

Notes on Basic Mathematics

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1 Numbers

1.1 The Integers

Definition 1.1. Positive Integers \mathbb{Z}^+ : 1,2,3,4,...

Definition 1.2. Origin: 0

Definition 1.3. Natural Numbers \mathbb{N}^+ : 0,1,2,3,...

Definition 1.4. Negative Integers \mathbb{Z}^- : -1,-2,-3,...

Theorem 1.1. $0+a=a+0=a$

Theorem 1.2. $a+(-a)=-a+a=0$

Definition 1.5. Additive Inverse of a : $-a$

Addition (1.1). Commutativity:
 $\forall a,b \in \mathbb{Z}, a+b=b+a$

Addition (1.2). Associativity:
 $\forall a,b,c \in \mathbb{Z}, (a+b)+c=a+(b+c)$

Theorem 1.3. $\forall a,b \in \mathbb{Z}, a+b=0 \implies (b=-a) \wedge (a=-b)$

Theorem 1.4. $\forall a \in \mathbb{Z}, a=-(-a)$

Theorem 1.5. $\forall a,b \in \mathbb{Z}, -(a+b)=-a-b$

Addition (1.3). $\forall a,b \in \mathbb{Z}^+ \implies a+b \in \mathbb{Z}^+$

Addition (1.4). $\forall a,b \in \mathbb{Z}^- \implies a+b \in \mathbb{Z}^-$

1.2 Rules for Multiplication

Multiplication 1.1. Commutativity:
 $\forall a,b \in \mathbb{Z}, ab=ba$

Multiplication 1.2. Associativity:
 $\forall a,b,c \in \mathbb{Z}, (ab)c=a(bc)$

Theorem 1.6. $\forall a \in \mathbb{Z}, 1a=a$ and $\forall a \in \mathbb{Z}, 0a=0$

Multiplication 1.3. Distributivity
 $\forall a,b,c \in \mathbb{Z}, a(b+c)=ab+ac$

and

$$\forall a,b,c \in \mathbb{Z}, (b+c)a=ba+bc$$

Theorem 1.7. $\forall a \in \mathbb{Z}, (-1)a=-a$

Theorem 1.8. $\forall a,b \in \mathbb{Z}, -(ab)=(-a)b$

Theorem 1.9. $\forall a,b \in \mathbb{Z}, -(ab)=a(-b)$

Theorem 1.10. $\forall a,b \in \mathbb{Z}, (-a)(-b)=ab$

Definition 1.6. n -th power of a is $a^n = aaa...a$ (n times)

Theorem 1.11. $\forall m,n \in \mathbb{Z}^+, a^{m+n}=a^m a^n$

Theorem 1.12. $\forall m,n \in \mathbb{Z}^+, (a^m)^n=a^{mn}$

Example 1.1.

$$\begin{aligned}(ab)^n &= a^n b^n \\ (ab)^n &= abab...ab \\ &= aa...abb...b \\ &= a^n b^n\end{aligned}$$

Multiplication 1.4.

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2 \\ (a-b)^2 &= a^2 - 2ab + b^2 \\ (a+b)(a-b) &= a^2 - b^2\end{aligned}$$

1.3 Even and Odd Integers; Divisibility

Definition 1.7. $\forall n \in \mathbb{Z}^+, \exists m \in \mathbb{Z}^+, n=2m \implies n$ is even

Definition 1.8. $\forall n \in \mathbb{Z}, \exists m \in \mathbb{N}^+$ (including 0), $n=2m+1 \implies n$ is odd

Theorem 1.13. Let $E(x)$ be the predicate for x is even and let $O(x)$ be the predicate for x is odd.

$$\begin{aligned}\forall a,b \in \mathbb{Z}^+, E(a) \wedge E(b) &\implies E(a+b) \\ \forall a,b \in \mathbb{Z}^+, E(a) \wedge O(b) &\implies O(a+b) \\ \forall a,b \in \mathbb{Z}^+, O(a) \wedge E(b) &\implies O(a+b) \\ \forall a,b \in \mathbb{Z}^+, O(a) \wedge O(b) &\implies E(a+b)\end{aligned}$$

Theorem 1.14.

$$\begin{aligned}\forall a \in \mathbb{Z}^+, E(a) &\implies E(a^2) \\ \forall a \in \mathbb{Z}^+, O(a) &\implies O(a^2)\end{aligned}$$

