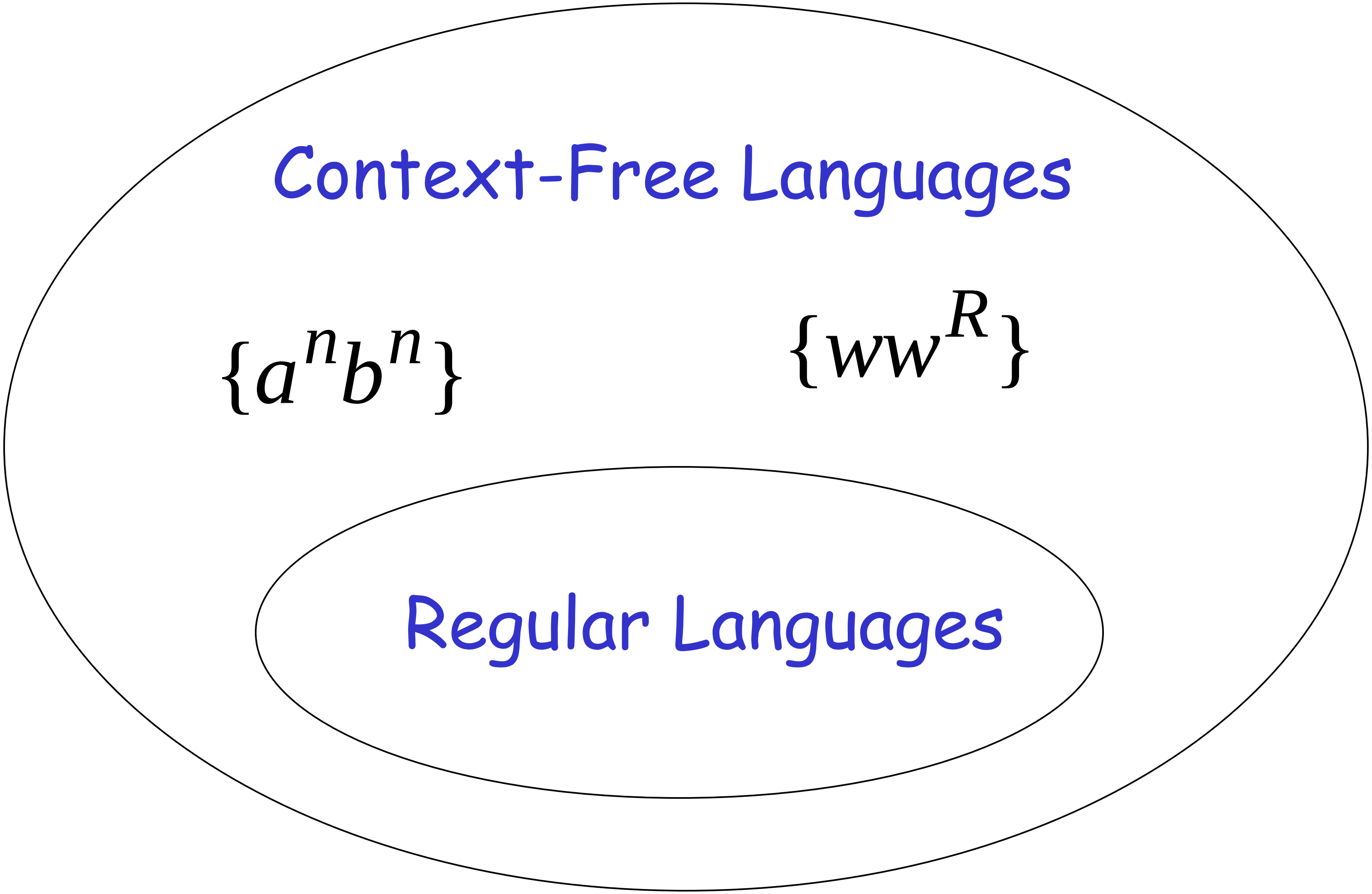
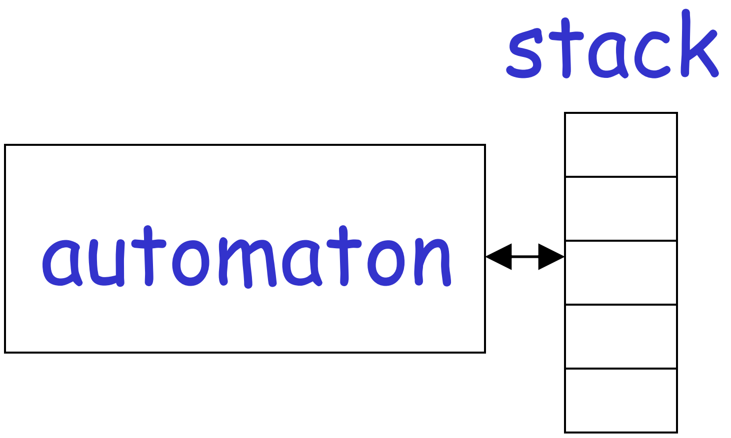
Context-Free Languages


Context-Free Languages
$\{a^n b^n\}$
$\{ww^R\}$
Regular Languages

Context-Free Languages

Context-Free Grammars

Pushdown Automata



Context-Free Grammars

Example

A context-free grammar G:

$$S \to aSb$$

$$S \to \lambda$$

Derivations:

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \Rightarrow aaabbb$$

$$S \to aSb$$

$$S \to \lambda$$

$$L(G) = \{a^n b^n : n \geq 0\}$$

Describes parentheses: (((( ))))

Example

A context-free grammar G:

$$S \rightarrow aSa$$

$$S \rightarrow bSb$$

$$S \rightarrow \lambda$$

Derivations:

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abba$$

$$S \Rightarrow aSa \Rightarrow abSba \Rightarrow abaSaba \Rightarrow abaaba$$

$$S \to aSa$$

$$S \to bSb$$

$$S \to \lambda$$

$$L(G) = \{ww^R : \quad w \in \{a,b\}^*\}$$

Example

A context-free grammar G:

$$S \to aSb$$

$$S \to SS$$

$$S \to \lambda$$

Derivations:

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow ab$$

$$S \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSb \Rightarrow abab$$

$$S \to aSb$$

$$S \to SS$$

$$S \to \lambda$$

$$L(G) = \{w \ : n_a(w) = n_b(w),$$

$$\text{and } n_a(v) \geq n_b(v)$$

$$\text{in any prefix } v\}$$

Describes matched parentheses: () ((( ))) (( ))

Definition: Context-Free Grammars

Grammar $G=(V,T,S,P)$

Variables

Terminal symbols

Start variable

Productions of the form:

$$A\rightarrow x$$

Variable

String of variables and terminals

Definition: Context-Free Languages

A language L is context-free if and only if

there is a context-free grammar G
with $L = L(G)$

where $G = (V, T, S, P)$ and

$$L(G) = \{w: \quad S \overset{*}{\Longrightarrow} w, \quad w \in T^*\}$$

Derivation Order

$1.\ S \to AB$ $\quad$ $2.\ A \to aaA$ $\quad$ $4.\ B \to Bb$

$3.\ A \to \lambda$ $\quad$ $5.\ B \to \lambda$

Leftmost derivation:

$$S \overset{1}{\Rightarrow} AB \overset{2}{\Rightarrow} aaAB \overset{3}{\Rightarrow} aaB \overset{4}{\Rightarrow} aaBb \overset{5}{\Rightarrow} aab$$

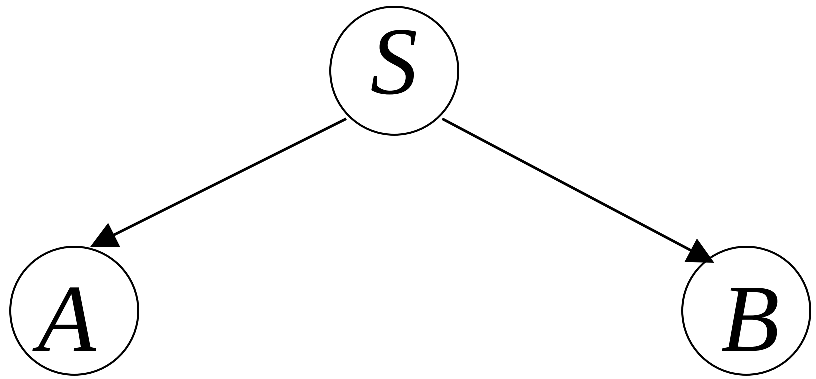
Rightmost derivation:

$$S \overset{1}{\Rightarrow} AB \overset{4}{\Rightarrow} ABb \overset{5}{\Rightarrow} Ab \overset{2}{\Rightarrow} aaAb \overset{3}{\Rightarrow} aab$$

