Notes on Basic Mathematics

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1 Numbers

1.1 The Integers

Definition 1.1. Positive Integers $\mathbb{Z}^+$: 1,2,3,4,...

Definition 1.2. Origin: 0

Definition 1.3. Natural Numbers $\mathbb{N}^+$: 0,1,2,3,...

Definition 1.4. Negative Integers $\mathbb{Z}^-$: $-1,-2,-3,...$

Theorem 1.1. $0+a=a+0=a$

Theorem 1.2. $a+(-a)=-a+a=0$

Definition 1.5. Additive Inverse of a: $-a$

Addition (1.1)**.** Commutativity:

$$\forall a,b\in\mathbb{Z}, a+b=b+a$$

Addition (1.2)**.** Associativity:

$$\forall a,b,c\in\mathbb{Z},(a+b)+c=a+(b+c)$$

Theorem 1.3. $\forall a,b\in\mathbb{Z}, a+b=0 \implies (b=-a)\wedge(a=-b)$

Theorem 1.4. $\forall a\in\mathbb{Z}, a=-(-a)$

Theorem 1.5. $\forall a,b\in\mathbb{Z},-(a+b)=-a-b$

Addition (1.3)**.** $\forall a,b\in\mathbb{Z}^+ \implies a+b\in\mathbb{Z}^+$

Addition (1.4)**.** $\forall a,b\in\mathbb{Z}^- \implies a+b\in\mathbb{Z}^-$

1.2 Rules for Multiplication

Multiplication 1.1. *Commutativity:*

$$\forall a,b\in\mathbb{Z}, ab=ba$$

Multiplication 1.2. *Associativity:*

$$\forall a,b,c\in\mathbb{Z},(ab)c=a(bc)$$

Theorem 1.6. $\forall a\in\mathbb{Z}, 1a=a$ *and* $\forall a\in\mathbb{Z}, 0a=0$

Multiplication 1.3. *Distributivity*

$$\forall a,b,c\in\mathbb{Z}, a(b+c)=ab+ac$$

and

$$\forall a,b,c\in\mathbb{Z},(b+c)a=ba+bc$$

Theorem 1.7. $\forall a\in\mathbb{Z},(-1)a=-a$

Theorem 1.8. $\forall a,b\in\mathbb{Z},-(ab)=(-a)b$

Theorem 1.9. $\forall a,b\in\mathbb{Z},-(ab)=a(-b)$

Theorem 1.10. $\forall a,b\in\mathbb{Z},(-a)(-b)=ab$

Definition 1.6. n-th power of a is $a^n = aaa...a$ (n times)

Theorem 1.11. $\forall m,n\in\mathbb{Z}^+, a^{m+n}=a^m a^n$

Theorem 1.12. $\forall m,n\in\mathbb{Z}^+,(a^m)^n=a^{mn}$

Example 1.1.

$$\begin{aligned}(ab)^n &= a^n b^n\\ (ab)^n &= abab...ab\\ &= aa...abb...b\\ &= a^n b^n\end{aligned}$$

Multiplication 1.4.

$$\begin{aligned}(a+b)^2 &= a^2+2ab+b^2\\ (a-b)^2 &= a^2-2ab+b^2\\ (a+b)(a-b) &= a^2-b^2\end{aligned}$$

1.3 Even and Odd Integers; Divisibility

Definition 1.7. $\forall n\in\mathbb{Z}^+,\exists m\in\mathbb{Z}^+, n=2m \implies n$ is even

Definition 1.8. $\forall n\in\mathbb{Z},\exists m\in\mathbb{N}^+$ (including 0)$, n=2m+1 \implies n$ is odd

Theorem 1.13. *Let $E(x)$ be the predicate for x is even and let $O(x)$ be the predicate for x is odd.*

$$\begin{aligned}&\forall a,b\in\mathbb{Z}^+, E(a)\wedge E(b) \implies E(a+b)\\ &\forall a,b\in\mathbb{Z}^+, E(a)\wedge O(b) \implies O(a+b)\\ &\forall a,b\in\mathbb{Z}^+, O(a)\wedge E(b) \implies O(a+b)\\ &\forall a,b\in\mathbb{Z}^+, O(a)\wedge O(b) \implies E(a+b)\end{aligned}$$

Theorem 1.14.

$$\begin{aligned}&\forall a\in\mathbb{Z}^+, E(a) \implies E(a^2)\\ &\forall a\in\mathbb{Z}^+, O(a) \implies O(a^2)\end{aligned}$$

Corallary 1.1.

$$\begin{aligned}&\forall a\in\mathbb{Z}^+, E(a^2) \implies E(a)\\ &\forall a\in\mathbb{Z}^+, O(a^2) \implies O(a)\end{aligned}$$

Definition 1.9. d divides n or n is divisible by d if $n=dk$ for some integer k

1.4 Rational Numbers

Definition 1.10. Rational Number $\mathbb{Q}$: can be written in the form $\frac{m}{n}$, where $m,n \in \mathbb{Z}$ and $n \neq 0$

Rational (1.1)**.** Rule for cross-multiplying.

$$\forall m,n,r,s \in \mathbb{Q}, n \neq 0, s \neq 0 \implies \frac{m}{n} = \frac{r}{s} \iff ms = rn$$

Rational (1.2)**.** Cancellation rule for fractions

$$\forall a,m,n \in \mathbb{Q}, a \neq 0, n \neq 0 \implies \frac{am}{an} = \frac{m}{n}$$

