Volatility Persistence and Apparent Scaling Laws in Finance

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Abstract

Recent evidence has shown possible scaling and self-similarity in high frequency financial time series. This paper demonstrates that many of these graphical scaling results could have been generated by a simple stochastic volatility model. This casts doubt on the power of these tests to discern between true scaling and simple highly dependent stochastic processes.

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1 Introduction

Self-similarity and scaling laws in financial prices have attracted attention for many years. Since the 1930's technical analysts have looked for patterns that get repeated at different time scales. During the 1960's studies of the stable distribution as a model for stock prices provided the first formal application of self-similarity in finance. Notions of scaling reappeared in the 1980's with tremendous excitement in fractals and chaos coming from the physical sciences. Now there has been a resurgence of interest in scaling and power laws coming from the recent availability of high-frequency price series. This has enabled researchers to look for patterns at time horizons often as short as 1 minute. Most of the previous research has failed to find strong evidence in support of self-similarity. This study suggests some caution is advised in interpreting some of the most recent results.

The idea that socio-economic series might have similar properties at different time/length scales is very interesting. Economists have looked across many different series to find the possibility of scaling laws.[1] Famous rules such as "Zip's Law" on the scaling of cities remain part of the lore in the area, and continue to attract attention. Even though the main purpose of this paper is to show that estimating scaling laws is a difficult job, the search for them should not stop. If such features existed they could be a tremendous guide for economic theorizing, and a strong empirical discipline, in that they force models to fit phenomena at many different horizons.

Finance has a long history of searching for scaling laws. The best documented beginning was probably the work of Elliot in the 1930's. Elliot's work emphasized the appearance of patterns at different time horizons, and forms the basis of one domain of technical analysis today.[2] Much of this work was qualitative in nature, and still remains untested in terms of formal statistical procedures.

In the 1960's scaling made a return in the work on stable distributions for stock returns as relatively high frequency data (daily), and the computing power to analyze it became available. Random walks and gaussian distributions were the models of choice in this period. It is important to realize that gaussian distributions and brownian motions are probably the most important self-similar objects of all time. Appropriately rescaled sums of gaussians follow the same distribution. Two important papers by Fama (1963) and Mandelbrot (1963) addressed the fact that daily prices were clearly not coming from a gaussian distribution. They explored a wider class of distributions, the stable class, which shares the self-similarity features of the gaussian, but allowed for some interesting other properties, such as the nonexistence of second moments.

[1] See Brock (1999*a*) for a survey of scaling in economics.

[2] The works of Elliot have be surveyed in many books such as Frost & R. R. Prechter (1978).

Mandelbrot was probably stronger in his arguments for self-similarity, and performed a forerunner to many of the tests done here in his famous plot of cotton prices.[3] Further research has generally shown that longer horizon return distributions do look different from the shorter horizon ones, and they generally are converging to gaussian as predicted by the central limit theorem.[4] Also, during this time period another feature was noted that is crucial to this study. The persistence of volatility, or clumping of large moves in returns was first reported in Mandelbrot (1963). This led to the conjecture that trading time might not move with clock time, and the application of subordinated processes in finance (Clark 1973).

The next wave of interest in scaling came from dynamical systems and chaos in the 1980's. Nonlinear chaotic processes both generate very interesting dynamics, and they can produce self-similar features. Although a few papers showed possibilities for some dynamic scaling behavior, the final results were inconclusive.[5] Chaos remains an interesting, but often elusive concept for researchers looking at economic, and other real world time series.[6]

The recent burst of interest in scaling laws can be attributed to several things. First, the availability of high frequency data has opened a new domain of time horizons over which scaling can be tested. Second, high speed computing and cheap data storage makes moving these large data sets possible. Finally, there has be a renewed excitement in cross disciplinary research. An entire new field, econophysics, has developed around some of these results. It is hard to trace some of the origins of this work, but probably the earliest results on scaling laws using high frequency data come from the group at Olsen and Associates (www.olsen.ch).[7] Good examples of these results are in Muller, Dacorogna, Dave, Pictet, Olsen & Ward (1995). With great foresight they put large computing resources into carefully storing high frequency foreign exchange time series. Along with this they produced several interesting studies on their new data set. Several others followed such as Mantegna & Stanley (1996) and Ghashghaie, Brewmann, Peinke, Talkner & Dodge (1996), and now the field has many examples of scaling and power laws detected in many high frequency financial series.[8] [9] Most of

[3]This originally appeared in Mandelbrot (1963). Much of Mandelbrot's old work along with some of his new work on self-similarity in finance is contained in Mandelbrot (1997). Also, see his recent paper Mandelbrot (1999) for a good summary.

[4]See Blattberg & Gonedes (1974),Tucker (1992),Lau, Lau & Wingender (1990), and LeBaron, Lo & Taylor (1993) for some examples.

[5]LeBaron (1994) is still a relevant survey to the literature today, and emphasizes just exactly what was found and what some of the difficulties were. Also, *stanley.feldberg.brandeis.edu/∼blebaron/id18.htm* gives some more recent thoughts on chaos and the social sciences.

[6]See Brock (forthcoming 1999*b*) and Ruelle (1994) for thoughts on the practicality of nonlinear modeling.

[7]Scaling is now more than simply a theoretical question. Some policy makers are recommending using scaling adjustments to estimate risks at longer horizons. Diebold, Hickman, Inoue & Schuermann (January, 1998) present a critical look at this practice.

[8]Financial series are not the only ones under study. Also, firm growth rates, and country growth rates have been analyzed as well. This study concentrates on the results from the finance related studies only.

[9]A complete list of references would be very lengthy, but a useful website containing many papers and pointers is at *http://www.unifr.ch/econophysics*.

this work documents interesting features from financial time series which exhibit scaling like characteristics. These features persist over many different time scales, and present a possible new set of empirical facts. Whether real or not, scaling, as mentioned before, is important and needs to be looked at very carefully.

This paper is a study of just what might be responsible for the observed scaling laws. It shows that a simple stochastic volatility model calibrated to actual data is capable of giving a visual appearance of power law scaling types of results. This appearance is merely an illusion since theoretically the process used is not scale invariant. The primary reason for the difficulty is that financial time series are far from independent at high frequencies, and this dependence only gets stronger as one moves to very high frequencies. This dependence can slow down ones intuitive notion of how fast convergence should occur, and apparently can appear as a type of self similarity. Section 2 describes the stochastic volatility model and calibrates it to a foreign exchange futures time series. Section 3 presents the scaling results, and section 4 concludes.

2 Stochastic Volatility

In this section a stochastic volatility model is calibrated to match some general features of a price series from the DM futures market. The actual data set comes from the Chicago Merchantile Exchange, and contains 4254 daily observations including the close and the daily high/low ranges.

The simulated data will be created using a stochastic volatility model which will be thought of as generating high frequency data at 5 minute intervals. Given the 6.5 hour trading day on the futures market this corresponds to an aggregation level of roughly 80 of the simulated intervals to one day of actual clock time. The simulation is set to generate 800,000 5 minute intervals which corresponds to 10,000 days. Over each simulated day the high/low range statistics are recorded and they will be used to align with those from the actual data along with other more traditional daily return measures.

The log price process is assumed to follow a martingale difference sequence where returns are given by,

$$r_t = \log(P_t) - \log(P_{t-1}) \tag{1}$$

$$r_t = e^{(1/2)v_t}\epsilon_t \tag{2}$$

and the log return volatility is given by,

$$v_t = \mu + \rho(v_{t-1} - \mu) + \beta\eta_{t-1} + \eta_t \tag{3}$$

where ϵ_t and η_t are both independent, and drawn from gaussian distributions. Furthermore, ϵ_t has a variance of 1. This is a type of stochastic volatility model which has been used extensively in modeling financial series.[10] They are similar to the more common ARCH/GARCH models, but they have advantages with respect to temporal aggregation. Generally, GARCH models only temporally aggregate under certain restricted conditions.[11] Stochastic volatility models aggregate (in terms of staying in the same class of model) for a wider range of structures.

