Stochastic Volatility as a Simple Generator of Financial Power-laws and Long Memory

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Abstract

There has been renewed interest in power-laws and various types of self-similarity in many financial time series. Most of these tests are visual in nature, and do not consider a wide range of possible candidate stochastic models capable of generating the observed results. This paper presents a relatively simple stochastic volatility model which is able to display power laws and scale invariance similar to actual financial data even though it constructed to have none of these properties. The primary mechanism is that volatility is assumed to have a driving process with a half life that is long relative to the tested aggregation ranges. It is argued that this might be a reasonable feature for financial, and other macroeconomic time series.

Keywords: Volatility, Power-laws, Long Memory

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1 Introduction

There is a long history of looking at financial markets from the standpoint of self-similarity. The pioneer in this field is unquestionably Benoit Mandelbrot.[1] Recently, the availability of high frequency data, and results on critical phenomena and scaling laws from physics have lead to a renewed interest in the ability of scaling relations and power-laws to uncover the underlying nature of socioeconomic processes.[2] The appeal of this research is the possibility that it will bring forth a new set of stylized facts guiding economic theorizing in many sub-fields including finance, industrial organization, labor economics, and growth.

This paper tests the ability to empirically discern between processes with many different length scales, versus those with only a few. It also asks about the ability to detect true long memory versus the simple combination of dynamics operating on a small number of different time scales including one that is long relative to the sample length. This latter process could be a proxy for slowly changing policies or fluctuations at business cycle frequencies.[3] In this sense this is a study of the statistical power of visual tests to reject scaling behavior when it is not present. It is also a test to see if there exist simple models of persistent volatility which can replicate the visual features from actual data.[4]

Section 2 introduces the class of stochastic models that will be used, and the calibration of their parameters to actual data. The following three sections, 3, 4, and 5, examine the ability of this process to reproduce a set of stylized facts. Section 3 looks at scaling laws in the tails of the unconditional distributions, and their stability over different time scales. Section 4 explores the long memory features of the simulated stochastic volatility process. Section 5 looks at tests for potential multi-fractal relationships in the return time series, and shows reasonable alignments with actual series. Finally, section 6 summarizes, concludes, and puts the results in perspective for both skeptics and non skeptics of power-laws.

[1]See Mandelbrot (1963) for his original classic, and also Mandelbrot (1997) for a collection of his works in finance.

[2]The earliest applications of high frequency data and scaling can be found in Muller, Dacorogna, Olsen, Pictet, Schwarz & Morgenegg (1990) which are also summarized in the book, Dacorogna, Gencay, Muller, Olsen & Pictet (2001). There have been many recent surveys on this subject including Bouchaud (2001), Brock (1999), Cont (2001), Farmer (November-December 1999), and Stanley, Amaral, Canning, Gopikrishnan, Lee & Liu (1999).

[3]There is a growing literature emphasizing the practical overlap in features coming from true long memory processes and processes with occasional regime shifts. These include Diebold & Inoue (1999), Gourieroux & Jasiak (2001), Granger & Hyung (1999), Liu (1995), and Mikosch & Starica (2000). Also related is the work of Granger & Ding (1996) which shows that many processes make exhibit long memory.

[4]This last objective is similar in spirit to that of Mantegna & Stanley (1998) which looks at the properties of certain dependent volatility models.

2 Three factor stochastic volatility

The primary stochastic process for comparison with actual stock returns will be a three factor stochastic volatility process. Stochastic volatility models generate returns series which display persistent volatility features similar to actual returns.[5] The model assumes zero autocorrelations for returns at all lags, and adds dependent structure only through the magnitude of these uncorrelated returns.

Returns are given by

$$r_t = e^{\gamma x_t + \mu} \epsilon_t \tag{1}$$

where ϵ_t is an independent, normal random variable with mean zero, and unit variance. x_t is the log of the changing volatility level. It is assumed to follow a process which is the sum of three different linear processes each operating on a different time frame.[6]

$$x_t = a_1 y_{t,1} + a_2 y_{t,2} + a_3 y_{t,3} \tag{2}$$

The scale factors are constrained so that

$$a_1^2 + a_2^2 + a_3^2 = 1. \tag{3}$$

This constraint is for convenience in setting up the process, but is actually not binding in the final model setup.

The first processes, $y_{t,1}$ is designed to capture relatively long range, but still exponentially decaying dependence. It is assumed to follow and AR(1) process,

$$y_{t+1,1} = \rho_1 y_{t,1} + \eta_{t+1,1}, \tag{4}$$

with $\rho_1 = 0.999$ which gives a half life of about 2.7 years. This captures an aspect of volatility which moves at roughly business cycle frequency. Much empirical evidence suggests that this is not an unreasonable feature

[5]For a recent survey see Ghysels, Harvey & Renault (1996). They are an alternative to the ARCH and GARCH models often used in finance and summarized in Bollerslev, Engle & Nelson (1995). There is also a spirit in which these models are attempting to capture the feature of changing economic time relative to clock time. This is often called "time deformation" and was suggested in Mandelbrot & Taylor (1967) and Clark (1973). It is also implemented for business cycles in Stock (1987).

[6]This model is inspired by the early two factor GARCH model in Engle & Lee (1993) which looks at a permanent and transitory component for volatility. Empirical support for a two factor volatility model similar in spirit to the three factor model used here can be found in Alizadeh, Brandt & Diebold (2001). These authors look at this model from a different testing perspective, but the basic premise of needing a short and long range persistent volatility component is similar.

for financial volatility.[7] The error term, $\eta_{t+1,1}$ is independent, Gaussian, and adjusted so that $\text{var}(y_{t,1}) = 1$. The second process, $y_{t,2}$ models a higher frequency component for volatility. It is built from another AR(1) process given by,

$$y_{t+1,2} = \rho_2 y_{t,2} + \eta_{t+1,2} \tag{5}$$

with $\rho_2 = 0.95$ which gives a half life of 2.5 weeks. This captures a middle range of volatility. The error term, $\eta_{t,2}$ is independent, Gaussian, and adjusted so that $\text{var}(y_{t,2}) = 1$. Finally, $y_{t,3}$ is drawn from an independent Gaussian with mean zero, and unit variance. This final disturbance allows for a volatility shock that persists for only 1 day. This might be a reasonable representation for news events which cause trading activity on a day, but are quickly absorbed into the price. These three time frames for volatility seem reasonable economically, and empirically. They were chosen with both the intention of lining up with the data, but also keeping to a reasonable picture of what the different volatility time frames might be.

The parameter μ is calibrated to set the unconditional variance of the simulated process equal to that from the actual stock returns series. Finally, the parameters, a_1, a_2, and γ were set to line up with the autocorrelations, and distributional features of the actual data. They are $a_1 = 0.678$, $a_2 = 0.548$, $a_3 = 0.490$ and $\gamma = 0.75$. The choice for these parameters was done primarily through trial and error. More formal estimation procedures would be possible using simulation based estimators such as simulated method of moments, but this experiment is not about finding the optimal volatility model. It is simply an exploration to see how closely a model in this class can replicate various features of the data considered interesting. The process will be simulated to demonstrate that it does a fair job in aligning with the empirical features of interest.

It is important to note that this process already has built into it two features which are probably at odds with the actual financial time series. First, it is a martingale with

$$E_t(r_{t+1}) = 0. \tag{6}$$

There is a large literature in finance which finds that this relationship is not true.[8] Empirically, estimating the conditional mean for financial returns is difficult, and the magnitude of this predictability is small.[9]

[7] Schwert (1989) gives some early evidence for a business cycle component for volatility. Chauvet & Potter (forthcoming 2001) and Hamilton & Lin (1996) give recent evidence on the connection between business cycles and stock market volatility. Franses & van Dijk (2000) show many examples with slowly changing regimes for financial data.

[8] See for example, Lo & MacKinlay (1988), Campbell & Shiller (1988), and Brock, Lakonishok & LeBaron (1992) for different perspectives on this same subject. The literature in this area is huge, and many references can be found in textbooks such as Campbell, Lo & MacKinlay (1996).

[9] Recently the stability of many predictors has also been questioned. See for example, Ghysels (1998), LeBaron (2000), and

A second feature that this process ignores is the connection between current returns and future volatility. Known as the "Leverage Effect" this captures the fact that as prices fall, volatility tends to rise.[10] The general assumption here is that both these features are second order to the broad stylized facts on volatility persistence and distributional scaling behavior that will be analyzed.

Another crucial fact about the simulated stochastic volatility process is that it is not long memory, and it has thin tails in that all higher order moments exist.[11] These theoretical properties are important since many of the visual tests will display features indicating that the process looks either fat tailed, or long memory, or both.

The volatility process will be compared with daily returns from the Dow Jones Industrial average sampled from January 1st, 1897 through February 3rd, 1999. Returns are calculated from the log difference of the price level for the Dow. This series has a total of 28094 daily returns. The daily Dow was used to give a very good estimate of long range properties for long memory scaling laws.

3 Unconditional distributions

The first stylized facts to be analyzed come from the unconditional return distributions. Since the early paper of Mandelbrot (1963), the characteristics of unconditional stock return distributions have been carefully scrutinized. Over the nearly half century since his important early work several stylized features have emerged:

1. At relatively high frequencies (less than 6 months) stock returns do not follow a Gaussian distribution. They possess tails with significantly larger probability mass than fitted Gaussian distributions.

2. As one looks at returns at lower frequencies they appear closer to Gaussian. However, this convergence appears to be very slow. So slow, that in certain short horizon ranges it is almost impossible to see.

3. The tails of these distributions exhibit power law like scaling behavior. The exponents are consistent with the existence of variances, but the existence of higher moments are not guaranteed.

The analysis begins with a simple plot of two different time series in figure 1. The upper panel shows the daily returns of the Dow over the last century, and the lower panel displays the stochastic volatility

Sullivan, Timmerman & White (1999).

[10] This was documented in Black (1976), and has recently been diagnosed extensively in Wu (forthcoming 2001).

