Calibrating an Agent-Based Financial Market

Blake LeBaron *
Brandeis University

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Abstract

This paper explores some of the empirical features generated in an agent-based computational stock market with market participants adapting and evolving over time. Investors view differing lengths of past information as being relevant to their investment decision making process. The interaction of these memory lengths in determining market prices creates a kind of market ecology in which it is difficult for the more stable, longer horizon agents to take over the market. What occurs is a dynamically changing market in which different types of strategies arrive and depart depending on their current relative performance, and the fraction of wealth following them. This paper analyzes several key time series features of such a market. It is calibrated to the variability and growth of dividend payments in the United States. The market generates some features which are remarkably similar to those from actual data. These include magnifying the volatility from the dividend process, inducing persistence in volatility and volume, and generating fat tailed return distributions. The market is simulated at the weekly frequency, but returns are also examined at longer horizons where they are compared with other long range results.

Keywords: Finance, Agent-based markets, Neural networks, Volatility

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1 Introduction

Financial markets are an area of economics where relatively precise data are plentiful. It is frustrating that these series provide some of the biggest puzzles in economics. Features for which a well accepted explanation does not exist include price volatility properties, excess kurtosis, equity premia, large Sharpe ratios, and predictable deviations from fundamentals. This paper addresses all of these features using an agent-based approach, where individual behavior is modeled as a collection of learning and adapting agents joined together in a simple financial market setting. In this way it is an early attempt at calibrating an agent-based model to financial and macro-economic time series. Models which attempt to represent the economy with a heterogeneous group of learning agents have appeared in many areas of economics and other social sciences.[1] The results show that in some dimensions agent-based markets show great promise in solving some of these puzzles, while in others more work remains.

In finance a large number of these models have appeared.[2] This plethora of market frameworks is somewhat driven by the fact that most of these models deviate from strict notions of rationality and equilibrium. The world of boundedly rational strategies is, of course, quite large. The best restriction one can make on model structures would be to use well known experimental features from economics, cognitive science, and psychology. However, some of this evidence is either in the early stages, or not applicable. Without this information one should at least demand a few things of an agent-based market. First, it should quantitatively replicate features in a reasonable calibration exercise. Second, the form of irrationality used should be relatively simple and transparent. Finally, the model structure should be a reasonable representation of a financial trading situation. The first of these will be addressed directly in this paper. It will also be argued that the model used meets the second two criteria, but these are obviously more subjective.

Simple quantitative calibration exercises in finance were emphasized most clearly in Mehra & Prescott (1985). This paper drew attention to the problems in quantitatively accounting for the spreads between risky and risk free assets in the economy, and showed that a stylized representative agent could not match these features.[3] In Campbell & Cochrane (1999) the authors take the financial calibration exercise much farther. They develop a representative agent framework with habit persistent preferences which is able to fit a large number of empirical facts. Features such as the dynamics of dividend/price ratios, and long range return correlations are added to the simple equity premium result. It should be noted that they try to tackle

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[1] A good summary of this research can be found at the web site maintained by Leigh Tesfatsion (http://www.econ.iastate.edu/tesfatsi/ace.htm).

[2] See the surveys of LeBaron (2000) and LeBaron (2001a) for references to this growing literature.

[3] See Mehra (2003) for a recent update on the large literature spawned by the original paper.

features at many different time scales which should be viewed as an important constraint to keep in mind. This sophisticated model is admittedly carefully "engineered" to fit these facts, but there is no question that this paper stands as an important benchmark in the calibration world. The model used here is less "engineered" since the impact of various parameters is less clear, but it will try to address many of the same features both at relatively high frequencies as well as longer horizons.

The model used is a relatively new agent-based market. Many of its details are developed and presented in LeBaron (2001b). Although in spirit it is similar to the Santa Fe artificial stock market it is a radically different design built to be more tractable, and closer to the world of more traditional macro finance models except for its heterogeneous agent framework.[4] The market stresses many of the coevolutionary features that are an interesting part of agent-based modeling. In a coevolutionary setting the fitness of strategies depends critically on the current population of other strategies. In this market rules and agents are evolved, and compete with each other in their trading activities. The wealthier agents survive, along with rules that have been actively used. Agents' decision making processes differ in a very special way designed to replicate heterogeneity in the real world. They use differing amounts of past data in deciding on their optimal trading strategies. Some traders take a perspective of looking back 30 years to evaluate a trading rule, while others view only the previous 6 months as being important. This allows for interesting evolutionary races across these different players, and helps to answer some questions about whether there is a way to judge the most rational, and to predict who should end up wealthier in the long run.[5] This bounded memory perspective on past information has been approached in many different contexts, and is related to constant gain learning algorithms.[6]

Section 2 summarizes the model. Section 3 gives results from some computer experiments, and the final section summarizes and proposes future directions.

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[4] Results on the SFI artificial market have been presented in Arthur, Holland, LeBaron, Palmer & Tayler (1997), and LeBaron, Arthur & Palmer (1999).

[5] The debate about the survival and impact of irrational strategies has a long history in finance going back at least to Friedman (1953) who claimed that less than rational strategies would be eliminated. Recent additions have added to this debate and provided interesting theoretical frameworks to test this hypothesis. These include DeLong, Shleifer, Summers & Waldmann (1991), Sandroni (2000), Blume & Easley (2001), and Kogan, Ross, Wang & Westerfield (2003). The noise trader approach of DeLong et al. (1991) is very close in spirit, but different in structure, to the dynamics in this paper, where less rational strategies keep the more rational types from taking over the market by generating enough excess volatility to confuse their learning algorithms.

[6] See Honkapohja & Mitra (2000), Sargent (1999), and Mullainathan (1998) for related work on bounded memory. There is also a connection to the issue of rationality of adaptive expectation forecasts as in Muth (1960).

2 Market Structure

The market consists of several different pieces, some of which are more or less standard in economics and finance, and others which are new. Agents chose from a common set of rules in an attempt to maximize their own objectives. Rules are evolved separately using only a weak notion of popularity as their fitness function. This allows for some aspects of social learning and information transfer across agents.[7] It also allows for an endogenous measure of belief dispersions coming from the number of different rules or strategies in use at a given time.

The following sections describe the market in greater detail. Since this market is calibrated to actual data, time increments are meaningful. The basic time interval of the market is taken to be 1 week which is a middle range between high frequency and longer horizon studies in finance.

2.1 Securities

The market is a partial equilibrium model with two securities, a risk free asset in infinite supply paying a constant interest rate, r_f, and a risky security paying investors a random dividend each period. It is available in a fixed supply of 1 share for the population, d_t. If s_i are the share holdings of agent i, the following constraint must be met in all periods,

$$1 = \sum_{i=1}^{I} s_i, \tag{1}$$

where I is the number of agents. The log dividend follows a random walk, with an annualized growth rate of 1.7 percent, and an annual standard deviation of 5.4 percent. This corresponds to the annual growth rate and volatility of the S&P real dividend from Shiller's annual data series from 1947-2001.[8] The model will be calibrated and compared with the postwar data. This is done for two reasons. First, this is one of the most difficult periods in terms of relatively quiet dividend dynamics. If the model can magnify this quieter period then other periods should not be difficult. Second, stock return volatility was unusually high during the 1930's. Trying to fit to this unusual period in terms of volatility dynamics might be a mistake without more higher frequency data. The paper will occasionally compare with longer horizon data when appropriate. The constant interest rate, r_f, is set to a value of 0 percent. This may seem exceedingly low, but there are some good arguments for doing this. First, it is not far off the actual long term real rate of

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[7] See Vriend (2000) for some comparisons of social versus individual learning.

[8] This data is available at http://aida.econ.yale.edu/~shiller/.

about 1 percent per year. More importantly, it is set to zero so that risk free assets are not injecting extra funds into the market. The only source of outside resources is coming from the dividend process.

2.2 Agents and rules

Agents are defined by intertemporal constant relative risk aversion preferences of logarithmic form,

$$u_{i,t} = E_t \sum_{s=0}^{\infty} \beta^s \log c_{i,t+s}, \tag{2}$$

subject to the intertemporal budget constraint,

$$w_{i,t} = p_t s_{i,t} + b_{i,t} + c_{i,t} = (p_t + d_t)s_{i,t-1} + (1 + r_f)b_{i,t-1}. \tag{3}$$

$s_{i,t}$ and $b_{i,t}$ are the risky and risk free asset holdings respectively, and r_f is the risk free rate of return. r_t will represent the risky asset return at time t. p_t is the price of the risky security at time t. $w_{t,i}$ and $c_{i,t}$ are the wealth and consumption of agent i. Finally, d_t is the dividend paid at time t. These heavily restricted preferences are used for tractability. It is well known that for logarithmic utility the agent's optimal consumption choice can be separated from the portfolio composition and is a constant proportion of wealth,[9]

$$c_{i,t} = (1 - \beta)w_{i,t}. \tag{4}$$

The time rate of discount, β, is set to $(0.975)^{1/52}$ which corresponds to 0.975 at an annual rate.

A second property of logarithmic preferences is that the portfolio decision is myopic in that agents maximize the logarithm of next period's portfolio return. Agents will concentrate their learning efforts on this optimal portfolio decision. They are interested in finding a rule that will maximize the expected logarithm of the portfolio return from a dynamic strategy. The strategy recommends a fraction of savings, $\beta w_{i,t}$, to invest in the risky asset as a function of current information, z_t. The objective is to maximize

$$\max_{\alpha_t} E_t \log[1 + \alpha_t r_{t+1} + (1 - \alpha_t)r_f], \tag{5}$$

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[9]This result is well known in dynamic financial models. See Merton (1969) and Samuelson (1969) for early derivations. Future versions of this model will generalize these preferences, but this brings in the added dimension of trying to determine optimal consumption given current information. Also related are the analytic policy rules for time varying returns in Barberis (2000) and Campbell & Viceira (1999). Finally, Lettau & Uhlig (1999) and Allen & Carroll (2001) present some results showing some of the difficulties in building agents with the capabilities of learning dynamic consumption plans. Campbell & Viceira (2002) is a recent survey of the dynamic portfolio choice literature.

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where α_t is a portfolio strategy which can depend on information known up to and including time t. In general it would be impossible for agents to run this optimization each period, since the above expectation depends on the state of all other agents in the market, along with the dividend state. The portfolio decision will therefore be replaced with a simple rule which will be continually tested against other candidate rules. This continual testing forms a key part of the learning going on the market.

The trading rules, or dynamic strategies, α_j, should be thought of as being separate from the actual agents. The best analogy is to that of an investment advisor or mutual fund. A population of rules is maintained, and agents select from this set as they might chose an adviser. One difference here from the world of investment advisers is that the rule is a simple function, $\alpha(z_t; \omega_j)$, where z_t is time t information, and ω_j are parameters specific to rule j. The functional form used for each rule is a feedforward neural network with a single hidden unit with restricted inputs. It is given by

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\begin{aligned}
h_k &= g(\omega_{1,k} z_{t,k} + \omega_{0,k}) & (6) \\
\alpha(z_t) &= 0.5 * (1 + g(\omega_2 + \sum_{k=1}^{m} \omega_{3,k} h_k)) & (7) \\
g(u) &= \tanh(u) & (8)
\end{aligned}
$$

This framework restricts the portfolio to positive weights between zero and one, so there is no short selling or borrowing allowed. The model needs to set borrowing constraints to keep it off nonstationary bubble trajectories, and to avoid having to unwind debt positions when the agents go bankrupt. With these restrictions in place agents' wealth can go to zero, but they cannot end up with negative wealth. It is also important to note that this is a very restrictive neural network structure. Each hidden unit, h_k, is connected to only one information variable. This differs from standard neural networks which let all inputs influence each hidden unit. This step was taken to enhance tractability of the learned rules, but it does limit the generality of how agents can combine input information. It also provides some intuition for why this might be a sensible and tractable way to build dynamic trading strategies. Each information input is translated into a variable between -1 and 1. This could be interpreted as a kind of buy/sell signal. The next stage of the network could then be thought of as a kind of "or" gate across these signals by summing a weighted version of them together. This creates a rule which can respond by moving the portfolio into stocks when any number of individual trading signals are active which seems like a sensible picture of actual trading behavior. The weights are stored in a population table along with information on performance of this rule

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in the recent past. A simple real valued vector, ω_j, completely describes each dynamic trading strategy.

Agents chose the rule to use in the current period based on its performance in the past. They evaluate rules using their own specific perspective on the past. Some agents take a "long memory" approach, and weigh the near and distant pasts relatively equally. Others follow a "short memory" perspective, and weigh the distant past much less than the near past. These are a somewhat irrational type of trader who's view is that the world is constantly changing, and only recent information is relevant to investment decision making.

This is implemented using history lengths for decision making. Agent i will consider the following portfolio objective,

$$h_{ij} = \hat{E}(r_p) = \frac{1}{T_i} \sum_{k=1}^{T_i} \log[1 + \alpha(z_{t-k}; \omega_j)r_{t-k+1} + (1 - \alpha(z_{t-k}; \omega_j))r_f]. \tag{9}$$

The variables indexed by j correspond to the different possible trading rules. The only feature driving heterogeneity across agents' decisions is their memory, T_i.

