Positive Properties
of
Context-Free languages

Union

Context-free languages

are closed under: **Union**

L_1 is context free

L_2 is context free

$\Rightarrow$ $L_1 \cup L_2$ is context-free

Example

Language **Grammar**

$L_1 = \{a^n b^n\}$ $S_1 \to aS_1b \mid \lambda$

$L_2 = \{ww^R\}$ $S_2 \to aS_2a \mid bS_2b \mid \lambda$

Union

$L = \{a^n b^n\} \cup \{ww^R\}$ $S \to S_1 \mid S_2$

Concatenation

Context-free languages are closed under: **Concatenation**

L_1 is context free

L_2 is context free

$L_1 L_2$ is context-free

Example

Language	Grammar
$L_1 = \{a^n b^n\}$	$S_1 \rightarrow aS_1b \mid \lambda$
$L_2 = \{ww^R\}$	$S_2 \rightarrow aS_2a \mid bS_2b \mid \lambda$

Concatenation

$L = \{a^n b^n\}\{ww^R\}$	$S \rightarrow S_1S_2$

Star Operation

Context-free languages
are closed under: **Star-operation**

L is context free ⇒ L^* is context-free

Example

Language

Grammar

$L = \{a^n b^n\}$

$S \to aSb \mid \lambda$

Star Operation

$L = \{a^n b^n\}^*$

$S_1 \to SS_1 \mid \lambda$

Negative Properties
of
Context-Free Languages

Intersection

Context-free languages
are **not** closed under: **intersection**

L_1 is context free

L_2 is context free

$\Rightarrow$ $L_1 \cap L_2$

not necessarily
context-free

Example

$$L_1 = \{a^n b^n c^m\}$$

Context-free:

$$S \to AC$$

$$A \to aAb \mid \lambda$$

$$C \to cC \mid \lambda$$

$$L_2 = \{a^n b^m c^m\}$$

Context-free:

$$S \to AB$$

$$A \to aA \mid \lambda$$

$$B \to bBc \mid \lambda$$

Intersection

$L_1 \cap L_2 = \{a^n b^n c^n\}$ **NOT** context-free

Complement

Context-free languages
are **not** closed under: **complement**

L is context free $\Longrightarrow$ $\overline{L}$ **not** necessarily context-free

Example

$$L_1 = \{a^n b^n c^m\}$$

Context-free:

$$S \rightarrow AC$$

$$A \rightarrow aAb \mid \lambda$$

$$C \rightarrow cC \mid \lambda$$

$$L_2 = \{a^n b^m c^m\}$$

Context-free:

$$S \rightarrow AB$$

$$A \rightarrow aA \mid \lambda$$

$$B \rightarrow bBc \mid \lambda$$

Complement

$$\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2 = \{a^n b^n c^n\}$$

NOT context-free

« Exception »

The intersection of

a context-free language and

a regular language

is a context-free language

L_1 context free

L_2 regular

$$\left.\begin{array}{l} L_1 \text{ context free} \\ L_2 \text{ regular} \end{array}\right\} \Longrightarrow L_1 \cap L_2$$

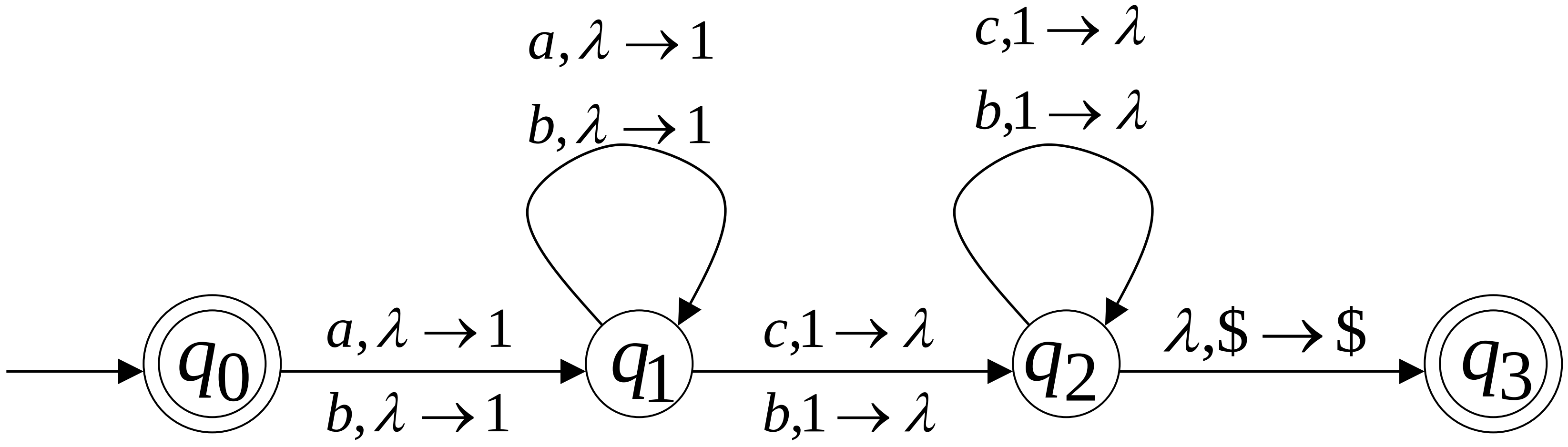
context-free

Example:

context-free

$$L_1 = \{w_1 w_2 : |w_1| = |w_2|, w_1 \in \{a,b\}^*, w_2 \in \{c,d\}^*\}$$

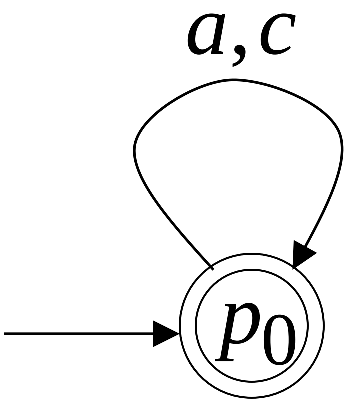
NPDA M_1



regular

$$L_2 = \{a, c\}^*$$

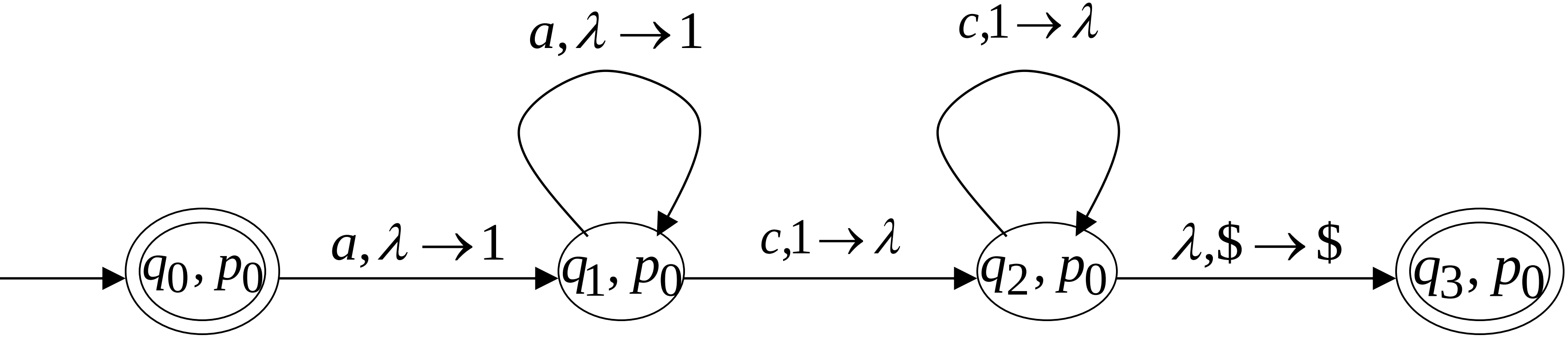
DFA M_2



context-free

Automaton for: $L_1 \cap L_2 = \{a^n c^n : \; n \geq 0\}$

NPDA M



An Application of Regular Closure

Prove that: $L = \{a^n b^n : \quad n \neq 100, n \geq 0\}$

is context-free

We know:

$\{a^n b^n : n \geq 0\}$ is context-free

We also know:

$L_1 = \{a^{100} b^{100}\}$ is regular

$$\Downarrow$$

$\overline{L_1} = \{(a+b)^*\} - \{a^{100} b^{100}\}$ is regular

$\{a^n b^n\}$ $\overline{L_1} = \{(a+b)^*\} - \{a^{100} b^{100}\}$

context-free regular

(regular closure) $\{a^n b^n\} \cap \overline{L_1}$ context-free

$$\{a^n b^n\} \cap \overline{L_1} = \{a^n b^n : \quad n \neq 100, n \geq 0\} = L$$

is context-free

Another Application of Regular Closure

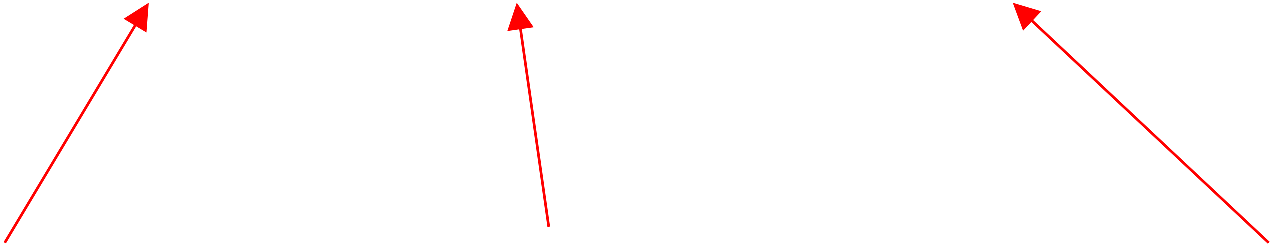
Prove that: $L = \{w : \quad n_a = n_b = n_c\}$

is **not** context-free

If $L = \{w: \quad n_a = n_b = n_c\}$ is context-free

(regular closure)

