

# From Matrix-Vector Multiplication to Matrix-Matrix Multiplication

Jirasak Sittigorn

Department of Computer Engineering  
Faculty of Engineering  
King Mongkut's Institute of Technology Ladkrabang

## Opening Remarks

- Predicting the Weather

If today is cloudy,

what is the probability that tomorrow is

- sunny?
- cloudy?
- rainy?



		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

## Opening Remarks

- Predicting the Weather

The following table tells us how the weather for any day (e.g., today) predicts the weather for the next day (e.g., tomorrow):

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

## Opening Remarks

- Predicting the Weather

If today is sunny,

what is the probability that the day after tomorrow is

- sunny?
- cloudy?
- rainy?

The probability

$$\text{sunny} \rightarrow \text{sunny} \rightarrow \text{sunny} = 0.4 \times 0.4$$

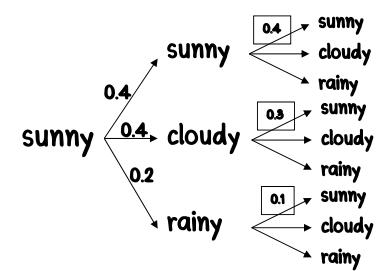
$$\text{sunny} \rightarrow \text{cloudy} \rightarrow \text{sunny} = 0.4 \times 0.3$$

$$\text{sunny} \rightarrow \text{rainy} \rightarrow \text{sunny} = 0.2 \times 0.1$$

The probability that the day after tomorrow is :  $0.4 \times 0.4 + 0.4 \times 0.3 + 0.2 \times 0.1$



		Tomorrow		
		sunny	cloudy	rainy
The day after tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3



## Opening Remarks

- Predicting the Weather

- When things get messy, it helps to introduce some notation.

- Let  $\chi_s^{(k)}$  denote the probability that it will be sunny  $k$  days from now (on day  $k$ ).
- Let  $\chi_c^{(k)}$  denote the probability that it will be cloudy  $k$  days from now.
- Let  $\chi_r^{(k)}$  denote the probability that it will be rainy  $k$  days from now.

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

## Opening Remarks

- Predicting the Weather

- What is the probability that tomorrow is

- sunny?
- cloudy?
- rainy?

- The discussion so far motivate the equations ( $k$  : today  $k+1$  : tomorrow)

- $\chi_s^{(k+1)} = 0.4 \times \chi_s^{(k)} + 0.3 \times \chi_c^{(k)} + 0.1 \times \chi_r^{(k)}$
- $\chi_c^{(k+1)} = 0.4 \times \chi_s^{(k)} + 0.3 \times \chi_c^{(k)} + 0.6 \times \chi_r^{(k)}$
- $\chi_r^{(k+1)} = 0.2 \times \chi_s^{(k)} + 0.4 \times \chi_c^{(k)} + 0.3 \times \chi_r^{(k)}$

## Opening Remarks

- Predicting the Weather

- If today is cloudy, what is the probability that tomorrow is sunny? cloudy? rainy?

- today is cloudy :  $\chi_s^{(0)} = 0$ ,  $\chi_c^{(0)} = 1$  and  $\chi_r^{(0)} = 0$
- $\chi_s^{(1)} = 0.4 \times \chi_s^{(0)} + 0.3 \times \chi_c^{(0)} + 0.1 \times \chi_r^{(0)}$   
 $= 0.4 \times 0 + 0.3 \times 1 + 0.1 \times 0 = 0.3$
- $\chi_c^{(1)} = 0.4 \times \chi_s^{(0)} + 0.3 \times \chi_c^{(0)} + 0.6 \times \chi_r^{(0)}$   
 $= 0.4 \times 0 + 0.3 \times 1 + 0.6 \times 0 = 0.3$
- $\chi_r^{(1)} = 0.2 \times \chi_s^{(0)} + 0.4 \times \chi_c^{(0)} + 0.3 \times \chi_r^{(0)}$   
 $= 0.2 \times 0 + 0.4 \times 1 + 0.3 \times 0 = 0.4$

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

## Opening Remarks

- Predicting the Weather

- The probabilities that denote what the weather may be on day  $k$  and the table that summarizes the probabilities are often represented as a (state) vector,  $x^{(k)}$ , and (transition) matrix,  $P$ , respectively:

$$\bullet x^{(k)} = \begin{pmatrix} \chi_s^{(k)} \\ \chi_c^{(k)} \\ \chi_r^{(k)} \end{pmatrix} \text{ and } P = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix}$$

- The transition from day  $k$  to day  $k+1$  is then written as the matrix-vector product (multiplication)

$$\bullet x^{(k+1)} = \begin{pmatrix} \chi_s^{(k+1)} \\ \chi_c^{(k+1)} \\ \chi_r^{(k+1)} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} \chi_s^{(k)} \\ \chi_c^{(k)} \\ \chi_r^{(k)} \end{pmatrix}$$

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

# Opening Remarks

## Predicting the Weather

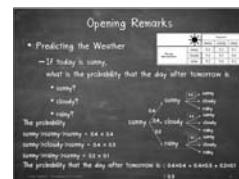
- If today is cloudy, what is the probability that tomorrow is sunny? cloudy? rainy?

$$\text{today is cloudy : } x^{(0)} = \begin{pmatrix} x_s^{(0)} \\ x_c^{(0)} \\ x_r^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x^{(1)} &= \begin{pmatrix} x_s^{(1)} \\ x_c^{(1)} \\ x_r^{(1)} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} x_s^{(0)} \\ x_c^{(0)} \\ x_r^{(0)} \end{pmatrix} \\ &= \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.4 \times 0 + 0.3 \times 1 + 0.1 \times 0 \\ 0.4 \times 0 + 0.3 \times 1 + 0.6 \times 0 \\ 0.2 \times 0 + 0.4 \times 1 + 0.3 \times 0 \end{pmatrix} \\ &= \begin{pmatrix} 0.3 \\ 0.4 \\ 0.3 \end{pmatrix} \end{aligned}$$

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

Linear Algebra - Foundations to Frontiers



# Opening Remarks (MATLAB)

```
>> P = [
    0.4 0.3 0.1
    0.4 0.3 0.6
    0.2 0.4 0.3
];
>> x0 = [
    1
    0
    0
];
>> x1 = P * x0
x1 =
    0.4000
    0.4000
    0.2000
;
>> x2 = P * x1
x2 =
    0.3000
    0.4000
    0.3000
;
>> x3 = P * x2
x3 =
    0.2700
    0.4200
    0.3100
;
>> x4 = P * x3
x4 =
    0.2650
    0.4200
    0.3150
;
>> x5 = P * x4
x5 =
    0.2635
    0.4210
    0.3155
;
>> x6 = P * x5
x6 =
    0.2633
    0.4210
    0.3158
;
>> x7 = P * x6
x7 =
    0.2632
    0.4211
    0.3158
;
>>
```

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# Opening Remarks

## Predicting the Weather

- If today is sunny, what is the probability that the day after tomorrow is sunny? cloudy? rainy?

$$\text{today is sunny : } x^{(0)} = \begin{pmatrix} x_s^{(0)} \\ x_c^{(0)} \\ x_r^{(0)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} x^{(2)} &= \begin{pmatrix} x_s^{(2)} \\ x_c^{(2)} \\ x_r^{(2)} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} x_s^{(1)} \\ x_c^{(1)} \\ x_r^{(1)} \end{pmatrix} \\ &= \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} &= \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} x_s^{(0)} \\ x_c^{(0)} \\ x_r^{(0)} \end{pmatrix} \\ &= \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.4 \\ 0.4 \\ 0.2 \end{pmatrix} \\ &= \begin{pmatrix} 0.4 \times 0.4 + 0.3 \times 0.4 + 0.1 \times 0.2 \\ 0.4 \times 0.4 + 0.3 \times 0.4 + 0.6 \times 0.2 \\ 0.2 \times 0.2 + 0.4 \times 0.4 + 0.3 \times 0.2 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.4 \\ 0.3 \end{pmatrix} \end{aligned}$$