Each period a random fraction of the population is queried to see if they want to change rules. This is set to 10 percent per week which corresponds to each agent checking their strategies about once per quarter. Decision making agents are presented with a new "challenger" rule (k) chosen at random from subset of active rules.[10] It is compared with the current rule (j) using the change in the objective, $\Delta h = h_{ik} - h_{ij}$. An easy mechanism at this point would be to switch rules if $\Delta h > 0$. However, this does not allow for unobserved heterogeneity in the decision making process. A richer approach is to use a discrete choice framework where the probability of a rule change (j to k) is given by

$$p_{jk} = \frac{exp(\phi \Delta h)}{1 + exp(\phi \Delta h)}, \tag{10}$$

where ϕ is the intensity of choice parameter. For very large ϕ this becomes the simple rule of switching to the better rule. For small ϕ the switch probability approaches 0.5 for any performance difference. In this way ϕ summarizes individual randomness in the decision making process, and will be a critical parameter.

2.3 Information

Agents trading strategies are based on simple information structures which are input to the neural network, and used to generate the trading strategies, $\alpha(z_t; \omega_j)$. Obviously, the choice of the information set, z_t, is important. This set will be chosen to encompass reasonable predictors that are commonly used in real markets. The information set will include, returns, past returns, the price dividend ratio, and trend following

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[10]This subset is composed of the best half of the rule set as defined by the agent's objective function, and memory length.

technical trading indicators. In the current version, the only types of technical rules used are exponential moving averages. The moving average is formed as

$$m_{k,t} = \rho m_{k,t-1} + (1 - \rho) p_t. \tag{11}$$

Formally, the information set, z_t, will consist of 6 items.

1. $r_t = \log(\frac{p_t + d_t - p_{t-1}}{p_{t-1}})$

2. r_{t-1}

3. r_{t-2}

4. $\log(\frac{d_t}{p_t})$

5. $\log(\frac{p_t}{m_{1,t}})$

6. $\log(\frac{p_t}{m_{2,t}})$

Several of the items are logged to make the relative units sensible. The two moving average indicators, $m_{1,t}$ and $m_{2,t}$, correspond to values of $\rho = 0.95$ and $\rho = 0.99$ respectively.

It is important to think about the timing of information as this will be important to the trading mechanisms covered in the next section. As trading begins at time t, all $t - 1$ and earlier information is known. Also, the dividend at time t has been revealed and paid. This means that α_j can be written as a function of p_t and information that is known at time t,[11]

$$\alpha_j = \alpha_j(p_t; I_t). \tag{12}$$

All variables in I_t are known before trading begins in period t. p_t will then be determined endogenously to clear the market.

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[11] The impact of p_t on the current information vector, z_t, is taken into account as well.

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2.4　Trading

Trading is performed by finding the aggregate demand for shares, and setting it equal to the fixed aggregate supply of 1 share. Given the strategy space each agent's demand for shares, $s_{i,t}$, at time t can be written as,

$$s_{i,t}(p_t) = \frac{\alpha_i(p_t; I_t)\beta w_{i,t}}{p_t} \tag{13}$$

$$w_{i,t} = (p_t + d_t)s_{i,t-1} + (1 + r_f)b_{i,t-1}, \tag{14}$$

where $w_{t,i}$ is the total wealth of agent i, and $b_{i,t-1}$ are the bond holdings from the previous period. Summing these demands gives an aggregate demand function,

$$D(p_t) = \sum_{i=1}^{I} s_{i,t}(p_t). \tag{15}$$

Setting $D(p_t) = 1$ will find the equilibrium price, p_t. This is essentially a simple Walrasian auction in the market for the risky asset. Unfortunately, there is no analytic way to do this given the complex nonlinear demand functions. This operation will be performed numerically. Also, it is not clear that the equilibrium price at time t is unique. Given the large number of nonlinear demand functions involved it probably is not. A nonlinear search procedure will start at p_{t-1} as its initial value, and stop at the first price that sets excess demand to zero.[12]

It is important to remember the equilibrium is found by taking the current set of trading strategies as given. Once p_t is revealed it is possible that agent i might want to change to a different trading rule. It is in this sense that the equilibrium is only temporary.[13]

2.5　Evolution

Around this structure of rules, agents, and markets is an evolutionary dynamic that controls adaptation and learning in the entire system. Agents acquiring more wealth will have a greater impact on prices. This alone is the primary evolutionary force behind the agent population. Occasionally agents may go bankrupt, and in these rare cases they are removed. Their debts are shared across the remaining agents, and a new agent is created with the median share and cash position. In practice, given the restrictions on short selling and

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[12]The search uses the matlab built in function fzero. Also, the results are not sensitive to starting exactly at p_{t-1}. Some experiments have been performed starting the search at p_{t-1} plus a small amount of noise. The results were for these experiments were not different from those starting the search at p_{t-1} exactly.

[13]This is similar to the types of learning equilibrium surveyed in Grandmont (1998).

borrowing, bankruptcy is a very rare event, and this form of evolution rarely occurs.

Rule evolution is more complicated. Rules are evolved using a genetic algorithm.[14] This method tries to evolve the population using biologically inspired operators that take useful rules, and either modify them a little (mutation), or combine them with parts of other rules (crossover).

One of the crucial aspects of evolutionary learning is the fitness criterion which is used to select good parents from the current generation. It is not clear what makes a rule "fit" in a multi-agent market. For example, it would be tempting to evolve the rules based on the historical performance on a fixed history of past data, but this would not capture the fact that agents are looking at different history lengths. To try to account for agent diversity a very weak selection criterion is used. A rule can be a parent for the next generation if at least one agent is currently using it. Rules that are not currently in use are marked for replacement with probability 0.005. This allows for some persistence in rules that are not in use in the current period. More complicated rule optimization would require a well defined objective surface which would be impossible to define for sets of heterogeneous agents. This method seems like a reasonable compromise in terms of building relatively robust rules that are not too finely tuned to the preferences of any one agent.[15]

Evolution proceeds as follows. First, the set of rules to be eliminated is identified. Then for each rule to be replaced the algorithm chooses between three methods with equal probability :

1. **Mutation:** Choose one rule from the parent set, and add a uniform random variable to one of the network weights, ω. The random increment is distributed uniform $[-0.25, 0.25]$.

2. **New weight:** Chose one rule from the parent set, chose one weight at random, and replace it with a new value chosen uniformly from $[-1, 1]$. This is the same distribution used at startup.

3. **Crossover:** Take two parents at random from the set of good rules. Take all weights from one parent, and replace one set of weights corresponding to one input with the weights from the other parent. This amounts to replacing the two weights that affect the input directly (linear and bias), plus the weight on the corresponding hidden unit. Visually this is equivalent to chopping off a branch of the network for parent one, and replacing it with a branch from parent two.

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[14]The genetic algorithm, Holland (1975), is a widely used technique in computational learning. Goldberg (1989) provides a good overview, and Mitchell (1996) gives a recent perspective. There are many evolutionary techniques, and this modified algorithm also contains inspiration from many of these others. Fogel (1995) provides a broad perspective to the complete set of methods.

[15]Hill climbing alone would also have to tackle the problem of over fitting in some efficient and sensible fashion. There is no easy solution to this problem.

A new rule is initialized by evaluating its performance over the past history of prices and information. Agents will use this performance history to decide on whether this rule should be used as they do with the others.

It would be difficult to argue that there is any particular magic to this procedure for evolving rules, and it goes without saying that these mechanisms are ad hoc. However, the objective is to produce new and interesting strategies that must then survive the competition with the other rules in terms of forecasting. Experiments have shown that the results are robust to different minor modifications of these mechanisms.

2.6   Equilibrium

It is useful in multi-agent financial simulations to have a benchmark with which the results can be compared. For multi-agent simulations the homogeneous agent world is often the appropriate benchmark. It turns out that in this model, for the given calibrated parameters, there is a homogeneous equilibrium in which agents all hold only the risky asset. Prices, dividends, and consumption all grow at the same expected growth rate. In this equilibrium stock returns should be unpredictable, and trading volume should be zero. In this sense it matches a classic efficient market situation where all information is contained in prices, and agents agree on functions mapping dividends into prices. Setting current consumption equal to dividends, and assuming all agents are the same gives the function mapping dividends to prices of

$$(1 - \beta)(p_t + d_t) = c_t = d_t \tag{16}$$

$$p_t = \frac{\beta}{1 - \beta} d_t \tag{17}$$

The existence of such an equilibrium provides an important benchmark for the model. Given the complexity of agent based financial markets it is not simply enough to match empirical facts. The model should also be able to show that for some region of the parameters it can do something consistent with existing economic theory.

2.7   Timing

The timing of the market is crucial since it is not an equilibrium model where everything happens simultaneously. A specific ordering must be prescribed to events. The following list shows how things proceed.

1. Dividends, d_t, are revealed and paid.

2. The new equilibrium price, p_t, is determined, and trades are made.

3. Rules are evolved.

4. Agents update their rule selection using the latest information.

5. Agents are evolved.

Although this appears to be a sensible ordering, it is not clear if other sequences might give different results. The fact that there needs to be an ordering is limited by usual computing tools. The best situation would be for things to be happening asynchronously, and software tools are becoming available to tackle this problem. However, this would open some very difficult problems in terms of trading and price determination.

2.8   Initialization and parameters

While this market is intended to be relatively streamlined, it still involves a fair number of parameters which may not have as much economic content as one would like. The first of these are the initialization parameters which control the agent and rule structure at startup. Rules are started with parameters, ω, drawn from a uniform $[-1, 1]$ distribution for each neural network weight. Agents begin with a memory, T_i, drawn from a specified distribution which will be set differently in various experiments. Bond holding levels are set to 0.1. The shares are divided equally among all agents. Finally, initial price and dividend series are generated using the stochastic dividend process, setting $p_t = d_t/(1 - \beta)$.

There are several other parameters that will remain fixed in these runs, but which may be interesting to change in the future. The number of agents is set to 500, the number of rules to 250, and the maximum history, T is set to 1500 weeks.

3   Results

The computer experiments presented here emphasize the key differences between two cases. In the first case, agents of many memory lengths, T_i, are allowed to interact in the market. This is referred to as the *all memory* case. Initial agents are drawn with T_i uniform over the range $[26, 1500]$ which corresponds to 6 months to about 30 years. This experiment is designed to explore the dynamics of the completely heterogeneous market setup. A comparison experiment, referred to as the *long memory* case starts the market with a set of only longer memory agents with T_i drawn from a uniform $[1350, 1500]$ distribution.

3.1 Run Summaries

Figure 1 presents plots of log prices and trading volume for a 25000 period run of the all memory case. Remember that periods are being calibrated to be one week in actual calendar time, so the 25000 period represents nearly 500 years of real price data. The figure shows a strong upward trend which is due to the trending random walk of the underlying dividend process. It also displays a large amount of variability about this trend with some very dramatic dips, and sharp rises. The corresponding volume series shows a moderate amount of trading activity with turnover rates of nearly 5 percent per week. There also appears to be some clumping to volume activity along with a connection between volume and large price moves. These features will be demonstrated in future sections. Finally, the lower panel displays a plot of the wealth weighted average of the agent memory lengths. This value moves around, but does not appear to be biased above or below the center of the uniform $[26, 1500]$ distribution. There is no indication of wealth moving to the longer memory strategies in this figure.

Figure 2 shows the same features for the long memory case. This displays a price series which appears to be following a random walk, and a volume series which is nearly zero. In the homogeneous equilibrium the price series should be proportional to the dividend series, and therefore follow a random walk as well. In the equilibrium all agents are in agreement on valuations, so trading volume should be zero. The simulation can occasionally generate some minor blips in trading as several agents may explore some out of equilibrium strategies, but they are quickly convinced to come back and join the rest of the crowd.

Figure 3 compares a short price series from the all memory run with the homogeneous equilibrium price,

$$p_t^e = \frac{\beta}{1 - \beta} d_t. \tag{18}$$

This is plotted along with the actual market price from the all memory market for the final 5000 periods. This plots shows the price slowly rising and breaking through the equilibrium, and then suddenly dropping below the equilibrium. The important feature is that the market is spending significant amounts of time far from the fundamental. The annual dividend yield (d/p ratio) is shown in the lower panel, and its fluctuations are consistent with the equilibrium price deviations in the upper panel.

A similar picture corresponding to the S&P 500 is shown in figure 4. This picture is drawn from the Shiller annual data, and uses Shiller's constant dividend discount price, P^*, as a comparison.[16] Obviously, there are many choices for a "rational price" in the actual data, but P^* is a good first comparison. The

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[16]See Shiller (1981), and Shiller (2002) for a recent analysis.

figure shows swings around P^* which are similar to the all memory simulation. The frequency of large price swings is a little smaller than in the actual data, but the patterns are similar. The lower panel displays the dividend yield again. This also shows large swings which are slightly less frequent then in the simulation. Finally, figure 5 repeats the previous pictures for the long memory only case. In this case there is little visible distinction between the true price and the homogeneous equilibrium price. Also, the dividend yield (d/p) is nearly constant. In this case the price does not vary much from the equilibrium fundamental price.