Then $L \cap \{a^*b^*c^*\} = \{a^n b^n c^n\}$



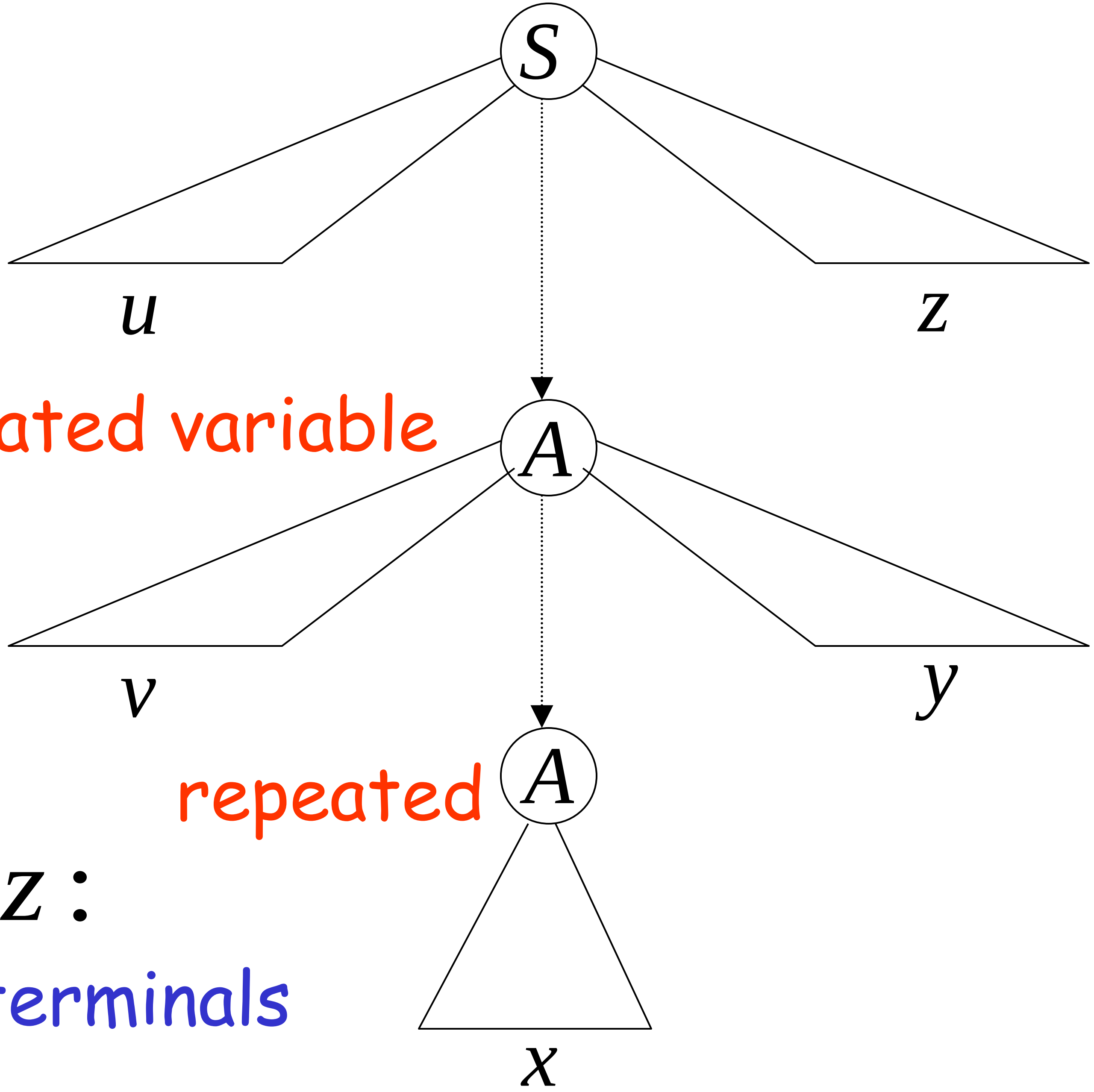
context-free regular context-free

Impossible!!!

Therefore, L is **not** context free

The Pumping Lemma
for
Context-Free Languages

Derivation tree of string w



Last repeated variable

repeated

$$w = uvxyz$$

$$u, v, x, y, z:$$

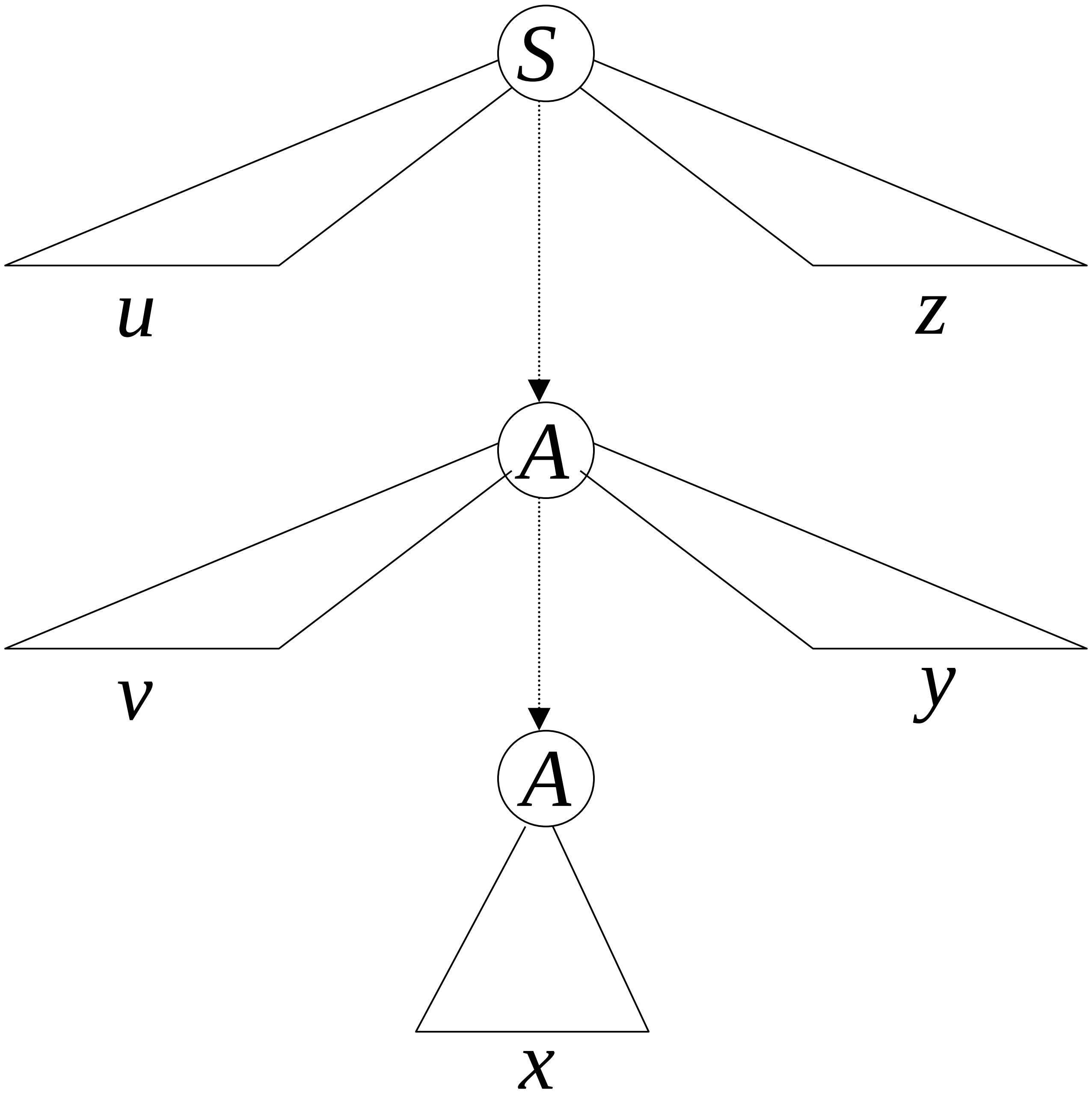
Strings of terminals

Possible derivations:

$$S \overset{*}{\Rightarrow} uAz$$

$$A \overset{*}{\Rightarrow} vAy$$

$$A \overset{*}{\Rightarrow} x$$



We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uxz$$

$$uv^0xy^0z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvxyz$$

The original $w = uv^1xy^1z$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow} uvvxyyz$$

$$uv^2xy^2z$$

We know:

$$S \overset{*}{\Rightarrow} uAz \qquad A \overset{*}{\Rightarrow} vAy \qquad A \overset{*}{\Rightarrow} x$$