Derivation Trees

$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

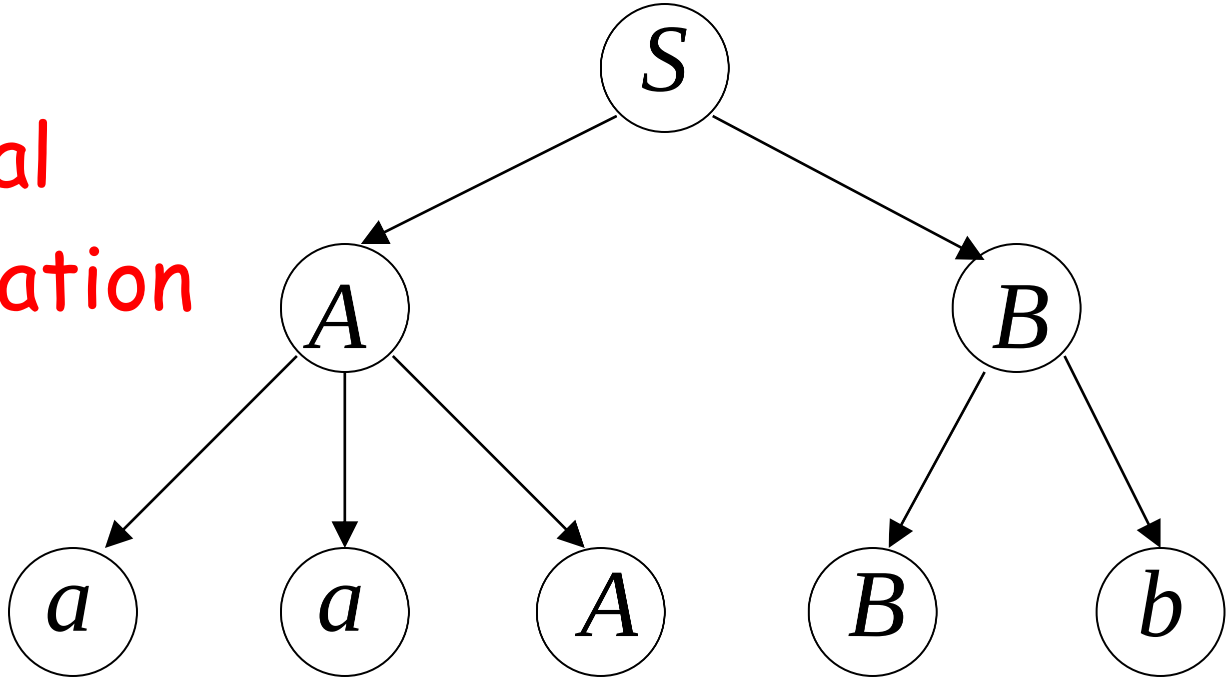
$$S \Rightarrow AB$$



$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb$$

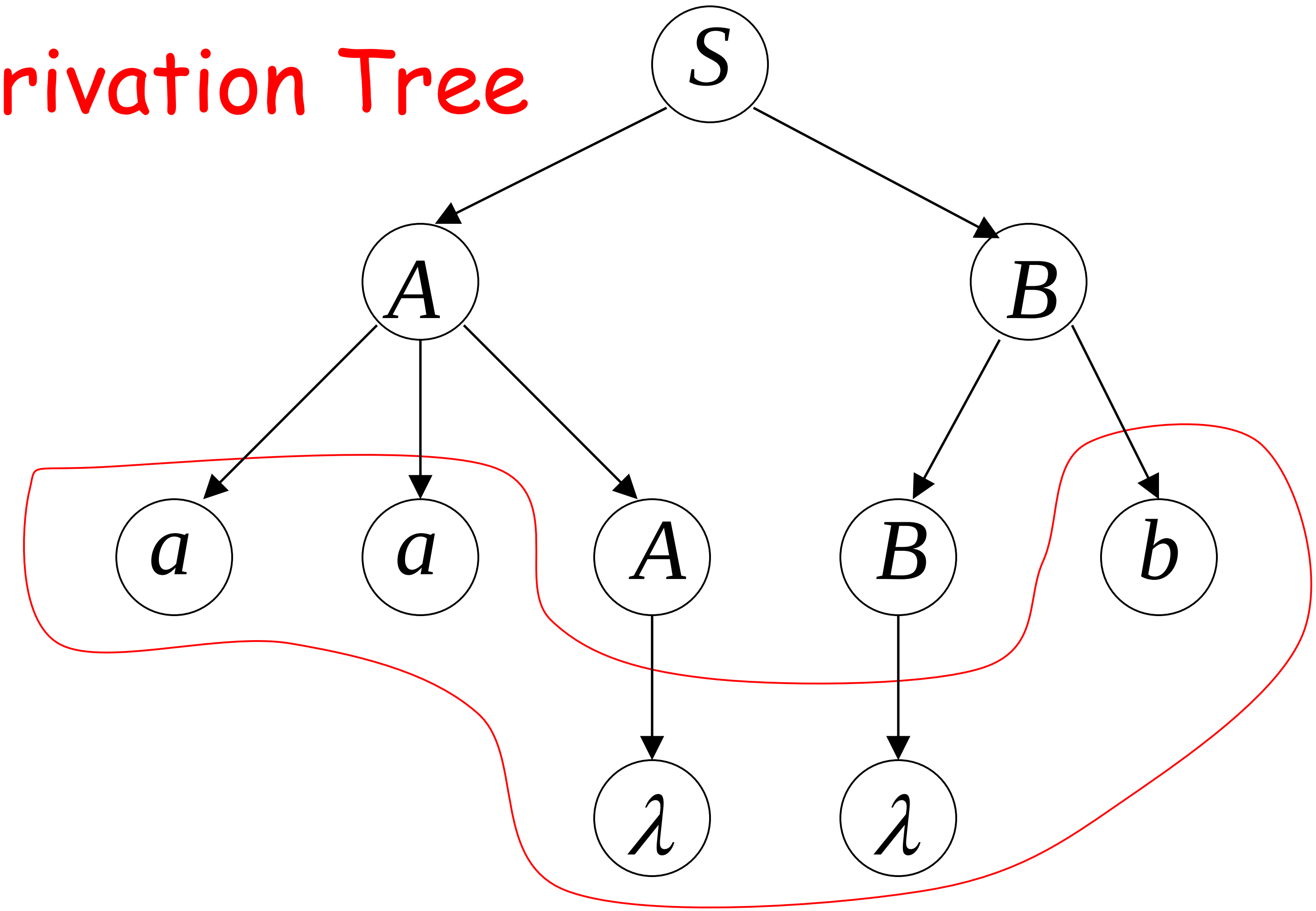
Partial derivation tree



$$S \to AB \qquad A \to aaA \mid \lambda \qquad B \to Bb \mid \lambda$$

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaABb \Rightarrow aaBb \Rightarrow aab$$

Derivation Tree



yield

$aa\lambda\lambda b$

$= aab$

Sometimes, derivation order doesn't matter

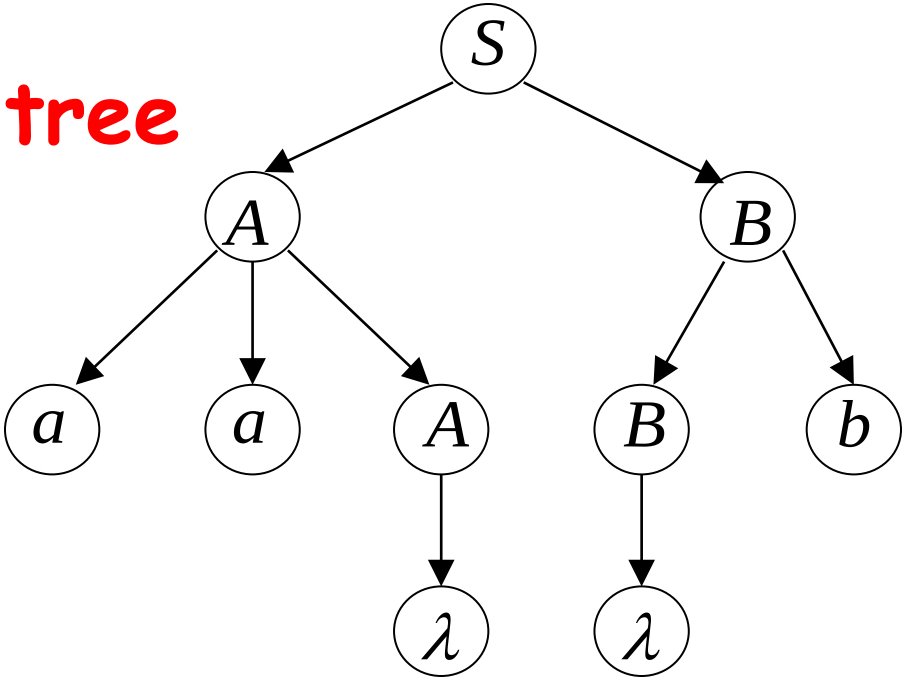
Leftmost:

$$S \Rightarrow AB \Rightarrow aaAB \Rightarrow aaB \Rightarrow aaBb \Rightarrow aab$$

Rightmost:

$$S \Rightarrow AB \Rightarrow ABb \Rightarrow Ab \Rightarrow aaAb \Rightarrow aab$$

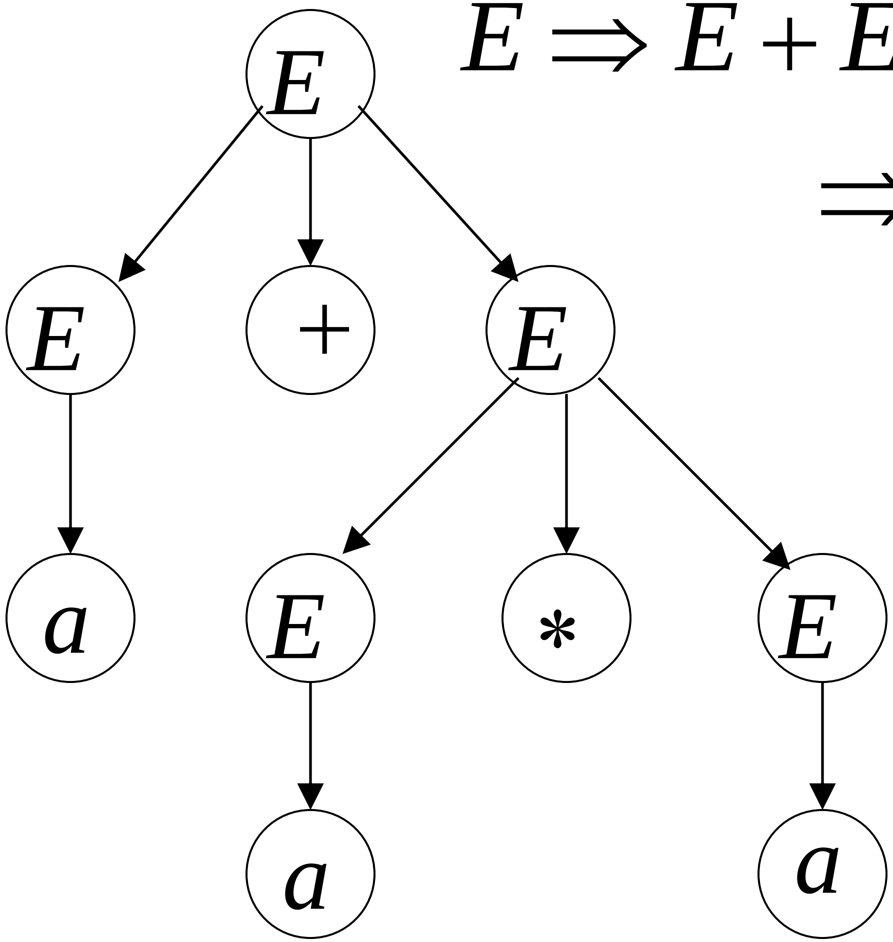
Same derivation tree



Ambiguity

$$E \to E+E \;\mid\; E*E \;\mid\; (E) \;\mid\; a$$

$$a+a*a$$



$$E \Rightarrow E+E \Rightarrow a+E \Rightarrow a+E*E$$
$$\Rightarrow a+a*E \Rightarrow a+a*a$$

