe that set. It is extremely important to realize that **order does not matter**: $\{x,y,z\}$ is the same set as $\{y,z,x\}$, and this is true for all ways in which you can order the objects.

A set itself is an object and hence once can have a set of sets, which has elements that are sets.

Definition 9.4 *The size of a set equals the number of distinct objects in the set.*

This size can be finite or infinite. If S denotes a set, then its size is denoted by $|S|$.

Definition 9.5 *Let S and T be sets. Then S is a subset of T if all elements of S are also elements of T. We use the notation $S \subset T$ or $T \supset S$ to indicate that S is a subset of T.*

Mathematically, we can state this as

$$(S \subset T) \Leftrightarrow (x \in S \Rightarrow x \in T).$$

(S is a subset of T if and only if every element in S is also an element in T.)

Definition 9.6 *Let S and T be sets. Then S is a proper subset of T if all S is a subset of T and there is an element in T that is not in S. We use the notation $S \subsetneq T$ or $T \supsetneq S$ to indicate that S is a proper subset of T.*

Some texts will use the symbol $\subset$ to mean "proper subset" and $\subseteq$ to mean "subset". Get used to it! You'll have to figure out from context what they mean.

Review of Sets

- Examples

Example 9.7 The integers $1, 2, 3$ are a collection of three objects (the given integers). The set formed by these three objects is given by $\{1, 2, 3\}$ (again, emphasizing that order doesn't matter). The size of this set is $|\{1, 2, 3, \}| = 3$.

Example 9.8 The collection of all integers is a set. It is typically denoted by $\mathbb{Z}$ and sometimes written as $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. Its size is infinite: $|\mathbb{Z}| = \infty$.

Example 9.9 The collection of all real numbers is a set that we have already encountered in our course. It is denoted by $\mathbb{R}$. Its size is infinite: $|\mathbb{R}| = \infty$. We cannot enumerate it (it is uncountably infinite, which is the subject of other courses).

Example 9.10 The set of all vectors of size n whose components are real valued is denoted by $\mathbb{R}^n$.

Review of Sets

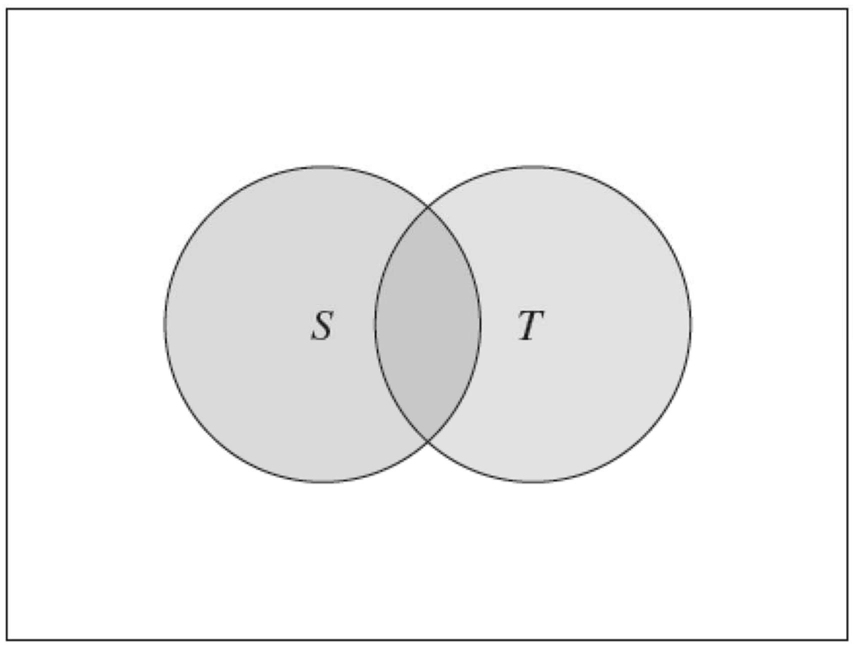
- Operations with Sets

Definition 9.11 *The* **union** *of two sets S and T is the set of all elements that are in S* **or** *in T. This union is denoted by $S \cup T$.*