3.2   Stock returns

Weekly excess returns with dividends are sampled from periods 20000 to 25000, and are given by

$$r_t = \frac{p_t + d_t - p_{t-1}}{p_{t-1}}. \tag{19}$$

Sampling far out into the simulation run allows the system to move beyond the initial learning phase during which time some of the worst randomly initialized strategies are removed. Table 1 presents summary statistics for these returns in the two different cases along with comparison numbers for the S&P 500 index.[17] The first four columns correspond to the series mean, standard deviation, skewness, and kurtosis. For the mean return, the all memory simulation gives the highest value with 0.11 percent per week. The long memory cases generates a much lower weekly return of 0.08 percent. The values for the two S&P weekly subsamples are larger. The means for these values should not be compared with the simulations since they are nominal equity returns, and are not adjusted by inflation or the risk free rate. Further analysis of longer range data will allow for these comparisons. A similar pattern emerges for the standard deviation, kurtosis, and VaR. In all three of these the all memory case generates the most volatility, and the largest deviations from normality. The S&P is a little smaller on all dimensions, and finally the long memory case is the smallest. In the long memory case, the returns show no deviation from normality. Finally, the table presents Engle's test for ARCH.[18] The all memory and S&P display strong evidence for ARCH, while the long memory case shows none. This is consistent with this case generating returns are are close to independent Gaussians.

These results can also be observed in figures 6 and 7. These figures compare the all memory, and weekly S&P returns. The histograms in figure 7 are superimposed with a fitted Gaussian distribution. Both the all memory and S&P returns are similar in that they both exhibit fat tails, and the time series plots

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[17]The S&P numbers are sampled from 1947-2000, and 1928-2000. They are from Schwert (1990) and Datastream.

[18]The details of this specification test can be found in Engle (1982), or in any of the many surveys on ARCH/GARCH processes such as Bollerslev, Engle & Nelson (1995).

show some indication of pockets of large volatility that are clumped in time. Figure 8 compares the return histograms directly. There are some small differences which are consistent with the higher volatility levels in the simulations, but these differences do not stand out when examining the entire distributions.

The dynamic properties of the return series are compared in figures 9 and 10. These compare the autocorrelations between the two simulated market returns and the S&P weekly return series. In figure 9 the autocorrelations of the returns are estimated. For both the S&P and the all memory cases they are close to zero which indicates that short run linear predictability in these series is small. Put differently, the all memory case replicates a short run element of the efficient markets hypothesis in that short run returns are close to uncorrelated.

Figure 10 shows one of the most well documented features in financial markets, the persistence of volatility. While returns are relatively uncorrelated, the magnitude of returns is very persistent over time. Figure 10 shows the autocorrelations of the absolute value of returns. For both the all memory, and S&P these are positive, and continue out for many lags. For the long memory cases they are near zero for all lags. The simulation and the data are remarkably close in this comparison out to 52 weeks. This long range persistence in volatility has been well documented.[19] It is not clear why volatility shows such persistence.[20]

Figure 11 displays the cross correlations between today's returns and future and lagged volatility. Both the S&P weekly returns, and the all memory simulations display some negative correlations between current returns and future volatility.[21] The actual data shows a much stronger and pronounced swing as the cross correlation moves to leading volatility. However, both are negative and of a similar magnitude. The long memory displays a completely different pattern with near zero cross correlations except for a large positive contemporaneous cross correlation.

3.3   Trading Volume

An important feature of most agent-based markets is that they generate trading volume time series which can be compared with actual trading volume. The level of trading volume has already been noted in table 1. This section looks at the dynamics of volume, and its relation with returns. A single stock volume time series is used for this purpose since it is more appropriate for comparing with the single stock agent-based market. Figure 12 plots the returns (top panel) and trading volume (bottom panel) for IBM from 1990-2002.

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[19]See for example Ding, Granger & Engle (1993) and Baillie, Bollerslev & Mikkelsen (1996).

[20]LeBaron (2001*b*) shows that the observed long range persistence could be generated by adding a small number of short range volatility processes. Also, Kirman & Teyssiere (1998) presents a theoretical model generating long memory in volatility.

[21]This feature was first documented in Black (1976) and Christie (1982), and has recently been explored in Ang & Chen (2002).

For this short volume series it appears reasonable not to detrend the volume time series.

Figure 13 shows the persistence of trading volume in both the all memory simulations and the IBM series. It shows the autocorrelation pattern going out 52 weeks. The overall pattern is remarkably similar, although the IBM series looks very noisy. This should be expected since the length of the IBM series is 681 weeks which is short to look at the long horizon correlations estimated in figure 13. The simulation sample again corresponds to the longer 5000 length period at the end of a 25,000 period run.[22]

Figure 14 displays the cross correlations between returns at time t, and trading volume at $t + j$. There is some weak evidence of a negative correlation between current returns and future trading volume in both the simulation and the IBM data. However, the magnitude is quite small and probably insignificant.[23] What is interesting is that the same sign pattern appears for both the simulated and actual series.

The most dramatic alignment between the simulated and real data occurs in figure 15 which examines the connection between absolute returns and trading volume. For both series there is a strong contemporaneous cross correlation. Also, there is an interesting asymmetry in both series revealing a larger correlation from current volatility to future volume, than to past volume. It is clear that the shape of the cross-correlations from the data is well replicated by the market simulation.

3.4   Annual Returns

Calibrating to weekly series is only part of a reasonable model exploration. Simulated models should be able to replicate features at all horizons. This section looks at annual series and compares some well known benchmarks across the simulations and data. The comparisons performed here are admittedly casual. Statistical properties of the estimated moments are not estimated. This is done to keep the comparisons simple at this point, and to admit that much intuitive judgment is still needed in such a broad calibration. However, the smaller sample sizes at annual frequencies imply much larger standard errors around these estimated moments.

Annual excess returns are generated by compounding the weekly returns. The final 5000 weeks are used to generate 96 years of simulated annual returns. This is done for both the all memory and long memory cases. These series are compared with excess S&P returns from Shiller's annual data series. The log price/dividend

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[22] The long memory experiments are not included in any of the volume examples. Trading volume in the long memory case was so small that it generated unusual numerical instabilities in all these tests.

[23] The asymptotic confidence bands for the IBM series would be $1/\sqrt{681} = 0.04$ which is large relative to most of the IBM values. The bands for the simulations are $1/\sqrt{5000} = 0.014$, which are smaller, but are still large.

ratio is estimated by sampling the price and dividend once a year, and finding

$$p_t - d_t = \log(P_t/D_t). \tag{20}$$

The dividend series, D_t, is formed by taking the sum of the weekly dividend payments over the previous year. This is the same method used in constructing the S&P series.

Table 2 presents some simple summary statistics comparing the simulations with the actual data. The all memory simulation generates an excess return of 6.8 percent which is comparable to the value of 7.1 percent from the postwar S&P. Two other subsamples are also presented. One corresponds to the longer weekly series starting in 1928, and another represents the entire Shiller annual series starting in 1871. The excess returns in these are also comparable to the all memory values. The long memory case generates a lower value of 4.5 percent which should be expected given the larger price levels and lower dividend yields in this case. The variability of annual returns is compared in the second column. The all memory case generates an annual standard deviation which is slightly larger than the postwar S&P. It is 20 percent compared with 15 percent for the postwar S&P. However, annual variability is higher in the two longer subsamples. The 28-2001 sample displays an annual standard deviation of 19 percent. The third column compares the annual Sharpe ratios.[24] The all memory case, with a value of 0.33, is slightly lower than the postwar data, but is close to the longer sample values of 0.36 and 0.32. The long memory values of 0.71 is much larger than any other series.[25] The next column looks at the kurtosis of the excess returns. In actual returns the deviations from normality diminish as the sampling interval increases. This can be seen in the table for all the series. In both the simulated and real cases, excess kurtosis at annual frequencies is not large.

The final two columns show the mean and variability of the price/dividend ratios. The all memory and long memory simulations both display average price dividend ratios which are larger than many of the S&P subsamples, but they are similar in magnitude. The all memory case shows an average price/dividend ratio of 37 as compared to 28 for the postwar S&P. This is one case where the all and long memory runs are very similar. Much of the price/dividend level in the simulations is controlled by the consumer time rate of discount, β. The simulations are run with a value which is high, 0.975. A lower value of β would reduce the price/dividend ratios.[26] The final column displays the annual variability in the log price/dividend ratio. For the long memory simulations this value is basically zero, as has been shown figure 5. The all memory

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[24]Sharpe ratios are estimated as $(r_e - r_f)/\sigma(r_e)$.

[25]The long memory benchmark is curious in that it generates a Sharpe ratio which is too high. This is a feature of the model being at a constrained equilibrium, where agents would borrow funds to invest in equity markets if they could.

[26]It would also increase the excess returns as well.

case generates too little variability in the ratio with a standard deviation of 0.17 as compared to 0.42 for the postwar sample.

Table 3 displays some of the dynamic features of the annual excess returns and price/dividend ratios. The first panel shows the autocorrelations for the log price/dividend ratio. The all memory simulation displays some positive correlations at the one year lag, but it goes away at lags of two or more. The S&P series all display positive and very persistent autocorrelation properties extending out at least 4 years. This does suggest a different magnitude of persistence in the actual data where much longer horizon features appear to be at work. There is more evidence of this in the autocorrelation patterns for annual returns. The all memory case shows very strong reversals at the 1 and 2 year horizons, as demonstrated by the large negative autocorrelations. In the S&P series there is some evidence for negative correlations at the 2 year horizon, but it is extremely weak.

The final panel examines the cross-correlations from the current price/dividend ratio to future excess returns. This is related to the large literature on long horizon forecasting of returns.[27] The all memory simulation displays a strong negative correlation one year in the future. This is comparable to the postwar S&P series which shows a similar pattern. These patterns are weaker, but similar, in the longer range samples. The simulated series would most likely generate useful forecasts from price/dividend regressions. The long memory case shows little cross correlations with future returns.

The annual return results are not as strong as the weekly results. It is clear that the agent-based simulation is better at replicating shorter horizon features. However, it is able to get the basic moments at the long horizon pretty close to the actual series. Where it is weakest is in the dynamic properties at the longer horizons. This problem is foreshadowed in some of the earlier figures which display the changing dividend yields, and price deviations from equilibrium pricing. These figures show some visual evidence for more frequent fluctuations in the simulations than in the actual data.

3.5   Consumption

Beyond equity market dynamics this model also generates consumption series. Table 4 presents summary statistics on annual consumption growth rates. For the market simulations these are generated by aggregating the weekly consumption series. The table shows a clear weakness of the model. While it is able to magnify the variability from the dividend process into a reasonably volatile price series, the fact that consumption moves in proportion to wealth carries this volatility into the consumption series as well. The all memory

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[27]See Campbell & Shiller (1988) and Campbell & Shiller (2001) for examples.

runs generate a consumption series which is more than eight times as volatile as the actual U.S. consumption series. The long memory run also displays an increase in consumption variability which is due to the fact that the runs are calibrated to aggregated dividends which are more volatile than consumption.

Figure 16 displays the total consumption/dividend ratio for the two simulations. It is clear that the long memory case yields a consumption value which is close to the equilibrium where consumers will consume 100 percent of their dividends. In the all memory case the consumption level relative to the dividend varies a lot. Often going as much as 50 percent above or below the dividend level.

4   Conclusions

The agent-based model is capable of quantitatively replicating many features of actual financial markets. Comparisons show favorable results for returns and volatility and their persistence. The data also replicates the well known feature of excess kurtosis in the returns series. It also shows promise in generating the joint dynamics of returns and trading volume along with the persistence of trading volume itself. Given that the market is forced to rely on a dividend process fitted to the U.S. aggregate, and to keep within the bounds of well defined intertemporal preferences, these successes are quite remarkable.

In addition to these features, there were places where the market appears weak. The biggest of these is consumption. As one could easily predict given the proportionality of consumption to wealth, consumption moves around considerably given asset price volatility. This shows that while agent-based markets can be viewed as a type of volatility generating engine, they cannot immediately solve one of the basic problems of macroeconomics and finance, the dichotomy between return and consumption variability. Other preferences and consumption rules will need to be considered to solve this puzzle.

The alignment with the actual financial serious at the annual frequency is acceptable, but displays some differences. The most important of these is a much less persistent price/dividend ratio in the simulations than the actual series. This is probably due to the fact that the large swings in prices occur slightly more frequently in the simulated market than in real world data. Future simulations will need to address this problem.

A crucial test of the reasonableness of many of the learning assumptions, and the importance of the short memory form of irrationality was the fact that when short memory agents are eliminated the market converges to a well defined homogeneous equilibrium. This shows us the learning machinery in use has the power to drive the behavior to an equilibrium, but in most of the experiments it is hindered by the behavior

of short memory types.

There are also some theoretical shortcomings to the model that will need to be addressed in future versions. Although reasonable, the market is still only a partial equilibrium model with a fixed risk free interest rate. This allows it to completely avoid issues about getting the risk free rate right in terms of levels or volatility. Future versions will need to endogenize the risk free market. Related to this is the fact that the equilibrium the long memory agents converge to is constrained in that they would like to borrow more at the zero risk free rate. Whether the long memory agents alone could learn an interior equilibrium is an interesting question. Finally, the two asset world is simple, and allows for many useful comparisons. However, if one wants to eventually get serious about calibrating both levels as well as the dynamics of trading volume, then a richer asset environment will be needed.

This market can only be viewed as an initial test of an emerging technology for finance and economics. Previous simplified analytic models have not fared well in matching financial data, and the time and technology have now arrived to turn toward agent-based approaches. However, several important questions still remain. The model presented is complicated, and contains many deep parameters controlling evolution and learning for which we have only very weak notions of what their values should be. These apparent degrees of freedom could potentially be used to fit just about any feature of the data. On the other hand, these models appear to fit many features which more traditional models do not even consider.