The parameters used will be $\mu = -16.4$, $\rho = 0.9998$, and $\beta = -0.994$. μ simply is used to align the mean of the process. However, the other two should cause some alarm. The AR parameter, ρ, is very close to one. This is driven by the fact that volatility is very persistent, and the process is designed to model returns over a very short horizon. To gain some further intuition it is important to remember that this ARMA(1,1) representation corresponds to a process such as,

$$v_t = u_t + \epsilon_t \tag{4}$$

$$u_t = \rho u_{t-1} + g_t. \tag{5}$$

The variance follows a process with two components. A slow moving drift component, u_t, along with a independent white noise component, ϵ_t.[12] ρ corresponds exactly to the ρ from the ARMA representation. Therefore, the response to a shock to volatility h periods in the future is ρ^h. For daily returns, $h = 80$, and $\rho^{80} = 0.98$, and for one week, $\rho^{400} = 0.92$. These values are not far off those that have been fit to daily and weekly data using ARCH/GARCH models, and as will be seen, they generate reasonable time series for daily returns.[13]

The key aspect of this calibration is to fit the dependence in volatility. This dependence is much larger than any changes in conditional mean that might be in the data, and for the purposes of this paper changes in conditional mean are ignored. Without high frequency data the volatility process is difficult to estimate. Measurements of correlations in daily squared returns provide only a crude measure of volatility over a day. The best situation is to use high frequency data to build a measure of volatility as in Andersen, Bollerslev, Diebold & Labys (1999), but without high frequency data, the next best approach is to use daily high/low

[10](Taylor 1986) and (Andersen 1994) are good examples of related earlier versions of this type of model. See Ghysels, Harvey & Renault (1996) for a recent survey of stochastic volatility models.

[11]See Drost & Nijman (1993) for temporal aggregation properties in the GARCH model class.

[12]See Taylor (1980) or LeBaron (1992) for examples using this to models changes in conditional means.

[13]A direct comparison between ρ, and GARCH parameters is not really possible here. However, this was mentioned to show that they are in a reasonable range. The calibration will be based on other features of the data.

ranges.

Garman & Klass (1980), Parkinson (1980), and Rogers & Satchell (1991), show that several high/low ranges will improve on the efficiency of close to close volatility estimators. One of the proposed estimators uses

$$\hat{\sigma}_t^2 = \frac{(\log(H_t) - \log(L_t))^2}{4\log 2} \tag{6}$$

as an estimator for daily volatility. H_t and L_t are the daily high and low respectively, and logarithms are all base e. We will apply this both to the actual and simulated series for comparisons in the following table.

We will try to match several features of the actual returns series. Table 1 presents summary statistics for the DM futures series along with the simulated 5 minute process. The processes has been aggregated to daily for the summary moments. Variance and kurtosis are estimated on the daily series. Log (σ_t^2) refers to the log of the daily volatility estimate given by the high/low ranges as in equation 6. It is clear from the table that this simulated process is giving a good representation of the basic moments of the actual series.

Table 1: *Return time series*

Series	Interval	Variance	Kurtosis	Mean($\log \sigma_t^2$)	var($\log \sigma_t^2$)	kurtosis($\log \sigma_t^2$)
DM Futures	1 Day	5.5e-5	5.14	-10.96	1.05	3.24
Simulation	1 Day	4.0e-5	5.64	-10.94	1.05	2.98

Futures versus simulated returns. Simulated returns are generated at the 5 minute horizon and aggregated to daily by summing 80 5 minute intervals. The daily variance, σ_t^2 is estimated using high/low ranges for both the simulated and actual series.

The critical feature that needs to be replicated is the autocorrelation of volatility itself. Figure 1 shows the autocorrelation for $\log \sigma_t^2$ estimated both for the futures series and for the simulated process. Both show a similar long range persistence in volatility. A critical look might suggest some differences in that the simulated series has a slightly higher correlation at the lower lags and lower at longer horizons, but for calibration purposes it still appears reasonable.[14] Figures 2 and 3 repeat the autocorrelations for the more common measures of volatility, squared and absolute returns, measured at the daily horizon. Once again the simulated process appears to be a reasonable replication of the dependence in the series.

The second test of how closely this simulated series is matching well known features of financial data is to fit a GARCH(1,1), (Bollerslev 1986), and to analyze the residuals. Table 2 presents these results for the daily DM futures series, and the simulated 5 minute returns series aggregated into simulated days. It shows a very similar pattern, common to other financial series. Two common tests for dependence are presented.

[14] Some authors have shown persistence at much longer horizons Ding, Granger & Engle (1993). This was not present in this futures series. It may be due to the shorter horizon of the overall series, or other features specific to FX futures markets such as the presence of price limits prior to February 1985.

First the usual Engle (1982) test for ARCH is shown in the row labeled ARCH(5), and the BDS, (Brock, Dechert, Scheinkman & LeBaron 1996), test for dependence is presented in the row labeled BDS(2). The numbers in parenthesis give asymptotic p-values for each test. The GARCH(1,1) eliminates traces of ARCH, and dependence in the returns series as it should if it were correctly specified. However, it leaves a residual series that still shows excess kurtosis. These results are very common for many financial series, and have led to the development of ARCH/GARCH models that depart from gaussian innovations.[15]

Table 2: *GARCH(1,1) Residual Diagnostics*

Diagnostic	DM Daily	Simulated Daily	DM Daily GARCH Residuals	Simulated Daily GARCH Residuals
ARCH(5)	69 (0.00)	263 (0.00)	3.21 (0.65)	1.40 (0.92)
BDS(2)	4.49 (0.00)	10.03 (0.00)	0.63 (0.53)	1.27 (0.20)
Kurtosis	5.14	4.82	4.39	4.41

Futures versus simulated returns. Simulated returns are generated at the 5 minute horizon and aggregated to daily by summing 80 5 minute intervals. Residuals are standardized returns from an estimated GARCH(1,1) for each series. Returns are divided by the estimated conditional standard deviation from the GARCH model. ARCH(5) is the Engle test for the presence of ARCH (Engle (1982)), and BDS is the BDS test for dependence (Brock et al. (1996)). The numbers in parenthesis are asymptotic p-values. Kurtosis is the sample kurtosis which should be 3 under a gaussian distribution.

3 Scaling Results

For many of the results in this section will use the notation of a time aggregation function. This is described by,

$$[r_t]_h = \sum_{i=1}^{h} r_{h(t-1)+i}. \tag{7}$$

The aggregator function is crucial to looking at returns over progressively longer horizons, and analyzing scaling properties.

The key property considered here is self-similarity. A stochastic process is said to be self-afine, or self-similar if

$$([r_t]_h, [r_{t-1}], [r_{t-1}]_h, \ldots) \stackrel{d}{=} (h^d r_t, h^d r_{t-1}, h^d r_{t-2}, \ldots). \tag{8}$$

d is the scaling exponent. Examples of this are the independent stable class, where $d = 1/\alpha$, and its most famous special case, the gaussian where $d = 1/2$. Fractional brownian motions are another large class of

[15]For some examples from other series see Brock, Hsieh & LeBaron (1991),

self-similar processes.

3.1 Histogram Shapes

Several authors have presented evidence on scaling and the unconditional distributions of high frequency asset returns. This literature concentrates on the deviations from normality evident in high frequency returns, and the persistence of this feature under time aggregation. This section explores some of these features using the simulated data from the previous sections.

Results in papers such as Mantegna & Stanley (1996), Ghashghaie et al. (1996), and Galluccio, Caldarelli, Marsili & Zhang (1997), show a pattern when histograms of unconditional returns are plotted in semi-log space. First, the returns distribution is clearly nongaussian with a pronounced peak, and this feature holds over many different time scaling horizons.[16] Also, there is some evidence for self-similarity across scaling ranges, but this feature is not as strong as the former.