This string is also generated:

$$S \overset{*}{\Rightarrow} uAz \overset{*}{\Rightarrow} uvAyz \overset{*}{\Rightarrow} uvvAyyz \overset{*}{\Rightarrow}$$

$$\overset{*}{\Rightarrow} uvvvAyyyz \overset{*}{\Rightarrow} \ldots$$

$$\overset{*}{\Rightarrow} uvvv \cdots vAy \cdots yyyz \overset{*}{\Rightarrow}$$

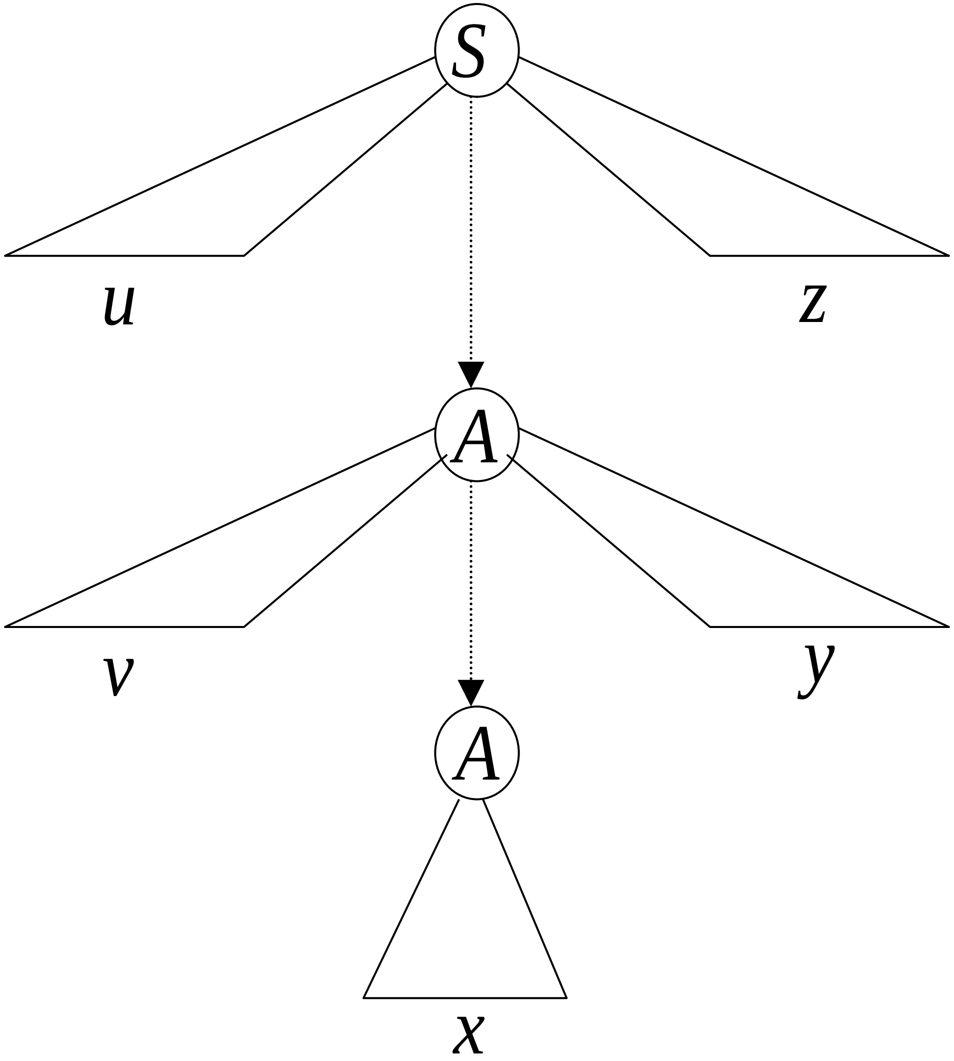
$$\overset{*}{\Rightarrow} uvvv \cdots vxy \cdots yyyz$$

$$uv^i xy^i z$$

Therefore, any string of the form

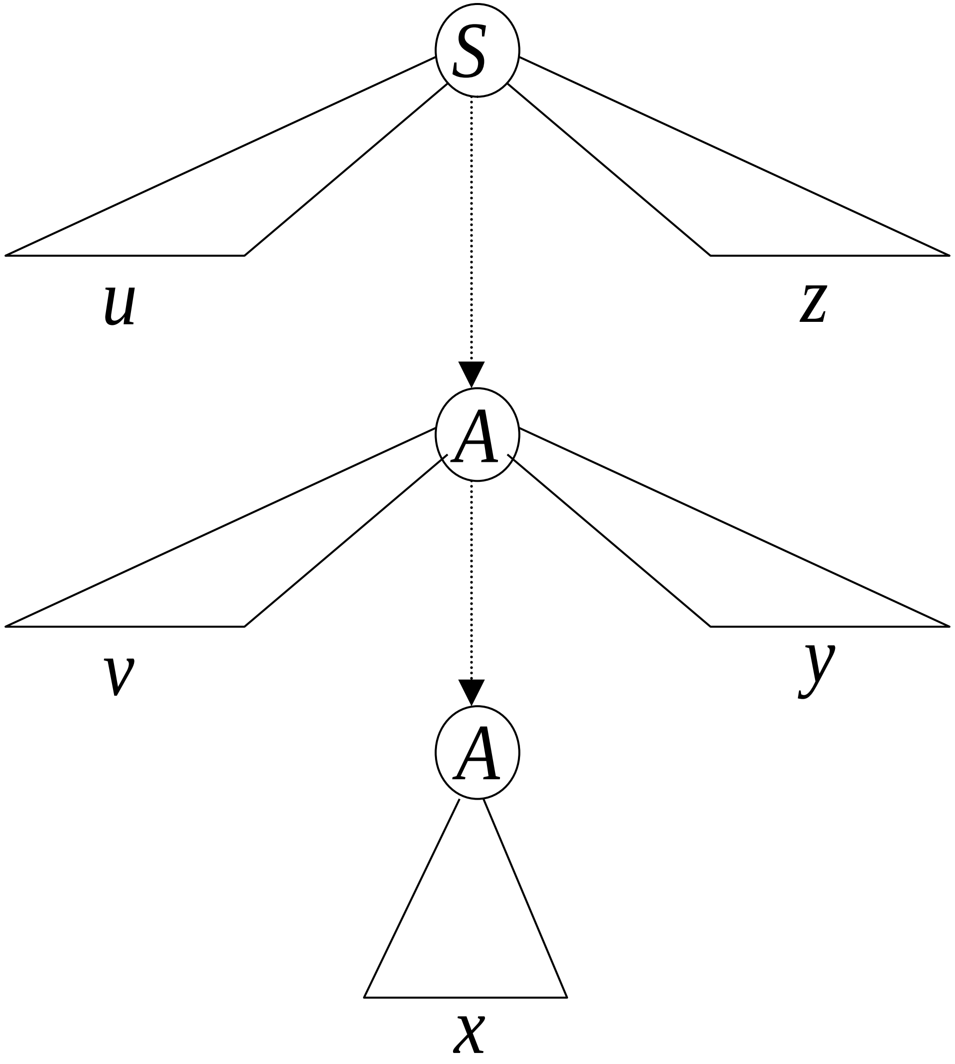
$$uv^i xy^i z \qquad i \geq 0$$

is generated by the grammar G



Observation: $|vxy| \leq m$

m is the number of states in PDA



Observation: $|vy| \geq 1$

Since repetitions are done on both v and y

The Pumping Lemma:

For infinite context-free language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

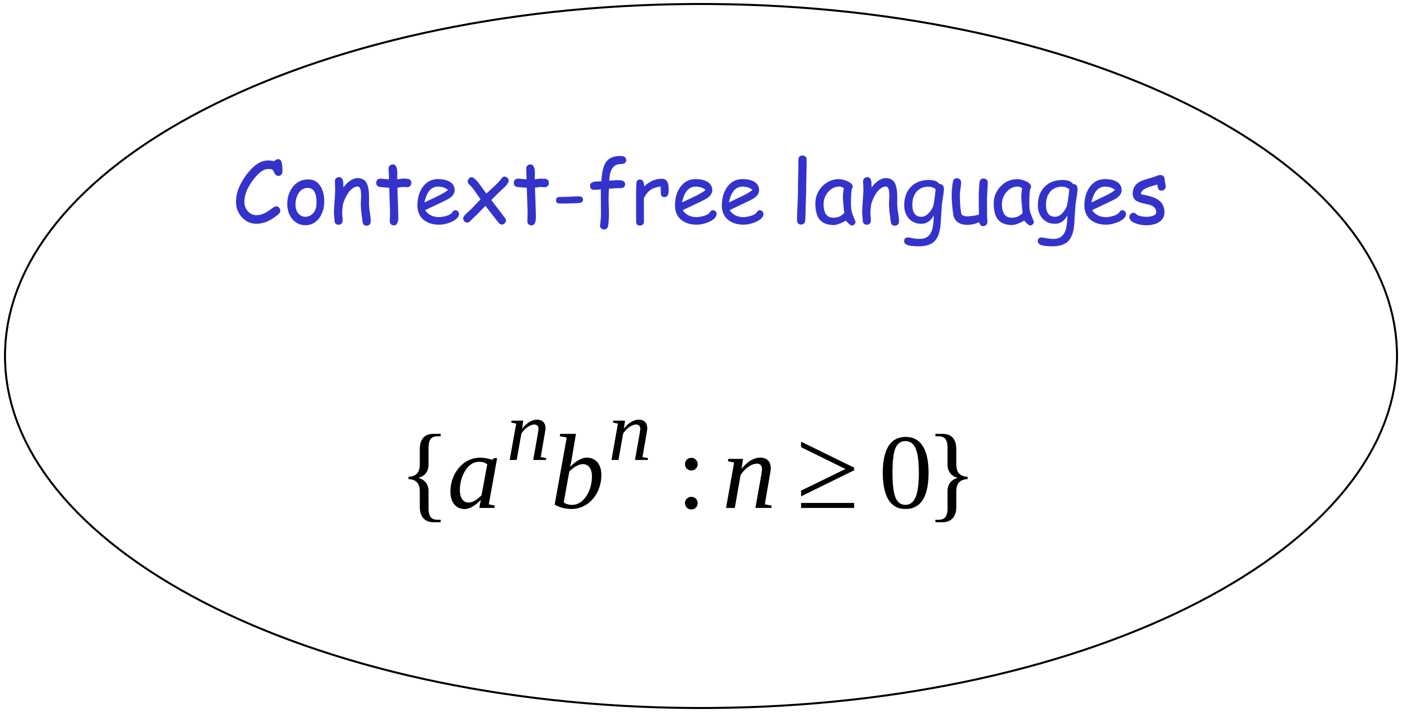
and it must be:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Applications
of
The Pumping Lemma

