Notes on Basic Mathematics

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1 Numbers

1.1 The Integers

Definition 1.1. Positive Integers $\mathbb{Z}^+$: 1,2,3,4,...

Definition 1.2. Origin: 0

Definition 1.3. Natural Numbers $\mathbb{N}^+$: 0,1,2,3,...

Definition 1.4. Negative Integers $\mathbb{Z}^-$: $-1,-2,-3,...$

Theorem 1.1. $0+a=a+0=a$

Theorem 1.2. $a+(-a)=-a+a=0$

Definition 1.5. Additive Inverse of a: $-a$

Addition (1.1)**.** Commutativity:
$$\forall a,b\in\mathbb{Z},a+b=b+a$$

Addition (1.2)**.** Associativity:
$$\forall a,b,c\in\mathbb{Z},(a+b)+c=a+(b+c)$$

Theorem 1.3. $\forall a,b\in\mathbb{Z},a+b=0 \implies (b=-a)\wedge(a=-b)$

Theorem 1.4. $\forall a\in\mathbb{Z},a=-(-a)$

Theorem 1.5. $\forall a,b\in\mathbb{Z},-(a+b)=-a-b$

Addition (1.3)**.** $\forall a,b\in\mathbb{Z}^+ \implies a+b\in\mathbb{Z}^+$

Addition (1.4)**.** $\forall a,b\in\mathbb{Z}^- \implies a+b\in\mathbb{Z}^-$

1.2 Rules for Multiplication

Multiplication 1.1. *Commutativity:*
$$\forall a,b\in\mathbb{Z},ab=ba$$

Multiplication 1.2. *Associativity:*
$$\forall a,b,c\in\mathbb{Z},(ab)c=a(bc)$$

Theorem 1.6. $\forall a\in\mathbb{Z},1a=a$ *and* $\forall a\in\mathbb{Z},0a=0$

Multiplication 1.3. *Distributivity*
$$\forall a,b,c\in\mathbb{Z},a(b+c)=ab+ac$$
and
$$\forall a,b,c\in\mathbb{Z},(b+c)a=ba+bc$$

Theorem 1.7. $\forall a\in\mathbb{Z},(-1)a=-a$

Theorem 1.8. $\forall a,b\in\mathbb{Z},-(ab)=(-a)b$

Theorem 1.9. $\forall a,b\in\mathbb{Z},-(ab)=a(-b)$

Theorem 1.10. $\forall a,b\in\mathbb{Z},(-a)(-b)=ab$

Definition 1.6. n-th power of a is $a^n=aaa...a$ (n times)

Theorem 1.11. $\forall m,n\in\mathbb{Z}^+,a^{m+n}=a^m a^n$

Theorem 1.12. $\forall m,n\in\mathbb{Z}^+,(a^m)^n=a^{mn}$

Example 1.1.
$$\begin{aligned}(ab)^n&=a^n b^n\\(ab)^n&=abab...ab\\&=aa...abb...b\\&=a^n b^n\end{aligned}$$

Multiplication 1.4.
$$\begin{gathered}(a+b)^2=a^2+2ab+b^2\\(a-b)^2=a^2-2ab+b^2\\(a+b)(a-b)=a^2-b^2\end{gathered}$$

1.3 Even and Odd Integers; Divisibility

Definition 1.7. $\forall n\in\mathbb{Z}^+,\exists m\in\mathbb{Z}^+,n=2m \implies n$ is even

Definition 1.8. $\forall n\in\mathbb{Z},\exists m\in\mathbb{N}^+$ (including 0)$,n=2m+1 \implies n$ is odd

Theorem 1.13. *Let $E(x)$ be the predicate for x is even and let $O(x)$ be the predicate for x is odd.*
$$\begin{gathered}\forall a,b\in\mathbb{Z}^+,E(a)\wedge E(b)\implies E(a+b)\\\forall a,b\in\mathbb{Z}^+,E(a)\wedge O(b)\implies O(a+b)\\\forall a,b\in\mathbb{Z}^+,O(a)\wedge E(b)\implies O(a+b)\\\forall a,b\in\mathbb{Z}^+,O(a)\wedge O(b)\implies E(a+b)\end{gathered}$$

