Backpropagation on the loss function

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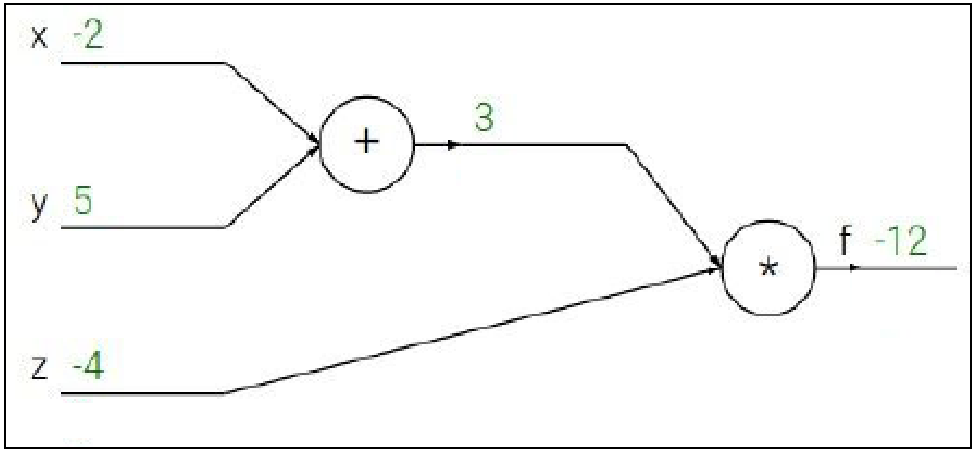
(Slides are adapted from cs231n @Stanford University)

Backpropagation(Revisited)

Backpropagation: a simple example

$$f(x, y, z) = (x + y)z$$

e.g. x = -2, y = 5, z = -4



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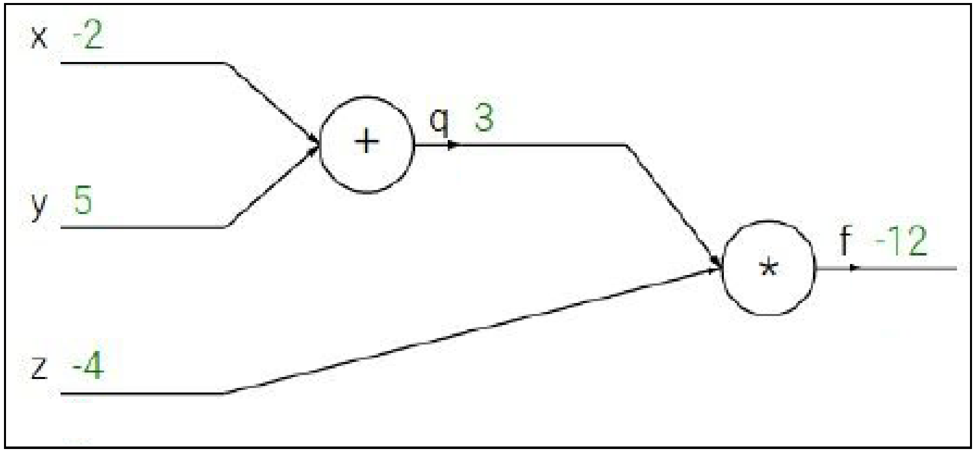
Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



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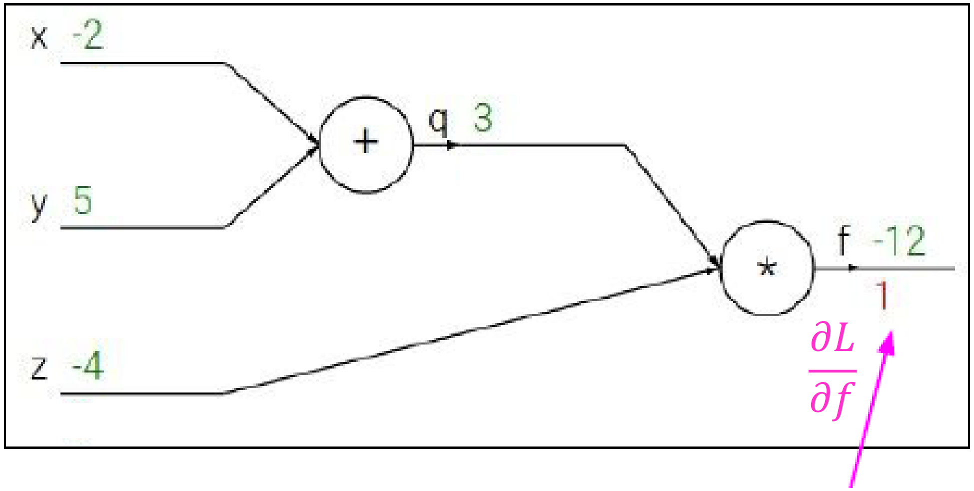

Suppose gradient of Loss wrt f equals to 1

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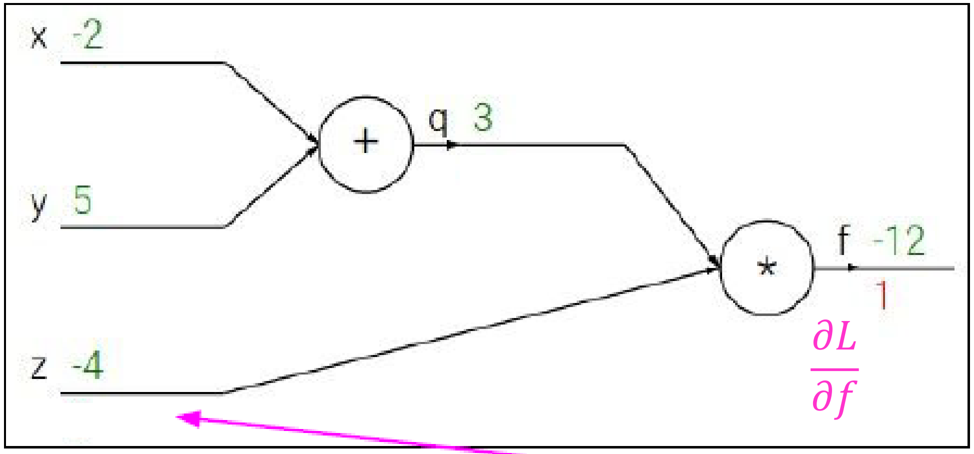

$$\frac{\partial L}{\partial z} = \frac{\partial L}{\partial f} \cdot \frac{\partial f}{\partial z}$$

Chain rule:

$$\boxed{\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q} \frac{\partial q}{\partial y}}$$

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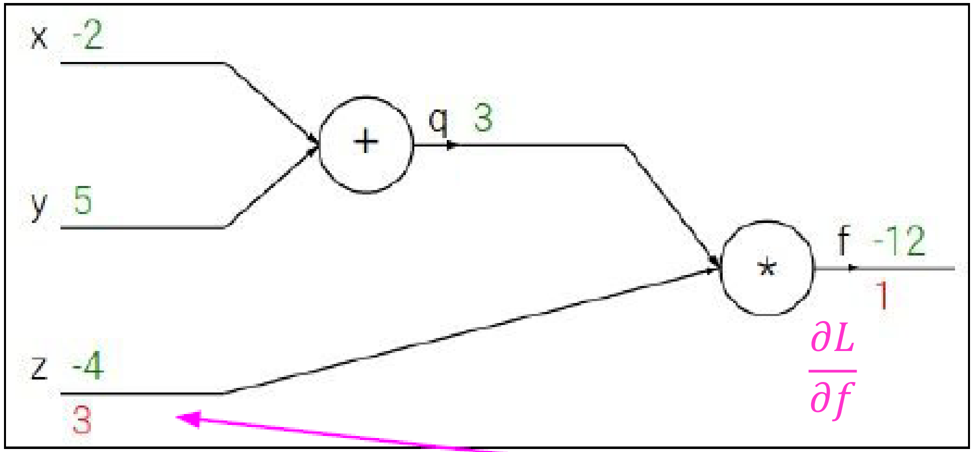

$$\frac{\partial L}{\partial z} = 1.\frac{\partial f}{\partial z}$$

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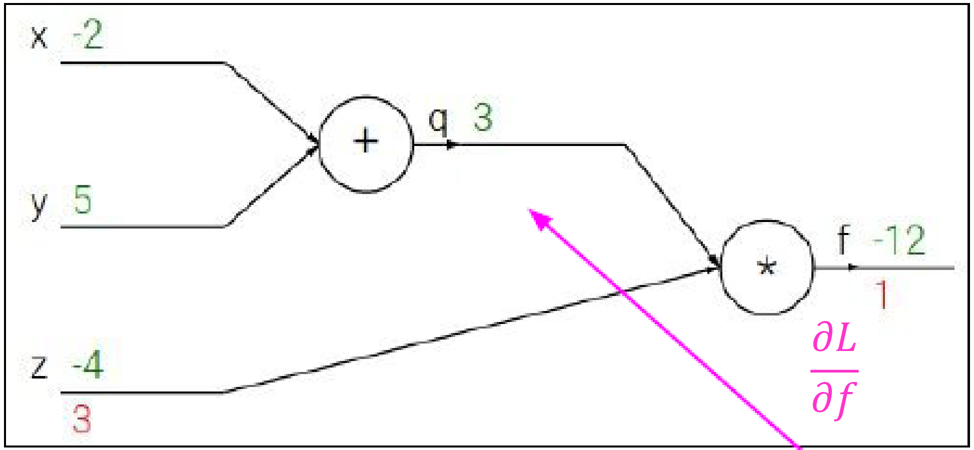

$\frac{\partial L}{\partial f}$

$$\frac{\partial L}{\partial q} = \frac{\partial L}{\partial f} \cdot \frac{\partial f}{\partial q}$$

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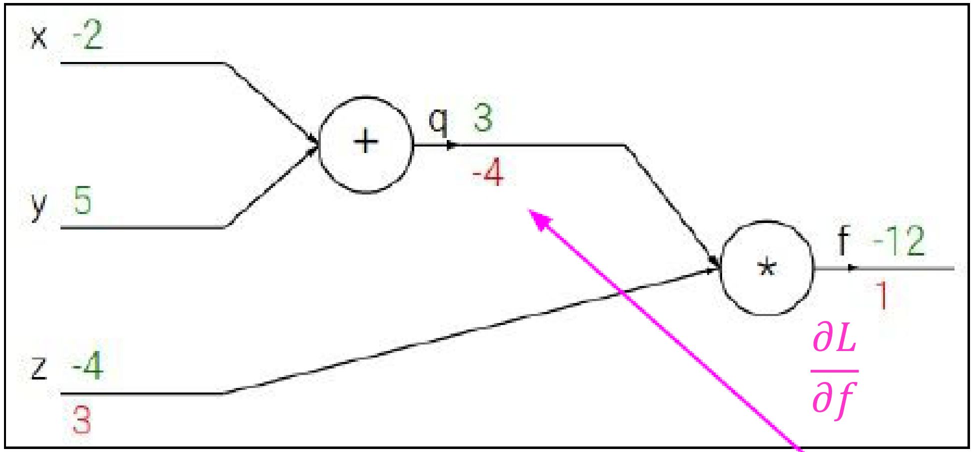

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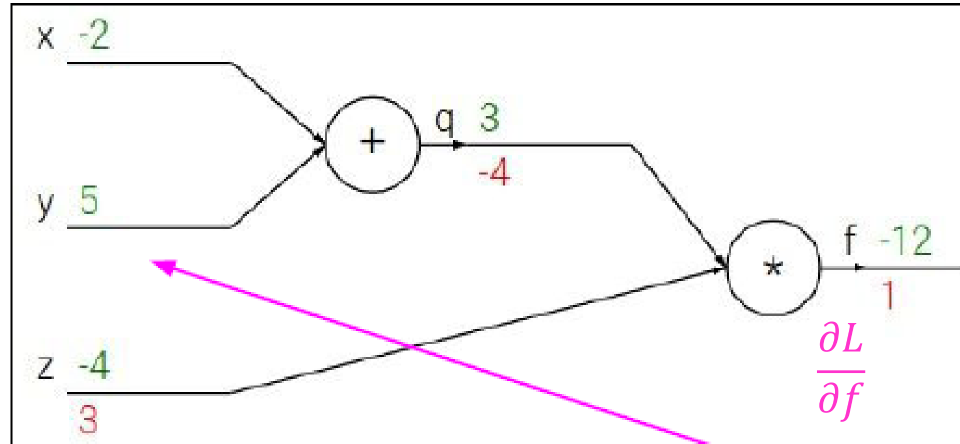

$$\frac{\partial L}{\partial y} = \frac{\partial L}{\partial q} . \frac{\partial q}{\partial y}$$

