Vector Spaces, Orthogonality, and Linear Least Squares

Jirasak Sittigorn

Department of Computer Engineering
Faculty of Engineering
King Mongkut's Institute of Technology Ladkrabang

Opening Remarks

- The polynomial equation (in week 9)

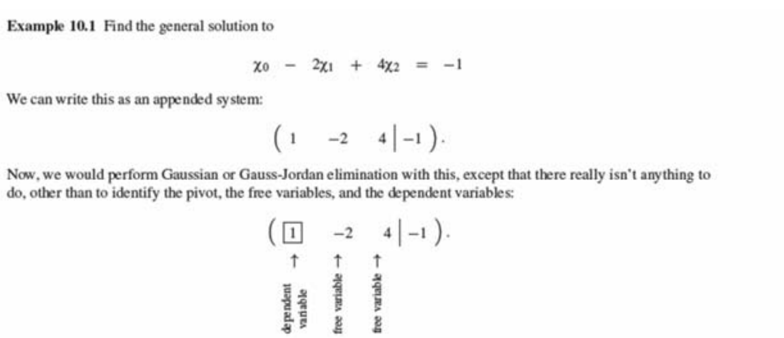
$$\begin{array}{rcrcrcr} \chi_0 & - & 2\chi_1 & + & 4\gamma_2 & = & -1 \\ \chi_0 & & & & & = & 2 \\ \chi_0 & + & 2\chi_1 & + & 4\chi_2 & = & 3 \end{array}$$

  - Solution

$$\begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -0.25 \end{pmatrix}$$

Opening Remarks

- Visualizing Planes, Lines, and Solutions



Example 10.1 Find the general solution to

$$\chi_0 - 2\chi_1 + 4\chi_2 = -1$$

We can write this as an appended system:

$$\left(\begin{array}{ccc|c} 1 & -2 & 4 & -1 \end{array}\right).$$

Now, we would perform Gaussian or Gauss-Jordan elimination with this, except that there really isn't anything to do, other than to identify the pivot, the free variables, and the dependent variables:

$$\left(\begin{array}{ccc|c} \boxed{1} & -2 & 4 & -1 \end{array}\right).$$

Opening Remarks

- Visualizing Planes, Lines, and Solutions
  - When Linear Systems Have Many Solutions (Week 9)

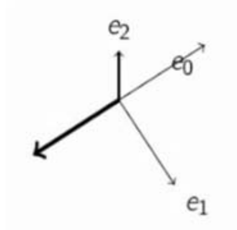
$$Ax = b$$

  - Toward a Systematic Approach to Finding All Solutions

$$A(x_s + \beta x_n)$$

    - $Ax_s = b$
    - $Ax_n = 0$

Opening Remarks



- **Visualizing Planes, Lines, and Solutions**
  - **Identify a specific solution**

$$\chi_0 - 2\chi_1 + 4\chi_2 = -1$$

$$x_s = \begin{pmatrix} \chi_0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Substituting $\chi_1 = \chi_2 = 0$ into the equation gives us

$$\chi_0 - 2(0) + 4(0) = -1.$$

or $\chi_0 = -1$ so that

$$x_s = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Opening Remarks

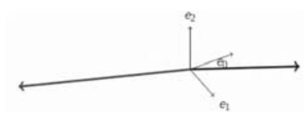
- **Visualizing Planes, Lines, and Solutions**
  - **Identifying a basis in the null space**

$$\chi_0 - 2\chi_1 + 4\chi_2 = 0.$$

If we write this as an appended system, we get

$$\begin{array}{ccc|c} 1 & -2 & 4 & 0 \\ \uparrow & \uparrow & \uparrow & \\ \text{dependent variable} & \text{free variable} & \text{free variable} & \end{array}$$

Opening Remarks



- **Visualizing Planes, Lines, and Solutions**
  - **Identify linearly independent vectors in the null space**

$$\chi_0 - 2\chi_1 + 4\chi_2 = 0.$$

We look for solutions in the form

$$x_{n_0} = \begin{pmatrix} \chi_0 \\ 1 \\ 0 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Substituting $\chi_1 = 1$ and $\chi_2 = 0$ into the equation gives us

$$\chi_0 - 2(1) + 4(0) = 0.$$

or $\chi_0 = 2$.

and

$$x_{n_1} = \begin{pmatrix} \chi_0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Dependent variable to be computed} \\ \longleftarrow \text{Free variable} \\ \longleftarrow \text{Free variable} \end{array}$$

Substituting $\chi_1 = 0$ and $\chi_2 = 1$ into the equation gives us

$$\chi_0 - 2(0) + 4(1) = 0.$$

or $\chi_0 = -4$.