		Tomorrow		
		sunny	cloudy	rainy
The day after tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

```
>> P = [
    0.4 0.3 0.1
    0.4 0.3 0.6
    0.2 0.4 0.3
];
>> x0 = [
    1
    0
    0
];
>> x1 = P*x0
x1 =
    0.4000
    0.4000
    0.3000
;
>> x2 = P*(P*x0)
x2 =
    0.3000
    0.4000
    0.3000
;
>> x3 = P*(P*x1)
x3 =
    0.2635
    0.4210
    0.3155
;
>> x4 = P*(P*x2)
x4 =
    0.2633
    0.4210
    0.3158
;
>> x5 = P*(P*x3)
x5 =
    0.2632
    0.4211
    0.3158
;
>>
```

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# Opening Remarks (MATLAB)

```
>> P = [
    0.4 0.3 0.1
    0.4 0.3 0.6
    0.2 0.4 0.3
];
>> x = [
    1
    0
    0
];
>> for i=1:7
    i =
        1
    x =
        0.3000
    i =
        2
    x =
        0.2635
    i =
        3
    x =
        0.2633
    i =
        4
    x =
        0.2632
    i =
        5
    x =
        0.2631
    i =
        6
    x =
        0.2630
    i =
        7
    x =
        0.2629
;
>>
```

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## Opening Remarks (MATLAB)

```
for i=1:365      for i=1:365
    i                  i;
    x = P * x        x = P * x;
end                end
                    i, x
```

## Opening Remarks

Homework 4.1.1.4 Given

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

fill in the following table, which predicts the weather the day after tomorrow given the weather today:

		Today		
		sunny	cloudy	rainy
Day after Tomorrow	sunny			
	cloudy			
	rainy			

## Preparation

- Partitioned Matrix-Vector
- Transposing a Partitioned Matrix
- Matrix-Vector Multiplication, Again

## Preparation

- Partitioned Matrix-Vector Multiplication

### — Motivation

$$A = \left( \begin{array}{c|cc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) = \left( \begin{array}{ccc|cc} -1 & 2 & 4 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ \hline 2 & -1 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 & 3 \\ \hline -1 & -2 & 0 & 1 & 2 \end{array} \right),$$

### • Consider

$$x = \begin{pmatrix} x_0 \\ \chi_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \quad \text{and} \quad y = \begin{pmatrix} y_0 \\ \psi_1 \\ y_2 \end{pmatrix},$$

— where  $y_0, y_2 \in \mathbb{R}^2$ . Then  $y := Ax$  means that

$$y = \begin{pmatrix} y_0 \\ \psi_1 \\ y_2 \end{pmatrix} = \left( \begin{array}{c|cc} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) \begin{pmatrix} x_0 \\ \chi_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{00}x_0 + a_{01}\chi_1 + A_{02}x_2 \\ a_{10}^T x_0 + \alpha_{11}\chi_1 + a_{12}^T x_2 \\ A_{20}x_0 + a_{21}\chi_1 + A_{22}x_2 \end{pmatrix}$$

## Preparation

$$A = \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & a_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) = \left( \begin{array}{ccc|cc} -1 & 2 & 4 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ \hline 2 & -1 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 & 3 \\ \hline -1 & -2 & 0 & 1 & 2 \end{array} \right), \quad y = \begin{pmatrix} y_0 \\ \Psi_1 \\ y_2 \end{pmatrix} = \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & a_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) \begin{pmatrix} x_0 \\ \chi_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{00}x_0 + a_{01}\chi_1 + A_{02}x_2 \\ a_{10}^T x_0 + a_{11}\chi_1 + a_{12}^T x_2 \\ A_{20}x_0 + a_{21}\chi_1 + A_{22}x_2 \end{pmatrix}$$

$$x = \begin{pmatrix} x_0 \\ \chi_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}, \quad \text{and } y = \begin{pmatrix} y_0 \\ \Psi_1 \\ y_2 \end{pmatrix}.$$

$$= \begin{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} 3 + \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ \begin{pmatrix} 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \end{pmatrix} 3 + \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \end{pmatrix} 3 + \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix} \end{pmatrix} =$$

$$\begin{pmatrix} (-1) \times (1) + (2) \times (2) \\ (1) \times (1) + (0) \times (2) \\ (2) \times (1) + (-1) \times (2) \end{pmatrix} + \begin{pmatrix} (4) \times (3) \\ (-1) \times (3) \\ (3) \times (3) \end{pmatrix} + \begin{pmatrix} (1) \times (4) + (0) \times (5) \\ (-2) \times (4) + (1) \times (5) \\ (1) \times (4) + (2) \times (5) \end{pmatrix} =$$

$$\begin{pmatrix} (-1) \times (1) + (2) \times (2) + (4) \times (3) + (1) \times (4) + (0) \times (5) \\ (1) \times (1) + (0) \times (2) + (-1) \times (3) + (-2) \times (4) + (1) \times (5) \\ (2) \times (1) + (-1) \times (2) + (3) \times (3) + (1) \times (4) + (2) \times (5) \end{pmatrix} =$$

$$\begin{pmatrix} (1) \times (1) + (2) \times (2) + (3) \times (3) + (4) \times (4) + (3) \times (5) \\ (-1) \times (1) + (-2) \times (2) + (0) \times (3) + (1) \times (4) + (2) \times (5) \end{pmatrix} =$$

$$\begin{pmatrix} 19 \\ -5 \\ 23 \\ 45 \\ 9 \end{pmatrix}$$

## Preparation

**Homework 4.2.1.1** Consider

$$A = \left( \begin{array}{ccccc} -1 & 2 & 4 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ 2 & -1 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 & 3 \\ -1 & -2 & 0 & 1 & 2 \end{array} \right) \quad \text{and} \quad x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix},$$

and partition these into submatrices (regions) as follows:

$$\left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & a_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right) \quad \text{and} \quad \begin{pmatrix} x_0 \\ \chi_1 \\ x_2 \end{pmatrix},$$

where  $A_{00} \in \mathbb{R}^{3 \times 3}$ ,  $x_0 \in \mathbb{R}^3$ ,  $a_{11}$  is a scalar, and  $\chi_1$  is a scalar. Show with lines how  $A$  and  $x$  are partitioned:

$$\left( \begin{array}{ccc|cc} -1 & 2 & 4 & 1 & 0 \\ 1 & 0 & -1 & -2 & 1 \\ 2 & -1 & 3 & 1 & 2 \\ \hline 1 & 2 & 3 & 4 & 3 \\ -1 & -2 & 0 & 1 & 2 \end{array} \right) \quad \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}.$$

## Preparation

### • Partitioned Matrix-Vector Multiplication

#### Theory

Let  $A \in \mathbb{R}^{m \times n}$ ,  $x \in \mathbb{R}^n$ , and  $y \in \mathbb{R}^m$ . Partition

$$A = \left( \begin{array}{c|c|c} A_{0,0} & A_{0,1} & \cdots & A_{0,N-1} \\ \hline A_{1,0} & A_{1,1} & \cdots & A_{1,N-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline A_{M-1,0} & A_{M-1,1} & \cdots & A_{M-1,N-1} \end{array} \right), \quad x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix}, \quad \text{and } y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{M-1} \end{pmatrix}$$

where

- $m = m_0 + m_1 + \cdots + m_{M-1}$ ,
- $m_i \geq 0$  for  $i = 0, \dots, M-1$ ,
- $n = n_0 + n_1 + \cdots + n_{N-1}$ ,
- $n_j \geq 0$  for  $j = 0, \dots, N-1$ , and
- $A_{i,j} \in \mathbb{R}^{m_i \times n_j}$ ,  $x_j \in \mathbb{R}^{n_j}$ , and  $y_i \in \mathbb{R}^{m_i}$ .