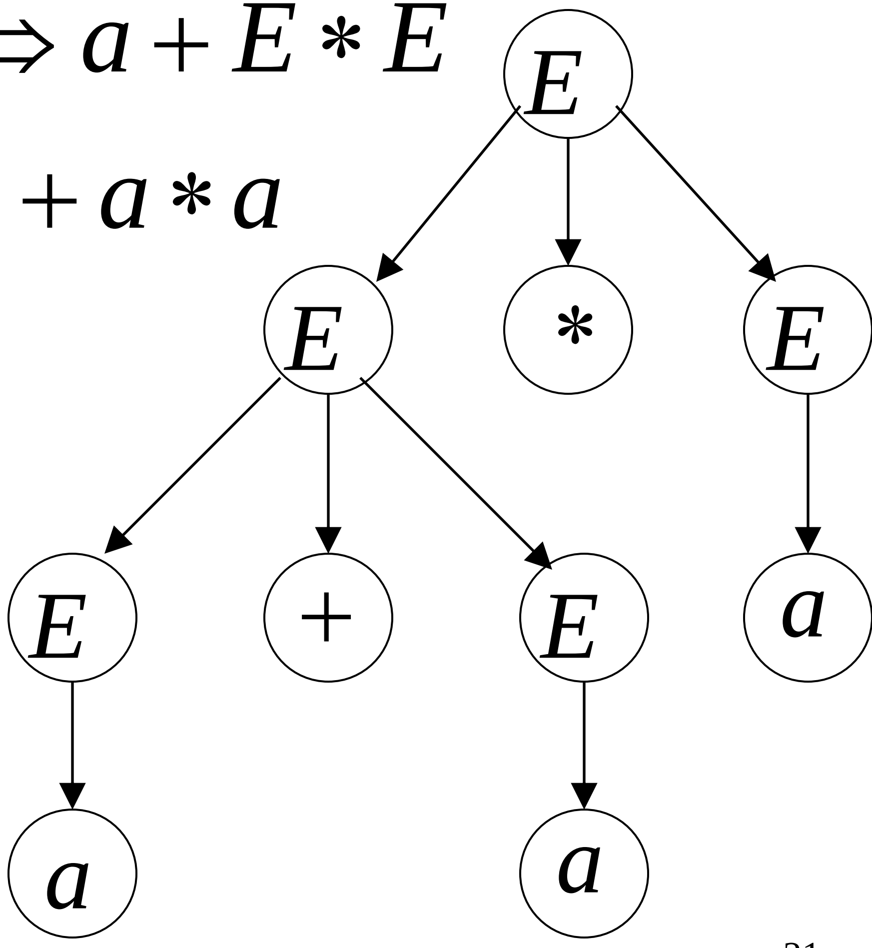
leftmost derivation

$$E \to E + E \;\mid\; E * E \;\mid\; (E) \;\mid\; a$$

$$a + a * a$$

$$E \Rightarrow E * E \Rightarrow E + E * E \Rightarrow a + E * E$$
$$\Rightarrow a + a * E \Rightarrow a + a * a$$

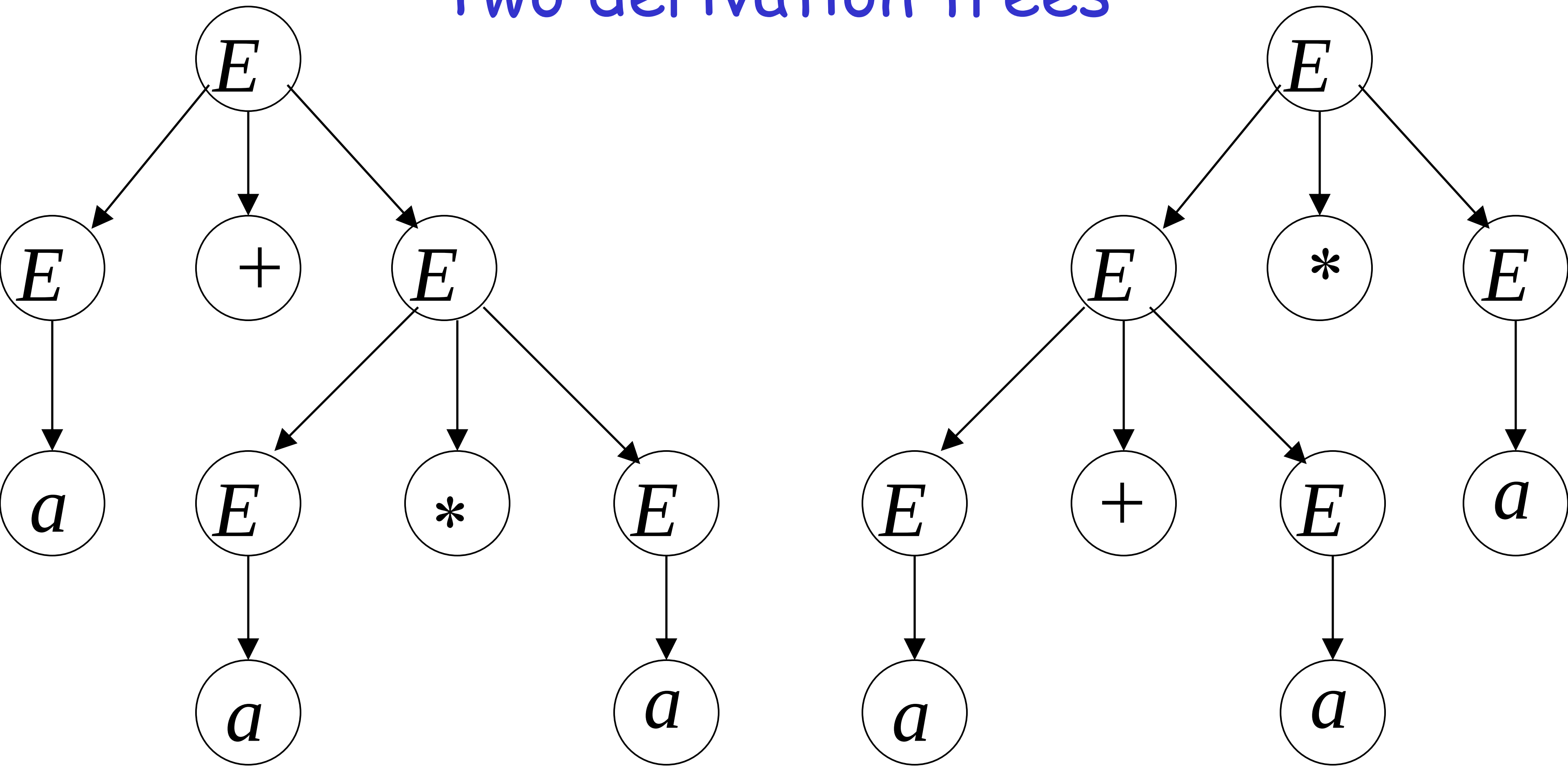
leftmost derivation



$$E \to E + E \ \mid \ E * E \ \mid \ (E) \ \mid \ a$$

$$a + a * a$$

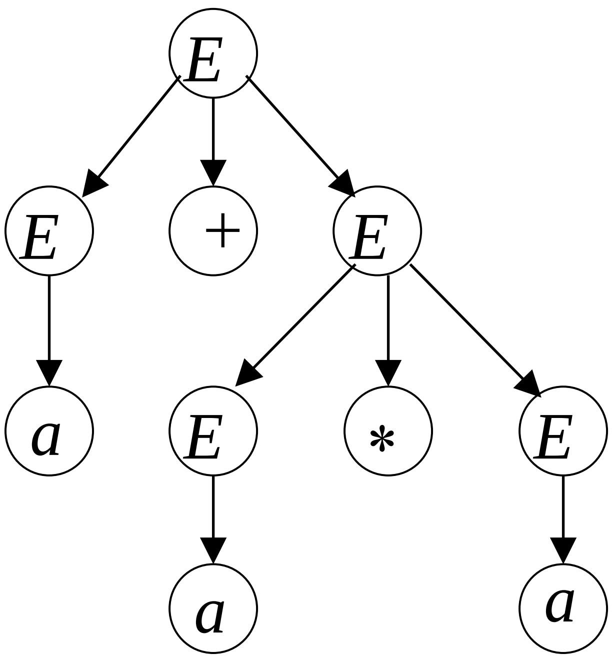
Two derivation trees

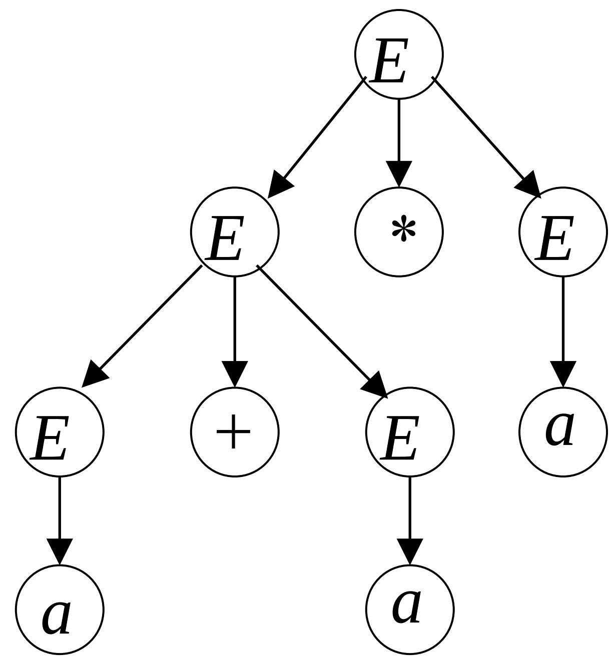


The grammar $E \to E + E \;\mid\; E * E \;\mid\; (E) \;\mid\; a$

is ambiguous:

string $a + a * a$ has two derivation trees





Definition:

