

Evolution and Time Horizons in an Agent Based Stock Market

Blake LeBaron *

Brandeis University

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Abstract

Recent research has shown the importance of time horizons in models of learning in finance. The dynamics of how agents adjust to believe that the world around them is stationary may be just as crucial in the convergence to a rational expectations equilibrium as getting parameters and model specifications correct in the learning process. This paper explores the process of this evolution in learning and time horizons in a simple agent based financial market. The results indicate that while the simple model structure used here replicates usual rational expectations results with long horizon agents, the route to evolving a population of both long and short horizon agents to long horizons alone may be difficult. Furthermore, populations with both short and long horizon agents increase return variability, and leave patterns in volatility and trading volume similar to actual financial markets.

*Graduate School of International Economics and Finance, Brandeis University, 415 South Street, Mailstop 32, Waltham, MA 02453 - 2728, blebaron@brandeis.edu, www.brandeis.edu/~blebaron. The author is also a faculty research fellow at the National Bureau of Economic Research, and an external faculty member at the Santa Fe Institute. The author is grateful to Jasmina Arifovic, Peter Bossaerts, James Bullard, John Duffy, Deborah Lucas, Carol Osler, an anonymous referee, seminar participants at Harvard University, The University of Iowa, and conference participants at The Econometric Society, The National Bureau of Economic Research, and CEF99 at Boston College for useful comments. This market was run using Matlab version 5.3.

1 Introduction

Traditional representative agent models for financial markets make many assumptions about the individuals that populate them. The problems these individuals have solved to reach an equilibrium may require enormous computing power, or are simply ill posed.¹ It is also possible that in a dynamically evolving world complicated strategies may prove brittle and give way to simpler, but changing short term policies.² This paper explores these questions in a multi-agent financial setting with learning and adaptation. Its main goal is to explore the importance of assumptions about stationarity used by learning agents. Stationarity of the economic environment is often viewed with skepticism by market participants. This is especially true in financial markets where one often hears of a “new economy” at work driving asset values to levels where old valuation ratios are irrelevant. This paper explores the evolutionary properties of a market populated with both long and short horizon investors in a well defined simple stock market. The objective is to find under what conditions dominance by the long horizon traders along with convergence to a rational expectations equilibrium is likely. A second goal is to understand what types of dynamics are possible outside of the equilibrium, and how they compare to features from actual financial time series.

Models with learning and forecasting agents require us to replace all knowing, fully rational agents with “artificially intelligent agents who behave like econometricians” (Sargent 1993).³ These artificially intelligent econometricians face the same sets of problems real econometricians do when exploring economic data. These include parameter estimation, model specification, and identification of the relevant sample of past data to use for estimation. In a stationary world all past observations are relevant, but if the data generating mechanism of the world is changing it may be advantageous only to look at recent observations. This problem still gets relatively little attention, even though it may be critical for how asset markets behave.⁴

¹See (Arthur 1994) and (Simon 1969) for discussions of this issue. Also, see Board (1994) for some measures of necessary computing power.

²See Lindgren (1992) for a good example of this from game theory. Axelrod (1984) is the classic example of work stressing the coevolutionary framework for social dynamics, and Bookstaber (1999) presents some direct examples of this related to risk management and finance.

³There are many papers on the subject of learning in macroeconomics. The recent work by Sargent (1993), and Marimon (1997) serve as good references to this large literature. (Cho & Sargent 1999) and (Sargent 1999) show the strong impact of constant gain sequences in learning dynamics which is related to limited past weighting done by agents in the markets considered here. Grandmont (1998) provides another synthesis of learning in economic systems, and states some basic features about instability that may have strong connections to the simulated markets analyzed here. Kurz (1997) proposes a weaker notion of equilibrium in a learning environment which also looks promising for understanding some financial puzzles as in Kurz & Beltratti (1995).

⁴Evans & Honkapohja (1995) and Honkapohja & Mitra (2000) consider the dynamics of learning with bounded memory which have similarities to the learning processes considered here. Early conjectures about the possible importance of traders with different relevant time scales are in Muller, Dacorogna, Dave, Pictet, Olsen & Ward (1995) and Peters (1994). Several papers in finance have started to deal with stationarity using very different approaches. Pastor & Stambaugh (1998) and Timmermann (1998) analyze financial series, and find that it is reasonable to view returns as having several structural breaks. Using a different framework from the one used in this paper Benink & Bossaerts (1996) consider the similar problem of agents learning the stationarity of the world in which they live. Also, Joshi (1998) uses a version of the Santa Fe Artificial Market to