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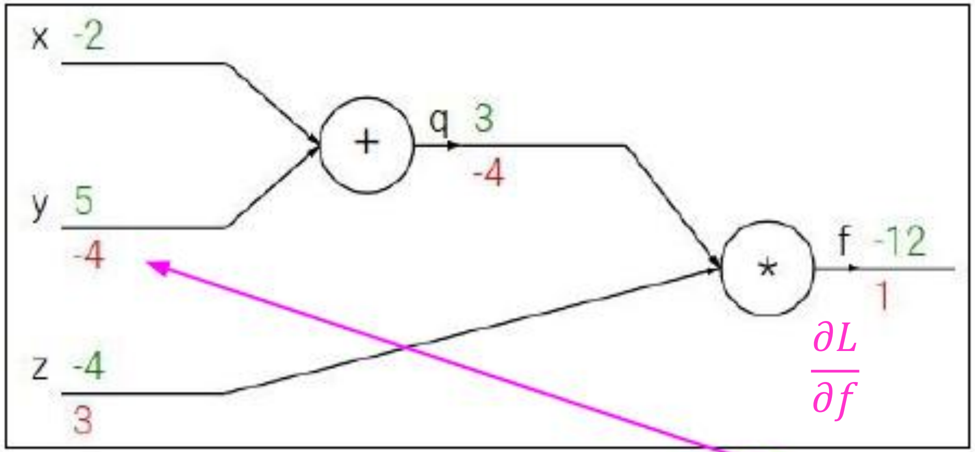

$$\frac{\partial L}{\partial y} = -4.\frac{\partial q}{\partial y}$$

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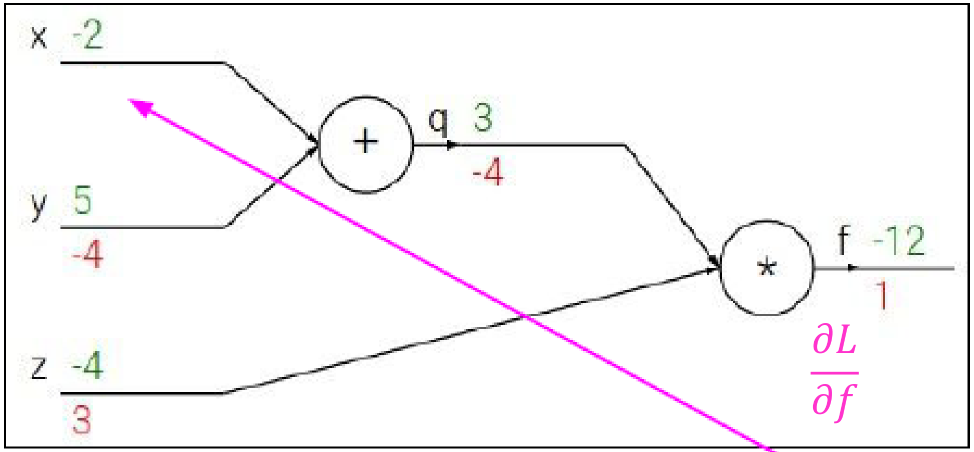

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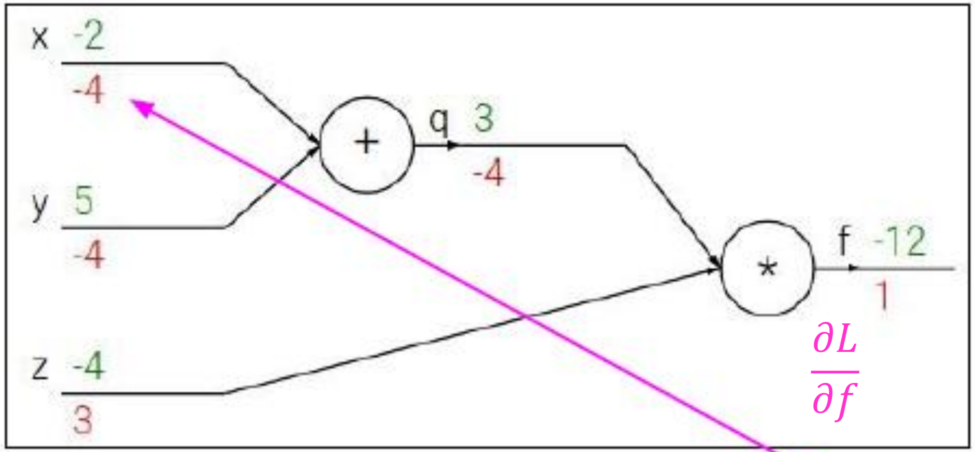

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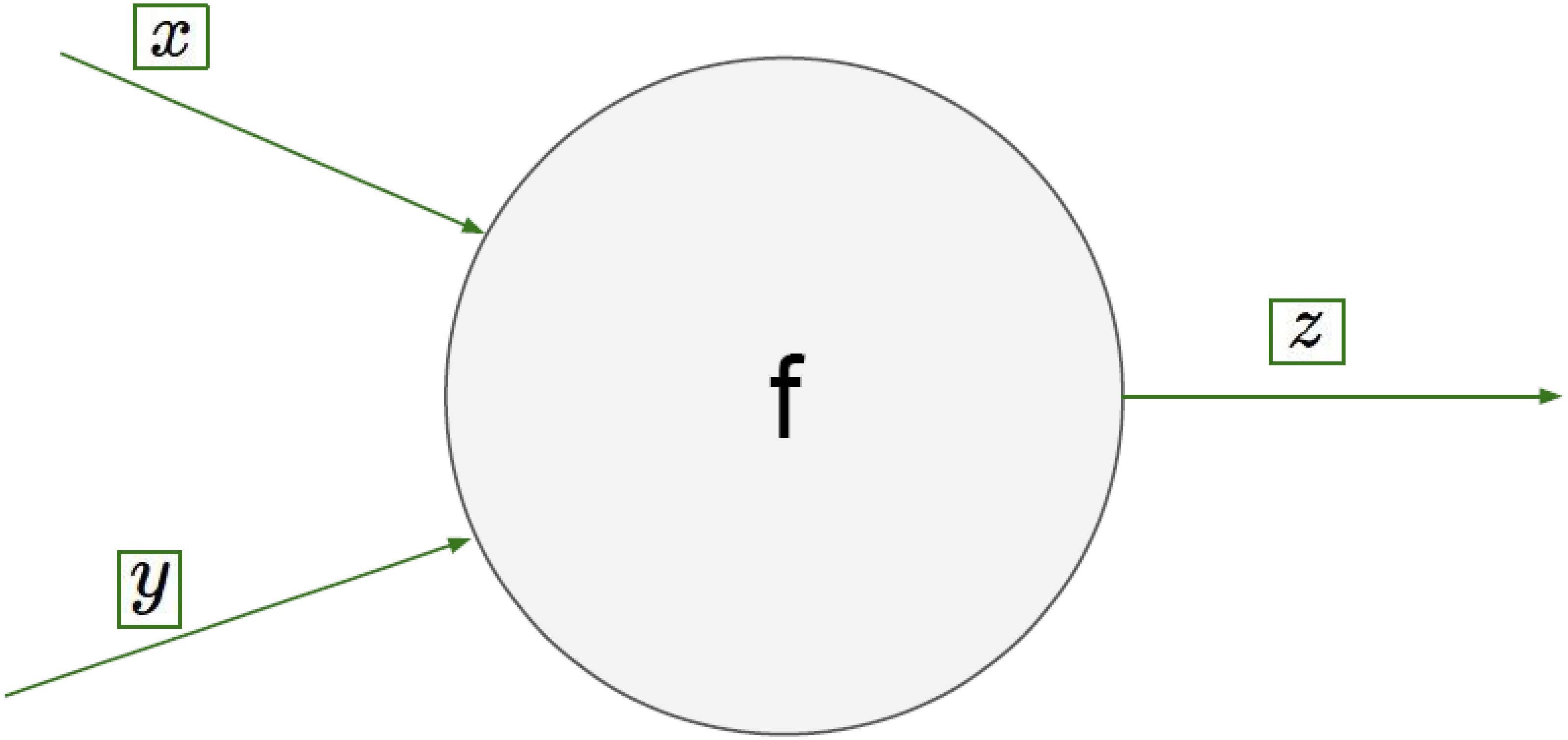
Backpropagation of gradient in general



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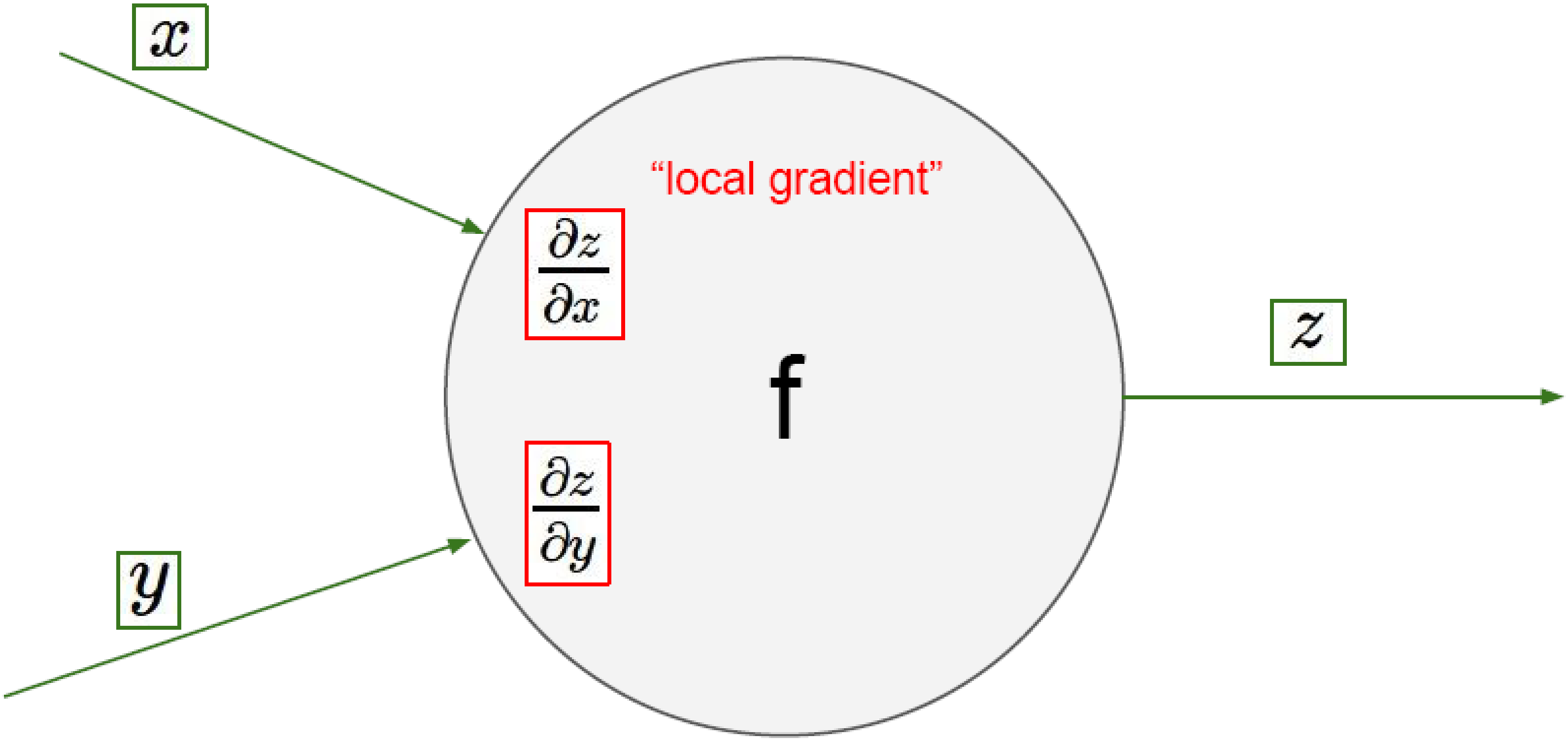

