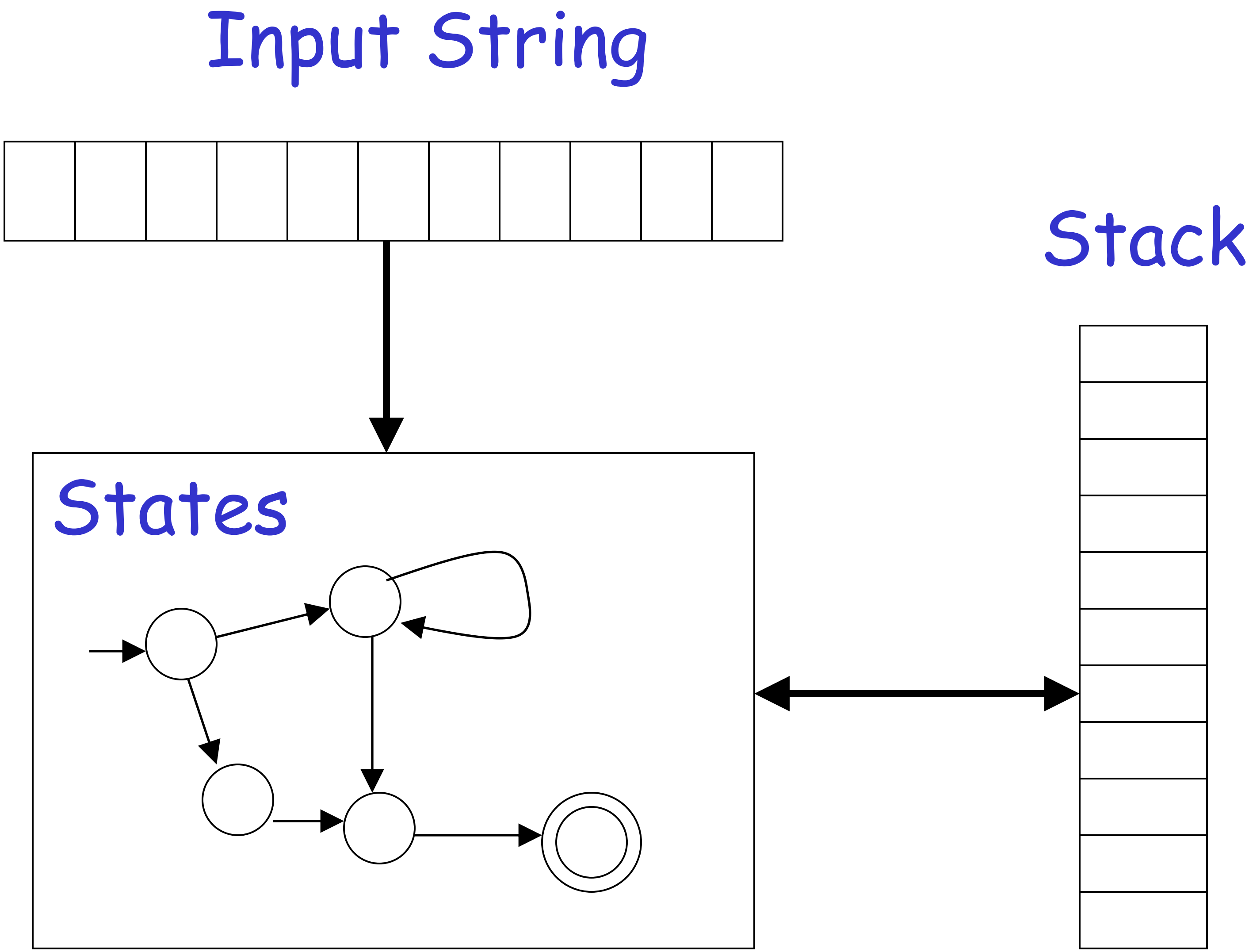
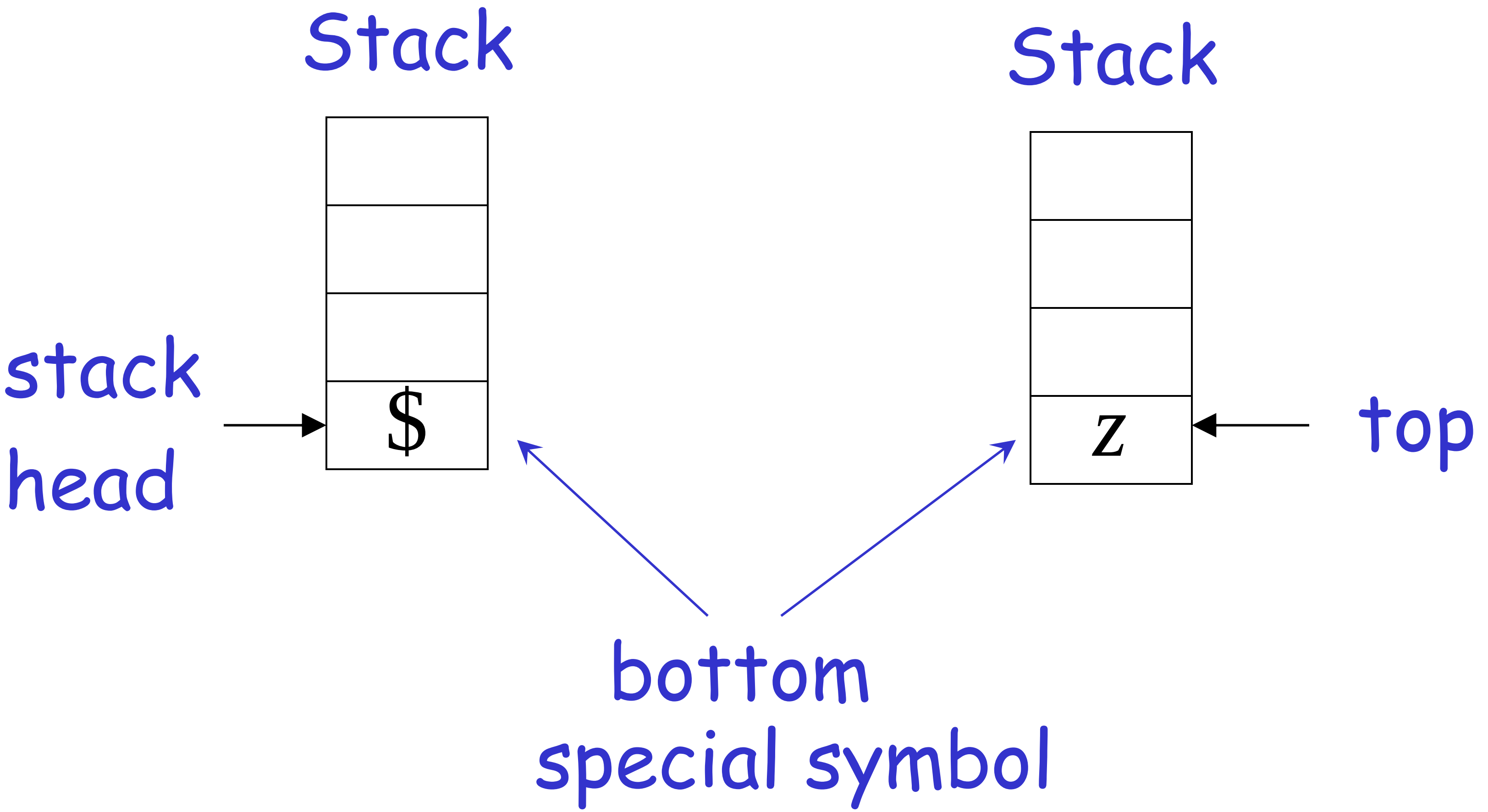
Pushdown Automata
PDAs

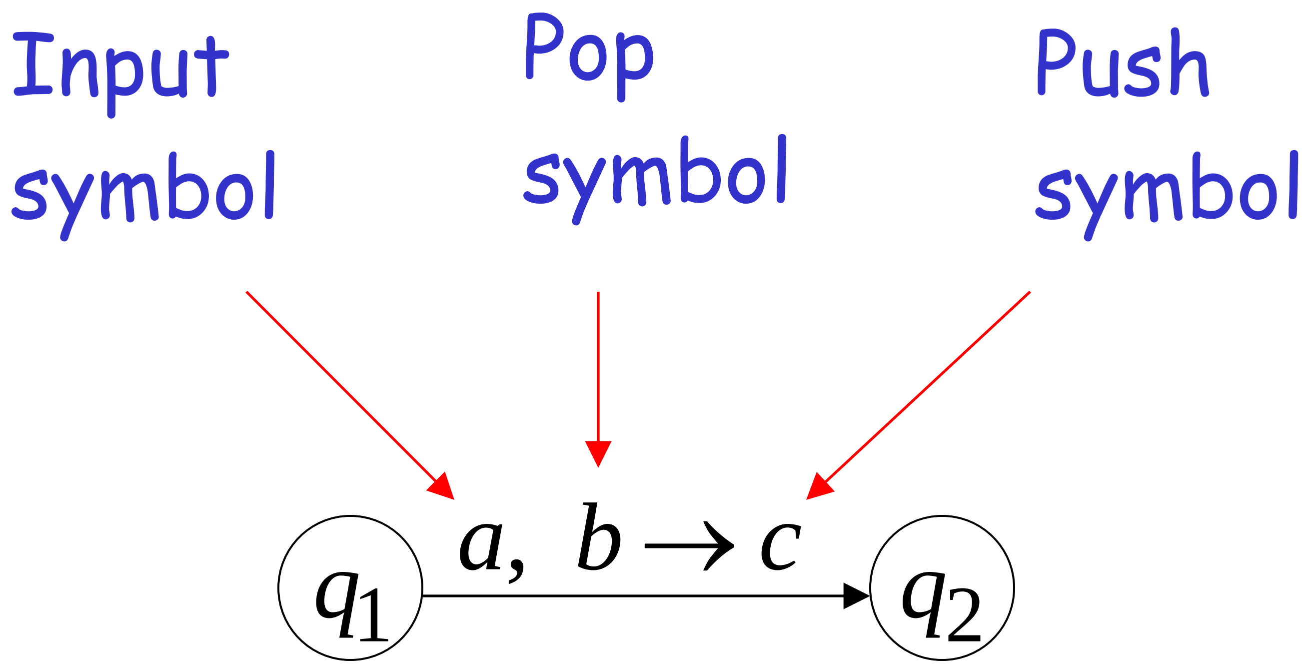
Pushdown Automaton -- PDA

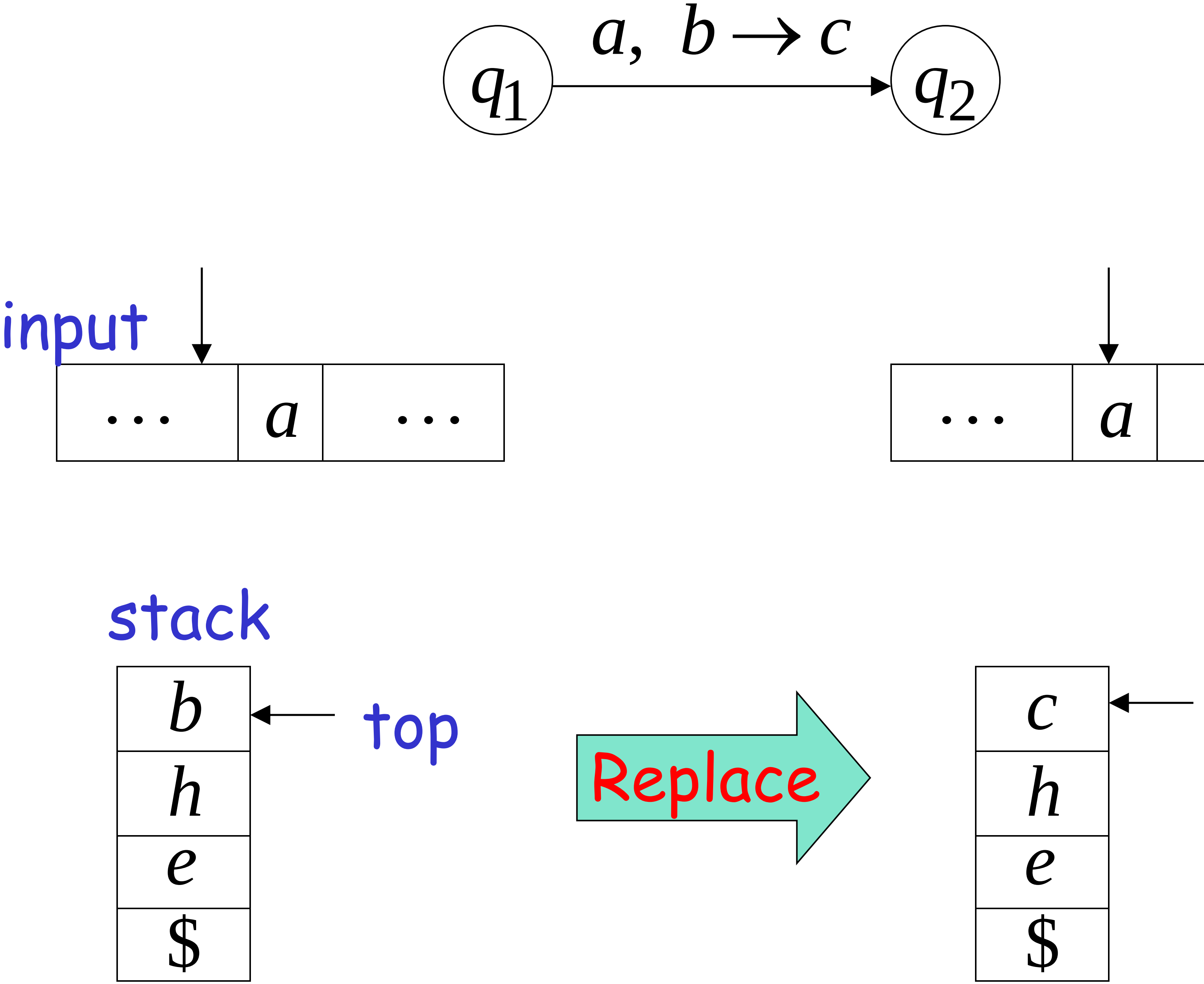


Initial Stack Symbol

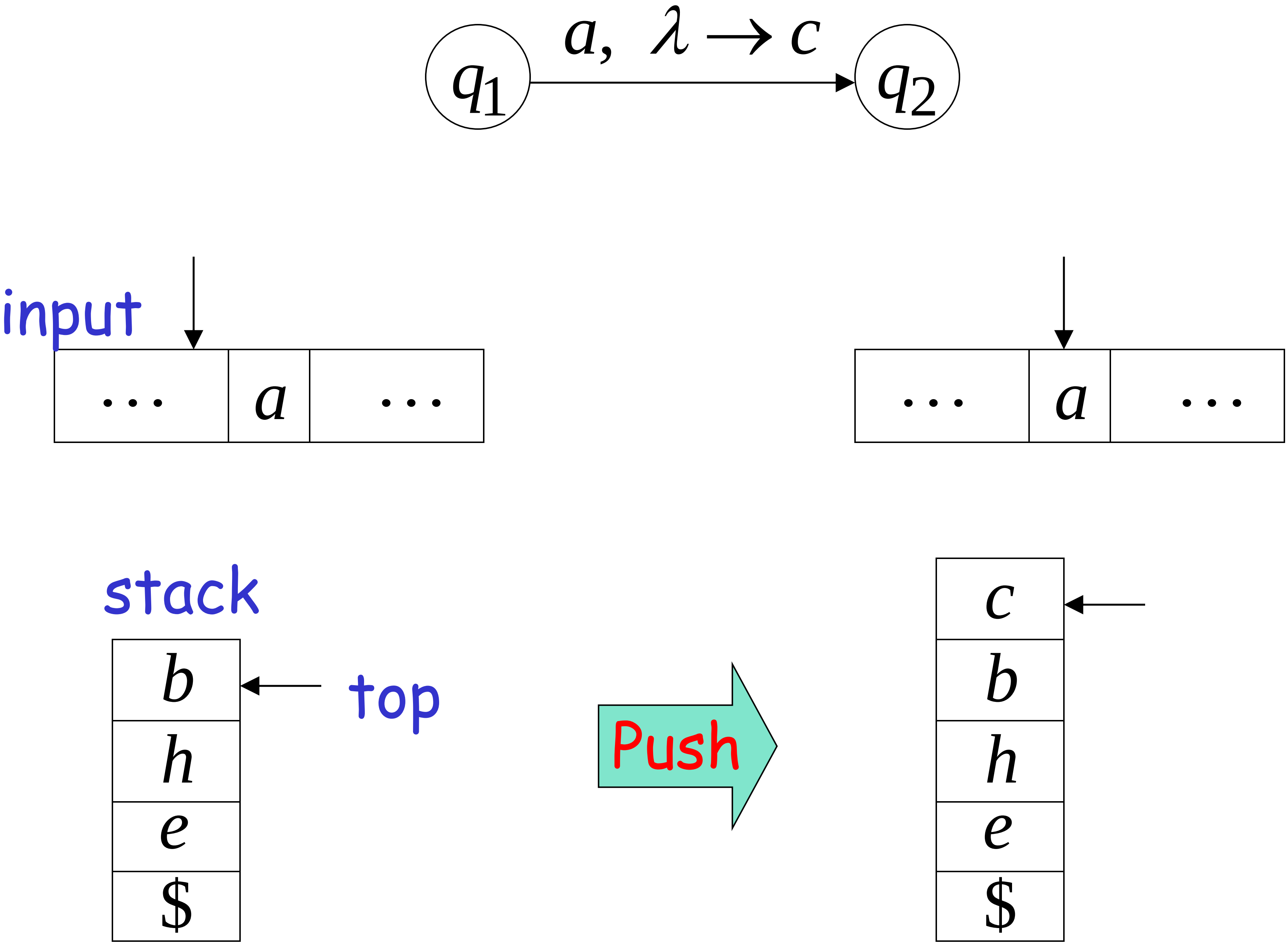


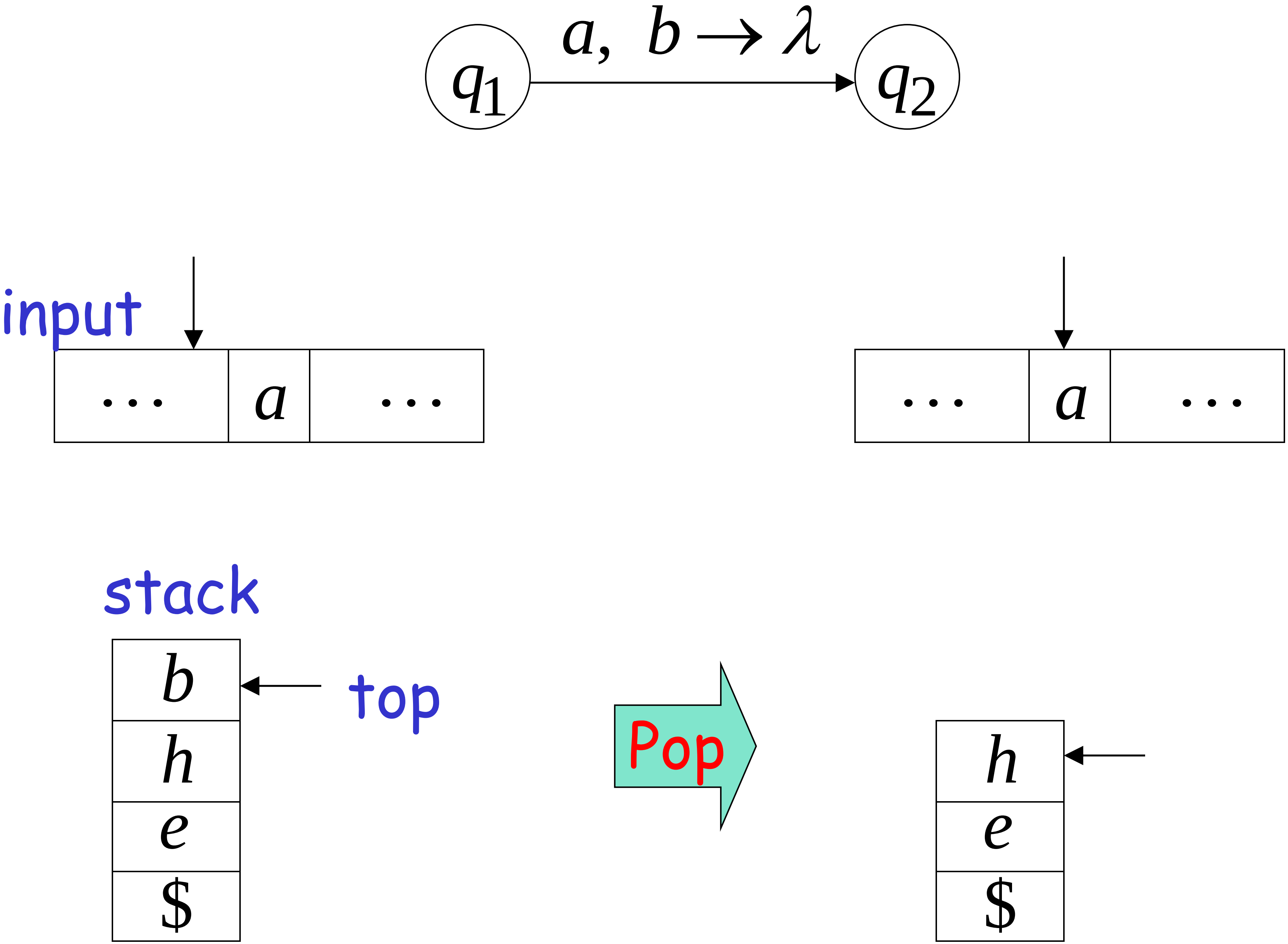
The States

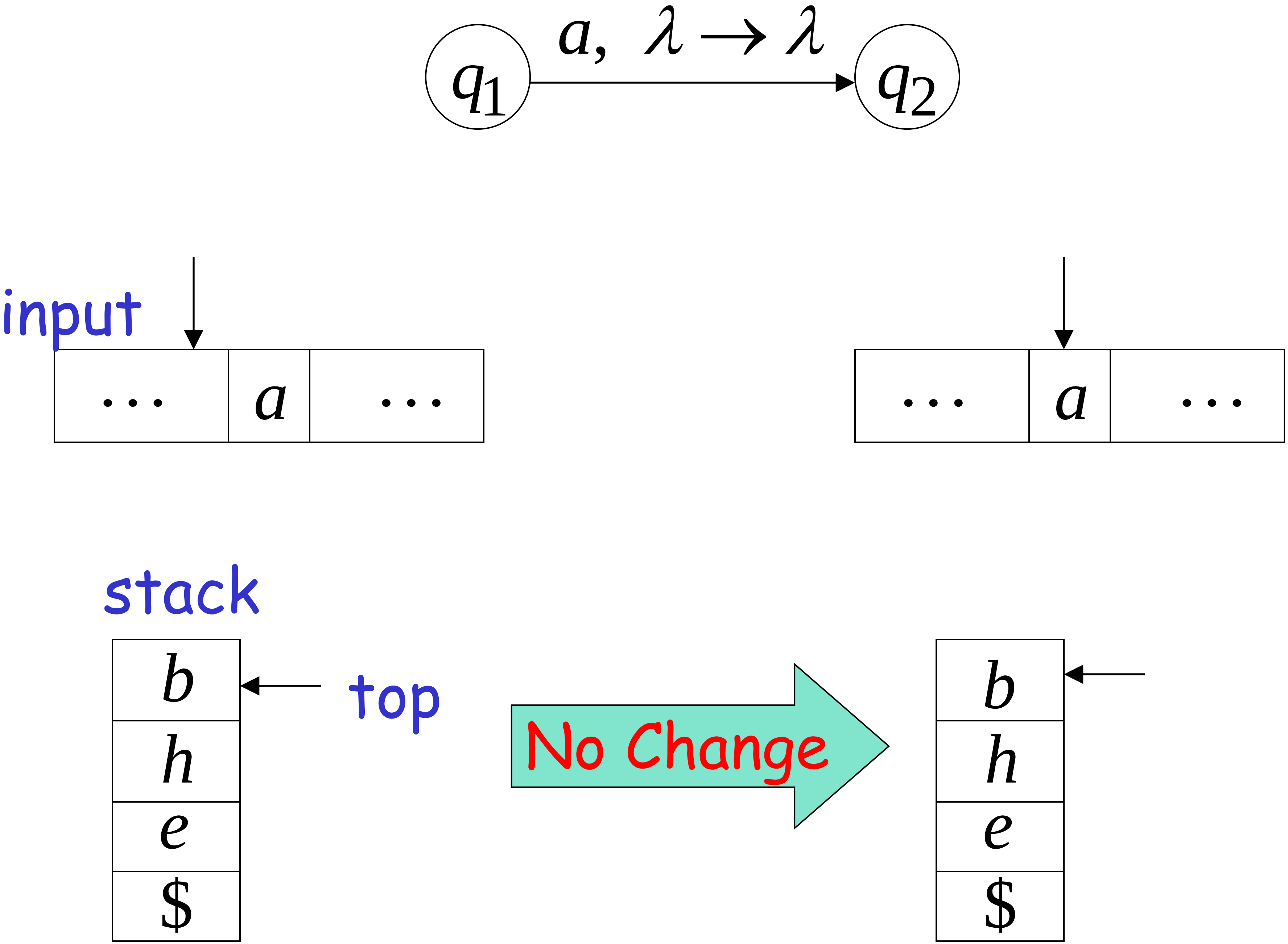




$a, \ b \to c$
q_1
q_2
input
... a ...
... a ...
stack
b
h
e
$
top
Replace
c
h
e
$

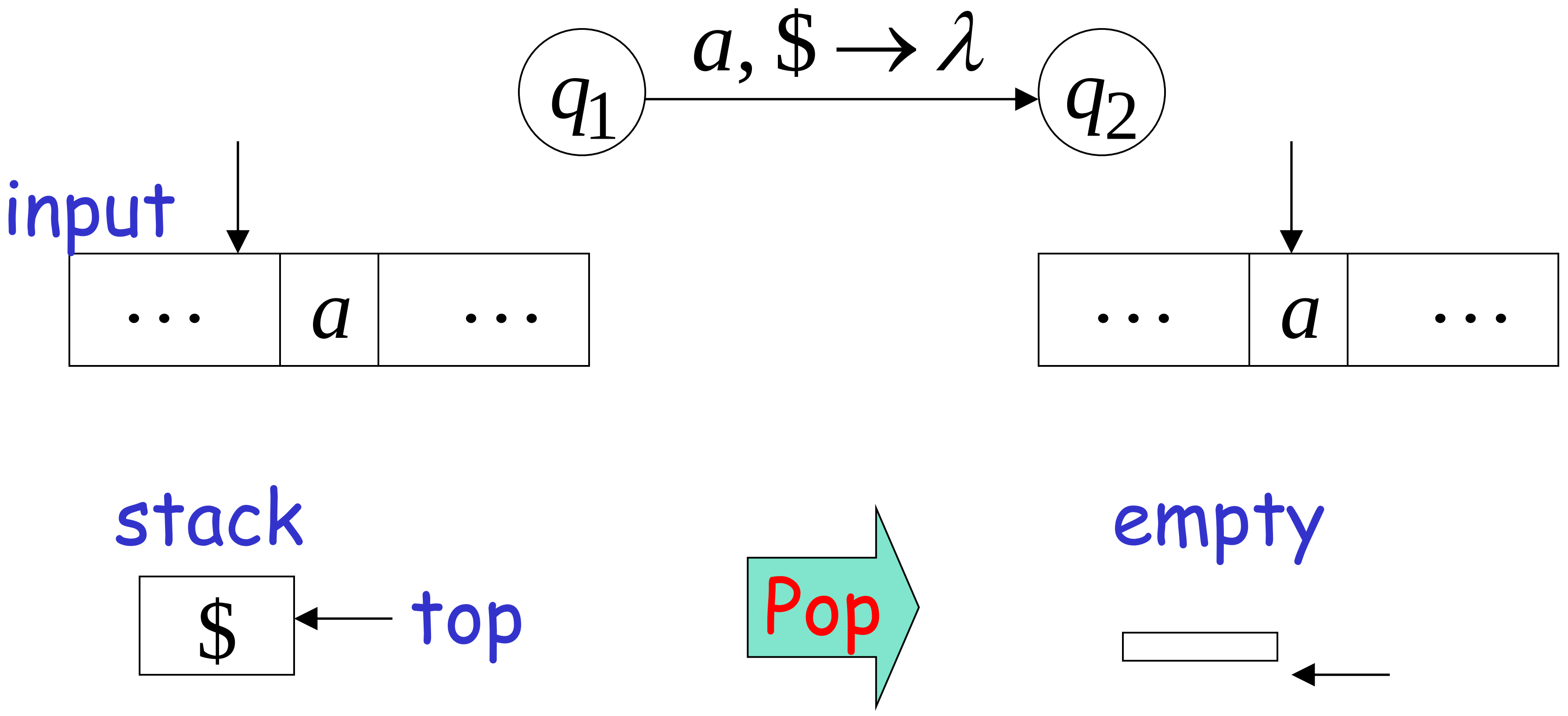

a, λ → c
q1
q2
input
...
a
...
...
a
...
stack
b
h
e
$
top
Push
c
b
h
e
$


$a, \; b \rightarrow \lambda$
q_1
q_2
input
$\cdots$
a
$\cdots$
$\cdots$
a
$\cdots$
stack
b
h
e
$\$$
top
Pop
h
e
$\$$

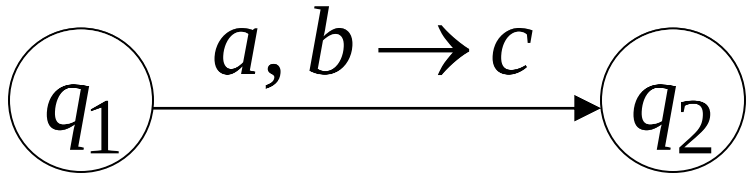


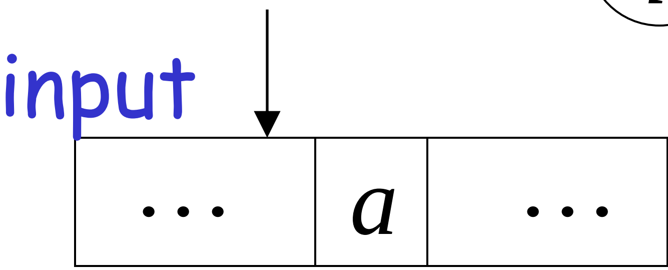
$a, \lambda \to \lambda$
q_1
q_2
input
$\cdots$ a $\cdots$
$\cdots$ a $\cdots$
stack
top
b
h
e
$\$$
No Change
b
h
e
$\$$