Non-context free languages

$$\{a^n b^n c^n : n \geq 0\}$$



Context-free languages

$$\{a^n b^n : n \geq 0\}$$

Theorem: The language

$$L = \{a^n b^n c^n : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^n b^n c^n : n \geq 0\}$$

Assume for contradiction that L is context-free

Pick any string $w \in L$ with length $|w| \geq m$

We pick: $w = a^m b^m c^m$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

We can write: $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$w = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

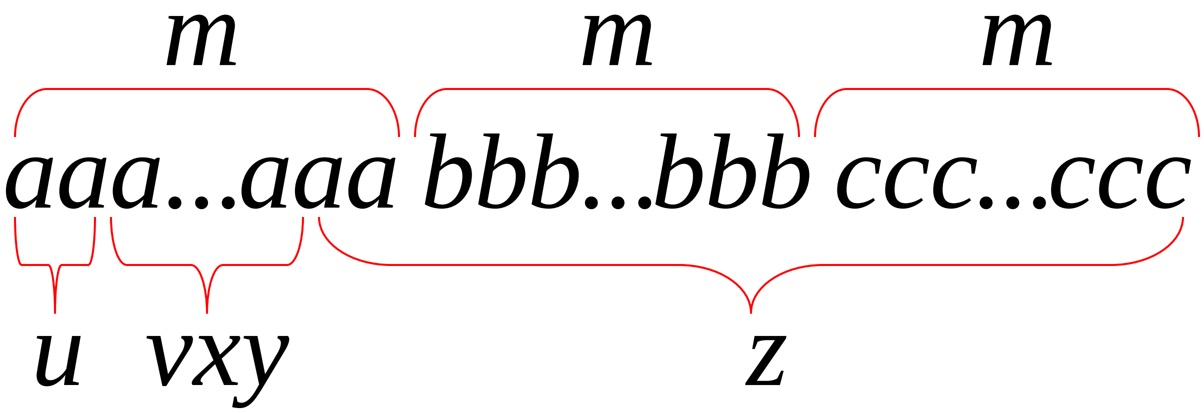
We examine **all** the possible locations of string vxy in w

$$L=\{a^n b^n c^n : n \geq 0\}$$

$$w=a^m b^m c^m$$

$$w=uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within a^m



$$\overbrace{aaa\ldots aaa}^{m}\;\overbrace{bbb\ldots bbb}^{m}\;\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{a}_{u}\underbrace{aa\ldots aa}_{vxy}\underbrace{a\;bbb\ldots bbb\;ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: Repeating v and y

$$k \geq 1$$

$$\overbrace{aaaaaa...aaaaaa}^{m+k}\,\overbrace{bbb...bbb}^{m}\,\overbrace{ccc...ccc}^{m}$$

$$\underbrace{a}_{u}\underbrace{aaaaa...aaaaa}_{v^2xy^2}\underbrace{a\,bbb...bbb\,ccc...ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$w = uvxyz$ $\quad |vxy| \leq m$ $\quad |vy| \geq 1$

Case 1: From Pumping Lemma: $uv^2xy^2z \in L$

$k \geq 1$

However: $uv^2xy^2z = a^{m+k} b^m c^m \notin L$

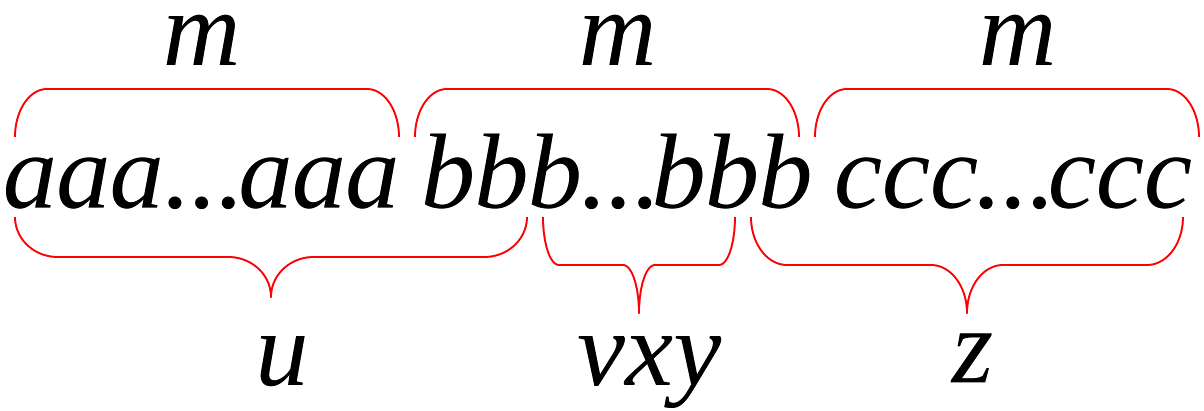
Contradiction!!!

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: vxy is within b^m

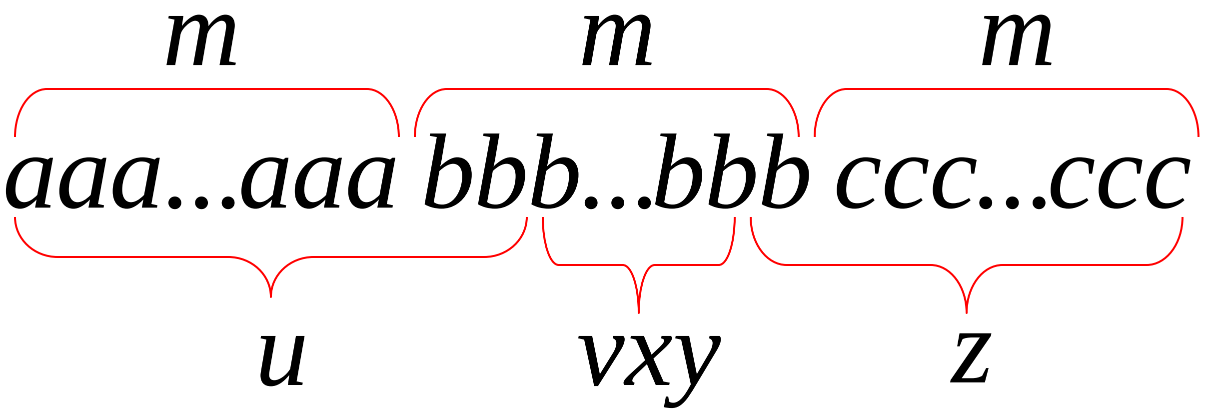


$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: Similar analysis with case 1

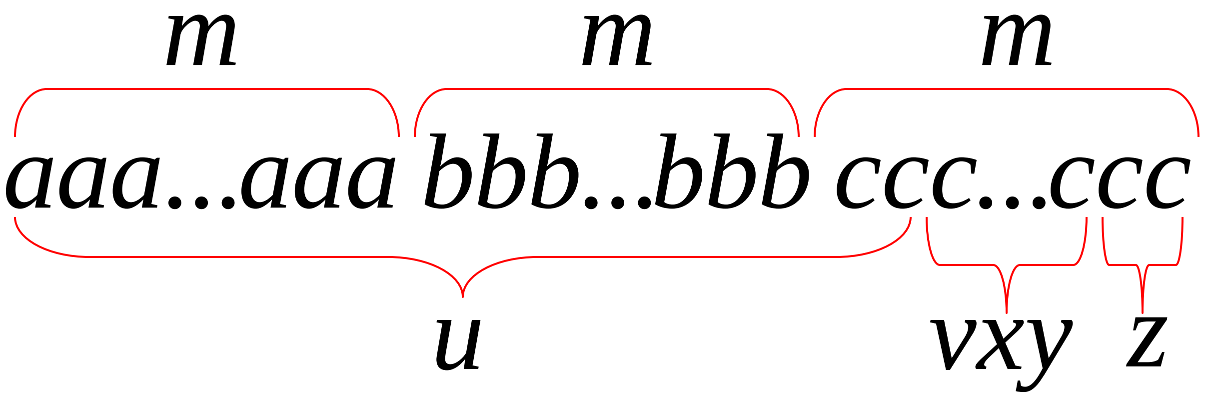


$$L=\{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: vxy is within c^m



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: Similar analysis with case 1