## Preparation

If  $y = Ax$  then

$$\begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,N-1} \\ \hline A_{1,0} & A_{1,1} & \cdots & A_{1,N-1} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline A_{M-1,0} & A_{M-1,1} & \cdots & A_{M-1,N-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix} = \begin{pmatrix} A_{0,0}x_0 + A_{0,1}x_1 + \cdots + A_{0,N-1}x_{N-1} \\ A_{1,0}x_0 + A_{1,1}x_1 + \cdots + A_{1,N-1}x_{N-1} \\ \vdots \\ A_{M-1,0}x_0 + A_{M-1,1}x_1 + \cdots + A_{M-1,N-1}x_{N-1} \end{pmatrix}.$$

In other words,

$$y_i = \sum_{j=0}^{N-1} A_{i,j}x_j.$$

# Preparation

- Transposing a Partitioned Matrix

- Motivation

- Consider

$$\begin{array}{c|c} \left( \begin{array}{ccc|cc} 1 & -1 & 3 & 2 \\ 2 & -2 & 1 & 0 \\ \hline 0 & -4 & 3 & 2 \end{array} \right)^T = \left( \begin{array}{ccc|c} 1 & -1 & 3 & 2 \\ 2 & -2 & 1 & 0 \\ \hline 0 & -4 & 3 & 2 \end{array} \right)^T \\ = \left( \begin{array}{c|c} \left( \begin{array}{ccc} 1 & -1 & 3 \\ 2 & -2 & 1 \end{array} \right)^T & \left( \begin{array}{c} 2 \\ 0 \end{array} \right)^T \\ \hline \left( \begin{array}{c} 2 \\ 0 \end{array} \right)^T & \left( \begin{array}{c} 2 \end{array} \right)^T \end{array} \right) \\ = \left( \begin{array}{c|c} \left( \begin{array}{cc} 1 & 2 \\ -1 & -2 \end{array} \right) & \left( \begin{array}{c} 0 \\ -4 \\ 3 \end{array} \right) \\ \hline \left( \begin{array}{c} 2 \\ 0 \end{array} \right) & \left( \begin{array}{c} 2 \end{array} \right) \end{array} \right) = \left( \begin{array}{cc|c} 1 & 2 & 0 \\ -1 & -2 & -4 \\ \hline 2 & 0 & 2 \end{array} \right). \end{array}$$

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# Preparation

- Transposing a Partitioned Matrix

- Theory

Let  $A \in \mathbb{R}^{m \times n}$  be partitioned as follows:

$$A = \left( \begin{array}{c|c|c|c} A_{0,0} & A_{0,1} & \cdots & A_{0,N-1} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,N-1} \\ \vdots & \vdots & & \vdots \\ A_{M-1,0} & A_{M-1,1} & \cdots & A_{M-1,N-1} \end{array} \right),$$

where  $A_{i,j} \in \mathbb{R}^{m_i \times n_j}$ . Then

$$A^T = \left( \begin{array}{c|c|c|c} A_{0,0}^T & A_{1,0}^T & \cdots & A_{M-1,0}^T \\ A_{0,1}^T & A_{1,1}^T & \cdots & A_{M-1,1}^T \\ \vdots & \vdots & & \vdots \\ A_{0,N-1}^T & A_{1,N-1}^T & \cdots & A_{M-1,N-1}^T \end{array} \right).$$

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# Preparation

- Transposing a Partitioned Matrix (Special cases)

Each submatrix is a scalar. If

$$A = \left( \begin{array}{c|c|c} a_{0,0} & a_{0,1} & \cdots & a_{0,N-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{1,N-1} \\ \vdots & \vdots & & \vdots \\ a_{M-1,0} & a_{M-1,1} & \cdots & a_{M-1,N-1} \end{array} \right)$$

then

$$A^T = \left( \begin{array}{c|c|c} a_{0,0}^T & a_{1,0}^T & \cdots & a_{M-1,0}^T \\ a_{0,1}^T & a_{1,1}^T & \cdots & a_{M-1,1}^T \\ \vdots & \vdots & & \vdots \\ a_{0,N-1}^T & a_{1,N-1}^T & \cdots & a_{M-1,N-1}^T \end{array} \right) = \left( \begin{array}{cccc} a_{0,0} & a_{1,0} & \cdots & a_{M-1,0} \\ a_{0,1} & a_{1,1} & \cdots & a_{M-1,1} \\ \vdots & \vdots & & \vdots \\ a_{0,N-1} & a_{1,N-1} & \cdots & a_{M-1,N-1} \end{array} \right),$$

This is because the transpose of a scalar is just that scalar.

$2 \times 2$  blocked partitioning. If

$$A = \left( \begin{array}{c|c} ATL & ATR \\ ABL & ABR \end{array} \right),$$

then

$$A^T = \left( \begin{array}{c|c} AT_L & A^T_R \\ A^T_L & A^T_B \end{array} \right).$$

$3 \times 3$  blocked partitioning. If

$$A = \left( \begin{array}{c|c|c} A_{0,0} & a_{0,1} & A_{0,2} \\ \hline A_{1,0} & a_{1,1} & a_{1,2} \\ \hline A_{2,0} & a_{2,1} & A_{2,2} \end{array} \right),$$

then

$$A^T = \left( \begin{array}{c|c|c} A_{0,0}^T & (a_{0,0}^T)^T & A_{0,2}^T \\ \hline A_{1,0}^T & a_{1,1}^T & a_{1,2}^T \\ \hline A_{2,0}^T & (a_{2,0}^T)^T & A_{2,2}^T \end{array} \right) = \left( \begin{array}{c|c|c} A_{0,0}^T & a_{0,1}^T & a_{0,2}^T \\ \hline A_{1,0}^T & a_{1,1}^T & a_{1,2}^T \\ \hline A_{2,0}^T & a_{2,1}^T & A_{2,2}^T \end{array} \right) = \left( \begin{array}{c|c|c} a_{0,0}^T & a_{0,1}^T & a_{0,2}^T \\ \hline a_{1,0}^T & a_{1,1}^T & a_{1,2}^T \\ \hline a_{2,0}^T & a_{2,1}^T & A_{2,2}^T \end{array} \right).$$

The matrix is partitioned by rows. If

$$A = \left( \begin{array}{c} \vec{a}_0^T \\ \vec{a}_1^T \\ \vdots \\ \vec{a}_{m-1}^T \end{array} \right),$$

where each  $\vec{a}_i^T$  is a row of  $A$ , then

$$A^T = \left( \begin{array}{c} \vec{a}_0^T \\ \vec{a}_1^T \\ \vdots \\ \vec{a}_{m-1}^T \end{array} \right)^T = \left( (\vec{a}_0^T)^T \mid (\vec{a}_1^T)^T \mid \cdots \mid (\vec{a}_{m-1}^T)^T \right) = \left( \vec{a}_0 \mid \vec{a}_1 \mid \cdots \mid \vec{a}_{m-1} \right).$$

This shows that rows of  $A$ ,  $\vec{a}_i^T$ , become columns of  $A^T$ :  $\vec{a}_i$ .

The matrix is partitioned by columns. If

$$A = \left( \begin{array}{c|c|c} a_0 & a_1 & \cdots & a_{n-1} \end{array} \right),$$

where each  $a_j$  is a column of  $A$ , then

# Preparation

- Matrix-Vector Multiplication, Again

- Consider  $y = Ax + b$

$$\bullet A = \begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}.$$

$$\bullet x = \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}, y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$(1 \quad -2 \quad 3 \quad 2 \quad -2) \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 1$$

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## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$(2 \quad -2 \quad 1 \quad 0 \quad -1) \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = -1$$

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## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$(0 \quad -4 \quad 3 \quad 2 \quad 1) \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 5$$

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## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$(3 \quad 1 \quad -2 \quad 1 \quad 0) \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = -8$$

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## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ 2 \end{pmatrix}$

$$(-1 \ 2 \ 1 \ -1 \ -2) \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 2$$

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## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ 2 \end{pmatrix}$

$$\psi_i := \alpha_i^T x + \psi_i$$



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## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$(1) \times (-1) + (-2 \ 3 \ 2 \ -2) \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 1$$

LINNAEUS - FOUNDATIONS TO FUNDAMENTALS

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## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$(2)(-1) + (2) \times (-1) + (1 \ 0 \ -1) \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = -1$$

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## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\left( \begin{array}{cc cc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ \hline 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array} \right)$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 5 \\ 0 \\ 0 \end{pmatrix}$

$$(0 \quad -4) \begin{pmatrix} -1 \\ 0 \end{pmatrix} + (3) \times (2) + (2 \quad 1) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 5$$

## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 1 \\ -1 \\ 5 \\ 0 \\ 0 \end{pmatrix}$	$\left( \begin{array}{ccc cc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ \hline 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array} \right)$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ 0 \end{pmatrix}$

$$(3 \quad 1 \quad -2) \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + (1) \times (-1) + (0)(-1) = -8$$

## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ 0 \end{pmatrix}$	$\left( \begin{array}{cccc c} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ \hline -1 & 2 & 1 & -1 & -2 \end{array} \right)$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ 2 \end{pmatrix}$

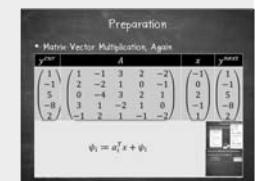
$$(-1 \quad 2 \quad 1 \quad -1) \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \end{pmatrix} + (-2) \times (1) = 2$$

## Preparation

- Matrix-Vector Multiplication, Again

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ 0 \end{pmatrix}$	$\left( \begin{array}{cccc c} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ \hline -1 & 2 & 1 & -1 & -2 \end{array} \right)$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ -1 \\ 5 \\ -8 \\ 2 \end{pmatrix}$

$$\psi_i := \alpha_{10}^T x_0 + \alpha_{11} \chi_1 + \alpha_{12}^T x_2 + \psi_i$$



## Preparation

- Matrix-Vector Multiplication, Again  
Original (Week3.pdf)

Algorithm:  $y := \text{MVMULT\_N\_UNB\_VAR1}(A, x, y)$

Partition  $A \rightarrow \begin{pmatrix} A_T \\ A_B \end{pmatrix}$ ,  $y \rightarrow \begin{pmatrix} y_T \\ y_B \end{pmatrix}$   
where  $A_T$  is  $0 \times n$  and  $y_T$  is  $0 \times 1$

while  $m(A_T) < m(A)$  do

Repartition

$$\begin{pmatrix} A_T \\ A_B \end{pmatrix} \rightarrow \begin{pmatrix} A_0 \\ a_1^T \\ A_2 \end{pmatrix} \cdot \begin{pmatrix} y_T \\ y_B \end{pmatrix} \rightarrow \begin{pmatrix} y_0 \\ \Psi_1 \\ y_2 \end{pmatrix}$$

where  $a_1$  is a row

$$\Psi_1 := a_1^T x + \Psi_1$$

Continue with

$$\begin{pmatrix} A_T \\ A_B \end{pmatrix} \leftarrow \begin{pmatrix} A_0 \\ a_1^T \\ A_2 \end{pmatrix}, \begin{pmatrix} y_T \\ y_B \end{pmatrix} \leftarrow \begin{pmatrix} y_0 \\ \Psi_1 \\ y_2 \end{pmatrix}$$

endwhile

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## Modified (Week4.pdf)

Algorithm:  $y := \text{MVMULT\_N\_UNB\_VAR1B}(A, x, y)$

Partition  $A \rightarrow \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix}$ ,

$x \rightarrow \begin{pmatrix} x_T \\ x_B \end{pmatrix}$ ,  $y \rightarrow \begin{pmatrix} y_T \\ y_B \end{pmatrix}$   
where  $A_{TL}$  is  $0 \times 0$ ,  $x_T, y_T$  are  $0 \times 1$   
while  $m(A_{TL}) < m(A)$  do

Repartition

$$\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \rightarrow \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ a_{10}^T & a_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{pmatrix}$$

$$\begin{pmatrix} x_T \\ x_B \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ \chi_1 \\ x_2 \end{pmatrix} \cdot \begin{pmatrix} y_T \\ y_B \end{pmatrix} \rightarrow \begin{pmatrix} y_0 \\ \Psi_1 \\ y_2 \end{pmatrix}$$

$$\Psi_1 := a_{10}^T x_0 + \alpha_{11} \chi_1 + a_{12}^T x_2 + \Psi_1$$

Continue with

$$\begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix} \leftarrow \begin{pmatrix} A_{00} & a_{01} & A_{02} \\ a_{10}^T & a_{11} & a_{12}^T \\ A_{20} & a_{21} & A_{22} \end{pmatrix}$$

$$\begin{pmatrix} x_T \\ x_B \end{pmatrix} \leftarrow \begin{pmatrix} x_0 \\ \chi_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_T \\ y_B \end{pmatrix} \leftarrow \begin{pmatrix} y_0 \\ \Psi_1 \\ y_2 \end{pmatrix}$$

endwhile

## Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication
- Triangular Matrix-Vector Multiplication
- Symmetric Matrix-Vector Multiplication .

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## Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication :  $A^T x + y$

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & 0 & 2 & 1 & 0 \\ 3 & 1 & -2 & 1 & -1 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & 0 & 2 & 1 & 0 \\ 3 & 1 & -2 & 1 & -1 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & 0 & 2 & 1 & 0 \\ 3 & 1 & -2 & 1 & -1 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & 0 & 2 & 1 & 0 \\ 3 & 1 & -2 & 1 & -1 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \\ -1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = -5$$

## Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication

$$-y := Ax + y$$

$$-y := A^T x + y$$

Preparation				
Matrix-Vector Multiplication, Again				
$y^{cur}$	$A$	$x$	$y^{next}$	
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & 0 & 2 & 1 & 0 \\ 3 & 1 & -2 & 1 & -1 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$		

$$(1 \quad -2 \quad 3 \quad 2 \quad -2) \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1$$

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## Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication :  $A^T x + y$

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} -5 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\left( \begin{array}{cc ccc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array} \right)$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -5 \\ -6 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1 \\ -2 \\ -4 \\ 1 \\ 2 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = -6$$

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## Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication :  $A^T x + y$

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} -5 \\ -6 \\ 6 \\ 0 \\ 0 \end{pmatrix}$	$\left( \begin{array}{cc ccc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array} \right)$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -5 \\ -6 \\ 6 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 0$$

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## Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication :  $A^T x + y$

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} -5 \\ -6 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\left( \begin{array}{cc ccc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array} \right)$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -5 \\ -6 \\ 6 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 3 \\ 1 \\ 3 \\ -2 \\ 1 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 6$$

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## Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication :  $A^T x + y$

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} -5 \\ -6 \\ 6 \\ 0 \\ 0 \end{pmatrix}$	$\left( \begin{array}{cc ccc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array} \right)$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -5 \\ -6 \\ 6 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \\ -2 \end{pmatrix}^T \begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 2$$

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## Matrix-Vector Multiplication with Special Matrices

## Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication :  $A^T x + y$

$y^{cur}$	$A$	$x$	$y^{next}$
$\begin{pmatrix} -5 \\ -6 \\ 6 \\ 0 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{pmatrix}$	$\begin{pmatrix} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -5 \\ -6 \\ 6 \\ 0 \\ 2 \end{pmatrix}$

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## Matrix-Vector Multiplication with Special Matrices

- Transpose Matrix-Vector Multiplication :  $A^T x + y$
- Original (Week3.pdf)

Algorithm:  $y := \text{MVMULT\_T\_UNB\_VAR1}(A, x, y)$

Partition  $A \rightarrow \begin{pmatrix} A_T \\ A_B \end{pmatrix}, y \rightarrow \begin{pmatrix} y_T \\ y_B \end{pmatrix}$   
where  $A_T$  is  $0 \times n$  and  $y_T$  is  $0 \times 1$   
while  $m(A_T) < m(A)$  do

Repartition

$$\left( \frac{A_T}{A_B} \right) \rightarrow \left( \frac{a_0}{a_1^T} \right), \left( \frac{y_T}{y_B} \right) \rightarrow \left( \frac{y_0}{y_1} \right)$$

where  $a_1$  is a row

---

$\Psi_1 := a_1^T x + \Psi_1$

---

Continue with

$$\left( \frac{A_T}{A_B} \right) \leftarrow \left( \frac{a_0}{a_1^T} \right), \left( \frac{y_T}{y_B} \right) \leftarrow \left( \frac{y_0}{\Psi_1} \right)$$

endwhile

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Algorithm:  $y := \text{MVMULT\_T\_UNB\_VAR1}(A, x, y)$