Sometimes behavior based on the recent past could be called irrational, but in the noisy world of financial time series it is not clear ex-ante which strategies can be ruled out. Furthermore, in an endogenous market setting the presence of short horizon agents will impact prices and may reinforce their behavior. In this way the short horizon traders are similar to noise traders. However, unlike noise traders, their actions may be driven by well defined policies, optimal with respect to their short horizon interpretation of the relevance of history, and to the current structure of the population of other traders. This paper takes the hard problem of deciding learning horizons and moves it to an agent based evolutionary setting.⁵

The literature on agent based computational economics is steadily growing and it would be impossible to cite the entire group of papers here.⁶ In the current framework agents with short and long horizons compete in an evolutionary struggle, trading a risky asset. It is tempting to assume that the obvious winner should be the long horizon agents since they will be using more information in their strategy learning process. However, in a dynamically evolving market this may not be the case since the long horizon agents will have to compete with a heterogeneous group of shorter horizon types who may end up dominating, or at least doing well enough to survive.⁷

This paper has several empirical as well as theoretical goals. Among these are the demonstration that a heterogeneous learning market can amplify the variability in a relatively stable fundamental series, and raise the risk premia for the risky asset.⁸ It also generates several other features commonly found in asset returns. Among these are fat tails, persistent volatility, and trading volume that is persistent and moves with volatility. Another important characteristic is that by changing the time horizons of the agents, many of

analyze this problem from a game theoretic perspective. Levy, Levy & Solomon (1994) and Mullainathan (1998) both stress limited memory about the past. The experiments in Lettau (1997) also suggest the importance of time, and sampling on the part of agents. There is also a connection to the way beliefs and actions are updated in reinforcement learning approaches in game theory and experimental economics. See for example Camerer & Ho (1999) and Erev & Roth (1998).

⁵DeLong, Schleifer, Summers & Waldmann (1991) and Hirshleifer & Luo (1999) are two examples showing that in some situations it may be possible for irrational strategies to survive.

⁶The interested reader should see the web site of Leigh Tesfatsion (<http://www.econ.iastate.edu/tesfatsi/ace.htm>). Also, the book by (Epstein & Axtell 1997) is an important unified framework for agent based modeling. In the area of finance the number of papers using agent based methods is also expanding. Several examples are Arifovic (1996), Chakrabarti (1999), Chiarella (1992), Hong & Stein (1999), Kim & Markowitz (Fall 1989), Lettau (1997), Lux (1997), Margarita & Beltratti (1993), Routledge (1999), Tay & Linn (1999) and Youssefmir & Huberman (1997). Examples looking at a continuum of agents in learning economies include papers by Brock & LeBaron (1996), Brock & Hommes (1997), and Brock & Hommes (1998). For a recent survey of the work in economics see LeBaron (2000). There has also been a recent set of papers coming out of the physics literature. A brief set of examples is: Bouchaud & Cont (1998), Bußhaus & Rieger (1999), Caldarelli, Marsili & Zhang (1997), Chang & Stauffer (1999), and Farmer (1998). The website at <http://www.brandeis.edu/~blebaron/acf> contains many references and links to agent based work in finance.

⁷A more analytic study of wealth dynamics is Blume & Easley (1990) which looks at the evolution of simple strategies. They find that “rationality” and evolutionary “fitness” do not necessarily go together in their financial market setting. In an interesting comparison piece, Sandroni (1996) shows that evolution may indeed select for accurate forecasters. That setup involves a common intertemporal preference setup as is used in this paper, but involves complete markets.

⁸This is the first step in an attempt to calibrate this model to relatively long horizon financial data, and to present a possible heterogeneous agent based solution to the equity premia puzzle (Mehra & Prescott 1985). Several other learning papers are much farther along in terms of calibrating learning models to financial data. Examples are Timmerman (1993) and Bullard & Duffy (1998).

these features disappear. It is important in agent based models not just to replicate features of real markets, but also to show which aspects of the model may have lead to them. In this case, forcing the world to be one made of traders using long time series histories reliably drives the model to a simple homogeneous rational expectations equilibrium.