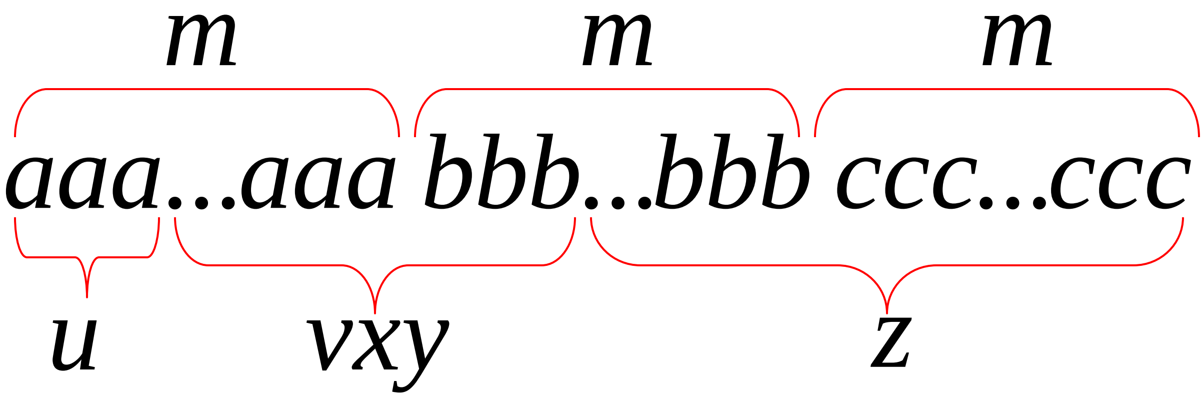
$$\underbrace{\overbrace{aaa\ldots aaa}^{m}\,\overbrace{bbb\ldots bbb}^{m}}_{u}\,\overbrace{\underbrace{ccc\ldots c}_{vxy}\underbrace{cc}_{z}}^{m}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: vxy overlaps a^m and b^m



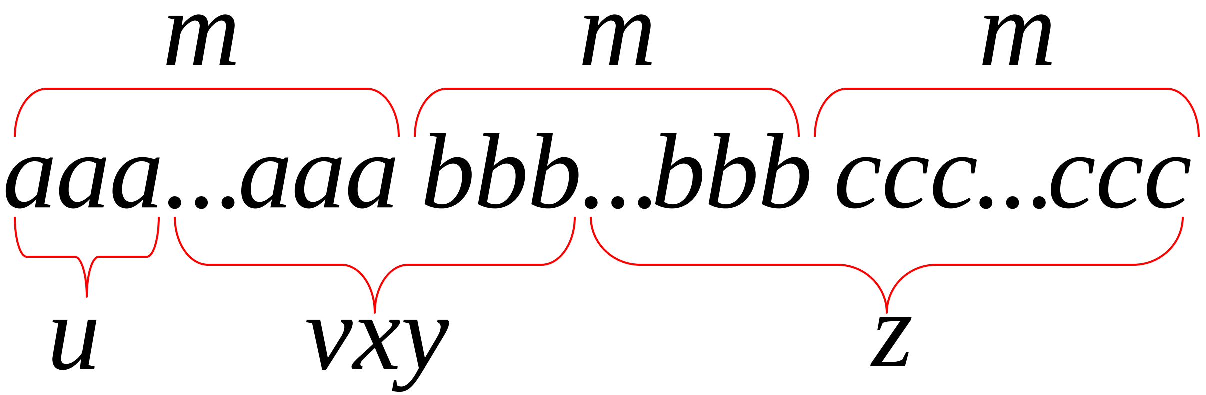
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 1: v contains only a

y contains only b



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 1: v contains only a

y contains only b

$$k_1 + k_2 \geq 1$$

$$\overbrace{aaa\ldots aaaaaaa}^{m+k_1}\;\overbrace{bbbbbbb\ldots bbb}^{m+k_2}\;\overbrace{ccc\ldots ccc}^{m}$$

$$\underbrace{aaa}_{u}\underbrace{\ldots aaaaaaa\;bbbbbbb}_{v^2xy^2}\underbrace{\ldots bbb\;ccc\ldots ccc}_{z}$$

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

$$k_1 + k_2 \geq 1$$

However: $uv^2xy^2z = a^{m+k_1} b^{m+k_2} c^m \notin L$

Contradiction!!!

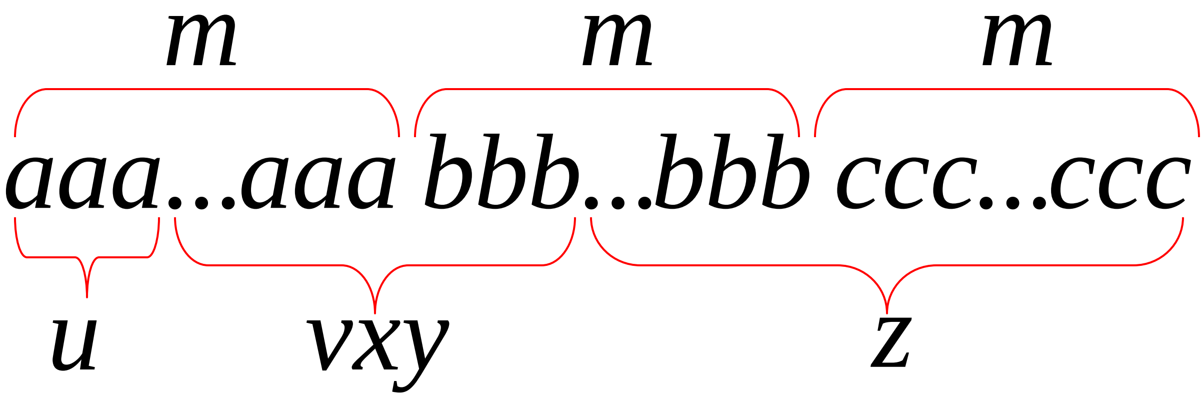
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 2: v contains a and b

y contains only b



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: From Pumping Lemma: $uv^2xy^2z \in L$

However: $k_1 + k_2 + k \geq 1$

$$uv^2xy^2z = a^m b^{k_1} a^{k_2} b^{m+k} c^m \notin L$$

Contradiction!!!

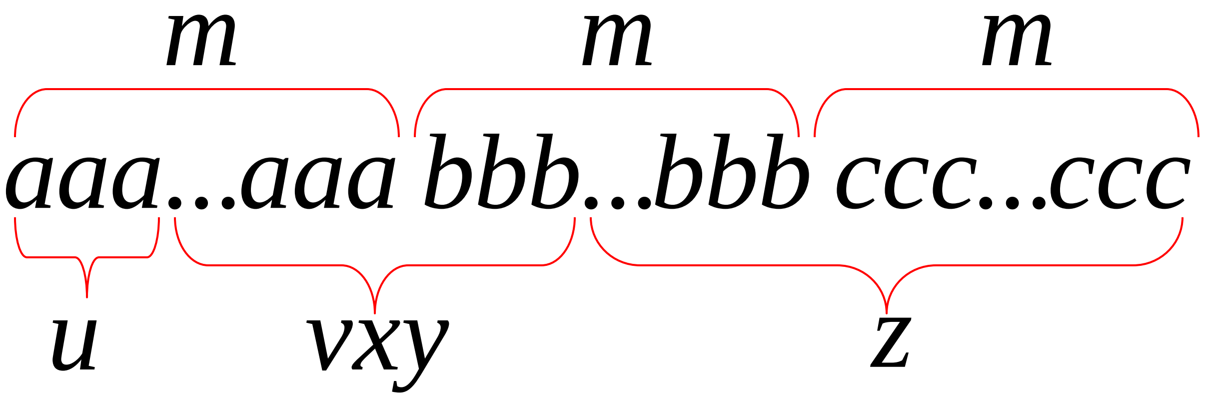
$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 4: Possibility 3: v contains only a

y contains a and b



$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$w = uvxyz$ $\qquad |vxy| \leq m$ $\qquad |vy| \geq 1$

Case 4: Possibility 3: v contains only a

y contains a and b

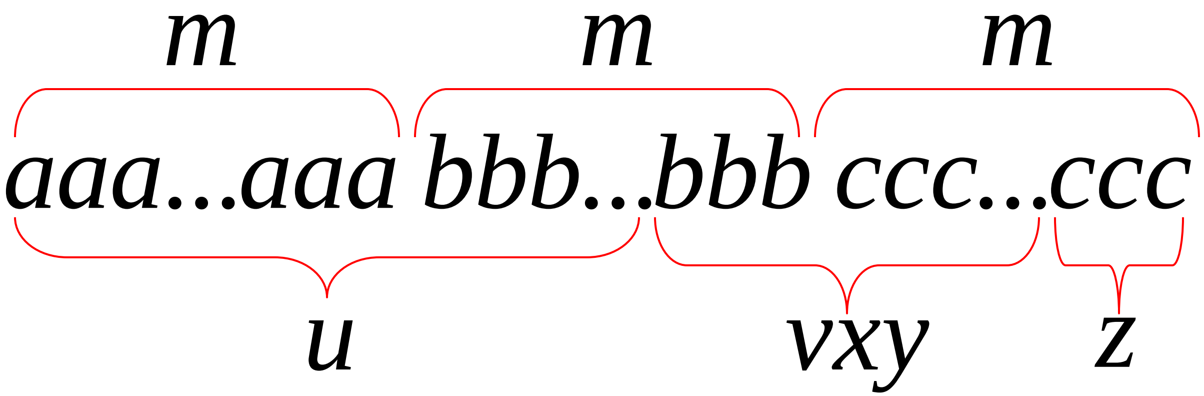
Similar analysis with Possibility 2

$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 5: vxy overlaps b^m and c^m