A Possible Transition

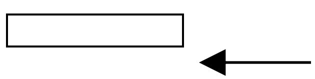


A Bad Transition





Empty stack

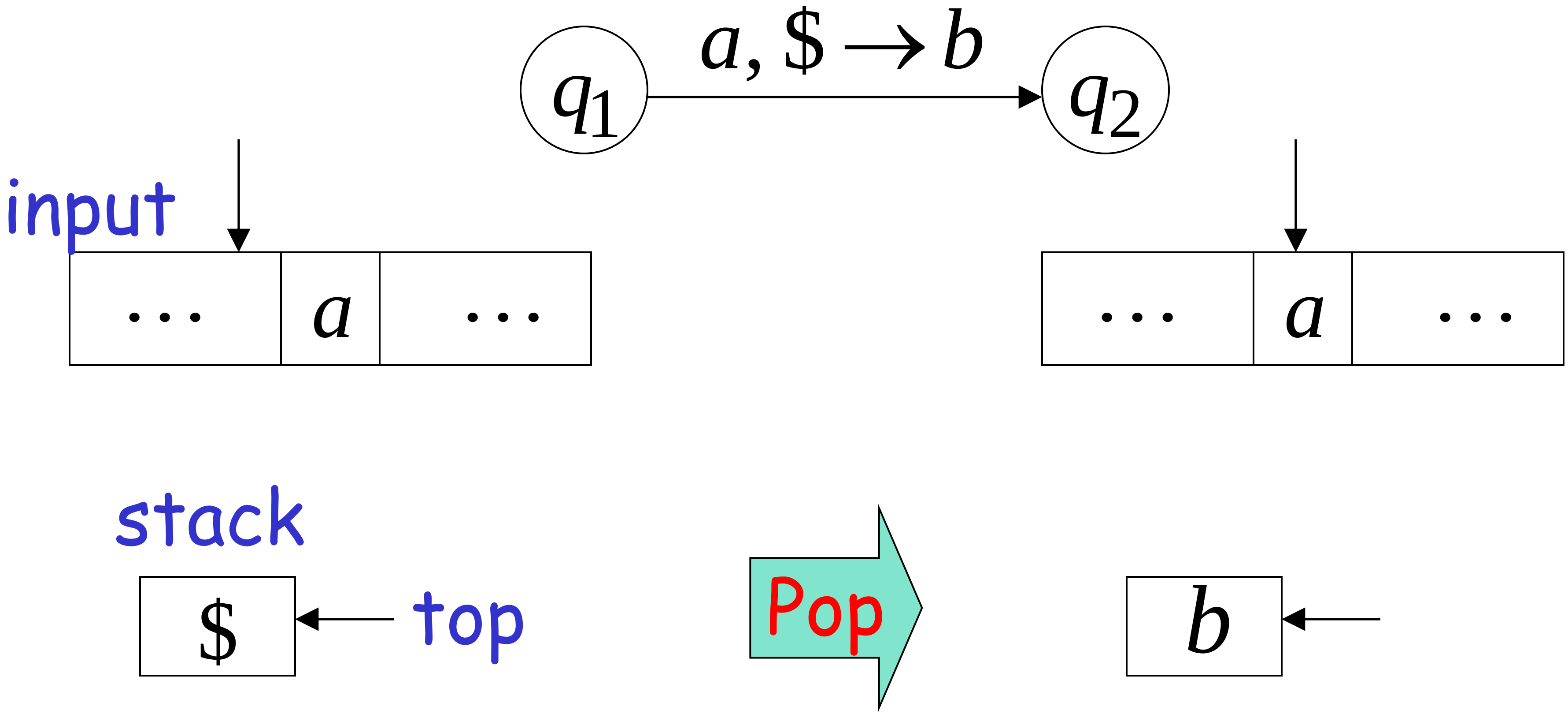


The automaton **Halts** in state q_1 and **Rejects** the input string

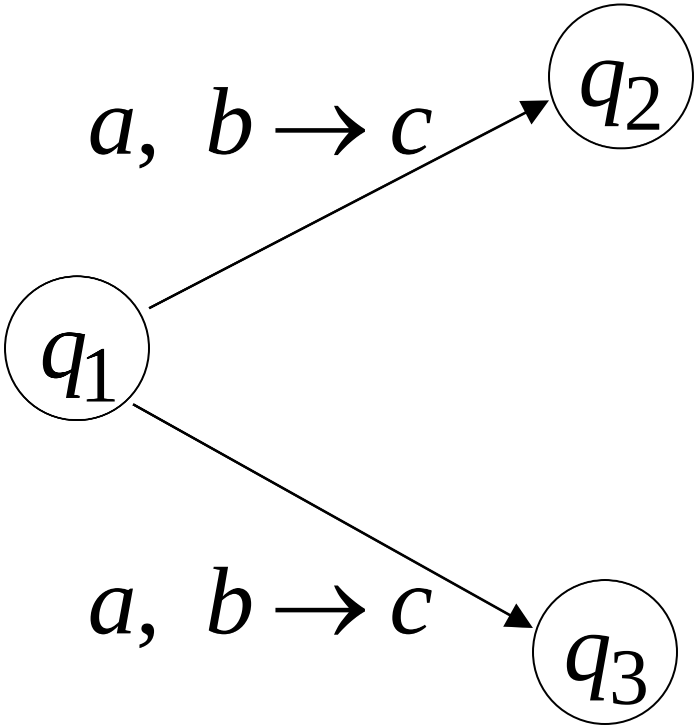
HALT

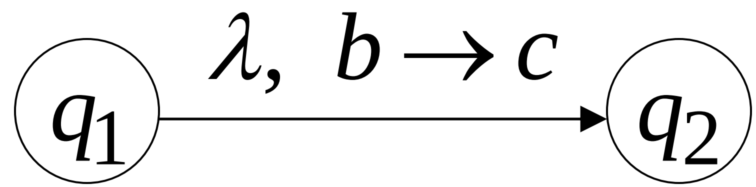
No transition is allowed to be followed when the stack is empty

Allowed Transition
(Not used in practice)



Non-Determinism



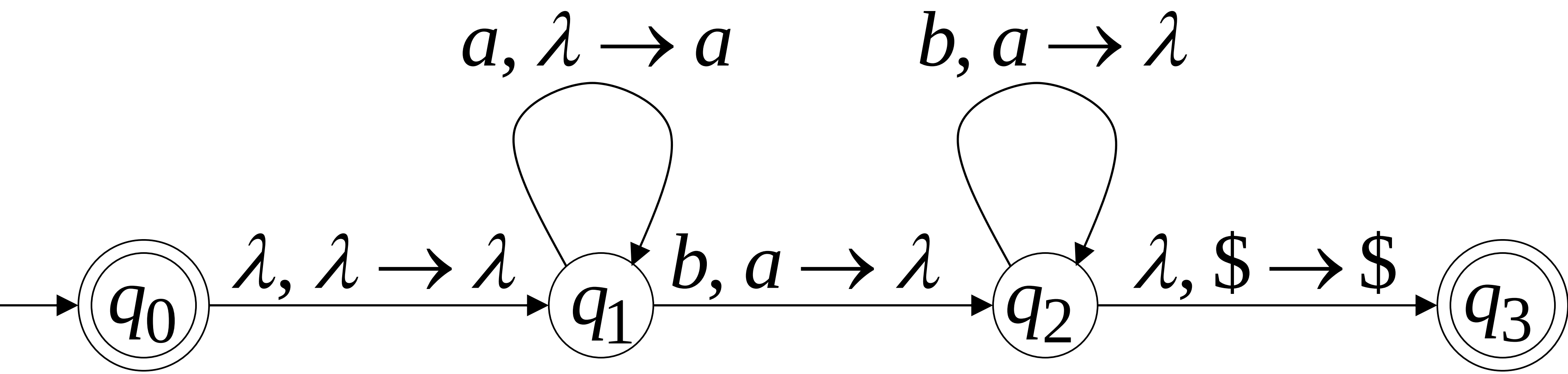


$\lambda -$ transition

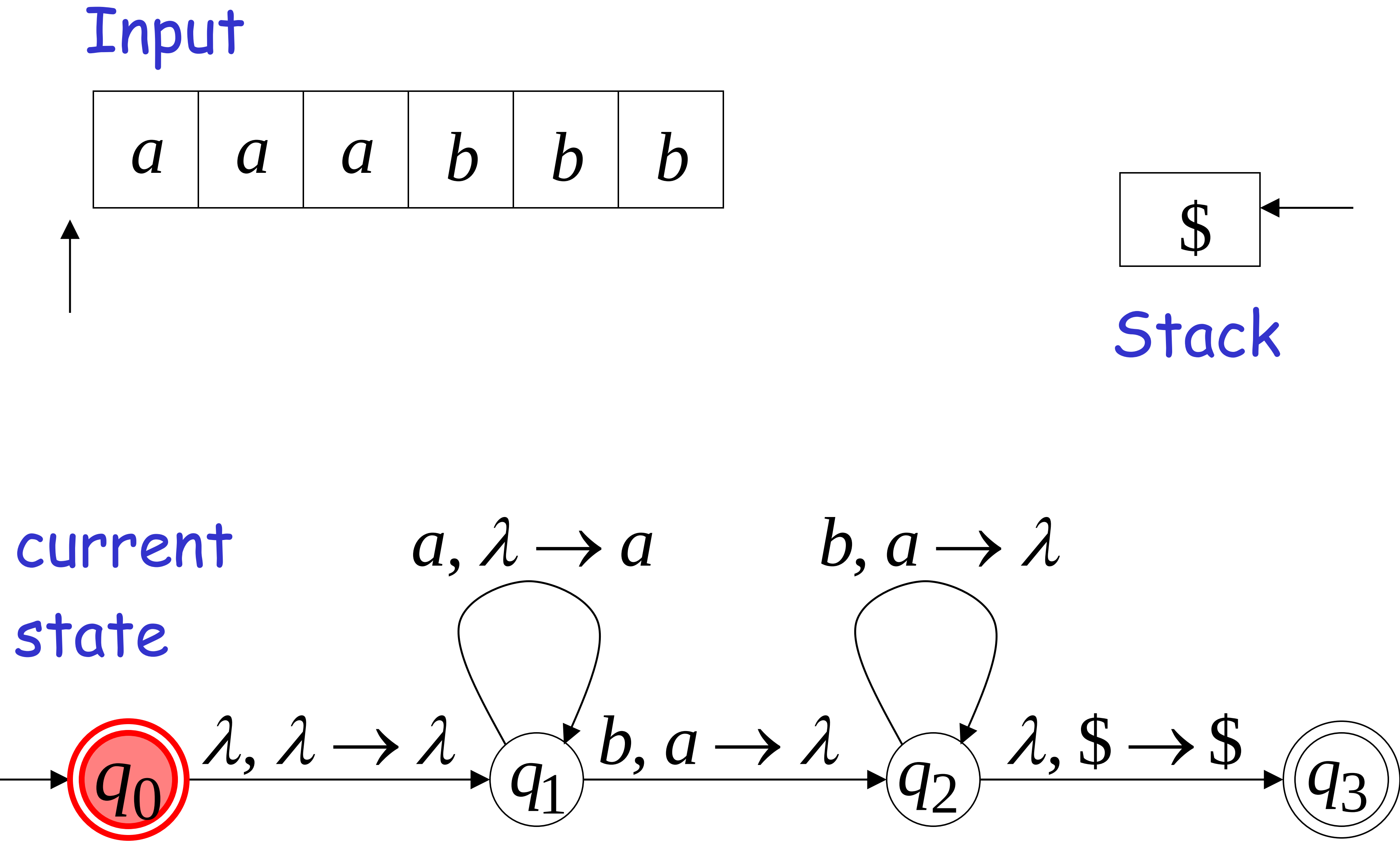
These are allowed transitions in a
Non-deterministic PDA (NPDA)

NPDA: Non-Deterministic PDA

Example:



Execution Example: Time 0



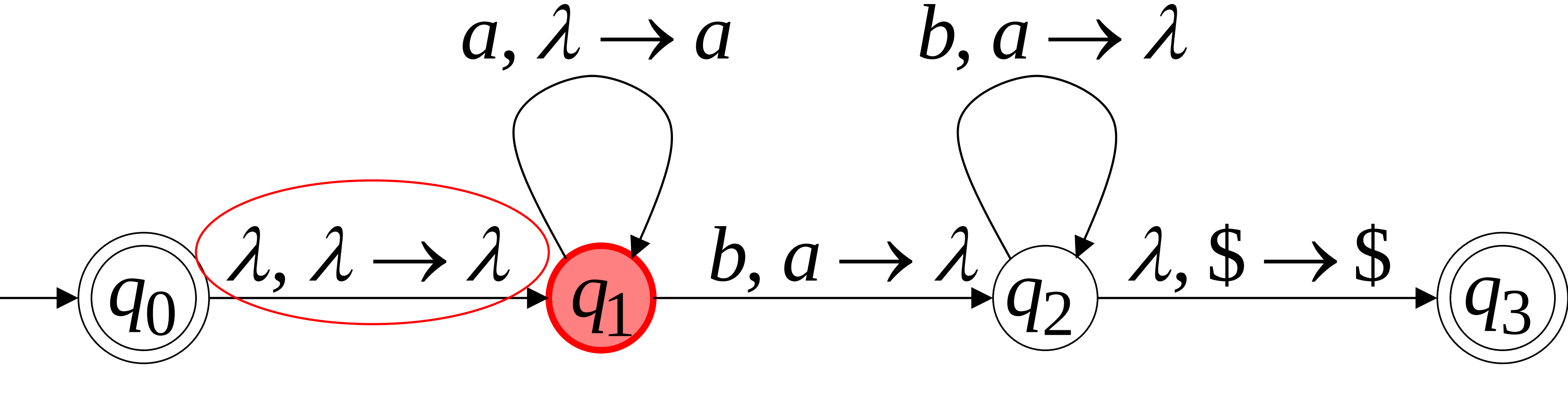
Time 1

Input

a	a	a	b	b	b

$\$$

Stack



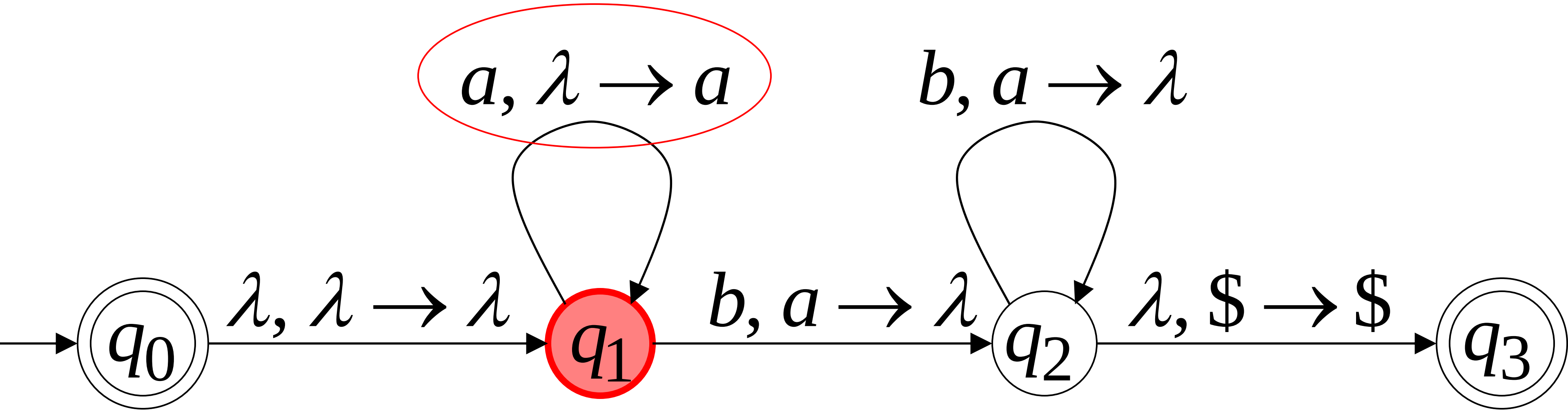
Time 2

Input

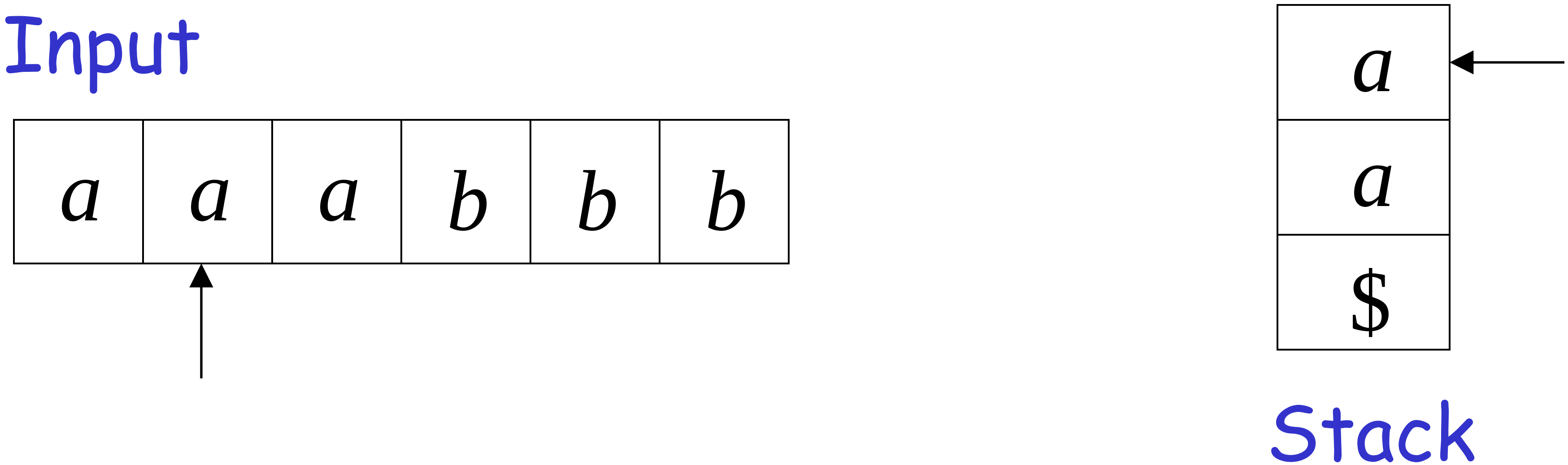
a	a	a	b	b	b

a
$\$$

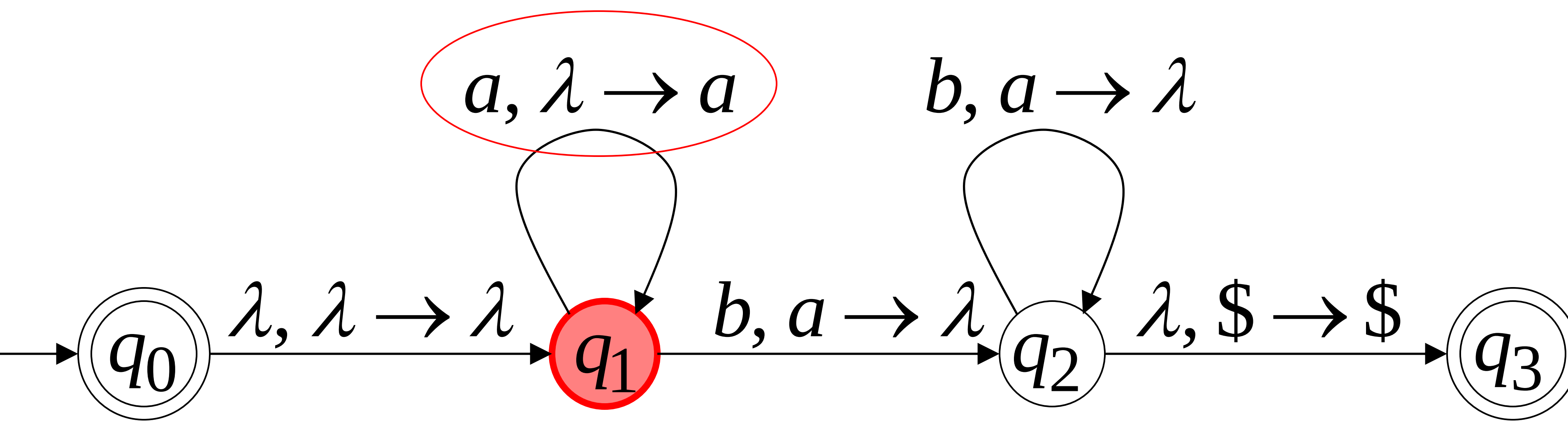
Stack



Time 3

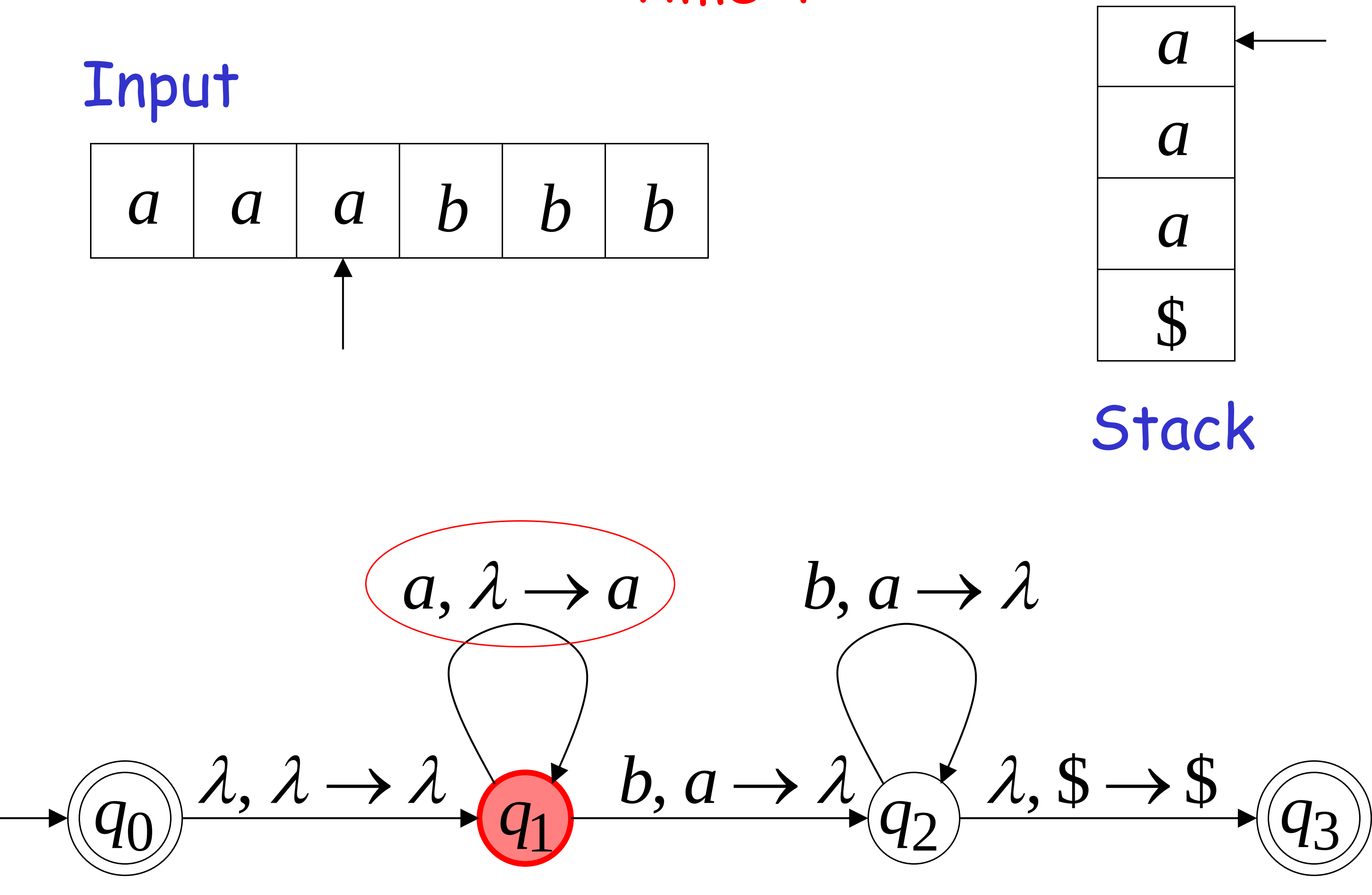


Input
a a a b b b
a
a
$
Stack

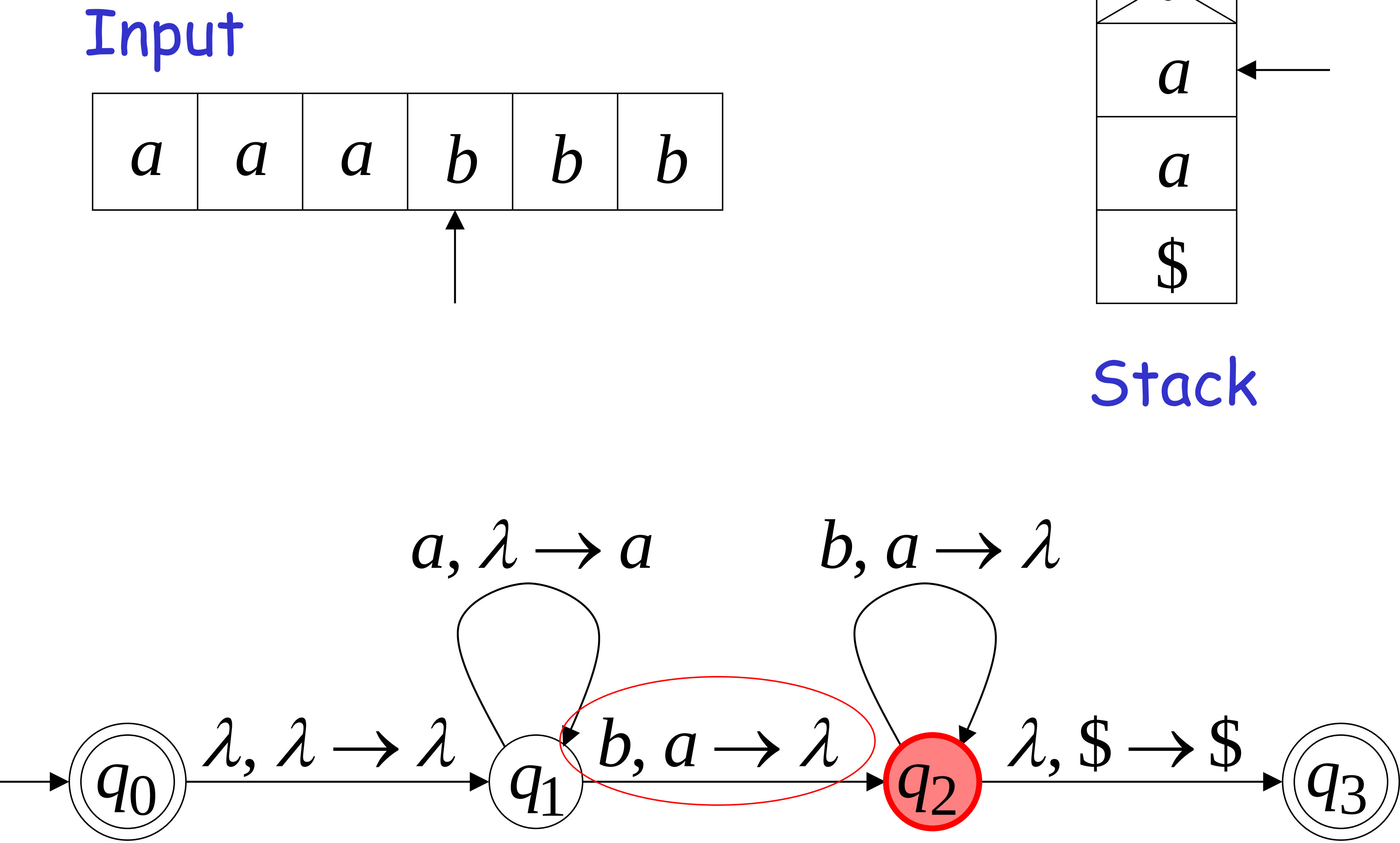


$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

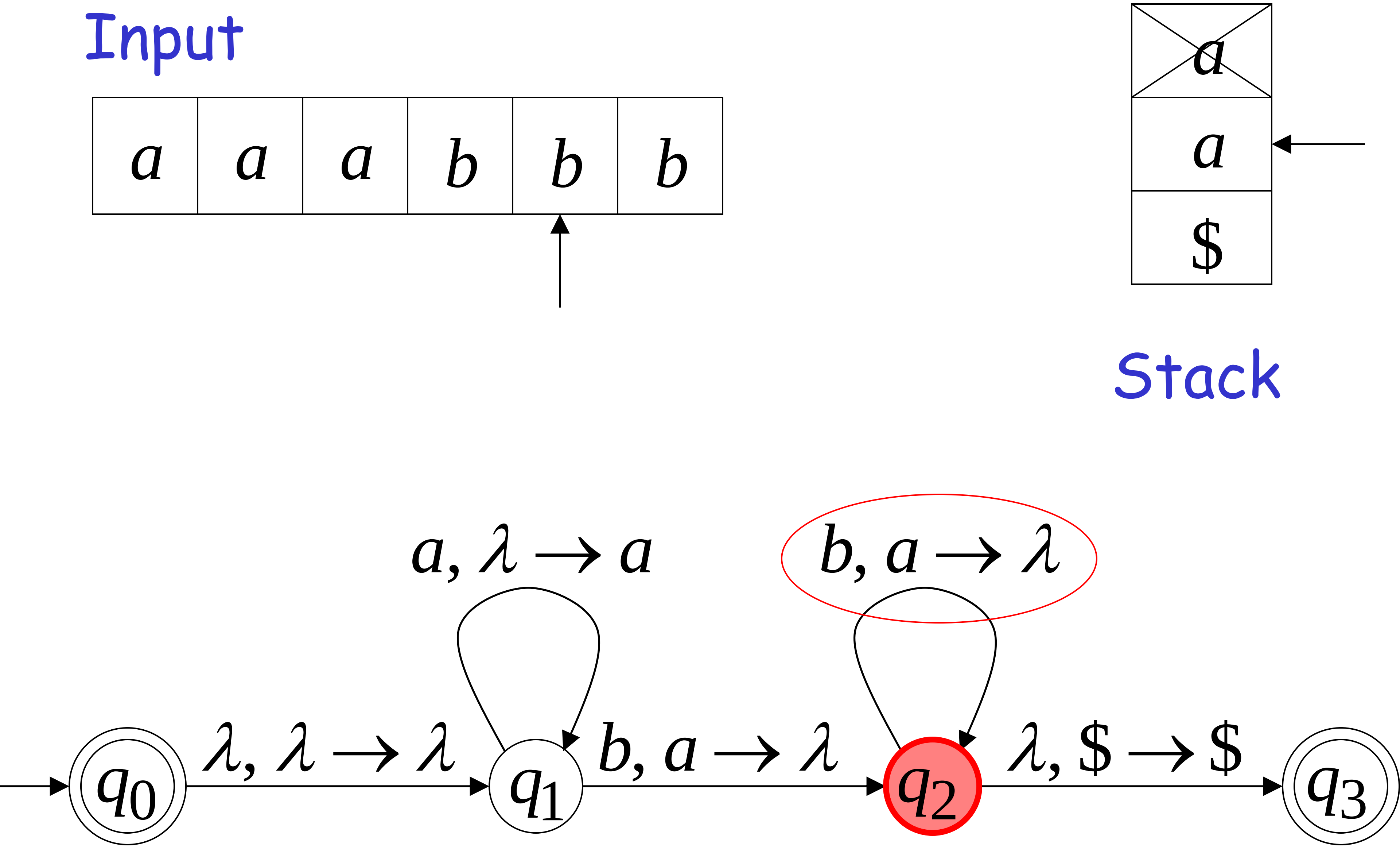
Time 4


Input
a a a b b b
a
a
a
$
Stack
$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, a \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

