# Notes on Basic Mathematics

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## 1 Numbers

### 1.1 The Integers

**Definition 1.1.** Positive Integers $\mathbb{Z}^+$: 1,2,3,4,...

**Definition 1.2.** Origin: 0

**Definition 1.3.** Natural Numbers $\mathbb{N}^+$: 0,1,2,3,...

**Definition 1.4.** Negative Integers $\mathbb{Z}^-$: $-1,-2,-3,...$

**Theorem 1.1.** $0+a=a+0=a$

**Theorem 1.2.** $a+(-a)=-a+a=0$

**Definition 1.5.** Additive Inverse of $a$: $-a$

***Addition*** (1.1)**.** Commutativity:

$$\forall a,b\in\mathbb{Z}, a+b=b+a$$

***Addition*** (1.2)**.** Associativity:

$$\forall a,b,c\in\mathbb{Z},(a+b)+c=a+(b+c)$$

**Theorem 1.3.** $\forall a,b\in\mathbb{Z}, a+b=0 \implies (b=-a)\wedge(a=-b)$

**Theorem 1.4.** $\forall a\in\mathbb{Z}, a=-(-a)$

**Theorem 1.5.** $\forall a,b\in\mathbb{Z},-(a+b)=-a-b$

***Addition*** (1.3)**.** $\forall a,b\in\mathbb{Z}^+ \implies a+b\in\mathbb{Z}^+$

***Addition*** (1.4)**.** $\forall a,b\in\mathbb{Z}^- \implies a+b\in\mathbb{Z}^-$

### 1.2 Rules for Multiplication

**Multiplication 1.1.** *Commutativity:*

$$\forall a,b\in\mathbb{Z}, ab=ba$$

**Multiplication 1.2.** *Associativity:*

$$\forall a,b,c\in\mathbb{Z},(ab)c=a(bc)$$

**Theorem 1.6.** $\forall a\in\mathbb{Z}, 1a=a$ *and* $\forall a\in\mathbb{Z}, 0a=0$

**Multiplication 1.3.** *Distributivity*

$$\forall a,b,c\in\mathbb{Z}, a(b+c)=ab+ac$$

*and*

$$\forall a,b,c\in\mathbb{Z},(b+c)a=ba+bc$$

**Theorem 1.7.** $\forall a\in\mathbb{Z},(-1)a=-a$

**Theorem 1.8.** $\forall a,b\in\mathbb{Z},-(ab)=(-a)b$

**Theorem 1.9.** $\forall a,b\in\mathbb{Z},-(ab)=a(-b)$

**Theorem 1.10.** $\forall a,b\in\mathbb{Z},(-a)(-b)=ab$

**Definition 1.6.** $n$-th power of $a$ is $a^n = aaa...a$ ($n$ times)

**Theorem 1.11.** $\forall m,n\in\mathbb{Z}^+, a^{m+n}=a^m a^n$

**Theorem 1.12.** $\forall m,n\in\mathbb{Z}^+,(a^m)^n=a^{mn}$

**Example 1.1.**

$$\begin{aligned}(ab)^n &= a^n b^n\\ (ab)^n &= abab...ab\\ &= aa...abb...b\\ &= a^n b^n\end{aligned}$$

**Multiplication 1.4.**

$$\begin{aligned}(a+b)^2&=a^2+2ab+b^2\\ (a-b)^2&=a^2-2ab+b^2\\ (a+b)(a-b)&=a^2-b^2\end{aligned}$$

### *1.3 Even and Odd Integers; Divisibility*

**Definition 1.7.** $\forall n\in\mathbb{Z}^+,\exists m\in\mathbb{Z}^+, n=2m \implies n$ is even

**Definition 1.8.** $\forall n\in\mathbb{Z},\exists m\in\mathbb{N}^+$ (including 0)$, n=2m+1 \implies n$ is odd

**Theorem 1.13.** *Let $E(x)$ be the predicate for $x$ is even and let $O(x)$ be the predicate for $x$ is odd.*

$$\begin{aligned}&\forall a,b\in\mathbb{Z}^+, E(a)\wedge E(b)\implies E(a+b)\\ &\forall a,b\in\mathbb{Z}^+, E(a)\wedge O(b)\implies O(a+b)\\ &\forall a,b\in\mathbb{Z}^+, O(a)\wedge E(b)\implies O(a+b)\\ &\forall a,b\in\mathbb{Z}^+, O(a)\wedge O(b)\implies E(a+b)\end{aligned}$$

**Theorem 1.14.**

$$\begin{aligned}&\forall a\in\mathbb{Z}^+, E(a)\implies E(a^2)\\ &\forall a\in\mathbb{Z}^+, O(a)\implies O(a^2)\end{aligned}$$

**Corallary 1.1.**

$$\begin{aligned}&\forall a\in\mathbb{Z}^+, E(a^2)\implies E(a)\\ &\forall a\in\mathbb{Z}^+, O(a^2)\implies O(a)\end{aligned}$$

**Definition 1.9.** $d$ divides $n$ or $n$ is divisible by $d$ if $n=dk$ for some integer $k$

### 1.4 Rational Numbers

**Definition 1.10.** Rational Number $\mathbb{Q}$: can be written in the form $\frac{m}{n}$, where $m,n \in \mathbb{Z}$ and $n \neq 0$

***Rational*** (1.1)**.** Rule for cross-multiplying.

$$\forall m,n,r,s \in \mathbb{Z} \wedge m,s \neq 0 \Longrightarrow \frac{m}{n} = \frac{r}{s} \Longleftrightarrow ms = rn$$