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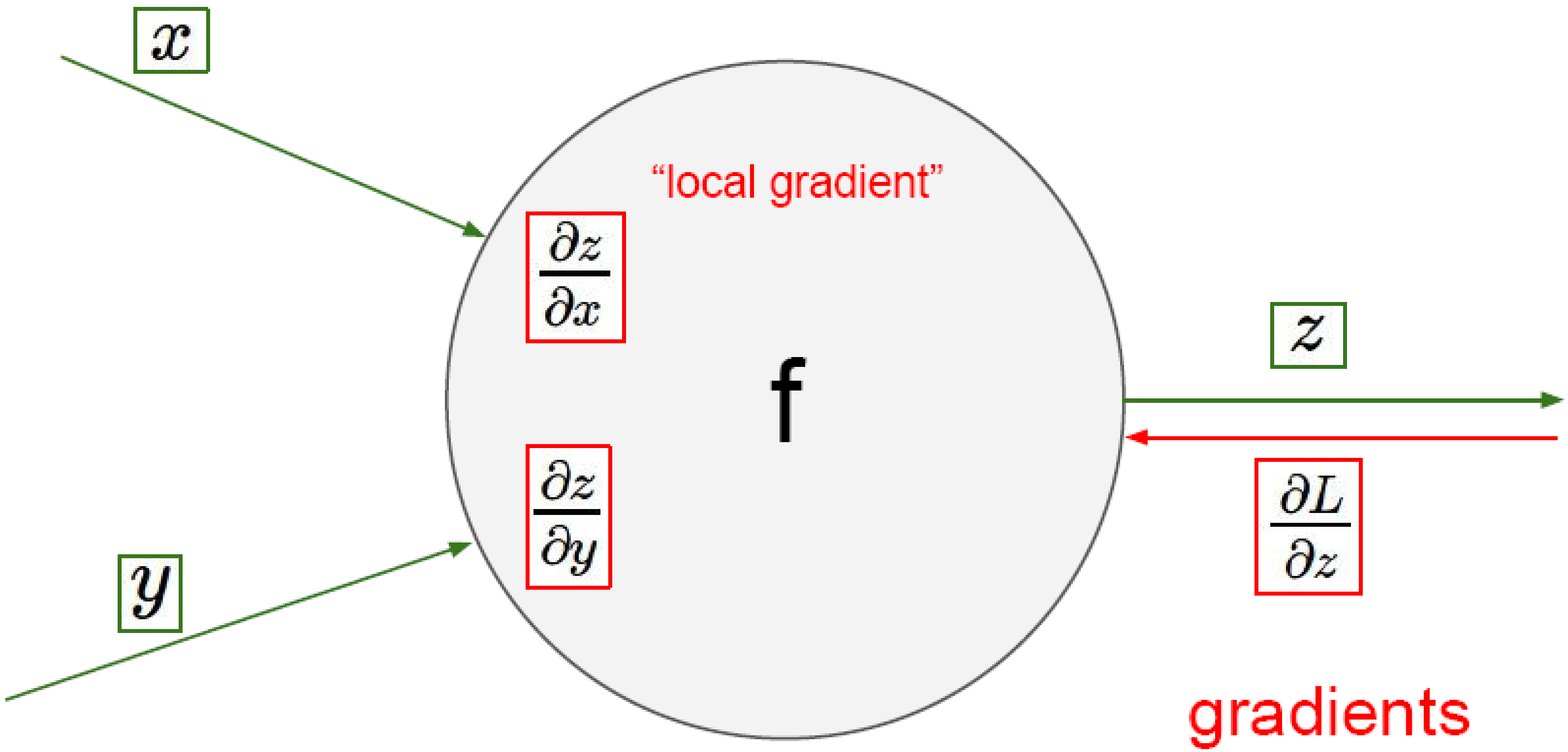

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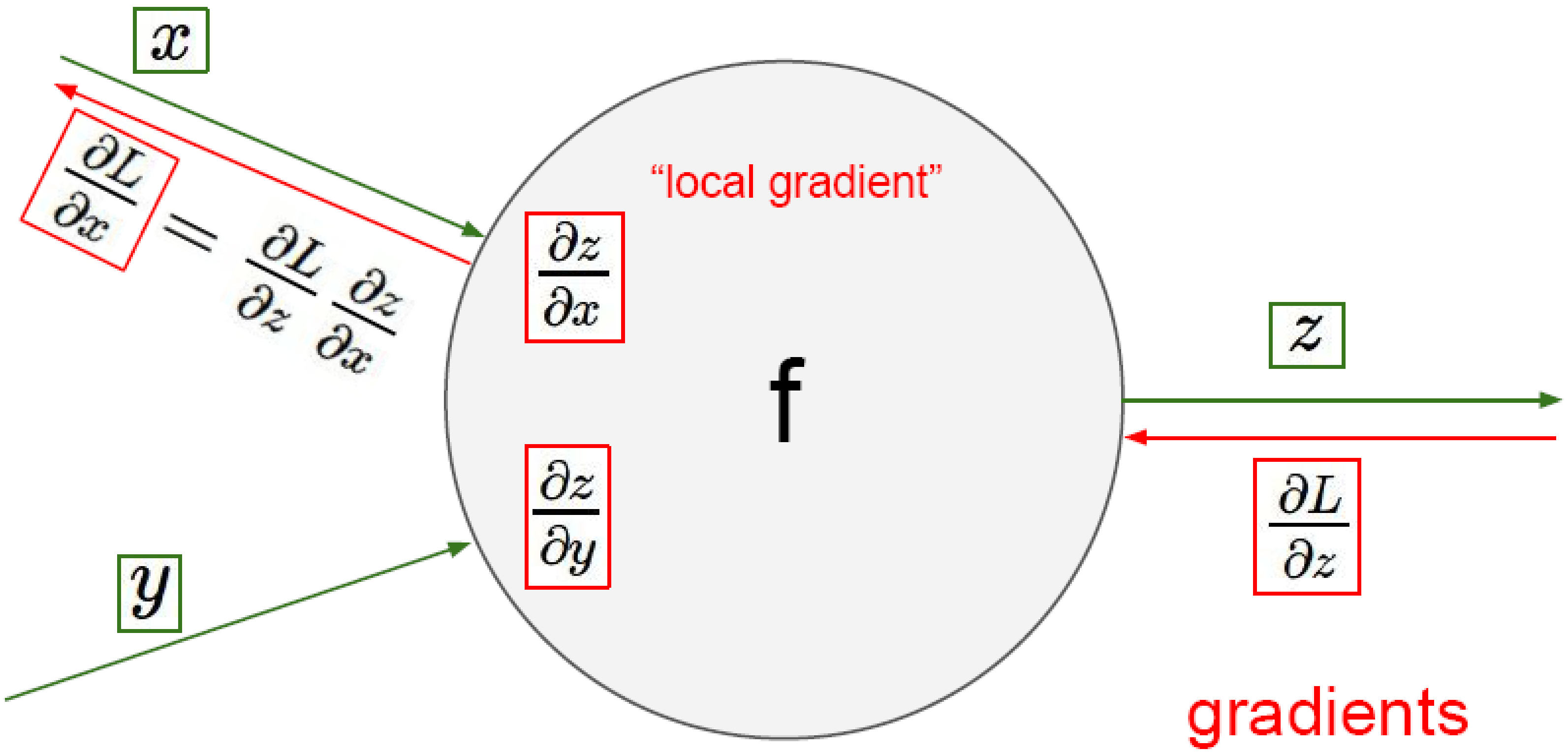

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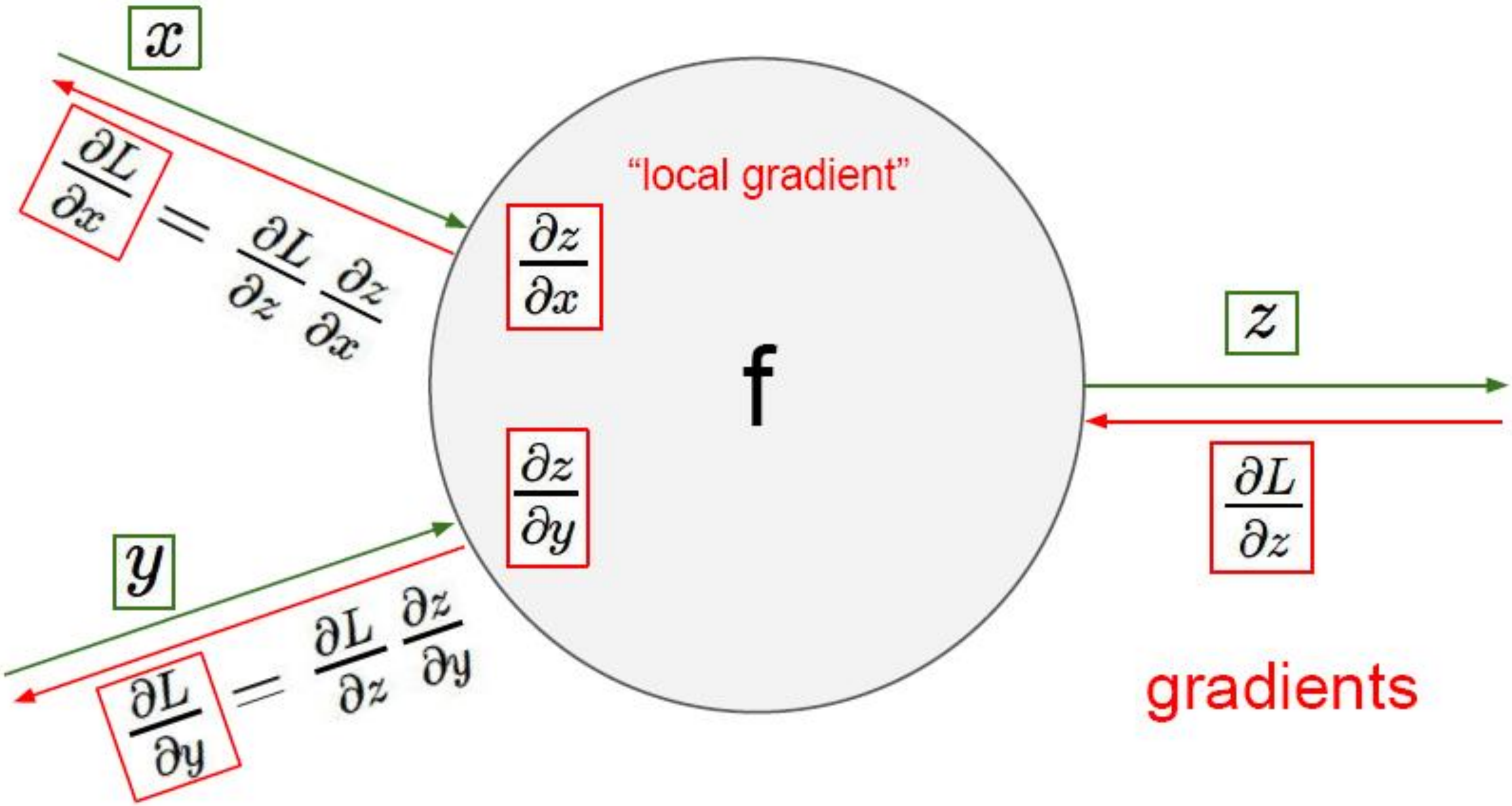
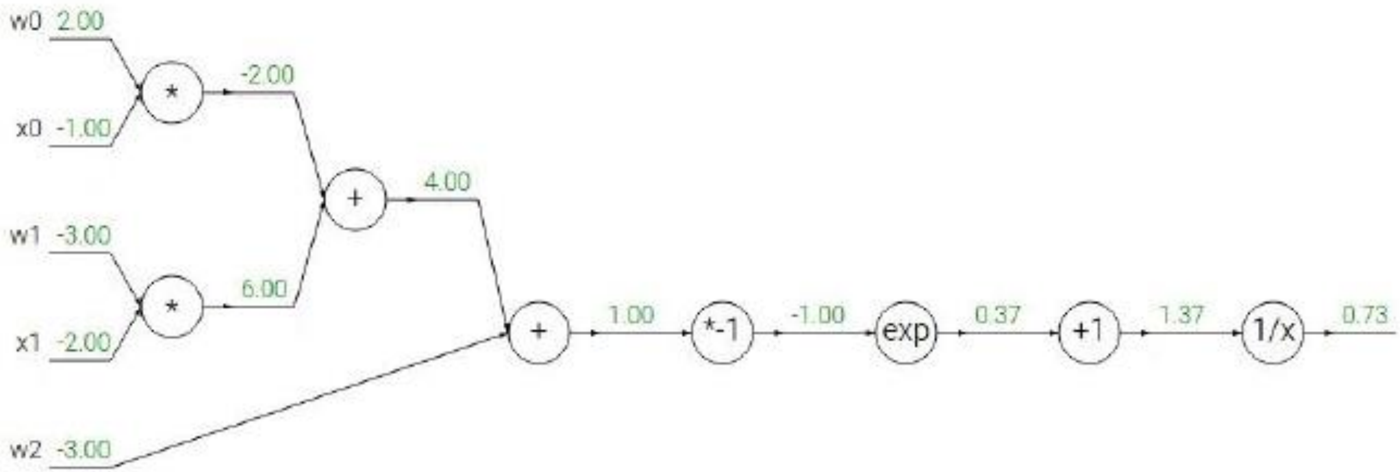
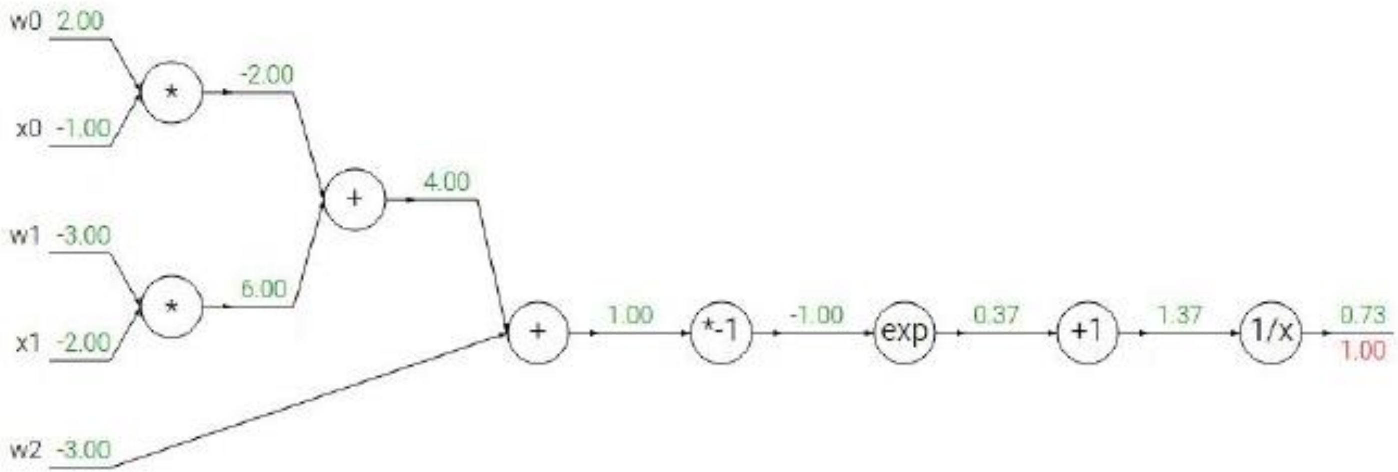