Time 5

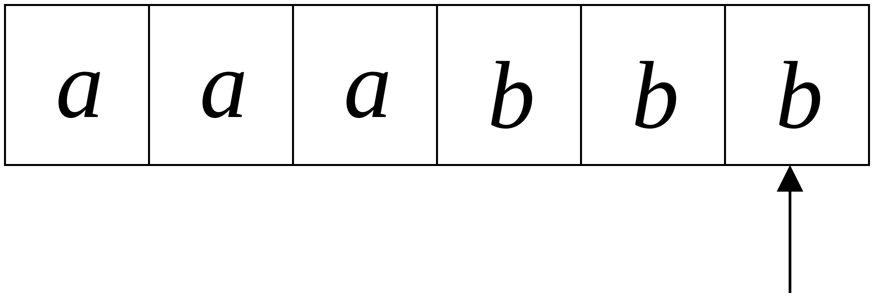


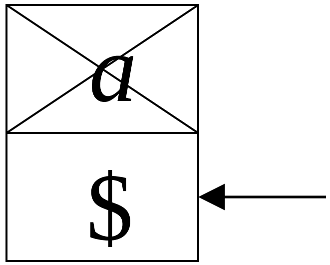
Time 6



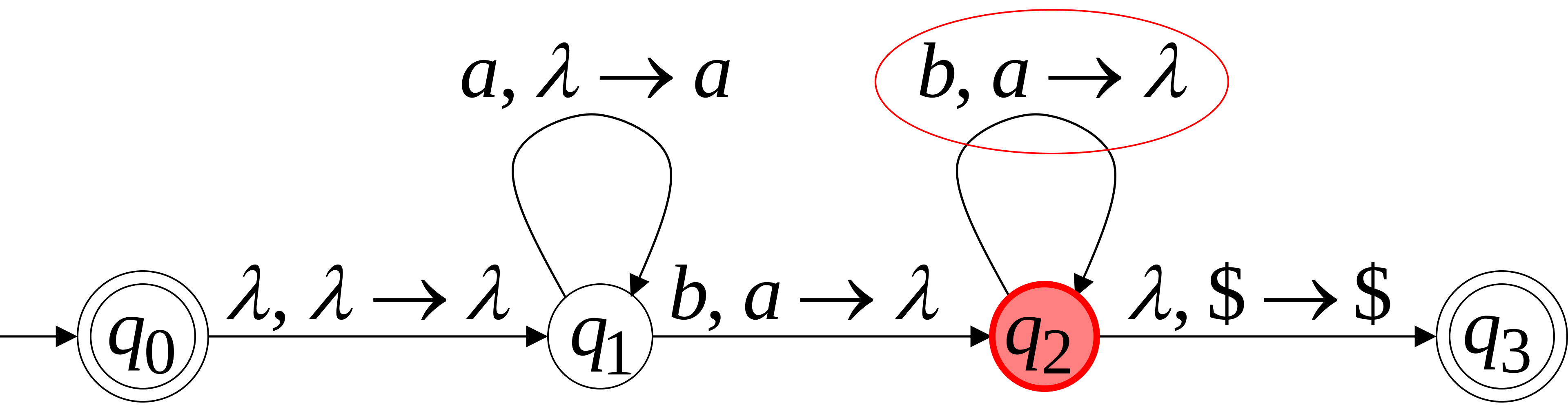
Time 7

Input



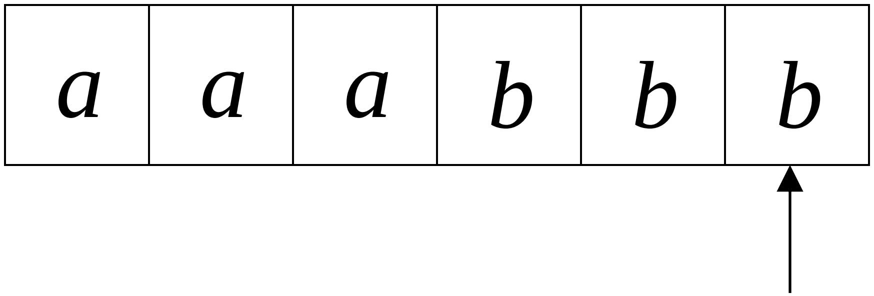


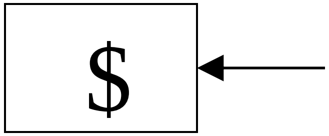
Stack



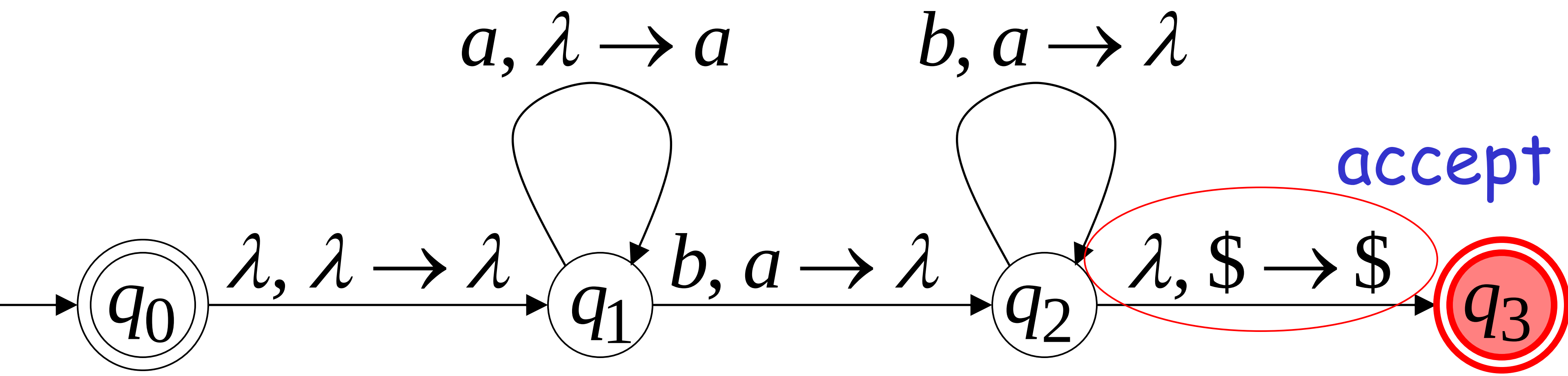
Time 8

Input





Stack



A string is accepted if there is
a computation such that:

All the input is consumed

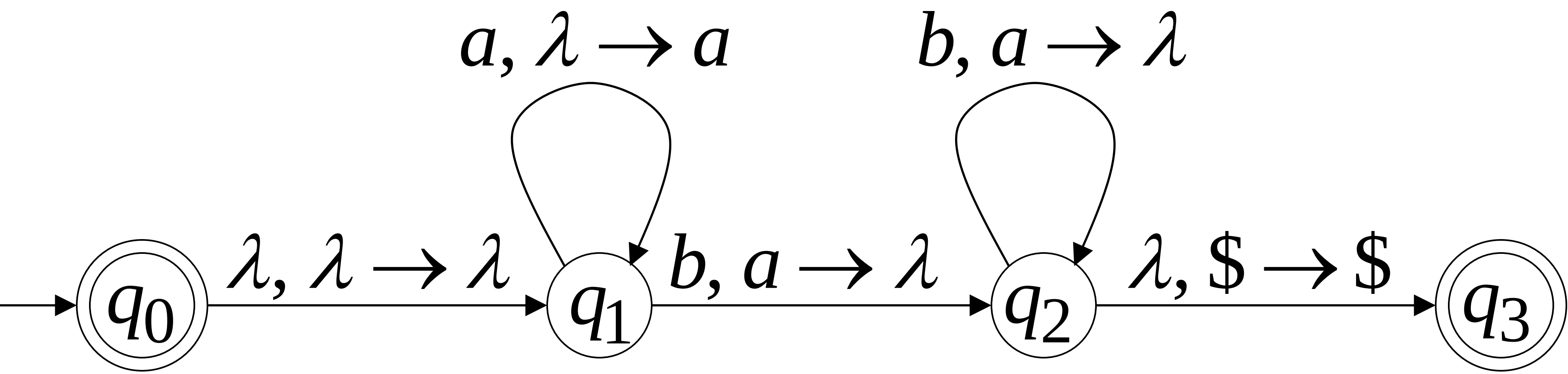
AND

The last state is a final state

At the end of the computation,
we do not care about the stack contents

$$L = \{a^n b^n : n \geq 0\}$$

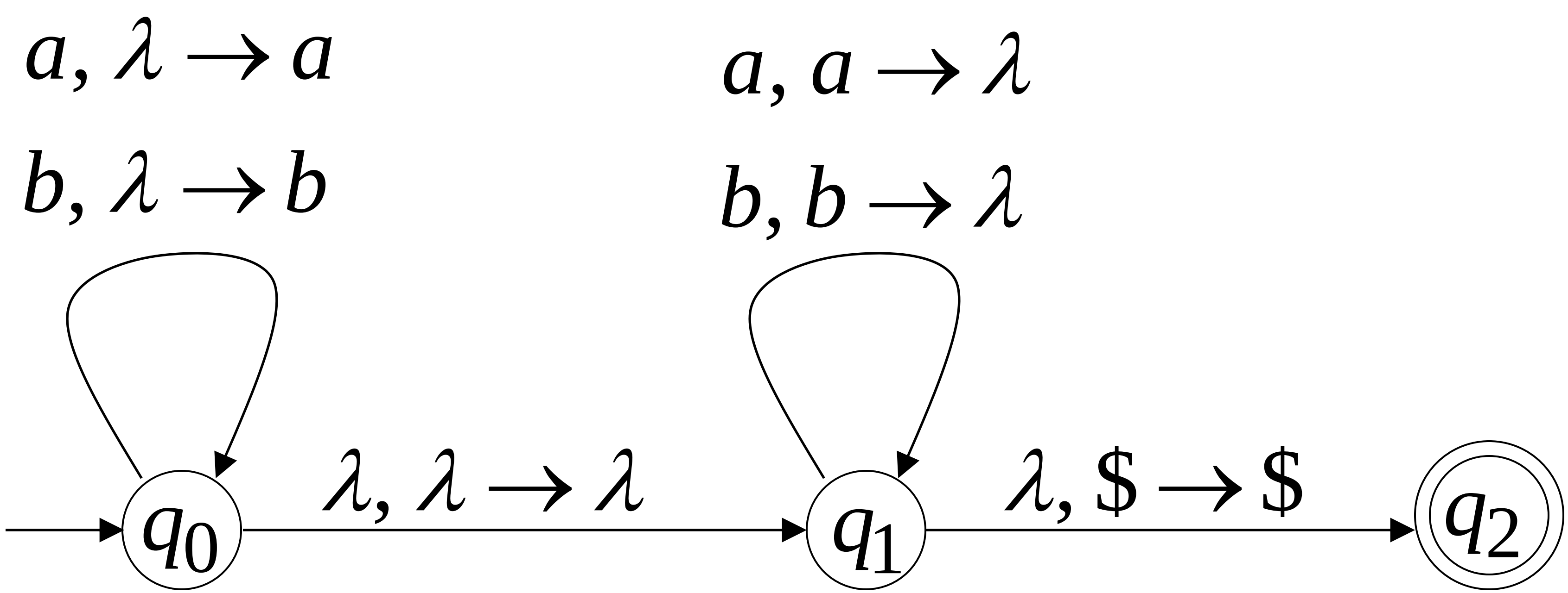
is the language accepted by the NPDA:



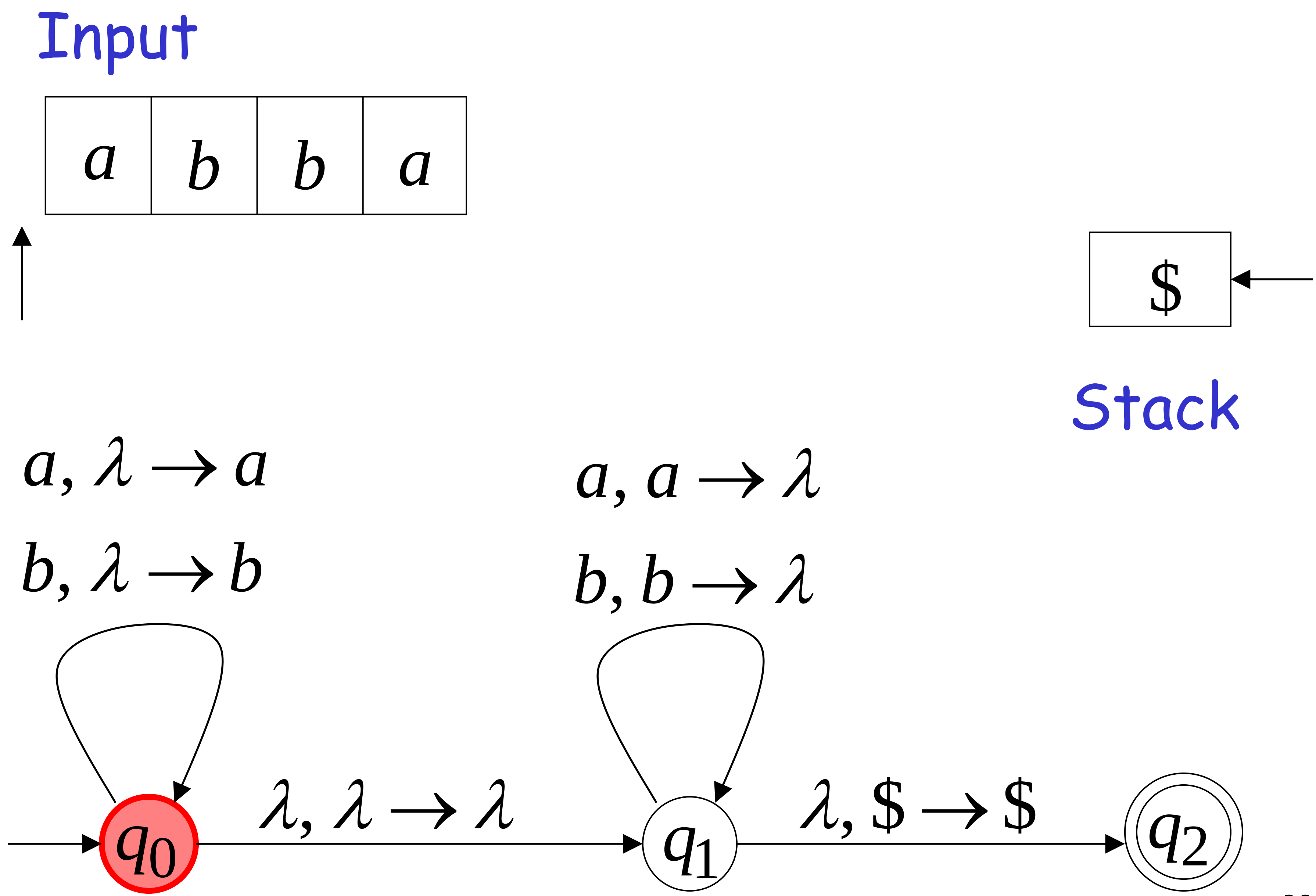
Another NPDA example

NPDA M

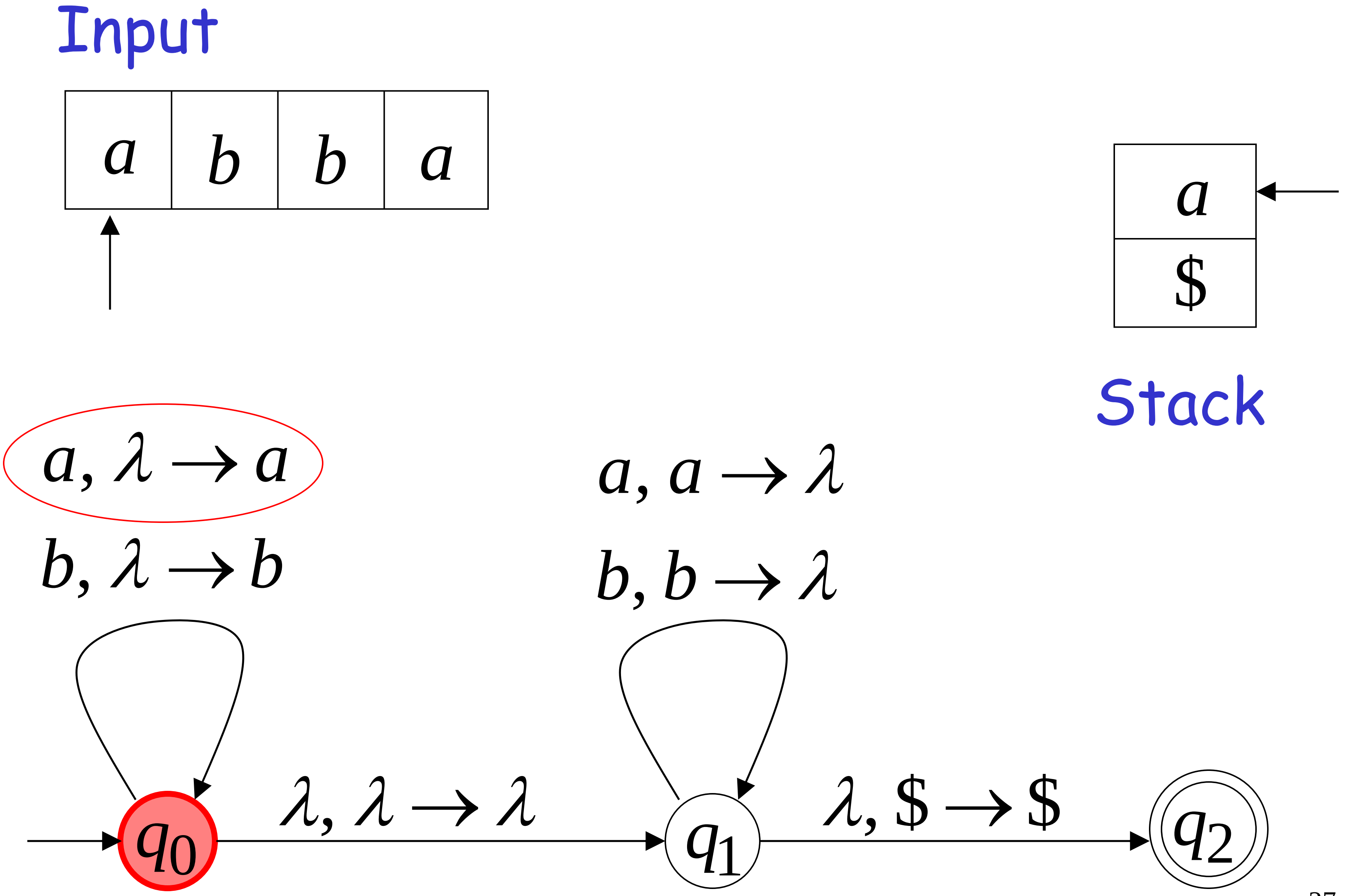
$$L(M) = \{ww^R\}$$



Execution Example: Time 0

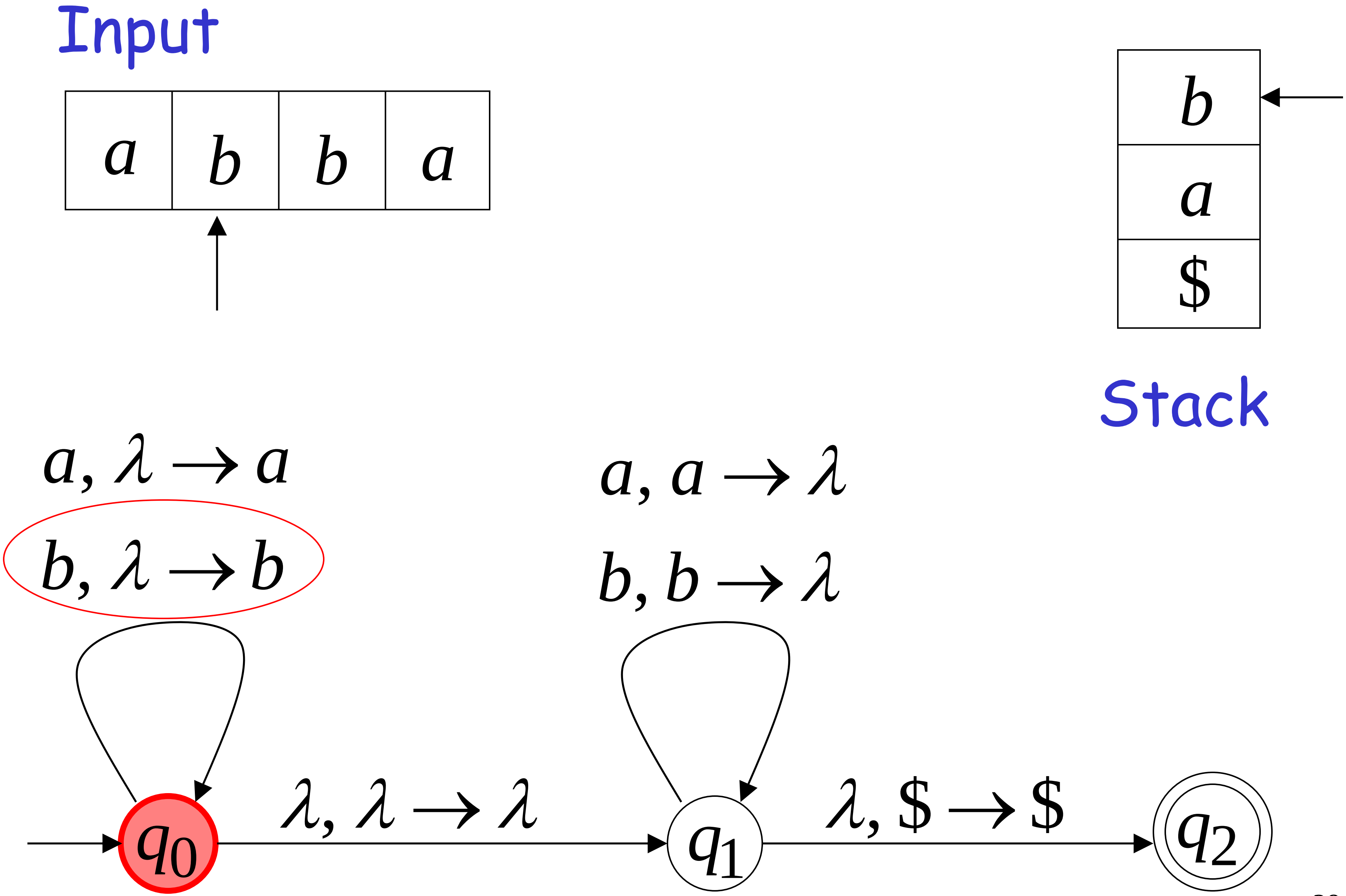


Time 1

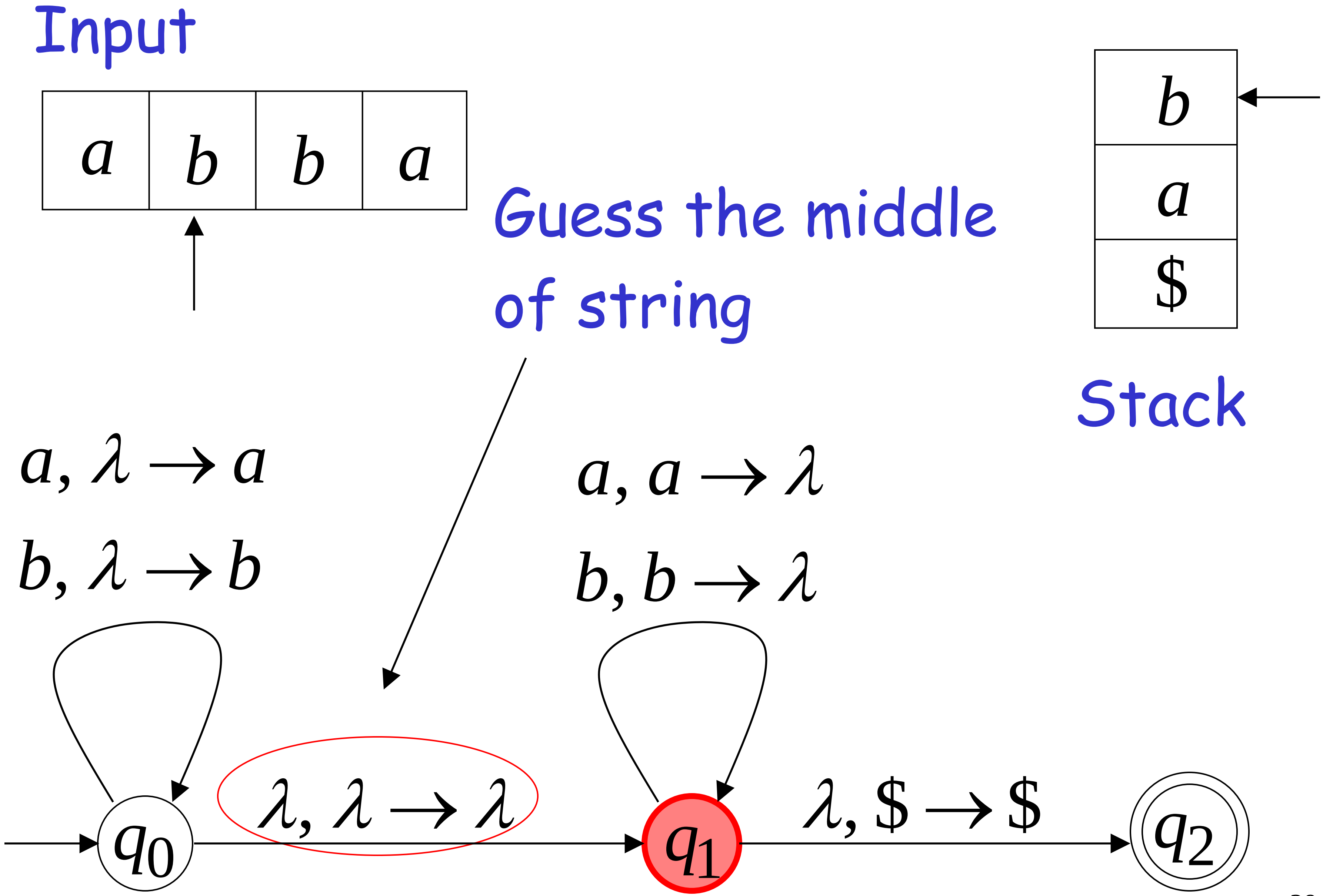


Input
a b b a
a
$\$$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 2

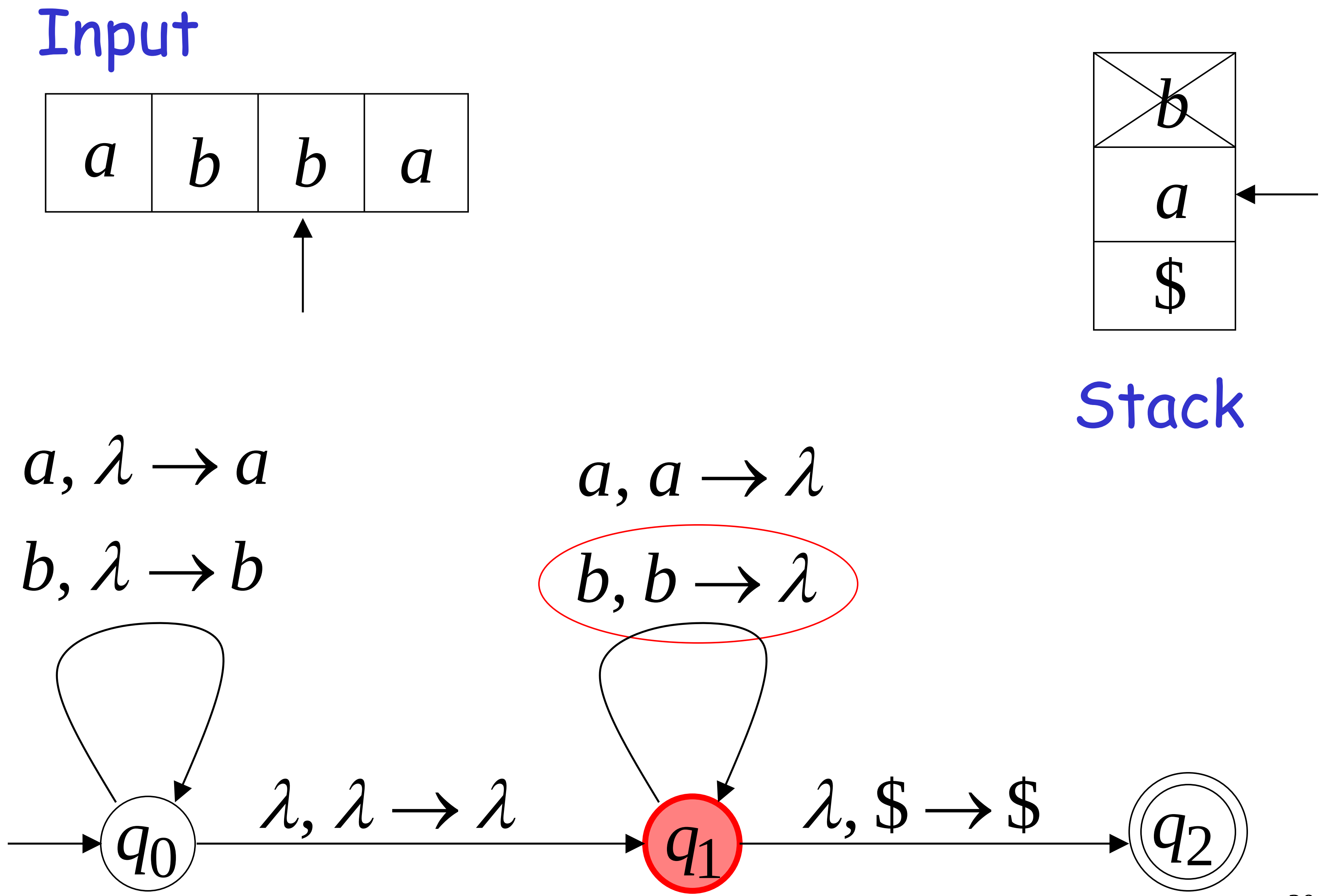

Input
a b b a
b
a
$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2

Time 3

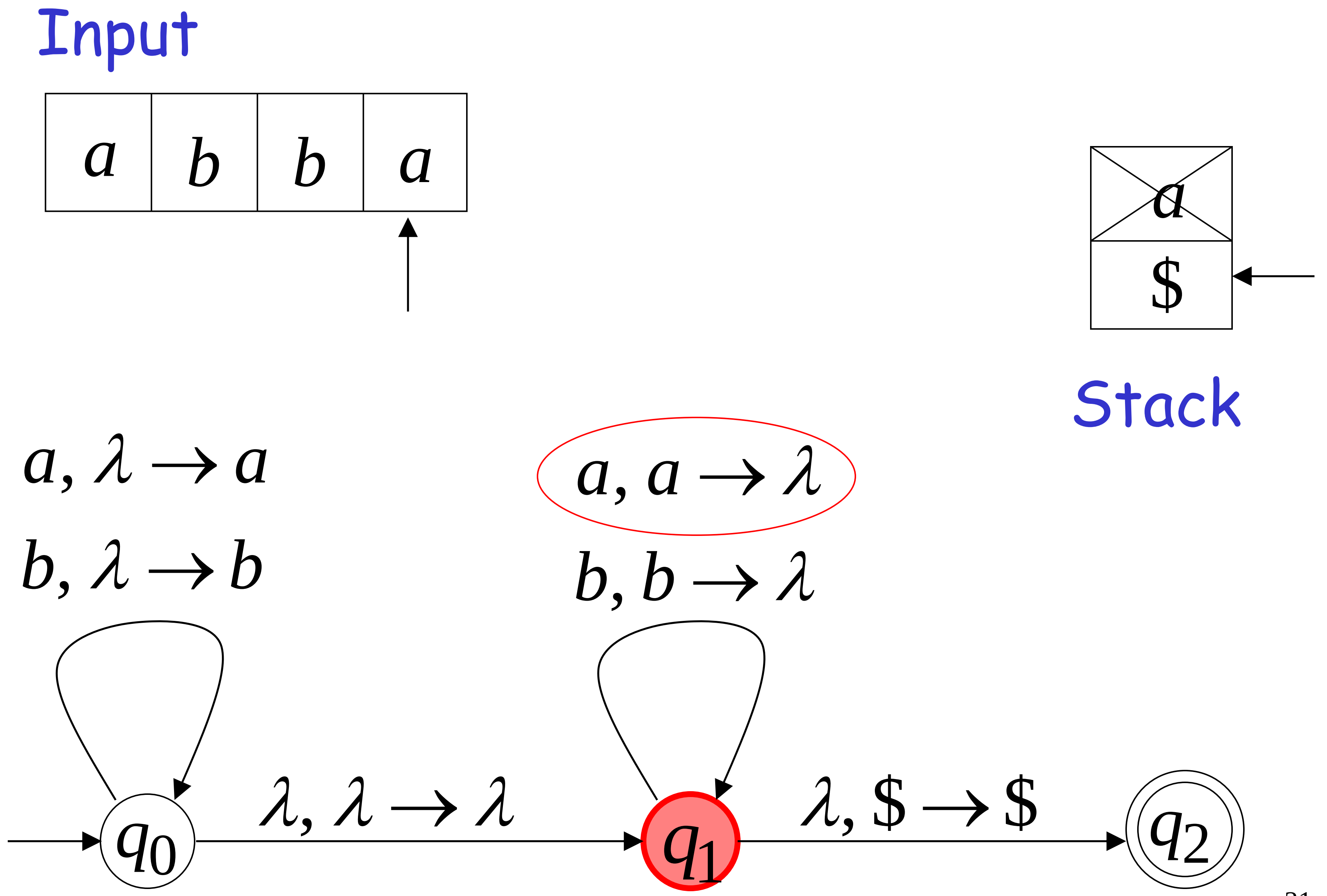


Input
a b b a
Guess the middle of string
b
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