Partition  $A \rightarrow \begin{pmatrix} A_L | A_R \end{pmatrix}, y \rightarrow \begin{pmatrix} y_T \\ y_B \end{pmatrix}$   
where  $A_L$  is  $m \times 0$  and  $y_T$  is  $0 \times 1$   
while  $m(y_T) < m(y)$  do

Repartition

$$(A_L | A_R) \rightarrow (A_0 | a_1 | A_2), \left( \frac{y_T}{y_B} \right) \rightarrow \left( \frac{y_0}{\Psi_1} \right)$$


---

$\Psi_1 := a_1^T x + \Psi_1$

---

Continue with

$$(A_L | A_R) \leftarrow (A_0 | a_1 | A_2), \left( \frac{y_T}{y_B} \right) \leftarrow \left( \frac{y_0}{\Psi_1} \right)$$

endwhile

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## Matrix-Vector Multiplication with Special Matrices

- Triangular Matrix-Vector Multiplication

Algorithms for  $y := Ax + y$  with triangular matrices

Discuss

$y := Ux + y$

Where  $U$  is upper triangular

$$Ux = \begin{pmatrix} U_{00} & u_{01} & U_{02} \\ u_{10}^T & u_{11} & u_{12}^T \\ U_{20} & u_{21} & U_{22} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 4 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 3 & 1 & 2 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x_T \\ x_B \end{pmatrix} \rightarrow \begin{pmatrix} x_0 \\ \chi_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_T \\ y_B \end{pmatrix} \rightarrow \begin{pmatrix} y_0 \\ \Psi_1 \end{pmatrix}$$


---

$\Psi_1 := a_{10}^T x_0 + \alpha_{11} \chi_1 + a_{12}^T x_2 + \Psi_1$



## Matrix-Vector Multiplication with Special Matrices

- Triangular Matrix-Vector Multiplication

Algorithm:  $y := \text{MVMULT\_N\_UNB\_VAR1}(A, x, y)$

Partition  $A \rightarrow \begin{pmatrix} A_{TL} & A_{TR} \\ A_{BL} & A_{BR} \end{pmatrix}, x \rightarrow \begin{pmatrix} x_T \\ x_B \end{pmatrix}, y \rightarrow \begin{pmatrix} y_T \\ y_B \end{pmatrix}$   
where  $A_{TL}$  is  $0 \times 0$ ,  $x_T, y_T$  are  $0 \times 1$   
while  $m(A_{TL}) < m(A)$  do

Repartition

$$\left( \frac{A_{TL}}{A_{BL}} \right) \rightarrow \left( \frac{a_{00}}{a_{10}^T} \right), \left( \frac{a_{01}}{a_{21}^T} \right) \rightarrow \left( \frac{a_{02}}{A_{22}} \right),$$

$$\left( \frac{x_T}{x_B} \right) \rightarrow \left( \frac{x_0}{\chi_1} \right), \left( \frac{y_T}{y_B} \right) \rightarrow \left( \frac{y_0}{\Psi_1} \right)$$


---

$\Psi_1 := a_{10}^T x_0 + \alpha_{11} \chi_1 + a_{12}^T x_2 + \Psi_1$

---

Continue with

$$\left( \frac{A_{TL}}{A_{BL}} \right) \leftarrow \left( \frac{a_{00}}{a_{10}^T} \right), \left( \frac{a_{01}}{a_{21}^T} \right) \leftarrow \left( \frac{a_{01}}{a_{22}} \right),$$

$$\left( \frac{x_T}{x_B} \right) \leftarrow \left( \frac{x_0}{\chi_1} \right), \left( \frac{y_T}{y_B} \right) \leftarrow \left( \frac{y_0}{\Psi_1} \right)$$

endwhile



Algorithm:  $y := \text{TRMV\_UN\_UNB\_VAR1}(U, x, y)$

Partition  $U \rightarrow \begin{pmatrix} U_{TL} & U_{TR} \\ U_{BL} & U_{BR} \end{pmatrix}, x \rightarrow \begin{pmatrix} x_T \\ x_B \end{pmatrix}, y \rightarrow \begin{pmatrix} y_T \\ y_B \end{pmatrix}$   
where  $U_{TL}$  is  $0 \times 0$ ,  $x_T, y_T$  are  $0 \times 1$   
while  $m(U_{TL}) < m(U)$  do

Repartition

$$\left( \frac{U_{TL}}{U_{BL}} \right) \rightarrow \left( \frac{u_{00}}{u_{10}^T} \right), \left( \frac{u_{01}}{u_{21}^T} \right) \rightarrow \left( \frac{u_{02}}{U_{22}} \right),$$

$$\left( \frac{x_T}{x_B} \right) \rightarrow \left( \frac{x_0}{\chi_1} \right), \left( \frac{y_T}{y_B} \right) \rightarrow \left( \frac{y_0}{\Psi_1} \right)$$


---

$\Psi_1 := a_{10}^T x_0 + \alpha_{11} \chi_1 + a_{12}^T x_2 + \Psi_1$

---

Continue with

$$\left( \frac{U_{TL}}{U_{BL}} \right) \leftarrow \left( \frac{u_{00}}{u_{10}^T} \right), \left( \frac{u_{01}}{u_{21}^T} \right) \leftarrow \left( \frac{u_{01}}{u_{22}} \right),$$

$$\left( \frac{x_T}{x_B} \right) \leftarrow \left( \frac{x_0}{\chi_1} \right), \left( \frac{y_T}{y_B} \right) \leftarrow \left( \frac{y_0}{\Psi_1} \right)$$

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## Matrix-Vector Multiplication with Special Matrices

- Triangular Matrix-Vector Multiplication

Algorithm:  $y := \text{TRMV}_P\text{UNB\_VAR1}(U, x, y)$

Partition  $U \rightarrow \left( \begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array} \right)$ .

$x \rightarrow \left( \begin{array}{c} x_T \\ x_B \end{array} \right)$ ,  $y \rightarrow \left( \begin{array}{c} y_T \\ y_B \end{array} \right)$   
where  $U_{TL}$  is  $0 \times 0$ ,  $x_T, y_T$  are  $0 \times 1$   
while  $m(U_{TL}) < m(U)$  do  
Repartition

$$\left( \begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & v_{11} & u_{12}^T \\ U_{20} & u_{21} & U_{22} \end{array} \right).$$

$$\left( \begin{array}{c} x_T \\ x_B \end{array} \right) \rightarrow \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right)$$

$$\left( \begin{array}{c} y_T \\ y_B \end{array} \right) \rightarrow \left( \begin{array}{c} y_0 \\ y_1 \\ y_2 \end{array} \right)$$


---

 $\psi_1 := u_{10}^T x_0 + v_{11} x_1 + u_{12}^T x_2 + \psi_1$ 


---

Continue with

$$\left( \begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & v_{11} & u_{12}^T \\ U_{20} & u_{21} & A_{22} \end{array} \right).$$

$$\left( \begin{array}{c} x_T \\ x_B \end{array} \right) \leftarrow \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right)$$

$$\left( \begin{array}{c} y_T \\ y_B \end{array} \right) \leftarrow \left( \begin{array}{c} y_0 \\ y_1 \\ y_2 \end{array} \right)$$


---

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endwhile

Algorithm:  $y := \text{TRMV}_P\text{UNB\_VAR2}(U, x, y)$

Partition  $U \rightarrow \left( \begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array} \right)$ .