There are close connections between this market and the Santa Fe Artificial Stock Market (Arthur, Holland, LeBaron, Palmer & Tayler 1997, LeBaron, Arthur & Palmer 1999). However, there are also many differences. The single period constant absolute risk aversion (CARA) agents in the SFI framework are now replaced with additively separable constant relative risk aversion agents. One feature that these agents bring to the market is that now wealthier agents will have a greater impact on market prices. This was not true in the SFI framework.⁹ Also, these preferences respect the fact that agents live for many periods and evaluate utility of many period uncertain consumption plans. Some of the computational details are changed and simplified from the Santa Fe market. Most importantly, the classifier based trading rules are replaced with neural networks. These reduce the number of parameters, and their continuity allows for certain market clearing mechanisms to be used. Though the mechanism has changed, many of the directions taken here are inspired from results found in the Santa Fe market. Specifically, it stressed the importance of time horizons for the learning agents. If they learned slowly and looked at relatively long periods of past data, then convergence to a rational expectations equilibrium was likely. If the speed at which agents were learning and adapting was increased, and the time horizons shortened, then this convergence was unlikely to occur, and the market appeared to move through continuously changing regimes. These results imply that there was a well defined benchmark for the market to converge to. This feature will be repeated here, and forms a useful focal point to think about how and when markets may function well, and when they may not. A final distinction from the SFI market framework is that trading or policy rules are separate from the agents. Trading rules can be thought of as hiring a mutual fund advisor, or an investment newsletter. In the SFI market, rules were local to each agent. A central pool of rules allows for the study of clustering on certain strategies by a large number of traders, and information about successful strategies can be transmitted from trader to trader.¹⁰ A future use of this may be as a mechanism to study how liquidity in a market can be reduced when traders all concentrate on the same strategy. The amount of heterogeneity across agents is easily analyzed through the use of a common rule set.

Section 2 describes the setup of the model. Section 3 presents results, and section 4 concludes, and makes

⁹This was an important part of the early heterogeneous agent work of Figlewski (1978).

¹⁰In Chen & Yeh (1999) this information transfer takes place through an educational mechanism.

suggestions for the future.

2 Market structure

2.1 Overview

The market simulation consists of three critical pieces, agents, rules, and securities. The underlying structure of the economy is a partial equilibrium model with a stock paying a stochastic dividend which is available in finite supply. There is also a risk free asset with a fixed interest rate. This financial structure is kept relatively simple to clarify the analysis of the market outcomes.

Agents are represented by well defined infinite horizon constant relative risk aversion (CRRA) preferences with a common discount factor. They are homogeneous except they each have different time horizons about how far back they should look into the past to estimate the performance of different trading strategies. In other words, the world is populated with traders with both short and long term perspectives in terms of how they view the past and its ability to tell them about the future. The set of agents does not remain static, and it will be evolved over time. The primary experiment is to see what the behavior of the diverse populations look like, and whether long horizon agents can eventually drive out short horizon ones.

The set of rules, or trading strategies, is a common pool that is shared across all agents. An appropriate analogy might be that the rules represent investment advisors, or mutual funds. In each period each agent will update their current rule or manager by choosing the one that has done best over their horizon in the past. Each rule is required to report its past performance over a long horizon, and agents then compare the rules over the past number of periods given by their horizon length. The rules are represented as an artificial neural network which is a flexible nonlinear function mapping past information into current portfolio weights.

Rules are evolved over time. Rules that are in use by more than zero agents are kept. Rules which have not been used by anyone for the past 10 periods are replaced. New rules are generated with a type of genetic algorithm that is modified to be most appropriate for the neural network information structures. Several pieces of currently available market information are used as trading rule inputs. These fixed inputs are not changed over the course of a simulation run. However, the rules are capable of ignoring any of these inputs.

Trading takes place after all agents have decided on a portfolio strategy. With the strategies in place, each agent's demand is a nonlinear function of the current price. This is totaled to give the market's excess demand, and the excess demand is set to zero using a numerical search procedure. There is no guarantee that the price that clears the market is unique. Therefore the search is started at the previous period's

clearing price with the intention of finding the nearest local equilibrium. After the current price has been set the period comes to an end. The new dividend is revealed and paid. At this point one of the worst performing agents in terms of wealth is removed, and replaced with a new one taking the departing agent's share holdings. Finally, agents reevaluate their rules, preparing for trading which begins when all rules to use have been chosen.

2.2 Securities

The market is a partial equilibrium model with two securities, a risk free asset in infinite supply paying a constant interest rate, r_f , and a risky security paying investors a random dividend each period. It is available in a fixed supply of 1 share for the population, so if s_i is the share holdings of agent i the following constraint must be met in all periods,

$$1 = \sum_{i=1}^I s_i, \quad (1)$$

where I is the number of agents. The dividend is driven by a two state Markov process, a_t , with symmetric transition probabilities of 0.9 of staying in the current state, and 0.1 of changing. Also, an independent noise term is added to the dividend. If we map the state variable, a_t , to zero and one, the current dividend is given by,

$$d_t = 1 + 0.05a_t + u_t, \quad (2)$$

where u_t is uniform random noise distributed $[-0.005, 0.005]$. Agents receive only three forms of income: dividends, interest payments, and capital gains from purchases and sales. These go to building wealth and current consumption. No other income streams are available.