[11] The existence of higher order moments is easy to show given the formula for the expectation of e^z where z is a Gaussian random variable.

process. From this figure alone it is difficult to discern much difference. Both show a characteristic of bursts of volatility which persist for long windows of time which are close to the order of magnitude of the entire time series length.[12]

Table 1 presents the summary statistics for the three factor volatility process along with values for the Dow returns. The three factor model is simulated 1000 times for sample lengths of 28,094 corresponding to the Dow return sample length. The first column of the table shows the mean of the sample variance for the three factor stochastic volatility process. The sample variance is estimated for each of the 1000 monte-carlo simulations. The second row displays the standard deviation of this estimate across the 1000 simulations, and the final row shows the estimated variance for the Dow. The results are very close, but this should not be surprising since the unconditional variance of the simulated process was calibrated to the stock return series. More surprising is the close agreement along the other moments. Parameters for the process were chosen to bring these close to their Dow counterparts, but no explicit calibration was done for any of these individually.

Table 1: *Summary Statistics*

	Variance $x10^4$	Skewness	Kurtosis	$E(\|r\|)$	$var(\|r\|)x10^5$
Stochastic Volatility	1.145	-0.068	22.34	0.0065	7.154
(Std.)	0.069	0.370	10.61	0.0002	0.652
Dow Returns	1.151	-0.620	26.47	0.0071	6.462

Skewness is near its true value of zero for the volatility process, and given the precision, is not far from the -0.62 value for the actual returns. Estimated kurtosis is clearly different from the Gaussian value of three for both the simulated processes, and the Dow, and both exhibit levels close to 20. The final two columns present the mean and variance of the absolute values of returns. In both cases the values are reasonably close to those from the Dow returns. For the expected absolute value the evidence suggests that the simulations are generating a value which is significantly smaller than the true value, but the value is still very close.

Figures 2 through 5 summarize the character of the simulated and actual return distributions for 1, 5, and 20 day intervals. In each case the three factor stochastic volatility process, indicated as Svol(3), is simulated for an extended series of $500,000$ observations which corresponds to roughly 1900 years of data. This is done to get an accurate picture of probability densities in the extreme tails of the process.

[12]The scale in both figures has been set so that the crash of 1987 does not appear which gives a better scale for comparing the figures.

Figure 2 displays the upper quartile of the distributions for the three aggregation levels, moving left to right. The figure is displayed on a double logarithmic scale to capture possible linear scaling regions in the tail. It displays a pretty good self-similarity, and also the simulated and actual distributions do not appear too far from each other.[13]

Figure 3 compares the stochastic volatility returns to those drawn from an independent Gaussian distribution with the same standard deviation. The Gaussian distribution shows the characteristic of increasing slopes as one moves farther out into the tail as it should. This figure also shows the deviations of the stochastic volatility process from the Gaussian. What is most amazing is that these deviations still appear quite strong at the 20 day aggregation level even though the stochastic volatility process has a finite variance, and must be converging to a Gaussian.[14] While the dependence in the process does not stop the central limit theorem from taking over, it significantly slows down the speed of convergence.[15]

Another property commonly observed in return distributions is a near linear scaling region in double log space. Heavy tailed cumulative distribution functions are characterized by,

$$F(x) \sim x^{-\alpha} L(x) \tag{7}$$

where α is the scaling parameter, and $L(x)$ is a slowly-varying function. Estimates of α for financial data have ranged from 2 to 5 for different series and time periods.[16] Since the stochastic volatility process has moments of all orders, it is not a heavy tailed process, and should not demonstrate power law scaling in the tails. Figure 4 shows the 5% right tail for both the stochastic volatility process, and the Dow. It shows a reasonable power law for both series at all three aggregation levels. The simulated process displays a slightly cleaner linear scaling region which is probably due to its much larger sample size. Linear regressions are fit to the 1% tail for the three aggregation levels yielding slope estimates of 2.98, 3.26, and 3.33 for the 1, 5, and 20 day aggregation levels respectively. Similar results are shown for the 5% left tail in figure 5. Here, the estimated slopes are 2.96, 3.31, and 3.40 for the 1, 5, and 20 day aggregation levels.

[13] In the tails it is important to remember that the sampling variation for the Dow is probably very large. For the 20 day aggregation level l, the sample size is only 1404 which makes tail probability estimation difficult.

[14] See White (1984) for an example of the central limit theorem for dependent processes.

[15] This slow convergence has been documented extensively in recent econophysics papers. It has lead to the proposal of a very different mechanism to generate this feature, the truncated Levy, process as in Mantegna & Stanley (1996).

[16] There is a large range of authors who have estimated power-law tails. Several examples are Koedijk, Schafgans & de Vries (1990) and Loretan & Phillips (1994). See Mills (1999) for a summary. The fact that $\alpha > 2$ is important since this guarantees the existence of second moments.

Tools also exist to estimate power law tails quantitatively. One such tool is the Hill estimator.[17] Table 2 presents tail exponent estimates based on the Hill estimator. The first row shows the mean estimates from 1000 simulations of the stochastic volatility process with sample size equal to the Dow returns. The values are very close to their corresponding estimates from the actual data. The second row shows the standard deviations, giving a good idea of the imprecision of the Hill estimator, which can be quite large as the sample size falls with increasing aggregation levels.

Table 2: *Estimated Tail Exponents: Hill estimator (1% tail)*

	Left tail			Right tail		
Aggregation	1	5	20	1	5	20
Stochastic Volatility	3.09	3.08	3.51	3.46	4.17	4.15
Std	0.25	0.24	0.51	0.49	1.19	1.23
Dow	2.98	3.11	3.15	3.41	3.81	3.94

In all cases the stochastic volatility model exhibits reasonably good power law scaling in the tails. The estimated magnitudes are close to the actual returns, and the range of values reported in the literature on distributional scaling. This is interesting for two reasons. First, it is another indication that the stochastic volatility model is matching these basic visual features of the data. Second, it is important to remember that theoretically the stochastic volatility model is thin tailed, and actually should not be exhibiting power law behavior in its tails. However, the visual pictures show a scaling region that appears to be at least as good as for the actual return series.

4 Volatility persistence

The second feature that will be examined is the persistence of volatility. First discovered in Mandelbrot (1963), this has become a key element of empirical work in modern finance. The fact that large moves of financial prices of either direction tend to be followed by large moves remains an important property that still lacks a satisfying explanation. This puzzle was deepened by Ding, Granger & Engle (1993) who showed that the decay of volatility autocorrelations remained positive out to very long time horizons. Also, it appears that the correlation patterns followed a hyperbolic decay which would be consistent with a long memory process for volatility.[18]

[17] The Hill estimator is described extensively in Embrechts, Kluppelberg & Mikosch (1997), section 6.4.2. These authors are also cautious in their recommended use of the Hill estimator, especially for dependent data.

[18] There have been both empirical and theoretical explorations in generating long memory in volatility. See Baillie, Bollerslev & Mikkelsen (1996) and and Breidt, Crato & de Lima (1998) for empirical examples, and Bouchaud, Giardina & Mezard (2001),

Figure 6 shows the autocorrelation function, ACF, for the absolute value of returns. Dow returns are from the daily returns series used in the previous sections. Svol(3) is again the three factor volatility model. In this picture it is simulated for 500000 time periods to get a precise estimate of the ACF. The figure shows a very good agreement between the simulated and actual data. A second process is simulated to demonstrate the difficulty of simpler volatility processes in replicating these facts. Svol(1) presents the ACF from a 1 factor volatility model. It is built off the middle range process, $y_{t,2}$, with an AR(1) parameter of 0.95. In other words, it is equivalent to setting $a_1 = 0$ and $a_3 = 0$. It is also simulated for 500000 periods. The ACF clearly shows that it will be difficult, if not impossible, to fit the actual autocorrelation pattern with a single factor model. The exponential drop off in the autocorrelation shows a value that is too large at the short horizon, and converges to zero for lags greater than 75. It is precisely this problem that has led researchers in the direction of long memory models.

The case for the 3 factor model is strengthened in figure 7. This shows that the single long simulation from figure 6 was not an anomaly. It presents the median and 5 to 95 quartile range for the stochastic volatility ACF estimated on sample sizes set equal to the daily Dow sample length. This shows that the general shape agreement is not a function of random sample variation, but is a general property of the stochastic volatility model. It also gives a picture of the sample variability in the individual ACF estimates, indicating that the deviations between the true and simulation values are not that large.[19]

If the volatility process were long memory we should expect to see the autocorrelation at lag k, ρ_k, follow

$$\rho_k \sim k^{-\gamma} \tag{8}$$

for large k. This hyperbolic decay is the signature for long memory.[20] It is also related to the self-similarity parameter, H, for a self-similar process through,

$$-\gamma = 2H - 2, \tag{9}$$

and Kirman & Teyssiere (1998) for theoretical examples.

[19]Some care should be taken in interpreting the bounds since they are compounded across many different draws of the stochastic volatility process. For example, it is not the case that 10% of the return autocorrelations should lie outside this bound since they are clearly not independent. For multiple draws of the true process, for a given lag, there should be 10% violations of the bounds.

[20]See Beran (1994) for exact definitions of long memory.

and the level of fractional differencing, for a fractionally integrated ARMA process by,

$$-\gamma = 2d - 1. \tag{10}$$

The self-similarity relation is particularly intuitive to the idea that there is no characteristic length scale to the process. A stochastic process is said to be self-similar if

$$(Y_{t1}, Y_{t2}, Y_{t3}, \ldots) \stackrel{d}{=} c^{-H}(Y_{ct1}, Y_{ct2}, Y_{ct3}, \ldots). \tag{11}$$

In words this means that sampling farther apart in time yields the same distribution for the process Y_t subject to a scale factor.