Corollary 1.1.

$$\begin{aligned}\forall a \in \mathbb{Z}^+, E(a^2) &\implies E(a) \\ \forall a \in \mathbb{Z}^+, O(a^2) &\implies O(a)\end{aligned}$$

Definition 1.9. d divides n or n is divisible by d if $n=dk$ for some integer k

1.4 Rational Numbers

Definition 1.10. Rational Number \mathbb{Q} : can be written in the form $\frac{m}{n}$, where $m, n \in \mathbb{Z}$ and $n \neq 0$

Rational (1.1). Rule for cross-multiplying.

$$\forall m, n, r, s \in \mathbb{Q}, n \neq 0, s \neq 0 \implies \frac{m}{n} = \frac{r}{s} \iff ms = rn$$

Rational (1.2). Cancellation rule for fractions

$$\forall a, m, n \in \mathbb{Q}, a \neq 0, n \neq 0 \implies \frac{am}{an} = \frac{m}{n}$$

Theorem 1.15. Any positive rational number has an expression as a fraction in lowest form

Rational (1.3). Addition rule for rational numbers

$$\forall a, b, d \in \mathbb{Q}, d \neq 0 \implies \frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}$$

Rational (1.4). General case of addition rule for rational numbers

$$\forall m, n, r, s \in \mathbb{Q}, n \neq 0, s \neq 0 \implies \frac{m}{n} + \frac{r}{s} = \frac{ms + rn}{ns}$$

Corollary 1.2. The sum of positive rational numbers is also positive

Corollary 1.3. $\forall a \in \mathbb{Q} \implies 0 + a = a + 0 = a$

Corollary 1.4. Addition of rational numbers satisfies the properties of commutativity and associativity

Rational (1.5). Multiplication of rational numbers

$$\forall m, n, r, s \in \mathbb{Q}, n \neq 0, s \neq 0 \implies \frac{m}{n} * \frac{r}{s} = \frac{mr}{ns}$$

Corollary 1.5. $\forall k \in \mathbb{Q}^+, n \neq 0 \implies a^k = \left(\frac{m}{n}\right)^k = \frac{m^k}{n^k}$

Theorem 1.16. There is no positive rational number whose square is 2

Definition 1.11. Irrational: a number which is not rational

Corollary 1.6. For any rational number a we have $1a = a$ and $0a = 0$. Furthermore, multiplication is associative, commutative, and distributive with respect to addition.

Rational (1.6). Rational numbers satisfy the property $\forall a \in \mathbb{Q}, a \neq 0 \implies \exists a^{-1}, a^{-1}a = aa^{-1} = 1$

Rational (1.7). Cross-multiplication

$$\forall a, b, c, d \in \mathbb{Q}, b \neq 0, d \neq 0 \implies \frac{a}{b} = \frac{c}{d}, ad = bc, ad = bc, \frac{a}{b} = \frac{c}{d}$$

Rational (1.8). Cancellation law for multiplication

$$\forall a, b, c \in \mathbb{Q}, a \neq 0, ab = ac \implies b = c$$

Rational (1.9). Common denominator

$$\forall a, b, c, d \in \mathbb{Q}, b \neq 0, d \neq 0 \implies \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

2 Chapter 2: Linear Equations

2.1 Equations In Two Unknowns

Linear Equations (2.1). General form for solving systems of linear equations in two unknowns.

$$\begin{aligned} \forall a, b, c, d, u, v \in \mathbb{R}, ab - bc \neq 0, ax + by = u, cx + dy = v \\ \implies x = \frac{du - bv}{ad - bc}, y = \frac{av - cu}{ad - bc} \end{aligned}$$

3 Chapter 3: Real Numbers

3.1 Addition and Multiplication

Addition (3.1). Addition is commutative and associative, meaning that

$$\forall a, b, c \in \mathbb{R}, a + b = b + a, a + (b + c) = (a + b) + c$$

Furthermore

$$0 + a = a$$

To each real number a there is an associated a denoted by $-a$ such that

$$a + (-a) = 0$$

Multiplication 3.1. Properties of multiplication, multiplication is commutative and associative, meaning that

$$\forall a, b, c \in \mathbb{R}, ab = ba, a(bc) = (ab)c$$

Furthermore

$$1a = a, 0a = 0$$

Multiplication is distributive with respect to addition meaning that

$$a(b + c) = ab + ac, (b + c)a = ba + ca$$

Corollary 3.1. For real numbers

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

Corollary 3.2. There is also the existence of a multiplicative inverse

$$\forall a \in \mathbb{R}, a \neq 0 \exists a^{-1} | a^{-1}a = aa^{-1} = 1$$

3.2 Real Numbers: Positivity

Positivity (3.1). If a, b are positive then so are the product ab and the sum $a+b$