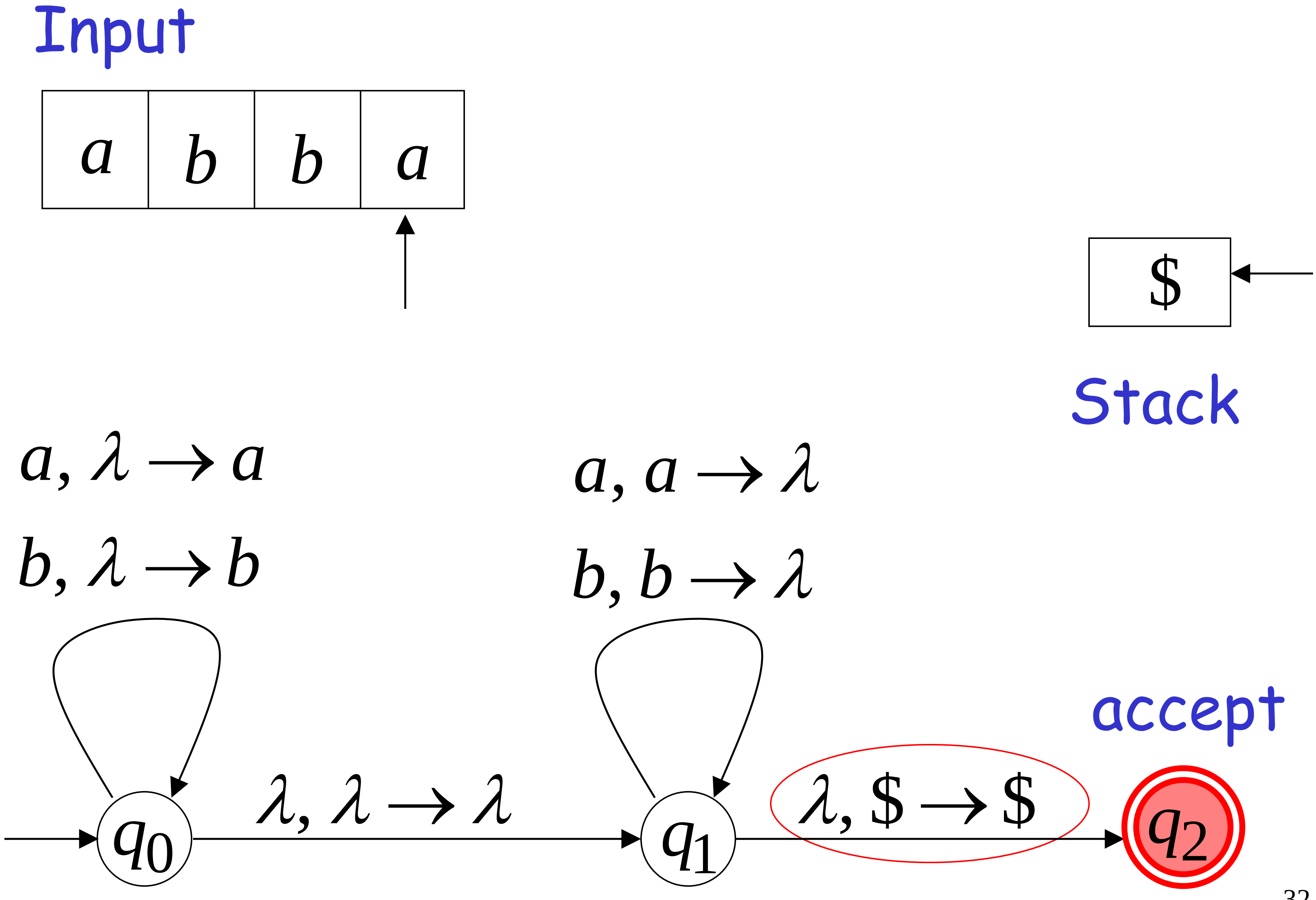
Time 4


Input
a
b
b
a
b
a
$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2

Time 5



Time 6


Input
a b b a
$
Stack
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
accept
q_0
q_1
q_2

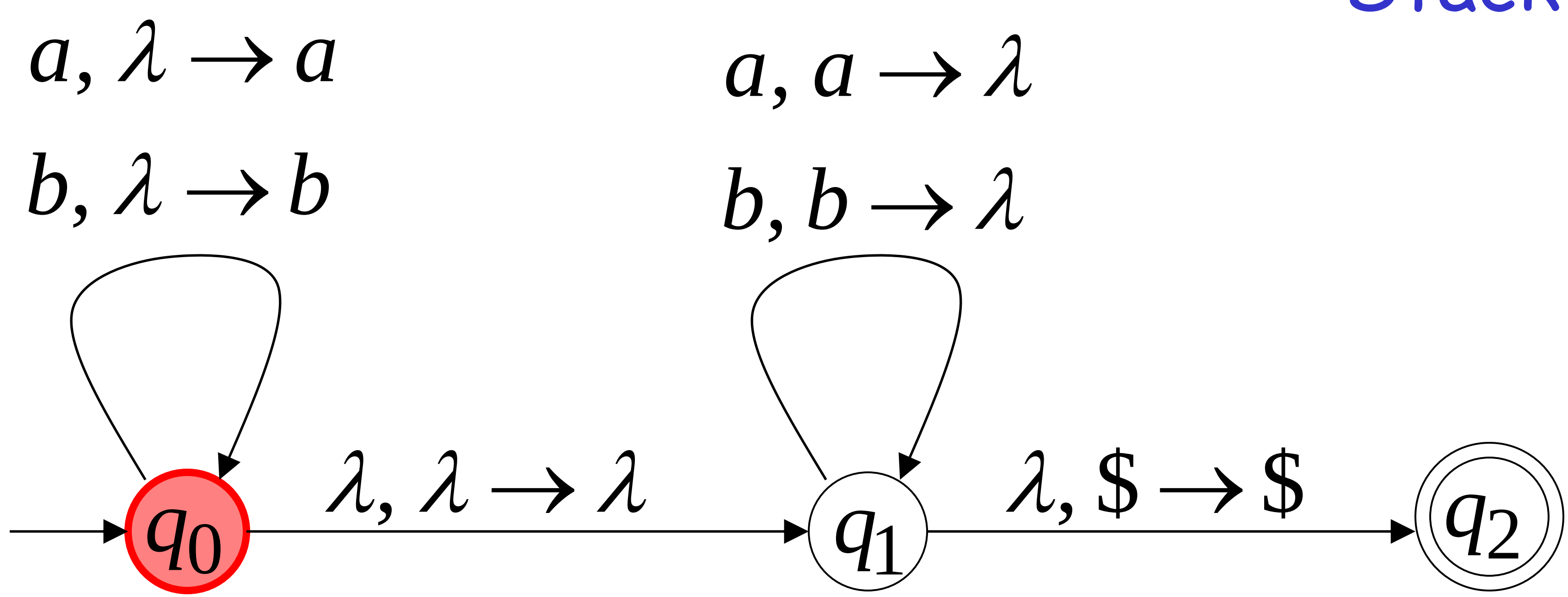
Rejection Example: Time 0

Input

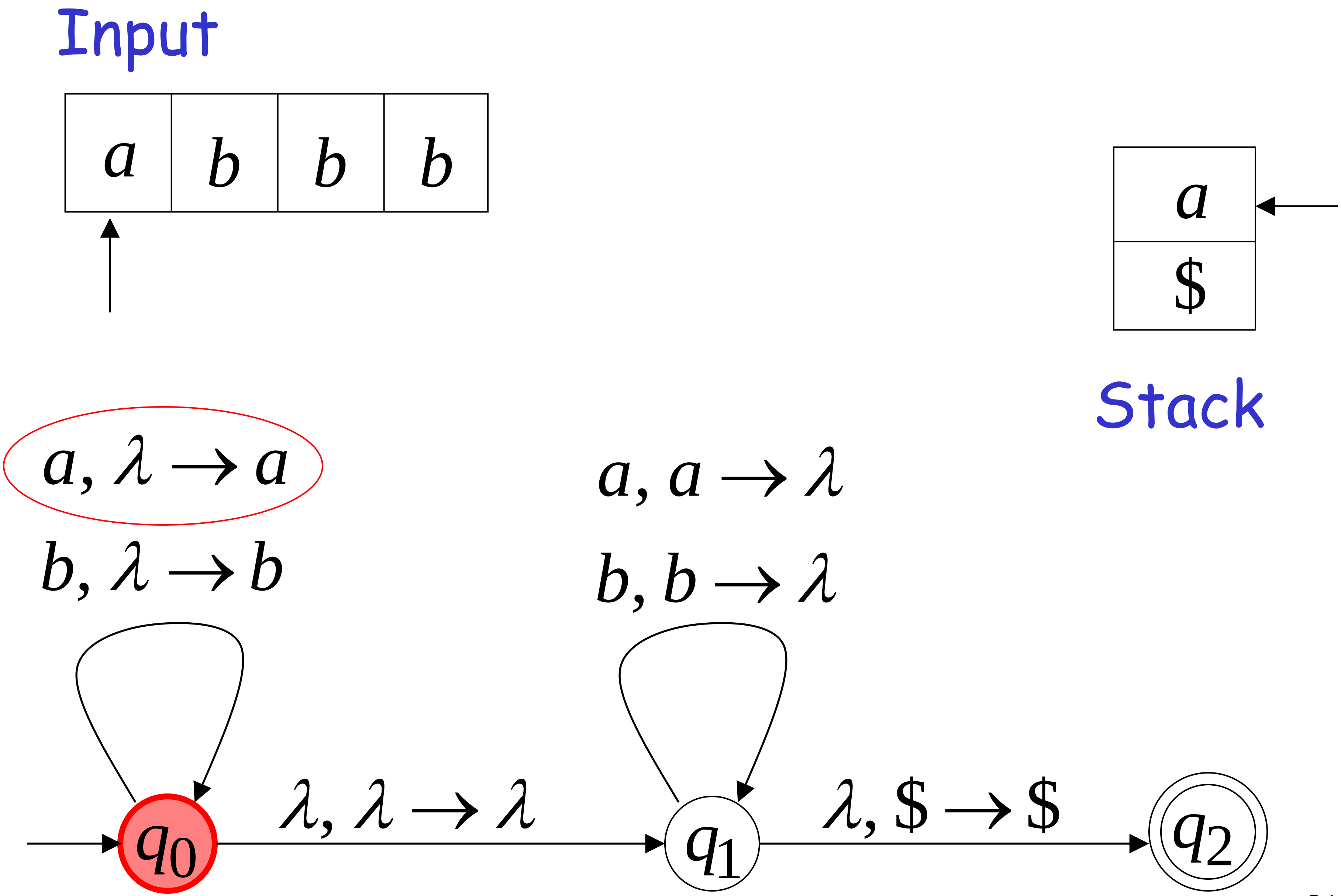
a	b	b	b

$\$$

Stack

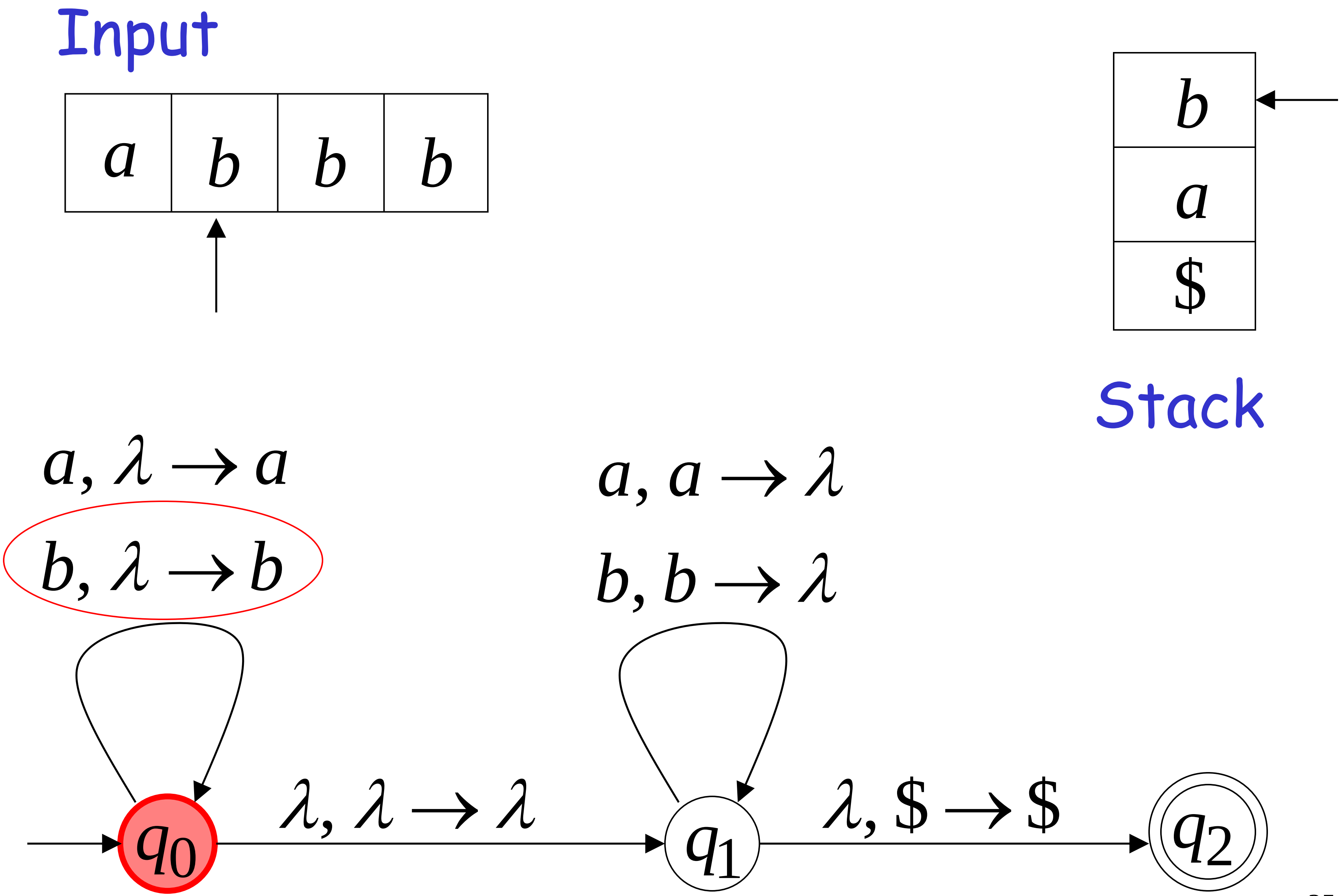


Time 1



Input
a b b b
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 2


Input
a b b b
b
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 3

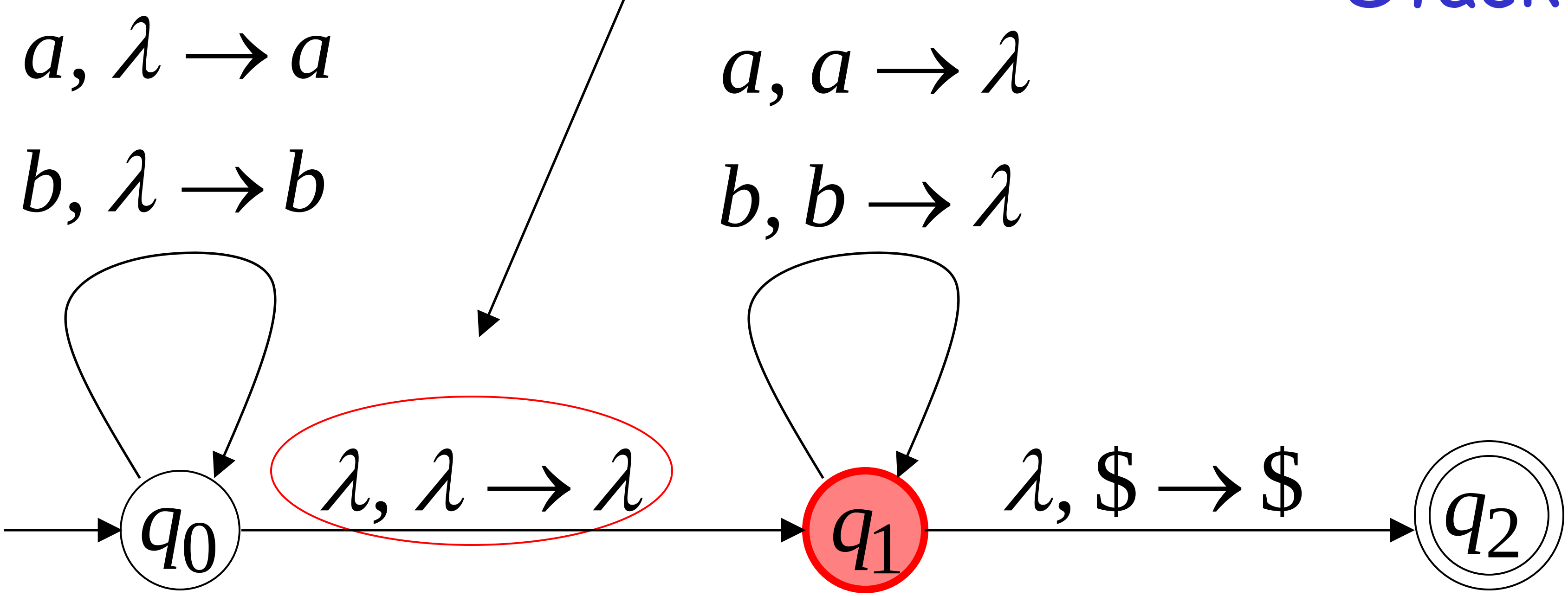
Input

a	b	b	b

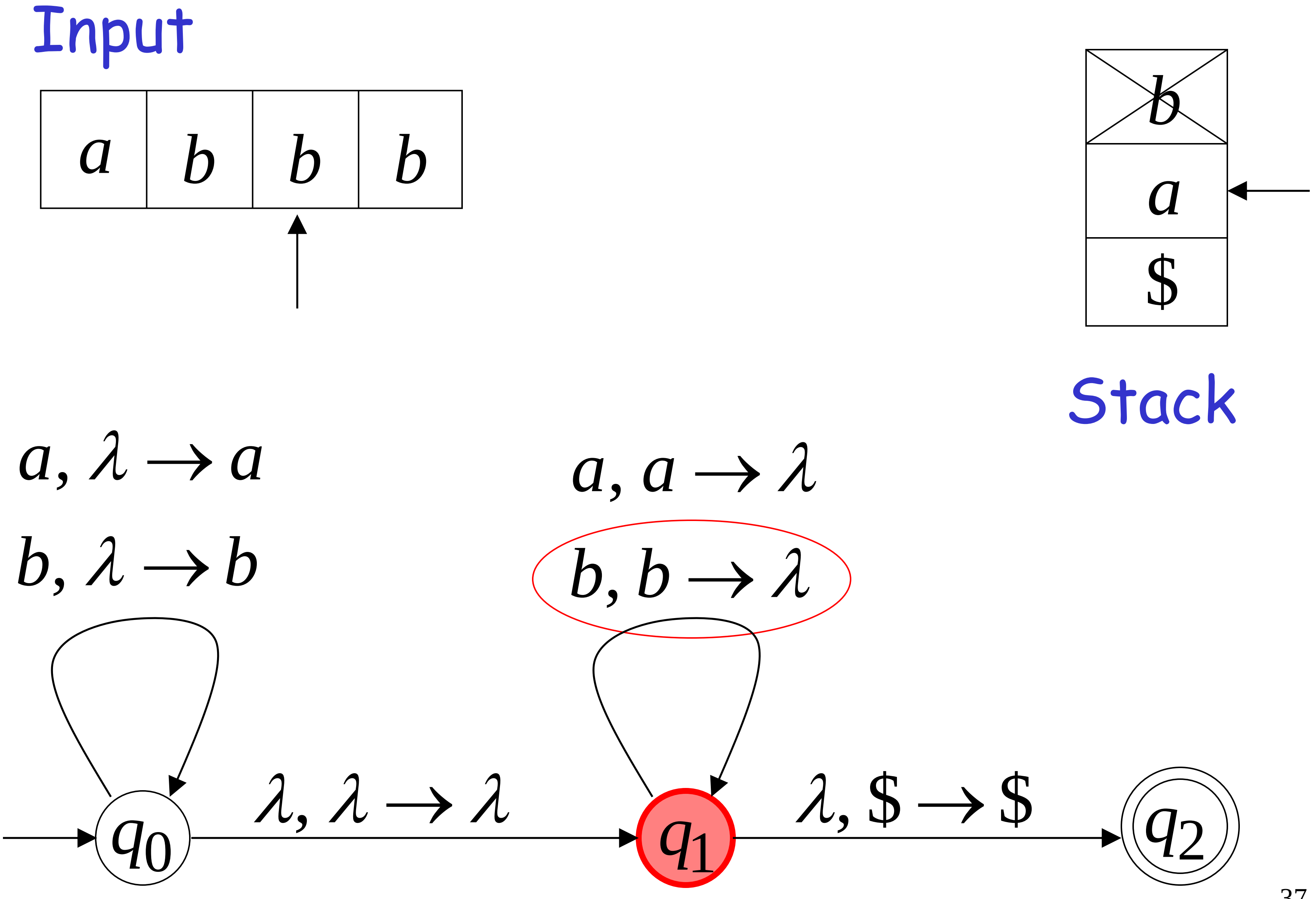
Guess the middle of string

b
a
$\$$

Stack



Time 4



Input
a
b
b
b
b
a
$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 5

There is no possible transition.