Another example:

$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$



Another example: $f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$



$f(x) = e^x$	$\rightarrow$	$\frac{df}{dx} = e^x$	$f(x) = \frac{1}{x}$	$\rightarrow$	$\frac{df}{dx} = -1/x^2$
$f_a(x) = ax$	$\rightarrow$	$\frac{df}{dx} = a$	$f_c(x) = c + x$	$\rightarrow$	$\frac{df}{dx} = 1$

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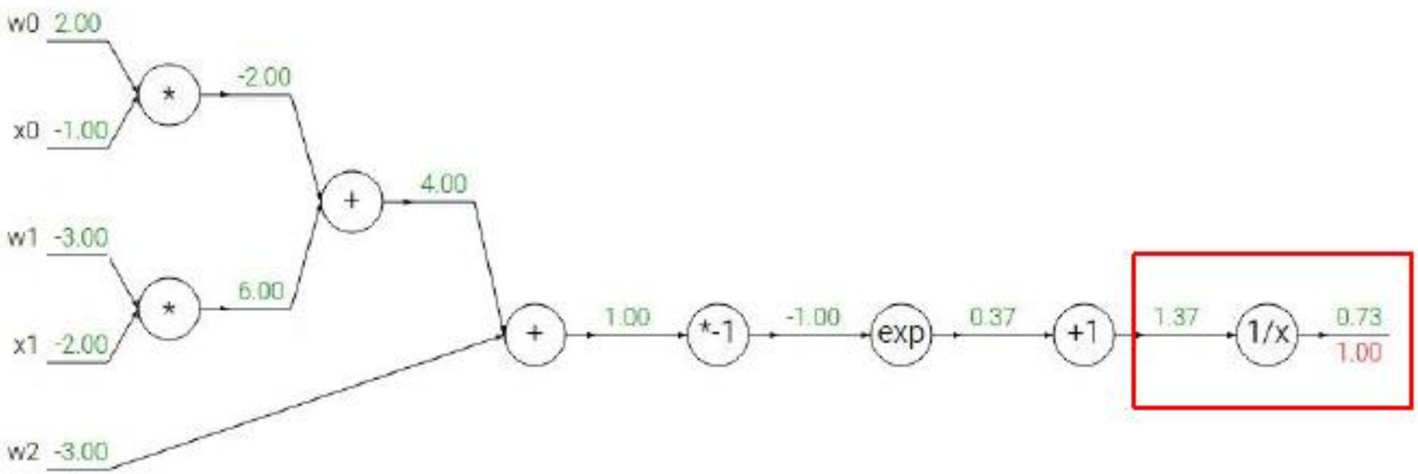

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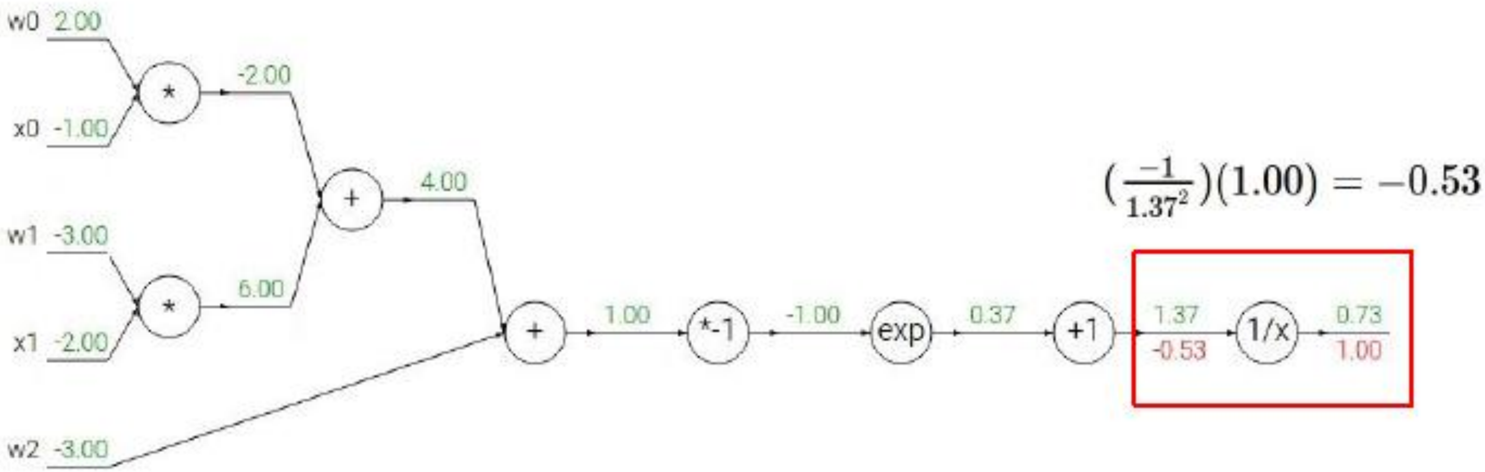

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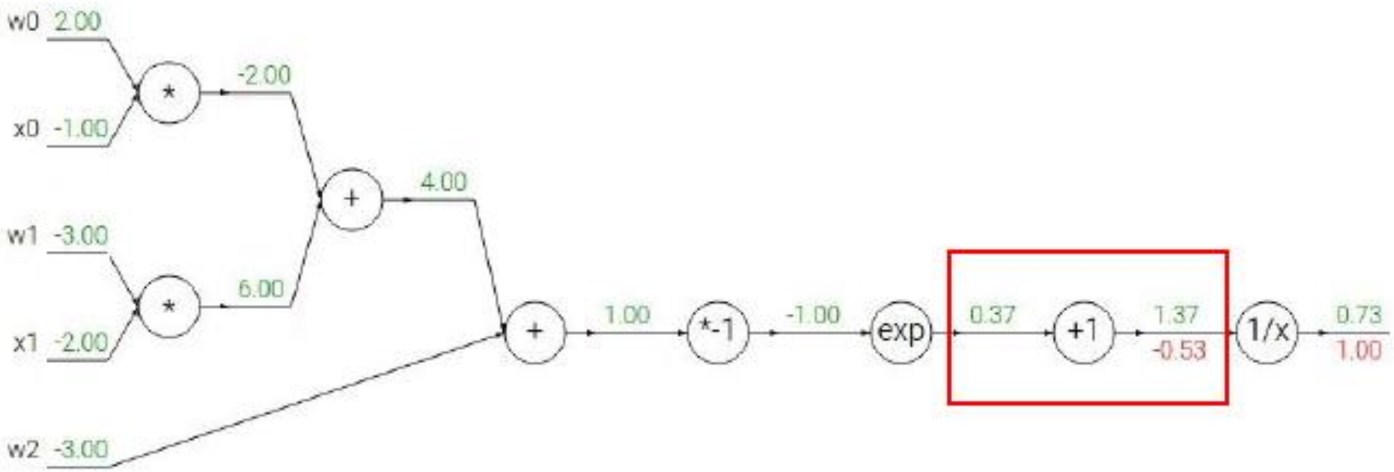

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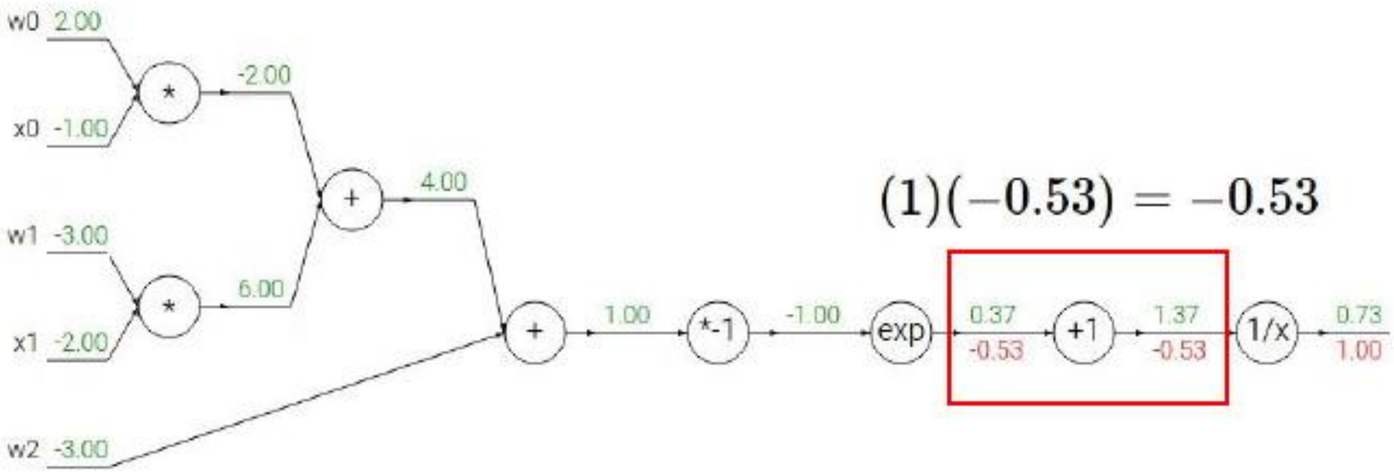
$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

$$(1)(-0.53) = -0.53$$



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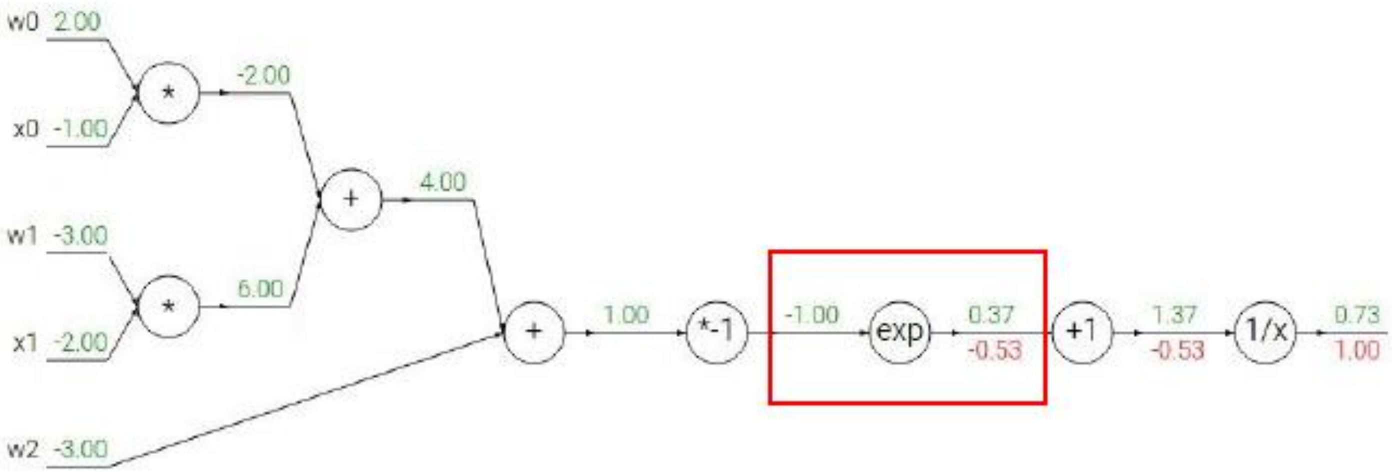