Figure 8 gives a direct test of the relation in 8 by plotting the ACF in log/log space. This shows a good linear scaling region for both the Dow and the three factor stochastic volatility model which is a good indication that long memory might be present in volatility. This fact is tested further by looking at the variance expansions of the volatility process. Returning to the previous notation, let x_t be the log volatility process. If this process were long memory then the variance of the partial sums will follow

$$\mathrm{var}(\sum_{i=1}^{n} x_i) \sim n^{2H} \tag{12}$$

for large n.[21] The best known case for this would be for no long memory where $d = 0$ and $H = 1/2$. In this case the variance of the partial sums scales in proportion to n. If x_t were observed an easy test would be to plot the variance and n in log/log space to estimate H and d. For the simulated process this could be done, but for the actual returns data it is not possible since x_t is not observed. Fortunately, one can approximate this by using the fact that,

$$\log(|r_t|) = x_t + \log(|\epsilon_t|). \tag{13}$$

Given that ϵ_t is independent over time, and independent of x_t, the variance of partial sums will scale as

$$\mathrm{var}(\sum_{i=1}^{n} log|r_i|) = \mathrm{var}(\sum_{i=1}^{n} x_i + \log(|\epsilon_i|)) = c_1 n^{2H} + c_2 n \tag{14}$$

Since $0.5 < H < 1$ for stationary long memory processes this will be dominated by the power in $2H$, so

[21] See Beran (1994) for this result. The variance of the increments of a self similar process will increase at this same rate, $\mathrm{var}(y_t - y_{t-n}) \sim n^{2H}$.

similar scaling pictures should hold for the log absolute returns as for the true volatility process, x_t.

Figure 9 plots the variance versus the n for the three factor stochastic volatility process, the Dow returns, and the one factor stochastic volatility process. All are shown in log/log space to visualize scaling. For the two simulated processes x_t is not long memory, giving theoretical values $d = 0$, and $H = 1/2$, which should lead to a slope of 1.

The estimated slopes in the scaling region from 10^2 to 10^3 for the Dow is 1.82, which gives a scaling parameter, $H = 0.91$, and $d = 0.41$. For the three factor stochastic volatility simulation the estimated slope is 1.77 which gives $H = 0.89$, and $d = 0.39$. For the one factor model the estimated slope is 1.08, giving $H = 0.54$, and $d = 0.04$ which is close to its theoretical short-memory value of $d = 0$. There are many other estimators for d. Most of these appeal directly to the self-similar aspect of fractionally integrated processes, so it is likely that they will generate similar results.

So far the tests have indicated the presence of long memory in the stochastic volatility process where none exists. The obvious conjecture for what is going on is that the aggregation levels have not been high enough to capture the asymptotic short memory features. It should therefore be true that if the partial sum aggregation amount were increased the true short memory of the three factor model would be revealed. This is demonstrated in figure 10 where the aggregation is extended to 10^5. This is too large for the Dow sample size, but still reasonable for the longer simulated series. The picture shows the slope of the stochastic volatility process finally diminishing. OLS estimation over the largest 1/3 of the aggregation levels gives an estimate of $d = 0.03$, or close to zero long memory. This figure demonstrates there is nothing wrong with the test. However, aggregation levels must be large relative to the lowest frequency component of the process to get a true picture of the dynamics.

5 Multi-fractal patterns

Multi-fractal patterns extend the idea of self-similarity to allow for different scaling relations for different functions of the series. Specifically, a multi-fractal process will display the following features. Let $p_t = \log(P_t)$ be a logged financial price series sampled at intervals of 1 day.[22] If returns are multi-fractal then

$$S_q(T, n) = \sum_{i=0}^{int(T/n)-1} |p_{n(i+1)} - p_{n(i)}|^q \sim (n)^{g(q)}. \tag{15}$$

[22]This example uses days as the underlying increment, but this could be any fixed time unit. Also, in this notation assume that the time subscript is adjusted to generate nonoverlapping return series.

The function $S_q(T, n)$ is known as the partition function. If the process is self-afine, then the general function $g(q)$ is simply Hq.[23]

Once again evidence in support of this feature can be found in log/log plots. Estimates of $g(q)$ can be obtained using log/log plots and plots of $S_q(T, n)$ for various aggregation levels. Combining these slope estimates gives a picture of the function $g(q)$. If the process is self-afine, then $g(q)$ will be linear with slope equal to H. Furthermore, if the process is Brownian, then the slope will be equal to 1/2.

Slope estimates using different time intervals are presented in figure 11. Estimated values for $S_q(T, n)$ are displayed for both the Dow returns, and the simulated 3 factor stochastic volatility process. In the latter case the sample is again expanded to 500,000 daily observations. The slopes are increasing as q increases in the figure. For $q = 1, 2$ the figure displays reasonably linear scaling regions for the actual and simulated data. Clean scaling breaks down as q increases which is true for actual data too. Figure 11 also shows pretty close agreement between the actual return data, and the simulations. There is some divergence at $q = 5$, but even for this value the scaling slopes appear similar.

The key feature of multi-fractals shows up in the changes of the slope estimates as q increases. These changes are plotted in figure 12. This figure displays, the Dow, the Dow with the crash of Oct. 19, 1987 removed, and simulated Gaussian returns. First, it is clear that the simulated Gaussians have a slope of about 1/2 as they should. The Dow series shows a clear bend to lower slopes for higher q values which is a common feature for financial data.[24] The figure compares the Dow with slope expansions from 25 simulated stochastic volatility models with length equal to the actual Dow return series. These slopes are plotted as well, and show several of them are capable of generating an expansion similar to the Dow. Furthermore, the Dow plot shape is very sensitive to one data point. When the crash of 1987 is removed the shape changes dramatically, but it is still within a reasonable range to be generated by the stochastic volatility model.

These results are summarized in figure 13 which displays the maximum and minimum bounds for 100 simulations of the stochastic volatility process. They repeat the message of the previous graph in that it doesn't look impossible for the actual data to be a draw from this process. The sensitivity to the crash is another feature which should make users of these plotting techniques a little more cautious of the results. Even though it was an actual data point, it is not good to see an interesting aspect of a time series so sensitive to its removal.

[23] The descriptions presented here on multi-fractal behavior are terse. More details on multi-fractals can be found in Calvet & Fisher (1996), Lux (1999), Mandelbrot (1997), and Pasquini & Serva (1999), and Vandewalle & Ausloos (1998). It is also important to note that this summation is slightly different from the one used in the previous section in that this is now the absolute value of longer horizon returns as opposed to the n length sum of the daily absolute values.

[24] See Calvet & Fisher (1996) and Lux (1999) for examples.

6 Conclusions

The introduction raised two questions about visual power-law tests. The first is concerned with the possibility that the visual scaling tests might display power-law, and long memory relationships when none exist. The results here suggest that this is a possibility. Most of the various power-law like results have been replicated with a fairly simple stochastic volatility model. The second, and related question is whether this stand in stochastic process is reasonable for financial markets. This single model matches many empirical features, but more importantly its calibration is based on the interaction of three reasonable time scales for volatility. The longest range of these is operating at business cycle frequency which is either reasonable in its own right, or might be a proxy for other long range features or possible nonstationarities.[25] A deeper related issue that this work raises is whether one can statistically discern the true multi-scale behavior of critical systems from, simple processes with more than one, but far from infinite time scales. Financial markets may prove particularly vexing for this issue, since they do not allow for the use of spatial as well as temporal correlations.

Hints at an explanation for these results were also presented. The stochastic volatility process requires a very persistent component for volatility. If the half life of this component is long relative to the aggregation levels used in testing, then the series may exhibit slow convergence to Gaussian distributions, and long memory like correlation properties. Furthermore, this very low frequency process is probably reasonable for financial time series.

The impact of these results on the financial scaling literature in general is more tentative. It does not say that power law results are wrong. It is only that they should be viewed as less conclusive than they often are, since there may be many explanations beyond those related to critical phenomena. It is important to augment power law related tests with the rich set of information contained in the conditional distributions of prices and trading volume. Furthermore, if the story on slowly changing policies or business cycle features is credible, then there is an argument for filtering out these features since most agent based models do not attempt to capture them.[26]

Both scaling laws and long memory may still have their place in financial analysis. It is important to remember that fundamentally one is interested in models that generate some kind of volatility persistent

[25]The appearance of long memory might not be surprising given the results of Granger (1980) which show that the sum of short memory processes can give long memory. However, it is surprising that it works well for only three different time scales. Further, Granger's results may prove troubling for the empirical testing of critical systems in that they emphasize very simple, noninteracting processes in the generation of long-memory correlation decay.

[26]This is another area where examining a broader set of facts would be useful. Two recent examples of pushing the data harder are Jefferies, Hart, Hui & Johnson (2000) which looks at out of sample predictability, and LeBaron (forthcoming 2001*a*) which examines many different conditional features of returns and trading volume.

features from actual agent behavior. Many agent based frameworks are capable of doing this, and some examples of these are based specifically on the interactions of agents of many different memory lengths.[27] Empirically it is possible that long memory models will prove most useful from the standpoint of forecasting. The experiments performed here were far from out of sample forecasting experiments. A simple fractionally integrated process with one parameter may prove to be a much more robust forecasting tool than multifactor models which will be difficult to identify and estimate.[28] Another practical application of scaling comes from using high frequency data to estimate distributional properties at lower frequencies. If this type of scaling holds relatively well, then from a practical standpoint it may still be useful to assume self-similarity from the 5 minute horizon to the 5 day horizon. Even though this result may not hold perfectly, its approximation could still be useful for those interested in getting detailed pictures of tail probabilities for risk management systems.[29]

It is also not fair to eliminate the rich concepts of scaling and critical system dynamics from the set of candidate theories for financial dynamics. A shared feature of most of these dynamical systems is that small scale interactions are able to generate large changes in the system at the macro level. These ideas need to remain part of the discussion in finance and economics in general. However, it is possible the empirical evidence will need to somehow move beyond power-laws alone. An example of what could be done in finance is the paper by Cutler, Poterba & Summers (1989) which shows that the connection between large stock market moves and major information events is tenuous. The ability of a large interconnected system of trading agents to self-organize in order to generate a crash still remains an enticing idea of physics for economics.

The search for reliable scaling laws in economics and finance should continue. However, the current state of the search must be viewed with some hesitation. The visual indication of a straight line going through some points should not be taken on its own as a "test for complexity", or critical behavior. The set of stochastic processes able to generate visually good scaling may be much larger than previously thought. It would be best not to abandon these concepts, but to improve statistical understanding of both the empirical tests, and the theoretical models under consideration.

[27]See for example, , and Levy, Levy & Solomon (1994), LeBaron (2001*b*). Also, see Peters (1994) and Dacorogna, Muller, Olsen & Pictet (2001) for some commentary on the construction of markets with time based heterogeneity.

[28]An example of this is Andersen, Diebold & Labys (2001).

[29]See Dacorogna, Muller, Pictet & de Vries (1998) for an example.