2.3 Agents and rules

Agents, indexed by i , are defined by intertemporal constant relative risk aversion preferences of logarithmic form,

$$u_{i,t} = E_t \sum_{s=0}^{\infty} \beta^s \log c_{i,t+s}. \quad (3)$$

Subject to the intertemporal budget constraint,

$$w_{i,t} = p_t s_{i,t} + b_{i,t} + c_t = (p_t + d_t) s_{i,t-1} + (1 + r_f) b_{i,t-1}. \quad (4)$$

$s_{i,t}$ and $b_{i,t}$ are the risky and risk free asset holdings respectively, and r_f is the risk free rate of return. r_t will represent the risky asset return at time t . These restricted preferences are used for tractability. It is well known that for logarithmic utility the agent's optimal consumption choice can be separated from the portfolio composition and is a constant proportion of wealth,¹¹

$$c_{i,t} = (1 - \beta)w_{i,t}. \quad (5)$$

The time rate of discount, β , is set to $\frac{1}{1+r}$ for $r = 0.01$. This r which is the interest rate corresponding to the discount rate is different from r_t , which represents the return on the risky asset, and r_f , which is the risk free rate.

A second property of logarithmic preferences is that the portfolio decision is myopic in that agents maximize the logarithm of next period's portfolio return. Agents will concentrate their learning efforts on this optimal portfolio decision. They are interested in finding a rule that will maximize the expected logarithm of the portfolio return from a dynamic strategy. The strategy recommends a fraction of savings, $\beta w_{i,t}$, to invest in the risky asset as a function of current information, z_t . The objective is to

$$\max_{\alpha_j} E_t \log[1 + \alpha_j r_{t+1} + (1 - \alpha_j) r_f], \quad (6)$$

for the set of all available rules, α_j . In general it would be impossible for agents to run this optimization each period, since the above expectation depends on the state of all other agents in the market, along with the dividend state. The portfolio decision will therefore be replaced with a simple rule which will be continually tested against other candidate rules. This continual testing forms a key part of the learning going on in the market.

The trading rules, α_j , should be thought of as being separate from the actual agents. The best analogy is to that of an investment advisor or mutual fund. A population of rules is maintained in the same way that real investors might chose an advisor or actively managed mutual fund. One difference here from the world of investment advisors is that the rule is a simple function, $\alpha(z_t; \omega_j)$, where z_t is time t information, and ω_j are parameters specific to rule j . The functional form used for each rule is a feedforward neural network

¹¹This result is well known in dynamic financial models. See Merton (1969) and Samuelson (1969) for early derivations. Altug & Labadie (1994) and Giovannini & Weil (1989) provide updated derivations in more general settings. Future versions of this model will generalize these preferences, but this brings in the added dimension of trying to determine optimal consumption given current information. It is possible that methods similar to those used in Brandt (1999) with conditional Euler equations may be useful in solving this problem. Also related are the analytic policy rules for time varying returns in Barberis (2000) and Campbell & Viceira (1996). Finally, Lettau & Uhlig (1999) present some results showing some of the difficulties in building agents with the capabilities of learning dynamic consumption plans.

with a single hidden unit with restricted inputs. It is given by

$$h_k = g_1(\omega_{1,k}z_{t,k} + \omega_{0,k}) \quad (7)$$

$$\alpha(z_t) = g_2(\omega_2 + \sum_{k=1}^m \omega_{3,k}h_k) \quad (8)$$

$$g_1(u) = \tanh(u) \quad (9)$$

$$g_2(u) = \frac{e^u}{1 + e^u}. \quad (10)$$

This framework restricts the portfolio to positive weights between zero and one, so there is no short selling or borrowing allowed. The model sets such severe borrowing constraints to avoid nonstationary bubble trajectories, and the problem of unwinding debt positions when agents go bankrupt. It is also important to note that this is a very restrictive neural network structure. Each hidden unit, h_k is connected to only one information variable. This differs from standard neural networks which let all inputs influence each hidden unit. This step was taken to enhance tractability of the learned rules, but it does limit the generality of how agents can combine input information. The weights are stored in a population table along with information on performance of this rule in the recent past. A simple real valued vector, ω_j , completely describes each dynamic trading strategy. All that is needed to be stored is a time series of the portfolio returns from each rule since the agent's objective only requires this as an input.