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Table 1: *Weekly Return Summary Statistics*

	Mean (%)	Std (%)	Skew	Kurtosis	1% VaR (%)	ARCH(5)	Volume
All Memory	0.114	2.51	-0.567	10.23	-7.47	315.3	0.065
Long Memory	0.083	0.75	-0.012	3.01	-1.69	2.7	$0.053x10^{-8}$
S&P (47-2000)	0.175	1.88	-0.380	5.92	-4.69	211.1	
S&P (28-2000)	0.140	2.56	-0.214	11.68	-7.38	738.6	

Summary statistics for weekly returns. Simulation returns include dividends. S&P returns are nominal without dividends. 1% VaR is the Value-at-Risk at the one percent level, or the 1% quantile of the return distribution. ARCH(5) is the Engle test for ARCH using a autoregression of squared returns on 5 lags. It is distributed asymptotically χ_5^2 with a 1% critical value of 15.1

Table 2: *Annual Return Summary Statistics*

	Excess Return (%)	Return Std (%)	Sharpe Ratio	Kurtosis	exp(mean($p-d$))	$\sigma(p-d)$
All Memory	6.8	20.5	0.33	3.49	36.7	0.17
Long Memory	4.5	6.2	0.71	2.96	39.5	$7.16x10^{-8}$
S&P 47-01	7.1	15.3	0.47	2.86	28.0	0.42
S&P 28-01	6.9	19.4	0.36	3.25	26.0	0.41
S&P 1871-2001	5.8	17.9	0.32	3.21	22.7	0.35

Annual summary statistics. All returns include dividends and are excess of a T-bill. Simulation returns are compounded over 52 weeks, and are non-overlapping. $p-d$ is the logged price dividend ratio.

Table 3: *Correlations*

Variable	Series	ρ_1	ρ_2	ρ_3	ρ_4
$p - d$	All Memory	0.31	-0.10	-0.13	0.16
	Long Memory	-0.07	-0.06	-0.16	0.00
	S&P 47-01	0.86	0.71	0.58	0.46
	S&P 28-01	0.85	0.71	0.59	0.50
	S&P 1871-2001	0.84	0.70	0.61	0.52
$r_e - r_f$	All Memory	-0.11	-0.37	-0.23	0.10
	Long Memory	0.00	-0.12	-0.21	-0.09
	S&P 47-01	0.00	-0.16	0.21	0.29
	S&P 28-01	0.04	-0.13	0.05	-0.14
	S&P 1871-2001	0.07	-0.17	0.08	-0.10
$(p - d)_t, (r_e - r_f)_{t+j}$	All Memory	-0.39	0.02	0.24	0.07
	Long Memory	-0.04	-0.17	-0.07	-0.01
	S&P 47-01	-0.20	-0.11	-0.14	-0.20
	S&P 28-01	-0.12	-0.11	-0.15	-0.14
	S&P 1871-2001	-0.10	-0.04	-0.08	-0.11

Autocorrelation and cross correlations. Panel 1, labeled $p - d$, are the autocorrelations of the log(price/dividend) ratio. Panel 2, labeled $r_e - r_f$, are the autocorrelations of excess returns. Panel 3, labeled $(p - d)_t, (r_e - r_f)_{t+j}$ are the cross correlations from the current price/dividend ratio to future excess returns. All time lags are in units of years.

Table 4: *Consumption*

Series	$E(\Delta c)$	$\sigma(\Delta c)$
All Memory	1.9	20.7
Long Memory	1.7	4.8
1947-2001	2.1	2.0
1928-2001	1.2	2.2

Consumption growth rates estimated by aggregating weekly consumption flows to annual series. Comparison values are real annual consumption values for the U.S. from Shiller's annual series.
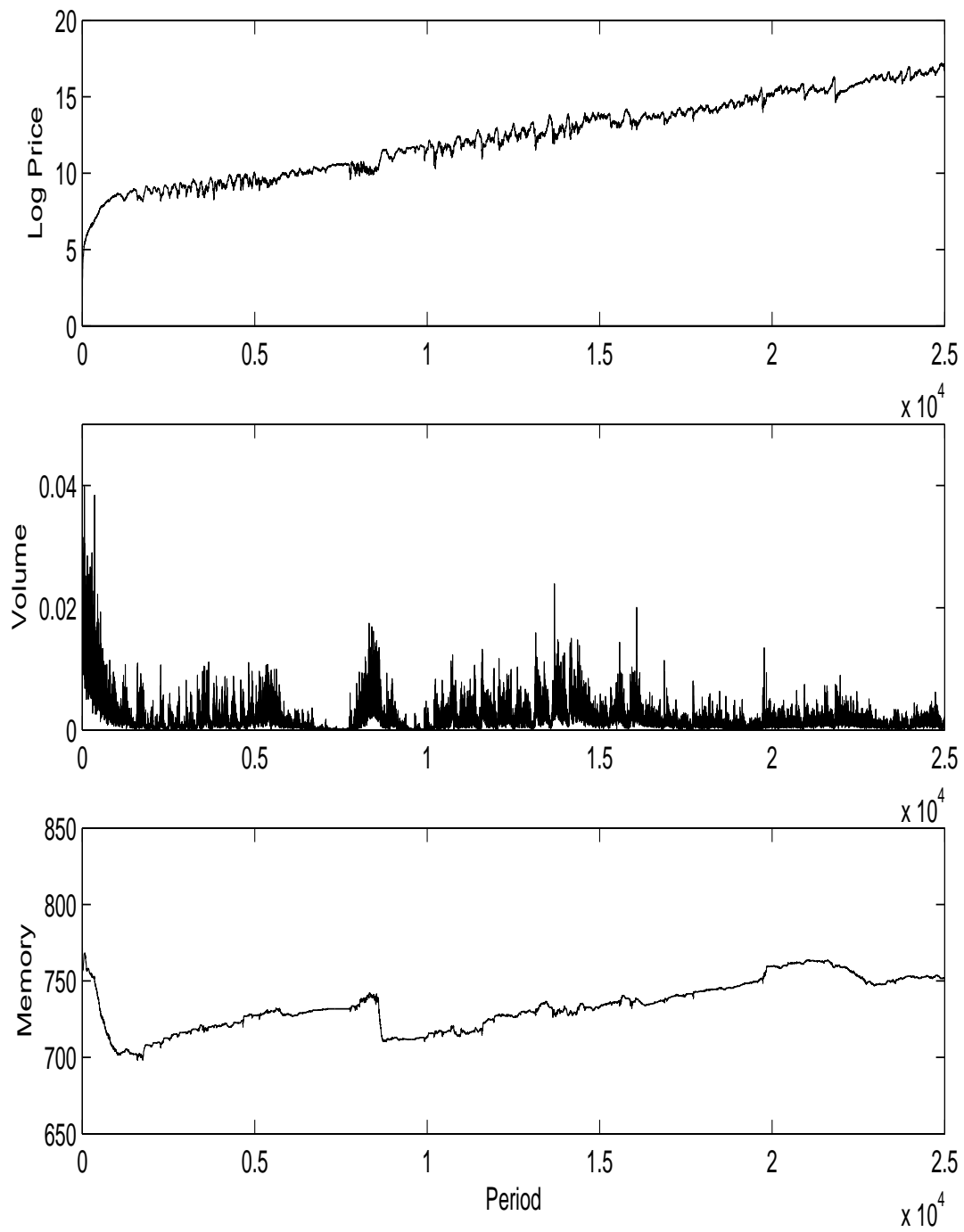
Figure 1: **Price/Trading Volume time series for all memory agents**