$x \rightarrow \left( \begin{array}{c} x_T \\ x_B \end{array} \right)$ ,  $y \rightarrow \left( \begin{array}{c} y_T \\ y_B \end{array} \right)$   
where  $U_{TL}$  is  $0 \times 0$ ,  $x_T, y_T$  are  $0 \times 1$   
while  $m(U_{TL}) < m(U)$  do  
Repartition

$$\left( \begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & v_{11} & u_{12}^T \\ U_{20} & u_{21} & U_{22} \end{array} \right).$$

$$\left( \begin{array}{c} x_T \\ x_B \end{array} \right) \rightarrow \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right)$$

$$\left( \begin{array}{c} y_T \\ y_B \end{array} \right) \rightarrow \left( \begin{array}{c} y_0 \\ y_1 \\ y_2 \end{array} \right)$$


---

 $y_0 := x_1 u_{01} + y_0$ 
 $\psi_1 := x_1 v_{11} + \psi_1$ 
 $y_1 := x_1 u_{21} + y_1$ 


---

Continue with

$$\left( \begin{array}{c|c} U_{TL} & U_{TR} \\ \hline U_{BL} & U_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & v_{11} & u_{12}^T \\ U_{20} & u_{21} & A_{22} \end{array} \right).$$

$$\left( \begin{array}{c} x_T \\ x_B \end{array} \right) \leftarrow \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right)$$

$$\left( \begin{array}{c} y_T \\ y_B \end{array} \right) \leftarrow \left( \begin{array}{c} y_0 \\ y_1 \\ y_2 \end{array} \right)$$


---

endwhile

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## Matrix-Vector Multiplication with Special Matrices

- Triangular Matrix-Vector Multiplication

$$y = Ux + y = \left( \begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & v_{11} & u_{12}^T \\ U_{20} & u_{21} & U_{22} \end{array} \right) \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right) + \left( \begin{array}{c} y_0 \\ \psi_1 \\ y_2 \end{array} \right)$$

- $u_{10}^T x_0 + v_{11} x_1 + u_{12}^T x_2 + \psi_1$
- $0 + 2 + 2(n-k-1)$
- $\sum_{k=0}^{n-1} (2 + 2(n-k-1))$
- $= \sum_{k=0}^{n-1} (2(n-k)) = 2 \sum_{k=0}^{n-1} n - k$
- $= 2(n + (n-1) + \dots + (n-(n-1)))$
- $= 2(n + (n-1) + \dots + (1)) = 2 \sum_{j=1}^n i = 2 \left( \frac{n(n+1)}{2} \right)$

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## Matrix-Vector Multiplication with Special Matrices

- Triangular Matrix-Vector Multiplication

$$y = Ux + y = \left( \begin{array}{c|c|c} U_{00} & u_{01} & U_{02} \\ \hline u_{10}^T & v_{11} & u_{12}^T \\ U_{20} & u_{21} & U_{22} \end{array} \right) \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right) + \left( \begin{array}{c} y_0 \\ \psi_1 \\ y_2 \end{array} \right)$$

- $y_0 = x_1 u_{01} + y_0$
- $\psi_1 = x_1 v_{11} + \psi_1$
- $y_2 = x_1 u_{21} + y_2$
- $2k+2+0$
- $\sum_{k=0}^{n-1} (2k+2) = 2 \sum_{k=0}^{n-1} (k+1) = 2 \sum_{k=1}^n k$

## Matrix-Vector Multiplication with Special Matrices

- Symmetric Matrix-Vector Multiplication

$$y = Ax + y = \left( \begin{array}{c|c|c} A_{00} & \alpha_{01} & A_{02} \\ \hline \alpha_{10}^T & \alpha_{11} & \alpha_{12}^T \\ A_{20} & \alpha_{21} & A_{22} \end{array} \right) \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right) + \left( \begin{array}{c} y_0 \\ \psi_1 \\ y_2 \end{array} \right)$$

$$= \left( \begin{array}{cc|cc} 1 & -1 & 3 & 2 & -2 \\ 2 & -2 & 1 & 0 & -1 \\ \hline 0 & -4 & 3 & 2 & 1 \\ 3 & 1 & -2 & 1 & 0 \\ -1 & 2 & 1 & -1 & -2 \end{array} \right) \left( \begin{array}{c} -1 \\ 0 \\ 2 \\ -1 \\ 1 \end{array} \right) + \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right)$$

## Matrix-Vector Multiplication with Special Matrices

- Matrix-Vector Multiplication

$$-y := Ax + y$$

$$-\psi_i := \alpha_{10}^T x_0 + \alpha_{11} x_1 + \alpha_{12}^T x_2 + \psi_i$$

- Symmetric Matrix-Vector Multiplication

$$-\psi_i := \alpha_{01}^T x_0 + \alpha_{11} x_1 + \alpha_{12}^T x_2 + \psi_i$$

$$\left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$$

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## Matrix-Vector Multiplication with Special Matrices

Algorithm:  $y := \text{SYMV\_U\_UNB\_VAR1}(A, x, y)$

Partition  $A \rightarrow \left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$ ,

$x \rightarrow \left( \begin{array}{c} x_T \\ x_B \end{array} \right)$ ,  $y \rightarrow \left( \begin{array}{c} y_T \\ y_B \end{array} \right)$

where  $A_{TL}$  is  $0 \times 0$ ,  $x_T, y_T$  are  $0 \times 1$

while  $m(A_{TL}) < m(A)$  do

Repartition

$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$ ,

$\left( \begin{array}{c} x_T \\ x_B \end{array} \right) \rightarrow \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right)$ ,  $\left( \begin{array}{c} y_T \\ y_B \end{array} \right) \rightarrow \left( \begin{array}{c} y_0 \\ y_1 \\ y_2 \end{array} \right)$

$\left( \begin{array}{c} a_{01}^T \\ a_{10}^T \end{array} \right) x_0 + \alpha_{11} x_1 + a_{12}^T x_2 + y_0 = y_1$

$\psi_1 := \underbrace{a_{01}^T}_{= a_{10}^T} x_0 + \alpha_{11} x_1 + a_{12}^T x_2 + \psi_1$

Continue with

$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$ ,

$\left( \begin{array}{c} x_T \\ x_B \end{array} \right) \leftarrow \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right)$ ,  $\left( \begin{array}{c} y_T \\ y_B \end{array} \right) \leftarrow \left( \begin{array}{c} y_0 \\ y_1 \\ y_2 \end{array} \right)$

endwhile

Algorithm:  $y := \text{SYMV\_U\_UNB\_VAR2}(A, x, y)$

Partition  $A \rightarrow \left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right)$ ,

$x \rightarrow \left( \begin{array}{c} x_T \\ x_B \end{array} \right)$ ,  $y \rightarrow \left( \begin{array}{c} y_T \\ y_B \end{array} \right)$

where  $A_{TL}$  is  $0 \times 0$ ,  $x_T, y_T$  are  $0 \times 1$

while  $m(A_{TL}) < m(A)$  do

Repartition

$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \rightarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$ ,

$\left( \begin{array}{c} x_T \\ x_B \end{array} \right) \rightarrow \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right)$ ,  $\left( \begin{array}{c} y_T \\ y_B \end{array} \right) \rightarrow \left( \begin{array}{c} y_0 \\ y_1 \\ y_2 \end{array} \right)$

$y_0 := x_1 a_{01} + y_0$   
 $\psi_1 := x_1 \alpha_{11} + \psi_1$   
 $y_2 := x_1 \frac{a_{11}}{a_{10}^T} + y_2$   
 $a_{12}$

Continue with

$\left( \begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array} \right) \leftarrow \left( \begin{array}{c|c|c} A_{00} & a_{01} & A_{02} \\ \hline a_{10}^T & \alpha_{11} & a_{12}^T \\ \hline A_{20} & a_{21} & A_{22} \end{array} \right)$ ,

$\left( \begin{array}{c} x_T \\ x_B \end{array} \right) \leftarrow \left( \begin{array}{c} x_0 \\ x_1 \\ x_2 \end{array} \right)$ ,  $\left( \begin{array}{c} y_T \\ y_B \end{array} \right) \leftarrow \left( \begin{array}{c} y_0 \\ y_1 \\ y_2 \end{array} \right)$

endwhile

## Matrix-Matrix Multiplication (Product)

- Motivation
- From Composing Linear Transformations to Matrix-Matrix Multiplication
- Computing the Matrix-Matrix Product
- Special Shapes

## Matrix-Matrix Multiplication (Product)

Given

		Today		
		sunny	cloudy	rainy
Tomorrow	sunny	0.4	0.3	0.1
	cloudy	0.4	0.3	0.6
	rainy	0.2	0.4	0.3

fill in the following table, which predicts the weather the day after tomorrow given the weather today:

		Today		
		sunny	cloudy	rainy
Day after Tomorrow	sunny			
	cloudy			
	rainy			

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## Matrix-Matrix Multiplication (Product)