Formally, we can give the union as

$$S \cup T = \{x | x \in S \vee x \in T\}$$

which is read as "The union of S and T equals the set of all elements x such that x is in S or x is in T." (The "|" (vertical bar) means "such that" and the $\vee$ is the logical "or" operator.) It can be depicted by the shaded area (blue, pink, and purple) in the following Venn diagram:



Review of Sets

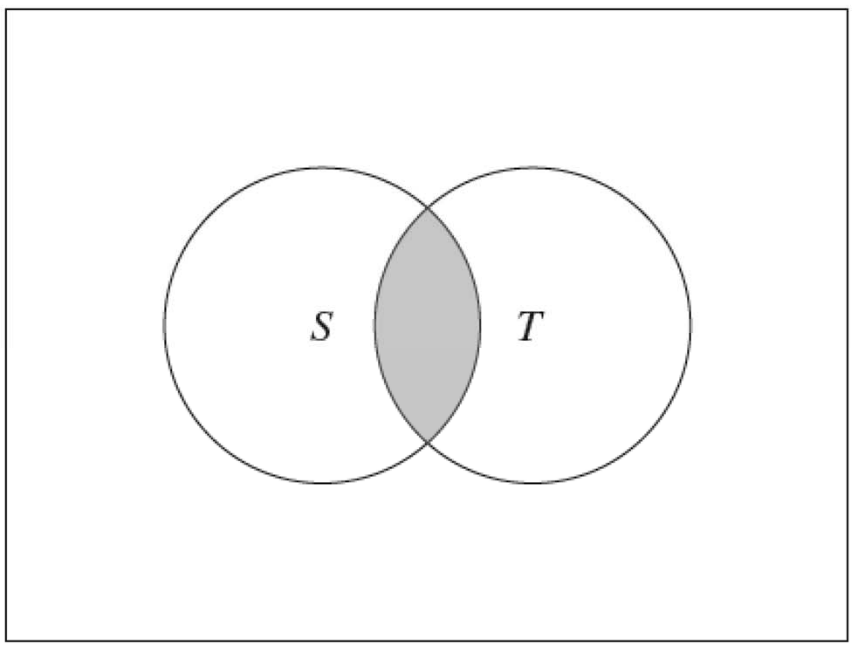
- Operations with Sets

Definition 9.13 *The* **intersection** *of two sets S and T is the set of all elements that are in S* **and** *in T. This intersection is denoted by $S \cap T$.*

Formally, we can give the intersection as

$$S \cap T = \{x | x \in S \wedge x \in T\}$$

which is read as "The intersection of S and T equals the set of all elements x such that x is in S and x is in T." (The "|" (vertical bar) means "such that" and the $\wedge$ is the logical "and" operator.) It can be depicted by the shaded area in the following Venn diagram:



Review of Sets

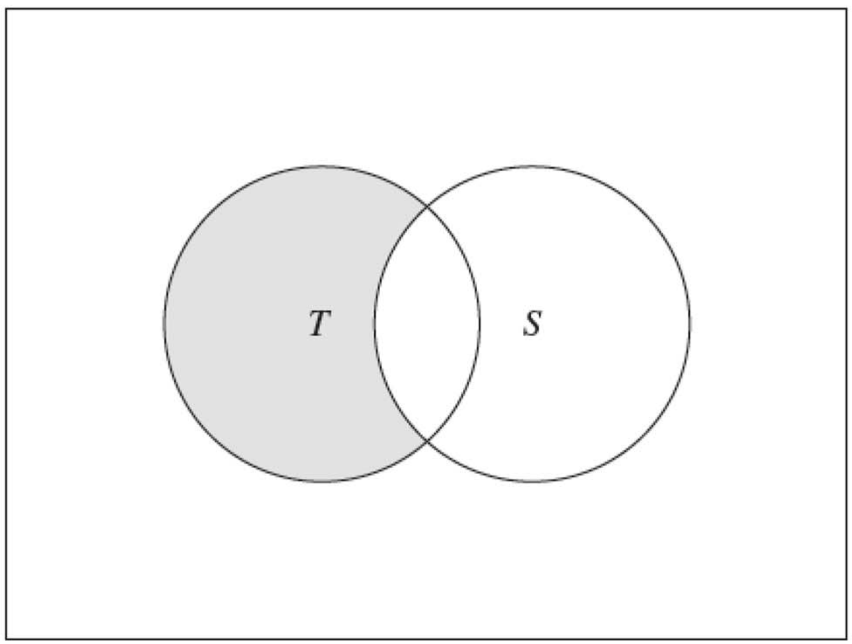
Definition 9.16 *The* **complement** *of set S with respect to set T is the set of all elements that are in T but are not in S. This complement is denoted by $T \backslash S$.*