Positivity (3.2). If $a \in \mathbb{R}$ then either a is positive or $a = 0$ or $-a$ is positive, and these possibilities are mutually exclusive.

Positivity (3.3). If a is positive and b is negative, then ab is negative

Positivity (3.4). If a is negative and b is negative, then ab is positive

Positivity (3.5). If a is positive, then $1/a$ is positive

Positivity (3.6). If a is negative, then $1/a$ is negative

Corollary 3.3. $\forall a \in \mathbb{R}, a > 0 \implies \exists b \in \mathbb{R} \mid b^2 = a$

Corollary 3.4. $\forall x, y \in \mathbb{R}, x^2 = y^2 \implies x = y \text{ or } x = -y$

Definition 3.1. Absolute value of x is $|x| = \sqrt{x^2}$

3.3 Powers and Roots

Definition 3.2. The product of a with itself n times is $\forall n \in \mathbb{Z}^+ a \in \mathbb{R} a^n$

Corollary 3.5. $\forall m, n \in \mathbb{Z}^+ a^{m+n} = a^m a^n$

Corollary 3.6. $\forall a \in \mathbb{R}^+, n \in \mathbb{Z}^+ \exists r \in \mathbb{R}^+ \mid r^n = a$

Definition 3.3. The n^{th} root of a is $a^{1/n}$ or $\sqrt[n]{a}$

Powers (3.1). $\forall a, b \in \mathbb{R}^+ (ab)^{1/n} = a^{1/n} b^{1/n}$

Corollary 3.7. Let a be a positive number. To each rational number x we can associate a positive number denoted by a^x , which is the n^{th} power of a when x is a positive integer n , the n^{th} root of a when $x = 1/n$, and satisfying the following conditions:

Powers (3.2). $\forall x, y \in \mathbb{Q} a^{x+y} = a^x a^y$

Powers (3.3). $\forall x, y \in \mathbb{Q} (a^x)^y = a^{xy}$

Powers (3.4). $\forall a, b \in \mathbb{R}^+ (ab)^x = a^x b^x$

Corollary 3.8. $\forall a \in \mathbb{Q} a^0 = 1$

Corollary 3.9. $\forall a \in \mathbb{R} x \in \mathbb{R}^+ a^{-x} = \frac{1}{a^x}$

Corollary 3.10. $\forall a \in \mathbb{R} m, n \in \mathbb{Z}^+ a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m$

3.4 Inequalities

Inequalities (3.1). $\forall a \in \mathbb{R}^+ \implies a > 0$

Inequalities (3.2). $\forall a, b \in \mathbb{R} a - b > 0 \implies a > b$

Inequalities (3.3). $\forall a \in \mathbb{R} -a > 0 \implies a < 0$

Inequalities (3.4). $\forall a, b \in \mathbb{R} a > b \implies b < a$

Inequalities (3.5). $\forall a, b \in \mathbb{R} a \geq b \implies a > b \vee a = b$

Inequalities (3.6). $\forall a, b, c \in \mathbb{R} a > b \wedge b > c \implies a > c$

Inequalities (3.7). $\forall a, b, c \in \mathbb{R} a > b \wedge c > 0 \implies ac > bc$

Inequalities (3.8). $\forall a, b, c \in \mathbb{R} a > b \wedge c < 0 \implies ac < bc$

Definition 3.4. open interval: $\forall x, a, b \in \mathbb{R} \mid a < x < b$

Definition 3.5. closed interval: $\forall x, a, b \in \mathbb{R} \mid a \leq x \leq b$

Definition 3.6. half open or half closed: $\forall x, a, b \in \mathbb{R} \mid a \leq x < b, a < x \leq b$

Definition 3.7. infinite interval: $\forall x, a \in \mathbb{R} \mid x < a \vee x > a \vee x \leq a \vee x \geq a$