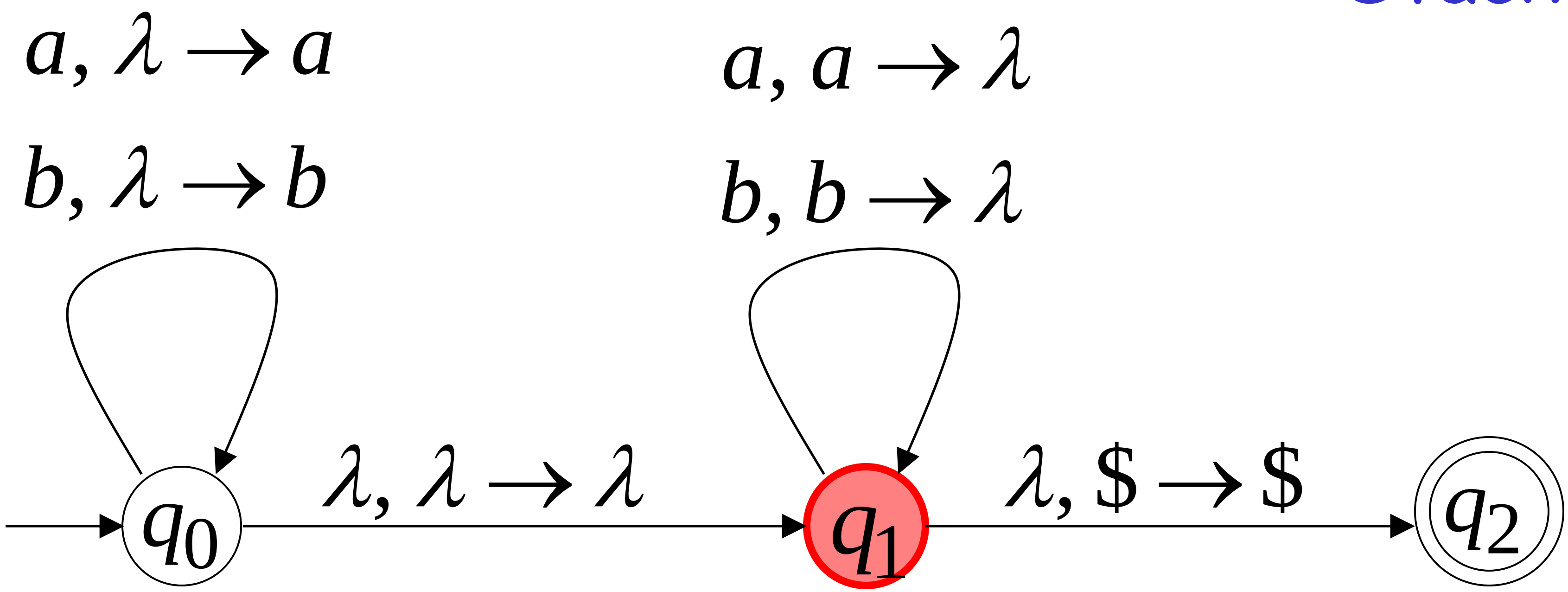
Input

a	b	b	b

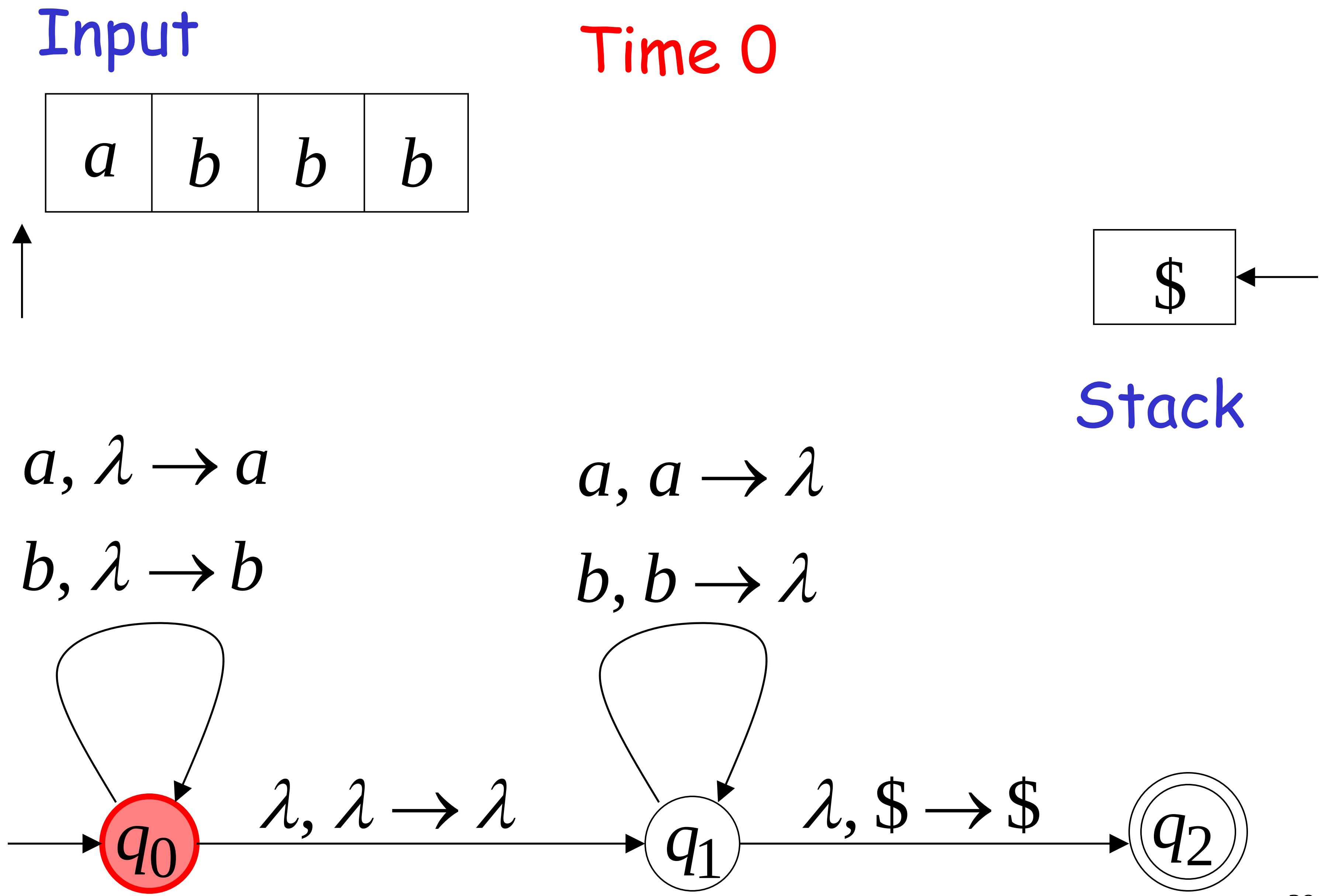
Input is not consumed

a
$\$$

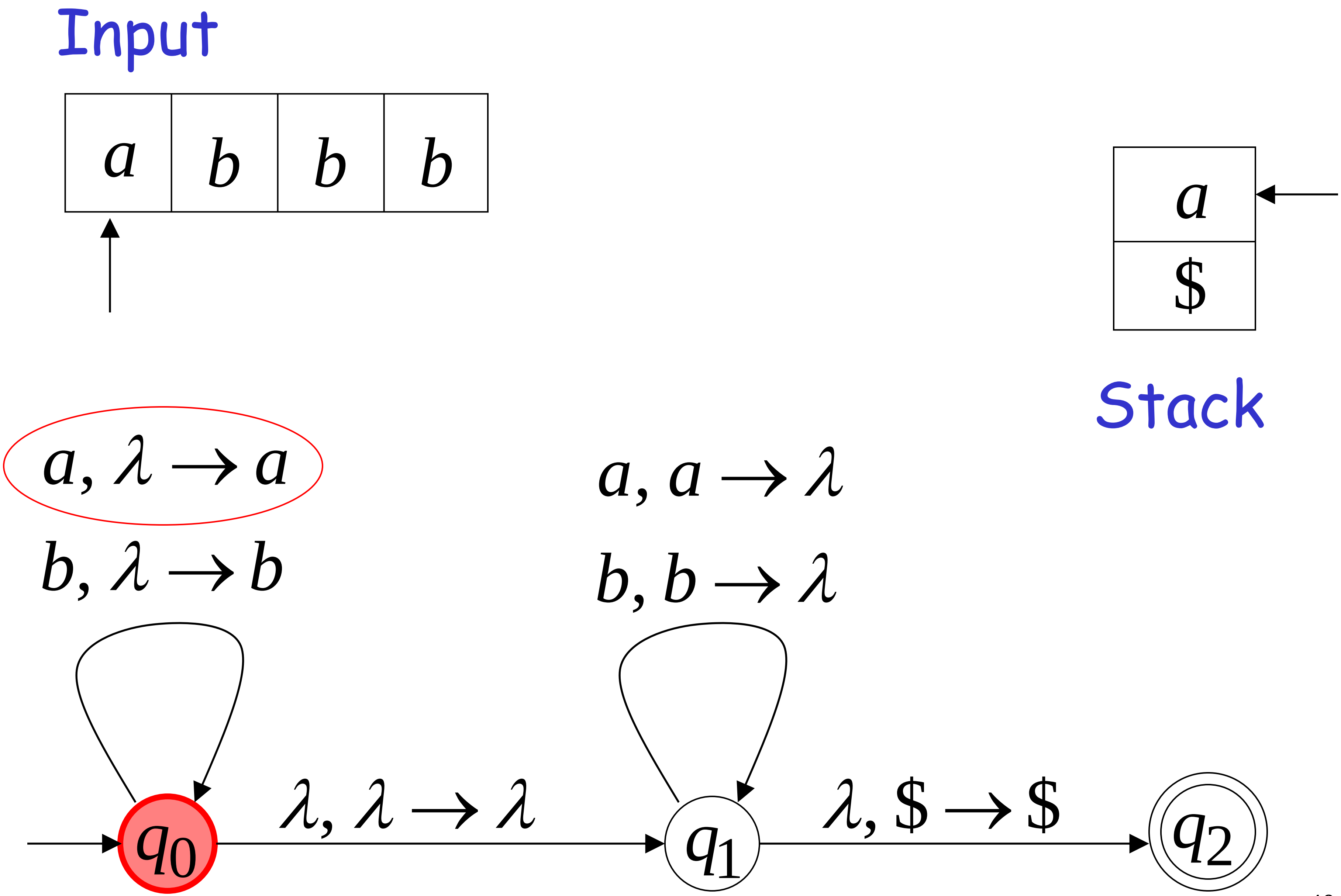
Stack



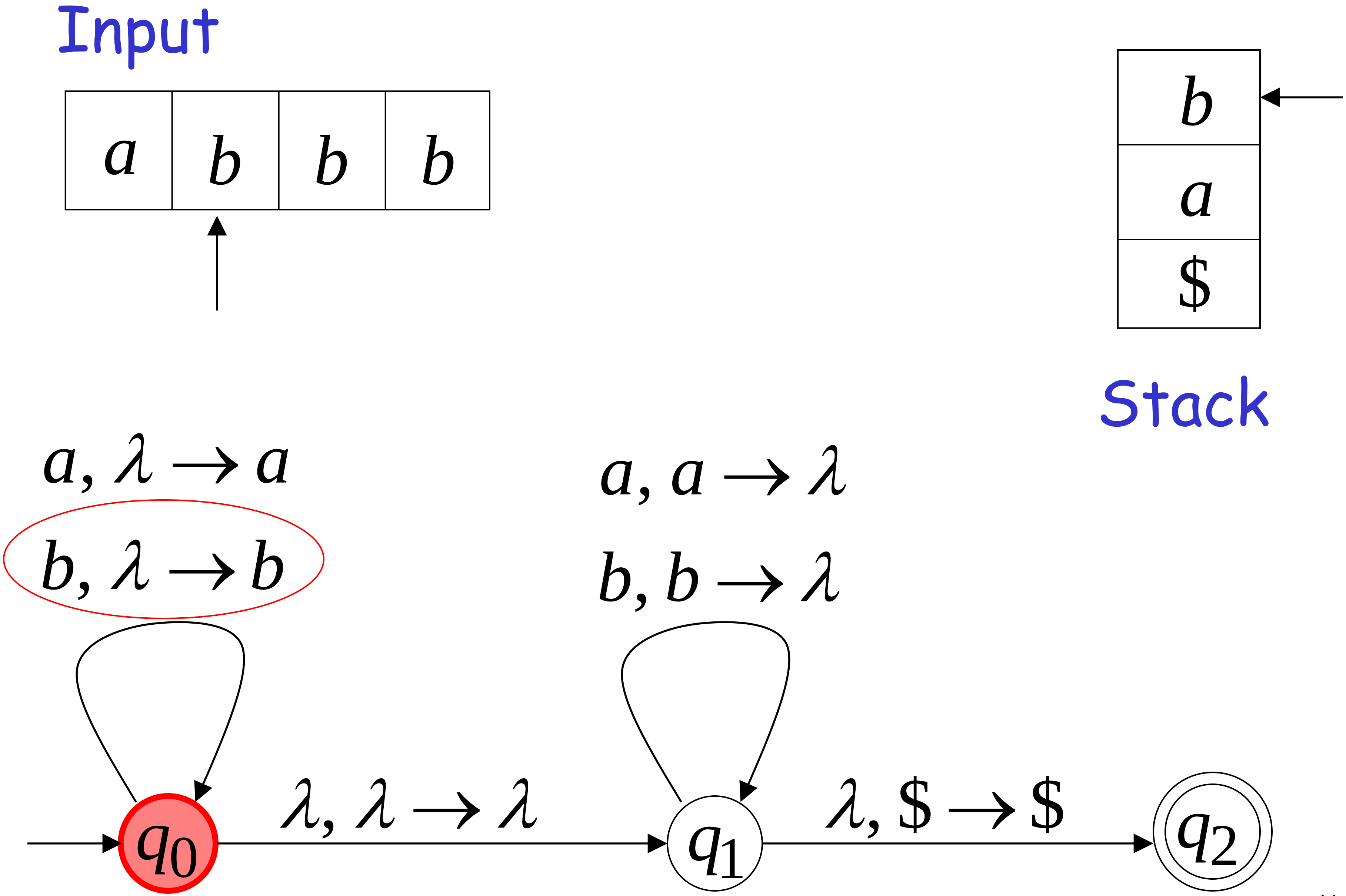
Another computation on same string:



Time 1

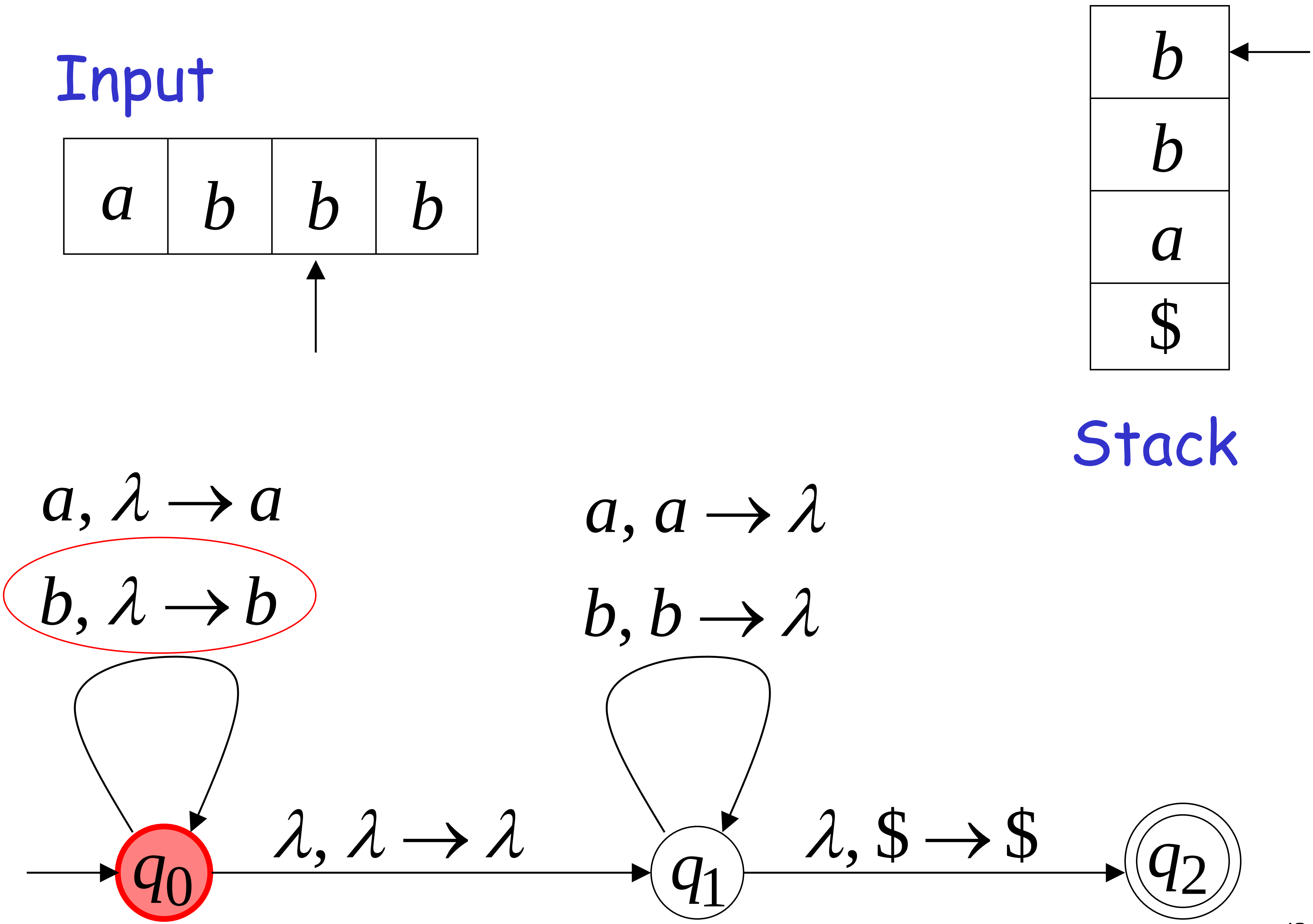

Input
a b b b
a
$\$$
Stack
$a, \lambda \rightarrow a$
$b, \lambda \rightarrow b$
$a, a \rightarrow \lambda$
$b, b \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \$ \rightarrow \$$
q_0
q_1
q_2

Time 2



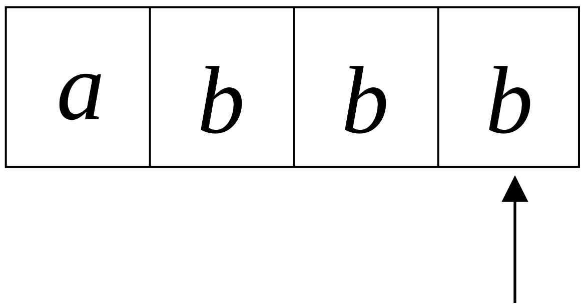
Input
a b b b
Stack
b
a
$
$a, \lambda \to a$
$b, \lambda \to b$
$a, a \to \lambda$
$b, b \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \$ \to \$$
q_0
q_1
q_2

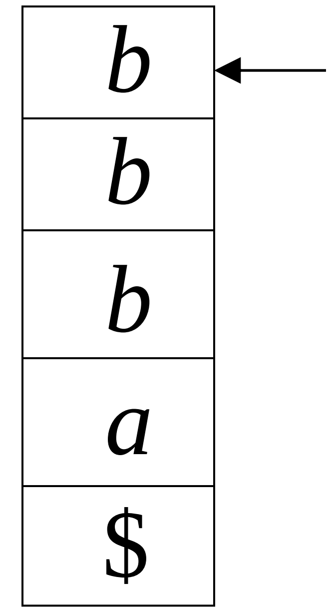
Time 3



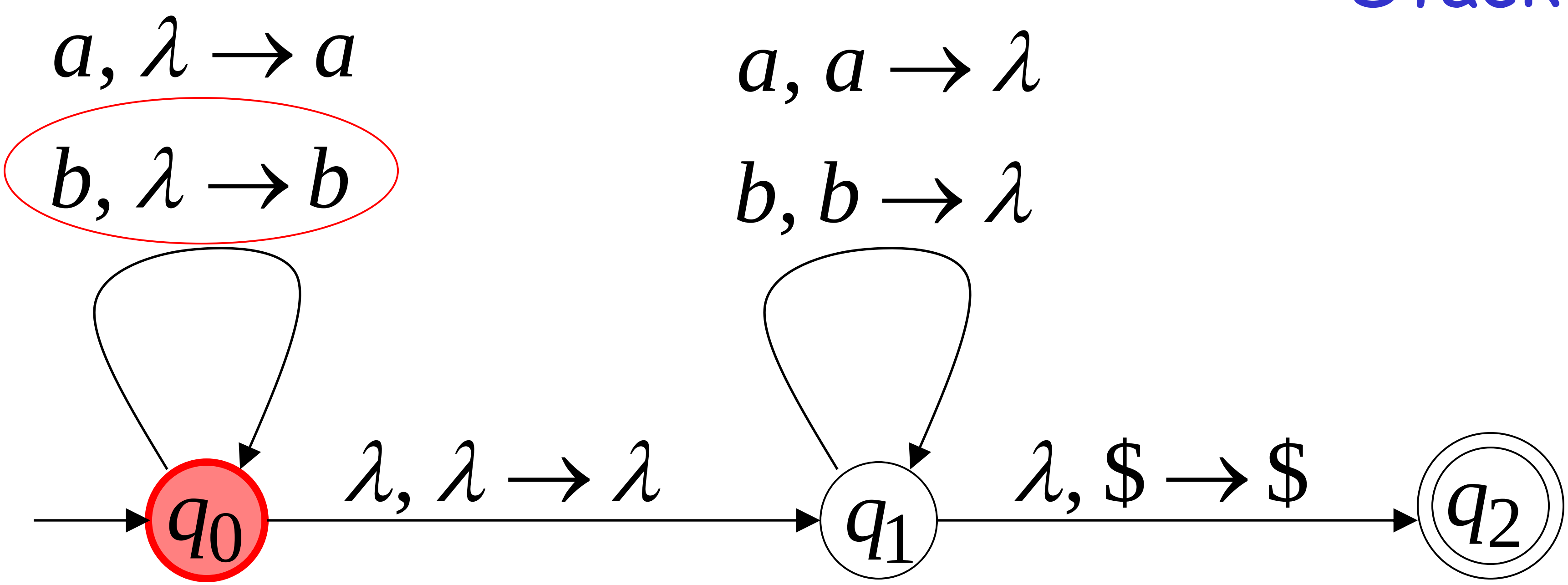
Time 4

Input



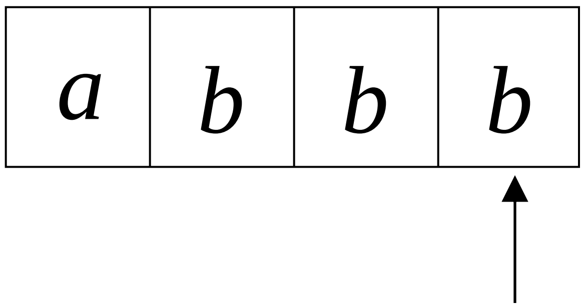


Stack



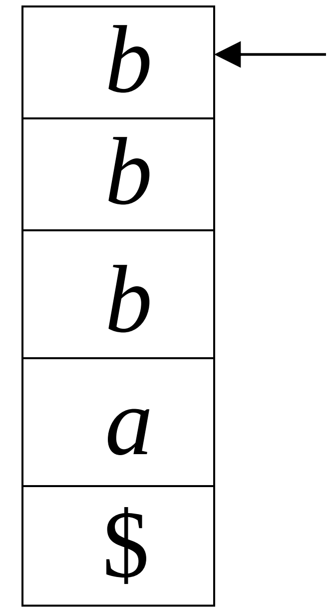
Time 5

Input

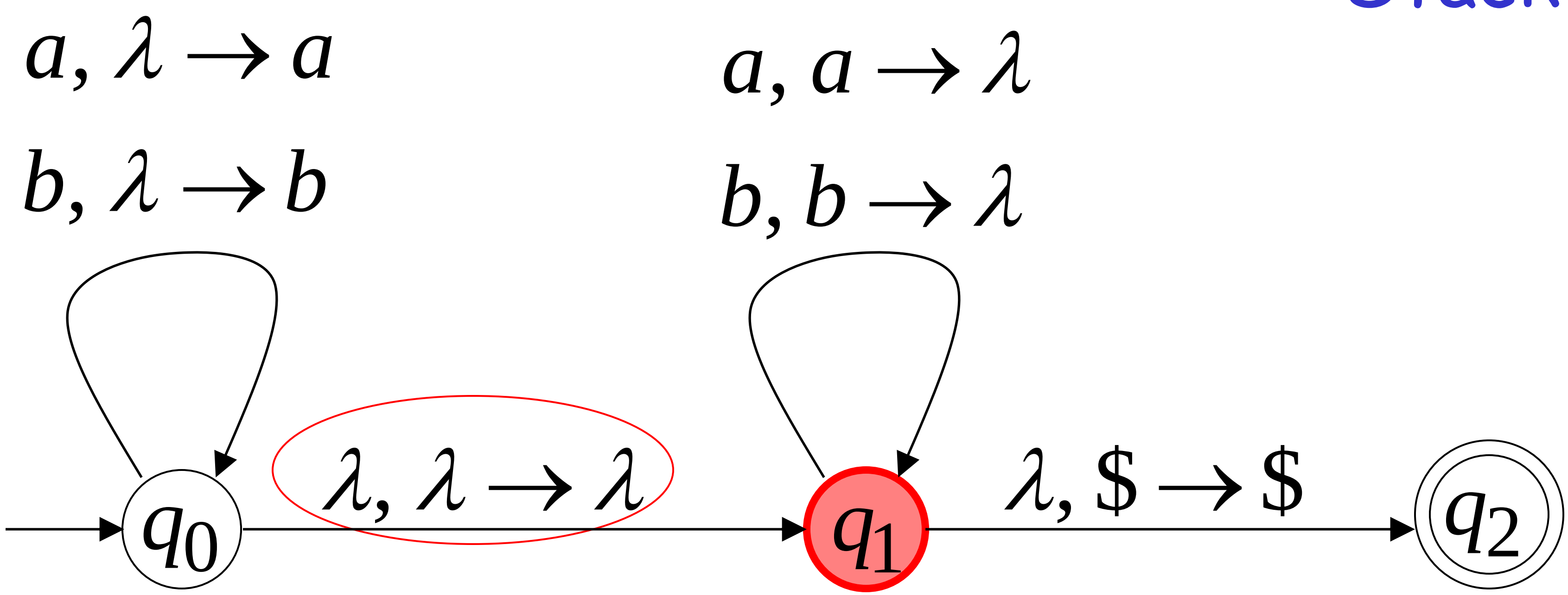


No final state

is reached

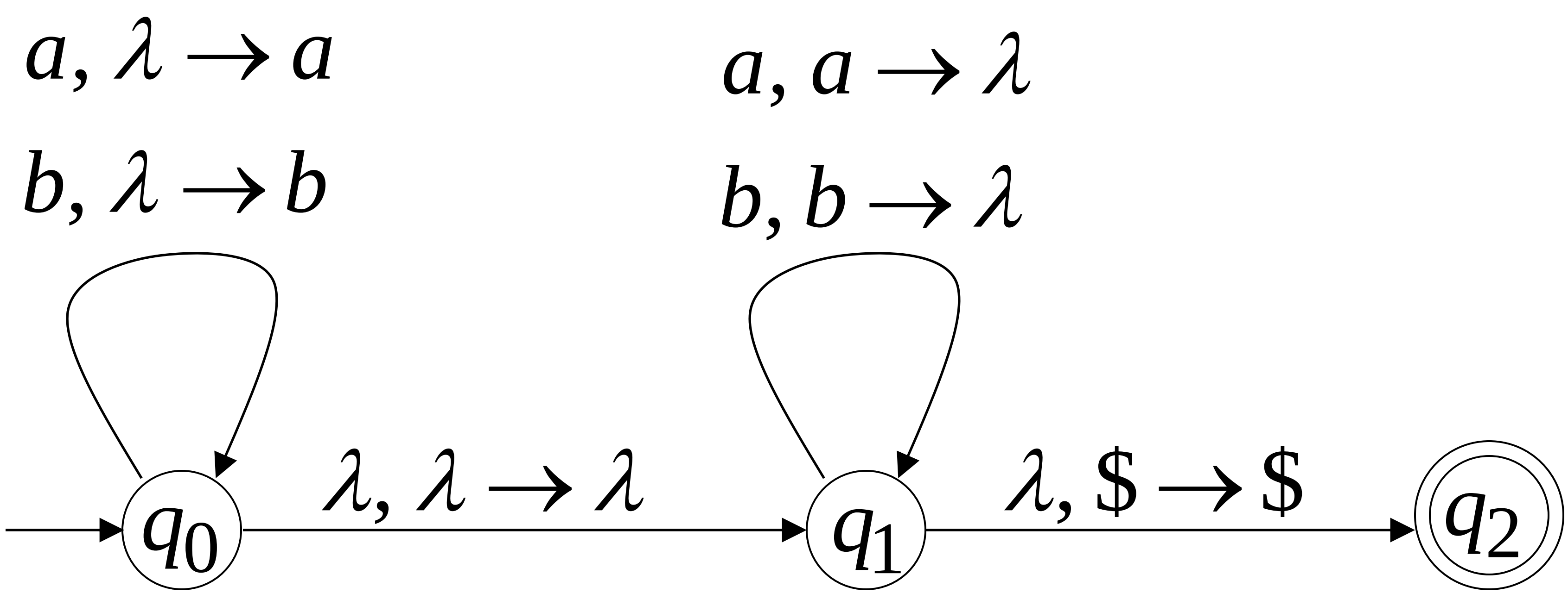


Stack



After executing all possible paths in NPDA, there is no computation that accepts string $abbb$.

$$abbb \notin L(M)$$



A string is rejected if there is no computation such that:

All the input is consumed

AND

The last state is a final state

At the end of the computation, we do not care about the stack contents

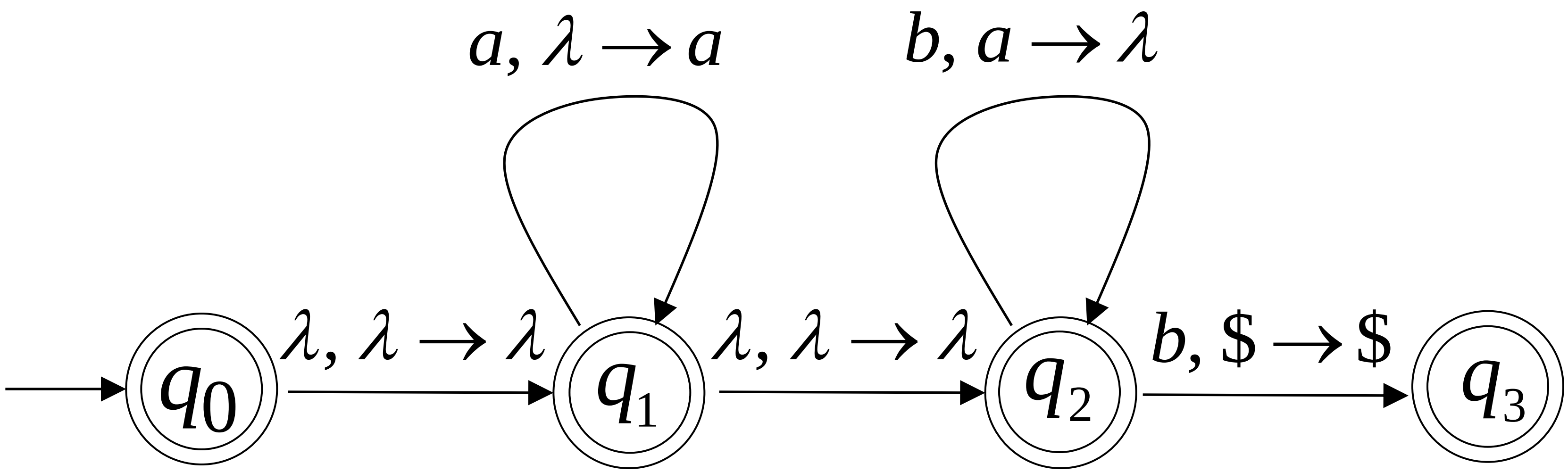
Another NPDA example

NPDA M

$$L(M) = \{a^n b^m \ : \ n \geq m-1\}$$

Note: - Minimum no. of b's = 0

- Maximum no. of b's = no. of a's + 1



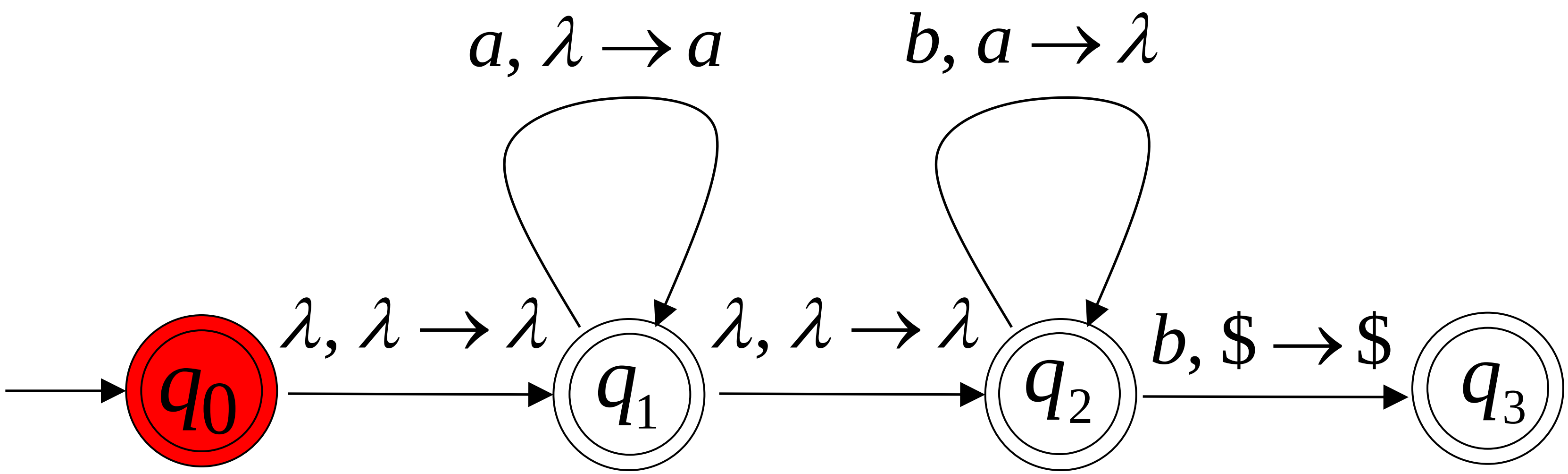
Execution Example: Time 0

Input

a	a	b

$\$$

Stack



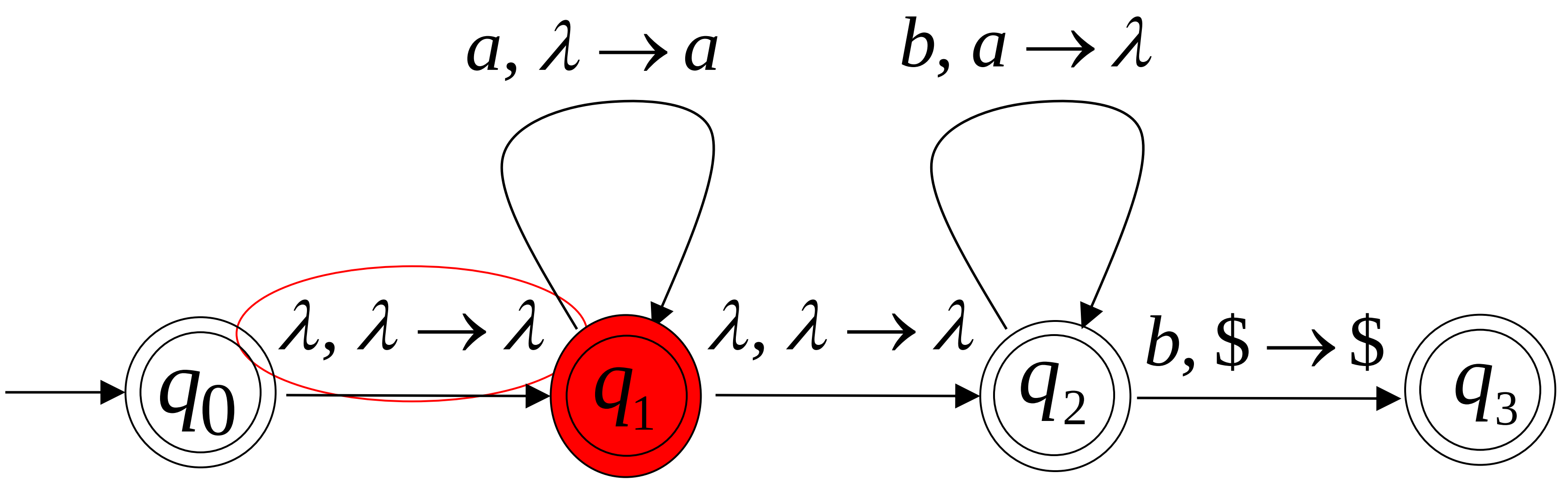
Time 1

Input

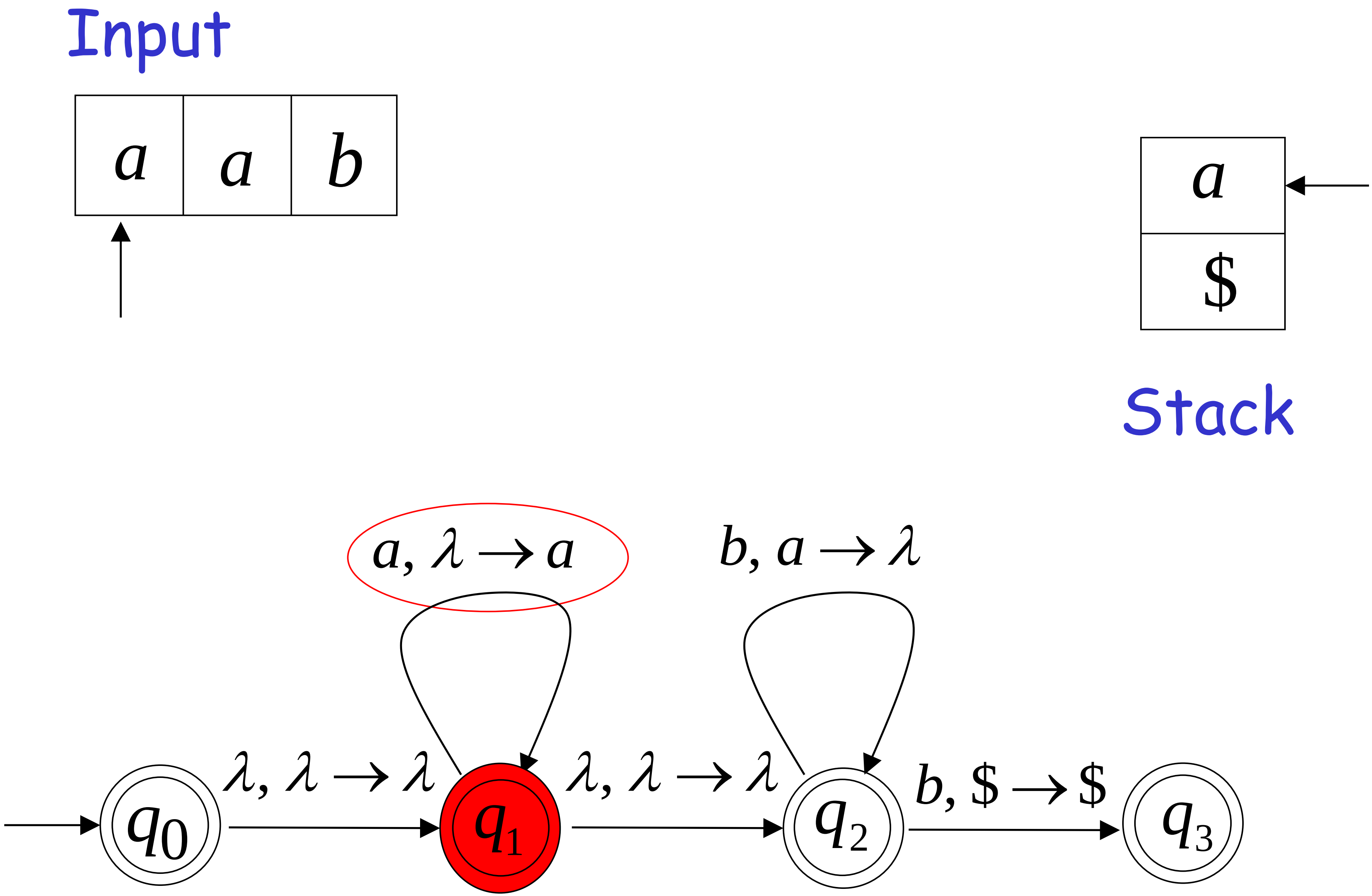
a	a	b

$\$$

Stack

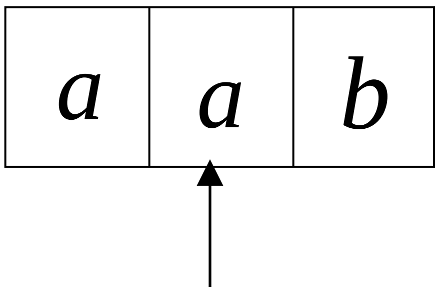


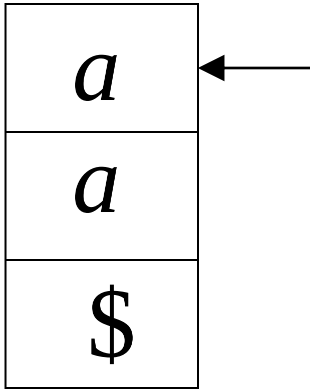
Time 2


Input
a
a
b
a
$
Stack
$a, \lambda \to a$
$b, a \to \lambda$
$\lambda, \lambda \to \lambda$
$\lambda, \lambda \to \lambda$
$b, \$ \to \$$
q_0
q_1
q_2
q_3