Example 9.17 Let $S = \{1,2,3\}$ and $T = \{2,3,5,8,9\}$. Then $T \backslash S = \{5,8,9\}$ and $S \backslash T = \{1\}$.

Formally, we can give the complement as

$$T \backslash S = \{x | x \notin S \wedge x \in T\}$$

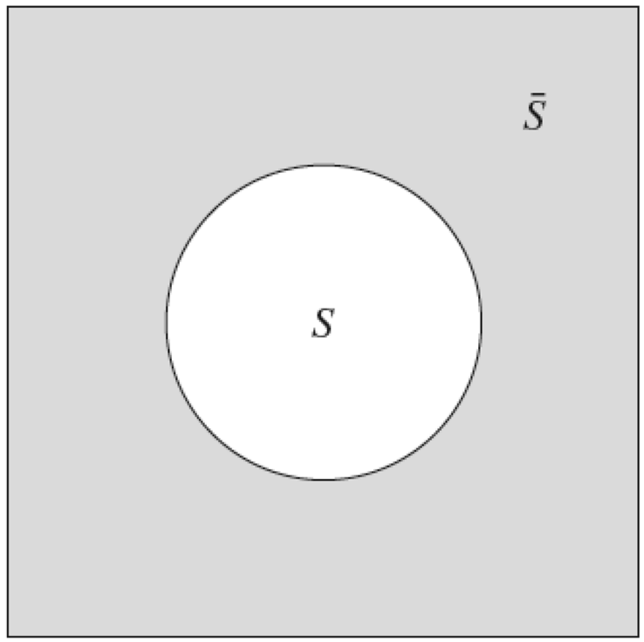
which is read as "The complement of S with respect to T equals the set of all elements x such that x is not in S and x is in T." (The "|" (vertical bar) means "such that", $\wedge$ is the logical "and" operator, and the $\notin$ means "is not an element in".) It can be depicted by the shaded area in the following Venn diagram:



Review of Sets

Sometimes, the notation $\bar{S}$ or S^c is used for the complement of set S. Here, the set with respect to which the complement is taken is "obvious from context".

For a single set S, the complement, $\bar{S}$ is shaded in the diagram below.



Homework 9.3.3.1 Let S and T be two sets. Then $S \subset S \cup T$.

Always/Sometimes/Never

Homework 9.3.3.2 Let S and T be two sets. Then $S \cap T \subset S$.

Always/Sometimes/Never

Vector Spaces

- **What is a Vector Space?**

For our purposes, a vector space is a subset, S, of $\mathbb{R}^n$ with the following properties:

- $0 \in S$ (the zero vector of size n is in the set S); and
- If $v, w \in S$ then $(v + w) \in S$; and
- If $\alpha \in \mathbb{R}$ and $v \in S$ then $\alpha v \in S$.

A mathematician would describe the last two properties as "S is closed under addition and scalar multiplication." All the results that we will encounter for such vector spaces carry over to the case where the components of vectors are complex valued.

Example 9.18 The set $\mathbb{R}^n$ is a vector space:

- $0 \in \mathbb{R}^n$.
- If $v, w \in \mathbb{R}^n$ then $v + w \in \mathbb{R}^n$.
- If $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ then $\alpha v \in \mathbb{R}^n$.

Vector Spaces

- Subspaces

Homework 9.4.2.1 Which of the following subsets of $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$?