A context-free grammar G is **ambiguous**

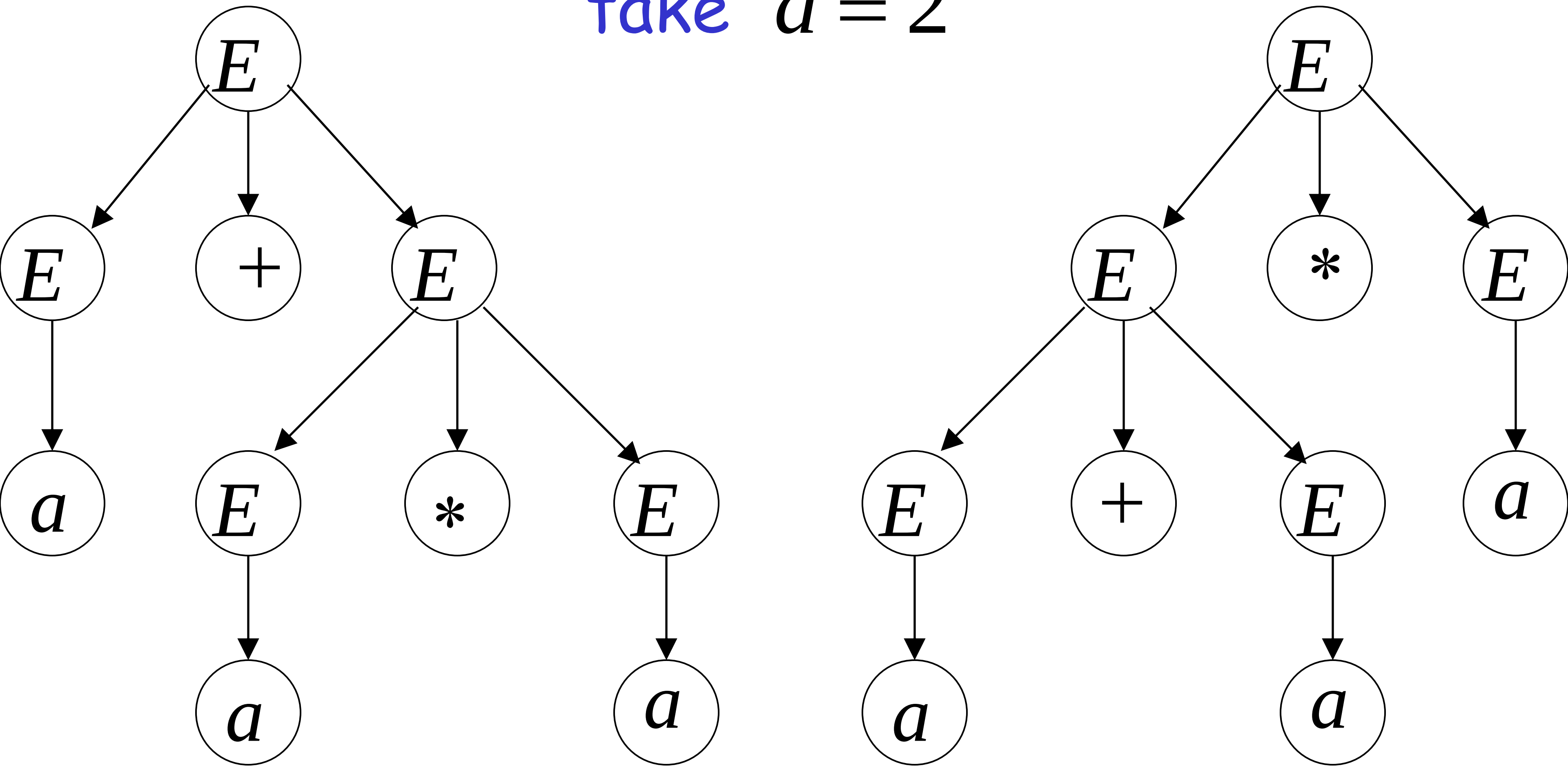
if some string $w \in L(G)$ has:

two or more derivation trees (derivations)

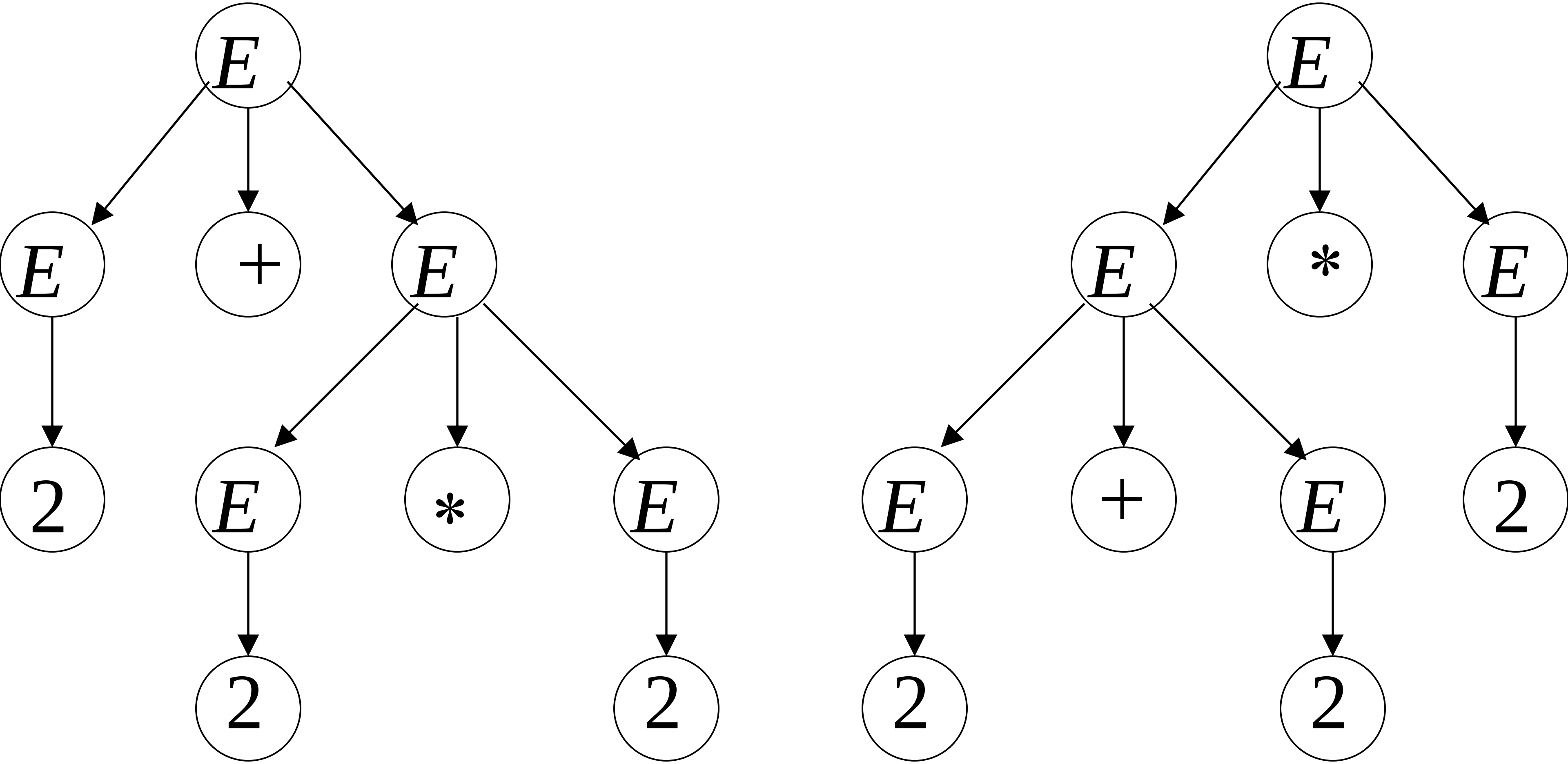
Why do we care about ambiguity?