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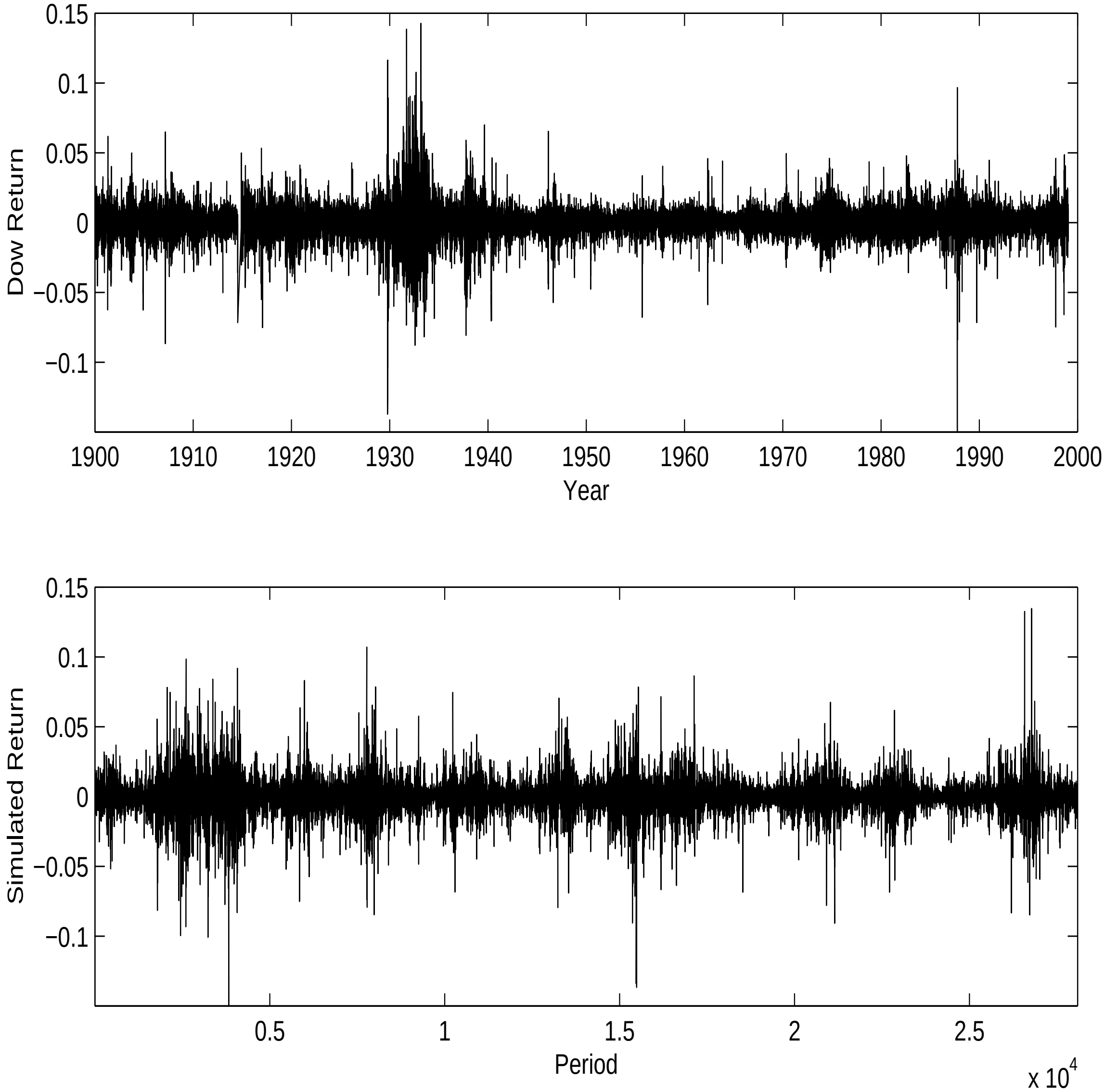

Figure 1: **Dow and Stochastic Volatility Time Series**

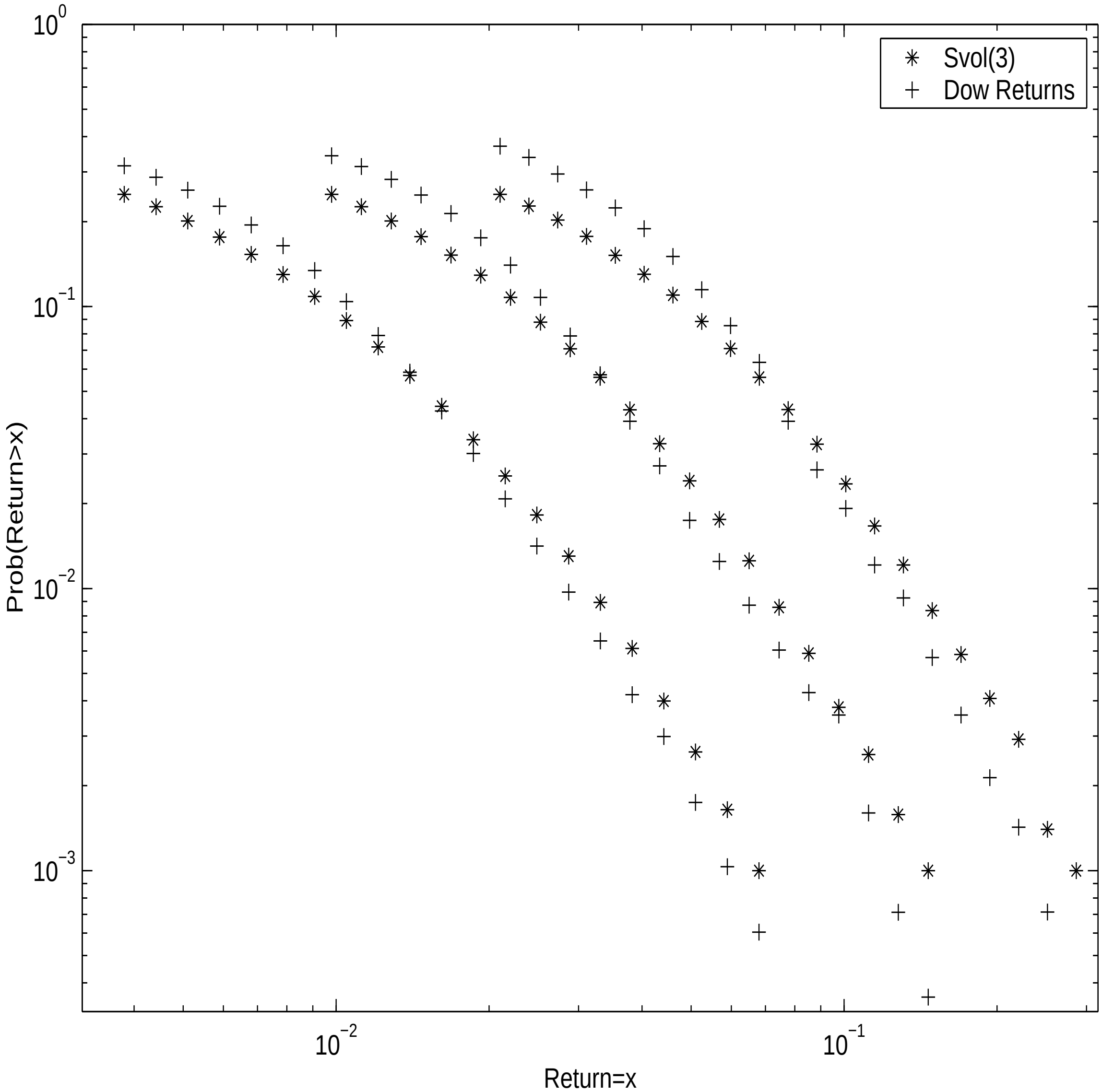

Figure 2: **Distribution Right Tail:** 1, 5, 20 days (left to right)