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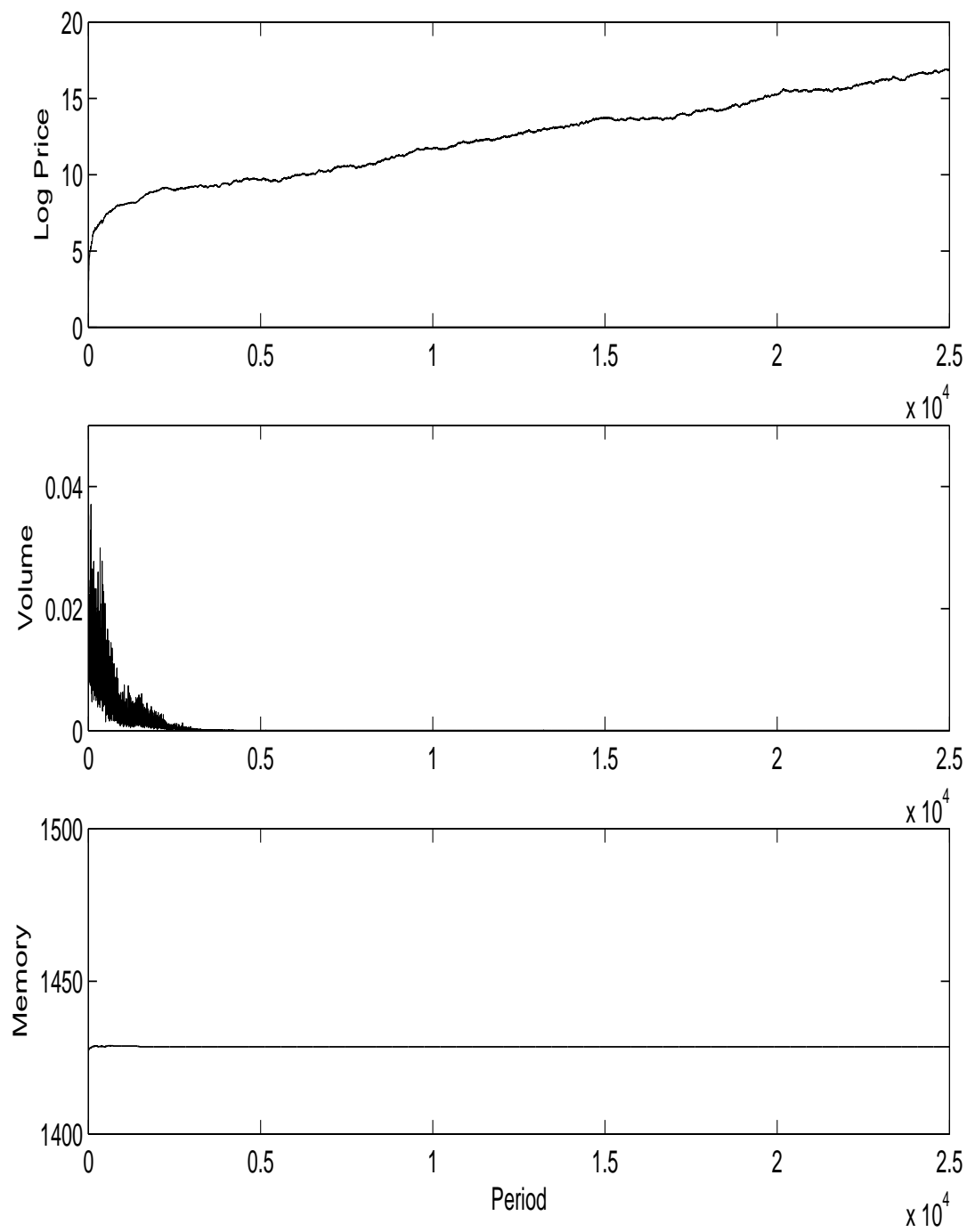
Figure 2: **Price/Trading Volume time series for long memory agents**
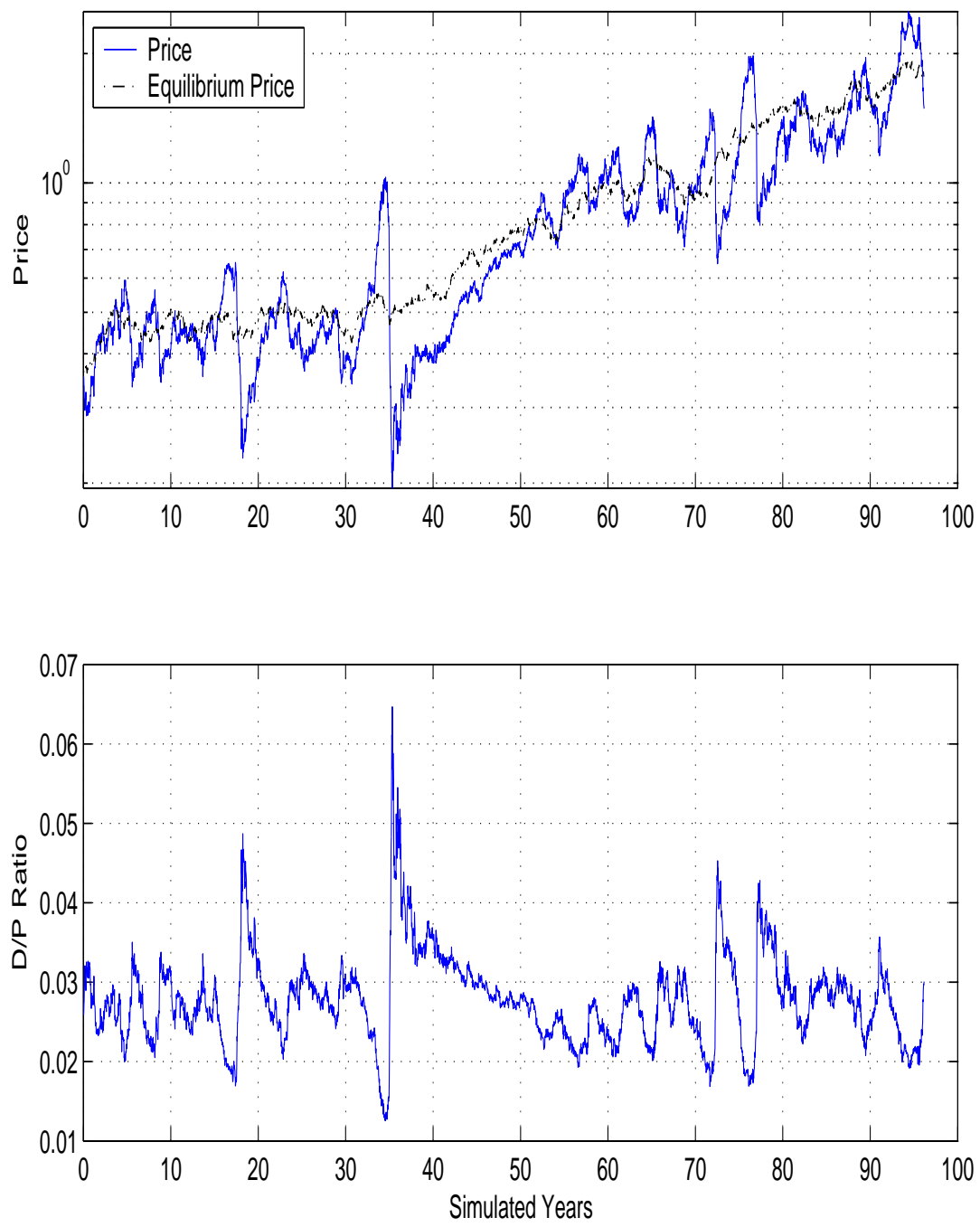
Figure 3: **Price, equilibrium price, and d/p ratio for all memory case**
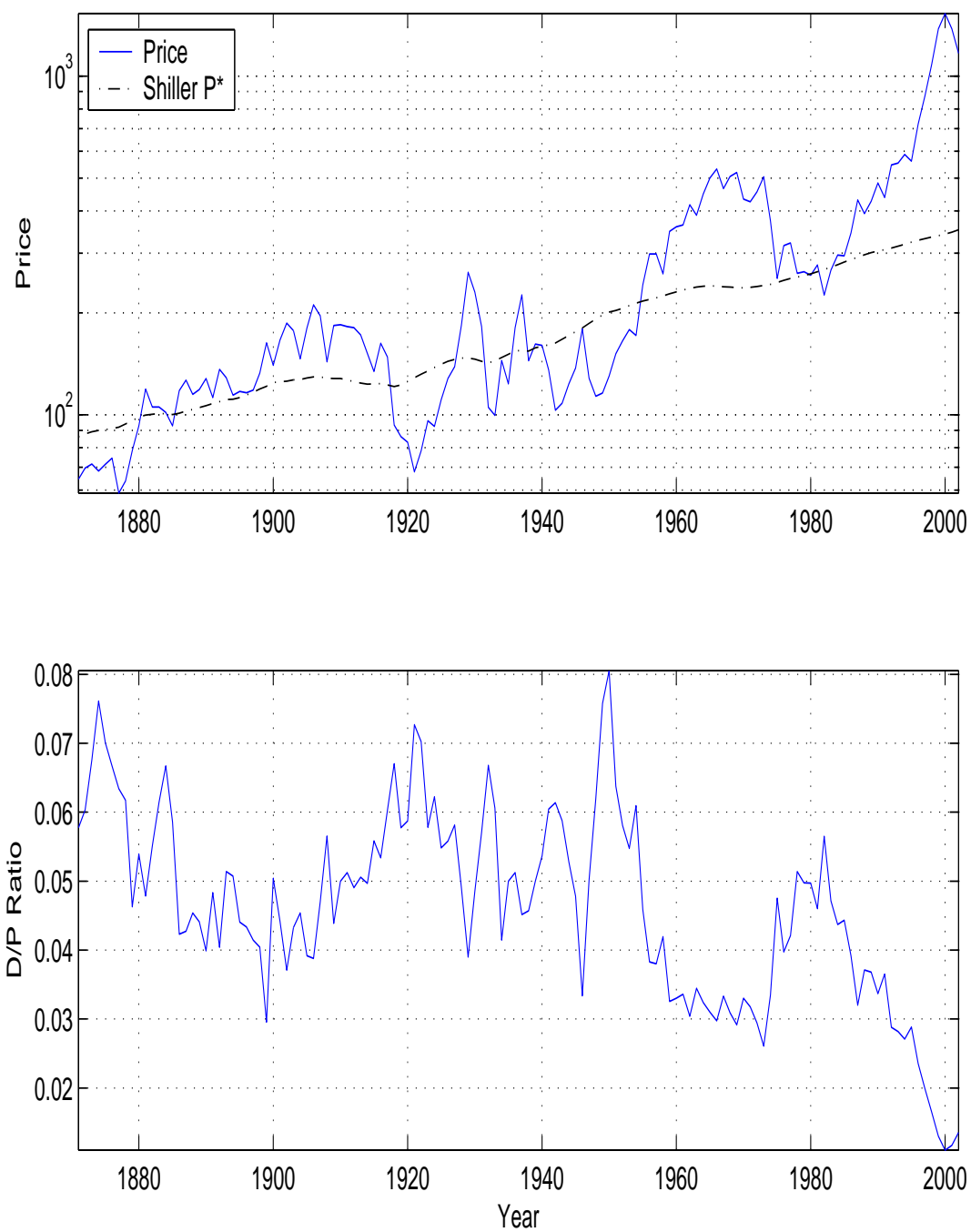
Figure 4: **Price, Shiller P*, and d/p ratio for S&P**
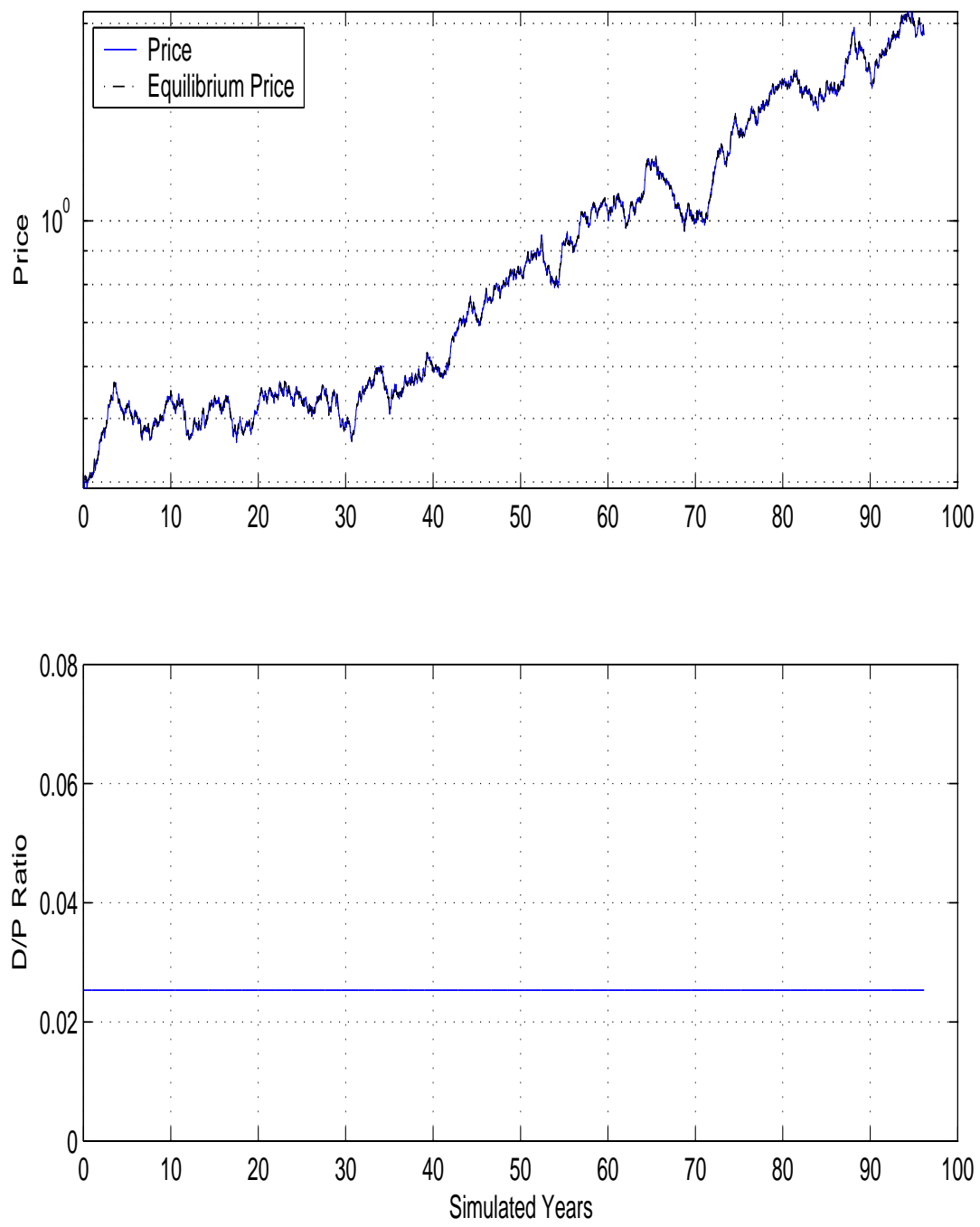
Figure 5: **Price, equilibrium price, and d/p ratio for long memory case**