$$a + a * a$$

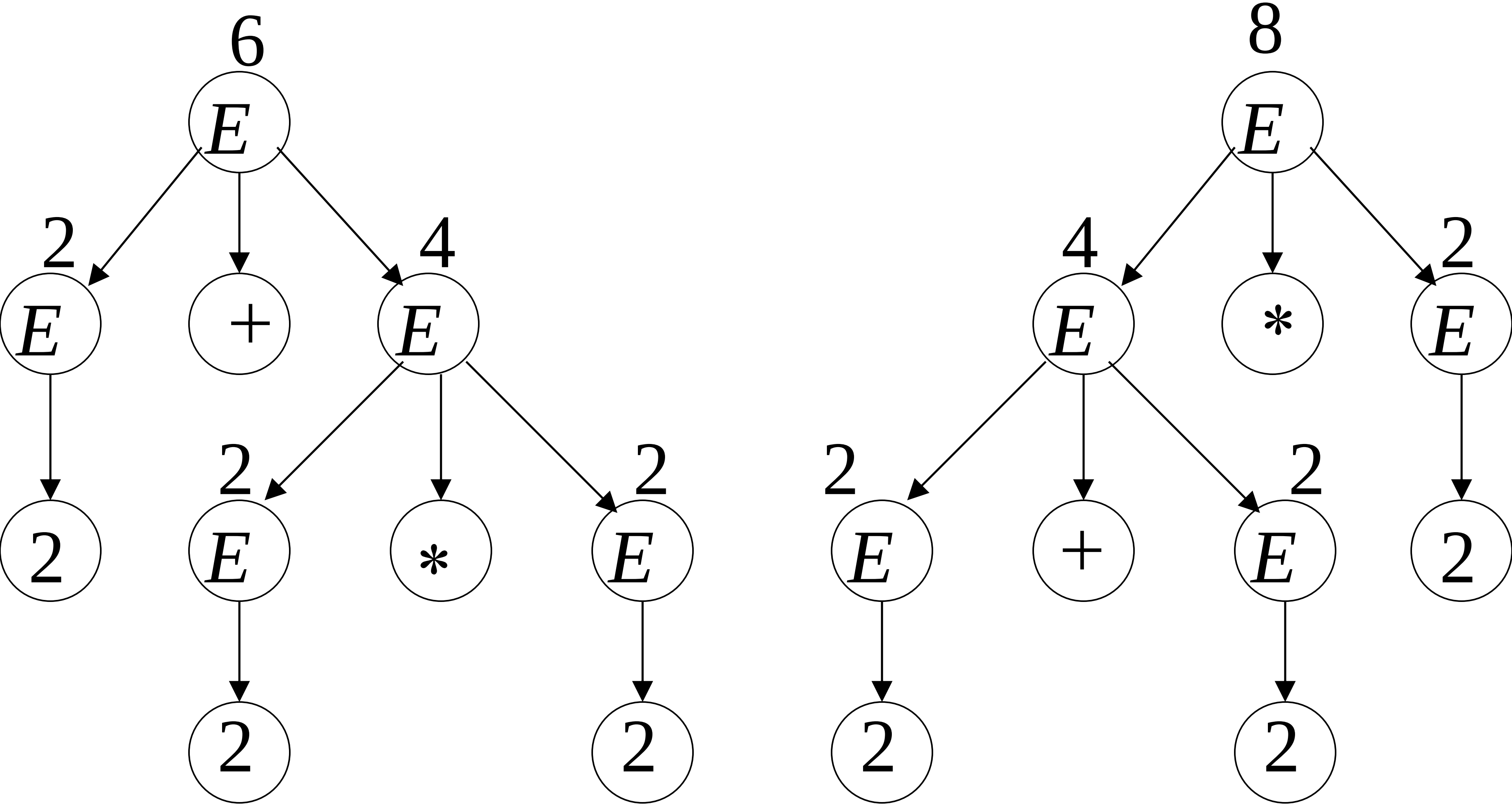
take $a = 2$



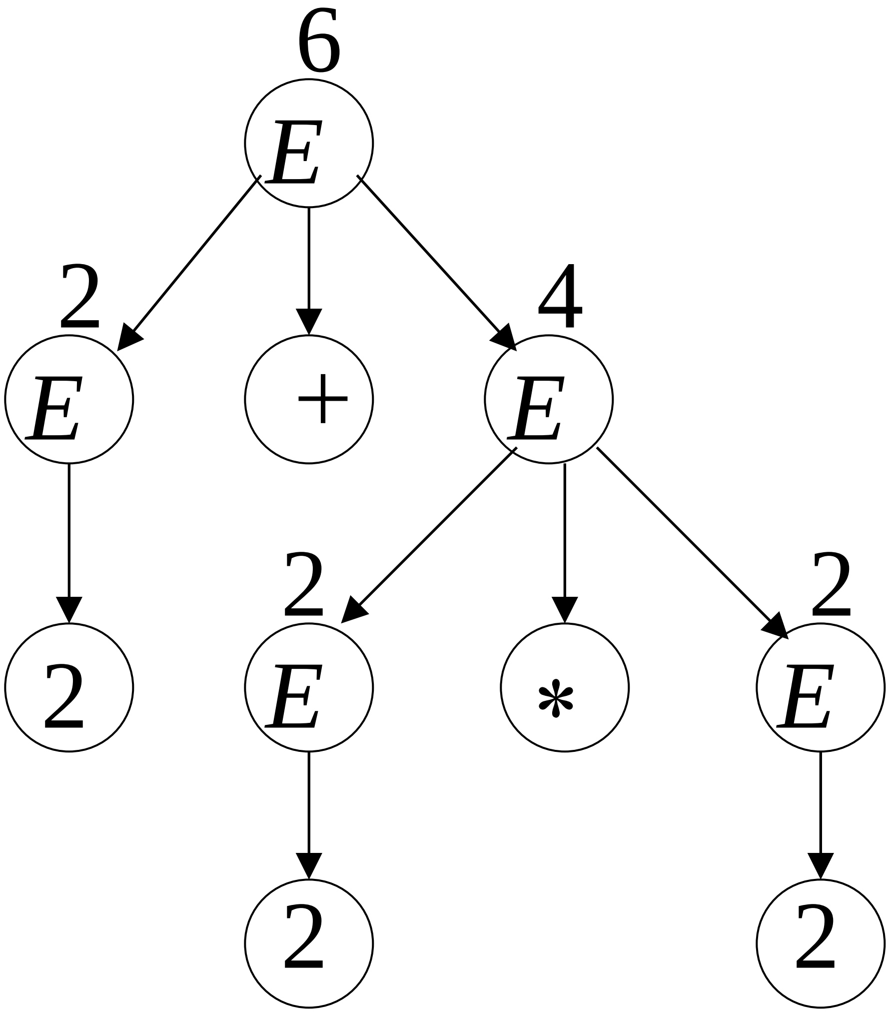
$$2 + 2 * 2$$



$2+2*2=6$ $\qquad\qquad$ $2+2*2=8$



Correct result: $2+2*2=6$



- Ambiguity is **bad** for programming languages

- We want to remove ambiguity

We fix the ambiguous grammar:

$$E \rightarrow E + E \mid E * E \mid (E) \mid a$$

New non-ambiguous grammar:

$$E \rightarrow E + T$$

$$E \rightarrow T$$

$$T \rightarrow T * F$$

$$T \rightarrow F$$

$$F \rightarrow (E)$$

$$F \rightarrow a$$

$$E \Rightarrow E+T \Rightarrow T+T \Rightarrow F+T \Rightarrow a+T \Rightarrow a+T*F$$

$$\Rightarrow a+F*F \Rightarrow a+a*F \Rightarrow a+a*a$$

$$a+a*a$$

$$E \rightarrow E+T$$

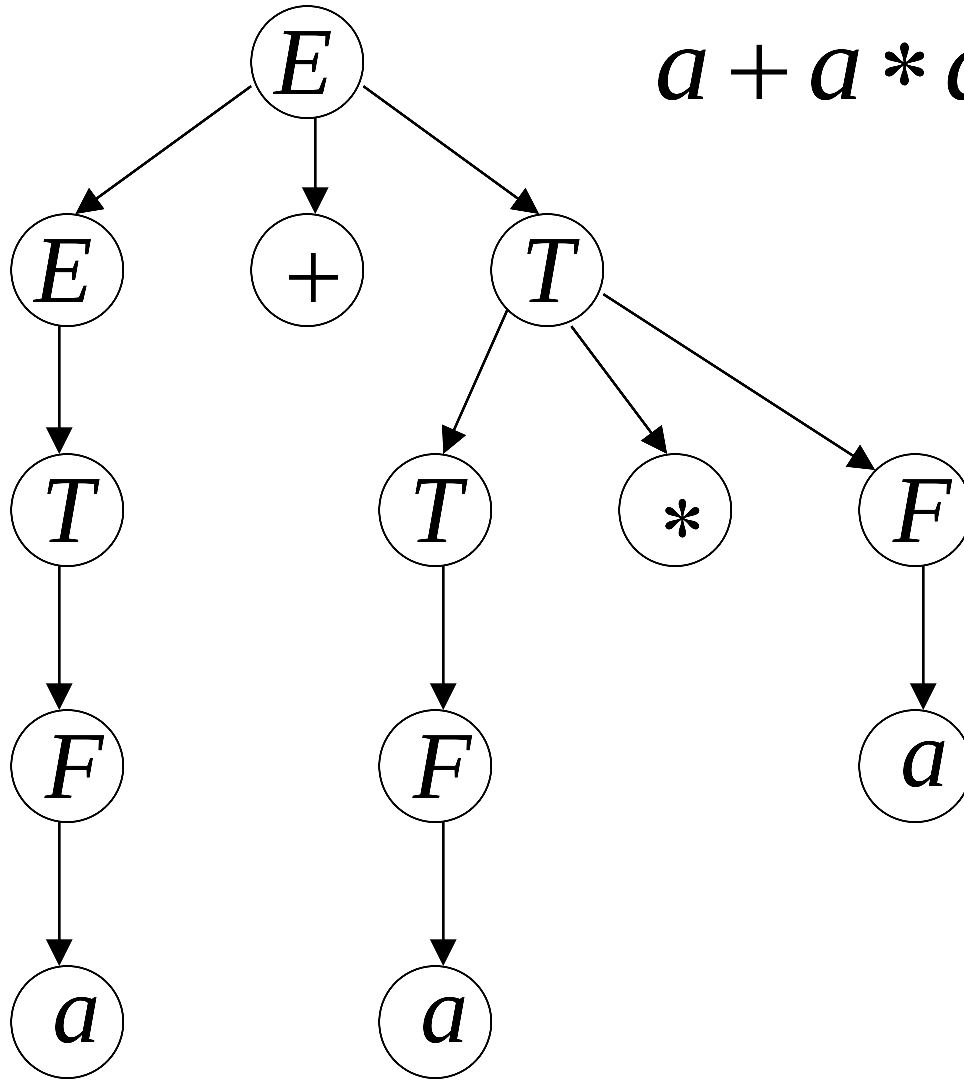
$$E \rightarrow T$$

$$T \rightarrow T*F$$

$$T \rightarrow F$$

$$F \rightarrow (E)$$

$$F \rightarrow a$$

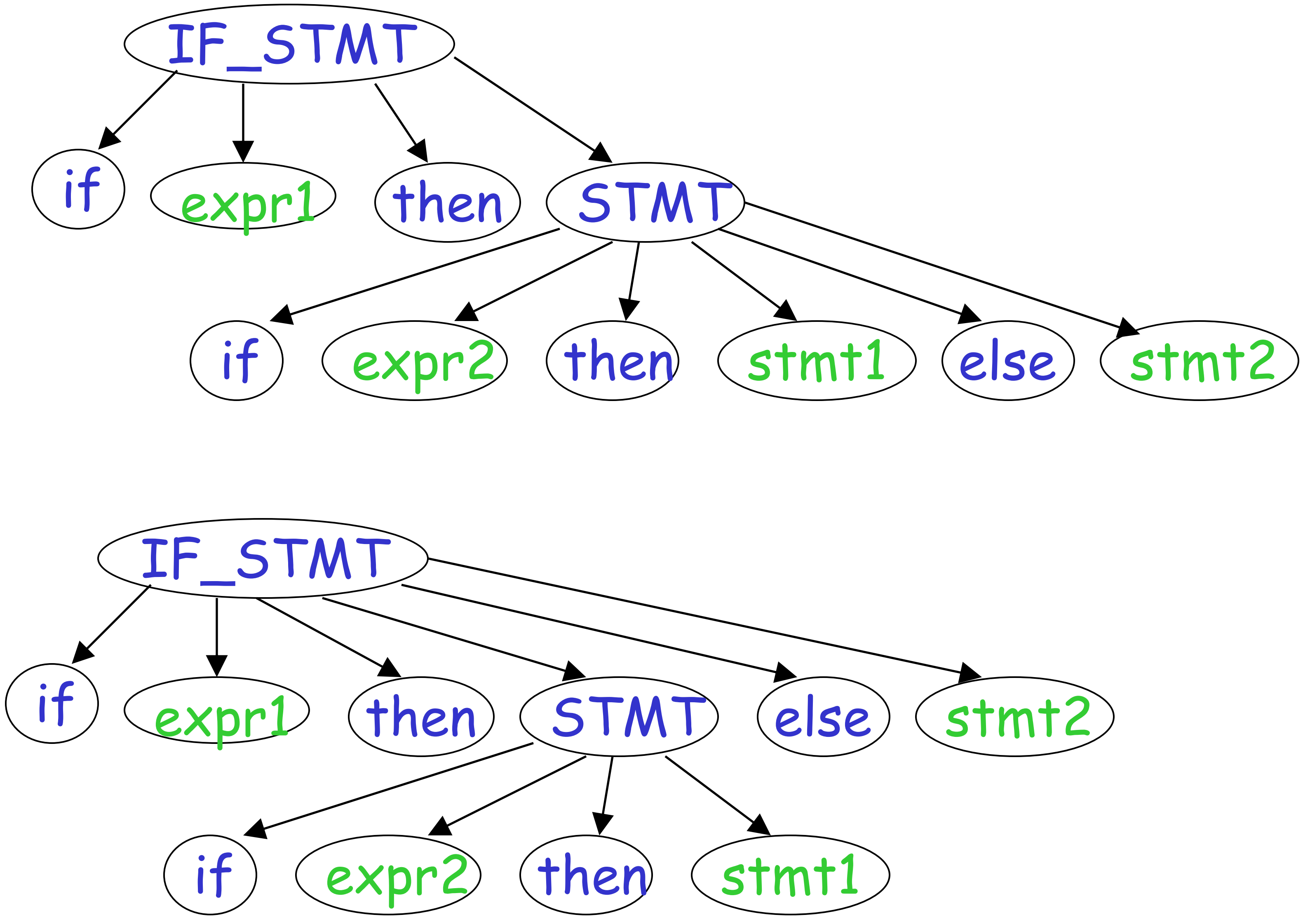


Unique derivation tree

Another Ambiguous Grammar

$$
\begin{aligned}
\text{IF\_STMT} \;\rightarrow\;& \text{if EXPR then STMT} \\
\mid\;& \text{if EXPR then STMT else STMT}
\end{aligned}
$$