Theorem 1.15. *Any positive rational number has an expression as a fraction in lowest form*

Rational (1.3)**.** Addition rule for rational numbers

$$\forall a,b,d \in \mathbb{Q}, d \neq 0 \implies \frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}$$

Rational (1.4)**.** General case of addition rule for rational numbers

$$\forall m,n,r,s \in \mathbb{Q}, n \neq 0, s \neq 0 \implies \frac{m}{n} + \frac{r}{s} = \frac{ms+rn}{ns}$$

Corallary 1.2. *The sum of positive rational numbers is also positive*

Corallary 1.3. $\forall a \in \mathbb{Q} \implies 0 + a = a + 0 = a$

Corallary 1.4. *Addition of rational numbers satisfies the properties of commutativity and associativity*

Rational (1.5)**.** Multiplication of rational numbers

$$\forall m,n,r,s \in \mathbb{Q}, n \neq 0, s \neq 0 \implies \frac{m}{n} * \frac{r}{s} = \frac{mr}{ns}$$

Corallary 1.5. $\forall k \in \mathbb{Q}^{+}, n \neq 0 \implies a^k = \left(\frac{m}{n}\right)^k = \frac{m^k}{n^k}$

Theorem 1.16. *There is no positive rational number whose square is 2*

Definition 1.11. Irrational: a number which is not rational

Corallary 1.6. *For any rational number a we have $1a = a$ and $0a = 0$. Furthermore, multiplication is associative, commutative, and distributive with respect to addition.*

Rational (1.6)**.** Rational numbers satisfy the property

$$\forall a \in \mathbb{Q}, a \neq 0 \implies \exists a^{-1}, a^{-1}a = aa^{-1} = 1$$

Rational (1.7)**.** Cross-multiplication

$$\forall a,b,c,d \in \mathbb{Q}, b \neq 0, d \neq 0 \implies \frac{a}{b} = \frac{c}{d}, ad = bc, ad = bc, \frac{a}{b} = \frac{c}{d}$$

Rational (1.8)**.** Cancellation law for multiplication

$$\forall a,b,c \in \mathbb{Q} a \neq 0, ab = ac \implies b = c$$

Rational (1.9)**.** Common denominator

$$\forall a,b,c,d \in \mathbb{Q}, b \neq 0, d \neq 0 \implies \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

2 *Chapter 2: Linear Equations*

2.1 Equations In Two Unknowns

Linear Equations (2.1)**.** General form for solving systems of linear equations in two unknowns.

$$\forall a,b,c,d,u,v \in \mathbb{R}, ab - bc \neq 0, ax + by = u, cx + dy = v$$

$$\implies x = \frac{du - bv}{ad - bc}, y = \frac{av - cu}{ad - bc}$$

3 *Chapter 3: Real Numbers*

3.1 Addition and Multiplication

Addition (3.1)**.** Addition is commutative and associative, meaning that

$$\forall a,b,c \in \mathbb{R}, a + b = b + a, a + (b + c) = (a + b) + c$$

Furthermore

$$0 + a = a$$

To each real number a there is an associated a denoted by $-a$ such that

$$a + (-a) = 0$$

Multiplication 3.1. *Properties of multiplication, multiplication is commutative and associative, meaning that*

$$\forall a,b,c \in \mathbb{R}, ab = ba, a(bc) = (ab)c$$

Furthermore

$$1a = a, 0a = 0$$

Multiplication is distributive with respect to addition meaning that

$$a(b + c) = ab + ac, (b + c)a = ba + ca$$

Corallary 3.1. *For real numbers*

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

Corallary 3.2. *There is also the existence of a multiplicative inverse*

$$\forall a \in \mathbb{R}, a \neq 0 \exists a^{-1} \mid a^{-1}a = aa^{-1} = 1$$

3.2 Real Numbers: Positivity

Positivity (3.1)**.** If a,b are positive then so are the product ab and the sum $a+b$

Positivity (3.2)**.** If $a\in\mathbb{R}$ then either a is positive or $a=0$ or $-a$ is positive, and these possibilities are mutually exclusive.

Positivity (3.3)**.** If a is positive and b is negative, then ab is negative

Positivity (3.4)**.** If a is negative and b is negative, then ab is positive

Positivity (3.5)**.** If a is positive, then $1/a$ is positive

Positivity (3.6)**.** If a is negative, then $1/a$ is negative

Corallary 3.3. $\forall a\in\mathbb{R}, a>0 \Longrightarrow \exists b\in\mathbb{R} \,|\, b^2=a$

Corallary 3.4. $\forall x,y\in\mathbb{R}, x^2=y^2 \Longrightarrow x=y \ \textit{or}\ x-y$

Definition 3.1. Absolute value of x is $|x|=\sqrt{x^2}$

3.3 Powers and Roots

Definition 3.2. The product of a with itself n times is $\forall n\in\mathbb{Z}^+ a\in\mathbb{R} a^n$

Corallary 3.5. $\forall m,n\in\mathbb{Z}^+ a^{m+n}=a^m a^n$

Corallary 3.6. $\forall a\in\mathbb{R}^+, n\in\mathbb{Z}^+ \exists r\in\mathbb{R}^+ \,|\, r^n=a$

Definition 3.3. The n^{th} root of a is $a^{1/n}$ or $\sqrt[n]{a}$

Powers (3.1)**.** $\forall a,b\in\mathbb{R}^+ (ab)^{1/n}=a^{1/n}b^{1/n}$