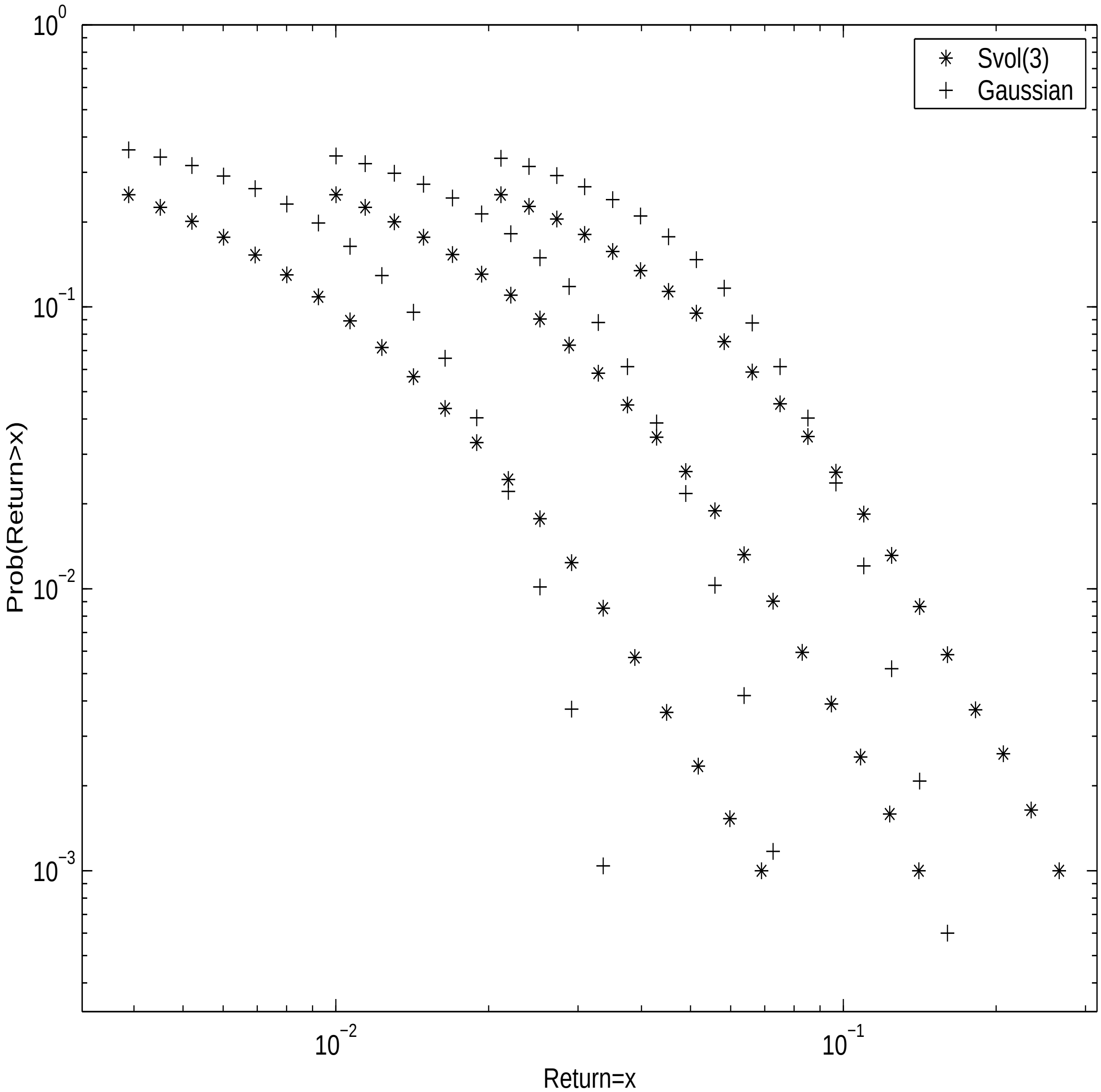

Figure 3: **Distribution Right Tail:** 1, 5, 20 days (left to right)

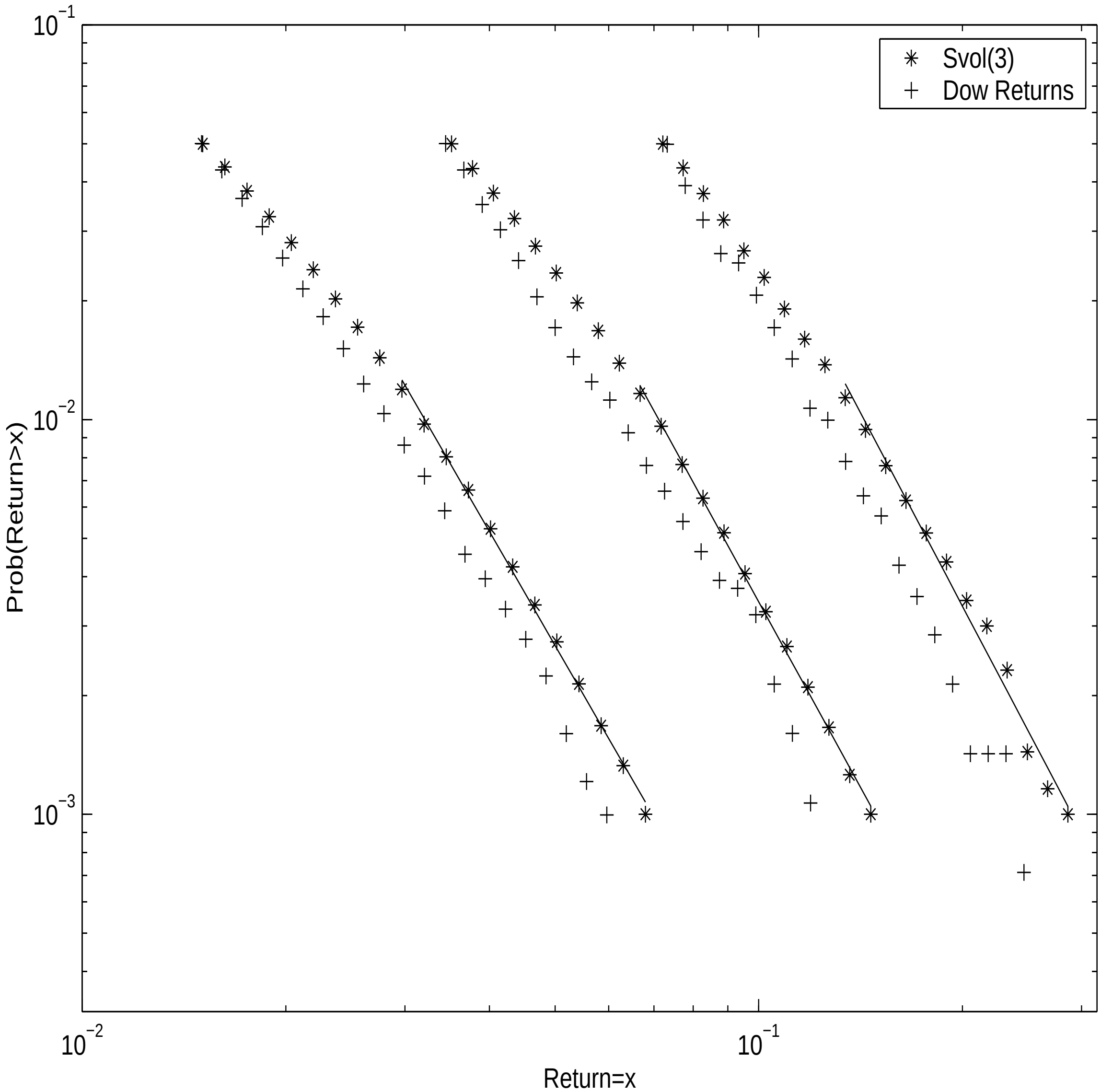

Figure 4: **Distribution Right Tail:** 1, 5, 20 days (left to right)

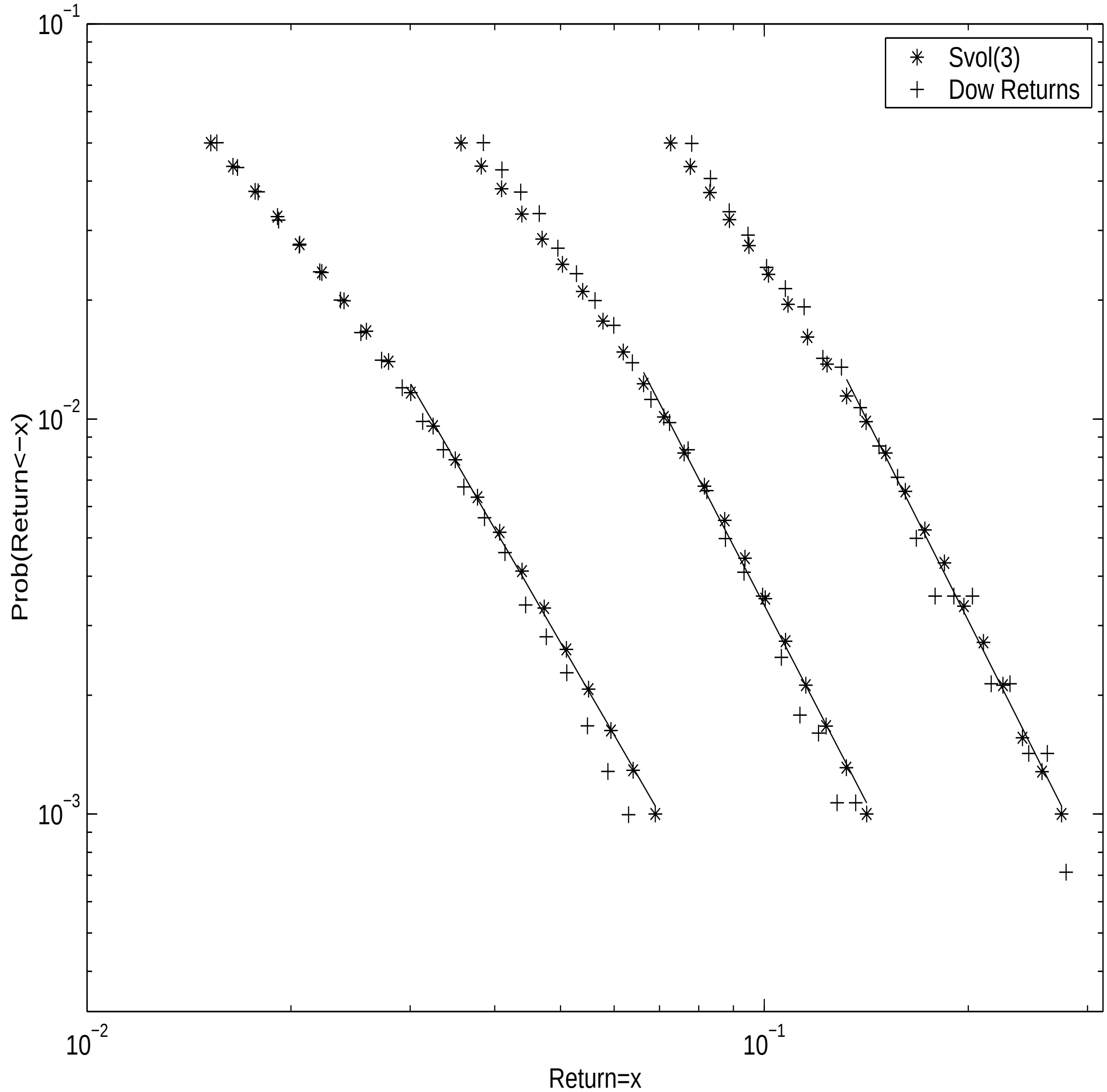

Figure 5: **Distribution Left Tail:** 1, 5, 20 days (left to right)