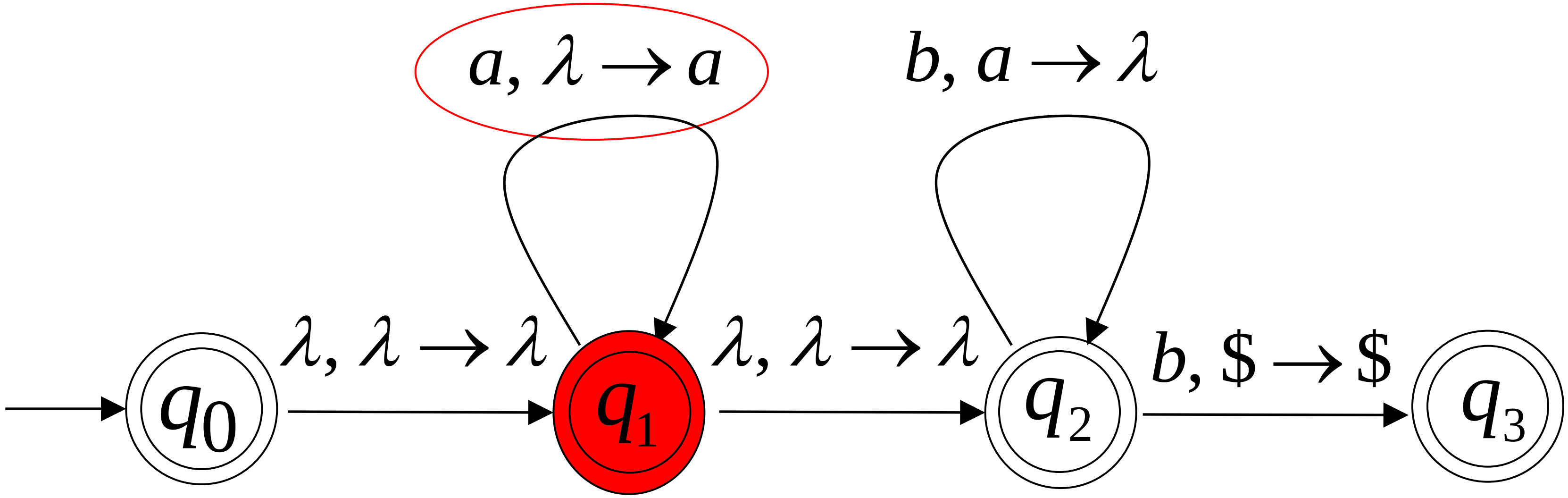
Time 3

Input





Stack



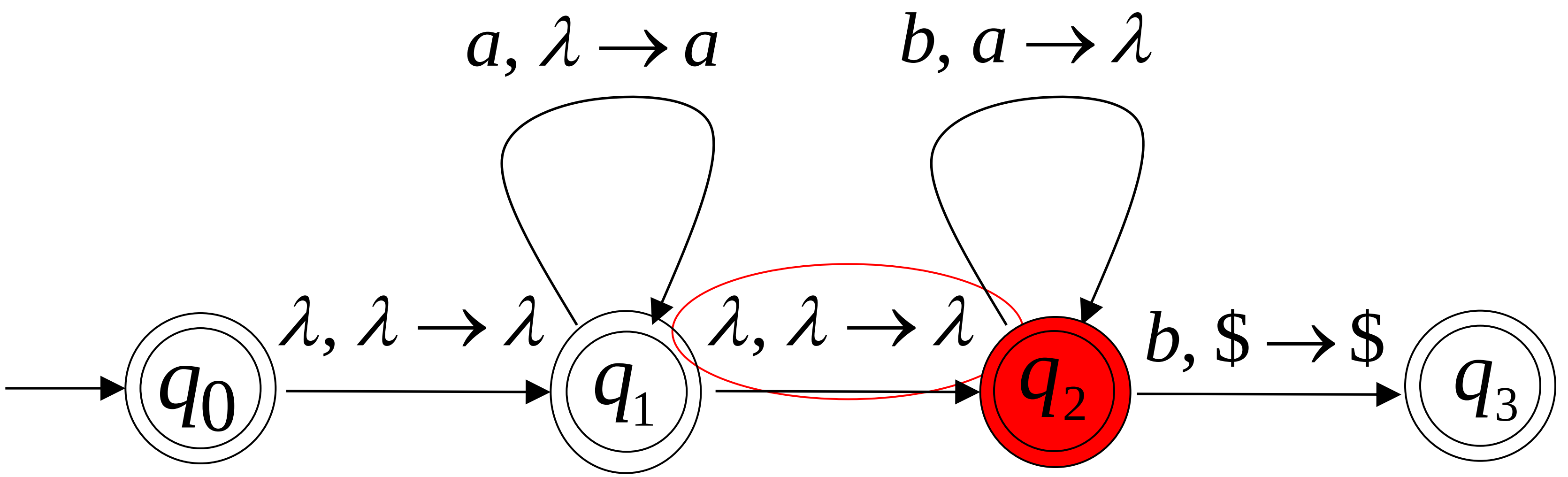
Time 4

Input

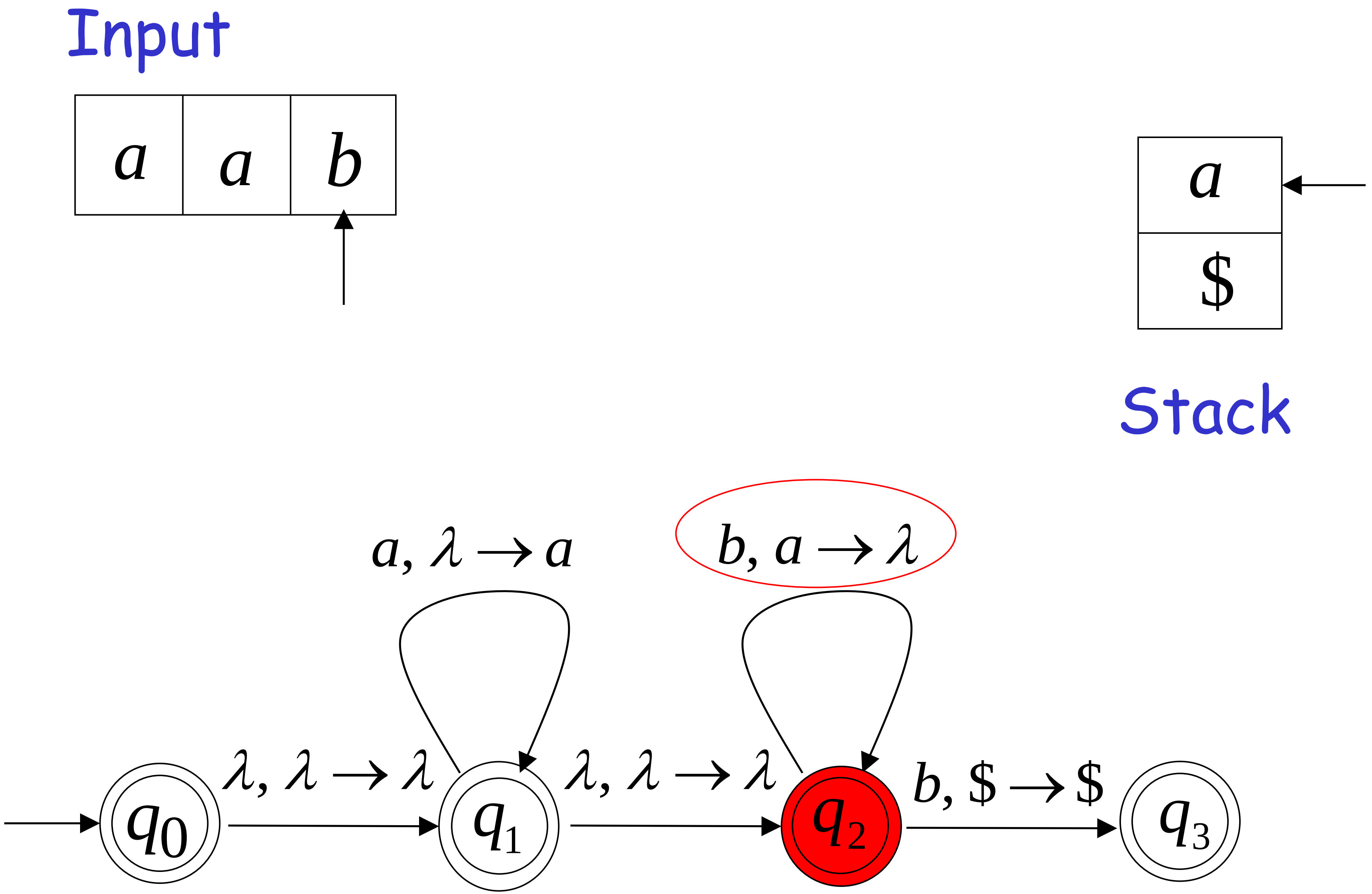
a	a	b

a
a
$\$$

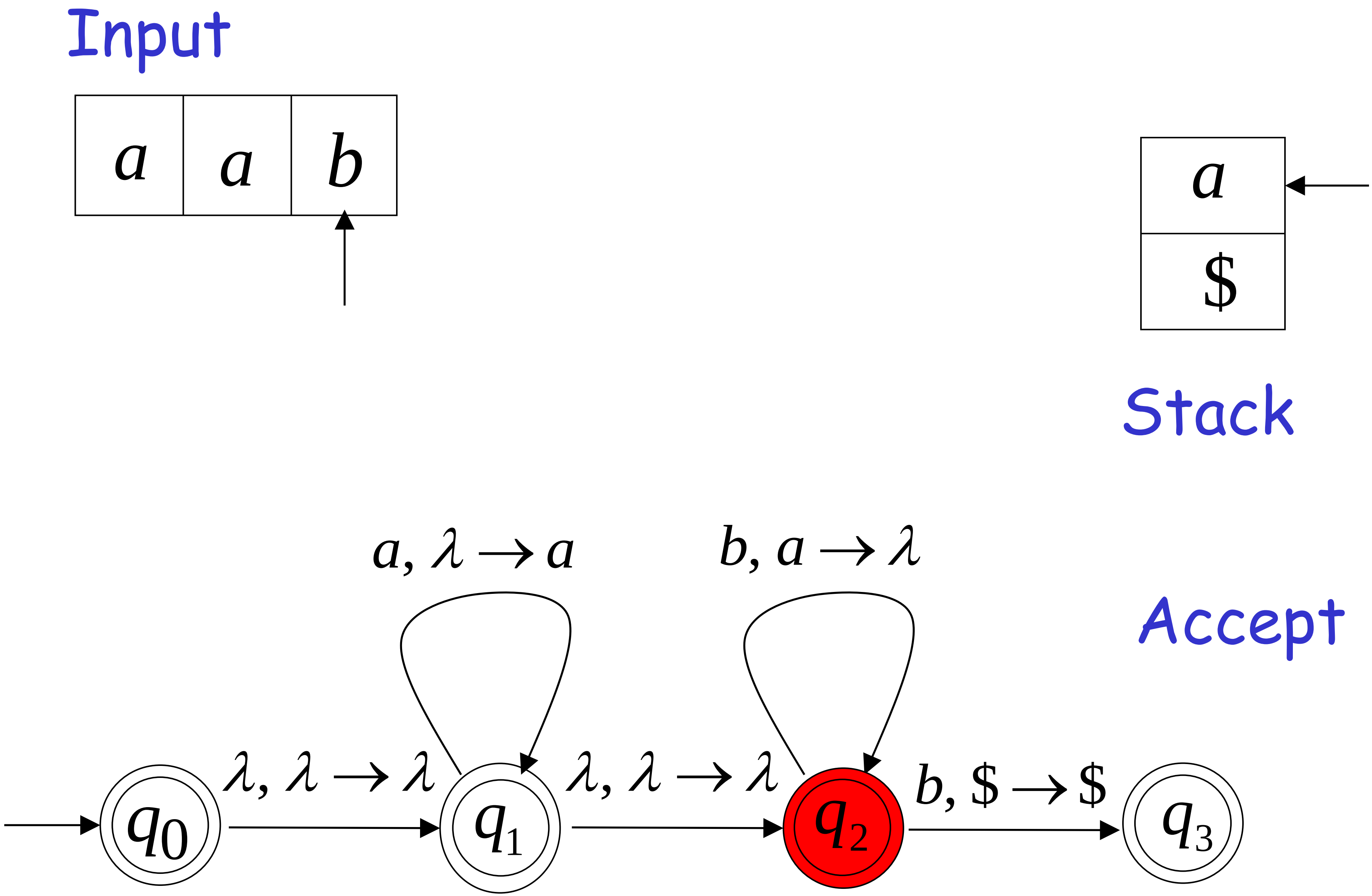
Stack



Time 5

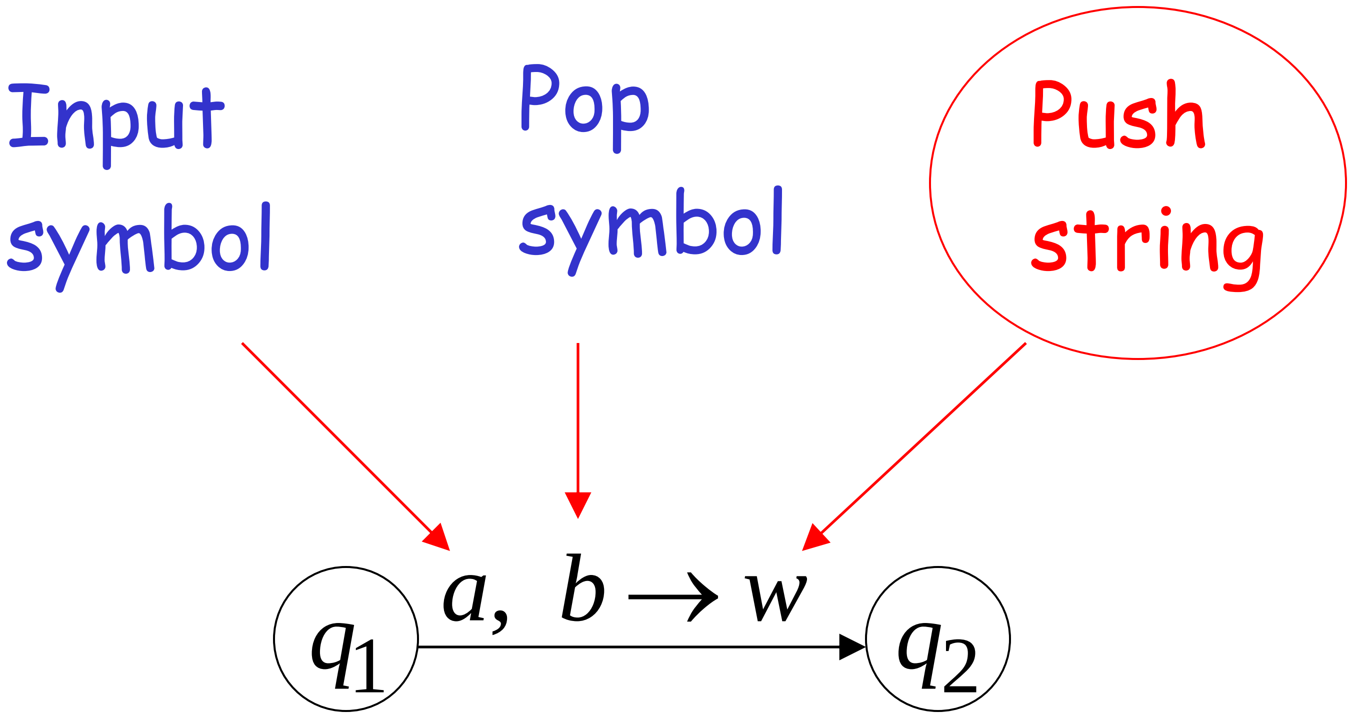


Time 5

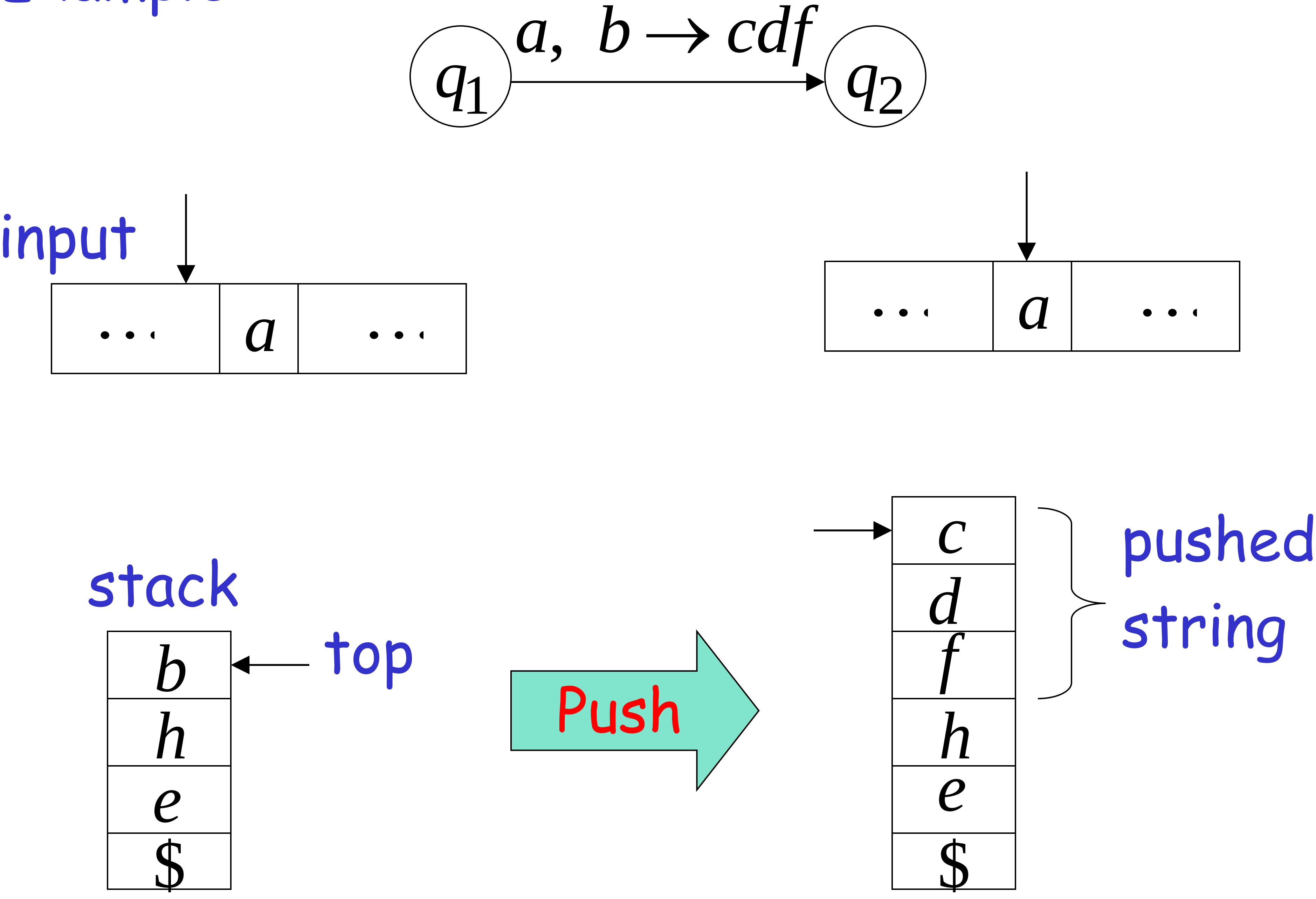


Input
a a b
a
$\$$
Stack
$a, \lambda \rightarrow a$
$b, a \rightarrow \lambda$
Accept
$\lambda, \lambda \rightarrow \lambda$
$\lambda, \lambda \rightarrow \lambda$
$b, \$ \rightarrow \$$
q_0
q_1
q_2
q_3

Pushing Strings



Example:



$a, \ b \rightarrow cdf$
q_1
q_2
input
a
stack
top
b
h
e
$\$$
Push
c
d
f
h
e
$\$$
pushed string

Another NPDA example

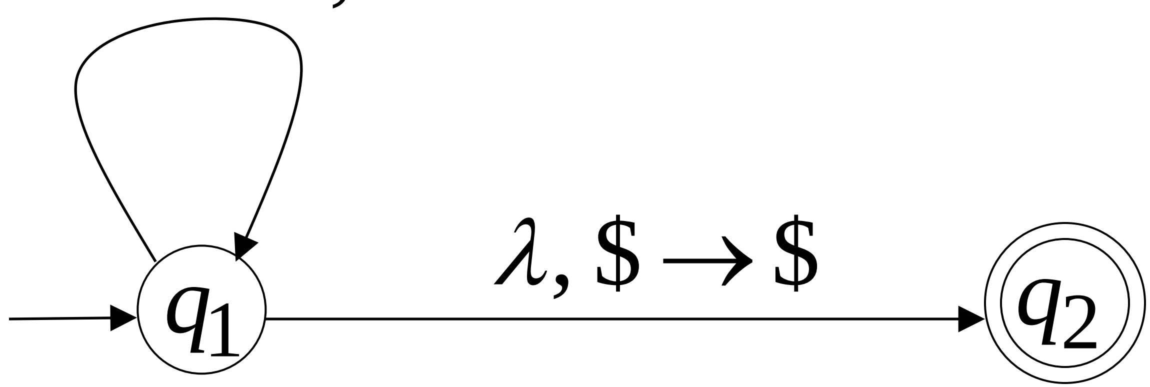
NPDA M

$$L(M) = \{w: \ n_a = n_b\}$$

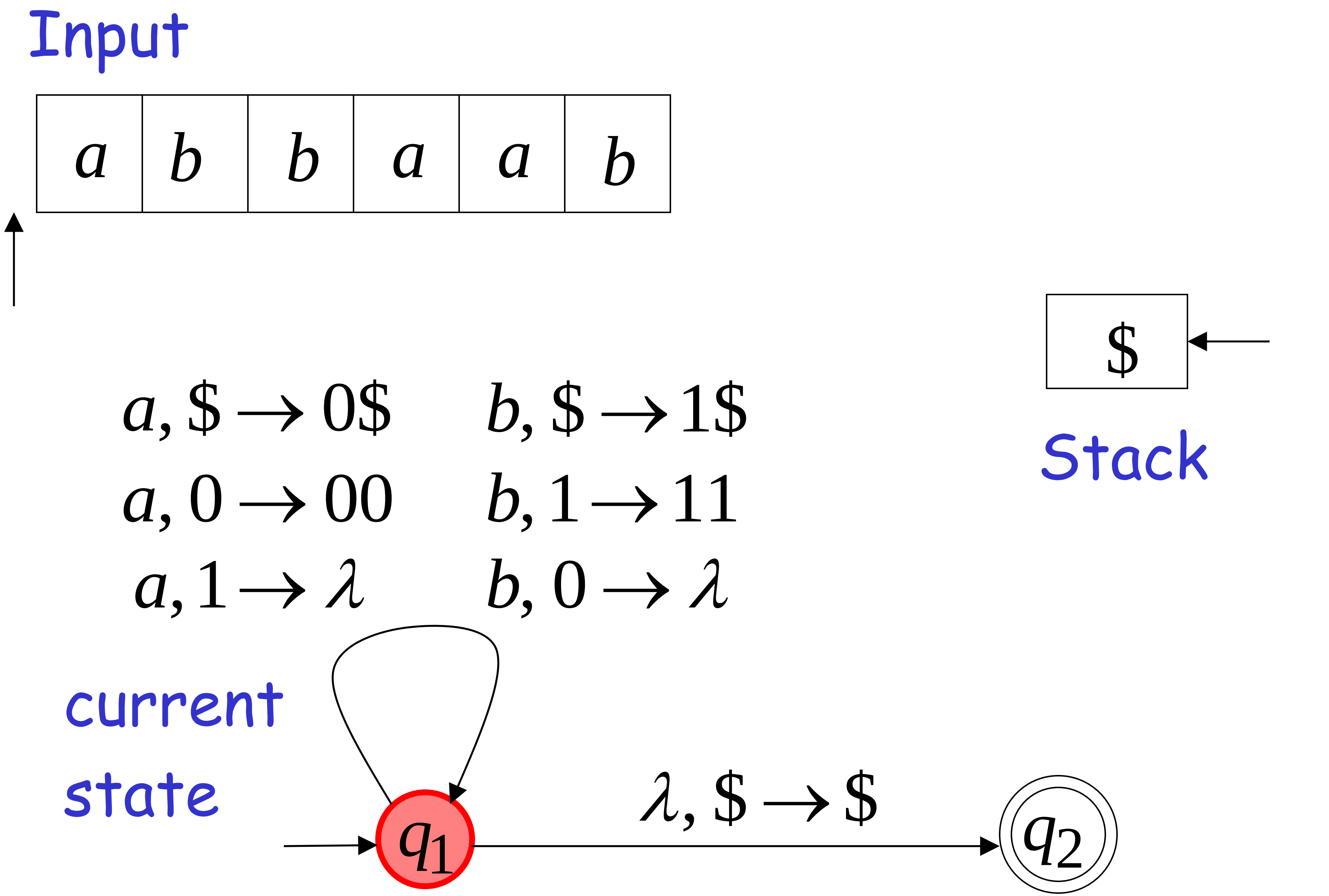
$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$



Execution Example: Time 0



Time 1

Input

a	b	b	a	a	b

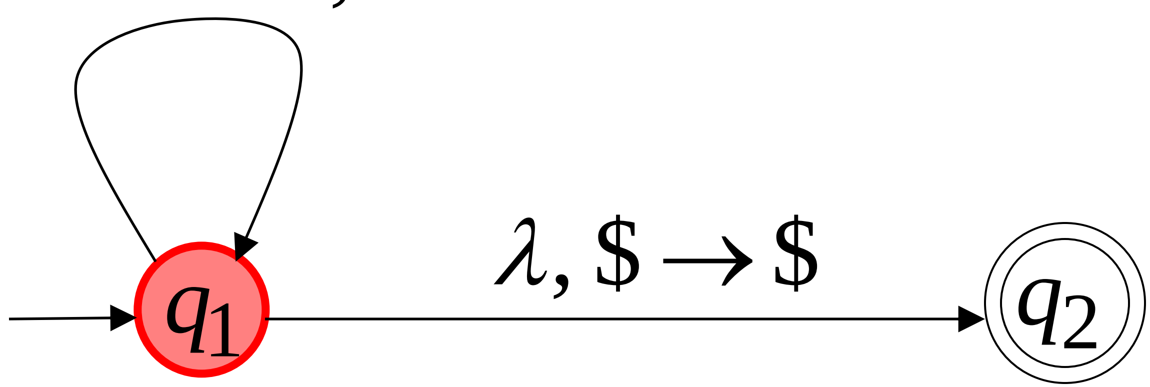
$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$