1. The plane of vectors $x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix}$ such that $\chi_0 = 0$. In other words, the set of all vectors $\left\{ x \in \mathbb{R}^3 \,\middle|\, x = \begin{pmatrix} 0 \\ \chi_1 \\ \chi_2 \end{pmatrix} \right\}$.
2. Similarly, the plane of vectors x with $\chi_0 = 1$: $\left\{ x \in \mathbb{R}^3 \,\middle|\, x = \begin{pmatrix} 1 \\ \chi_1 \\ \chi_2 \end{pmatrix} \right\}$.
3. $\left\{ x \in \mathbb{R}^3 \,\middle|\, x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} \wedge \chi_0\chi_1 = 0 \right\}$. (Recall, $\wedge$ is the logical "and" operator.)
4. $\left\{ x \in \mathbb{R}^3 \,\middle|\, x = \beta_0 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \text{ where } \beta_0, \beta_1 \in \mathbb{R} \right\}$.
5. $\left\{ x \in \mathbb{R}^3 \,\middle|\, x = \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} \wedge \chi_0 - \chi_1 + 3\chi_2 = 0 \right\}$.

Vector Spaces

- The Column Space

Definition 9.19 *Let $A \in \mathbb{R}^{m \times n}$. Then the* **column space** *of A equals the set*

$$\{Ax \mid x \in \mathbb{R}^n\}.$$

It is denoted by $\mathcal{C}(A)$.

The name "column space" comes from the observation (which we have made many times by now) that

$$Ax = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right) \left(\begin{array}{c} \chi_0 \\ \chi_1 \\ \vdots \\ \chi_{n-1} \end{array} \right) = \chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}.$$

Thus $\mathcal{C}(A)$ equals the set of all linear combinations of the columns of matrix A.

Vector Spaces

- ## The Column Space

Theorem 9.20 *The column space of* $A \in \mathbb{R}^{m \times n}$ *is a subspace of* $\mathbb{R}^m$.

Proof: The last exercise proved this.

Theorem 9.21 *Let* $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, *and* $b \in \mathbb{R}^m$. *Then* $Ax = b$ *has a solution if and only if* $b \in \mathcal{C}(A)$.

Proof: Recall that to prove an "if and only if" statement $P \Leftrightarrow Q$, you may want to instead separately prove $P \Rightarrow Q$ and $P \Leftarrow Q$.

($\Rightarrow$) Assume that $Ax = b$. Then $b \in \{Ax | x \in \mathbb{R}^n\}$. Hence b is in the column space of A.

($\Leftarrow$) Assume that b is in the column space of A. Then $b \in \{Ax | x \in \mathbb{R}^n\}$. But this means there exists a vector x such that $Ax = b$.

Vector Spaces

- **The Null Space**

Definition 9.22 *Let $A \in \mathbb{R}^{m \times n}$. Then the set of all vectors $x \in \mathbb{R}^n$ that have the property that $Ax = 0$ is called the null space of A and is denoted by*

$$\mathcal{N}(A) = \{x | Ax = 0\}.$$

Span, Linear Independence, and Bases

- Span

Definition 9.24 *Let $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$. Then the span of these vectors, $Span\{v_0, v_1, \cdots, v_{n-1}\}$, is said to be the set of all vectors that are a linear combination of the given set of vectors.*

We will later see that the vectors in this last example "span" the entire null space for the given matrix. But we are not quite ready to claim that.

We have learned three things in this course that relate to this discussion:

- Given a set of vectors $\{v_0, v_1, \ldots, v_{n-1}\} \subset \mathbb{R}^n$, we can create a matrix that has those vectors as its columns:

$$V = \left(\begin{array}{c|c|c|c} v_0 & v_1 & \cdots & v_{n-1} \end{array} \right).$$

- Given a matrix $V \in \mathbb{R}^{m \times n}$ and vector $x \in \mathbb{R}^n$,

$$Vx = \chi_0 v_0 + \chi_1 v_1 + \cdots + \chi_{n-1} v_{n-1}.$$

  In other words, Vx takes a linear combination of the columns of V.