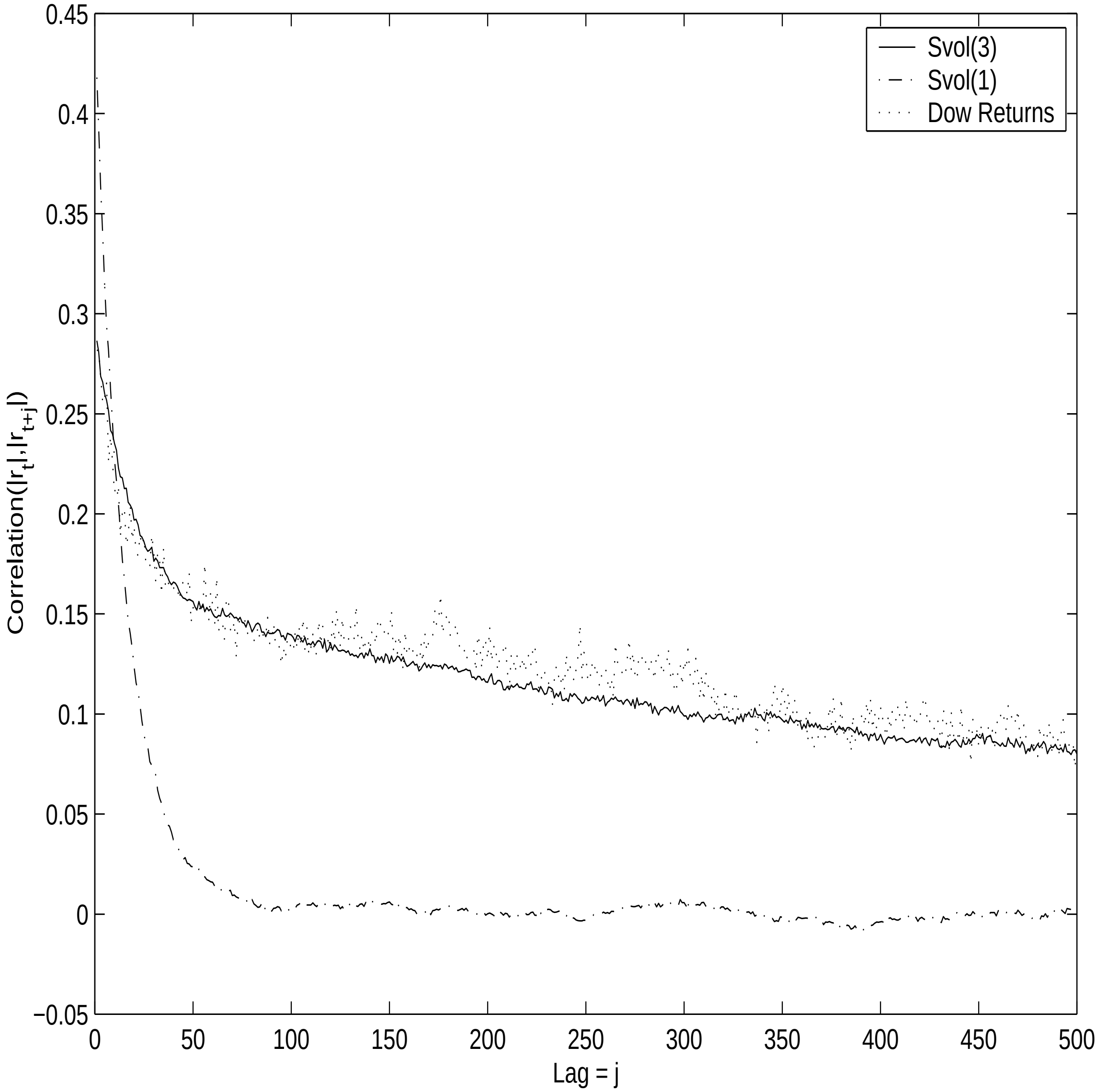

Figure 6: **Absolute Return Autocorrelations**

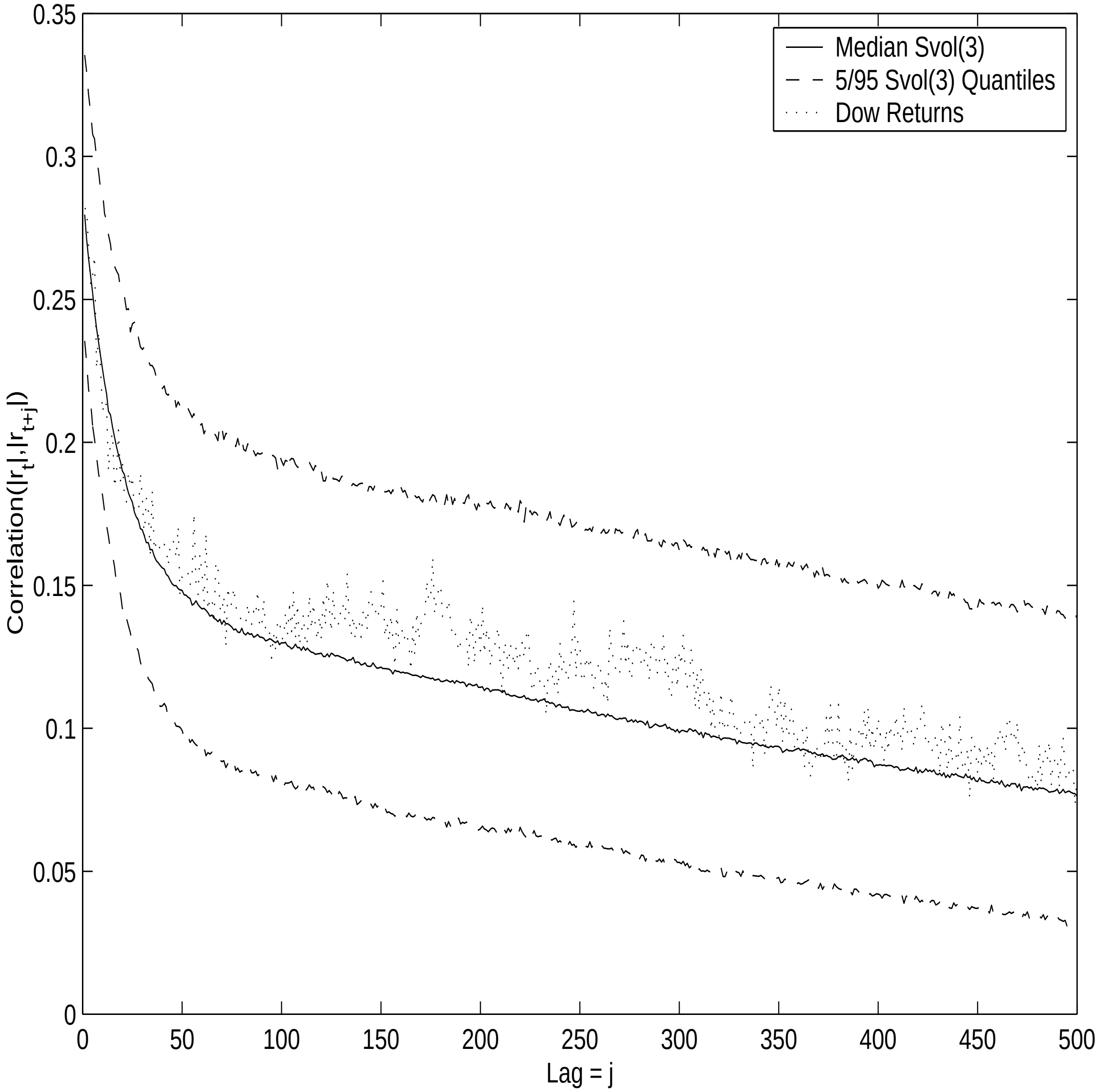

Figure 7: **Absolute Return Autocorrelations:** Dow and stochastic volatility quantiles