Agents chose the rule to use in the current period based on its performance in the past. They look over their own past horizon length, T_i , to evaluate the performance of the rule in the future using,

$$\max_j \hat{E}(r_p) = \frac{1}{T_i} \sum_{k=1}^{T_i} \log[1 + \alpha(z_{t-k}; \omega_j)r_{t-k+1} + (1 - \alpha(z_{t-k}; \omega_j))r_f]. \quad (11)$$

This limited memory feature to agent learning is similar to constant gain types of dynamic learning algorithms.¹² The only feature driving heterogeneity across agents' decisions is their time horizon, T_i . This can lead to relatively similar decision rules, and very unstable markets. A further element of random heterogeneity is implemented by adding noise to the rule decision process for each agent. Agents look over the set of rules and randomly chose one from the top 25 percent of all rules measured over the recent T_i time

¹²Examples of these are in (Cho & Sargent 1999) and (Sargent 1999). In both of these cases the constant gain algorithms in which agents discount the past generate escape routes from rational expectations equilibria. The setup here is more complicated in that it is analyzing agents with differing memories, or gains, in order to see if the agents using more history will eventually dominate the market. Another example of a modified past learning algorithm is in (Marcet & Nicolini 1997) where agents shift between least squares learning, and constant gain learning depending on recent forecast errors.

periods. This candidate rule is then compared to the current trading strategy, and it replaces this rule only if it is better. This selection method introduces some amount of unobserved heterogeneity into the investment process. For a few of the runs a second restriction will be made in terms of rule choices. Agents will change rules only if the new one exceeds the current one by a fixed percentage. This would be similar to a transaction cost in shifting advisors or changing mutual funds. It also takes the place of a certain statistical threshold for making an important investment decision. Both rules and agents are allowed to evolve over time and these mechanisms will be described in future sections.

2.4 Information

Agents trading strategies are based on simple information structures which are input to the neural network, and used to generate the trading strategies, $\alpha(z_t; \omega_j)$. Obviously, the choice of the information set, z_t , is important. This set will be chosen to encompass reasonable predictors that are commonly used in real markets. The information set will include, past dividends, returns, the price dividend ratio, and trend following technical trading indicators. In the current version, the only types of technical rules used are exponential moving averages. The moving average is formed as

$$m_{k,t} = \rho m_{k,t-1} + (1 - \rho)p_t. \quad (12)$$

Formally, the information set, z_t , will consist of 5 items.

1. d_t
2. $\left(\frac{p_t + d_t - p_{t-1}}{p_{t-1}}\right)$
3. $\log\left(\frac{rp_t}{d_t}\right)$
4. $\log\left(\frac{p_t}{m_{1,t}}\right)$
5. $\log\left(\frac{p_t}{m_{2,t}}\right)$

Several of the items are logged to make the relative units sensible. The dividend price ratio is normalized around a benchmark determined in the equilibrium presented in a future section. The two moving average indicators, $m_{1,t}$ and $m_{2,t}$, correspond to values of $\rho = 0.8$ and $\rho = 0.99$ respectively.

It is important to think about the timing of information as this will be important to the trading mechanisms covered in the next section. As trading begins at time t , all $t - 1$ and earlier information is known.

Also, the dividend at time t has been revealed and paid. This means that α_j can be written as a function of p_t and information that is known at time t ,¹³

$$\alpha_j = \alpha_j(p_t; I_t). \quad (13)$$

All variables in I_t are known before trading begins in period t . p_t will then be determined endogenously to clear the market.

2.5 Trading

Trading is performed by finding the aggregate demand for shares, and setting it equal to the fixed aggregate supply of 1 share. Given the strategy space each agent's demand for shares, $s_{i,t}$, at time t can be written as,

$$s_{i,t}(p_t) = \frac{\alpha_i(p_t; I_t)\beta w_{i,t}}{p_t} \quad (14)$$

$$w_{i,t} = (p_t + d_t)s_{i,t-1} + (1 + r_f)b_{i,t-1}, \quad (15)$$

where $w_{t,i}$ is the total wealth of agent i , and $b_{i,t-1}$ are the bond holdings from the previous period. Summing these demands gives an aggregate demand function,

$$D(p_t) = \sum_{i=1}^I s_{i,t}(p_t). \quad (16)$$

Setting $D(p_t) = 1$ will find the equilibrium price, p_t . Unfortunately, there is no analytic way to do this given the complex nonlinear demand functions. This operation will be performed numerically. Also, it is not clear that the equilibrium price at time t is unique. Given the large number of nonlinear demand functions involved it probably is not. A nonlinear search procedure will start at p_{t-1} as its initial value, and stop at the first price that sets excess demand to zero.¹⁴

It is important to remember the equilibrium is found by taking the current set of trading strategies as given. Once p_t is revealed then it is possible that agent i might want to change to a different trading rule. It is in this sense that the equilibrium is only temporary.¹⁵

¹³The impact of p_t on the current information vector, z_t , is taken into account as well.