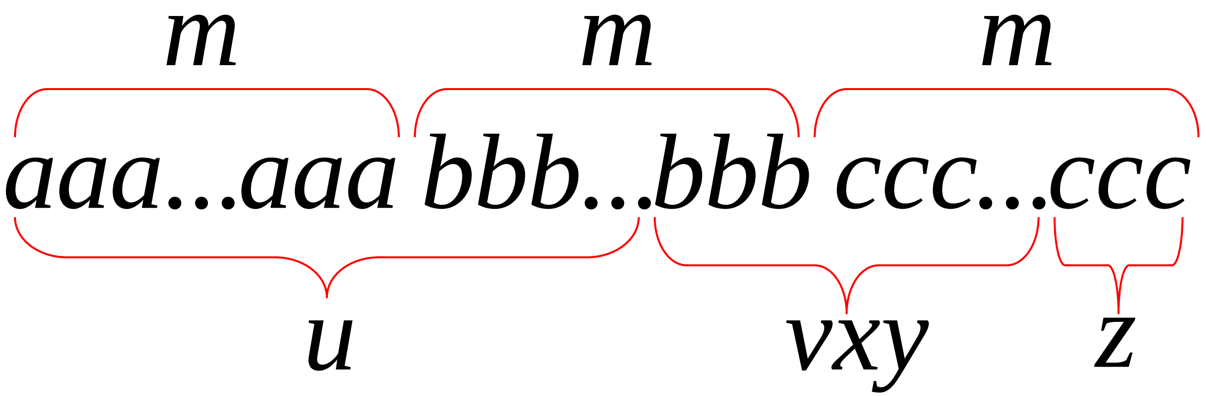


$$L = \{a^n b^n c^n : n \geq 0\}$$

$$w = a^m b^m c^m$$

$$w = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 5: Similar analysis with case 4



There are no other cases to consider

(since $|vxy| \leq m$, string vxy cannot

overlap a^m, b^m and c^m at the same time)

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{a^n b^n c^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free

The Pumping Lemma:

Reminder

For infinite context-free language L

there exists an integer m such that

for any string $w \in L, \quad |w| \geq m$

we can write $w = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

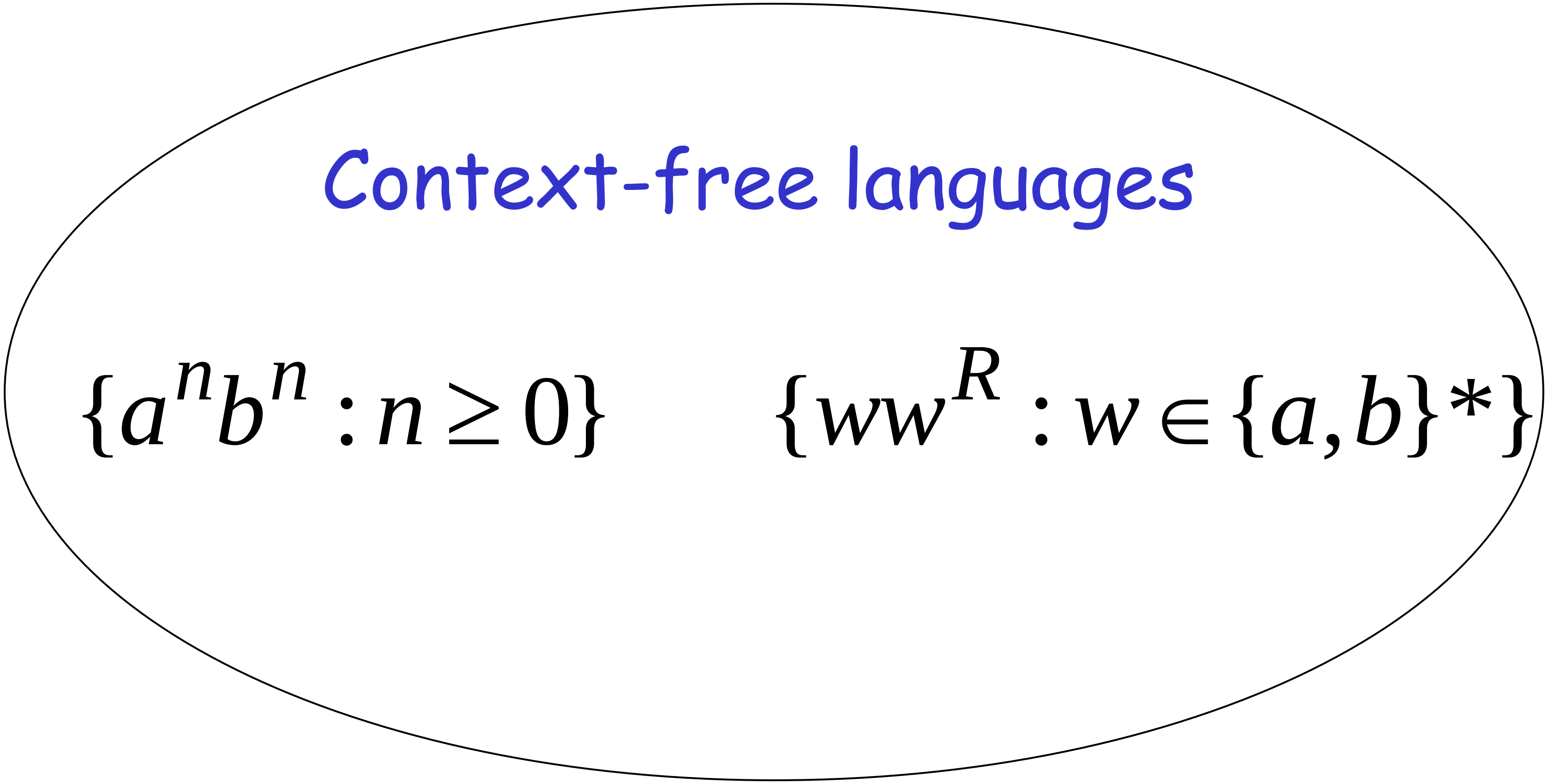
and it must be:

$$uv^i xy^i z \in L, \quad \text{for all } i \geq 0$$

Non-context free languages

$\{a^n b^n c^n : n \geq 0\}$

$\{vv : v \in \{a,b\}^*\}$



Context-free languages

$\{a^n b^n : n \geq 0\}$

$\{ww^R : w \in \{a,b\}^*\}$

Theorem: The language

$$L = \{vv : v \in \{a,b\}^*\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{vv : v \in \{a,b\}^*\}$$

Assume for contradiction that L is context-free

Pick any string of L with length at least m

we pick: $a^m b^m a^m b^m \in L$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

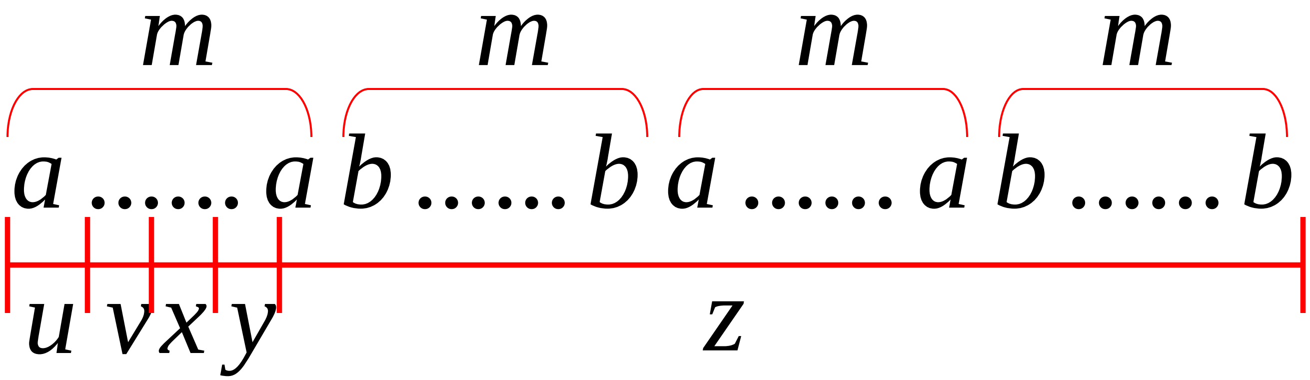
We examine **all** the possible locations of string vxy in $a^m b^m a^m b^m$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within the first a^m

$$v = a^{k_1} \qquad y = a^{k_2} \qquad k_1 + k_2 \geq 1$$



$$\overbrace{a \ldots\ldots a}^{m}\ \overbrace{b \ldots\ldots b}^{m}\ \overbrace{a \ldots\ldots a}^{m}\ \overbrace{b \ldots\ldots b}^{m}$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 1: vxy is within the first a^m

$$v = a^{k_1} \qquad y = a^{k_2} \qquad\qquad k_1 + k_2 \geq 1$$

$$\overbrace{a \ldots\ldots\ldots\ldots\ldots\ldots a}^{m+k_1+k_2}\ \overbrace{b \ldots\ldots b}^{m}\ \overbrace{a \ldots\ldots a}^{m}\ \overbrace{b \ldots\ldots b}^{m}$$