Thus two (nonunique) linearly independent vectors in the null space are given by

$$x_{n_0} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad x_{n_1} = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}.$$

Opening Remarks

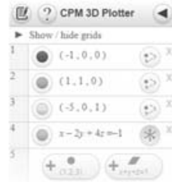
- **Visualizing Planes, Lines, and Solutions**
  - Toward a Systematic Approach to Finding All Solutions

$$A(x_s + \beta x_n)$$

  - Towards visualizing the plane of solutions

$$x_{general} = x_s + \beta_0 x_{n_0} + \beta_1 x_{n_0}$$

Opening Remarks



- Visualizing Planes, Lines, and Solutions
  - Problem

$$\chi_0 - 2\chi_1 + 4\chi_2 = -1$$

  - Towards visualizing the plane of solutions

$$x = x_s + \beta_0 x_{n_0} + \beta_1 x_{n_0}$$

$$x = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \beta_0 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$$

Opening Remarks

Homework 10.1.1.1 Consider, again, the equation from the last example:

$$\chi_0 - 2\chi_1 + 4\chi_2 = -1$$

Which of the following represent(s) a general solution to this equation? (Mark all)

Opening Remarks

Homework 10.1.1.2 Now you find the general solution for the **second** equation in the system of linear equations with which we started this unit. Consider

$$\chi_0 = 2$$

Which of the following is a true statement about this equation:

- $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ is a specific solution.
- $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is a specific solution.
- $\boxed{\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \beta_0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ is a general solution.}}$
- $\begin{pmatrix} 2 \\ 1 \\ -0.25 \end{pmatrix} + \beta_0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a general solution.
- $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \beta_0 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$ is a general solution.

Opening Remarks

Homework 10.1.1.4 We notice that it would be nice to put lines where planes meet. Now, let's start by focusing on the first two equations: Consider

$$\begin{array}{rcl} \chi_0 - 2\chi_1 + 4\chi_2 & = & -1 \\ \chi_0 & = & 2 \end{array}$$

Compute the general solution of this system with two equations in three unknowns and indicate which of the following is true about this system?

- $\begin{pmatrix} 2 \\ 1 \\ -0.25 \end{pmatrix}$ is a specific solution.
- $\begin{pmatrix} 2 \\ 3/2 \\ 0 \end{pmatrix}$ is a specific solution.
- $\begin{pmatrix} 2 \\ 3/2 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is a general solution.
- $\begin{pmatrix} 2 \\ 1 \\ -0.25 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ is a general solution.

How the Row Echelon Form Answers (Almost) Everything

- **Example**

Homework 10.2.1.1 Consider the linear system of equations

$$\underbrace{\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}}_{b}.$$

Write it as an appended system and reduce it to to row echelon form (but not reduced row echelon form). Identify the pivots, the free variables and the dependent variables.

How the Row Echelon Form Answers (Almost) Everything

- **The Important Attributes of a Linear System**
  - The row-echelon form of the system.
  - The pivots.
  - he free variables.
  - The dependent variables.
  - A specific solution : Often called a particular solution.
  - A general solution : Often called a complete solution.
  - A basis for the column space. : Something we should have mentioned before: The column space is often called the range of the matrix.
  - A basis for the null space. : Something we should have mentioned before: The null space is often called the kernel of the matrix.
  - A basis for the row space. : The row space is the subspace of all vectors that can be created by taking linear combinations of the rows of a matrix. In other words, the row space of A equals C(AT) (the column space of AT).
  - The dimension of the row and column space.
  - The rank of the matrix.
  - The dimension of the null space.