¹⁴The search uses the matlab built in function `fzero`.

¹⁵This is similar to the types of learning equilibrium surveyed in Grandmont (1998).

2.6 Evolution

Around this structure of rules, agents, and markets is an evolutionary dynamic that controls adaptation and learning in the entire system. Evolution of agents is performed in a very simple fashion based on accumulated wealth. Every period one of the 5 least wealthiest agents is chosen at random and removed. One new agent with a horizon length T_i drawn randomly replaces this agent. For most experiments the distribution for this draw will be uniform $[5, 100]$. It is initialized with the share holdings of the deleted agent, and receives bond holdings equal to the average over the population.

Rule evolution is more complicated. Rules are evolved using a genetic algorithm.¹⁶ This method tries to evolve the population using biologically inspired operators that take useful rules, and either modify them a little (mutation), or combine them with parts of other rules (crossover).

One of the crucial aspects of evolutionary learning is the fitness criterion which is used to select good parents from the current generation. It is not clear what makes a rule “fit” in a multiagent market. For example, it would be tempting to evolve the rules based on the historical performance on a fixed history of past data, but this would not capture the fact that agents are looking at different history lengths. To try to account for agent diversity a very weak selection criterion is used. A rule can be a parent for the next generation if at least one agent has used it over the last 10 periods. Rules that haven’t been used for 10 periods are marked for replacement. This is equivalent to eliminating all mutual fund managers with no customers.

Evolution proceeds as follows. First, the set of rules to be eliminated is identified. Then for each rule to be replaced the algorithm chooses between three methods with equal probability :

1. **Mutation:** Choose one rule from the parent set, and add a uniform random variable to one of the network weights, ω . The random increment is distributed uniformly $[-0.25, 0.25]$.
2. **New weight:** Choose one rule from the parent set, chose one weight at random, and replace it with a new value chosen uniformly from $[-1, 1]$. This is the same distribution used at startup.
3. **Crossover:** Take two parents at random from the set of good rules. Take all weights from one parent, and replace one set of weights corresponding to one input with the weights from the other parent. This amounts to replacing the two weights that affect the input directly (linear and bias), plus the weight

¹⁶The genetic algorithm, (Holland 1975), is a widely used technique in computational learning. Goldberg (1989) provides a good overview, and Mitchell (1996) gives a recent perspective.

on the corresponding hidden unit. Visually this is equivalent to chopping off a branch of the network for parent one, and replacing it with a branch from parent two.

A new rule is initialized by evaluating its performance over the past history of prices and information. Agents will use this performance history to decide on whether this rule should be used as they do with the others.

It would be difficult to argue that there is any particular magic to this procedure for evolving rules, and it goes without saying that these mechanisms are ad hoc. However, the objective is to produce new and interesting strategies that must then survive the competition with the other rules in terms of forecasting.

2.7 Equilibrium

It is useful in multi-agent financial simulations to have a benchmark with which the results can be compared. For multi-agent simulations the homogeneous agent world is often the appropriate benchmark. In this paper the parameters and processes are set to allow for a simple and tractable homogeneous rational expectations equilibrium to exist.

In the general equilibrium form of the representative agent asset pricing world prices are determined so that the representative agent is satisfied holding the entire share supply, and having no net borrowing or lending demand. Although this model is different, an attempt to follow this will be made here. Parameters will be chosen such that each agent is content with investing their entire wealth in the risky asset. Agents are allowed the option of lending, but in the homogeneous equilibrium they will not want to do this.¹⁷

The first stage in examining the equilibrium is to determine the price in a single agent world, where the portfolio is invested entirely in the risky asset.¹⁸ The demand for the risky asset would be

$$s_t = \frac{w_t - c_t}{p_t} \quad (17)$$

$$s_t = \frac{p_t s_{t-1} + (1 + r_f) b_{t-1} + s_{t-1} d_t - c_t}{p_t} \quad (18)$$

$$s_t = \frac{p_t + d_t - c_t}{p_t}. \quad (19)$$

It is obvious that in order to keep the share holdings equal to one, $s_t = 1$, consumption must be equal to

¹⁷The model allows for agents to put funds in a risk free asset which is available in infinite supply. A benchmark equilibrium where this value was nonzero was problematic since it leaves open determination of the price of the risk free asset which should be part of any calculation. A less restrictive equilibrium benchmark awaits the incorporation of endogenous borrowing and lending into the model. Only when this is complete can a true comparison be made with the traditional representative agent asset pricing literature.