$$\underbrace{\qquad}_{u}\underbrace{\qquad\quad}_{v^2}\underbrace{\ }_{x}\underbrace{\qquad\quad}_{y^2}\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{z}$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 1: vxy is within the first a^m

$$a^{m+k_1+k_2} b^m a^m b^m = uv^2xy^2z \quad \notin L$$

Contradiction!!! Since $k_1 + k_2 \ge 1$

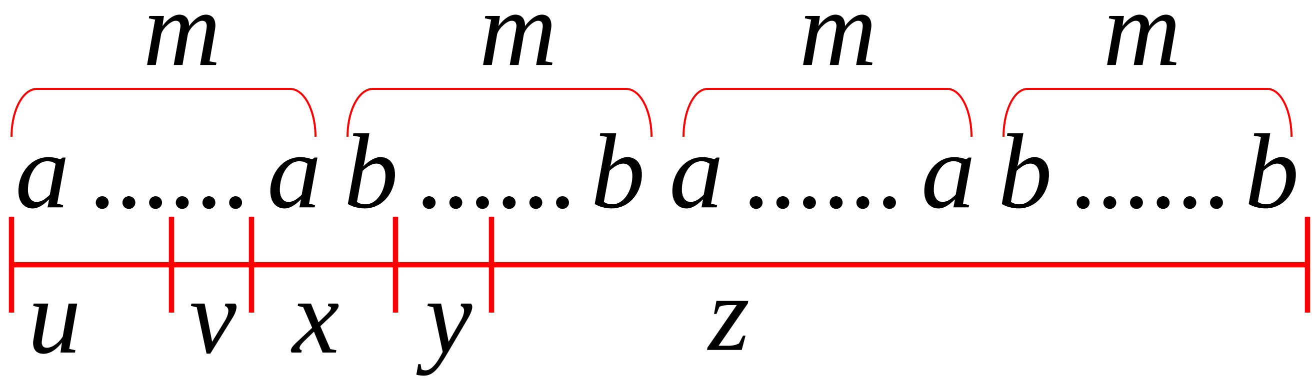
$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 2: v is in the first a^m

y is in the first b^m

$$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \ge 1$$



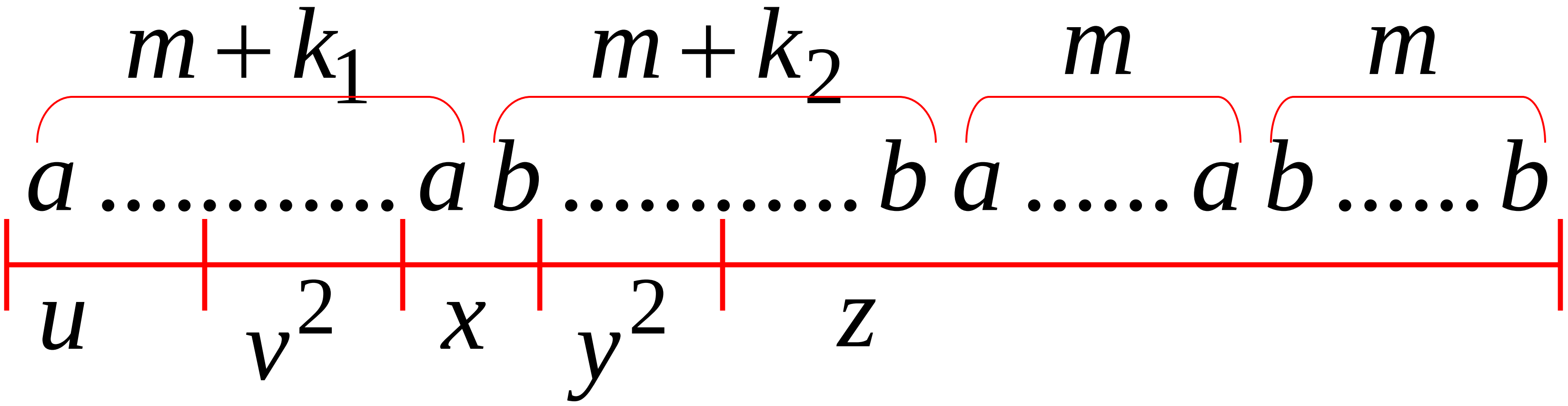
$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \le m \qquad |vy| \ge 1$$

Case 2: v is in the first a^m

y is in the first b^m

$$v = a^{k_1} \qquad y = b^{k_2} \qquad k_1 + k_2 \ge 1$$



$$\overbrace{a \ldots\ldots\ldots\ldots a}^{m+k_1}\,\overbrace{b \ldots\ldots\ldots\ldots b}^{m+k_2}\,\overbrace{a \ldots\ldots a}^{m}\,\overbrace{b \ldots\ldots b}^{m}$$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 2: v is in the first a^m

y is in the first b^m

$$a^{m+k_1} b^{m+k_2} a^m b^m = uv^2xy^2z \quad \notin L$$

Contradiction!!! **Since** $k_1 + k_2 \geq 1$

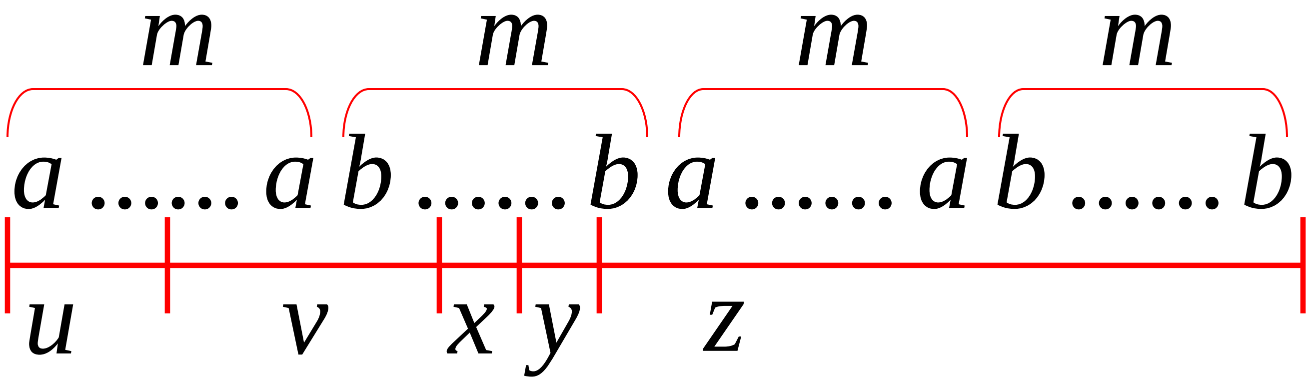
$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: v overlaps the first $a^m b^m$

y is in the first b^m

$$v = a^{k_1} b^{k_2} \qquad y = b^{k_3} \qquad k_1, k_2 \geq 1$$



$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: v overlaps the first $a^m b^m$

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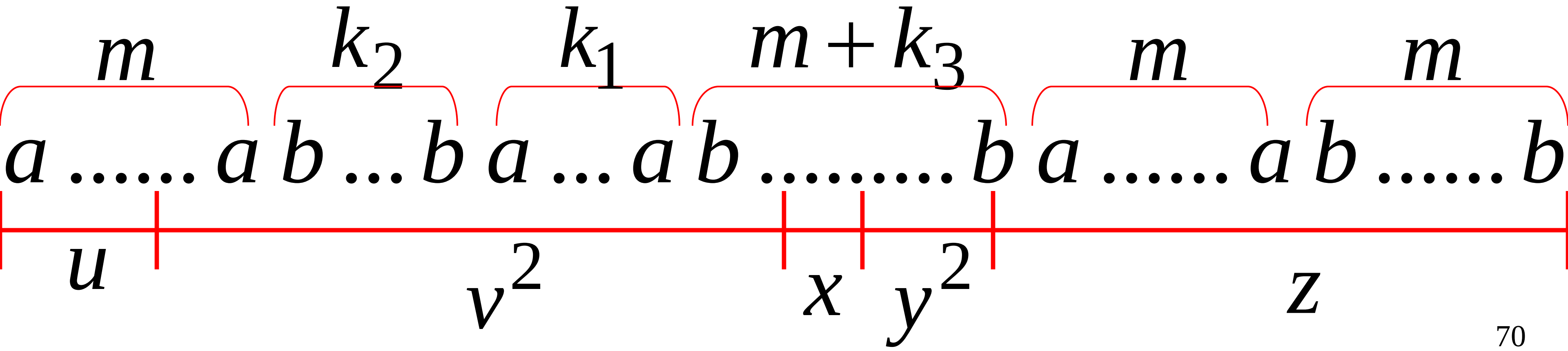
$$v = a^{k_1} b^{k_2} \qquad y = b^{k_3} \qquad k_1, k_2 \geq 1$$



$$\overbrace{a \ldots\ldots a}^{m}\,\overbrace{b \ldots b}^{k_2}\,\overbrace{a \ldots a}^{k_1}\,\overbrace{b \ldots\ldots\ldots b}^{m+k_3}\,\overbrace{a \ldots\ldots a}^{m}\,\overbrace{b \ldots\ldots b}^{m}$$

$u \qquad v^2 \qquad x \qquad y^2 \qquad z$

$$L = \{vv : v \in \{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Case 3: v overlaps the first $a^m b^m$

y is in the first b^m

$$a^m b^{k_2} a^{k_1} b^{m+k_3} a^m b^m = uv^2xy^2z \quad \notin L$$

Contradiction!!!