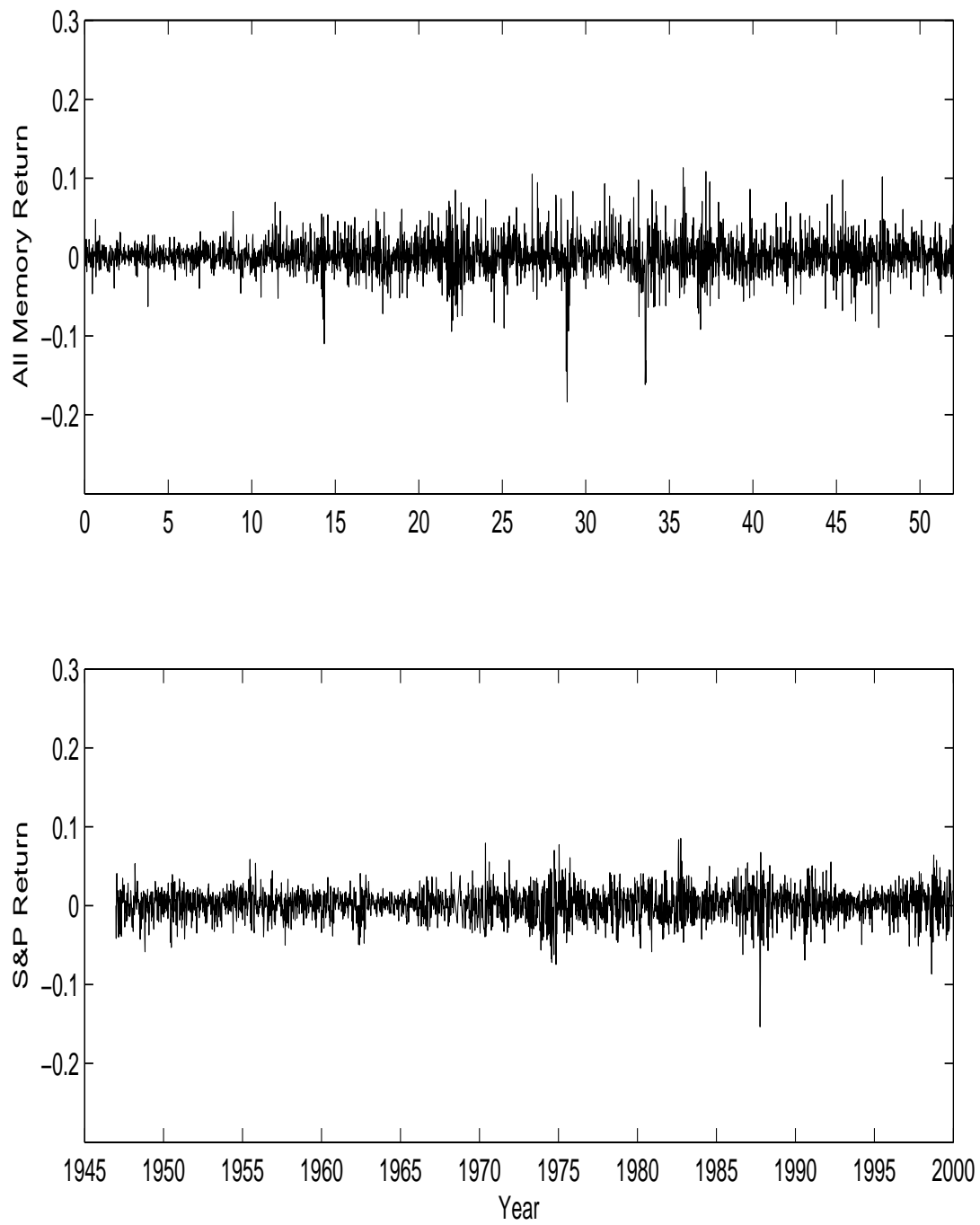
Figure 6: **Weekly Returns: Simulation (top), and S&P (bottom)**
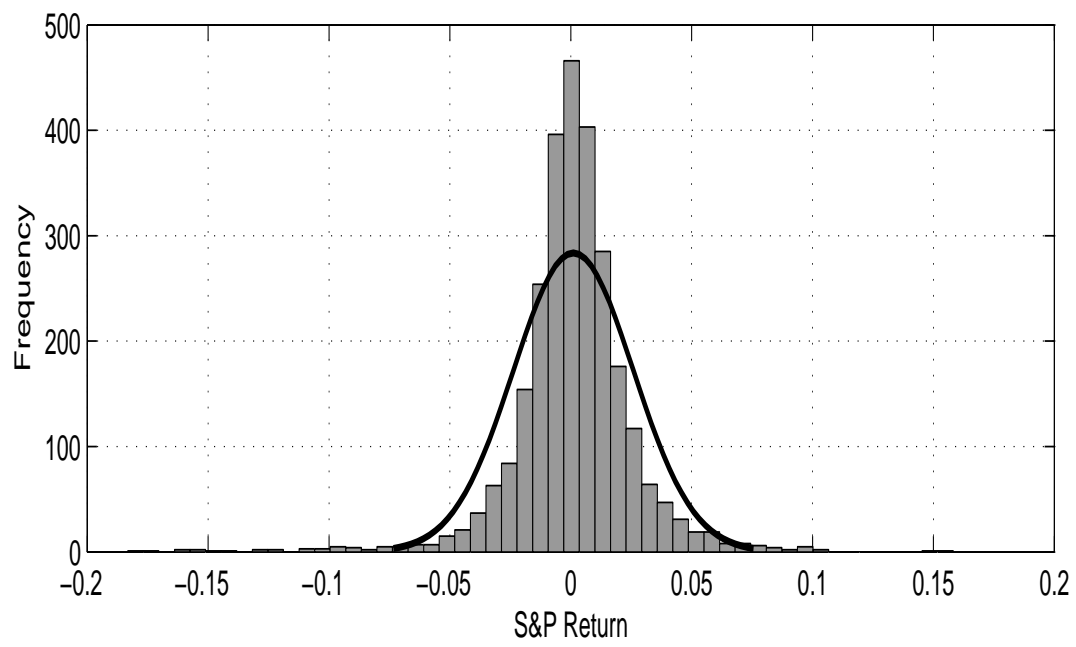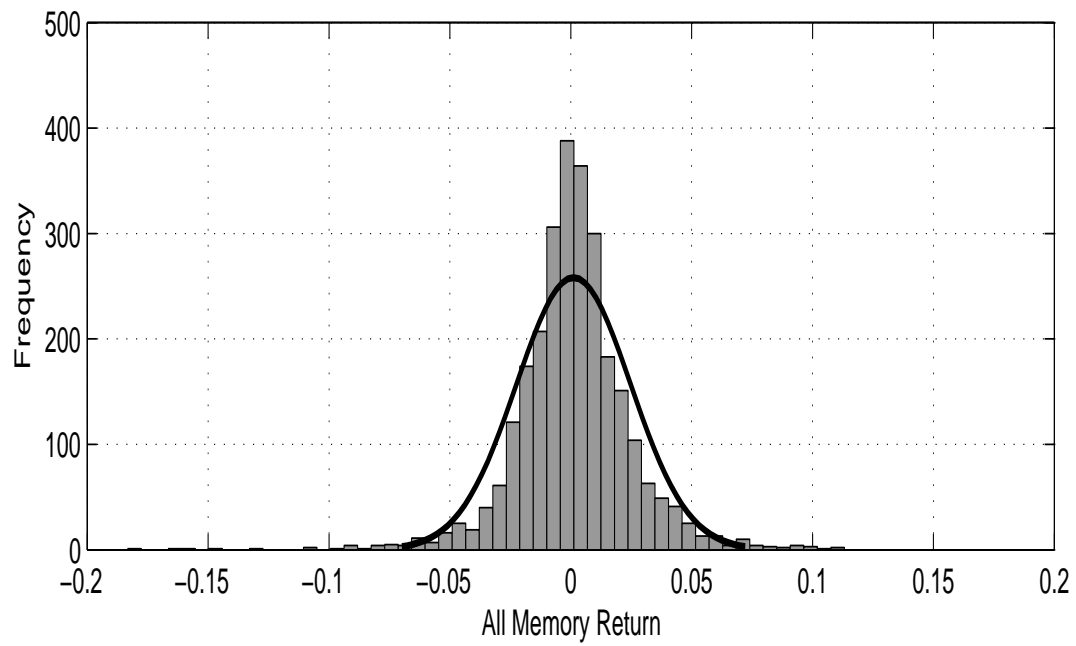
Figure 7: **Weekly Returns: Normal comparison, Simulation (top), and S&P (bottom)**
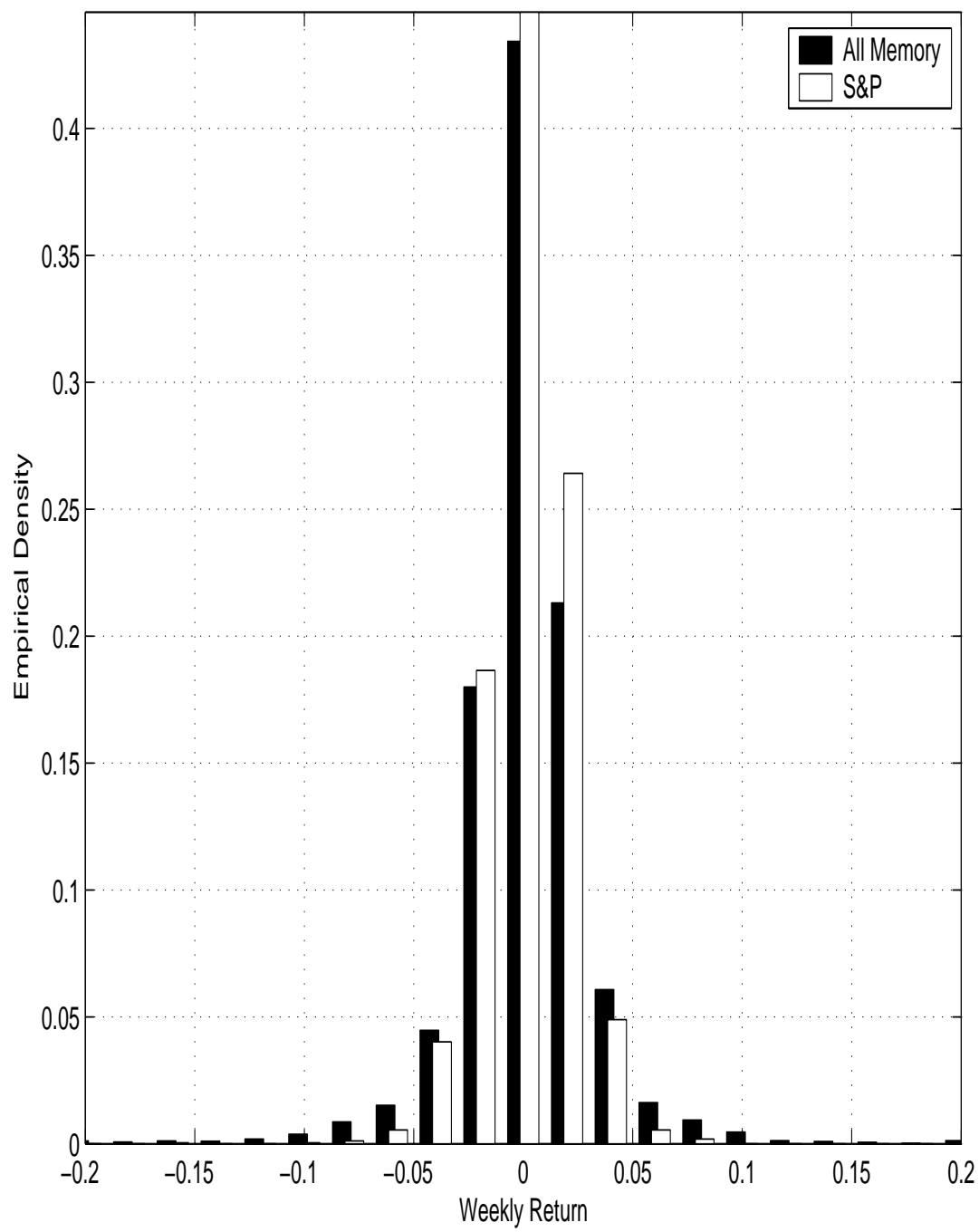
Figure 8: **Weekly Returns: 5000 All Memory and S&P**
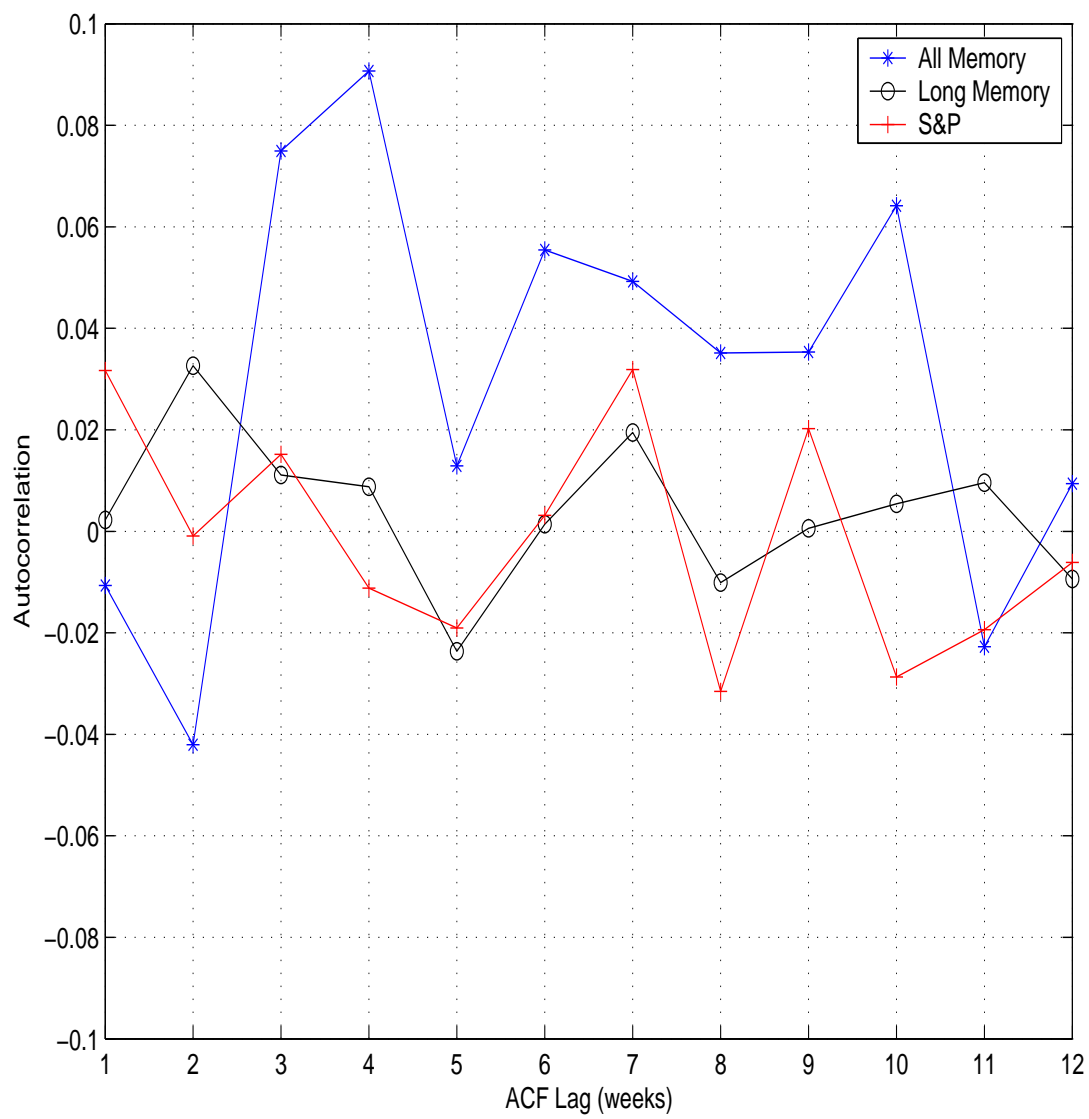
Figure 9: **Return Autocorrelations: All Memory, Long Memory, S&P**