¹⁸The homogeneous equilibrium allows this calculation to be made for a representative agent holding all assets where $s_t = 1$, $b_t = 0$.

current dividend payments, $c_t = d_t$. Therefore,

$$c_t = (1 - \beta)w_t = (1 - \beta)(p_t + d_t) \quad (20)$$

gives the equilibrium price,

$$p_t = \frac{\beta d_t}{1 - \beta}. \quad (21)$$

Since $\beta = 1/(1 + r)$ this gives $p_t = d_t/r$.

The second stage shows that putting all wealth into the risky asset is optimal in all possible states. This first requires specification of the dividend process. Since the dividend process is given by,

$$d_t = 1 + 0.05a_t + u_t, \quad (22)$$

with a_t a 0, 1 symmetric two state markov process with transition probability 0.1, and u_t a uniform random noise distributed $[-0.005, 0.005]$. This gives an expected dividend payment of 1.05 in one state, and 1.00 in the other. For simplicity, the lending interest rate, r_f , is assumed to be zero.¹⁹

Now the final step of the equilibrium determination is to show that in the equilibrium it is optimal to hold only the risky asset. This might appear obvious since the risk free rate is zero, and the stock pays a strictly positive dividend, but given the pricing mechanism in equilibrium there will be times in which the market will predict a capital loss, and it might be advantageous to move some investment elsewhere. Given the agents' preferences, and $r_f = 0$, they will be attempting to maximize

$$E_t \log(1 + \alpha r_{t+1}), \quad (23)$$

over portfolio choices α . In equilibrium, the worst state in terms of low conditional returns, occurs when the dividend is at its maximum value. This value can be found directly from the dividend process in equation 2. From this a distribution of the future dividend and equilibrium price can be determined, and the optimal portfolio can be determined numerically from 23. For the parameters used here the optimal portfolio was numerically determined to be $\alpha = 1$ for all states of the world. This calculation completes the equilibrium.

¹⁹Given a zero interest rate, and discounting, intertemporal optimizing agents would want to borrow. However, borrowing and short selling are not allowed.

2.8 Timing

The timing of the market is crucial since it is not an equilibrium model where everything happens simultaneously. A specific ordering must be prescribed to events. The following list shows how things proceed.

1. Dividends, d_t , are revealed and paid.
2. The new equilibrium price, p_t , is determined, and trades are made.
3. Rules are evolved.
4. Agents update their rule selection using the latest information.
5. Agents are evolved.

While this appears to be a sensible ordering, it is not clear if other sequences might give different results. The fact that there needs to be an ordering is limited by usual computing tools. The best situation would be for things to be happening asynchronously, and software tools are becoming available to tackle this problem. However, this would open some very difficult problems in terms of trading and price determination.

2.9 Initialization and parameters

While this market is intended to be relatively streamlined, it still involves a fair number of parameters which may not have as much economic content as one would like. The first of these are the initialization parameters which control the agent and rule structure at startup. Rules are started with parameters, ω , drawn from a uniform $[-1, 1]$ distribution for each neural network weight. Agents begin with a horizon, T_i , drawn from a specified distribution which will be set differently in various experiments. Bond holding levels are set to 0.1. The shares are equally divided among all agents. Finally, initial price and dividend series are generated using the stochastic dividend process, and $p_t = d_t/r$. Trading proceeds for 100 periods, during which time this initial price is contained in the past history. At time $t = 100$ all prices are live traded prices, and the agents are reinitialized to equal share and cash holdings. This minimises the impact of this arbitrary initial price sequence.

There are several other parameters that will remain fixed in these runs, but which may be interesting to change in the future. The number of agents is set to 250, and the number of rules to 250, and the maximum history is set to 100. The cost of changing a forecast rule is set to zero, but will be increased on several specific runs. The horizon after which a rule is deleted due to lack of use is fixed at 10 for all runs.

3 Results

3.1 Run Summary

Most of the analysis will center around two different runs of the market. In the first case agent horizons will be drawn uniformly between 5 and 100. This will be referred to as the “All Horizon” run. In the second case horizons will be drawn uniformly between 75 and 100. This will be referred to as the “Long Horizon” run. The objective here is to restrict the agents to longer horizons to see if this will be able to drive the market to its equilibrium pricing relationship.

Figure 1 gives a summary plot of an all horizon run. The upper panel displays the price series over a range of 10000 periods. It is clear that the price is not settling down, and it displays several large moves. The next panel shows the mean agent time horizon weighted by agent wealth. This shows no tendency to move to a longer horizon, and it stays slightly below the expected value of 52.5. The bottom panel shows trading volume, displaying a strong clumping pattern with periods of large activity followed by quiet periods. Figure 2 shows a very different situation for the long horizon traders. They converge after a brief period to what appears in this scale as a nearly constant price, and trading volume goes to zero.