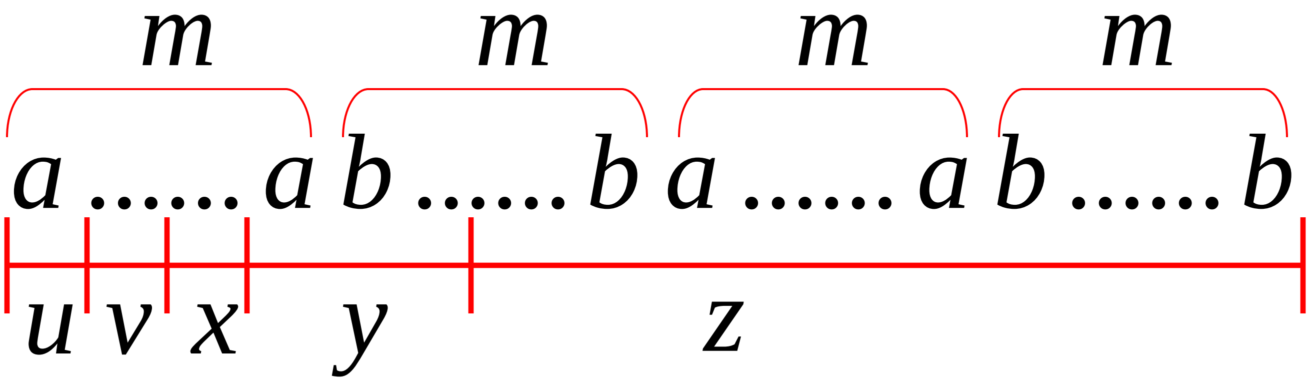
$$L=\{vv : v\in\{a,b\}^*\}$$

$$a^m b^m a^m b^m = uvxyz \qquad |vxy|\le m \qquad |vy|\ge 1$$

Case 4: v in the first a^m

y Overlaps the first $a^m b^m$

Analysis is similar to case 3



Other cases: vxy is within $a^m\boxed{b^m}a^mb^m$

or

$$a^mb^m\boxed{a^m}b^m$$

or

$$a^mb^ma^m\boxed{b^m}$$

Analysis is similar to case 1:

$$\boxed{a^m}b^ma^mb^m$$

More cases:

vxy overlaps $a^m\boxed{b^m a^m}b^m$

or

$a^m b^m\boxed{a^m b^m}$

Analysis is similar to cases 2,3,4:

$$\boxed{a^m b^m}a^m b^m$$

There are no other cases to consider

Since $|vxy| \leq m$, it is impossible vxy to overlap:

$$\boxed{a^m b^m a^m} b^m$$

nor

$$a^m \boxed{b^m a^m b^m}$$

nor

$$\boxed{a^m b^m a^m b^m}$$

In all cases we obtained a contradiction

Therefore: The original assumption that

$$L = \{vv : v \in \{a,b\}^*\}$$

is context-free must be wrong

Conclusion: L is not context-free

Theorem: The language

$$L = \{a^{n^2} b^n : n \geq 0\}$$

is **not** context free

Proof: Use the Pumping Lemma for context-free languages

$$L = \{a^{n^2} b^n : n \geq 0\}$$

Assume for contradiction that L

is context-free

Pick any string of L with length at least m

we pick: $a^{m^2} b^m \in L$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

We can write: $a^{m^2} b^m = uvxyz$

with lengths $|vxy| \leq m$ and $|vy| \geq 1$

Pumping Lemma says:

$uv^i xy^i z \in L$ for all $i \geq 0$

We examine **<u>all</u>** the possible locations of string vxy in $a^{m^2} b^m$

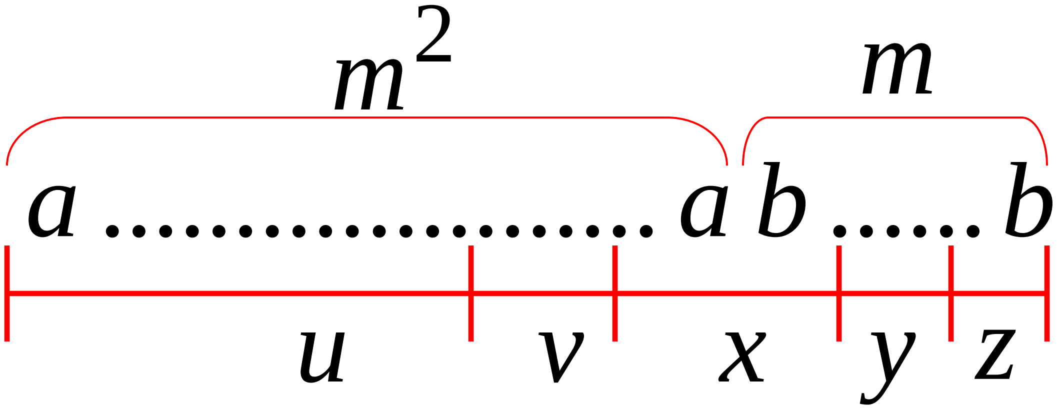
$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated case: v is in a^m

y is in b^m

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

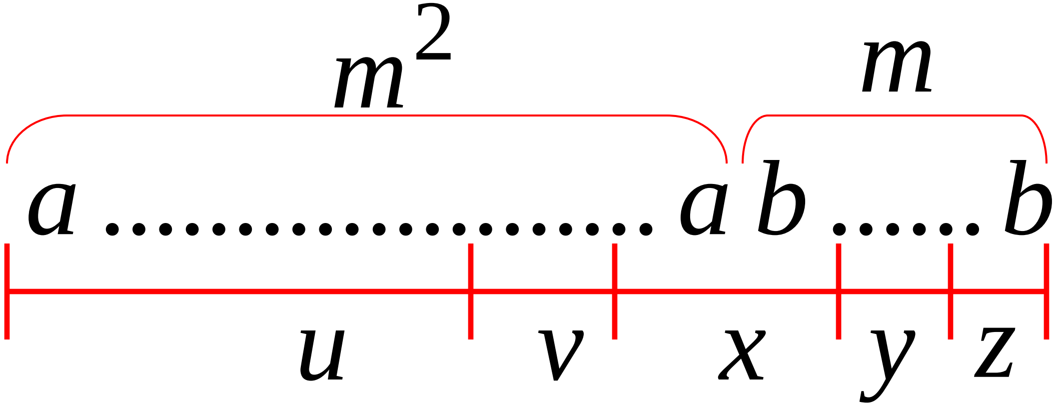


$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$



$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

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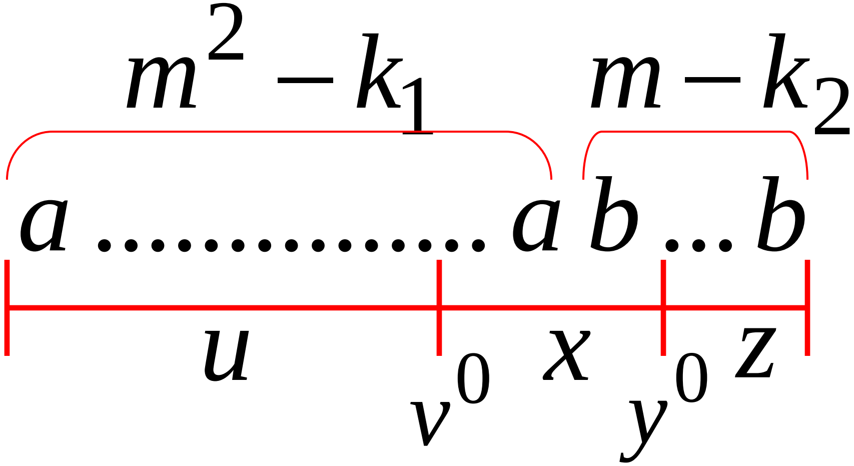

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

Most complicated sub-case: $k_1 \neq 0$ and $k_2 \neq 0$

$$v = a^{k_1} \qquad y = b^{k_2} \qquad 1 \leq k_1 + k_2 \leq m$$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z$$

Question: $m^2 - k_1 = (m - k_2)^2$?

$k_1 \neq 0$ and $k_2 \neq 0$ $\qquad$ $1 \leq k_1 + k_2 \leq m$

$$\Downarrow$$

$$\begin{aligned}(m - k_2)^2 &\leq (m-1)^2 \\ &\leq m^2 - 2m + 1 \\ &< m^2 - k_1\end{aligned}$$

$$\Downarrow$$

$$m^2 - k_1 \neq (m - k_2)^2$$

$$L = \{a^{n^2} b^n : n \geq 0\}$$

$$a^{m^2} b^m = uvxyz \qquad |vxy| \leq m \qquad |vy| \geq 1$$

$$m^2 - k_1 \neq (m - k_2)^2$$

$$\Downarrow$$

$$a^{m^2 - k_1} b^{m - k_2} = uv^0 xy^0 z \quad \notin L$$

Contradiction!!!

After examining all cases, we will obtain a contradiction

Therefore: The original assumption that

$$L = \{a^{n^2} b^n : n \geq 0\}$$

is context-free must be wrong

Conclusion: L is not context-free