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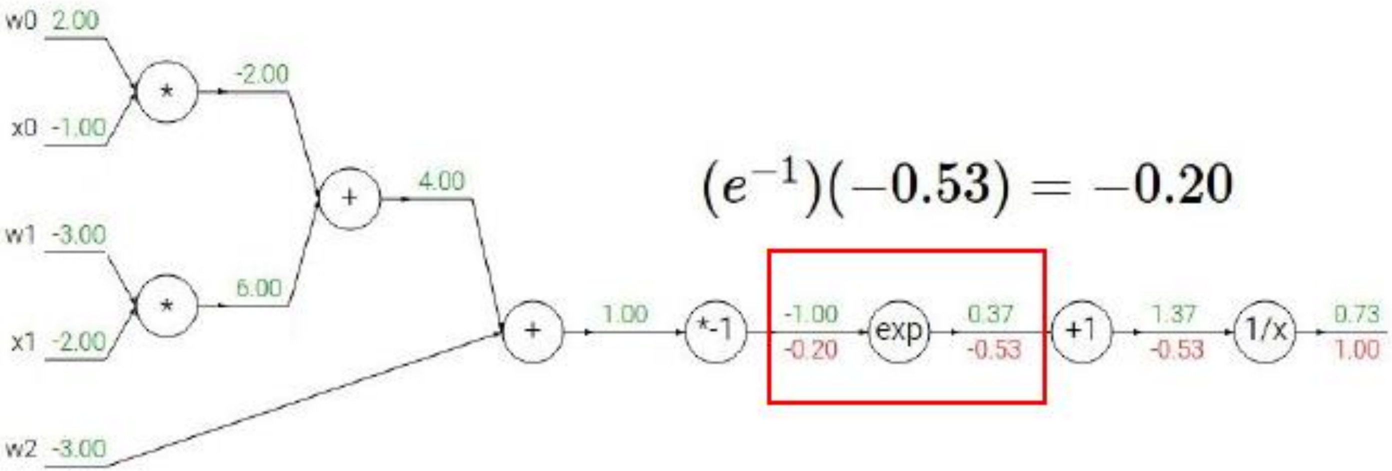

$$(e^{-1})(-0.53) = -0.20$$

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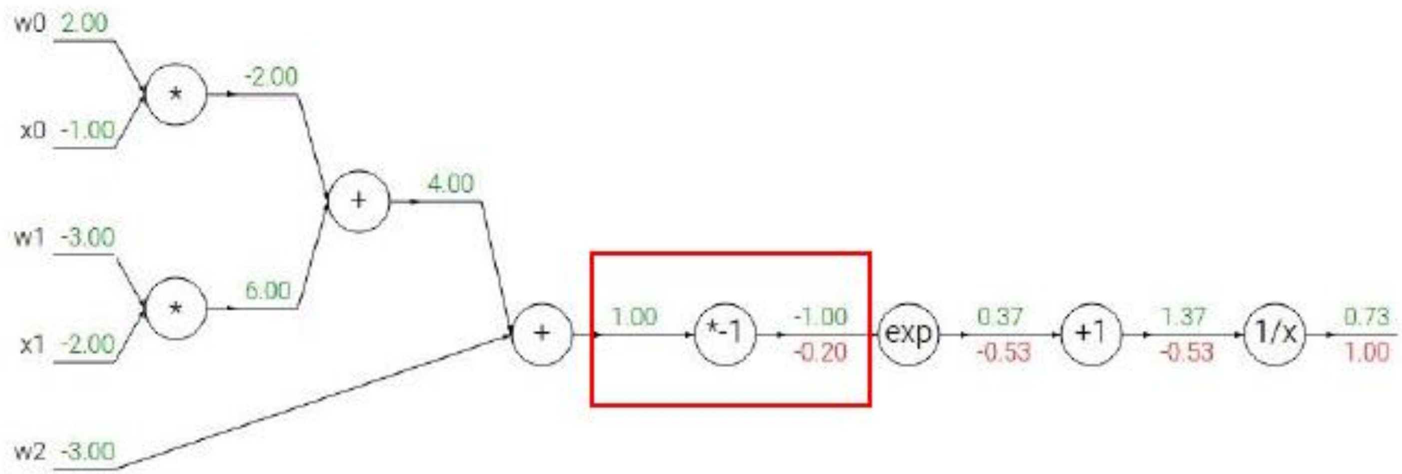

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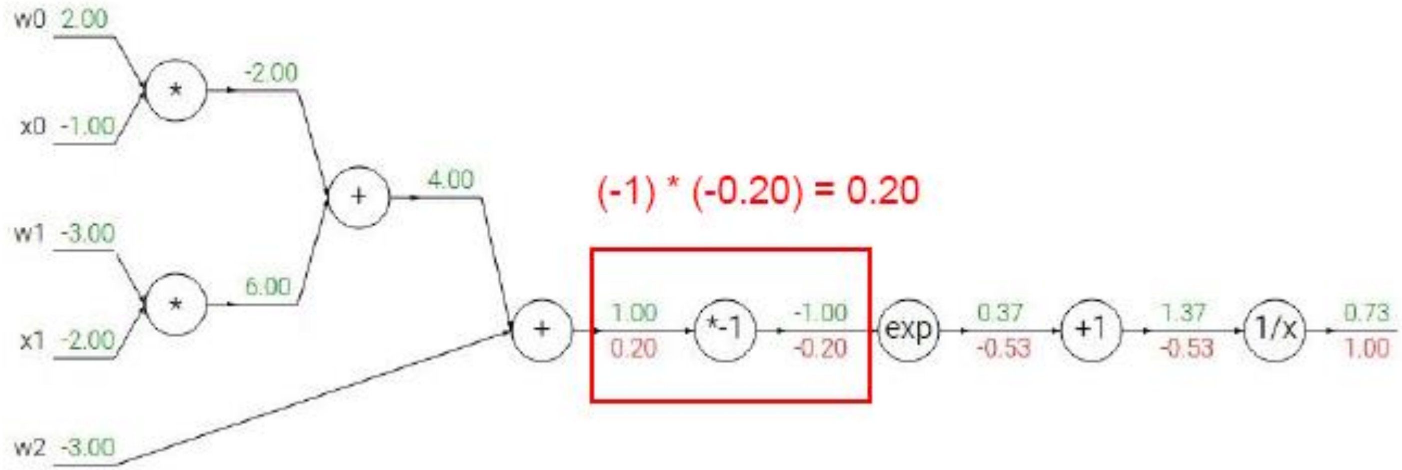

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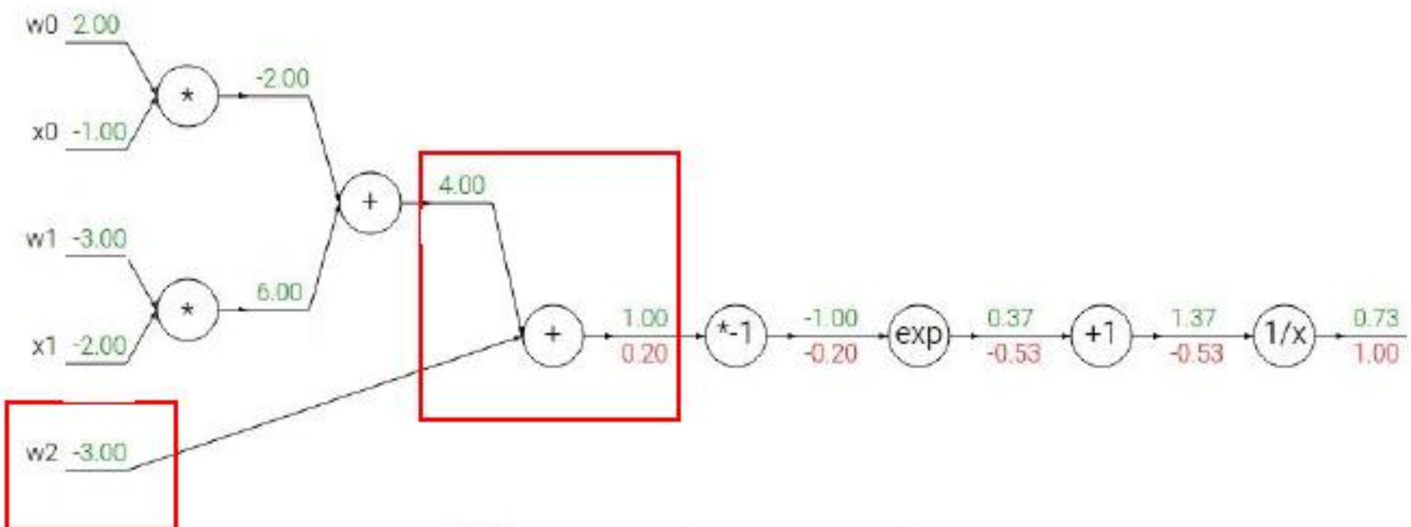

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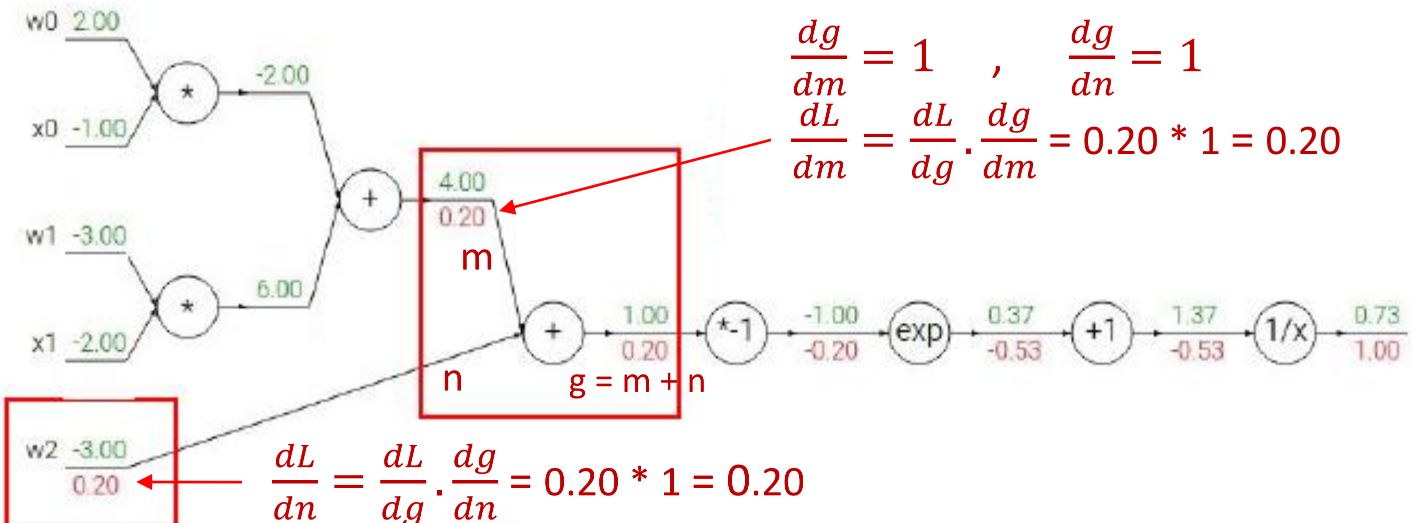

$$\frac{dg}{dm} = 1 \quad , \quad \frac{dg}{dn} = 1$$

$$\frac{dL}{dm} = \frac{dL}{dg} . \frac{dg}{dm} = 0.20 * 1 = 0.20$$

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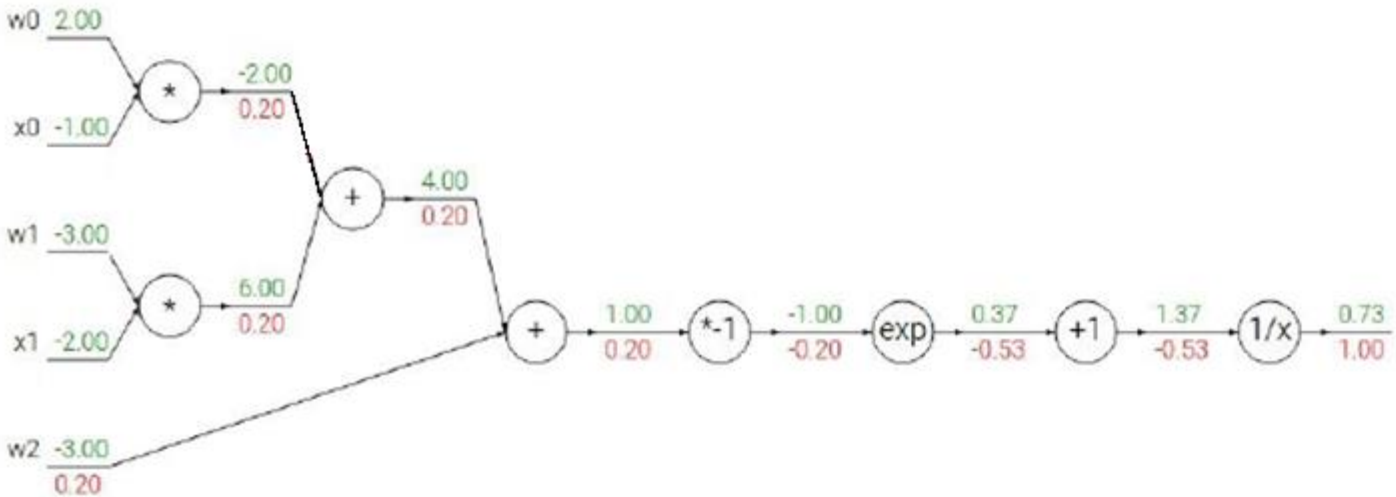