- The column space of V, $\mathcal{C}(V)$, is the set (subspace) of all linear combinations of the columns of V:

$$\mathcal{C}(V) = \{Vx | x \in \mathbb{R}^n\} = \{\chi_0 v_0 + \chi_1 v_1 + \cdots + \chi_{n-1} v_{n-1} | \chi_0, \chi_1, \ldots, \chi_{n-1} \in \mathbb{R}\}.$$

We conclude that

Definition 9.27 *A spanning set of a subspace S is a set of vectors $\{v_0, v_1, \ldots, v_{n-1}\}$ such that $Span(\{v_0, v_1, \ldots, v_{n-1}\}) = S$.*

Span, Linear Independence, and Bases

- **Linear Independence**

Definition 9.29 *Let* $\{v_0, \ldots, v_{n-1}\} \subset \mathbb{R}^m$*. Then this set of vectors is said to be linearly independent if* $\chi_0 v_0 + \chi_1 v_1 + \cdots + \chi_{n-1} v_{n-1} = 0$ *implies that* $\chi_0 = \cdots = \chi_{n-1} = 0$*. A set of vectors that is not linearly independent is said to be linearly dependent.*

Example 9.28 We show that $\mathrm{Span}\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}\right) = \mathrm{Span}\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}\right)$. One can either simply recognize that both sets equal all of $\mathbb{R}^2$, or one can reason it by realizing that in order to show that sets S and T are equal one can just show that both $S \subset T$ and $T \subset S$:

- $S \subset T$: Let $x \in \mathrm{Span}\left(\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right)$ Then there exist α_0 and α_1 such that $x = \alpha_0\begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha_1\begin{pmatrix}0\\1\\0\end{pmatrix}$. This in turn means that $x = \alpha_0\begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha_1\begin{pmatrix}0\\1\\0\end{pmatrix} + (0)\begin{pmatrix}1\\1\\0\end{pmatrix}$. Hence

$$x \in \mathrm{Span}\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}\right).$$

Span, Linear Independence, and Bases

- $T \subset S$: Let $x \in \mathrm{Span}\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}\right)$. Then there exist α_0, α_1, and α_2 such that $x = \alpha_0\begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha_1\begin{pmatrix}0\\1\\0\end{pmatrix} + \alpha_2\begin{pmatrix}1\\1\\0\end{pmatrix}$. But $\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}$. Hence

$$x = \alpha_0\begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha_1\begin{pmatrix}0\\1\\0\end{pmatrix} + \alpha_2\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix}\right) = (\alpha_0+\alpha_2)\begin{pmatrix}1\\0\\0\end{pmatrix} + (\alpha_1+\alpha_2)\begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Therefore $x \in \mathrm{Span}\left(\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}\right)$.

Span, Linear Independence, and Bases

Theorem 9.31 *Let $\{a_0, \ldots, a_{n-1}\} \subset \mathbb{R}^m$ and let $A = \left(\begin{array}{c|c|c} a_0 & \cdots & a_{n-1} \end{array} \right)$. Then the vectors $\{a_0, \ldots, a_{n-1}\}$ are linearly independent if and only if $\mathcal{N}(A) = \{0\}$.*

Proof:

($\Rightarrow$) Assume $\{a_0, \ldots, a_{n-1}\}$ are linearly independent. We need to show that $\mathcal{N}(A) = \{0\}$. Assume $x \in \mathcal{N}(A)$. Then $Ax = 0$ implies that

$$
\begin{aligned}
0 &= Ax = \left(\begin{array}{c|c|c} a_0 & \cdots & a_{n-1} \end{array} \right) \left(\begin{array}{c} \chi_0 \\ \vdots \\ \chi_{n-1} \end{array} \right) \\
&= \chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1}
\end{aligned}
$$

and hence $\chi_0 = \cdots = \chi_{n-1} = 0$. Hence $x = 0$.