If expr1 then if expr2 then stmt1 else stmt2



Inherent Ambiguity

Some context free languages
have only ambiguous grammars

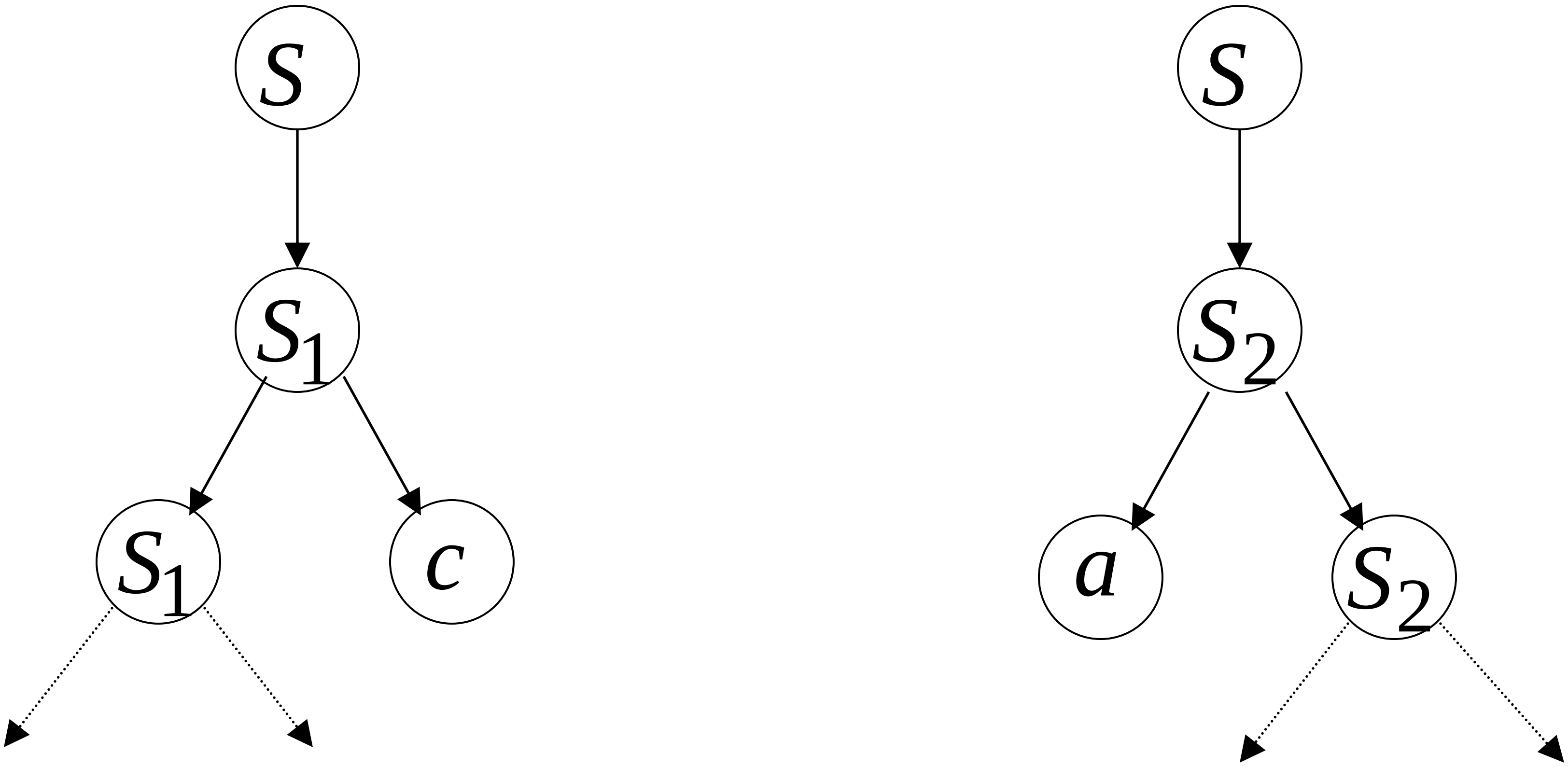
Example: $L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$

$$S \to S_1 \mid S_2 \qquad S_1 \to S_1 c \mid A \qquad S_2 \to a S_2 \mid B$$

$$A \to aAb \mid \lambda \qquad B \to bBc \mid \lambda$$

The string $a^n b^n c^n$

has two derivation trees



Compilers

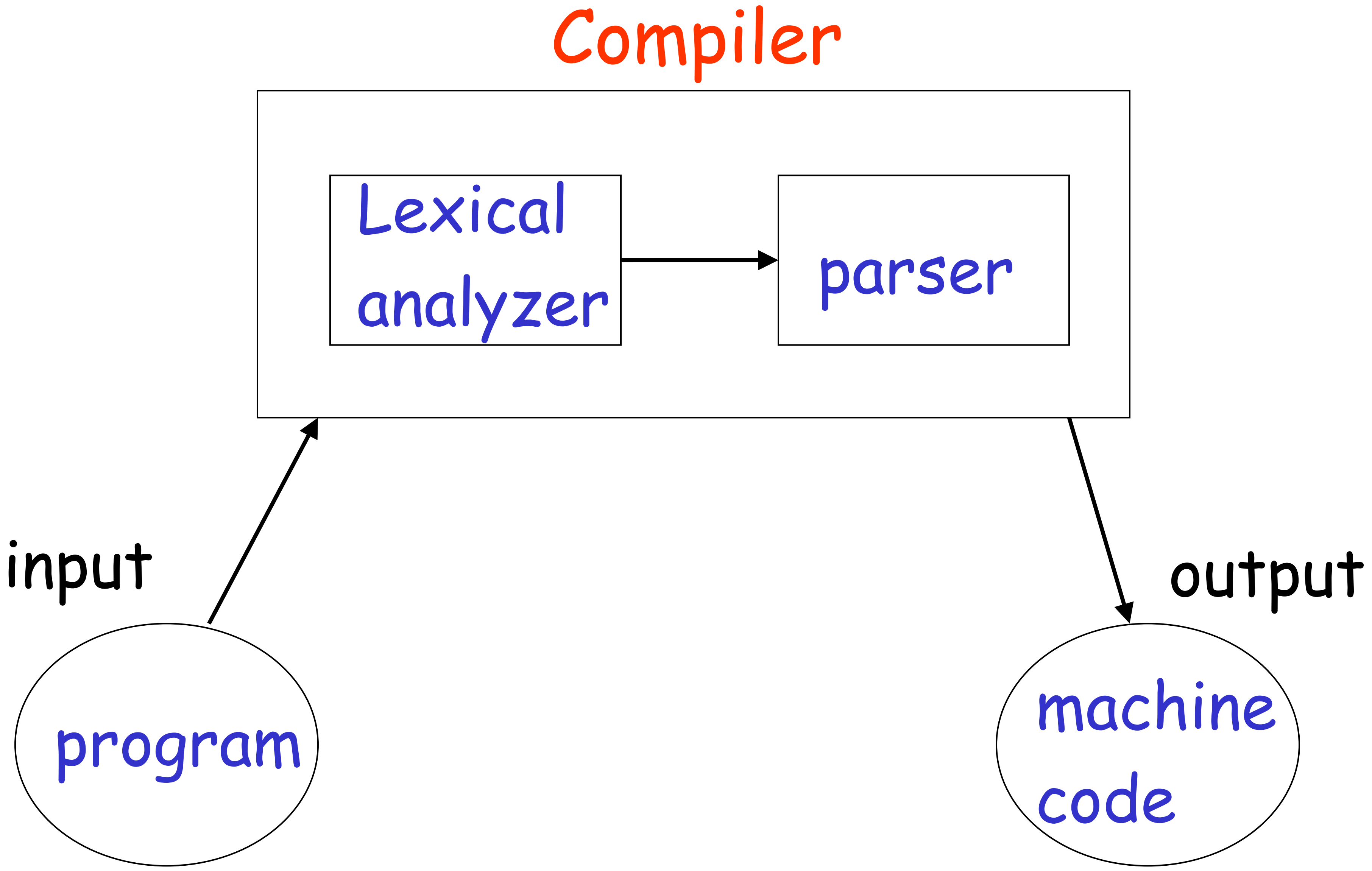
Program

```
v = 5;
if (v>5)
   x = 12 + v;
while (x !=3) {
  x = x - 3;
  v = 10;
}
......
```

→ Compiler →

Machine Code

```
Add v,v,0
cmp v,5
jmplt ELSE
THEN:
  add x, 12,v
ELSE:
WHILE:
cmp x,3
...
```