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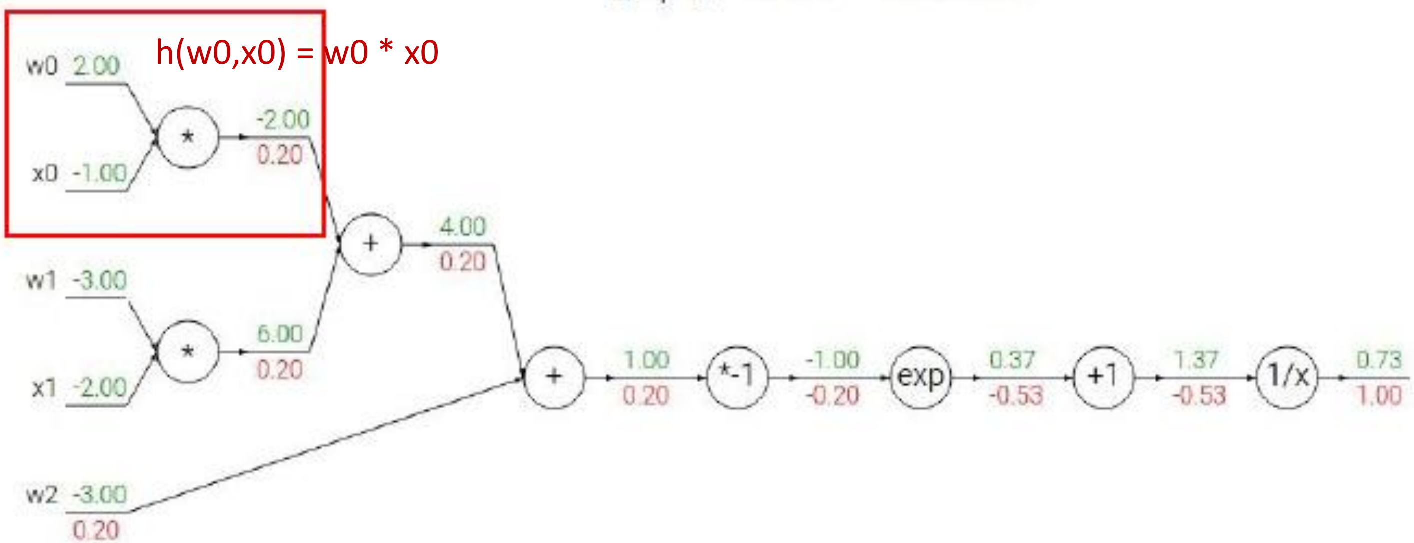

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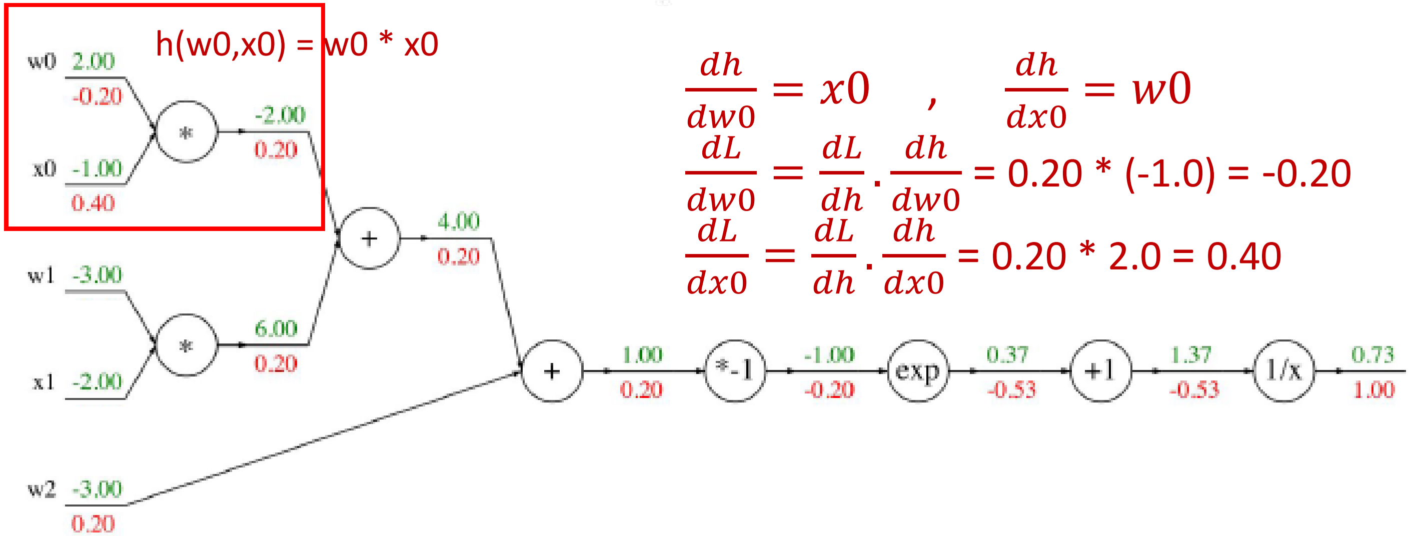

$$\frac{dh}{dw0} = x0 \quad , \quad \frac{dh}{dx0} = w0$$

$\frac{dL}{dw0} = \frac{dL}{dh} \cdot \frac{dh}{dw0}$ = 0.20 * (-1.0) = -0.20

$\frac{dL}{dx0} = \frac{dL}{dh} \cdot \frac{dh}{dx0}$ = 0.20 * 2.0 = 0.40

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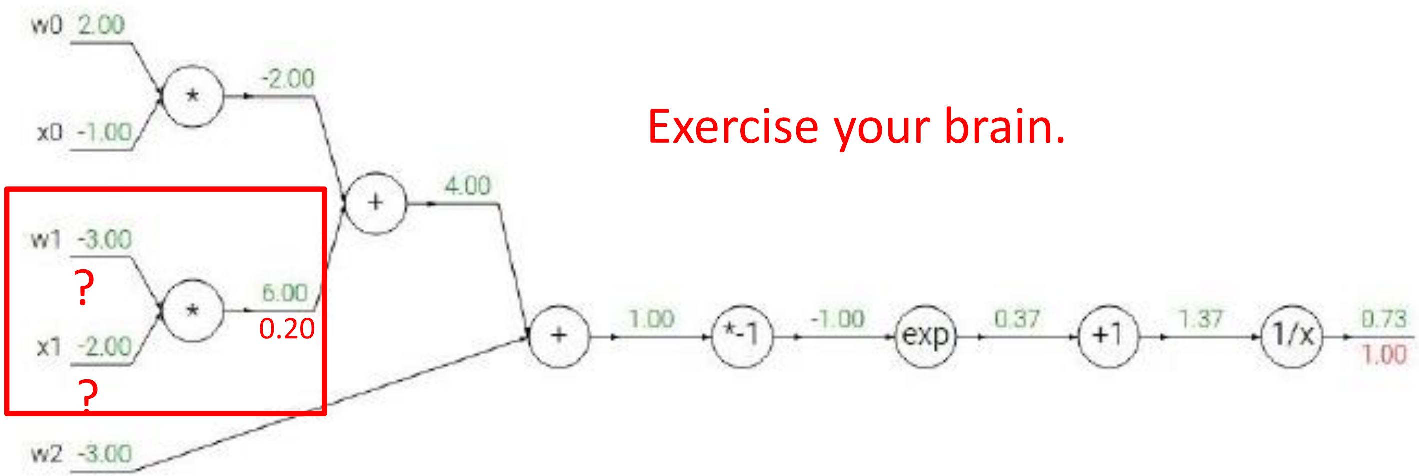

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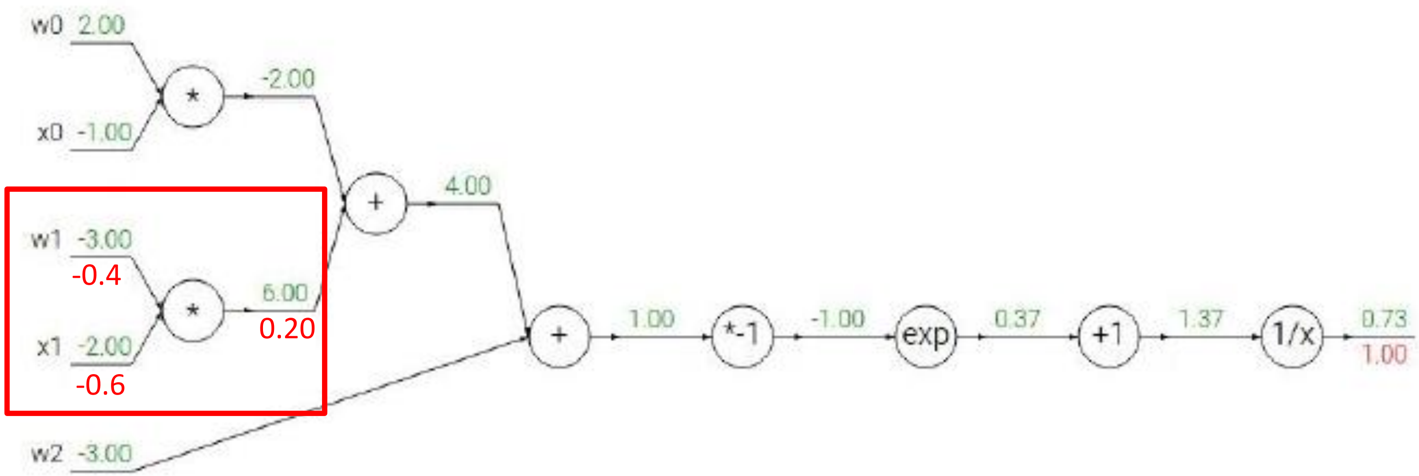


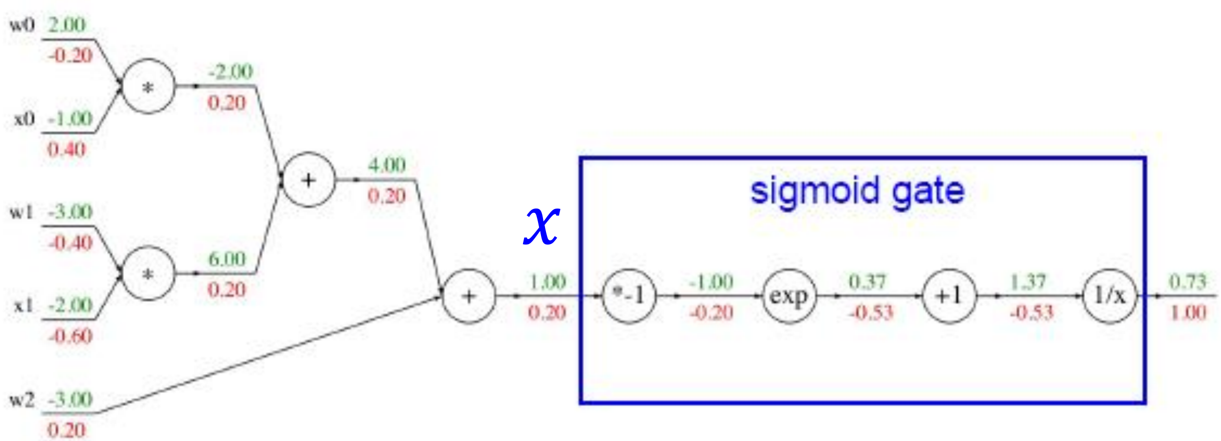
$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x \qquad\qquad f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

$$f_a(x) = ax \quad \rightarrow \quad \frac{df}{dx} = a \qquad\qquad f_c(x) = c + x \quad \rightarrow \quad \frac{df}{dx} = 1$$