The entries in the table turn out to be the entries in the transition matrix  $Q$  that was described just above the exercise:

$$\begin{pmatrix} \chi_s^{(2)} \\ \chi_c^{(2)} \\ \chi_r^{(2)} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} \chi_s^{(1)} \\ \chi_c^{(1)} \\ \chi_r^{(1)} \end{pmatrix}$$

$$= \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \left( \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix} \right) = Q \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix},$$

Now, those of you who remembered from, for example, some other course that

$$\begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \left( \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix} \right)$$

$$= \left( \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \right) \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix}$$

would recognize that

$$Q = \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.4 & 0.3 & 0.1 \\ 0.4 & 0.3 & 0.6 \\ 0.2 & 0.4 & 0.3 \end{pmatrix}.$$

## Matrix-Matrix Multiplication (Product)

- $L_P(x) = P \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix}$

- $L_Q(x) = Q \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix} = P \left( P \begin{pmatrix} \chi_s^{(0)} \\ \chi_c^{(0)} \\ \chi_r^{(0)} \end{pmatrix} \right) = L_P(L_P(x))$

## Matrix-Matrix Multiplication (Product)

- Question

- Is  $L_Q(x) = L_P(L_P(x))$  a linear transformation?
- Is  $L_C(x) = L_B(L_A(x))$  a linear transformation?
- If it is, how are matrices  $A$ ,  $B$  and  $C$  related?

## Matrix-Matrix Multiplication (Product)

- From Composing Linear Transformations to Matrix-Matrix Multiplication

- Homework

- Let  $L_A: \mathbb{R}^k \rightarrow \mathbb{R}^m$  and  $L_B: \mathbb{R}^n \rightarrow \mathbb{R}^k$  both be linear transformations and, for all  $x \in \mathbb{R}^n$ , define the function  $L_C: \mathbb{R}^n \rightarrow \mathbb{R}^m$  by  $L_C(x) = L_B(L_A(x))$ .  $L_C(x)$  is a linear transformations.

- Always

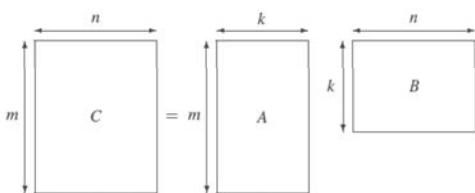
- Sometimes

- Never

# Matrix-Matrix Multiplication (Product)

- From Composing Linear Transformations to Matrix-Matrix Multiplication

If  $A$  is  $m_A \times n_A$  matrix,  $B$  is  $m_B \times n_B$  matrix, and  $C$  is  $m_C \times n_C$  matrix, then for  $C = AB$  to hold it must be the case that  $m_C = m_A$ ,  $n_C = n_B$ , and  $n_A = m_B$ . Usually, the integers  $m$  and  $n$  are used for the sizes of  $C$ :  $C \in \mathbb{R}^{m \times n}$  and  $k$  is used for the "other size":  $A \in \mathbb{R}^{m \times k}$  and  $B \in \mathbb{R}^{k \times n}$ .



# Matrix-Matrix Multiplication (Product)

- Summary

Let  $L_A: \mathbb{R}^k \rightarrow \mathbb{R}^m$  and  $L_B: \mathbb{R}^n \rightarrow \mathbb{R}^k$  are linear transformations and define  $L_C(x) = L_A(L_B(x))$ . Then

- $L_C: \mathbb{R}^n \rightarrow \mathbb{R}^m$  is linear transformations
- There are  $A \in \mathbb{R}^{m \times k}$  and  $B \in \mathbb{R}^{k \times n}$  that represent  $L_A$  and  $L_B$  respectively.
- There is a matrix,  $C \in \mathbb{R}^{m \times n}$  that represent  $L_C(x) = L_A(L_B(x)) = A(B(x))$ .
- The operation that computes  $C$  from  $A$  and  $B$  is called Matrix-Matrix Multiplication.
- Notation:  $C = AB$  and  $Cx = (AB)x = A(B(x))$

# Matrix-Matrix Multiplication (Product)

- Computing the Matrix-Matrix Product

Consider the following. Let

- $C \in \mathbb{R}^{m \times n}$ ,  $A \in \mathbb{R}^{m \times k}$  and  $B \in \mathbb{R}^{k \times n}$ ; and  $C = AB$ ; and
- $L_C: \mathbb{R}^n \rightarrow \mathbb{R}^m$  equal the linear transformation such that  $L_C(x) = Cx$ ; and
- $L_A: \mathbb{R}^k \rightarrow \mathbb{R}^m$  equal the linear transformation such that  $L_A(x) = Bx$ ; and
- $L_B: \mathbb{R}^n \rightarrow \mathbb{R}^k$  equal the linear transformation such that  $L_B(x) = Bx$ ; and
- $e_j$  denote the  $j$  th unit basis vector; and
- $c_j$  denote the  $j$  th column of  $C$ ; and
- $b_j$  denote the  $j$  th column of  $B$ .

$$c_j = Ce_j = L_C(e_j) = L_A(L_B(e_j)) = L_A(B(e_j)) = L_A(b_j) = Ab_j$$

# Matrix-Matrix Multiplication (Product)

From this we learn that

If  $C = AB$  then the  $j$ th column of  $C$ ,  $c_j$ , equals  $Ab_j$ , where  $b_j$  is the  $j$ th column of  $B$ .

Since by now you should be very comfortable with partitioning matrices by columns, we can summarize this as

$$\left( \begin{array}{c|c|c|c} c_0 & c_1 & \cdots & c_{n-1} \end{array} \right) = C = AB = A \left( \begin{array}{c|c|c|c} b_0 & b_1 & \cdots & b_{n-1} \end{array} \right) = \left( \begin{array}{c|c|c|c} Ab_0 & Ab_1 & \cdots & Ab_{n-1} \end{array} \right).$$

Now, let's expose the elements of  $C$ ,  $A$ , and  $B$ .

$$C = \begin{pmatrix} \gamma_{0,0} & \boxed{\gamma_{0,1}} & \cdots & \gamma_{0,n-1} \\ \gamma_{1,0} & \boxed{\gamma_{1,1}} & \cdots & \gamma_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m-1,0} & \boxed{\gamma_{m-1,1}} & \cdots & \gamma_{m-1,n-1} \end{pmatrix}, \quad A = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,k-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,k-1} \end{pmatrix},$$

$$\text{and } B = \begin{pmatrix} \beta_{0,0} & \boxed{\beta_{0,1}} & \cdots & \beta_{0,n-1} \\ \beta_{1,0} & \beta_{1,1} & \cdots & \beta_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k-1,0} & \beta_{k-1,1} & \cdots & \beta_{k-1,n-1} \end{pmatrix}.$$

We are going to show that

$$\gamma_{i,j} = \sum_{p=0}^{k-1} \alpha_{i,p} \beta_{p,j},$$

which you may have learned in a high school algebra course.

# Matrix-Matrix Multiplication (Product)

We reasoned that  $c_j = Ab_j$ :

$$\begin{pmatrix} \gamma_{0,j} \\ \gamma_{1,j} \\ \vdots \\ \boxed{\gamma_{i,j}} \\ \vdots \\ \gamma_{m-1,j} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,k-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{\alpha_{i,0}} & \alpha_{i,1} & \cdots & \alpha_{i,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,k-1} \end{pmatrix} \begin{pmatrix} \beta_{0,j} \\ \beta_{1,j} \\ \vdots \\ \beta_{k-1,j} \end{pmatrix}.$$

Here we highlight the  $i$ th element of  $c_j$ ,  $\gamma_{i,j}$ , and the  $i$ th row of  $A$ . We recall that the  $i$ th element of  $Ax$  equals the dot product of the  $i$ th row of  $A$  with the vector  $x$ . Thus,  $\gamma_{i,j}$  equals the dot product of the  $i$ th row of  $A$  with the vector  $b_j$ :

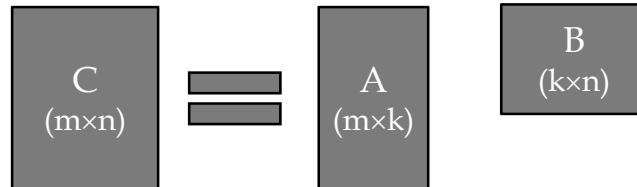
$$\gamma_{i,j} = \sum_{p=0}^{k-1} \alpha_{i,p} \beta_{p,j}.$$