Figure 4 plots histograms for the simulated series using aggregation levels of $h = 1, 10, 100, 200$ which correspond to time intervals in real clock time of $t = 5, 50, 500, 1000$ minutes, respectively. This lines up with the maximum time horizon used in Mantegna & Stanley (1996). The series are scaled using a scaling exponent of $d = 0.7$.[17] It provides a reasonable matching for the histograms as can be seen in the figure. The solid lines give the first three aggregation levels, and the dotted line represents $h = 200$. The central parts of the distributions appear quite similar and sharply peaked while some differences appear in the tails. Finally, a gaussian distribution is fit to $h = 200$ after it had been rescaled to the daily horizon. It is shown by the stars. It should line up exactly with the dashed line if the series was gaussian. The important fact is that a distinct deviation from the gaussian is still seen at this aggregation level.[18]

The next figure, figure 5, presents results similar to the cotton scaling plots of Mandelbrot (1963). In his figures a strong similarity was shown for cotton futures prices over several different time horizons in a log-log plot of the tail probabilities. This is repeated here for both the simulations (*), and the DM futures series (+). Under self-similarity the distribution shapes should be parallel shifts of each other. This figures show a good replication of this feature for both the simulated and actual series.

Now that the simulation has passed two ocular tests a more traditional measure will be used. Kurtosis is estimated over several different horizons ranging from five to five thousand minutes (125 days). Table 3 shows some amount of nongaussian behavior over the first 4 scaling ranges (up to about two weeks). It is

[16] Gaussian density plots should be quadratic in semilog space.
[17] This exponent was estimated by Mantegna & Stanley (1996).
[18] Although the aggregation level seems large, h = 200, or 1000 minutes still corresponds only to about 2.5 days.

clear that there is a slow convergence, but this doesn't appear until the very long horizon is reached. The kurtosis levels are also very close to those for the DM futures series. This is consistent with the results from the modern literature and high frequency scaling laws. Most all agree on a slow convergence at the very long horizon.

Table 3: *Simulation time series*

Interval	Kurtosis Simulation	Kurtosis DM Futures
5 minute	111	
50 minute	14.6	
500 minute	5.39	5.13
5000 minute (12.5 days)	3.88	3.46
50000 minute (125 days)	2.89	2.77

Simulation return kurtosis estimated over progressively longer sampling intervals. The sample size corresponds to 10,000 days of data for the simulation, and 4254 for the DM futures.

3.2 Absolute value scaling

Other results have demonstrated a more detailed scaling property in financial markets. For example, Muller et al. (1995) show that absolute returns exhibit linear scaling over a very large range of aggregation levels.[19] Figure 6 shows scaling over a similar time range.[20] Scaling clearly breaks down at the longer horizons due to the smaller data sets.[21] The slope of the line is 0.53 with an estimated standard error of 0.002.[22] This is lower than the estimates produced in Muller et al. (1995) which give a value of 0.586.

It is important to remember that the simulated series is not scale invariant. To keep this in perspective another simulation is run with a much less persistent series to see if it shows any type of scaling behavior. The AR coefficient is reduced to 0.9, and the MA coefficient is reduced to zero. The previous scaling results for this new simulated series are presented in figure 7 which clearly shows a deviation from linear scaling.

Power laws for higher order moments are also possible. Figure 8 reports scaling laws for $|[r]_h|^3$. This figure again shows a good scaling law with a slope of 1.48. It is easy to show that under an IID gaussian null hypothesis these scaling laws should move as $(1/2)n$ where n is the exponent that the absolute value is raised to. Figure 9 looks at scaling estimates for n varying from 1 to 10 for two experiments. The

[19]This is also similar to the results in Fisher, Calvet & Mandelbrot (1996). Also, more recent results with different scaling exponents are obtained in Weron, Weron & Weron (1999).

[20]The aggregation level starts at a larger value $e^8 \approx 3000$ seconds or about 10 of the 5 minute intervals. This is due to the fact that sampling from the simulation is done more coarsely than in the Olsen data set.

[21]At e^{17} seconds, which corresponds to about 1000 days, the sample is reduced to 10.

[22]Estimated OLS standard errors should be viewed with caution here. They are reported only to compare with previous results. However, they should not be considered a good estimate of the true standard error of a scaling slope.

simulated stochastic volatility data is compared to a simulated IID gaussian with the same length and standard deviation as the original series. The figure clearly shows that the simulated data does not scale as $(1/2)n$. However, as predicted, the gaussian scales as it should. Results such as this make it difficult to judge what less than 1/2 scaling really means in these cases. These results replicate scaling results that have been reported for actual financial data such as those in Galluccio et al. (1997).

3.3 Volatility Scaling

Figure 10 replicates the scaling tests performed in Andersen et al. (1999). These graphically explore the time scaling properties in the foreign exchange volatility process itself. In their study volatility is estimated using high frequency data to get a daily time series. They present strong evidence that the volatility process has a scale invariance property. If volatility itself, and not returns, scales temporally then

$$[v_t]_h = h^d [v_t]_1. \tag{9}$$

From this relation it is easy to show that the variance of the volatility process should scale as,[23]

$$\mathrm{var}([\mathrm{v_t}]_\mathrm{h}) = \mathrm{h}^{2\mathrm{d}}\mathrm{var}([\mathrm{v_t}]_1) \tag{10}$$

Since v_t is the logged variance, the same scaling relationship will hold for the standard deviation since it is $s_t = (1/2)v_t$. As in Andersen et al. (1999), the short horizon is defined to be one day, or 80 5-minute increments. This is used to build an estimate of the daily standard deviation which is then logged,

$$[s_t]_1 = (1/2)\log(\sum_{h=1}^{80} r_{t,h}^2). \tag{11}$$

In this notation h represents each 5 minute subperiod on day t.

The value $\mathrm{var}([\mathrm{s_t}]_\mathrm{h})$ is estimated over different horizons, h. These are then plotted in log/log space since this will give the slope from equation 10. Under scale invariance the figure will scale as a straight line. Figure 10 shows the simulation results. Also, presented are values from the actual futures series using the high/low estimator to get the daily standard deviation. It is clear that both show what appears to be a simple linear scaling law in log/log space. The slopes were estimated using OLS, and are 1.7 and 1.8, for the actual data and simulated series respectively. This compares with values estimated in Andersen et al. (1999) which were

[23] Variance scaling tests are described in Teverovsky & Taqqu (1995).

estimated at 1.779 for their high frequency DM series.[24]

4 Conclusions

These results should increase caution in interpreting purely graphical evidence on scaling and self-similarity. However, they should not put an end to research in these areas. As mentioned in the introduction, these results, if reliably defended would be very exciting for economics and finance. At this time it is still premature to accept self-similarity as an empirical fact.

The reader of this literature should be aware of two problems. The first is the traditional issue of how to interpret purely graphical evidence. It is not clear what is being tested, what the null hypothesis is, and what the standard errors should be for slope estimators, and scaling pictures. This paper reminds us that often in graphical spaces very different processes may appear similar to the eyeball. Experiments such as these are important in reminding us the amount of statistical power that is contained in the pictures. Also, with graphical tests it is the case that the user has some license to make the tests look as good as possible by choosing axes and scaling ranges. Graphical tests can and should be used as diagnostic tools, but they should not be the end product in hypothesis testing.

The second problem is that for a time series displaying strong dependence, obtaining a large number of points sampled at high frequency does not increase your sample size as much as you may believe. This is well known, but sometimes it is not clear how far things can differ from intuitive notions of what an independent series should look like. This paper has shown that dependence in these series can cause apparent scaling laws to appear when none actually exist. Although no theoretical results were given, the mechanism for this was hinted at in several places. In one example reducing the dependence in the volatility process caused a breakdown in the scaling results. This is suggestive that dependence may be the key cause in the apparent, but spurious scaling that is observed here.

This paper does not address the related literature on long-memory in volatility. Most of these studies use more reliable statistics than simple graphs for their work, and many use much longer time series than the one used here. The question of whether there is long memory in volatility remains an interesting, and open question, but it was not addressed here. It also should be stressed that this process is probably not the best for modeling returns and volatility. It is not clear that it captures all known puzzles from financial markets. It is simply a counter example to indicate that many processes may exhibit ocular sorts of scaling

[24] These correspond to a fractional difference level, D, where the slope equals $2D + 1$. This gives a value of $D = 0.4$ for the simulated series.

behavior.[25]

Finally, the scaling literature concentrates on mostly unconditional distributional features of these data, such as histograms, moments, and tail probabilities. There are also many conditional moments that change in interesting ways in financial series, and there are also important comovements with other variables such as trading volume. It would appear that these feature may be more useful and powerful in helping to discern between many of the new multi-agent bounded rationality models that are being proposed.[26]

[25] An interesting side question in financial markets is the behavior of large moves. This appears to be an integral part of all markets, and volatility dynamics. Do they behave like big versions of the small moves, or are they fundamentally different? This question remains unanswered. However, a recent study by Bakshi & Madan (1999) shows some new interesting results.