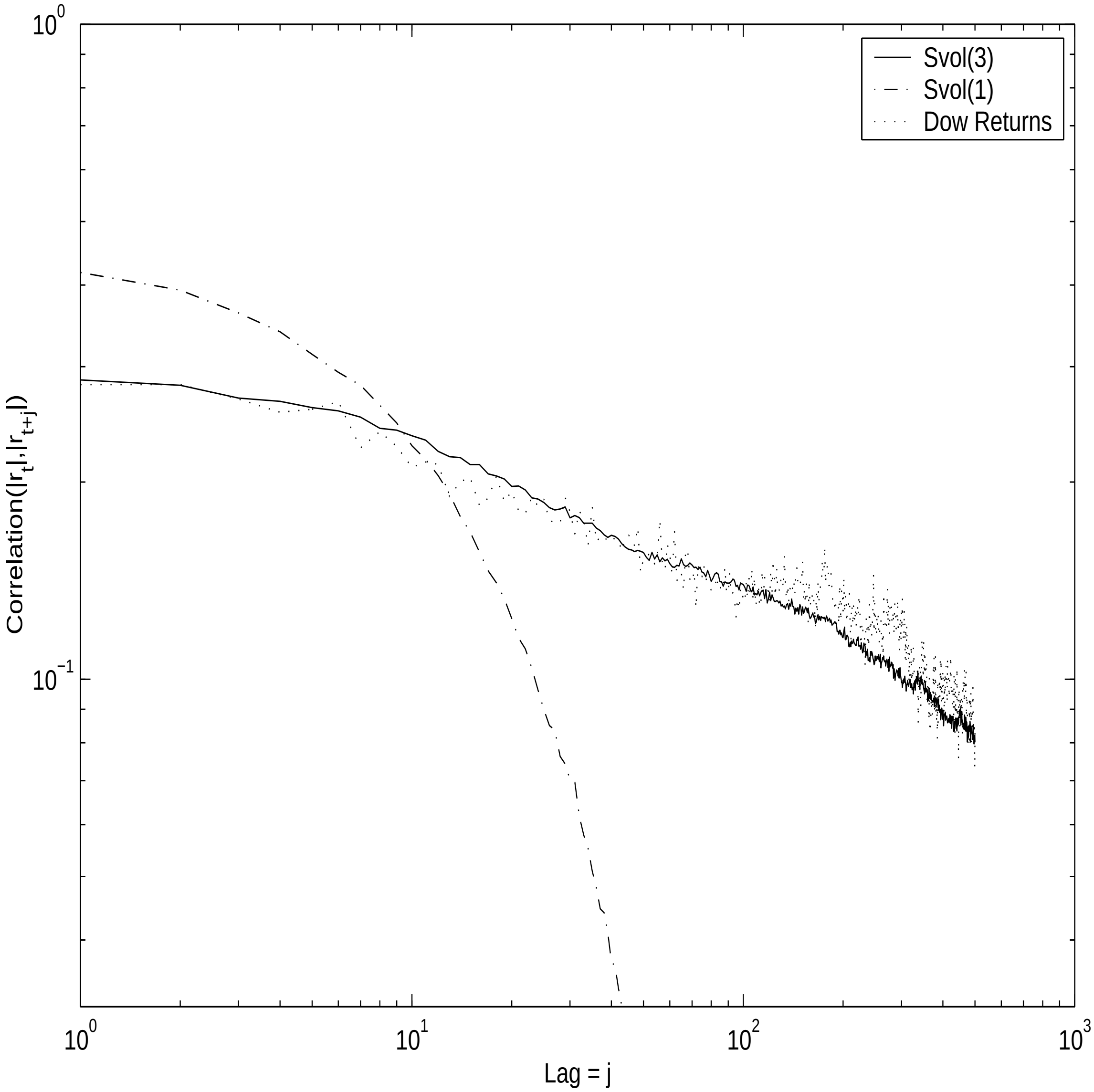

Figure 8: **Absolute Return Autocorrelations**

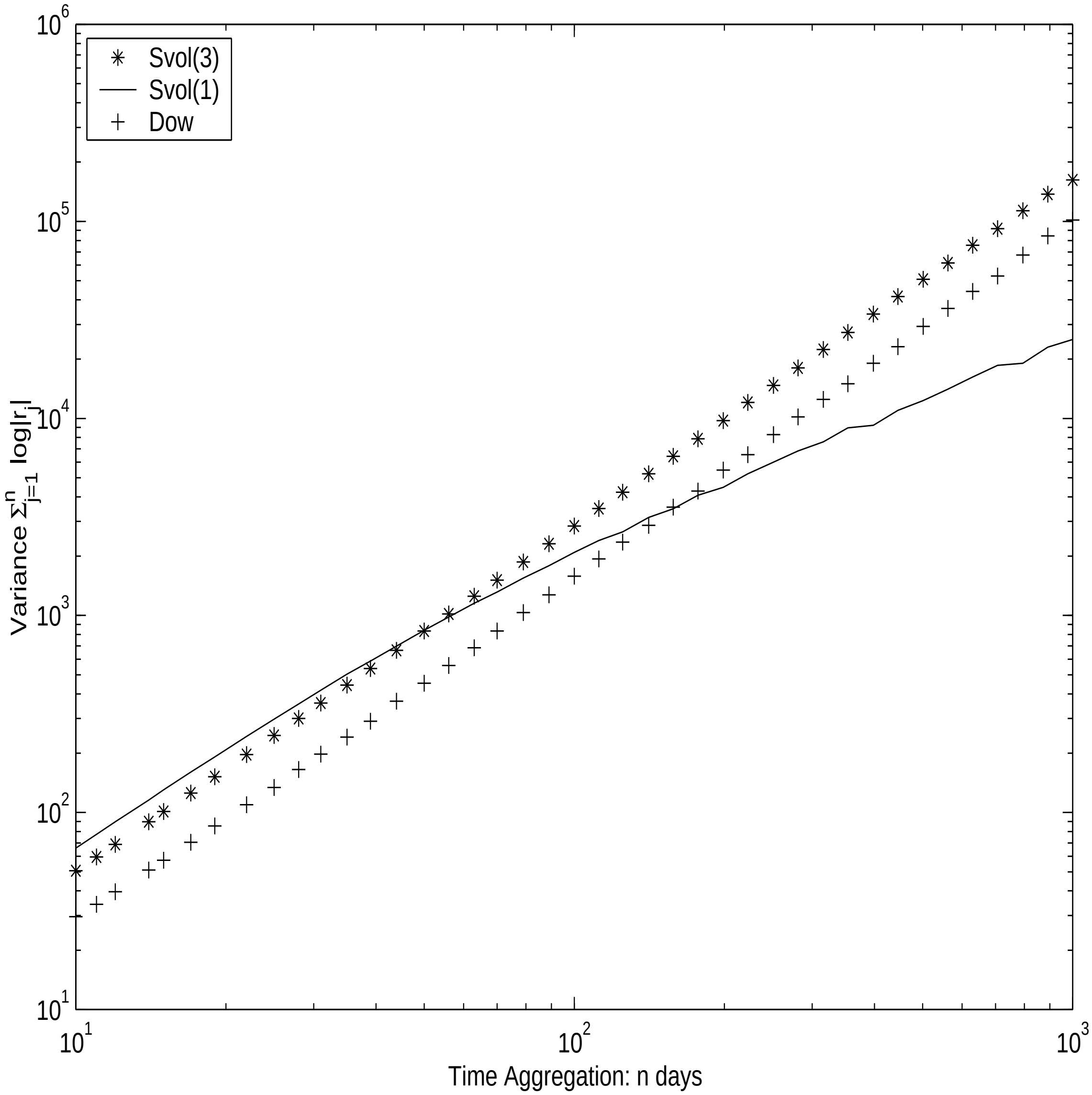

Figure 9: **Variance Aggregation:** n length partial sums of daily log absolute returns

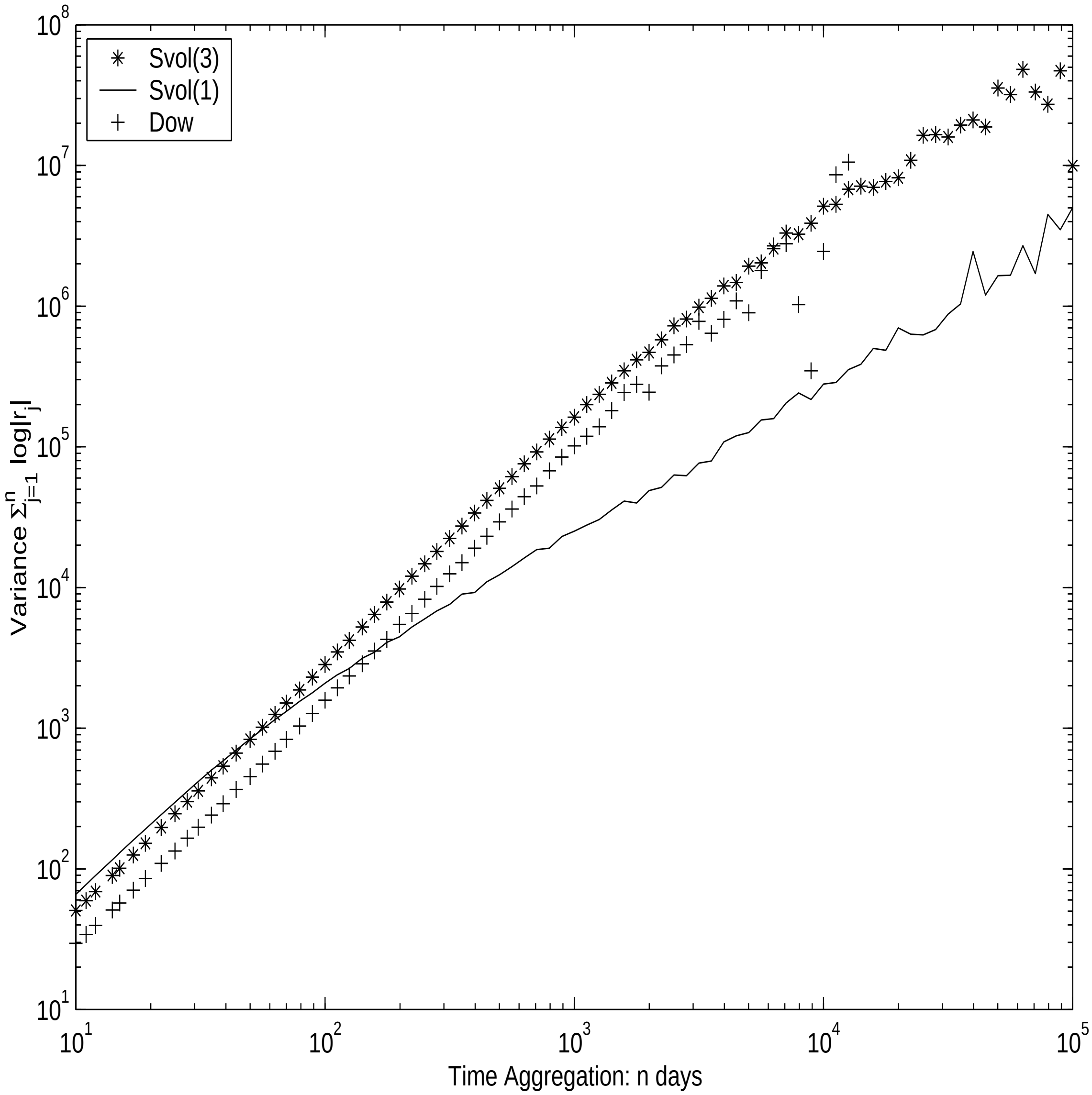

Figure 10: **Variance Aggregations:** n length partial sums of daily log absolute returns