$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

$$\boxed{\sigma(x) = \frac{1}{1+e^{-x}}}$$ sigmoid function

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} = \left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)\left(\frac{1}{1+e^{-x}}\right) = (1-\sigma(x))\,\sigma(x)$$



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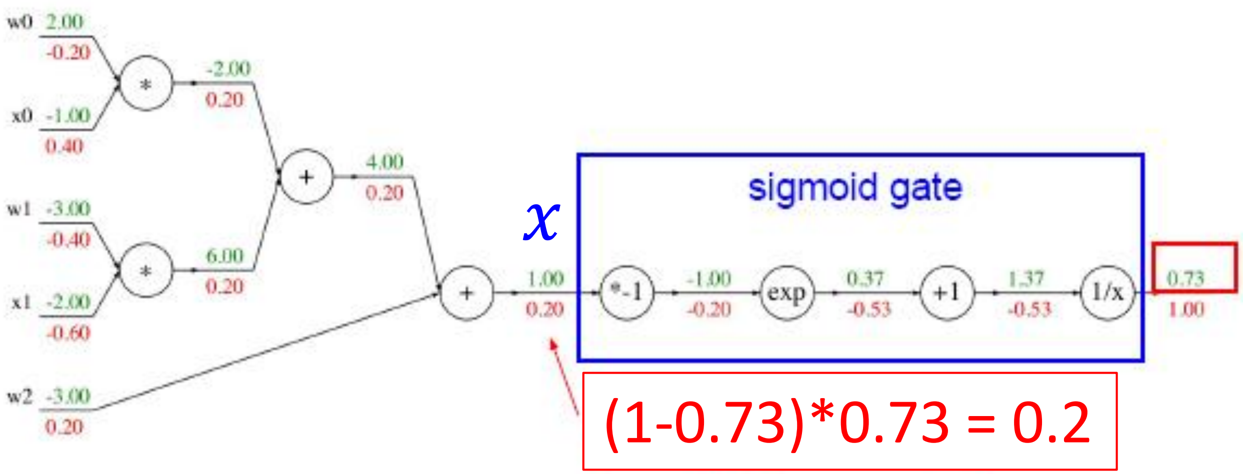


Exercise

In the context of artificial neural networks, the **rectifier** is an activation function defined as the positive part of its argument:

$f(x) = \max(0, x)$; this is also know as ReLU function

Let $x = w_0x_0 + w_1x_1 + b$

$w_0 = 0.2$, $w_1 = 0.7$, $x_0 = 1.0$, $x_1 = -1.2$, $b = 0.8$

1. Draw the complete computational graph for f(x).
2. Assume that $\frac{\partial L}{\partial f} = 1.0$
3. Calculate backpropagate gradient of L at every branch of the graph.

Computational graph

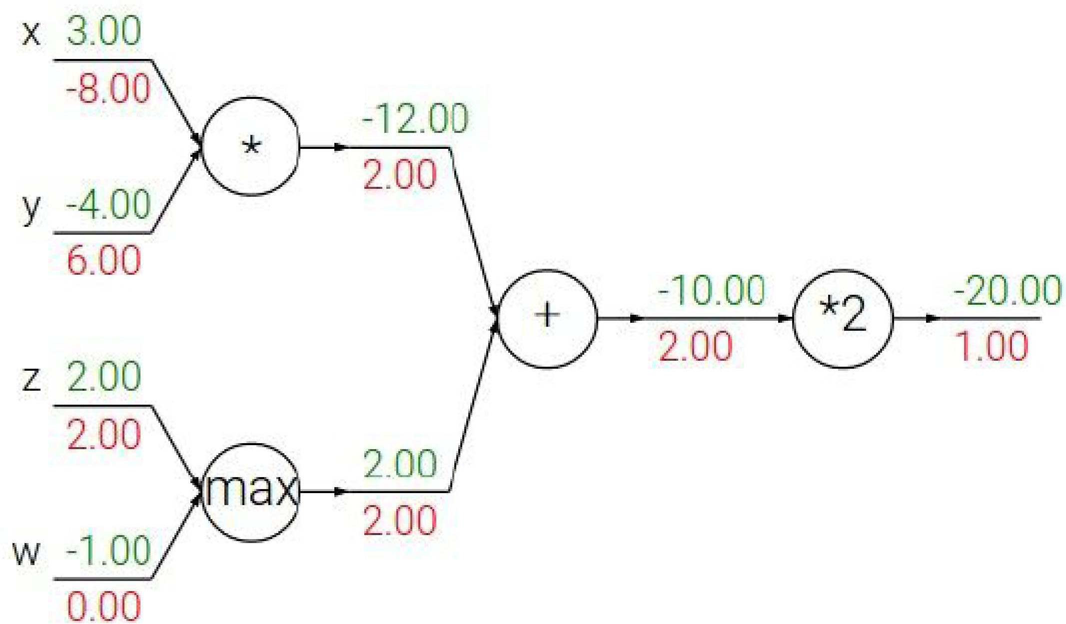
Gradients

Patterns in backward flow

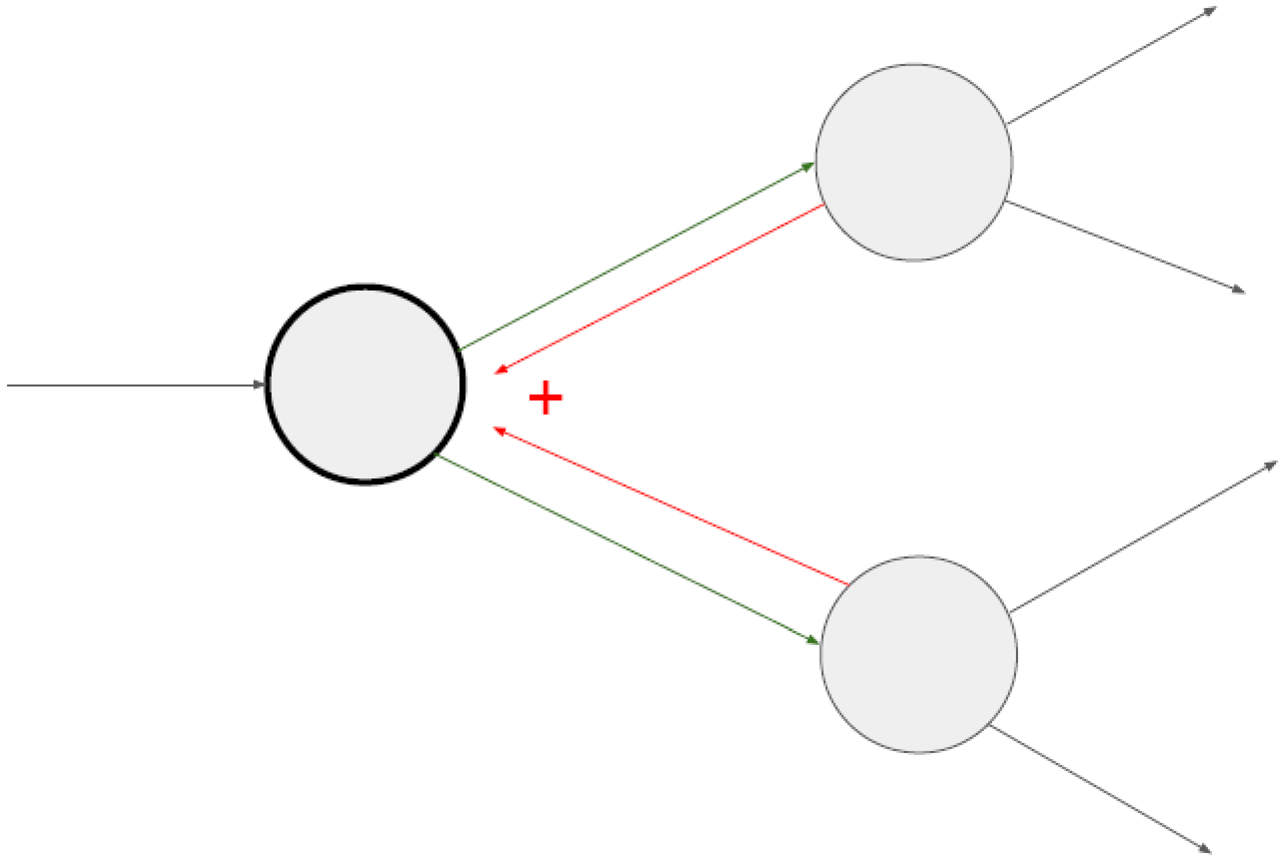
add gate: gradient distributor

max gate: gradient router

mul gate: gradient switcher



Gradients add at branches



Revisit: Backpropagation for multilayer NN

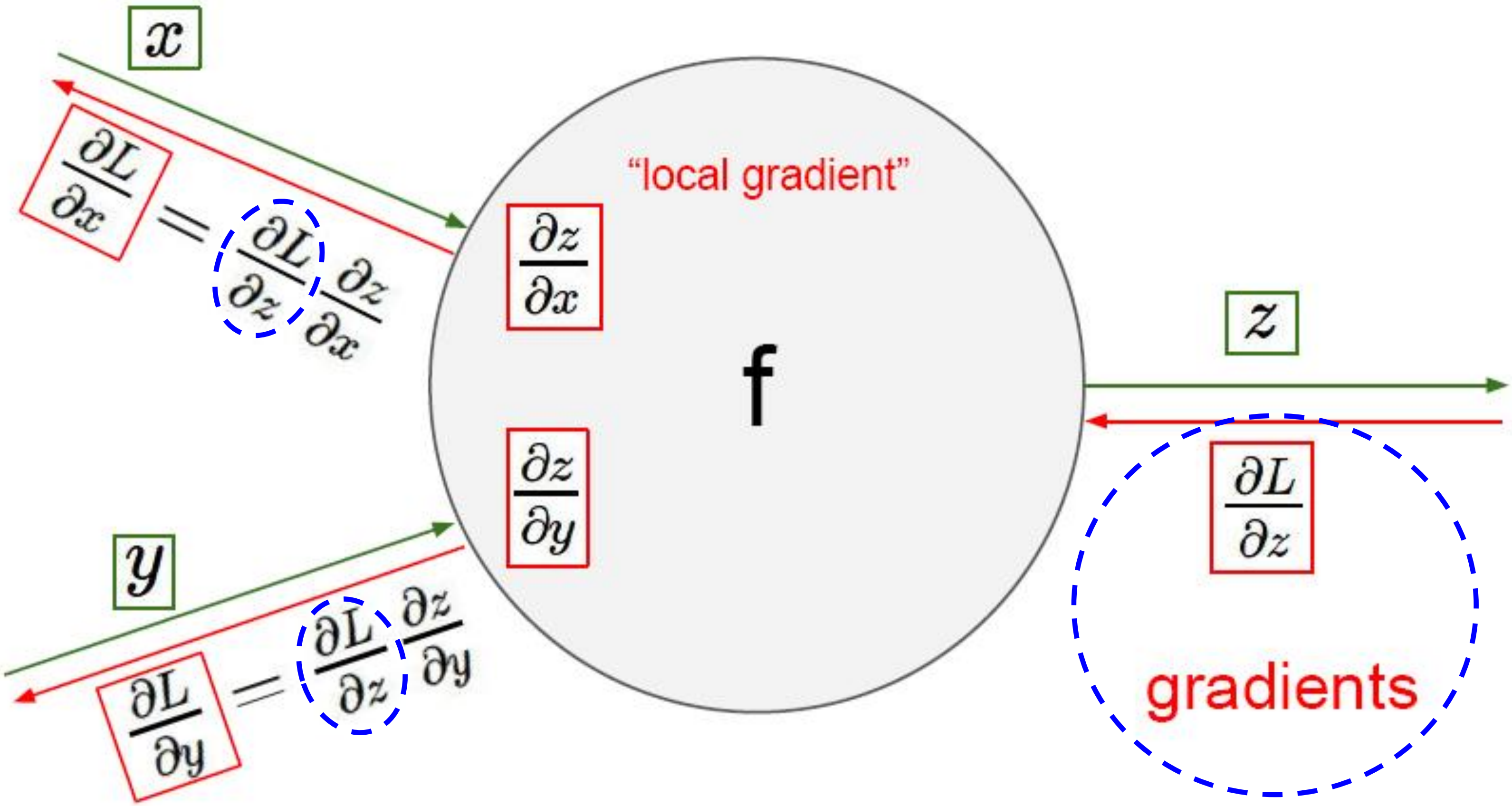
- To propagate error signals, we start at the output layer and work backward to the hidden layer.