0
\$

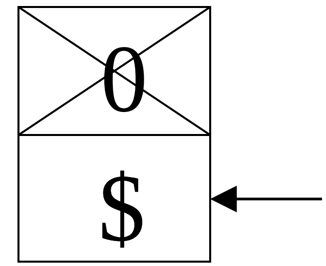
Stack



Time 3

Input

a	b	b	b	a	a

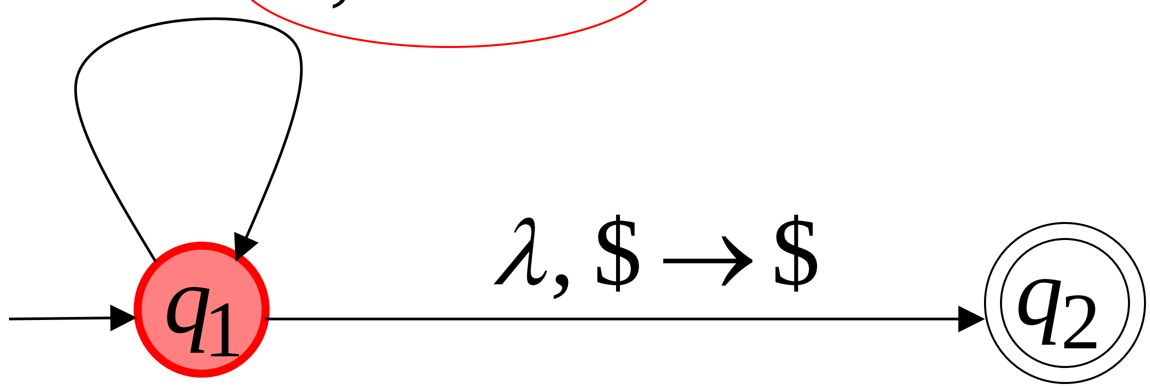


Stack

$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

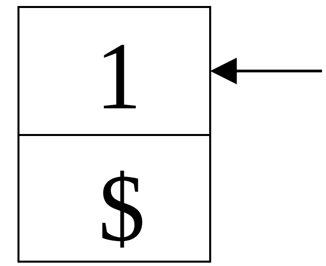
$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



Time 4

Input

a	b	b	b	a	a

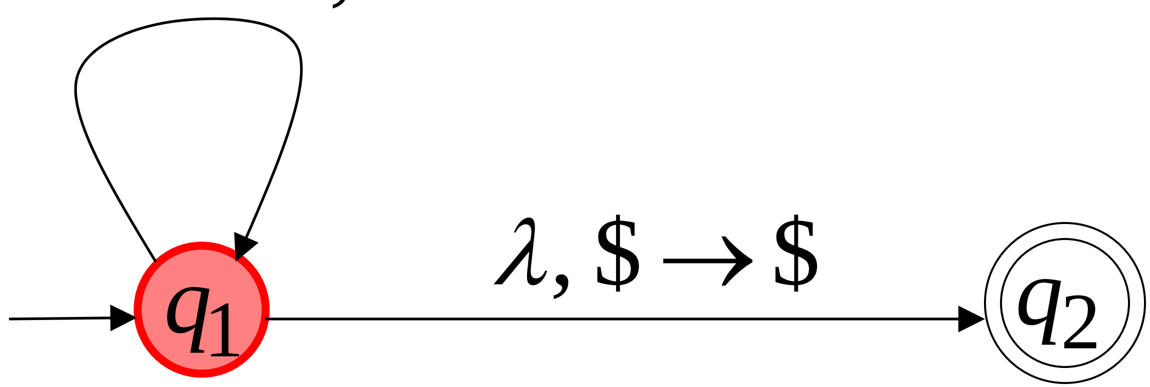


Stack

$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$



Time 5

Input

a	b	b	b	a	a

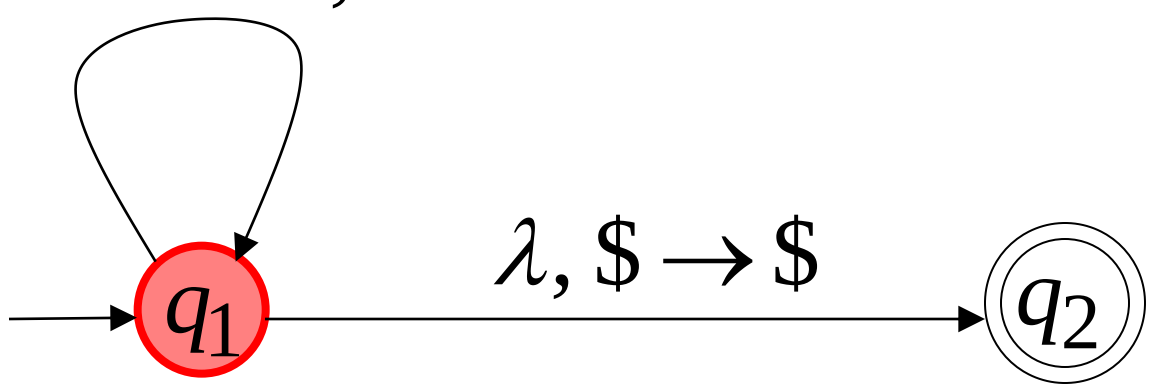
1
1
$

Stack

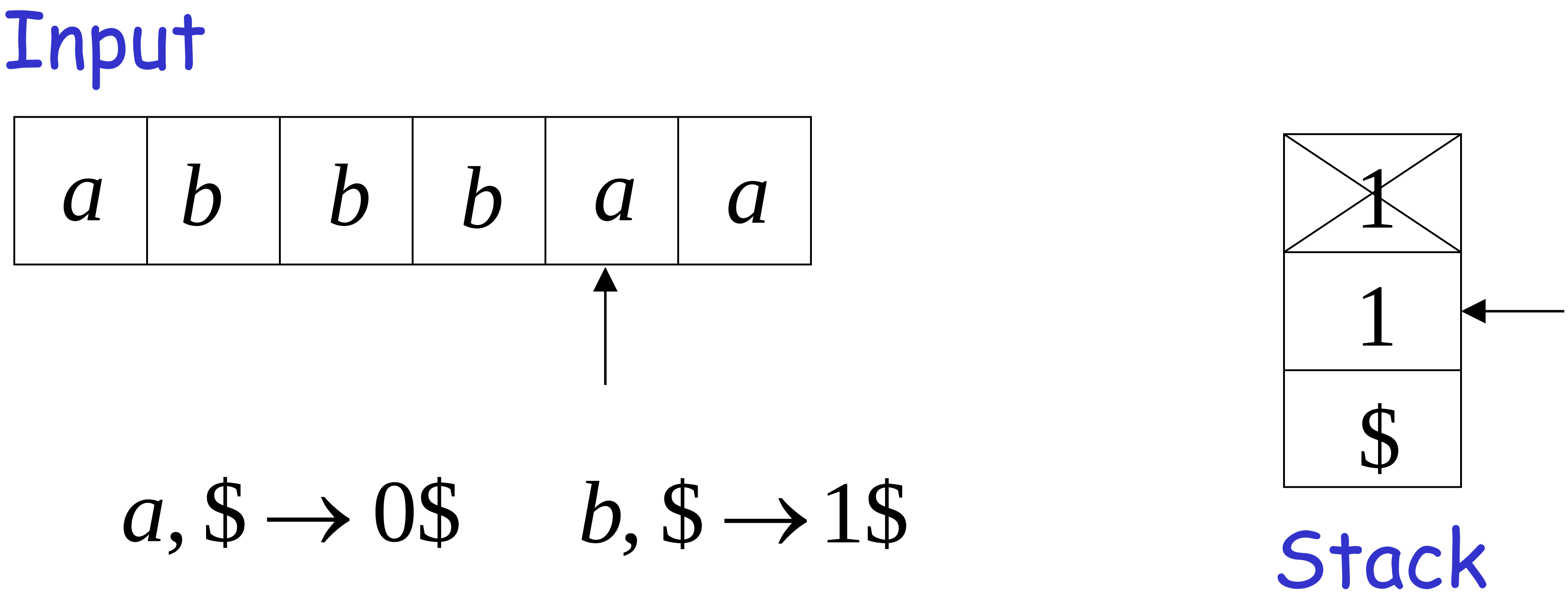
$a, \$ \rightarrow 0\$$ $\quad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\quad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\quad$ $b, 0 \rightarrow \lambda$



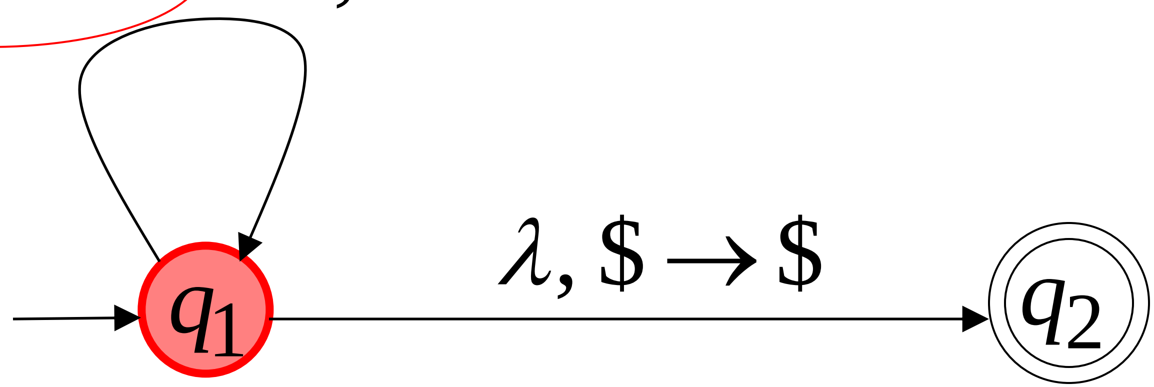
Time 6



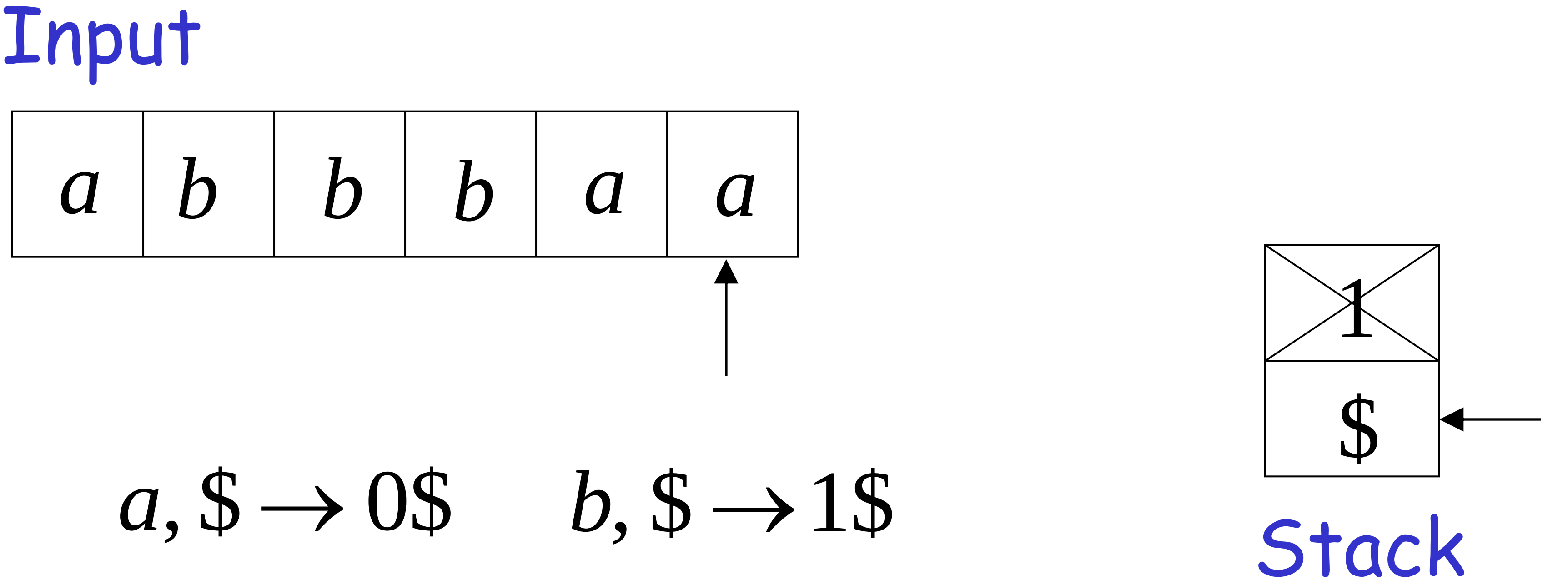
$$a, \$ \rightarrow 0\$ \qquad b, \$ \rightarrow 1\$$$

$$a, 0 \rightarrow 00 \qquad b, 1 \rightarrow 11$$

$$a, 1 \rightarrow \lambda \qquad b, 0 \rightarrow \lambda$$



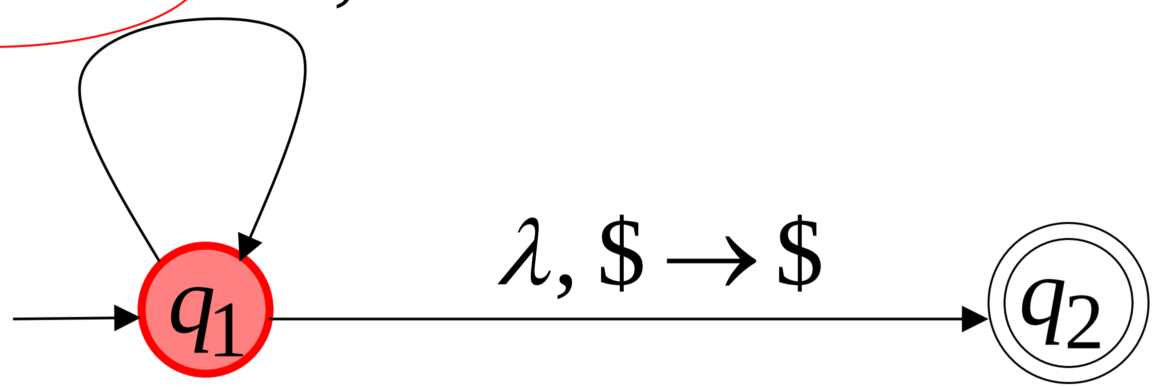
Time 7



$a, \$ \rightarrow 0\$$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $b, 0 \rightarrow \lambda$



Time 8

Input

a	b	b	b	a	a

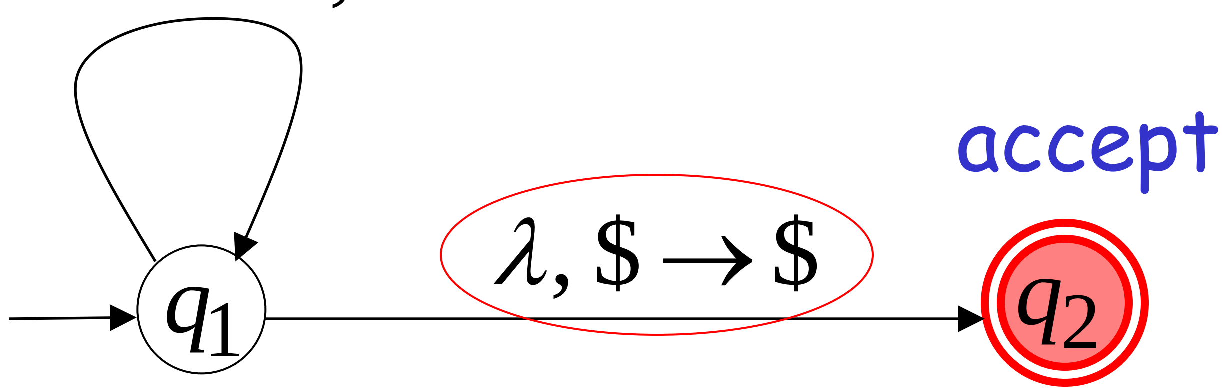
$\$$

Stack

$a, \$ \rightarrow 0\$$ $\qquad$ $b, \$ \rightarrow 1\$$

$a, 0 \rightarrow 00$ $\qquad$ $b, 1 \rightarrow 11$

$a, 1 \rightarrow \lambda$ $\qquad$ $b, 0 \rightarrow \lambda$



Formal Definition

Non-Deterministic Pushdown Automaton

NPDA

$$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$$

States

Input alphabet

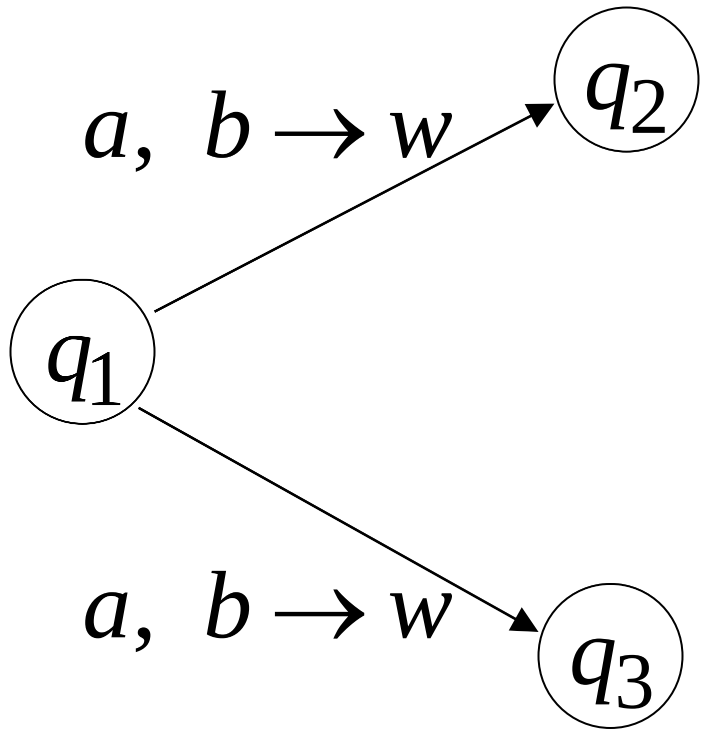
Stack alphabet

Transition function

Initial state

Stack start symbol

Final states



Transition function:

$$\delta(q_1, a, b) = \{(q_2, w), \ (q_3, w)\}$$

Instantaneous Description

$$(q, \ u, \ s)$$

Current state

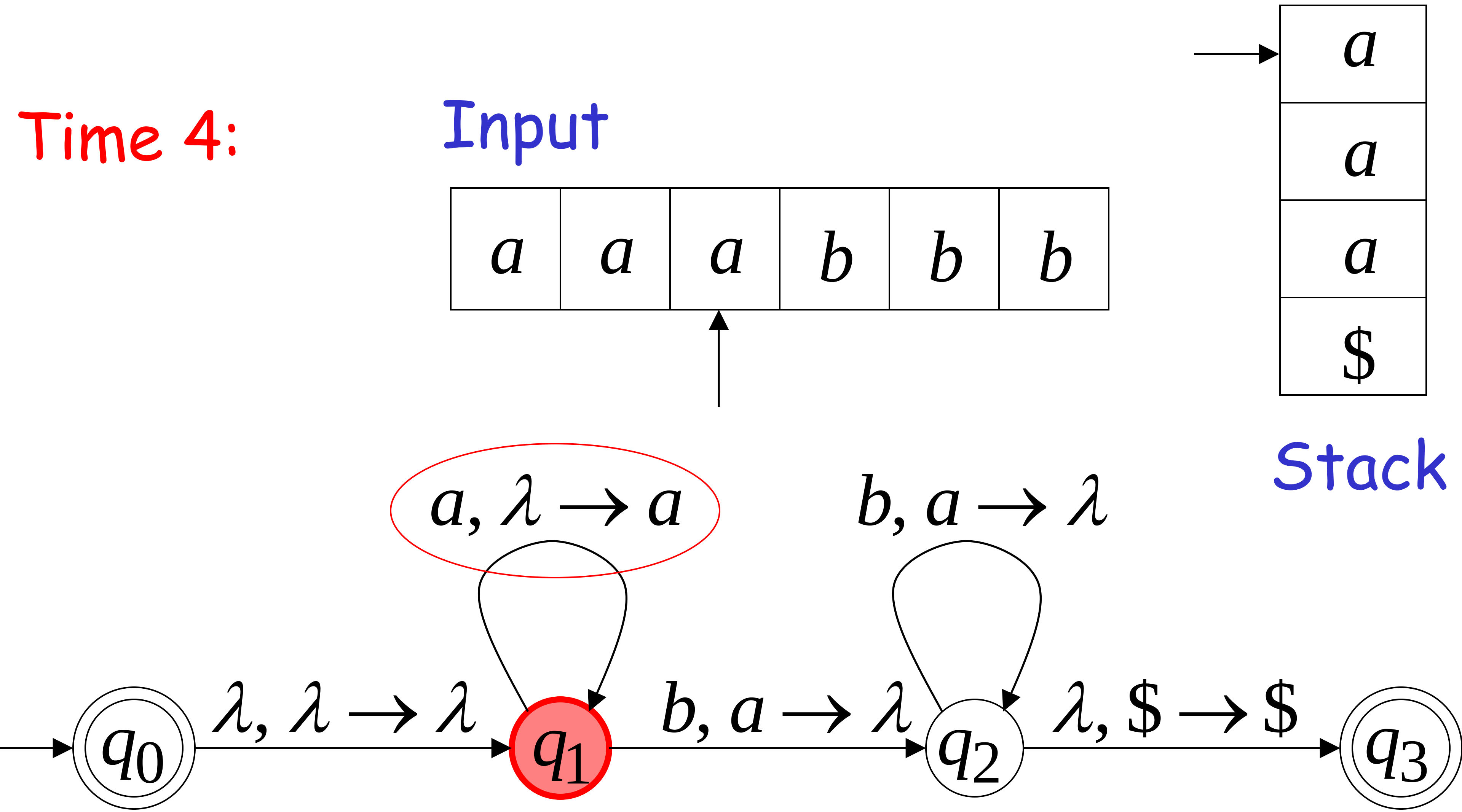
Remaining input

Current stack contents

Example: Instantaneous Description

$$(q_1, bbb, aaa\$)$$

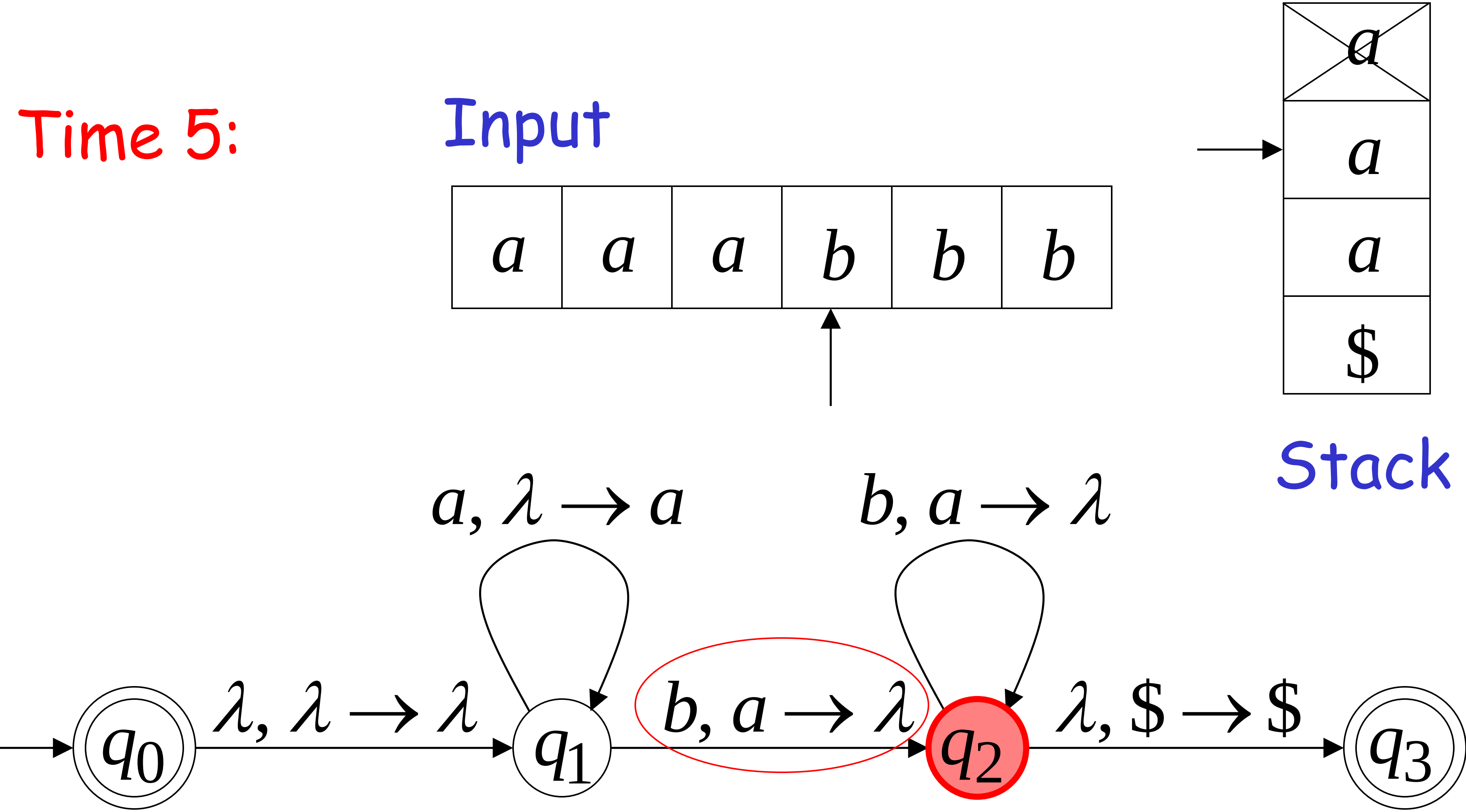
Time 4:



Example: **Instantaneous Description**

$$(q_2, bb, aa\$)$$

Time 5:



We write:

$$(q_1, bbb, aaa\$) \succ (q_2, bb, aa\$)$$

Time 4

Time 5

$$(q_0, aaabbb, \$) \succ (q_1, aaabbb, \$) \succ$$

$$(q_1, aabbb, a\$) \succ (q_1, abbb, aa\$) \succ (q_1, bbb, aaa\$) \succ$$

$$(q_2, bb, aa\$) \succ (q_2, b, a\$) \succ (q_2, \lambda, \$) \succ (q_3, \lambda, \$)$$

For convenience we write:

$$(q_0, aaabbb, \$) \overset{*}{\succ} (q_3, \lambda, \$)$$

Formal Definition

Language $L(M)$ of NPDA M:

$$L(M) = \{w : \ (q_0, w, s) \overset{*}{\succ} (q_f, \lambda, s')\}$$

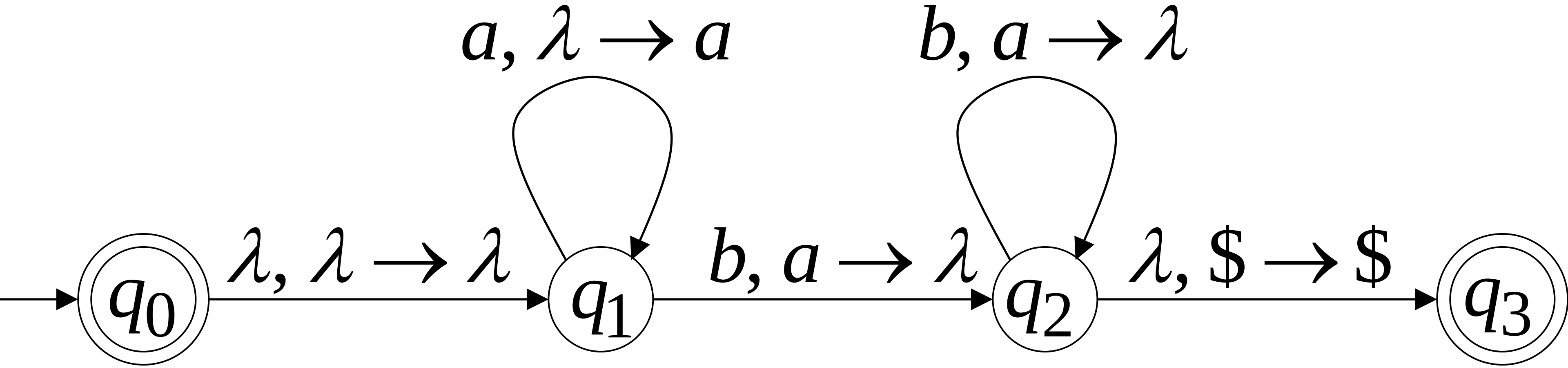
Initial state

Final state

Example: $L(M) = \{a^n b^n : n \geq 0\}$

Since, $(q_0, a^n b^n, \$) \overset{*}{\succ} (q_3, \lambda, \$)$

NPDA M :



NPDAs Accept Context-Free Languages

Theorem:

$$\left\{\begin{array}{c}\text{Context-Free}\\\text{Languages}\\\text{(Grammars)}\end{array}\right\} = \left\{\begin{array}{c}\text{Languages}\\\text{Accepted by}\\\text{NPDAs}\end{array}\right\}$$