A parser knows the grammar of the programming language

The parser finds the derivation of a particular input

input

10 + 2 * 5

Parser

E -> E + E
    | E * E
    | INT

derivation

E => E + E
  => E + E * E
  => 10 + E*E
  => 10 + 2 * E
  => 10 + 2 * 5

derivation tree

derivation

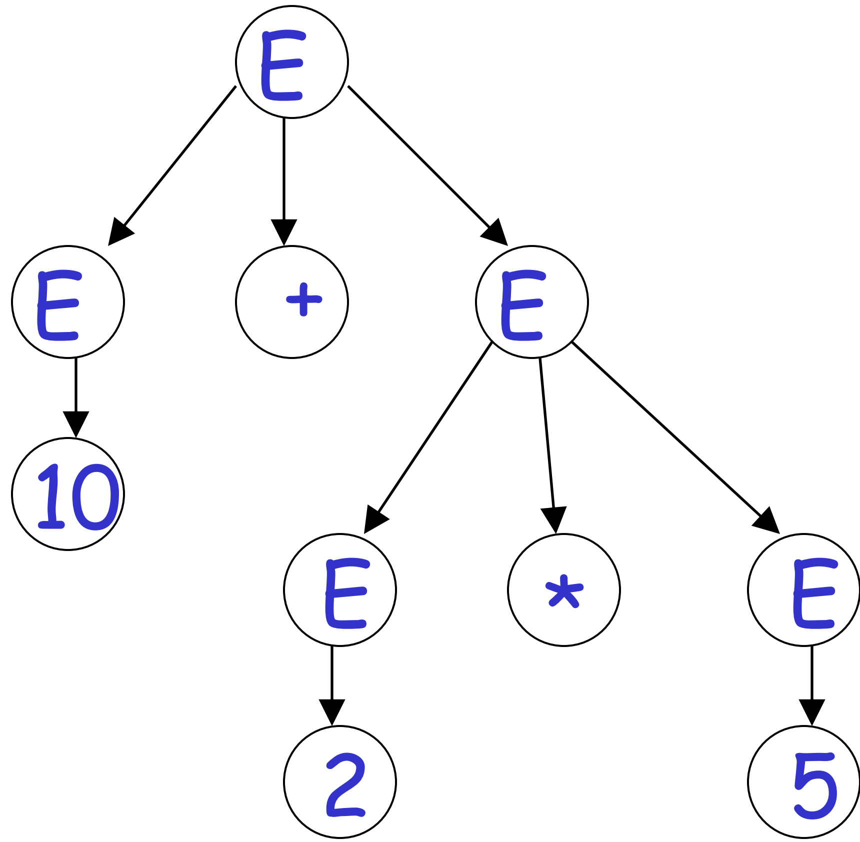
E => E + E

=> E + E * E

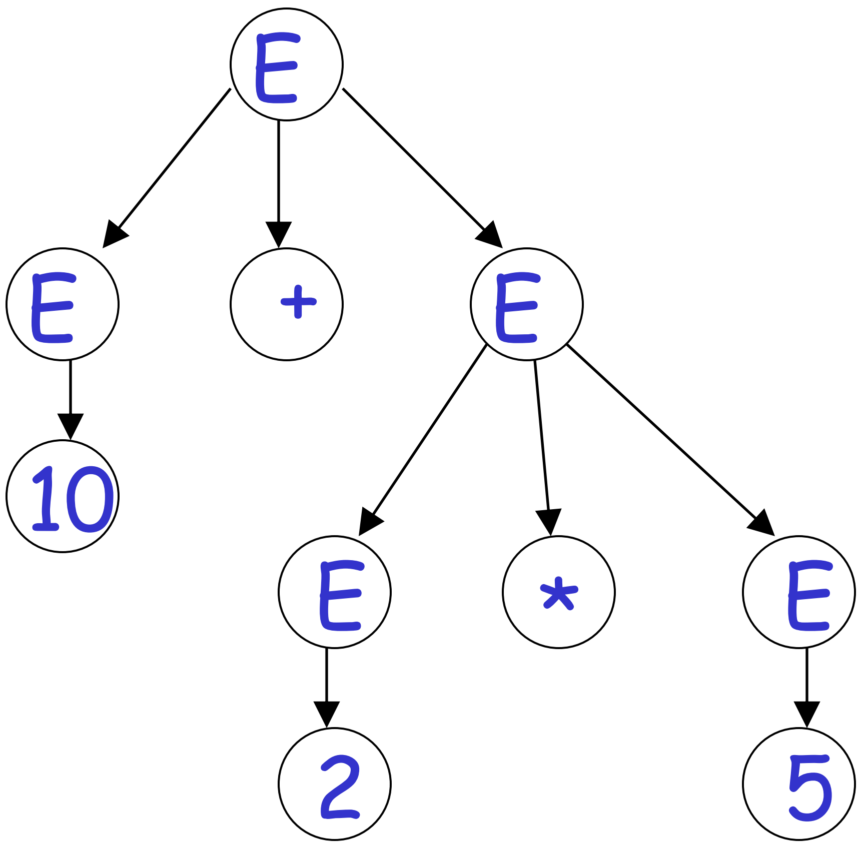
=> 10 + E*E

=> 10 + 2 * E

=> 10 + 2 * 5



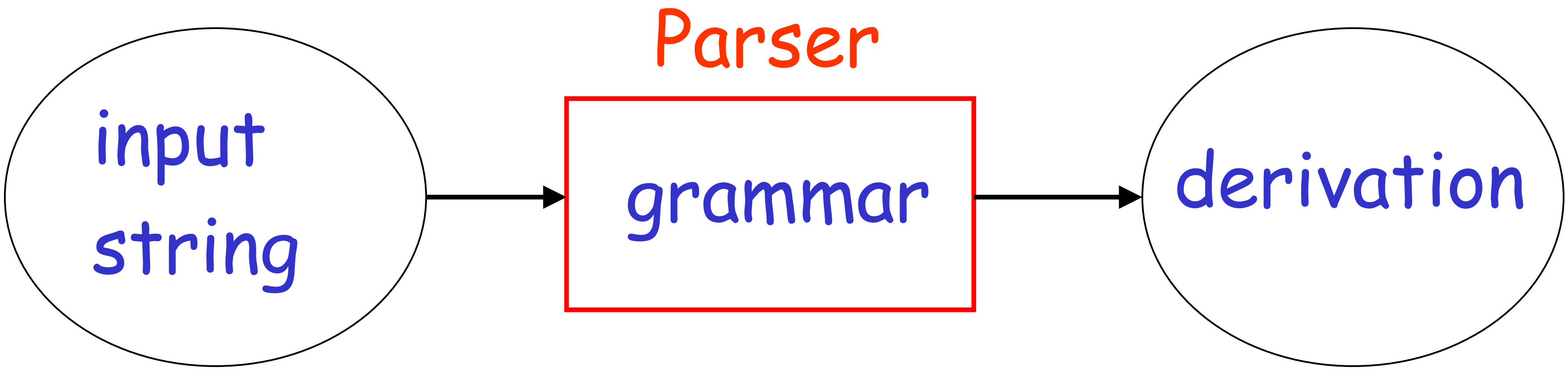
derivation tree



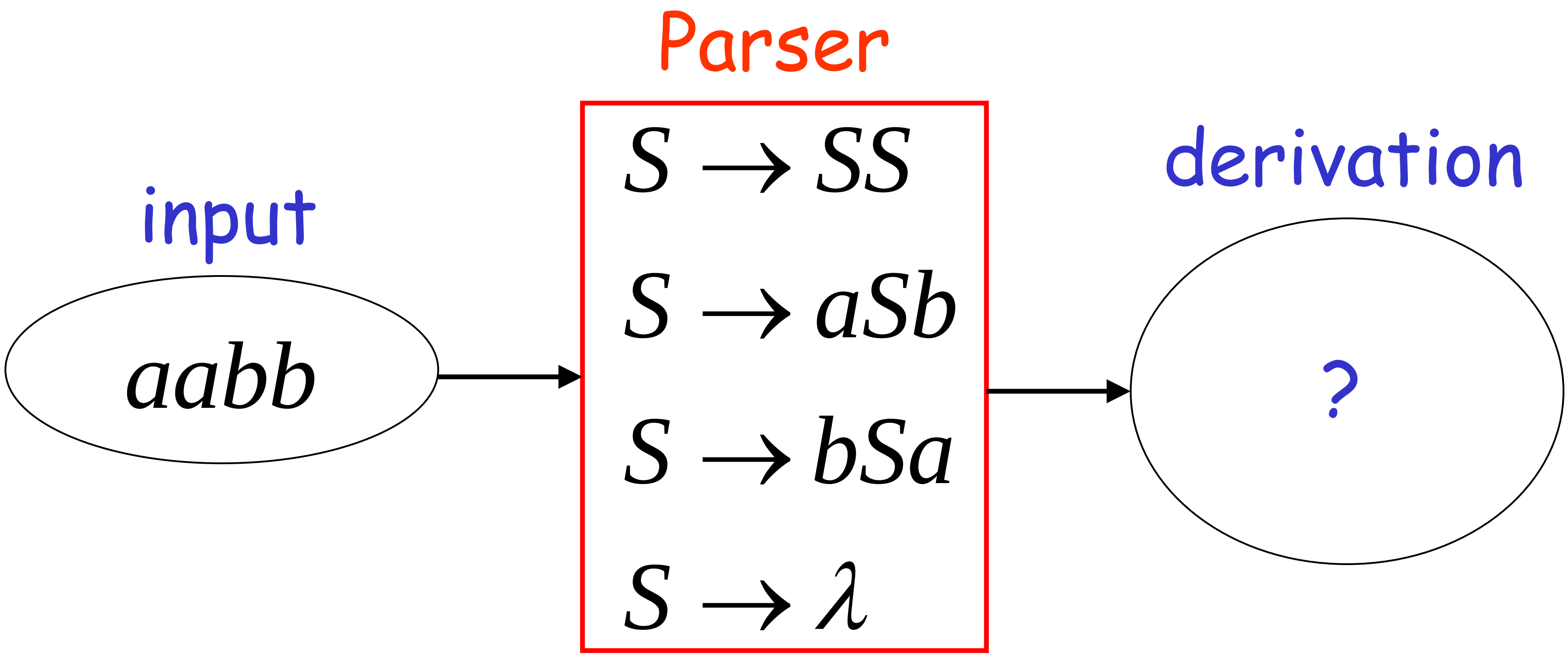
machine code

mult a, 2, 5
add b, 10, a

Parsing


Parser
input
string
grammar
derivation

Example:



Parser
input
$aabb$
$S \to SS$
$S \to aSb$
$S \to bSa$
$S \to \lambda$
derivation
?

Exhaustive Search

$$S \to SS \mid aSb \mid bSa \mid \lambda$$

Phase 1:

$$S \Rightarrow SS$$

$$S \Rightarrow aSb$$

$$S \Rightarrow bSa$$

$$S \Rightarrow \lambda$$

Find derivation of

$$aabb$$

All possible derivations of length 1

$$S \Rightarrow SS \qquad\qquad aabb$$

$$S \Rightarrow aSb$$

$$\cancel{S \Rightarrow bSa}$$

$$\cancel{S \Rightarrow \lambda}$$

Phase 2 $S \to SS \mid aSb \mid bSa \mid \lambda$

$S \Rightarrow SS \Rightarrow SSS$

$S \Rightarrow SS \Rightarrow aSbS$

$aabb$

Phase 1

$\cancel{S \Rightarrow SS \Rightarrow bSaS}$

$S \Rightarrow SS$

$S \Rightarrow SS \Rightarrow S$

$S \Rightarrow aSb$

$S \Rightarrow aSb \Rightarrow aSSb$

$S \Rightarrow aSb \Rightarrow aaSbb$

$\cancel{S \Rightarrow aSb \Rightarrow abSab}$

$\cancel{S \Rightarrow aSb \Rightarrow ab}$

$$S \to SS \mid aSb \mid bSa \mid \lambda$$

Phase 2

$$S \Rightarrow SS \Rightarrow SSS$$

$$S \Rightarrow SS \Rightarrow aSbS$$

$$aabb$$

$$S \Rightarrow SS \Rightarrow S$$

$$S \Rightarrow aSb \Rightarrow aSSb$$

$$S \Rightarrow aSb \Rightarrow aaSbb$$

Phase 3

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

Final result of exhaustive search (top-down parsing)

Parser

$$S \to SS$$
$$S \to aSb$$
$$S \to bSa$$
$$S \to \lambda$$

input

$aabb$

derivation

$$S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$$

Time complexity of exhaustive search

Suppose there are no productions of the form

$$A \rightarrow \lambda$$

$$A \rightarrow B$$

Number of phases for string w : $2|w|$

For grammar with k rules

Total time needed for string w :

$$k + k^2 + \cdots + k^{2|w|}$$

Extremely bad!!!