Let  $A \in \mathbb{R}^{m \times k}$ ,  $B \in \mathbb{R}^{k \times n}$ , and  $C \in \mathbb{R}^{m \times n}$ . Then the matrix-matrix multiplication (product)  $C = AB$  is computed by

$$\gamma_{i,j} = \sum_{p=0}^{k-1} \alpha_{i,p} \beta_{p,j} = \alpha_{i,0} \beta_{0,j} + \alpha_{i,1} \beta_{1,j} + \cdots + \alpha_{i,k-1} \beta_{k-1,j}.$$

# Matrix-Matrix Multiplication (Product)

- Special Shapes



# Matrix-Matrix Multiplication (Product)

- Special Shapes : m=n=k=1 (scalar multiplication)

$m = n = k = 1$  (scalar multiplication)

$$1 \uparrow \boxed{C} = 1 \uparrow \boxed{A} \quad 1 \uparrow \boxed{B}$$

In this case, all three matrices are actually scalars:

$$\begin{pmatrix} \gamma_{0,0} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} \end{pmatrix} \begin{pmatrix} \beta_{0,0} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} \beta_{0,0} \end{pmatrix}$$

so that matrix-matrix multiplication becomes scalar multiplication.

# Matrix-Matrix Multiplication (Product)

- Special Shapes : n=1, k=1 (SCAL)

$n = 1, k = 1$  (SCAL)

$$\begin{array}{c} 1 \\ \uparrow \\ \boxed{C} \end{array} = \begin{array}{c} m \\ \uparrow \\ \boxed{A} \end{array} \quad 1 \uparrow \boxed{B}$$

Now the matrices look like

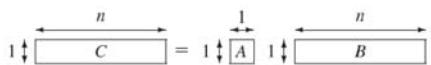
$$\begin{pmatrix} \gamma_{0,0} \\ \gamma_{1,0} \\ \vdots \\ \gamma_{m-1,0} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} \\ \alpha_{1,0} \\ \vdots \\ \alpha_{m-1,0} \end{pmatrix} \begin{pmatrix} \beta_{0,0} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} \beta_{0,0} \\ \alpha_{1,0} \beta_{0,0} \\ \vdots \\ \alpha_{m-1,0} \beta_{0,0} \end{pmatrix} = \begin{pmatrix} \beta_{0,0} \alpha_{0,0} \\ \beta_{0,0} \alpha_{1,0} \\ \vdots \\ \beta_{0,0} \alpha_{m-1,0} \end{pmatrix} = \beta_{0,0} \begin{pmatrix} \alpha_{0,0} \\ \alpha_{1,0} \\ \vdots \\ \alpha_{m-1,0} \end{pmatrix}.$$

In other words,  $C$  and  $A$  are vectors,  $B$  is a scalar, and the matrix-matrix multiplication becomes scaling of a vector.

## Matrix-Matrix Multiplication (Product)

- Special Shapes :  $m=1, k=1$  (SCAL)

$m=1, k=1$  (SCAL)



Now the matrices look like

$$\begin{pmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} \\ \alpha_{0,1} \\ \vdots \\ \alpha_{0,n-1} \end{pmatrix} \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \end{pmatrix}$$

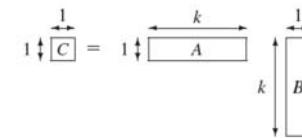
$$= \alpha_{0,0} \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{0,0}\beta_{0,0} & \alpha_{0,0}\beta_{0,1} & \cdots & \alpha_{0,0}\beta_{0,n-1} \end{pmatrix}.$$

In other words,  $C$  and  $B$  are just row vectors and  $A$  is a scalar. The vector  $C$  is computed by scaling the row vector  $B$  by the scalar  $A$ .

## Matrix-Matrix Multiplication (Product)

- Special Shapes :  $m=1, n=1$  (DOT)



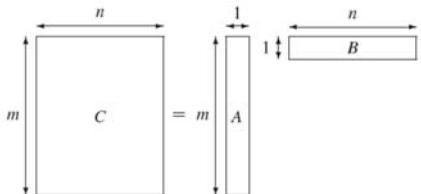
The matrices look like

$$\begin{pmatrix} \gamma_{0,0} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,k-1} \end{pmatrix} \begin{pmatrix} \beta_{0,0} \\ \beta_{1,0} \\ \vdots \\ \beta_{k-1,0} \end{pmatrix} = \sum_{p=0}^{k-1} \alpha_{0,p} \beta_{p,0}.$$

In other words,  $C$  is a scalar that is computed by taking the dot product of the one row that is  $A$  and the one column that is  $B$ .

## Matrix-Matrix Multiplication (Product)

- Special Shapes :  $k=1$  (outer product)

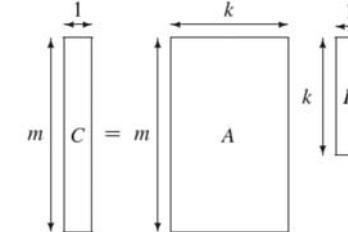


$$\begin{pmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n-1} \\ \gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{m-1,0} & \gamma_{m-1,1} & \cdots & \gamma_{m-1,n-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} \\ \alpha_{1,0} \\ \vdots \\ \alpha_{m-1,0} \end{pmatrix} \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_{0,0}\beta_{0,0} & \alpha_{0,0}\beta_{0,1} & \cdots & \alpha_{0,0}\beta_{0,n-1} \\ \alpha_{1,0}\beta_{0,0} & \alpha_{1,0}\beta_{0,1} & \cdots & \alpha_{1,0}\beta_{0,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1,0}\beta_{0,0} & \alpha_{m-1,0}\beta_{0,1} & \cdots & \alpha_{m-1,0}\beta_{0,n-1} \end{pmatrix}$$

## Matrix-Matrix Multiplication (Product)

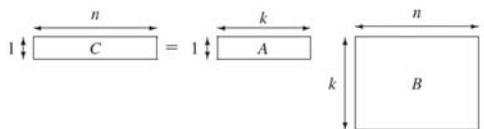
- Special Shapes :  $n=1$  (matrix-vector product)



$$\begin{pmatrix} \gamma_{0,0} \\ \gamma_{1,0} \\ \vdots \\ \gamma_{m-1,0} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,k-1} \\ \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1,0} & \alpha_{m-1,1} & \cdots & \alpha_{m-1,k-1} \end{pmatrix} \begin{pmatrix} \beta_{0,0} \\ \beta_{1,0} \\ \vdots \\ \beta_{k-1,0} \end{pmatrix}$$

## Matrix-Matrix Multiplication (Product)

- Special Shapes :  $m = 1$  (row vector-matrix product)



$$\begin{pmatrix} \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,k-1} \end{pmatrix} \begin{pmatrix} \beta_{0,0} & \beta_{0,1} & \cdots & \beta_{0,n-1} \\ \beta_{1,0} & \beta_{1,1} & \cdots & \beta_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k-1,0} & \beta_{k-1,1} & \cdots & \beta_{k-1,n-1} \end{pmatrix}$$

so that  $\gamma_{0,j} = \sum_{p=0}^{k-1} \alpha_{0,p} \beta_{p,j}$ . To emphasize how it relates to have matrix-matrix multiplication is computed, consider the following:

$$\begin{pmatrix} \gamma_{0,0} & \cdots & \boxed{\gamma_{0,j}} & \cdots & \gamma_{0,n-1} \end{pmatrix} = \begin{pmatrix} \alpha_{0,0} & \alpha_{0,1} & \cdots & \alpha_{0,k-1} \end{pmatrix} \begin{pmatrix} \beta_{0,0} & \cdots & \boxed{\beta_{0,j}} & \cdots & \beta_{0,n-1} \\ \beta_{1,0} & \cdots & \boxed{\beta_{1,j}} & \cdots & \beta_{1,n-1} \\ \vdots & & \vdots & & \vdots \\ \beta_{k-1,0} & \cdots & \boxed{\beta_{k-1,j}} & \cdots & \beta_{k-1,n-1} \end{pmatrix}.$$

## Questions and Answers