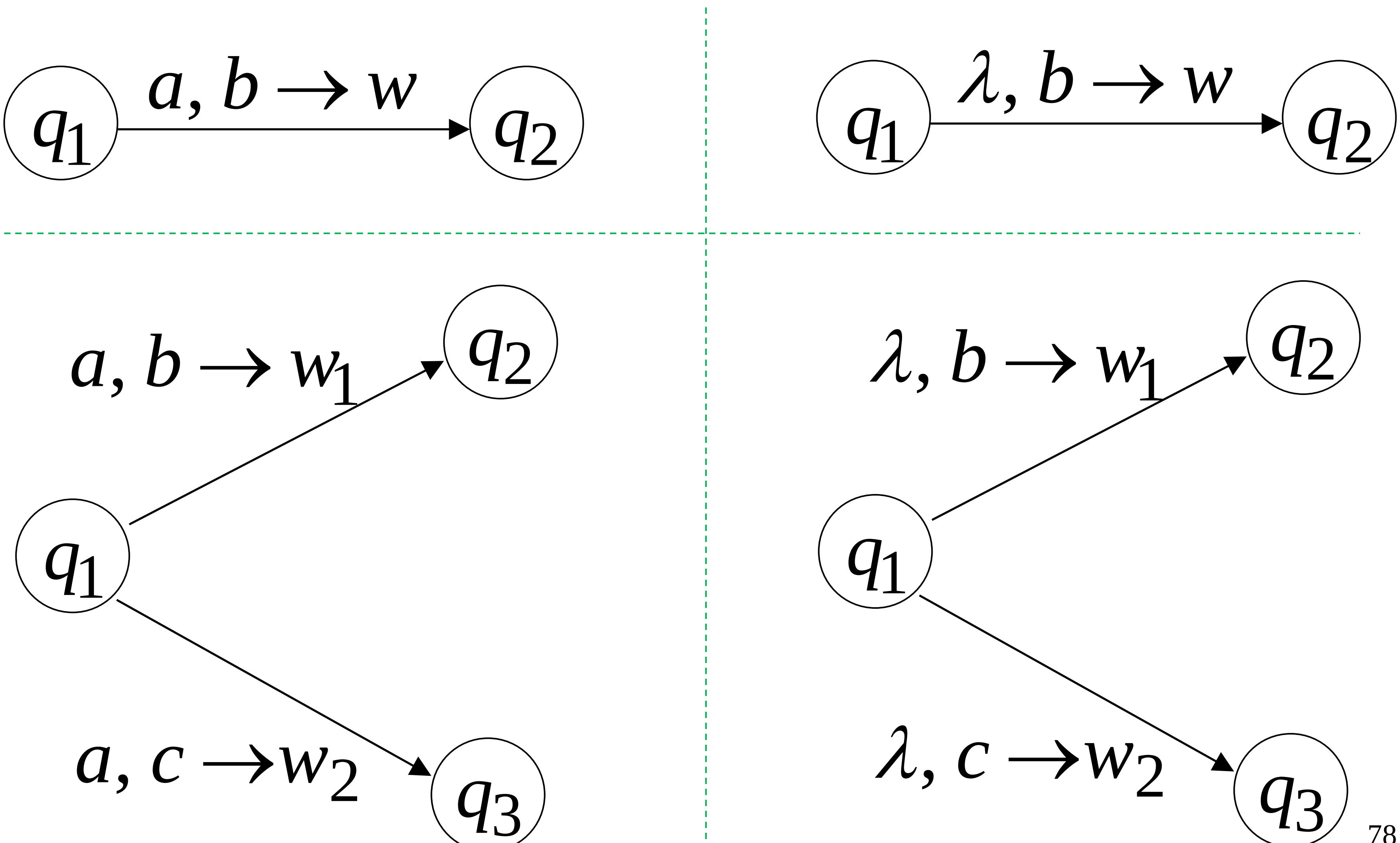
See proof in the text book

Deterministic PDA

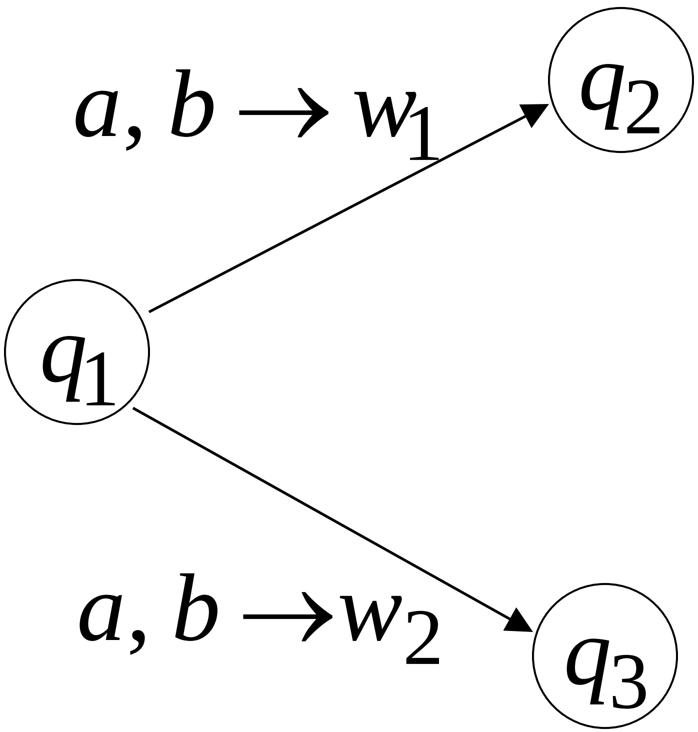
DPDA

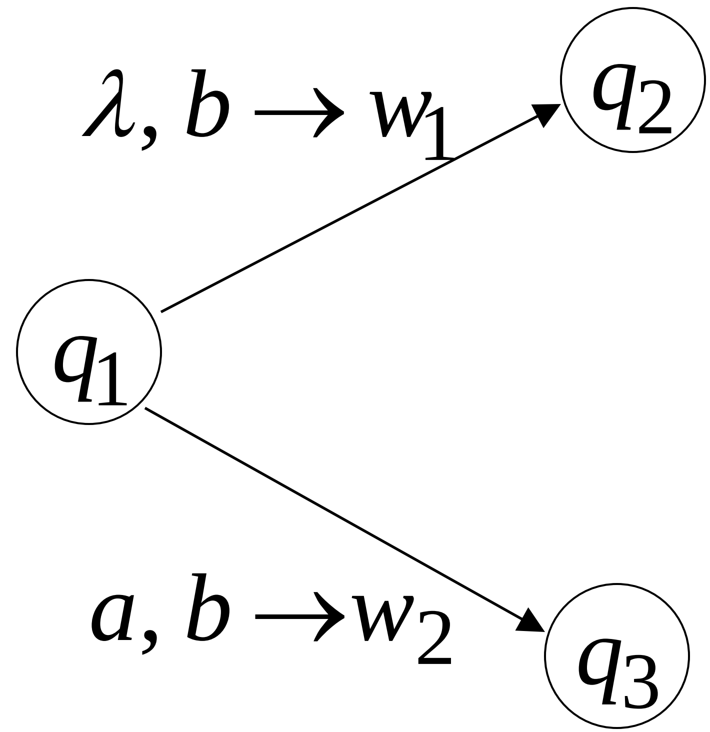
Deterministic PDA: DPDA

Allowed transitions:



Not allowed:





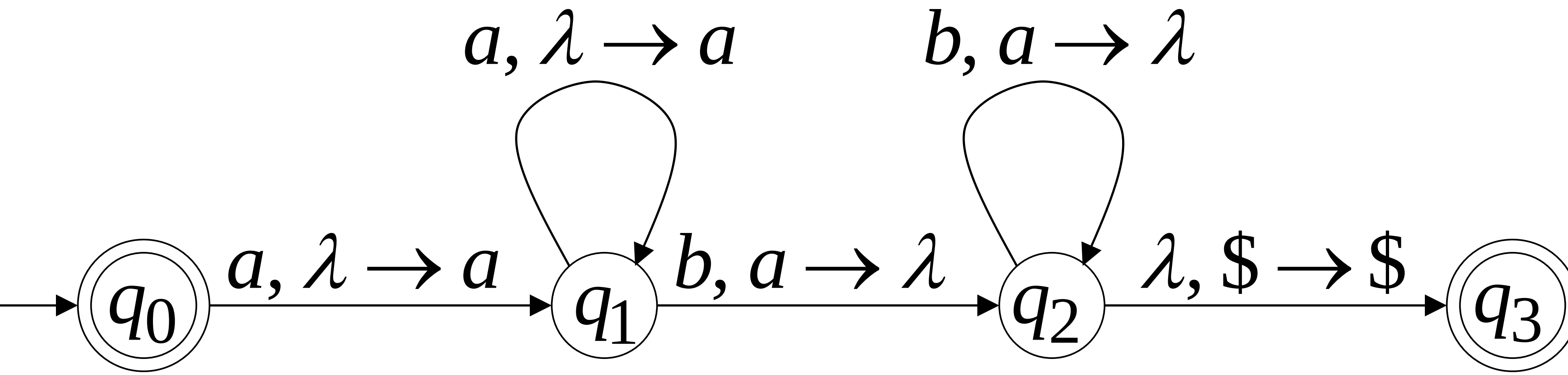
(non deterministic choices)

Definition:

A language L is **deterministic context-free** if there exists some DPDA that accepts it

DPDA example

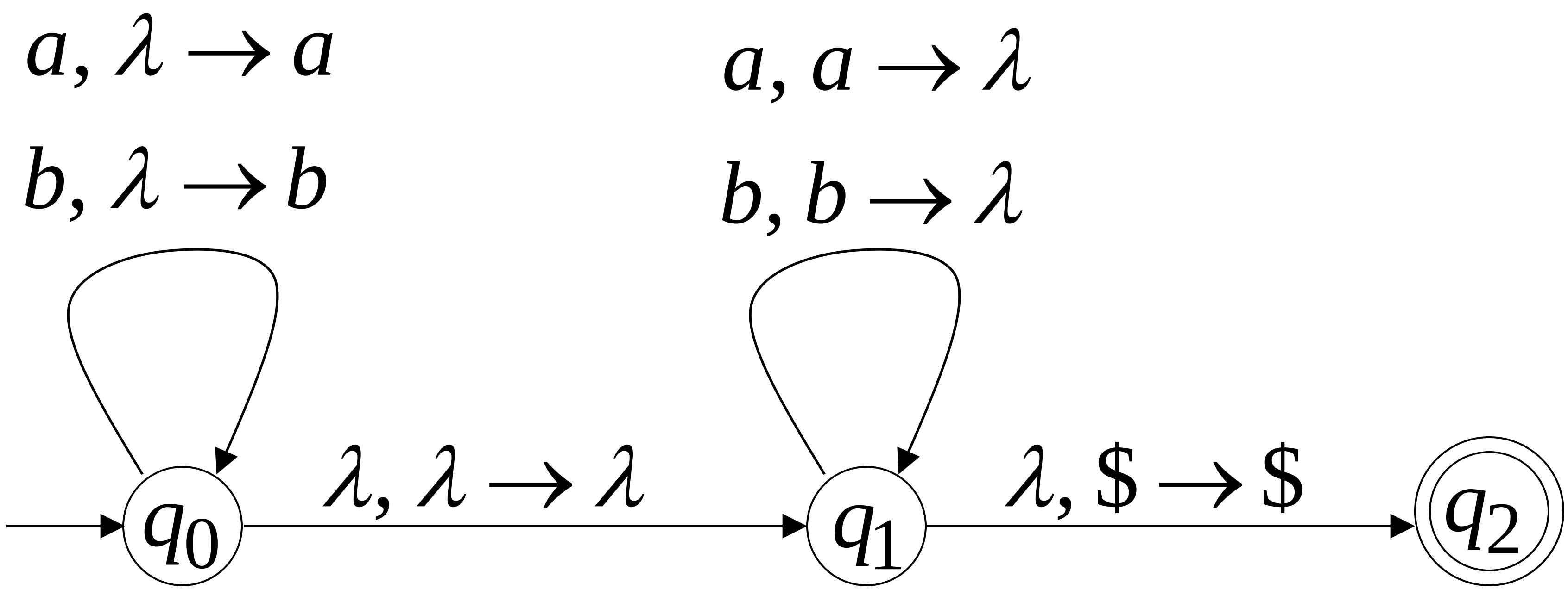
$$L(M) = \{a^n b^n : n \geq 0\}$$



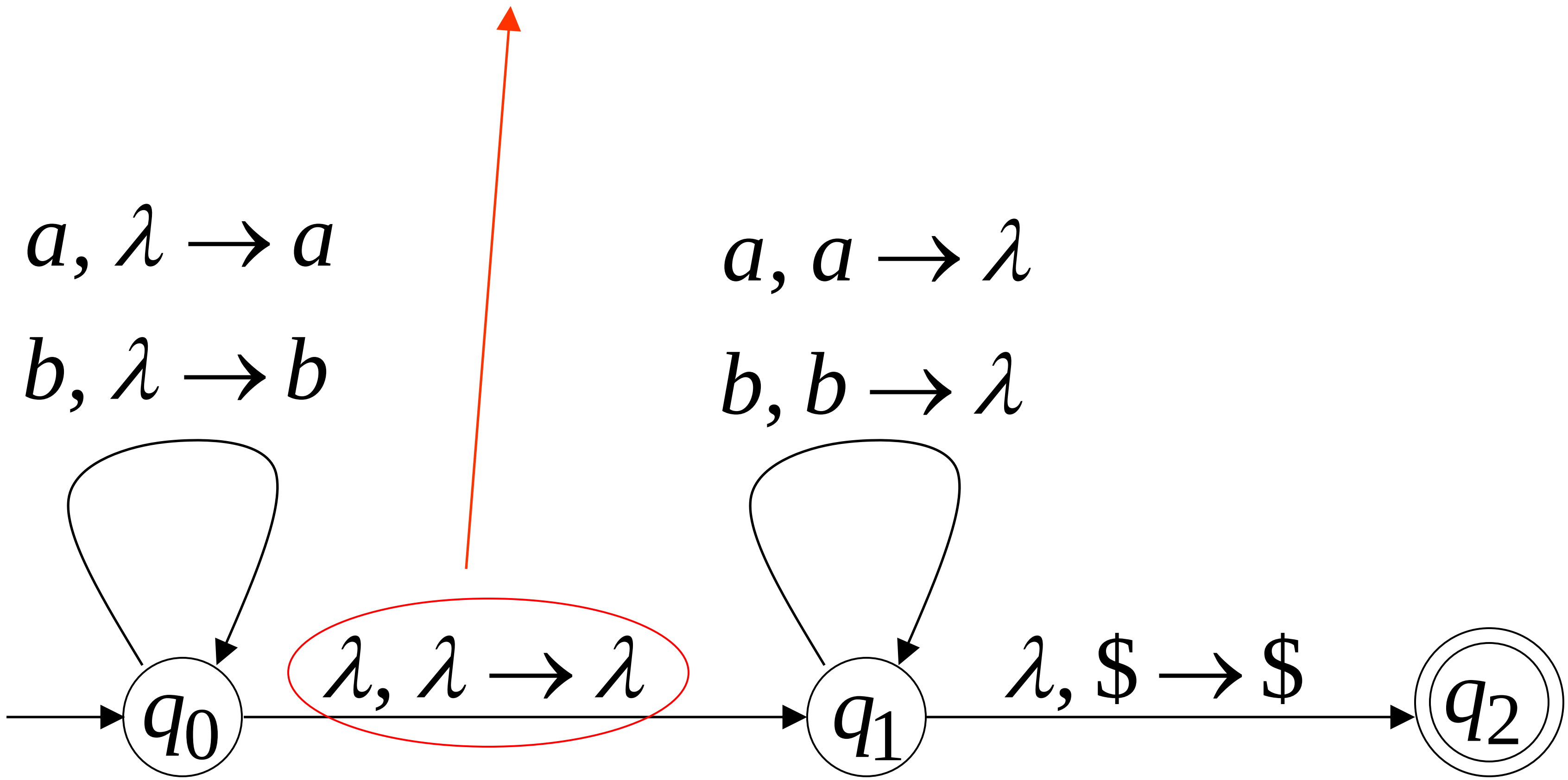
This language is **deterministic context-free**

Example of Non-DPDA (NPDA)

$$L(M) = \{ww^R\}$$



Not allowed in DPDAs



NPDAs

Have More Power than

DPDAs

We will show that:

$$\left\{\begin{array}{c}\text{Deterministic}\\\text{Context-Free}\\\text{Languages}\\\text{(DPDA)}\end{array}\right\} \subset \left\{\begin{array}{c}\text{Context-Free}\\\text{Languages}\\\text{(NPDA)}\end{array}\right\}$$

$L \notin$ $\qquad$ $L \in$

We will show that there exists a context-free language L which is not accepted by any DPDA

The language is:

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\} \qquad n \geq 0$$

We will show:

- L is context-free
- L is **not** deterministic context-free

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

Language L is context-free

Context-free grammar for L :

$$S \rightarrow S_1 \mid S_2 \qquad \{a^n b^n\} \cup \{a^n b^{2n}\}$$

$$S_1 \rightarrow aS_1b \mid \lambda \qquad \{a^n b^n\}$$

$$S_2 \rightarrow aS_2bb \mid \lambda \qquad \{a^n b^{2n}\}$$

Theorem:

The language $L = \{a^n b^n\} \cup \{a^n b^{2n}\}$

is **not** deterministic context-free

(there is **no** DPDA that accepts L)

Proof: Assume for contradiction that

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

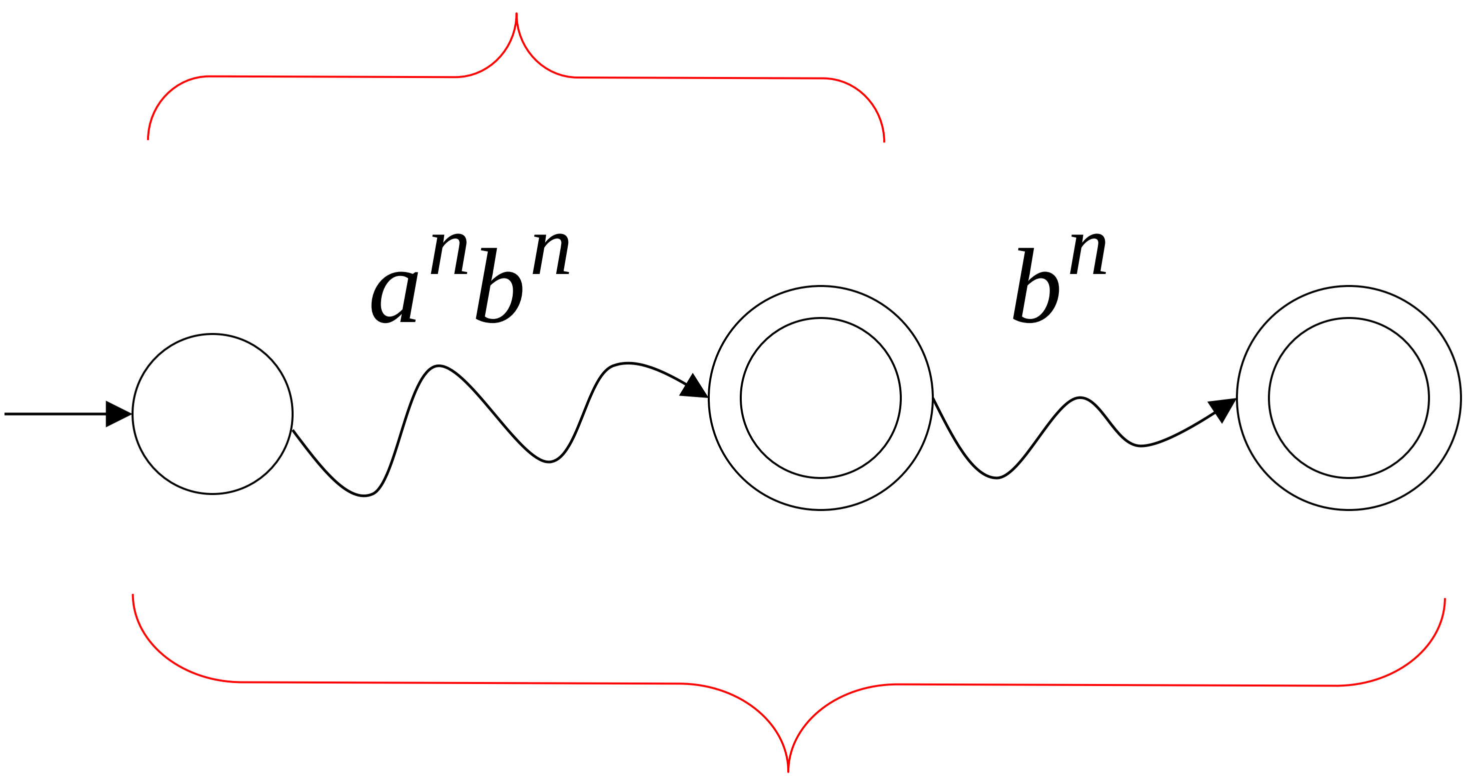
is deterministic context free

Therefore:

there is a DPDA M that accepts L

DPDA M with $L(M) = \{a^n b^n\} \cup \{a^n b^{2n}\}$

accepts $a^n b^n$

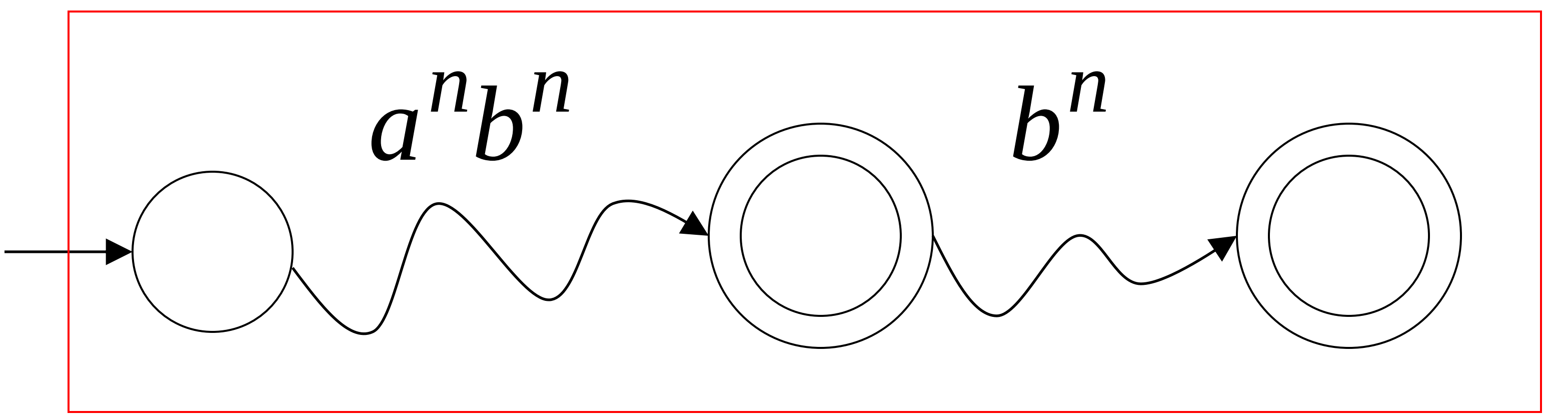


accepts $a^n b^{2n}$

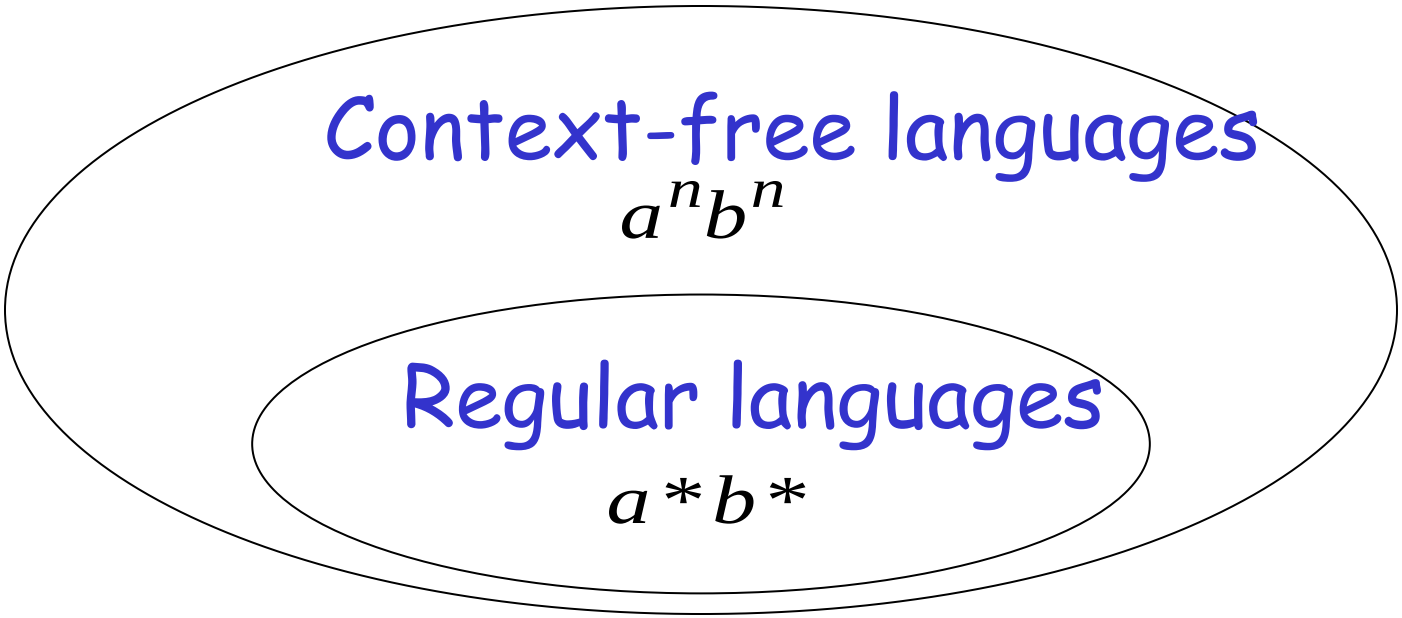
DPDA M with $L(M)=\{a^n b^n\} \cup \{a^n b^{2n}\}$

Such a path exists because of the determinism

M



Fact 1: The language $\{a^n b^n c^n\}$ is **not** context-free



(we will prove this at a later class using pumping lemma for context-free languages)

Fact 2: The language $L \cup \{a^n b^n c^n\}$

is **not** context-free

$$(L = \{a^n b^n\} \cup \{a^n b^{2n}\})$$

(we can prove this using pumping lemma for context-free languages)

We will construct a NPDA that accepts:

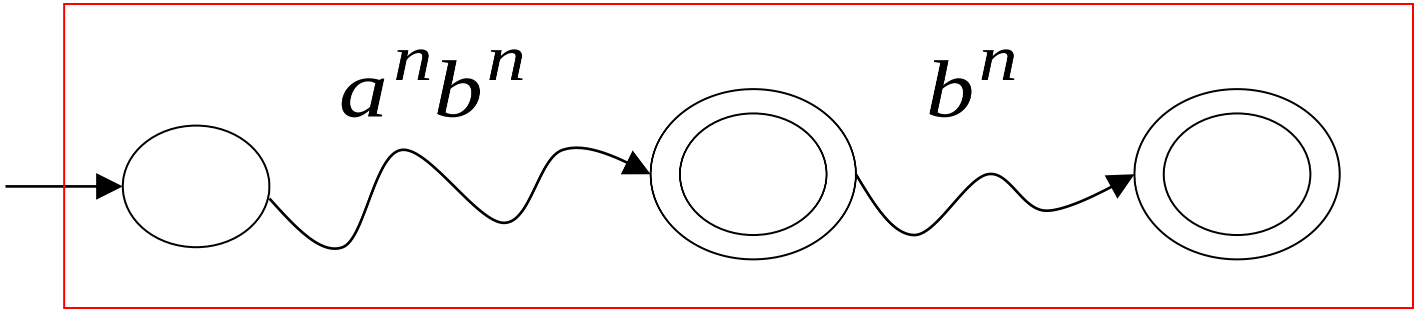
$$L \cup \{a^n b^n c^n\}$$

$$(L = \{a^n b^n\} \cup \{a^n b^{2n}\})$$

which is a contradiction!

M

$$L(M) = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

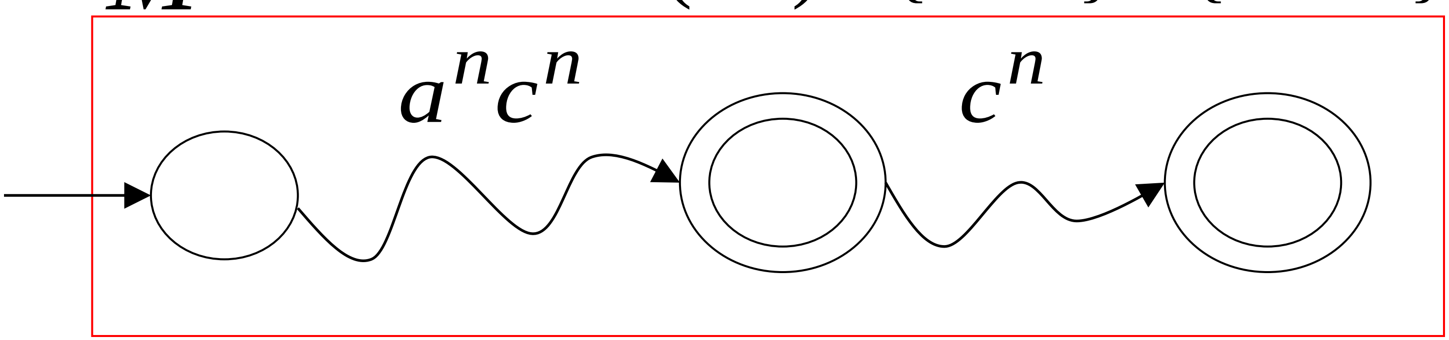


Modify M

Replace b with c

M'

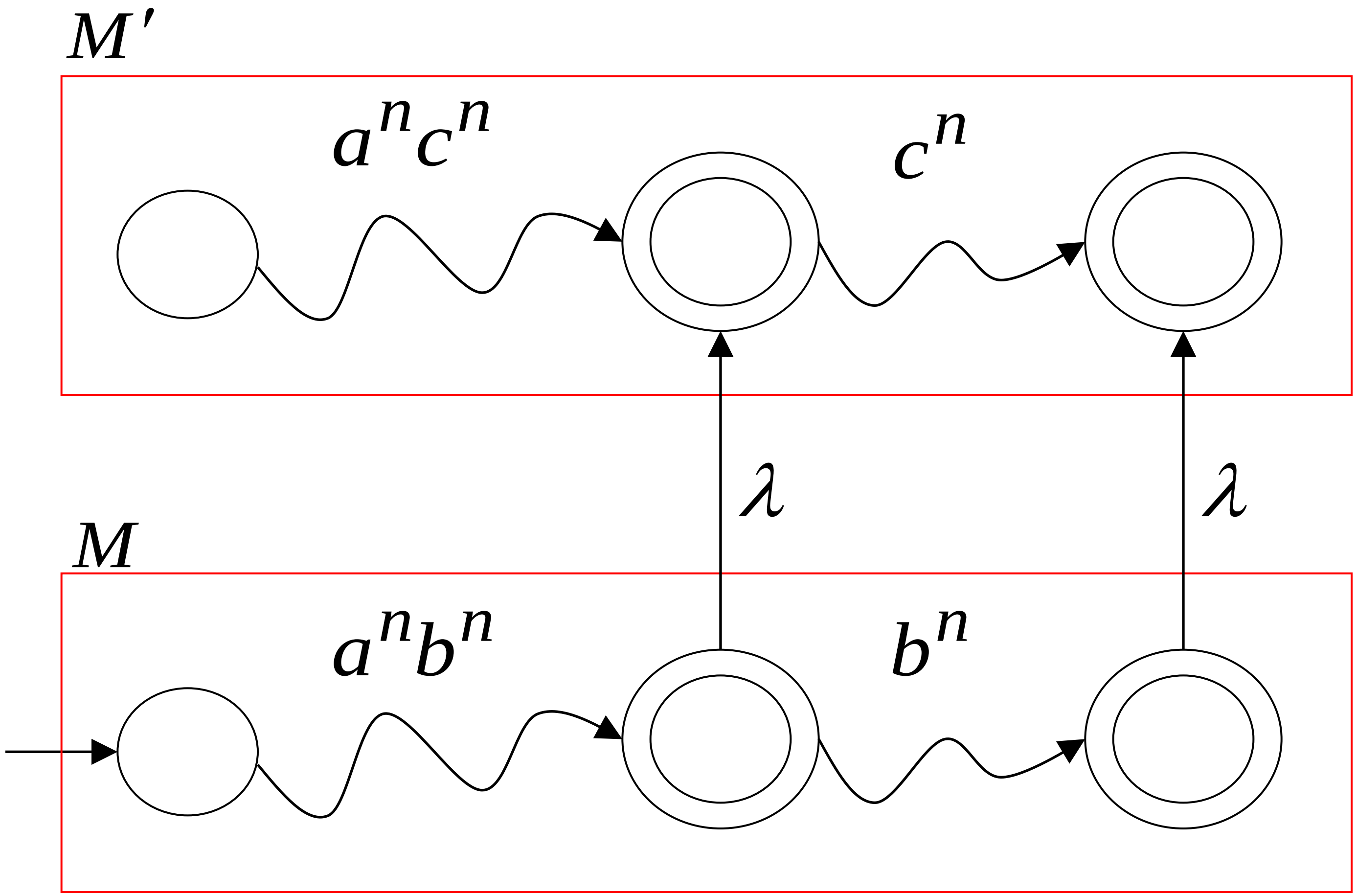
$$L(M') = \{a^n c^n\} \cup \{a^n c^{2n}\}$$



The NPDA that accepts $\{a^n b^n\} \cup \{a^n b^{2n}\} \cup \{a^n b^n c^n\}$

Connect final states of M'

with final states of M



Since $L \cup \{a^n b^n c^n\}$ is accepted by a NPDA

it is context-free

Contradiction!

(since $L \cup \{a^n b^n c^n\}$ is not context-free)

Therefore:

Not deterministic context free

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

There is **no** DPDA that accepts

End of Proof