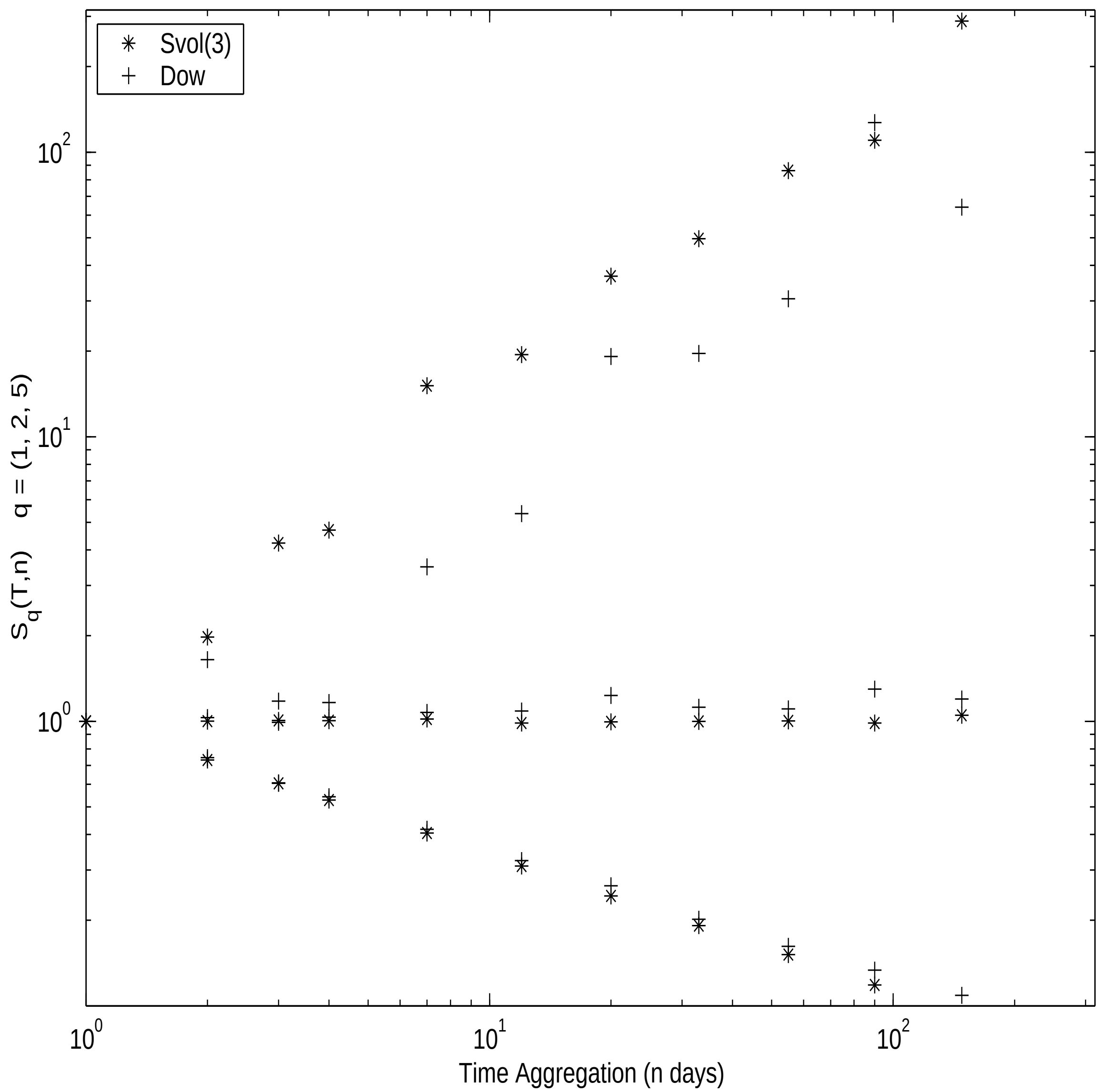

Figure 11: **Partition Function Time Aggregation Plots:** Increasing slope corresponds to increasing q

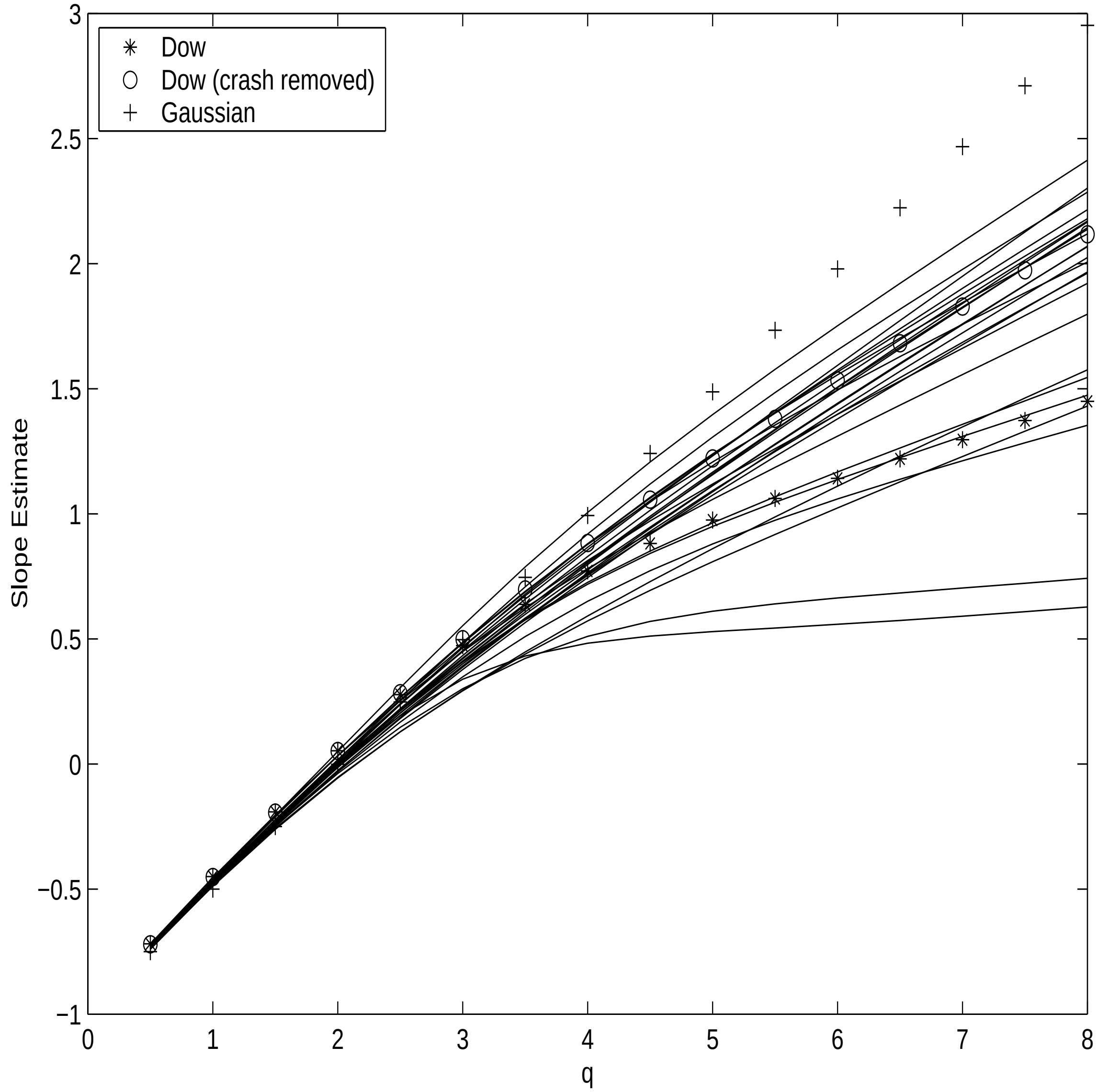

Figure 12: **Partition Function Slope Estimates**

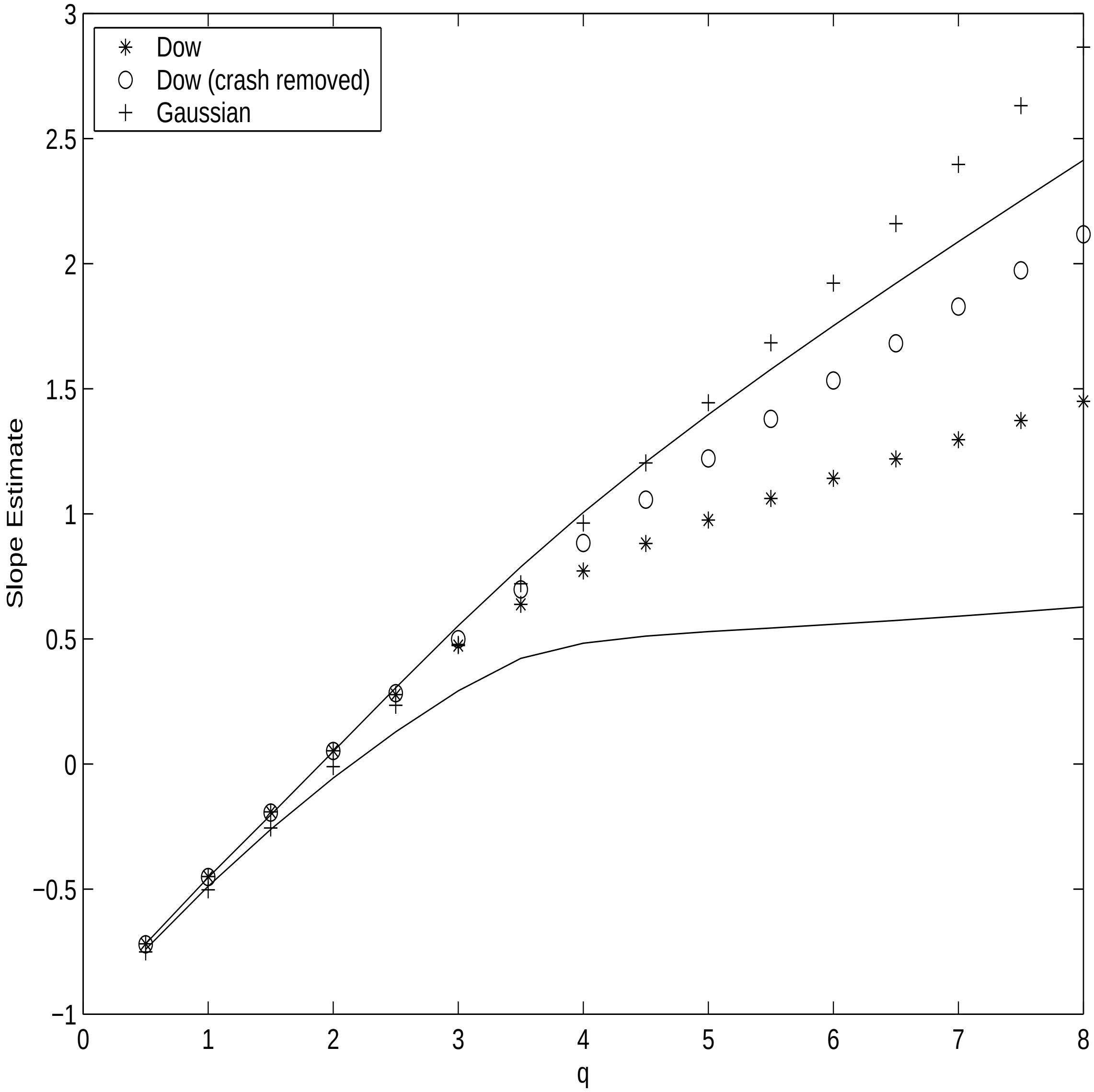

Figure 13: **Partition Function Slope Estimates**