[26] See Brock & LeBaron (1996) and LeBaron, Arthur & Palmer (1999) for examples.

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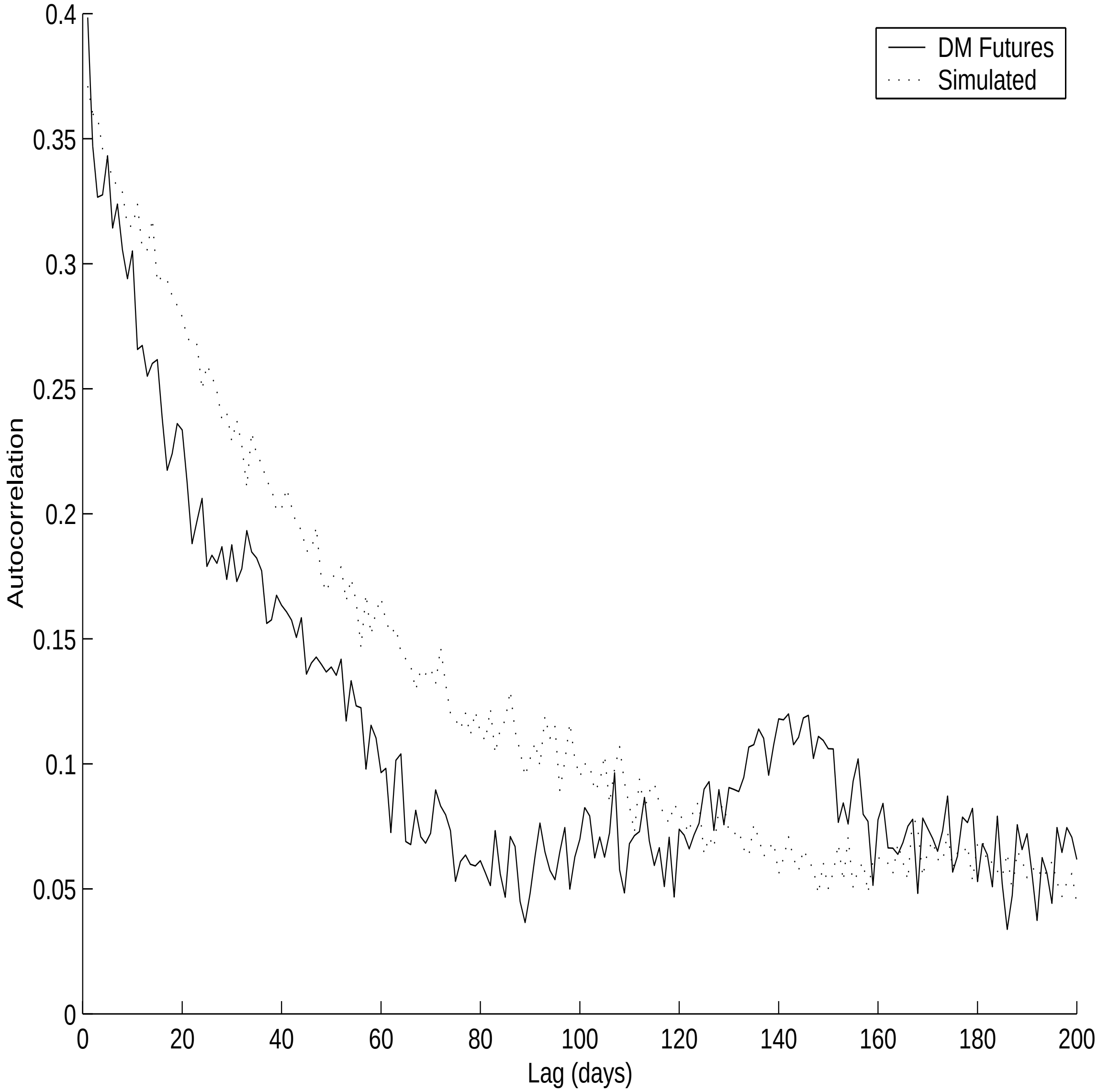

Figure 1: *Daily High/Low volatility ACF* : Correlation of log volatility process estimated using daily high/low ranges.