($\Leftarrow$) Notice that we are trying to prove $P \Leftarrow Q$, where P represents "the vectors $\{a_0, \ldots, a_{n-1}\}$ are linearly independent" and Q represents "$\mathcal{N}(A) = \{0\}$". It suffices to prove the **contrapositive**: $\neg P \Rightarrow \neg Q$. (Note that $\neg$ means "not") Assume that $\{a_0, \ldots, a_{n-1}\}$ are *not* linearly independent. Then there exist $\{\chi_0, \cdots, \chi_{n-1}\}$ with at least one $\chi_j \neq 0$ such that $\chi_0 a_0 + \chi_1 a_1 + \cdots + \chi_{n-1} a_{n-1} = 0$. Let $x = (\chi_0, \ldots, \chi_{n-1})^T$. Then $Ax = 0$ which means $x \in \mathcal{N}(A)$ and hence $\mathcal{N}(A) \neq \{0\}$.

Span, Linear Independence, and Bases

- **Bases for Subspaces**

Definition 9.38 *Let S be a subspace of $\mathbb{R}^m$. Then the set $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ is said to be a basis for S if (1) $\{v_0, v_1, \cdots, v_{n-1}\}$ are linearly independent and (2) Span$\{v_0, v_1, \cdots, v_{n-1}\} = S$.*

Homework 9.5.3.1 The vectors $\{e_0, e_1, \ldots, e_{n-1}\} \subset \mathbb{R}^n$ are a basis for $\mathbb{R}^n$.

True/False

Example 9.39 Let $\{a_0, \ldots, a_{n-1}\} \subset \mathbb{R}^n$ and let $A = \left(\begin{array}{c|c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right)$ be invertible. Then $\{a_0, \ldots, a_{n-1}\} \subset \mathbb{R}^n$ form a basis for $\mathbb{R}^n$.

Note: The fact that A is invertible means there exists A^{-1} such that $A^{-1}A = I$. Since $Ax = 0$ means $x = A^{-1}Ax = A^{-1}0 = 0$, the columns of A are linearly independent. Also, given any vector $y \in \mathbb{R}^n$, there exists a vector $x \in \mathbb{R}^n$ such that $Ax = y$ (namely $x = A^{-1}y$). Letting $x = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix}$ we find that $y = \chi_0 a_0 + \cdots + \chi_{n-1} a_{n-1}$ and hence every vector in $\mathbb{R}^n$ is a linear combination of the set $\{a_0, \ldots, a_{n-1}\} \subset \mathbb{R}^n$.

Span, Linear Independence, and Bases

- **The Dimension of a Subspace**

Theorem 9.42 *Let S be a subspace of $\mathbb{R}^m$ and let $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ and $\{w_0, w_1, \cdots, w_{k-1}\} \subset \mathbb{R}^m$ both be bases for S. Then $k = n$. In other words, the number of vectors in a basis is unique.*

Definition 9.43 *The dimension of a subspace S equals the number of vectors in a basis for that subspace.*

A basis for a subspace S can be derived from a spanning set of a subspace S by, one-to-one, removing vectors from the set that are dependent on other remaining vectors until the remaining set of vectors is linearly independent , as a consequence of the following observation:

Definition 9.44 *Let $A \in \mathbb{R}^{m\times n}$. The rank of A equals the number of vectors in a basis for the column space of A. We will let* $\text{rank}(A)$ *denote that rank.*

Theorem 9.45 *Let $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ be a spanning set for subspace S and assume that v_i equals a linear combination of the other vectors. Then $\{v_0, v_1, \cdots, v_{i-1}, v_{i+1}, \cdots, v_{n-1}\}$ is a spanning set of S.*

Similarly, a set of linearly independent vectors that are in a subspace S can be “built up” to be a basis by successively adding vectors that are in S to the set while maintaining that the vectors in the set remain linearly independent until the resulting is a basis for S.

Theorem 9.46 *Let $\{v_0, v_1, \cdots, v_{n-1}\} \subset \mathbb{R}^m$ be linearly independent and assume that $\{v_0, v_1, \cdots, v_{n-1}\} \subset S$ where S is a subspace. Then this set of vectors is either a spanning set for S or there exists $w \in S$ such that $\{v_0, v_1, \cdots, v_{n-1}, w\}$ are linearly independent.*

Questions and Answers