Theorem 1.14.
$$\begin{gathered}\forall a\in\mathbb{Z}^+,E(a)\implies E(a^2)\\\forall a\in\mathbb{Z}^+,O(a)\implies O(a^2)\end{gathered}$$

Corallary 1.1.
$$\begin{gathered}\forall a\in\mathbb{Z}^+,E(a^2)\implies E(a)\\\forall a\in\mathbb{Z}^+,O(a^2)\implies O(a)\end{gathered}$$

Definition 1.9. d divides n or n is divisible by d if $n=dk$ for some integer k

1.4 Rational Numbers

Definition 1.10. Rational Number $\mathbb{Q}$: can be written in the form $\frac{m}{n}$, where $m,n \in \mathbb{Z}$ and $n \neq 0$

Rational (1.1)**.** Rule for cross-multiplying.

$$\forall m,n,r,s \in \mathbb{Q}, n \neq 0, s \neq 0 \implies \frac{m}{n} = \frac{r}{s} \iff ms = rn$$

Rational (1.2)**.** Cancellation rule for fractions

$$\forall a,m,n \in \mathbb{Q}, a \neq 0, n \neq 0 \implies \frac{am}{an} = \frac{m}{n}$$

Theorem 1.15. *Any positive rational number has an expression as a fraction in lowest form*

Rational (1.3)**.** Addition rule for rational numbers

$$\forall a,b,d \in \mathbb{Q}, d \neq 0 \implies \frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}$$

Rational (1.4)**.** General case of addition rule for rational numbers

$$\forall m,n,r,s \in \mathbb{Q}, n \neq 0, s \neq 0 \implies \frac{m}{n} + \frac{r}{s} = \frac{ms+rn}{ns}$$

Corallary 1.2. *The sum of positive rational numbers is also positive*

Corallary 1.3. $\forall a \in \mathbb{Q} \implies 0+a = a+0 = a$

Corallary 1.4. *Addition of rational numbers satisfies the properties of commutativity and associativity*

Rational (1.5)**.** Multiplication of rational numbers

$$\forall m,n,r,s \in \mathbb{Q}, n \neq 0, s \neq 0 \implies \frac{m}{n} * \frac{r}{s} = \frac{mr}{ns}$$

Corallary 1.5. $\forall k \in \mathbb{Q}^{+}, n \neq 0 \implies a^k = \left(\frac{m}{n}\right)^k = \frac{m^k}{n^k}$

Theorem 1.16. *There is no positive rational number whose square is 2*

Definition 1.11. Irrational: a number which is not rational

Corallary 1.6. *For any rational number a we have* $1a = a$ *and* $0a = 0$. *Furthermore, multiplication is associative, commutative, and distributive with respect to addition.*

Rational (1.6)**.** Rational numbers satisfy the property

$$\forall a \in \mathbb{Q}, a \neq 0 \implies \exists a^{-1}, a^{-1}a = aa^{-1} = 1$$

Rational (1.7)**.** Cross-multiplication

$$\forall a,b,c,d \in \mathbb{Q}, b \neq 0, d \neq 0 \implies \frac{a}{b} = \frac{c}{d}, ad = bc, ad = bc, \frac{a}{b} = \frac{c}{d}$$

Rational (1.8)**.** Cancellation law for multiplication

$$\forall a,b,c \in \mathbb{Q} a \neq 0, ab = ac \implies b = c$$

Rational (1.9)**.** Common denominator

$$\forall a,b,c,d \in \mathbb{Q}, b \neq 0, d \neq 0 \implies \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

2 *Chapter 2: Linear Equations*

2.1 *Equations In Two Unknowns*

Linear Equations (2.1)**.** General form for solving systems of linear equations in two unknowns.