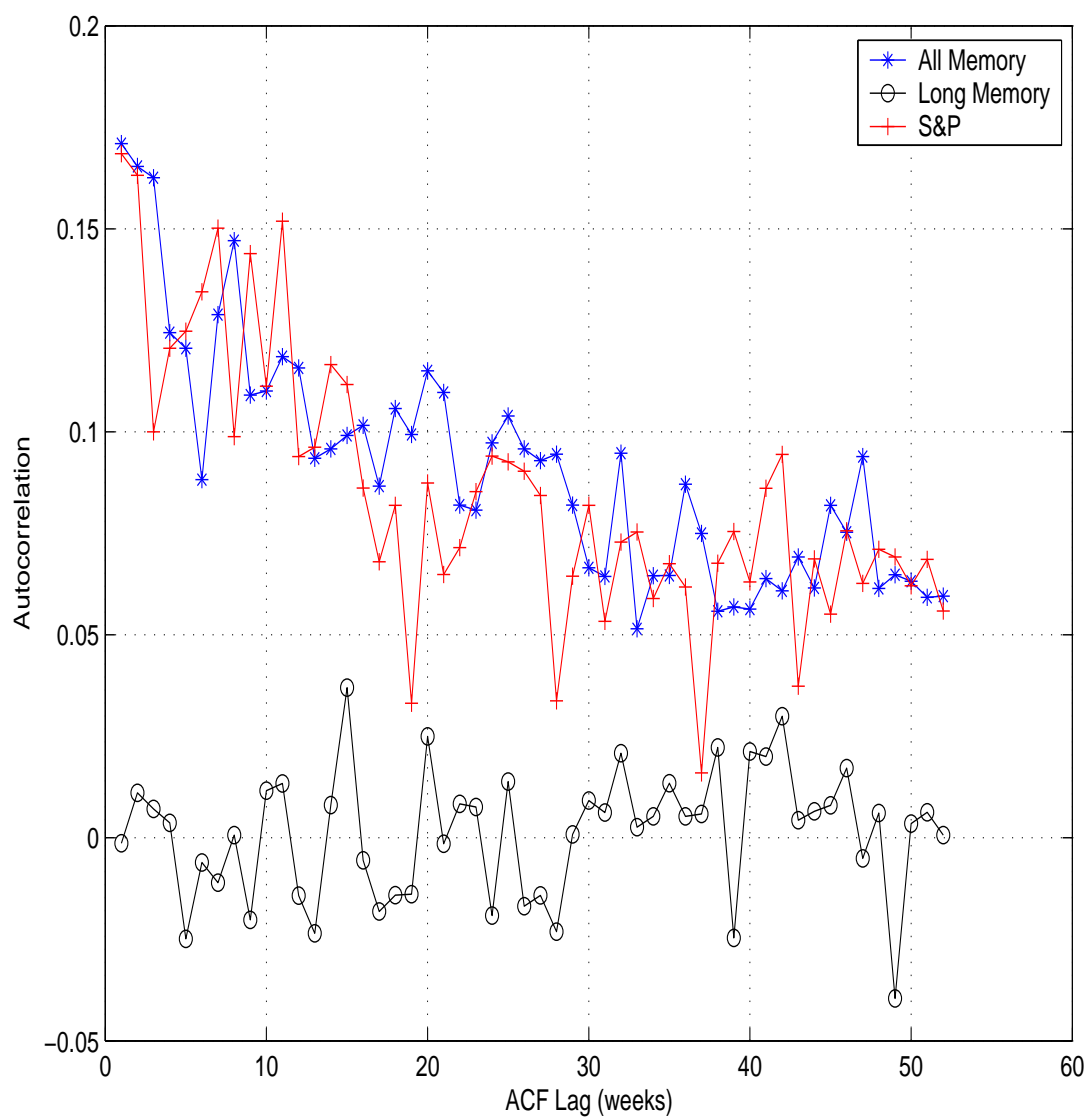
Figure 10: **Absolute Return Autocorrelations: All Memory, Long Memory, S&P**
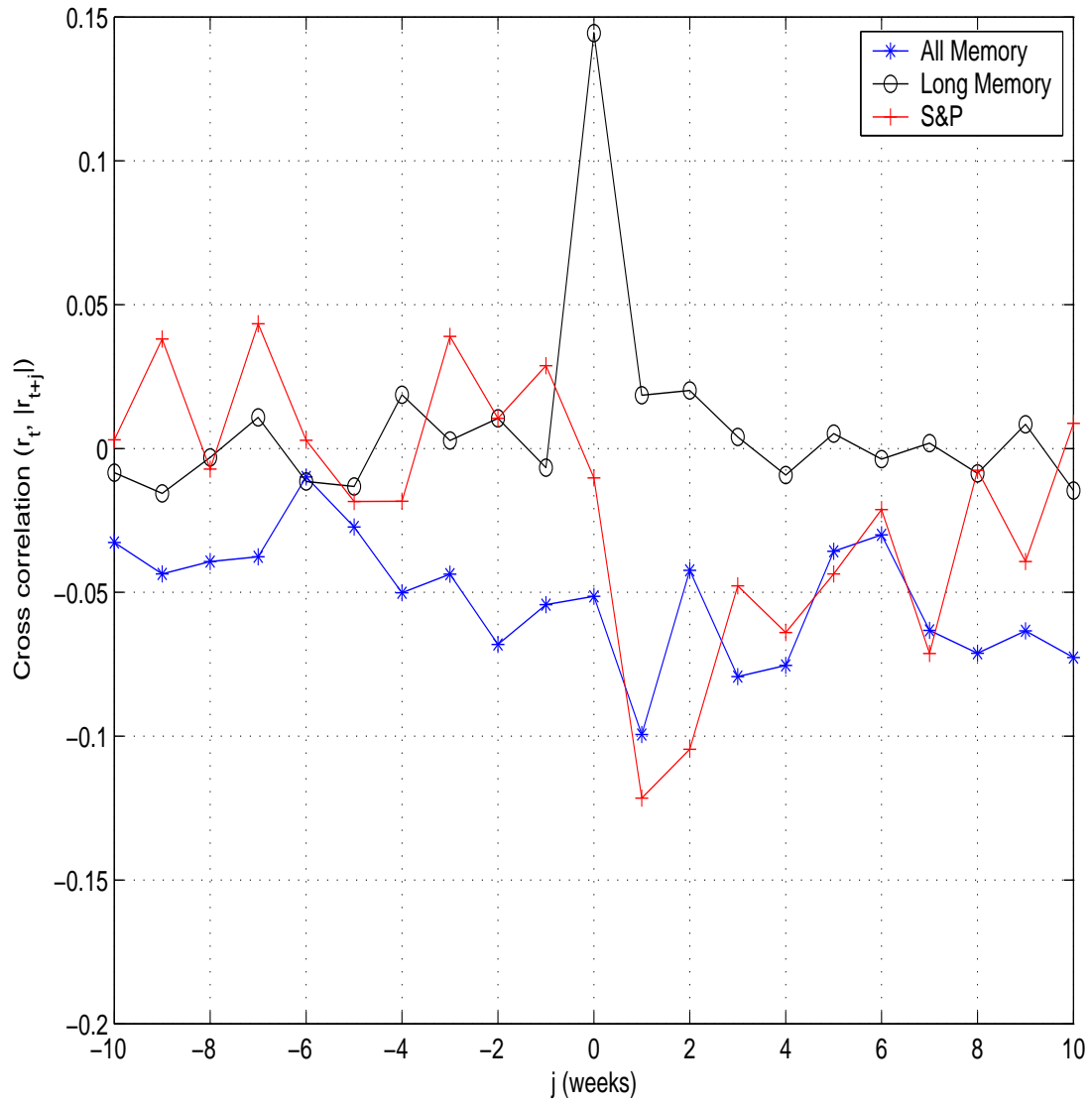
Figure 11: **Return/Absolute return crosscorrelations: All Memory, Long Memory, S&P**
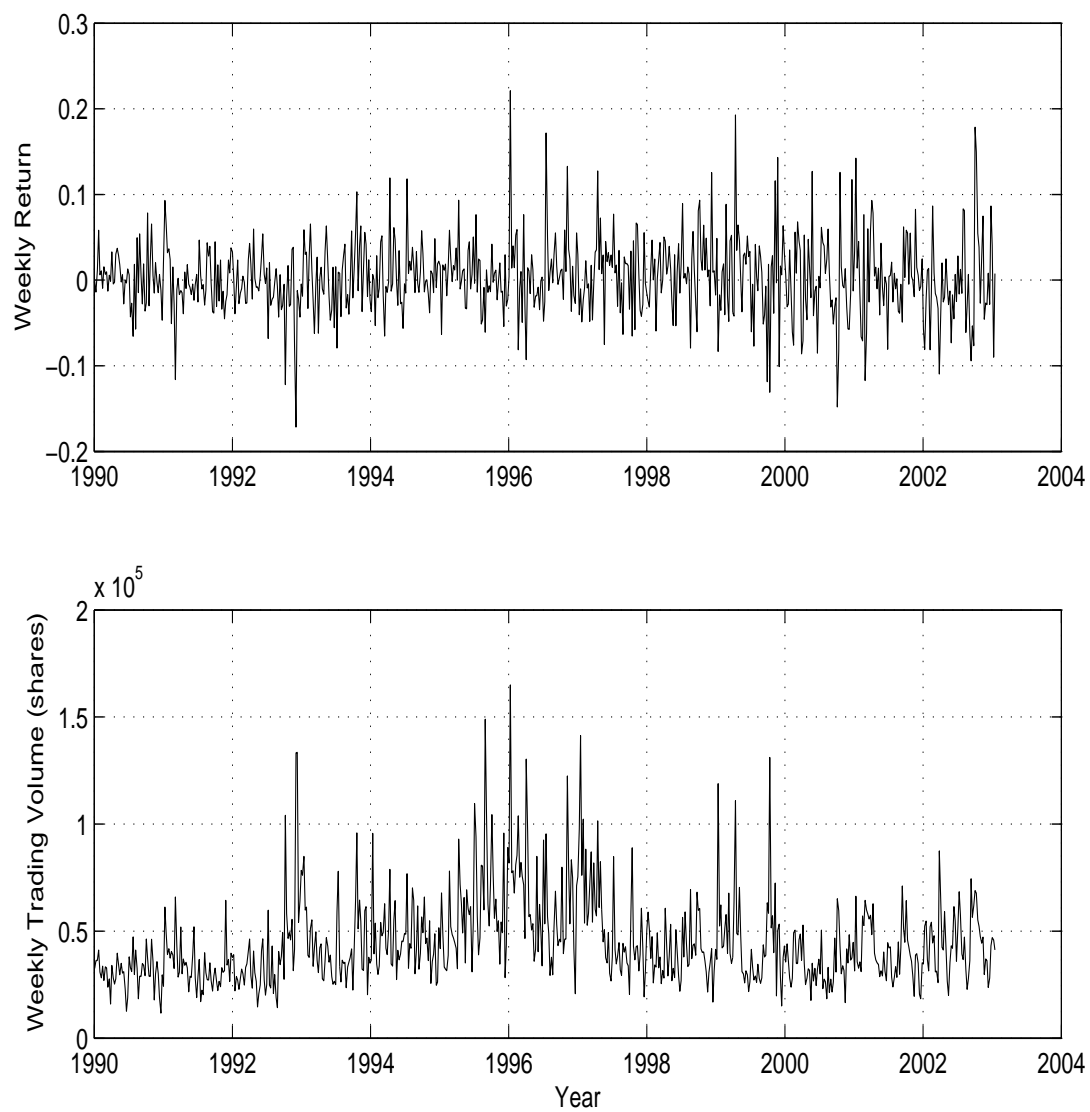
Figure 12: **IBM Returns (top), and Volume (bottom)**
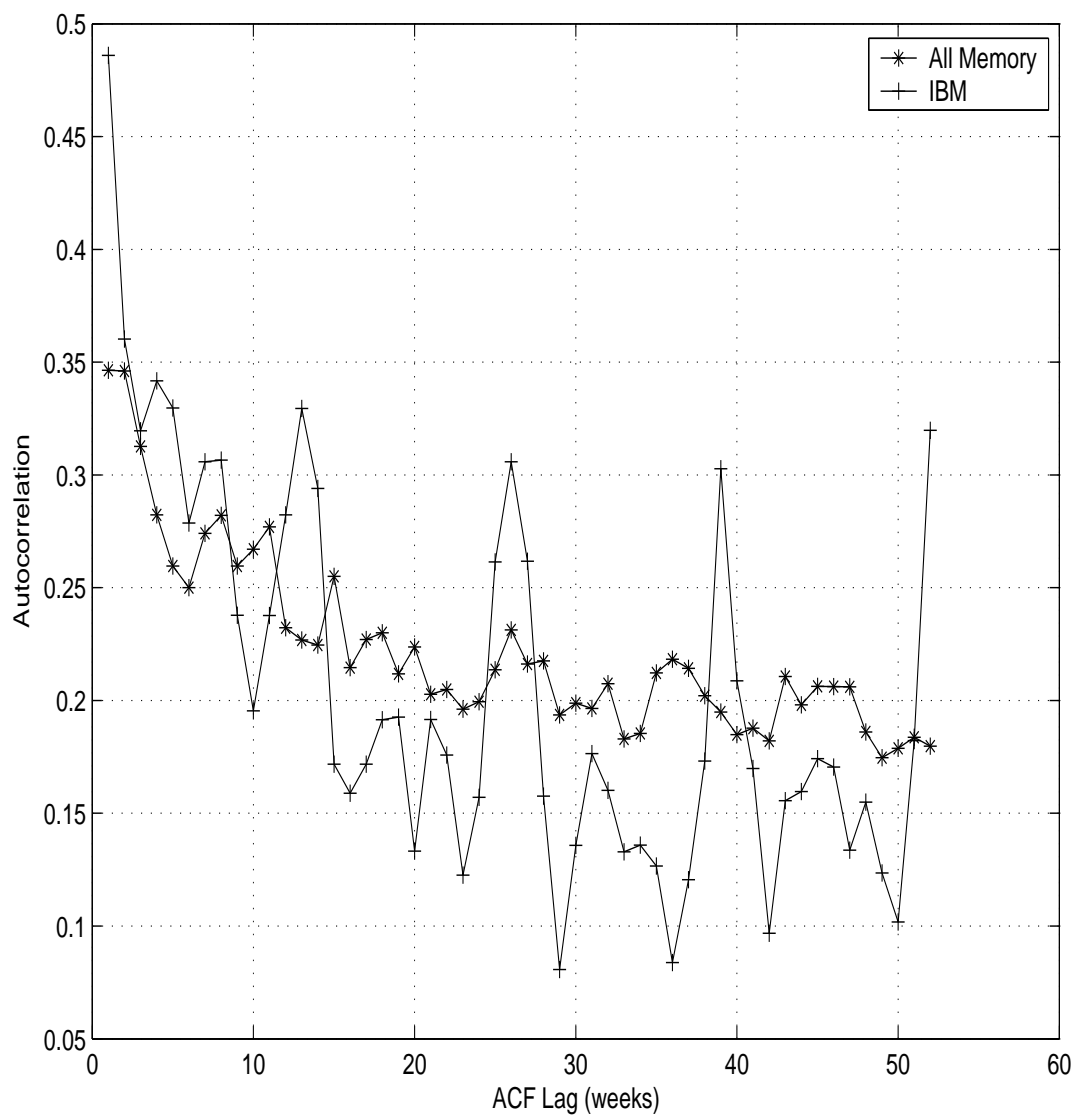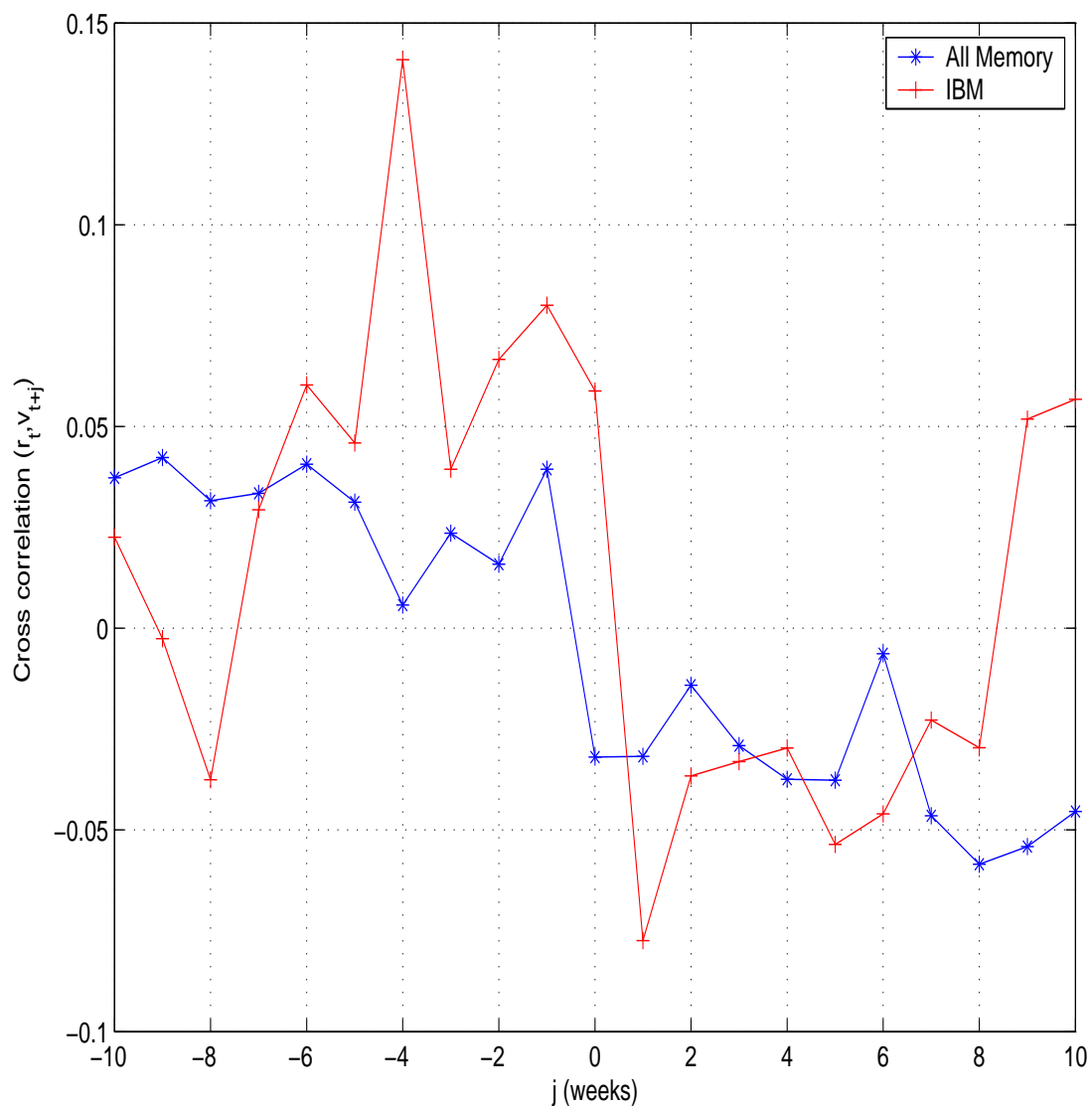
Figure 13: **Trading Volume Autocorrelations**