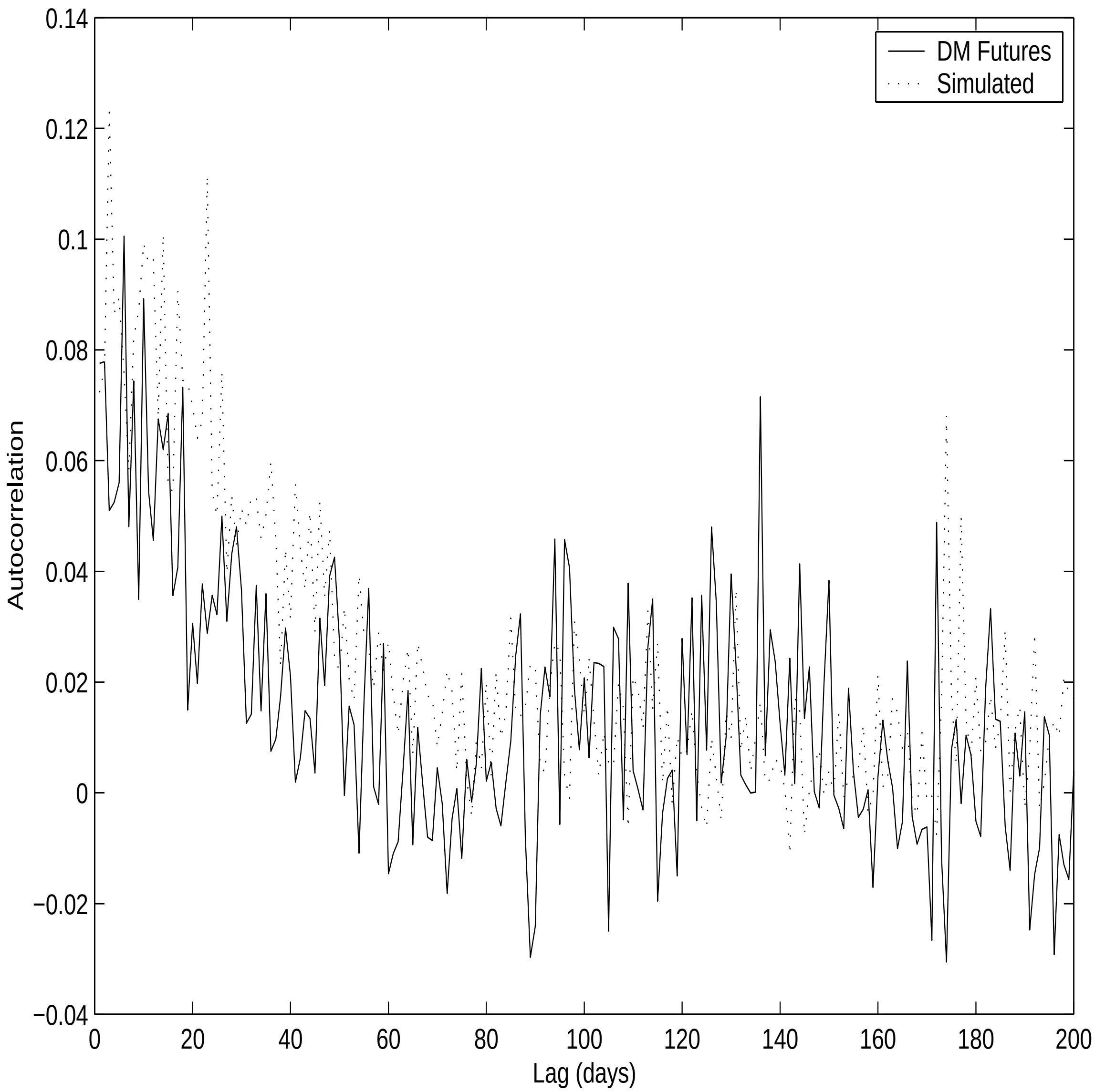

Figure 2: *Squared Daily Returns ACF*

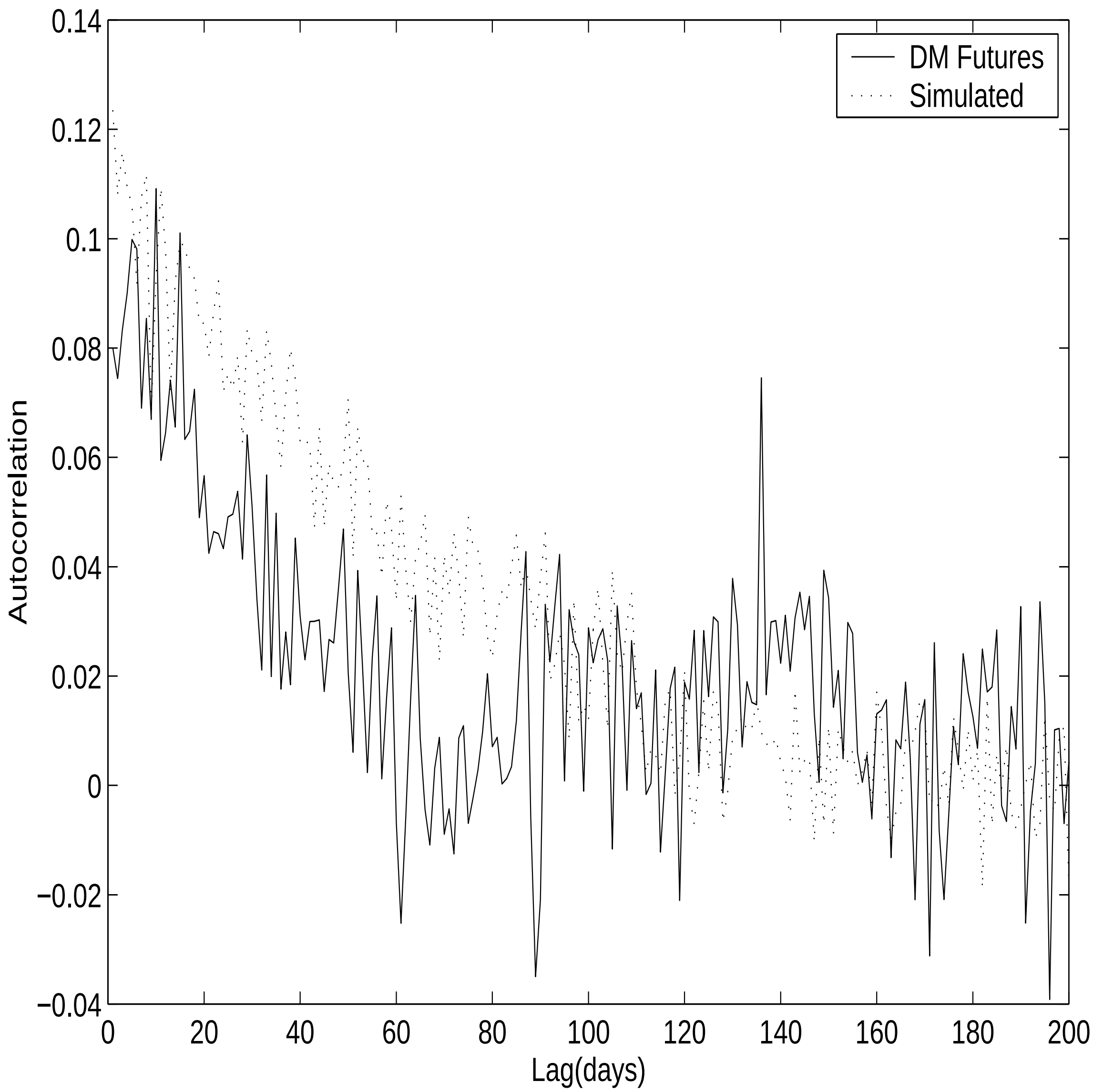

Figure 3: *Absolute Daily Returns ACF*

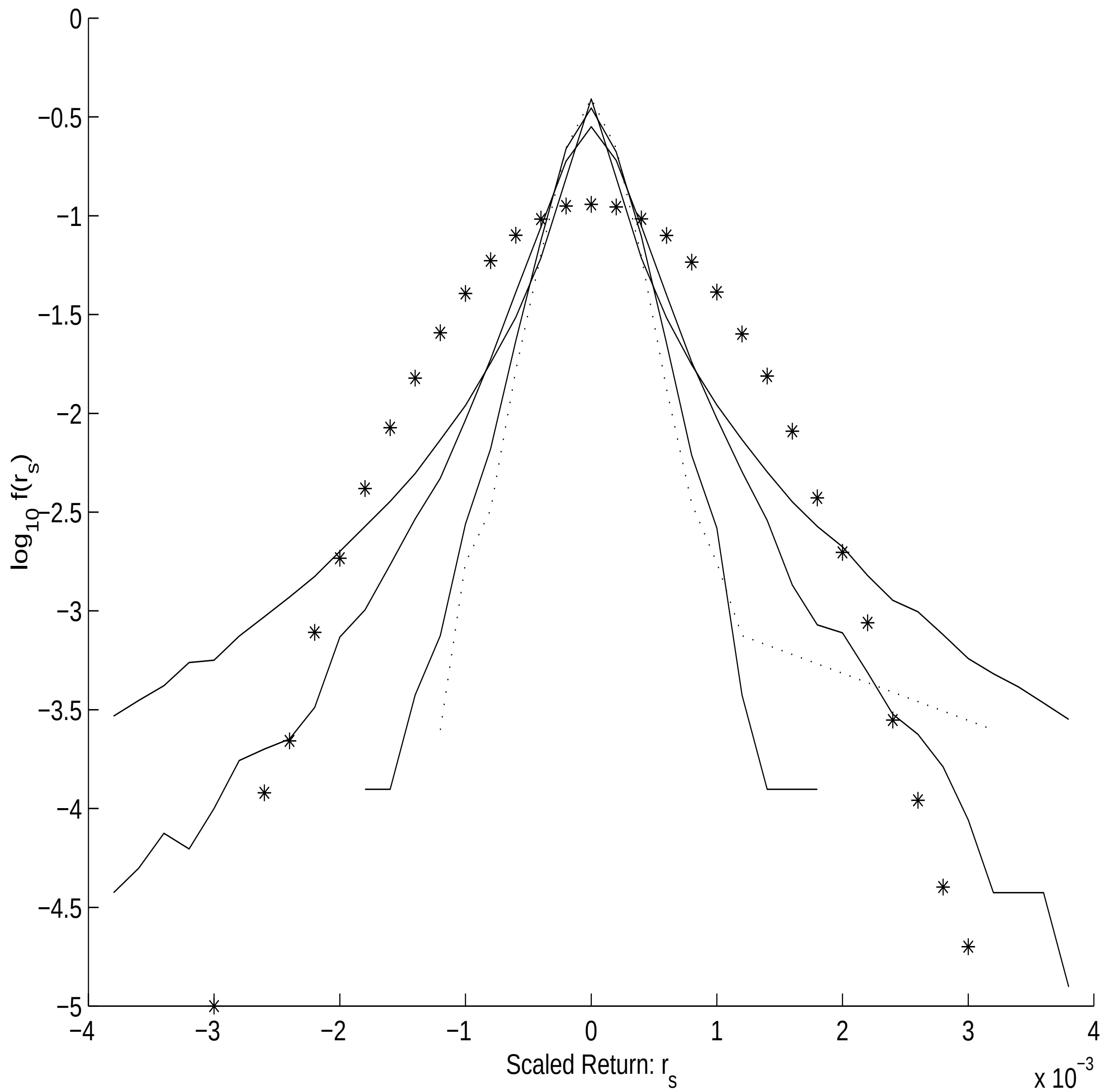

Figure 4: *Probability density histogram*: Aggregation levels of 5, 50, 500, 5000 minutes. All are solid lines except 5000 = dots. * are gaussian density fit to long horizon reuturns. Returns are rescaled using $r_s = (h)^{-0.7}[r_1]_h$.