$$\forall a,b,c,d,u,v \in \mathbb{R}, ab-bc \neq 0, ax+by = u, cx+dy = v$$

$$\implies x = \frac{du-bv}{ad-bc}, y = \frac{av-cu}{ad-bc}$$

3 *Chapter 3: Real Numbers*

3.1 *Addition and Multiplication*

Addition (3.1)**.** Addition is commutative and associative, meaning that

$$\forall a,b,c \in \mathbb{R}, a+b = b+a, a+(b+c) = (a+b)+c$$

Furthermore

$$0+a = a$$

To each real number a there is an associated a denoted by $-a$ such that

$$a+(-a) = 0$$

Multiplication 3.1. *Properties of multiplication, multiplication is commutative and associative, meaning that*

$$\forall a,b,c \in \mathbb{R}, ab = ba, a(bc) = (ab)c$$

Furthermore

$$1a = a, 0a = 0$$

Multiplication is distributive with respect to addition meaning that

$$a(b+c) = ab+ac, (b+c)a = ba+ca$$

Corallary 3.1. *For real numbers*

$$(a+b)^2 = a^2+2ab+b^2$$

$$(a-b)^2 = a^2-2ab+b^2$$

$$(a+b)(a-b) = a^2-b^2$$

Corallary 3.2. *There is also the existence of a multiplicative inverse*

$$\forall a \in \mathbb{R}, a \neq 0 \exists a^{-1} \mid a^{-1}a = aa^{-1} = 1$$

3.2 Real Numbers: Positivity

Positivity (3.1)**.** If a,b are positive then so are the product ab and the sum $a+b$

Positivity (3.2)**.** If $a \in \mathbb{R}$ then either a is positive or $a=0$ or $-a$ is positive, and these possibilities are mutually exclusive.

Positivity (3.3)**.** If a is positive and b is negative, then ab is negative

Positivity (3.4)**.** If a is negative and b is negative, then ab is positive

Positivity (3.5)**.** If a is positive, then $1/a$ is positive

Positivity (3.6)**.** If a is negative, then $1/a$ is negative

Corallary 3.3. $\forall a \in \mathbb{R}, a > 0 \implies \exists b \in \mathbb{R} \,|\, b^2 = a$

Corallary 3.4. $\forall x,y \in \mathbb{R}, x^2 = y^2 \implies x = y \text{ or } x - y$

Definition 3.1. Absolute value of x is $|x| = \sqrt{x^2}$

3.3 Powers and Roots

Definition 3.2. The product of a with itself n times is $\forall n \in \mathbb{Z}^+ a \in \mathbb{R} a^n$

Corallary 3.5. $\forall m,n \in \mathbb{Z}^+ a^{m+n} = a^m a^n$

Corallary 3.6. $\forall a \in \mathbb{R}^+, n \in \mathbb{Z}^+ \exists r \in \mathbb{R}^+ \,|\, r^n = a$

Definition 3.3. The n^{th} root of a is $a^{1/n}$ or $\sqrt[n]{a}$

Powers (3.1)**.** $\forall a,b \in \mathbb{R}^+ (ab)^{1/n} = a^{1/n} b^{1/n}$

Corallary 3.7. *Let a be a positive number. To each rational number x we can associate a positive number denoted by a^x, which is the n^{th} power of a when x is a positive integer n, the n^{th} root of a when $x = 1/n$, and satisfying the following conditions:*

Powers (3.2)**.** $\forall x,y \in \mathbb{Q} a^{x+y} = a^x a^y$

Powers (3.3)**.** $\forall x,y \in \mathbb{Q} (a^x)^y = a^{xy}$

Powers (3.4)**.** $\forall a,b \in \mathbb{R}^+ (ab)^x = a^x b^x$

Corallary 3.8. $\forall a \in \mathbb{Q} a^0 = 1$

Corallary 3.9. $\forall a \in \mathbb{R} x \in \mathbb{R}^+ a^{-x} = \dfrac{1}{a^x}$

Corallary 3.10. $\forall a \in \mathbb{R} m,n \in \mathbb{Z}^+ a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m$

3.4 Inequalities

Inequalities (3.1)**.** $\forall a \in \mathbb{R}^+ \implies a > 0$

Inequalities (3.2)**.** $\forall a,b \in \mathbb{R} a - b > 0 \implies a > b$

Inequalities (3.3)**.** $\forall a \in \mathbb{R} - a > 0 \implies a < 0$

Inequalities (3.4)**.** $\forall a,b \in \mathbb{R} a > b \implies b < a$

Inequalities (3.5)**.** $\forall a,b \in \mathbb{R} a \geq b \implies a > b \vee a = b$