- The error signal at the output of neuron k at iteration p is defined by:

$$e_k(p) = yd_k(p) - y_k(p)$$

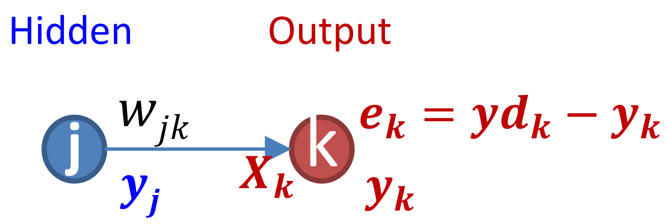
Where $yd_k(p)$ is the desired output of neuron k at iteration p.

Recall gradients backpropagation from page 18.



This can be substituted by e_k

- To update weights, the weight correction (Δw) is adjusted to the previous weights.



$$w_{jk}(p+1) = w_{jk}(p) - \Delta w_{jk}(p),$$

$$\Delta w_{jk}(p) = \alpha . y_j(p) . \delta_k(p),$$

$$\delta_k(p) = \frac{\partial y_k(p)}{\partial X_k(p)} . e_k(p),$$

How do we get these terms?

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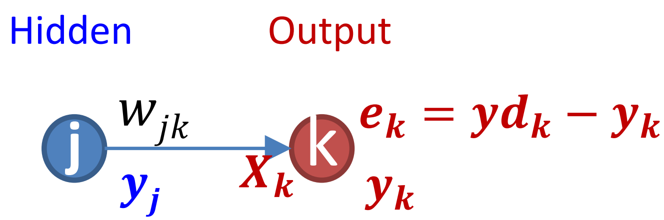

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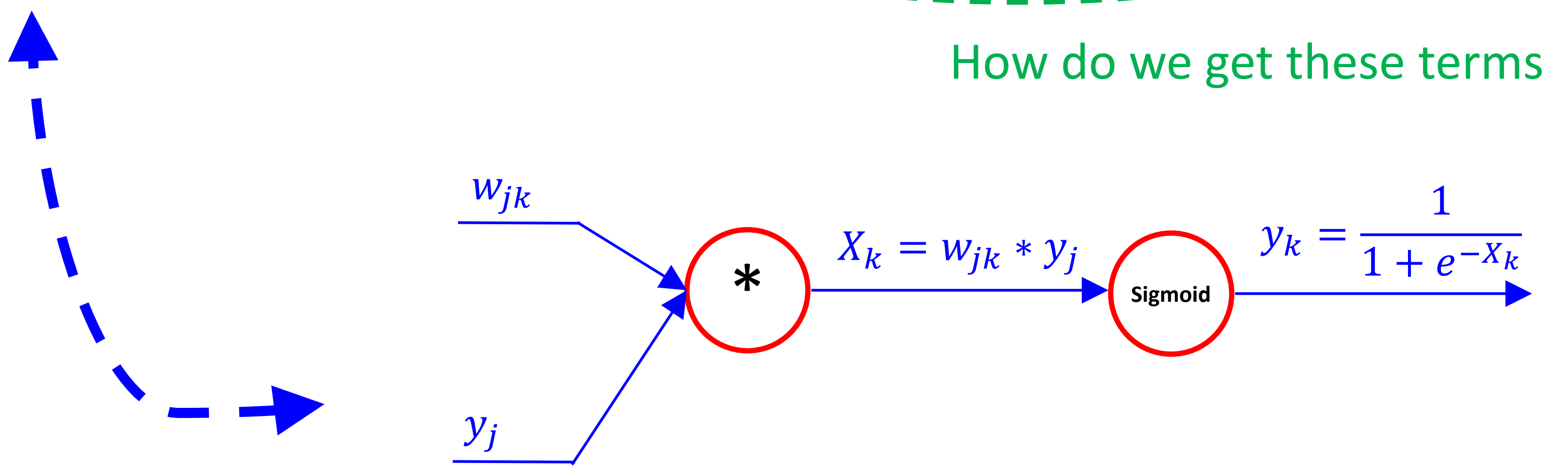

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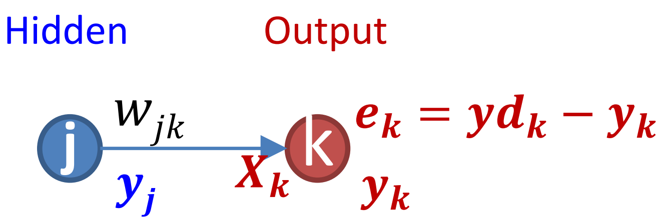

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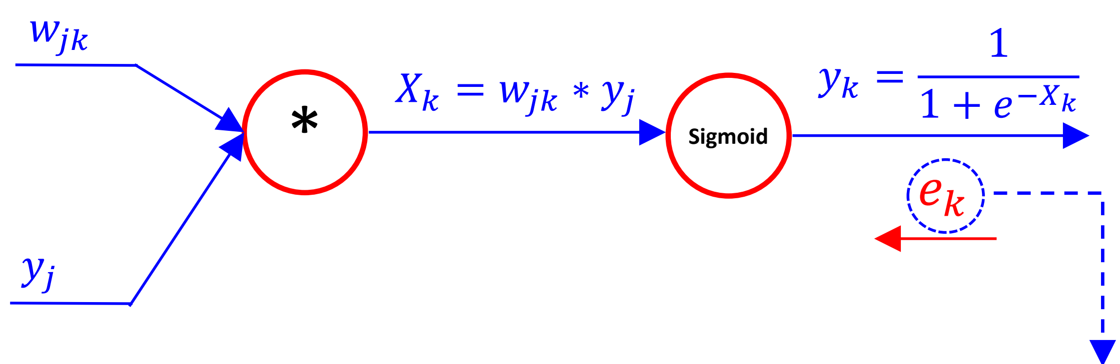

This is equivalent to $\frac{\partial e_k}{\partial y_k}$

(or $\frac{\partial L}{\partial z}$ in page 18)

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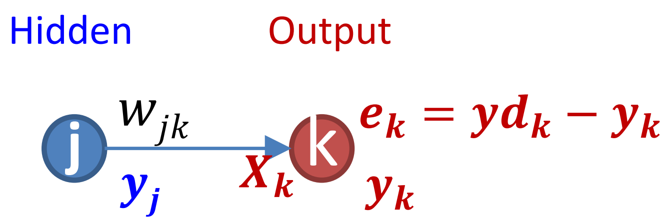

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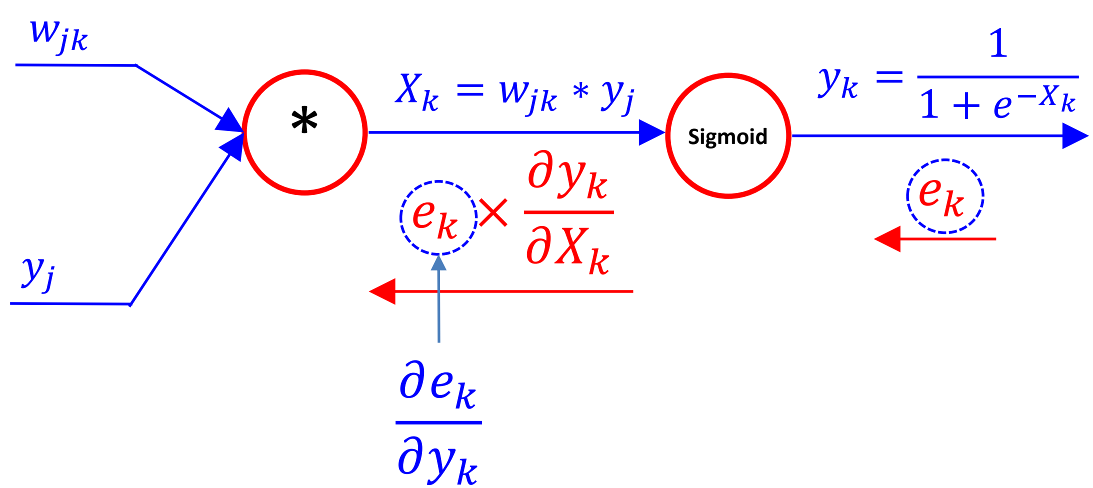

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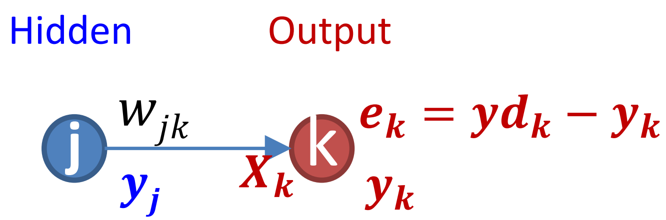

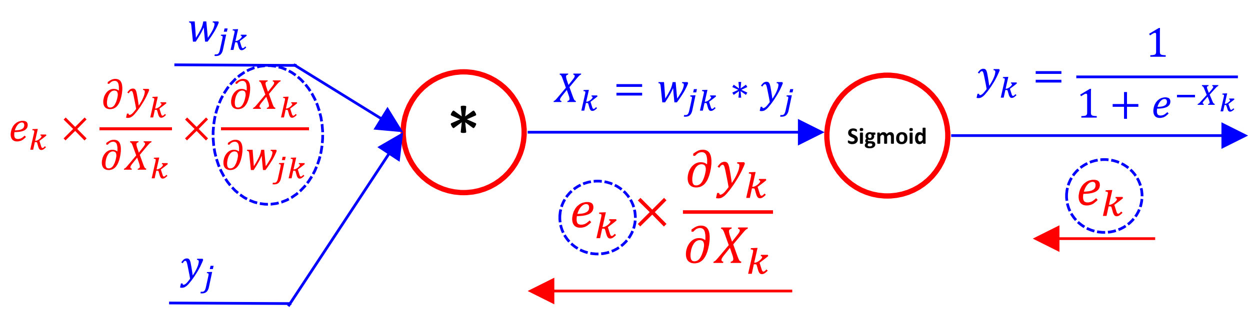

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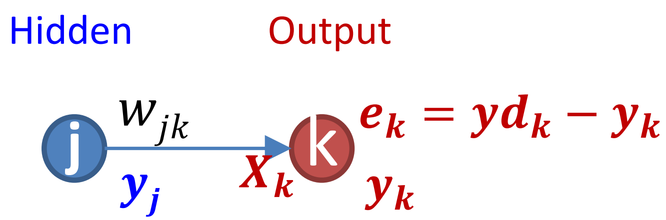

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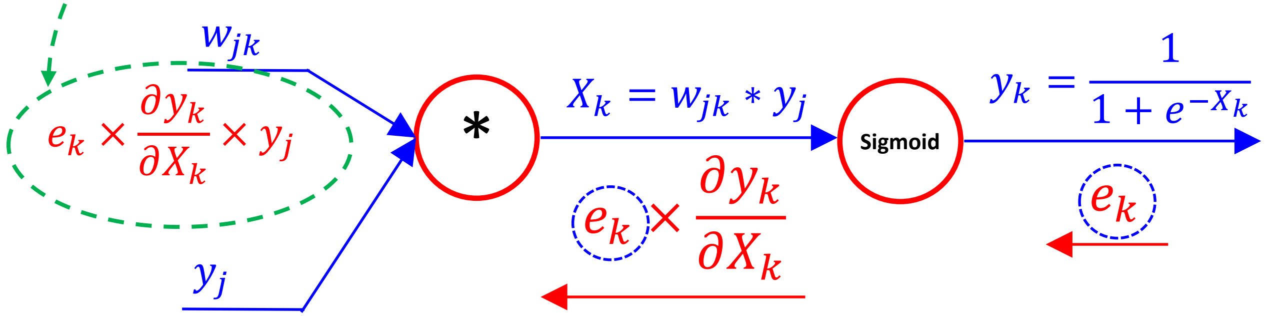

- Next class
  - Convolutional neural network.