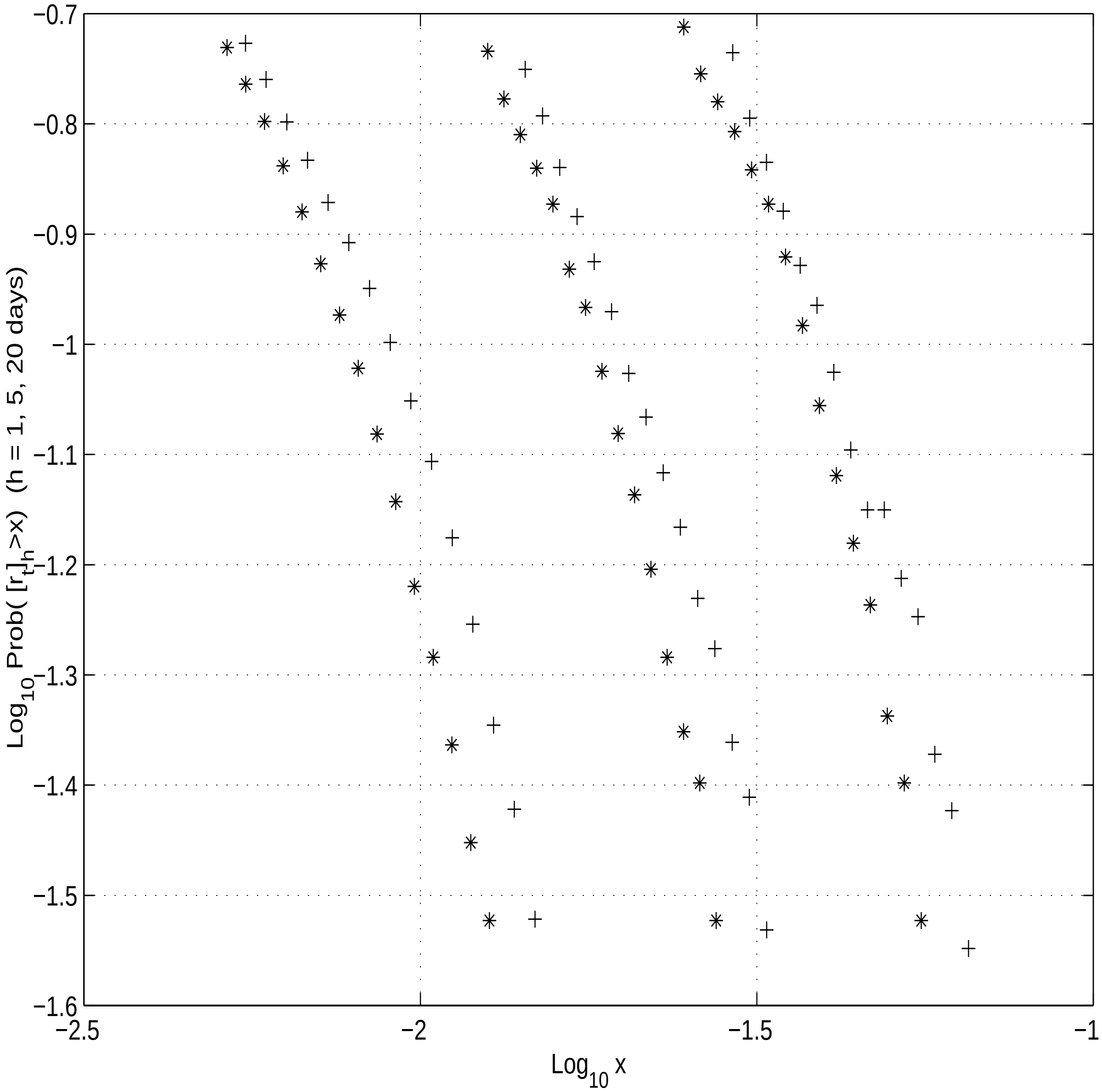

Figure 5: *Log CDF: Right tail. Aggregation = 1, 5, 20 days moving left to right. * = simulated process, + = DM Futures.*

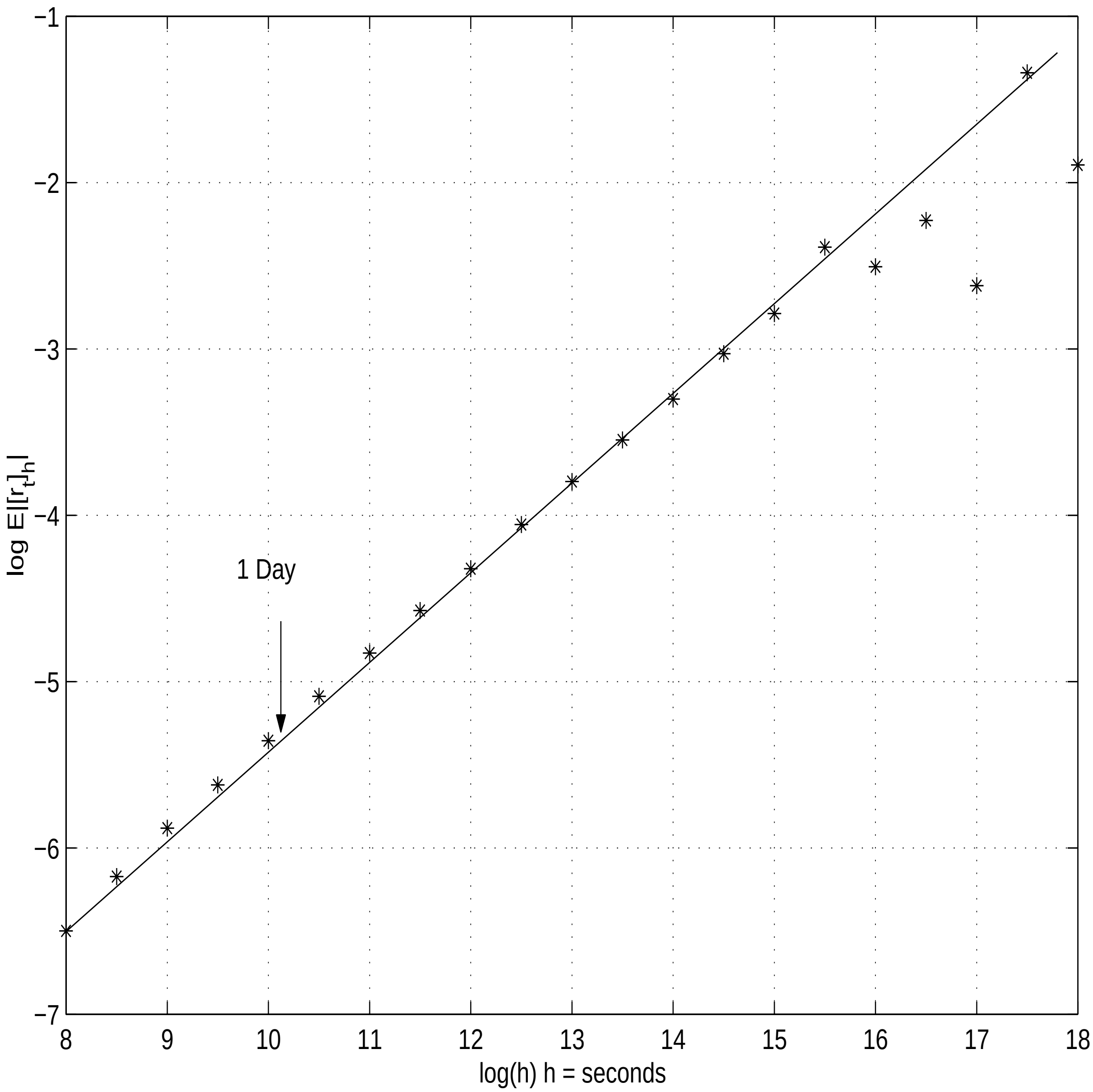

Figure 6: *Scaling in* $E|[r_t]_h|$ *for simulated data.* h *is adjusted to correspond to seconds of trading time.* $e^8 =$ *2900 seconds, or* 0.8 *hours.* e^{15} *corresponds to roughly 130 days.*