Inequalities (3.6)**.** $\forall a,b,c \in \mathbb{R} a > b \wedge b > c \implies a > c$

Inequalities (3.7)**.** $\forall a,b,c \in \mathbb{R} a > b \wedge c > 0 \implies ac > bc$

Inequalities (3.8)**.** $\forall a,b,c \in \mathbb{R} a > b \wedge c < 0 \implies ac < bc$

Definition 3.4. open interval: $\forall x,a,b \in \mathbb{R} \,|\, a < x < b$

Definition 3.5. closed interval: $\forall x,a,b \in \mathbb{R} \,|\, a \leq x \leq b$

Definition 3.6. half open or half closed: $\forall x,a,b \in \mathbb{R} \,|\, a \leq x < b, a < x \leq b$

Definition 3.7. infinite interval: $\forall x,a \in \mathbb{R} \,|\, x < a \vee x > a \vee x \leq a \vee x \geq a$

4 Chapter 4: Quadratic Equations

Quadratic (4.1)**.** Let a,b,c be real numbers and $a \neq 0$. The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

provided that $b^2 - 4ac$ is positive or 0. If $b^2 - 4ac$ is negative, then the equation has no solution in the real numbers.

5 Chapter 5: Distance and Angles

5.1 Distance

The distance between points P,Q in the plane by $d(P,Q)$. It is a number which satisfies the following properties.

Distance (5.1)**.** $\forall P,Q : d(P,Q) \iff P = Q$

Distance (5.2)**.** $\forall P,Q : d(P,Q) = d(Q,P)$

Distance (5.3)**.** Triangle inequality: $\forall P,Q,M : d(P,M) \leq d(P,Q) + d(Q,M)$

Definition 5.1. We assume that two distinct points P,Q lie on only one line denoted L_{PQ}

Definition 5.2. Segment: the portion of the line between two points P,Q, denoted by $\overline{PQ}$. If units of measure are selected then the length of the segment is equal to the distance $d(P,Q)$

Segment (5.1)**.** $\forall P, Q, M : d(P, M) = d(P,Q) + d(Q,M) \iff Q \in \overline{PQ}$

Segment (5.2)**.** $\forall P,M \,|\, d = d(P,M) : 0 \leq c \leq d \implies \exists Q \in \overline{PM} \,|\, d(P,Q) = c$

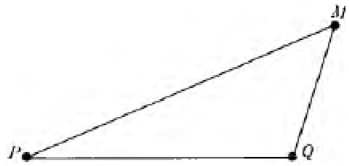

Figure 1: Triangle inequality

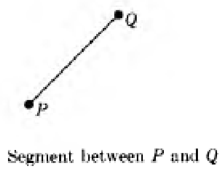

Figure 2: Segment

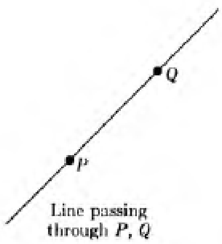

Figure 3: Line

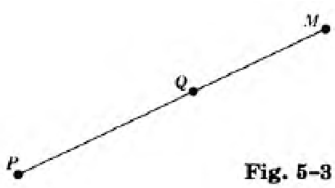

Figure 4: PQM Angle

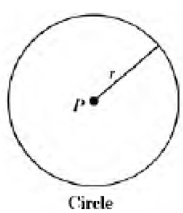

Figure 5: Circle

Figure 6: Disc

5.2 Angles

Corallary 5.1. $\forall (P,Q,P \neq Q) \exists ! L_{PQ}$

Corallary 5.2. $\forall L_1,\ L_2,\ \neg(L_1 \parallel L_2) \exists ! P \ | \ (P \in L_1) \wedge (P \in L_2)$

Corallary 5.3. $\forall L_1, P \exists ! L_2 \,|\, (P \in L_2) \wedge (L_1 \parallel L_2)$

Corallary 5.4. $\forall L_1, L_2, L_3, (L_1 \parallel L_2) \wedge (L_2 \parallel L_3) \implies (L_1 \parallel L_3)$