For general context-free grammars:

There exists a parsing algorithm
that parses a string $|w|$
in time $|w|^3$

(We will see it later in this class. We need to simplify our context-free grammar in order to use this algorithm)

Simplifications of Context-Free Grammars

A Substitution Rule

$$S \to aB$$
$$A \to aaA$$
$$A \to abBc$$
$$B \to aA$$
$$B \to b$$

Substitute
$$B \to b$$

Equivalent grammar

$$S \to aB \mid ab$$
$$A \to aaA$$
$$A \to abBc \mid abbc$$
$$B \to aA$$

A Substitution Rule

$$S \to aB \mid ab$$

$$A \to aaA$$

$$A \to abBc \mid abbc$$

$$B \to aA$$

Substitute

$$B \to aA$$

$$S \to \cancel{aB} \mid ab \mid aaA$$

$$A \to aaA$$

$$A \to \cancel{abBc} \mid abbc \mid abaAc$$

Equivalent grammar

In general:

$$A \to xBz$$

$$B \to y_1$$

Substitute $B \to y_1$

$$A \to xBz \mid xy_1z$$

equivalent grammar

Nullable Variables

$\lambda-\text{production}:$ $A \to \lambda$

Nullable Variable: $A \Rightarrow \ldots \Rightarrow \lambda$

Removing Nullable Variables

Example Grammar:

$$S \rightarrow aMb$$

$$M \rightarrow aMb$$

$$M \rightarrow \lambda$$

Nullable variable

$$S \to aMb$$

$$M \to aMb$$

$$\cancel{M \to \lambda}$$

Substitute

$$M \to \lambda$$

Final Grammar

$$S \to aMb$$

$$S \to ab$$

$$M \to aMb$$

$$M \to ab$$

Unit-Productions

Unit Production: $A \to B$

(a single variable in both sides)

Observation: $A \to A$

Is removed immediately

Example Grammar:

$$S \to aA$$

$$A \to a$$

$$A \to B$$

$$B \to A$$

$$B \to bb$$

$$S \to aA$$

$$A \to a$$

$$\cancel{A \to B}$$

$$B \to A$$

$$B \to bb$$

Substitute

$$A \to B$$

$$S \to aA \mid aB$$

$$A \to a$$

$$B \to A \mid B$$

$$B \to bb$$

$$S \to aA \mid aB$$

$$A \to a$$

$$B \to A \mid \cancel{B}$$

$$B \to bb$$

Remove

$$B \to B$$

$$S \to aA \mid aB$$

$$A \to a$$

$$B \to A$$

$$B \to bb$$

$$S \to aA \mid aB$$

$$A \to a$$

$$\cancel{B \to A}$$

$$B \to bb$$

Substitute

$$B \to A$$

$$S \to aA \mid aB \mid aA$$

$$A \to a$$

$$B \to bb$$

Remove repeated productions

$$S \to aA \mid aB \mid \cancel{aA}$$
$$A \to a$$
$$B \to bb$$

Final grammar

$$S \to aA \mid aB$$
$$A \to a$$
$$B \to bb$$

Useless Productions

$$S \rightarrow aSb$$

$$S \rightarrow \lambda$$

$$S \rightarrow A$$

$$A \rightarrow aA$$ Useless Production

Some derivations never terminate...

$$S \Rightarrow A \Rightarrow aA \Rightarrow aaA \Rightarrow \ldots \Rightarrow aa\ldots aA \Rightarrow \ldots$$

Another grammar:

$$S \to A$$

$$A \to aA$$

$$A \to \lambda$$

$$B \to bA$$ Useless Production

Not reachable from S

In general:

contains only terminals

if $S \Rightarrow \ldots \Rightarrow xAy \Rightarrow \ldots \Rightarrow w$

$w \in L(G)$

then variable A is useful

otherwise, variable A is useless

A production $A \to x$ is useless
if any of its variables is useless

$$S \to aSb$$

$$S \to \lambda$$

Productions

Variables $S \to A$ useless

useless $A \to aA$ useless

useless $B \to C$ useless

useless $C \to D$ useless

Removing Useless Productions

Example Grammar:

$$S \to aS \mid A \mid C$$

$$A \to a$$

$$B \to aa$$

$$C \to aCb$$

First: find all variables that can produce strings with only terminals

$$\left.\begin{array}{l} S \rightarrow aS \mid A \mid C \\ A \rightarrow a \\ B \rightarrow aa \\ C \rightarrow aCb \end{array}\right\} \quad \{A, B, S\}$$

Keep only the variables
that produce terminal symbols: $\{A, B, S\}$

(the rest variables are useless)

$$S \to aS \mid A \mid \cancel{C}$$

$$A \to a$$

$$B \to aa$$

$$\cancel{C \to aCb}$$

$$\Rightarrow$$

$$S \to aS \mid A$$

$$A \to a$$

$$B \to aa$$

Remove useless productions

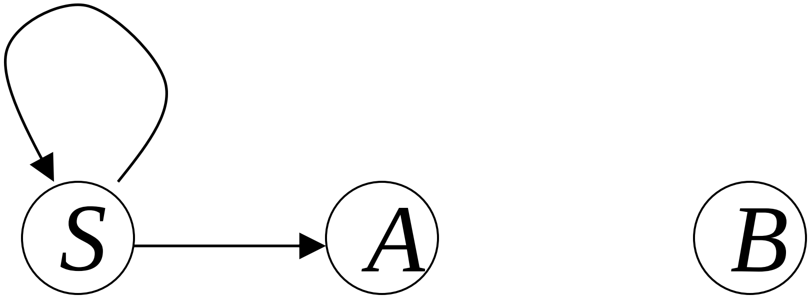
Second: Find all variables reachable from S

Use a Dependency Graph

$$S \rightarrow aS \mid A$$

$$A \rightarrow a$$

$$B \rightarrow aa$$



not reachable

Keep only the variables reachable from S

(the rest variables are useless)

Final Grammar

$$S \to aS \mid A$$

$$A \to a$$

$$\cancel{B \to aa}$$

$$S \to aS \mid A$$

$$A \to a$$

Remove useless productions

Removing All

Step 1: Remove Nullable Variables

Step 2: Remove Unit-Productions

Step 3: Remove Useless Variables

Normal Forms for Context-free Grammars

Chomsky Normal Form

Each productions has form:

$A \to BC$ or $A \to a$

variable variable terminal

- Chomsky normal forms are good for parsing and proving theorems

Examples:

$$S \to AS$$
$$S \to a$$
$$A \to SA$$
$$A \to b$$

Chomsky Normal Form

$$S \to AS$$
$$S \to AAS$$
$$A \to SA$$
$$A \to aa$$

Not Chomsky Normal Form

Convertion to Chomsky Normal Form

Example:

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

Not Chomsky
Normal Form

Make sure that there is no λ-production before we proceed.

Introduce variables for terminals: T_a, T_b, T_c

$$S \to ABa$$
$$A \to aab$$
$$B \to Ac$$

$$S \to ABT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Introduce intermediate variable: V_1

$$S \to ABT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

$\Rightarrow$

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Introduce intermediate variable: V_2

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_aT_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

$$S \to AV_1$$
$$V_1 \to BT_a$$
$$A \to T_aV_2$$
$$V_2 \to T_aT_b$$
$$B \to AT_c$$
$$T_a \to a$$
$$T_b \to b$$
$$T_c \to c$$

Final grammar in Chomsky Normal Form:

Initial grammar

$$S \to ABa$$

$$A \to aab$$

$$B \to Ac$$

$$S \to AV_1$$

$$V_1 \to BT_a$$

$$A \to T_a V_2$$

$$V_2 \to T_a T_b$$

$$B \to AT_c$$

$$T_a \to a$$

$$T_b \to b$$

$$T_c \to c$$

Theorem: For any context-free grammar (which doesn't produce λ) there is an equivalent grammar in Chomsky Normal Form

Greibach Normal Form

All productions have form:

$$A \to a\, V_1 V_2 \cdots V_k \qquad\qquad k \geq 0$$

symbol variables

- Greibach normal forms are very good for parsing

Examples:

$$S \rightarrow cAB$$

$$A \rightarrow aA \mid bB \mid b$$

$$B \rightarrow b$$

Greibach
Normal Form

$$S \rightarrow abSb$$

$$S \rightarrow aa$$

Not Greibach
Normal Form

Conversion to Greibach Normal Form:

$$S \to abSb$$

$$S \to aa$$

$$S \to aT_bST_b$$

$$S \to aT_a$$

$$T_a \to a$$

$$T_b \to b$$

Greibach
Normal Form

Theorem: For any context-free grammar (which doesn't produce λ) there is an equivalent grammar in Greibach Normal Form

The CYK Parser

The CYK Membership Algorithm

Input:

- Grammar G in Chomsky Normal Form
- String w

Output:

find if $w \in L(G)$

The Algorithm

Input example:

- Grammar G: $S \rightarrow AB$

  $A \rightarrow BB$

  $A \rightarrow a$

  $B \rightarrow AB$

  $B \rightarrow b$

- String w: $aabbb$

aabbb

a	a	b	b	b
aa	ab	bb	bb	
aab	abb	bbb		
aabb	abbb			
aabbb				

$$S \to AB$$

$$A \to BB$$

$$A \to a$$

$$B \to AB$$

$$B \to b$$

a	a	b	b	b
A	A	B	B	B
aa	ab	bb	bb	
aab	abb	bbb		
aabb	abbb			
aabbb				

$$S \to AB$$

$$A \to BB$$

$$A \to a$$

$$B \to AB$$

$$B \to b$$

a	a	b	b	b
A	A	B	B	B
aa	ab	bb	bb	
	S,B	A	A	
aab	abb	bbb		
aabb	abbb			
aabbb				

$S \to AB$

$A \to BB$

$A \to a$

$B \to AB$

$B \to b$

a	a	b	b	b
A	A	B	B	B
aa	ab	bb	bb	
	S,B	A	A	
aab	abb	bbb		
S,B	A	S,B		
aabb	abbb			
A	S,B			
aabbb				
S,B				

Therefore: $aabbb \in L(G)$

Time Complexity: $|w|^3$

Observation: The CYK algorithm can be easily converted to a parser (bottom up parser)