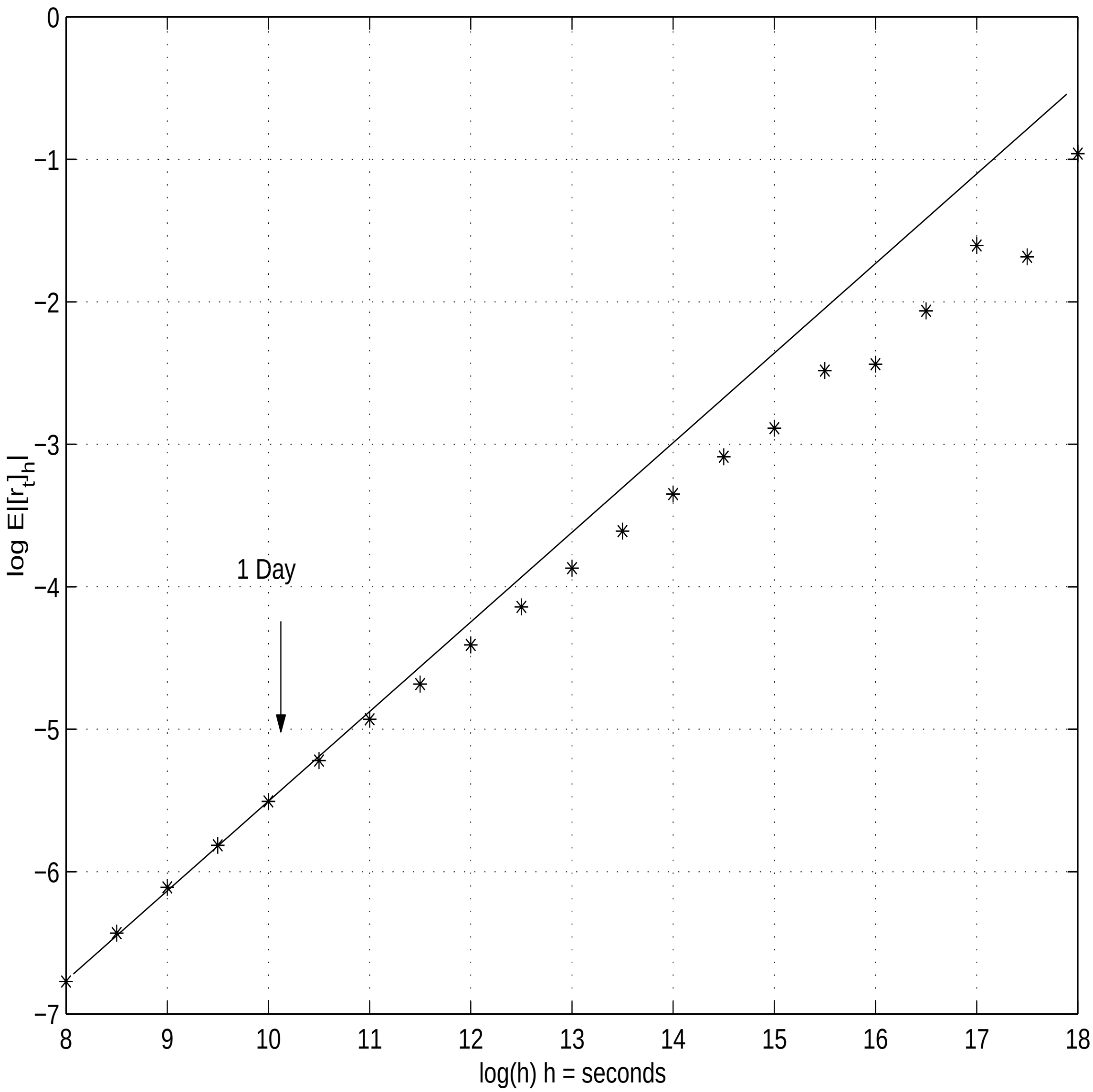

Figure 7: *Scaling in* $E|[r_t]_h|$ *for simulated data with lower persistence.*

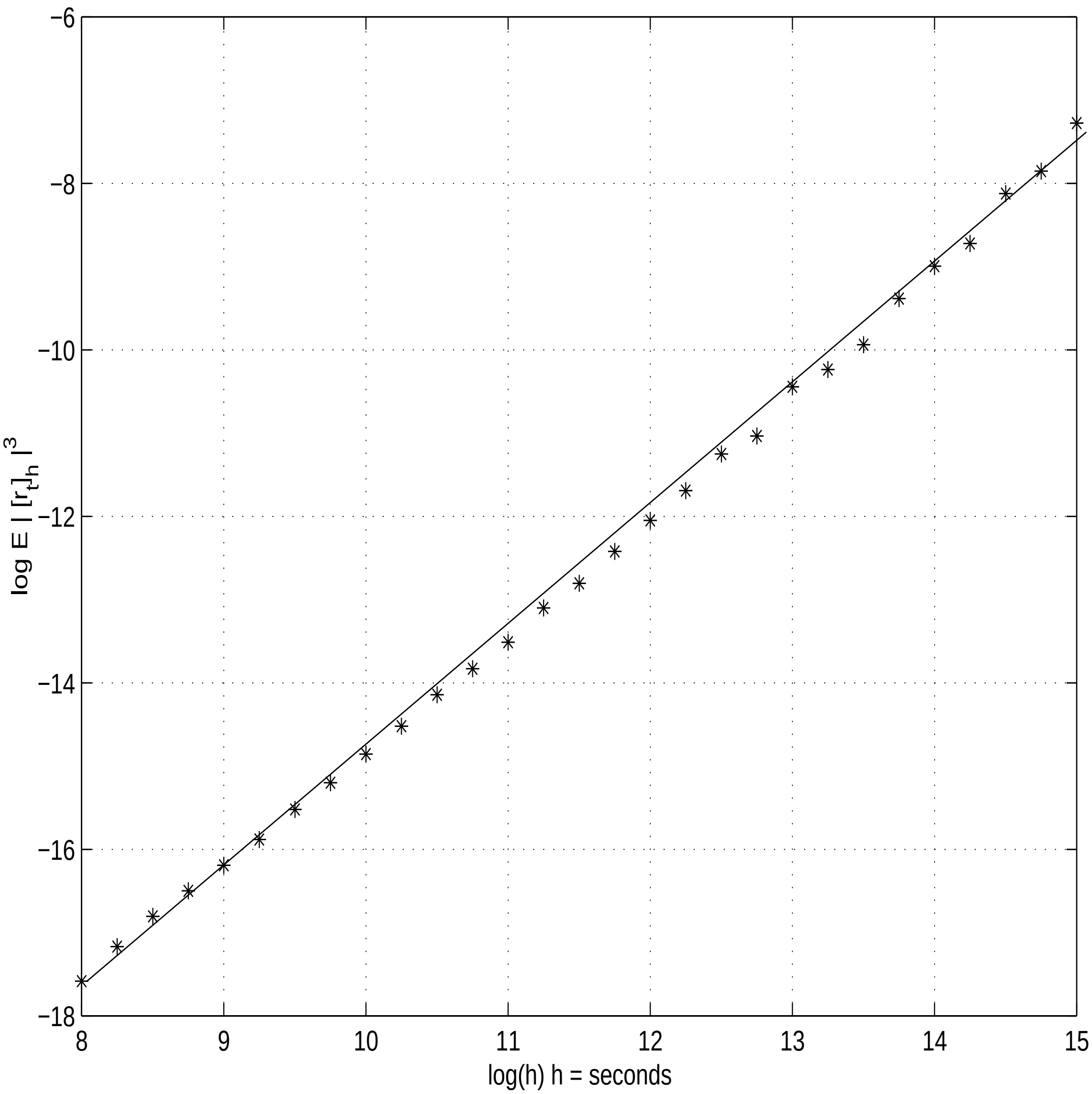

Figure 8: *Scaling in* $E|[x]_h|^3$

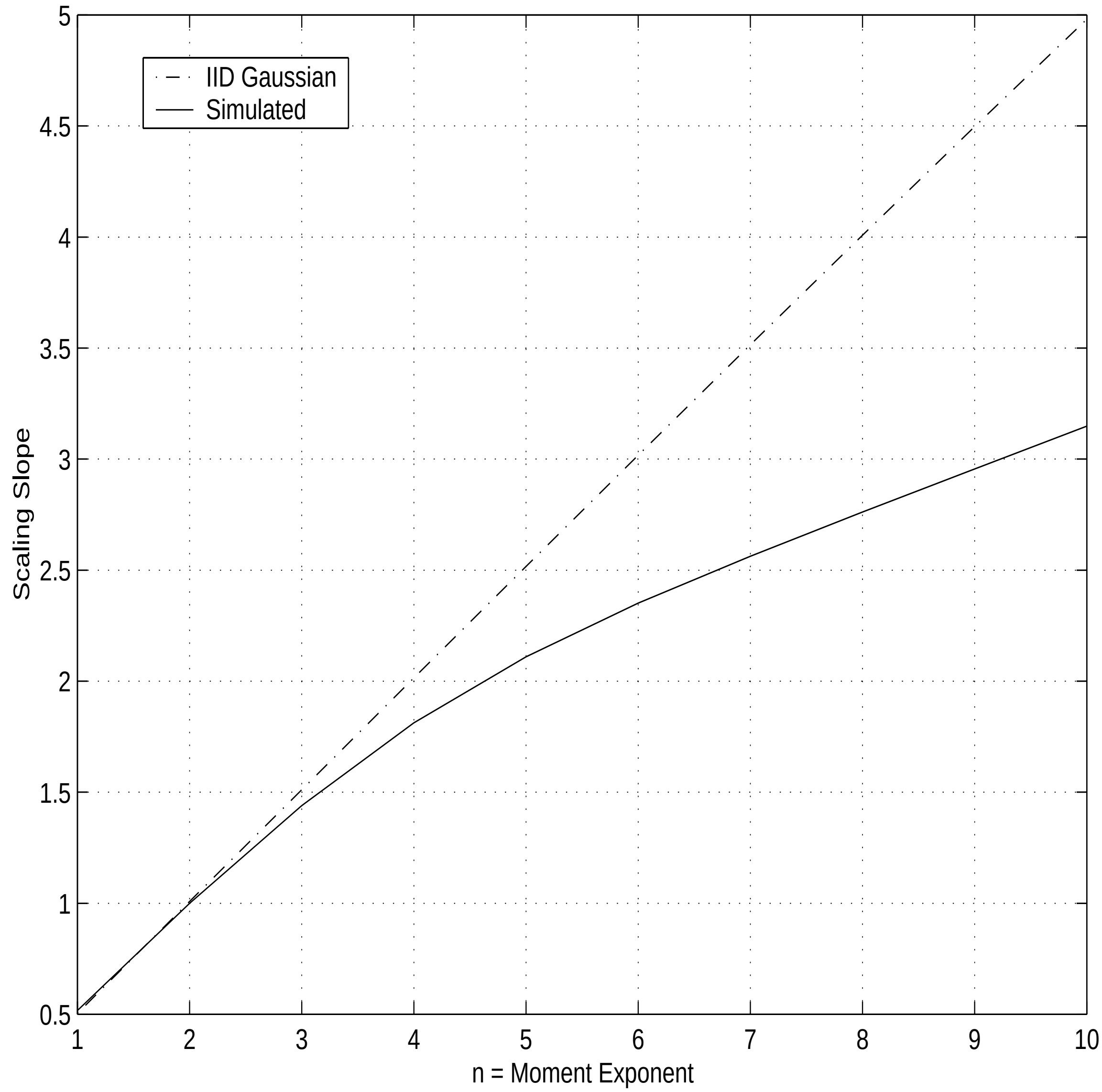

Figure 9: *Scaling slopes for* $E|[x]_h|^n$. n is