Corallary 5.5. $\forall L_1, P \exists ! L_2 \,|\, (P \in L_2) \wedge (L_1 \perp L_2)$

Corallary 5.6. $\forall L_1, L_2, L_3, (L_1 \perp L_2) \wedge (L_2 \parallel L_3) \implies (L_1 \perp L_3)$

Corallary 5.7. $\forall L_1, L_2, L_3, (L_1 \perp L_2) \wedge (L_2 \perp L_3) \implies (L_1 \parallel L_3)$

Definition 5.3. Ray: a ray is a line between two points P,Q such that the ray is composed of all possible points extending past Q infinitely on one side. A ray is determined by it's starting point and direction.

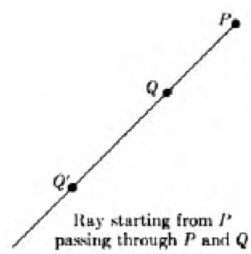

Figure 7: Ray

Definition 5.4. Vertex: the starting point of the ray.

Definition 5.5. Angle: The portion enclosed by two rays R_{PQ} and R_{PM}. The angle must be given additional information in order to determine which side of the resulting enclosure should be considered as the angle.

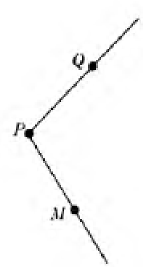

Figure 8: Angle

Definition 5.6. We shall determine the side of an angle by the portion enclosed by the clockwise enumeration of $\angle QPM$. In this case there are two rays R_{QP} R_{PM} and this angle would be the amount enclosed on the inside of Q,P,M in that order. The opposite side would be $\angle MPQ$.

Definition 5.7. Zero Angle: the angle enclosed by the line P,Q,M. 0 degrees.

Definition 5.8. Full Angle: the angle enclosed by the line on the opposite side P,Q,M. 360 degrees.

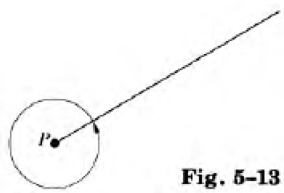

Figure 9: Full Angle

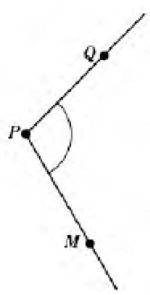

Figure 10: Inside Angle

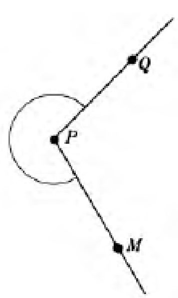

Figure 11: Outside Angle

Definition 5.9. Straight Angle: the angle enclosed by the line M,P,Q. 180 degrees.

Definition 5.10. Sector: The area inside a circle captured by an angle with the vertex in the center of the circle. (A slice of a circle).

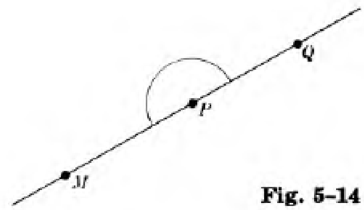

Figure 12: Straight Angle

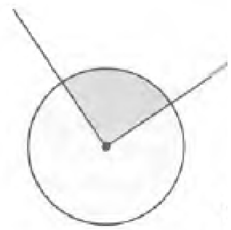

Figure 13: Sector

Corallary 5.8. *Angle* $x = 360\left(\frac{\text{area of } S}{\text{area of } D}\right)$ *where* S *is the sector and* D *is the disk.*

Definition 5.11. Right angle: 90 degrees.