How the Row Echelon Form Answers (Almost) Everything

Motivating example

Consider the example from the last unit.

$$\underbrace{\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix}}_{A} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

which, when reduced to row echelon form, yields

$$\left(\begin{array}{cccc|c} 1 & 3 & 1 & 2 & 1 \\ 2 & 6 & 4 & 8 & 3 \\ 0 & 0 & 2 & 4 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 2 & 1 \\ 0 & 0 & \boxed{2} & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Here the boxed entries are the pivots (the first nonzero entry in each row) and they identify that the corresponding variables (χ_0 and χ_2) are dependent variables while the other variables (χ_1 and χ_3) are free variables.

How the Row Echelon Form Answers (Almost) Everything

Various dimensions

Notice that inherently the matrix is $m \times n$. In this case

- $m = 3$ (the number of rows in the matrix which equals the number of equations in the linear system); and
- $n = 4$ (the number of columns in the matrix which equals the number of equations in the linear system).

Now

- There are two pivots. Let's say that in general there are k pivots, where here $k = 2$.
- There are two free variables. In general, there are $n-k$ free variables, corresponding to the columns in which no pivot reside. **This means that the null space dimension equals $n-k$, or two in this case.**
- There are two dependent variables. In general, there are k dependent variables, corresponding to the columns in which the pivots reside. **This means that the column space dimension equals k, or also two in this case. This also means that the row space dimension equals k, or also two in this case.**
- The dimension of the row space always equals the dimension of the column space which always equals the number of pivots in the row echelon form of the equation. This number, k, is called the **rank** of matrix A, rank(A).

How the Row Echelon Form Answers (Almost) Everything

Format of a general solution

To find a general solution to problem, you recognize that there are two free variables (χ_1 and χ_3) and a general solution can be given by

$$\begin{pmatrix} \square \\ 0 \\ \square \\ 0 \end{pmatrix} + \beta_0 \begin{pmatrix} \square \\ 1 \\ \square \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} \square \\ 0 \\ \square \\ 1 \end{pmatrix}.$$

$$x_p = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}, \qquad x_{n_0} = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_{n_1} = \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}. \qquad \mathcal{N}(A) = \text{Span}\left(\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\} \right).$$

A general solution

Thus, a general solution is given by

$$\begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} + \beta_0 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \beta_1 \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix},$$

where $\beta_0, \beta_1 \in \mathbb{R}$.

How the Row Echelon Form Answers (Almost) Everything

Finding a basis for the column space of the original matrix

To find the linearly independent columns, you look at the row echelon form of the matrix:

$$\begin{pmatrix} \boxed{1} & 3 & 1 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with the pivots highlighted. The columns that have pivots in them are linearly independent. The corresponding columns in the original matrix are also linearly independent:

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix}.$$

Thus, in our example, the answer is $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ (the first and third column).

Thus,

$$\mathcal{C}(A) = \text{Span}\left(\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\} \right).$$

How the Row Echelon Form Answers (Almost) Everything

Find a basis for the row space of the matrix.

The row space (we will see in the next chapter) is the space spanned by the rows of the matrix (viewed as column vectors). Reducing a matrix to row echelon form merely takes linear combinations of the rows of the matrix. What this means is that the space spanned by the rows of the original matrix is the same space as is spanned by the rows of the matrix in row echelon form. Thus, all you need to do is list the rows in the matrix in row echelon form, as column vectors.

For our example this means a basis for the row space of the matrix is given by

$$\mathcal{R}(A) = \text{Span}\left(\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 4 \end{pmatrix} \right\} \right).$$

How the Row Echelon Form Answers (Almost) Everything

Homework 10.2.2.1 Consider $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

- Reduce the system to row echelon form (but not reduced row echelon form).
- Identify the free variables.
- Identify the dependent variables.
- What is the dimension of the column space?
- What is the dimension of the row space?
- What is the dimension of the null space?
- Give a set of linearly independent vectors that span the column space
- Give a set of linearly independent vectors that span the row space.
- What is the rank of the matrix?
- Give a general solution.