varied from 1 to 10. IID Gaussians were matched in sample length mean and standard deviation to the simulated high frequency returns series. Slopes are estimated over the range of e^8 to e^{15} seconds.

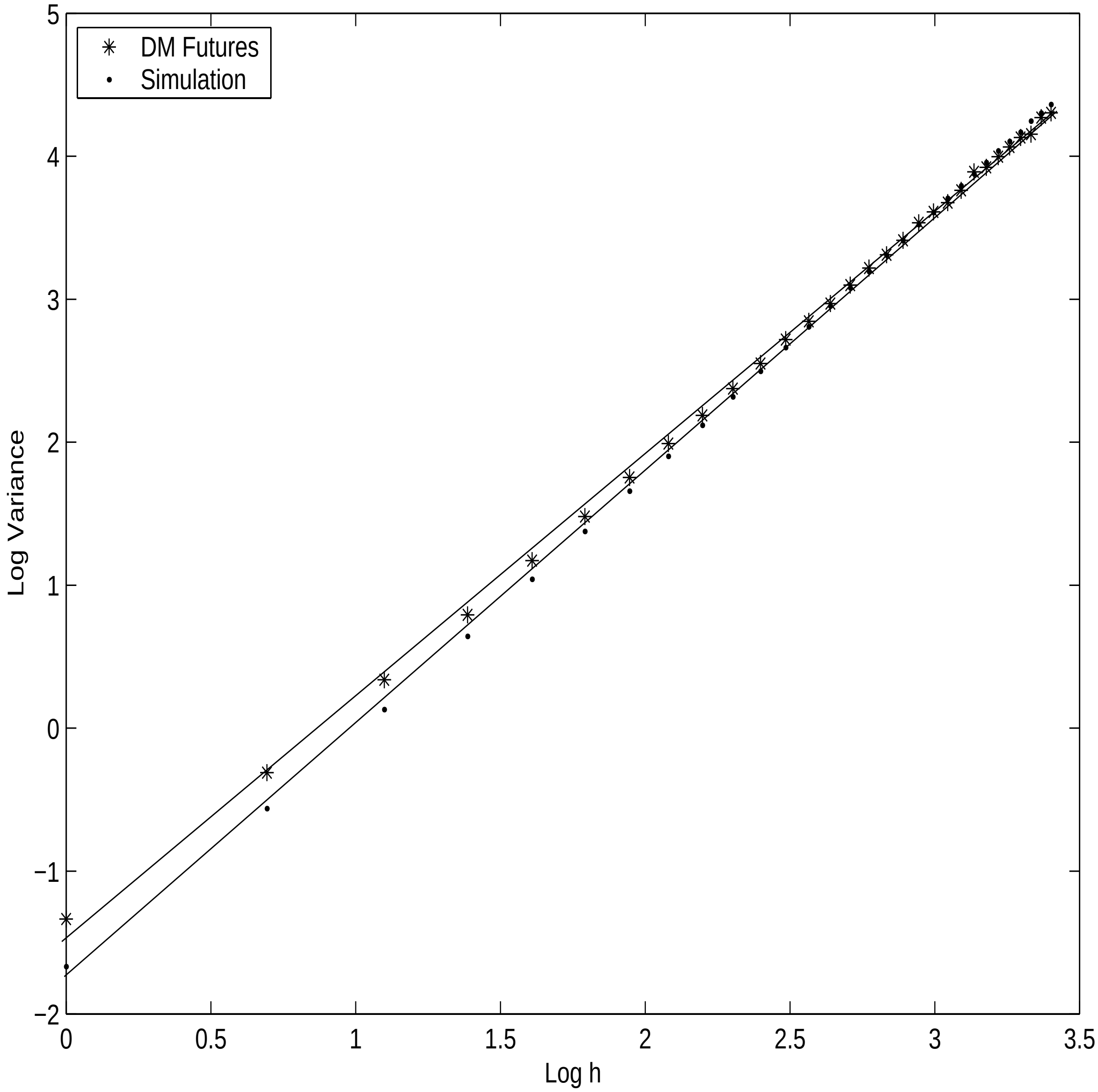

Figure 10: *Scaling in the variance of the volatility*