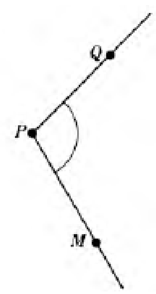

Figure 14: Right angle 90 degrees

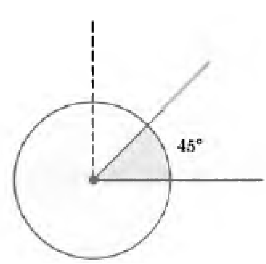

Figure 15: 45 degree angle

Corallary 5.9. *Area of* $S = \frac{\angle S}{360}\pi r^2$ *where* r *is the radius.*

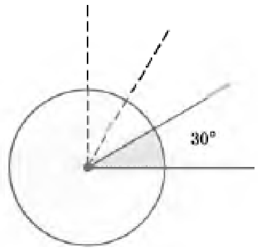

Figure 16: 3 degree angle

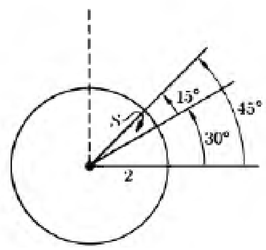

Figure 17: Degree example

5.3 The Pythagoras Theorem

Definition 5.12. Triangle: $\forall P,Q,M,(\overline{PQ},\overline{QM},\overline{PM}) \Longrightarrow \triangle PQM$

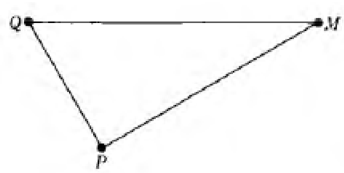

Figure 18: Triangle

Definition 5.13. Right-angle triangle: When one of the three sides of a triangle is 90°

Definition 5.14. Legs of right triangle: The sides of the triangle that meet at the 90° angle

Right Triangle (5.1)**.** If two right-triangles $\triangle PQM$ and $\triangle P'Q'M'$ have legs $\overline{PQ},\overline{PM}$ and $\overline{P'Q'},\overline{P'M'}$ of equal lengths

$$\text{length}\overline{PQ} = \text{length}\overline{P'Q'}$$
$$\text{length}\overline{PM} = \text{length}\overline{P'M'}$$

then the angles of the triangles have equal measure, their areas are equal and the length of $\overline{QM}$ is equal to $\overline{Q'M'}$.

Right Triangle (5.2)**.** $\forall L,L',(L \parallel L'),P,Q,(P \in L \wedge Q \in L)K_P,(K_P \perp L \wedge P \in K_P),P',(P' \in K_P \wedge P' \in L'),K_Q,(K_Q \perp L \wedge Q \in K_P),Q',(Q' \in K_Q \wedge Q' \in L') \Longrightarrow$ length$(\overline{PP'}) =$ length$(\overline{QQ'}) \Longrightarrow d(P,P') = d(Q,Q')$

Right Triangle (5.3)**.** Rectangle:
$\forall P,\ Q,\ M,\ N \ \mid\ F\ \left(\overline{PQ},\overline{QN},\overline{NM},\overline{MP}\right)\ ,\ (\overline{PQ} \parallel \overline{NM} \wedge \overline{QN} \parallel \overline{MP}) \wedge (\overline{PQ} \perp \overline{QN} \wedge \overline{NM} \perp \overline{NP}) \Longrightarrow \overline{PQ},\overline{QN},\overline{NM},\overline{MP} \wedge R(P,Q,N,M)$.
where:

- F is the predicate such that it's arguments all form sides
- R is the predicate such that all of it's arguments form a rectangle.

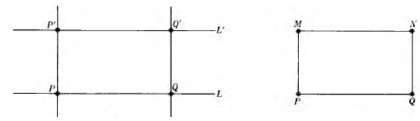

Figure 19: Rectangle

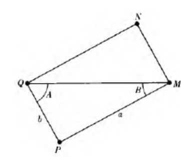

Figure 20: Triangle Rectangle

Theorem 5.1. *$\forall A,\ B \in \angle$Right Triangle, other than right angle then $\Longrightarrow$ $m(A)+m(B) = 90°$*

Theorem 5.2. *Area of right triangle $\mid$ legs a,b $\Longrightarrow$ $\dfrac{ab}{2}$*

Definition 5.15. Hypotenuse: the third side of a right-angle triangle which is not one of the legs

Theorem 5.3. *Let a,b be lengths of two legs of a right triangle and let c be the length of the hypotenuse then: $a^2+b^2=c^2$*

Corallary 5.10. *Let P, Q be distinct points in the plane, Let M be a point in the plane, then $d(P,M) = d(Q,M) \Longleftrightarrow M \in \perp$ bisec $\overline{PQ}$*

6 Isometries