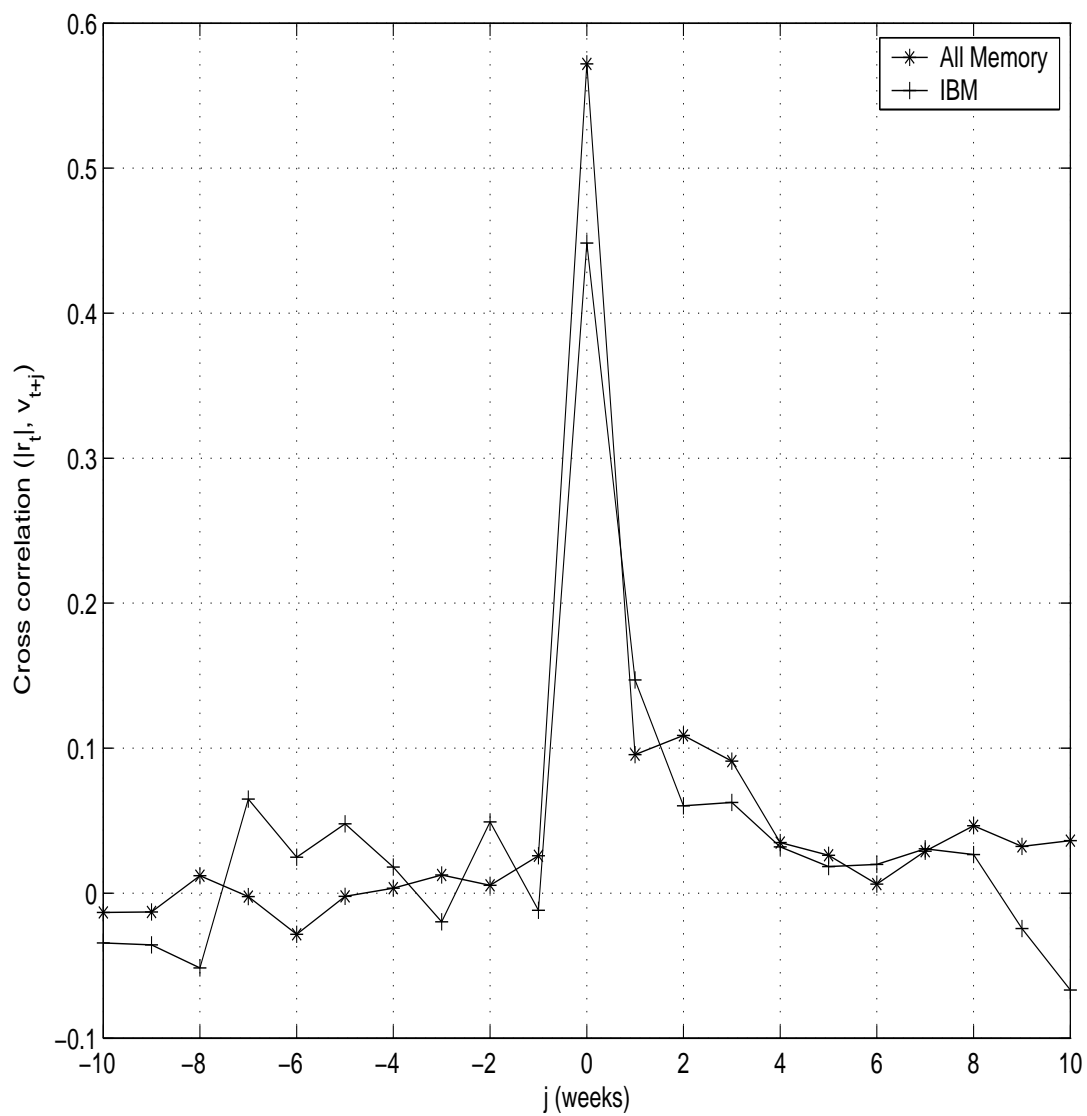Figure 14: **Return/volume cross correlations**
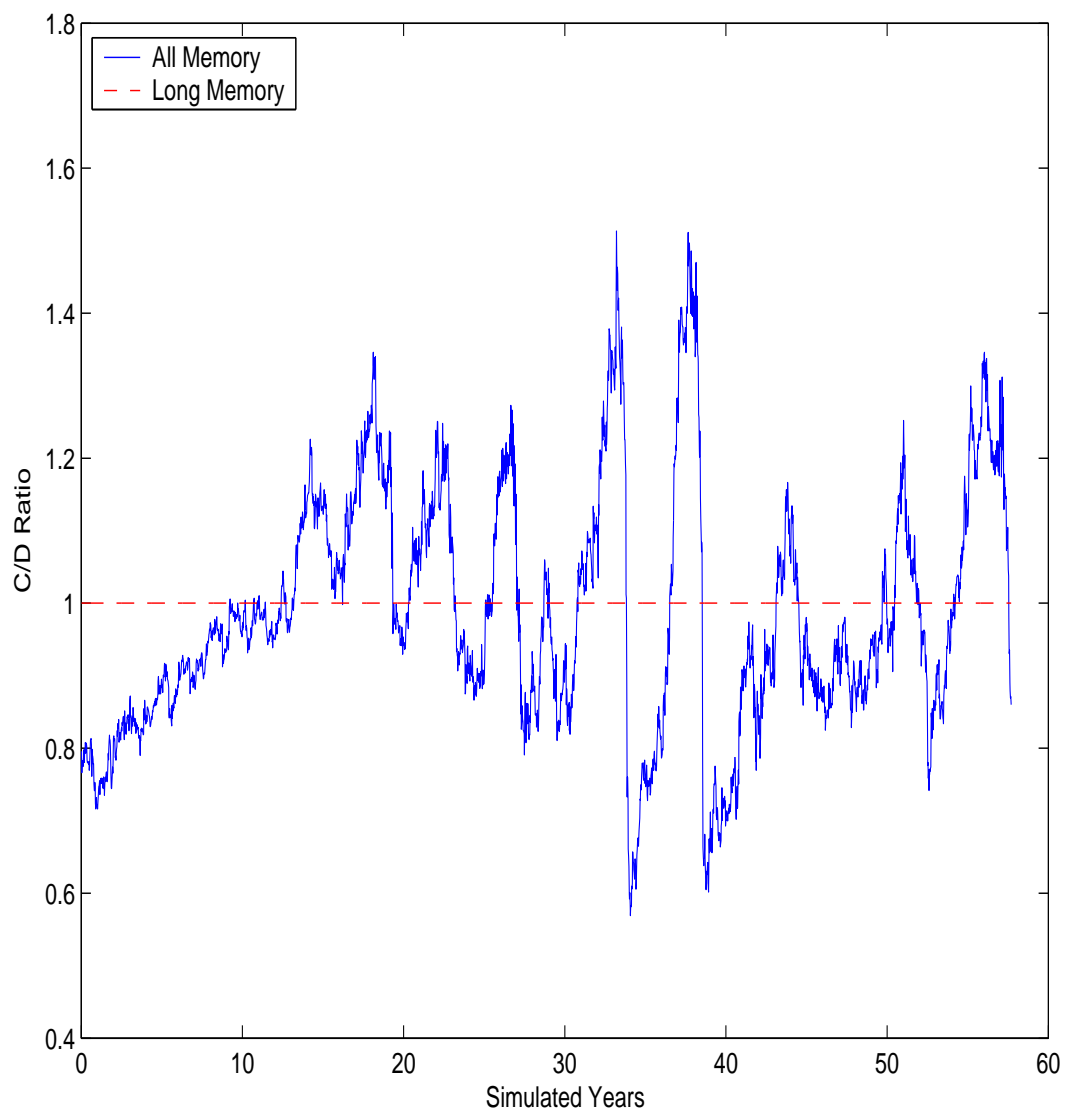
Figure 15: **Absolute return/volume cross correlations**

Figure 16: **Consumption/Dividend Ratios**