Figure 3 shows a detailed view of the final 500 periods of the 10000 period runs. The long horizon series settles to a switching process between the high and low prices as it should be in equilibrium. Given the dividend process and market equilibrium it should be a 2 state markov process with a small amount of random noise. The prices should be centered at 100 and 105 as they appear to be. The all horizon process moves around between prices, showing no obvious pattern. The picture clearly demonstrates that something very different is going on in these two cases.

3.2 Returns

Table 1 presents summary statistics for the two cases plus a simulated version of the equilibrium price process which is labeled “Benchmark”. This is a price and dividend process which follows the derived equilibrium. In all cases a snapshot series of 1000 returns is used where returns are defined inclusive of dividends,

$$r_t = \frac{p_t + d_t - p_{t-1}}{p_{t-1}}. \quad (24)$$

For the two market simulations this is taken from the final 1000 periods of a 10000 period run. The first two rows of the table show that the long horizon run has settled into a situation close to the equilibrium.

All the summary measures on the returns series are close, and the market simulation has introduced only a small amount of extra variability into the returns process. Both series show excess kurtosis which comes from the sudden shifts in regime in the dividend process.

The final row shows a very different situation for the all horizon runs. The mean return has increased from about 1 percent, to 3 percent, and the standard deviation has increased by over a factor of 20. The spikes from figure 1 appear here in the very large amount of kurtosis.²⁰ The interquartile range is reported as a more robust measure of return dispersion. The increase from the long horizon to all horizon case is by a factor of 10 indicating that the dispersion increase is not simply caused by a few outliers. The last column reports the amount of turnover in the two markets. The long horizon market is basically zero, while the all horizon market is about 4 percent per period. This market is both moving the price into a range where the expected returns are higher than they should be in equilibrium, and the volatility has been given a significant boost.

Further return structure is presented in several figures. Figure 4 shows the return autocorrelations for both the all horizon and long horizon cases along with the benchmark. The long horizon case again follows the benchmark in that there is some negative return correlation at lag 1. This is due to the fact that when the dividend shifts between states the price jumps up, causing a large return at time t which will be followed by a lower return at time $t + 1$. This slight amount of mean reversion as the states change is the cause for the negative correlation. Interestingly, the all horizon returns series shows a small amount of autocorrelation moving from positive to negative which dies off quickly.

A common feature of most financial markets is the persistence of volatility.²¹ This is tested in figure 5 which measures the autocorrelation between absolute values of returns.²² The long horizon series shows little evidence for any persistence in volatility, while the all horizon series shows strong persistence. Hints of this result can be seen in figure 1 which shows periods of large activity followed by periods of relative calm. Markets also show persistent trading volume as well as volatility. Figure 6, which presents the autocorrelation of trading volume, shows that again there is very strong persistence for the all horizon case, and very little in the long horizon case. Figure 7 examines the connection between trading volume and volatility. The all horizon case shows a strong contemporaneous connection between volume and volatility which appears in actual markets.²³ The long horizon case shows almost no relationship.

²⁰This kurtosis should make some of the other moments a little suspicious. A few tests with trimming have shown all of the results in the table are robust to the elimination of the most extreme outliers in the series.

²¹See Bollerslev, Chou, Jayaraman & Kroner (1990) for a survey.

²²Given the amount of kurtosis present this is probably safer than looking at squared returns.

²³See Karpov (1987) for a summary of stylized facts on trading volume, and Gallant, Rossi & Tauchen (1992) for more recent

3.3 Consumption

This market allows the analysis of both asset returns and consumption data. The next two tables present a brief summary and welfare analysis of the consumption processes generated in the simulations. The first two columns in table 2 examine aggregate properties of consumption. c_t is the per capita consumption level, and estimates are again taken from the final 1000 periods of the market. The first row shows the values from the benchmark simulation, and the second shows the long horizon market simulations. The mean and standard deviation of the two consumption processes are very close. For the all horizon case the mean percapita consumption level is still close to the other two cases, but the standard deviation is now more than 15 times as large. Mean consumption should not change since investors can only take out the income flows that are being supplied exogenously to the market. However, there is nothing that controls the variability of the consumption process. This shows that the increased volatility in the price series is directly reflected in consumption. This should not be surprising given that consumption is proportional to wealth.

The column labeled, CE, shows the percentage of the expected consumption level that the consumer would be willing to give up to remove all uncertainty. The mean of this value is taken over the 250 individual agents. The reported values of about 0.03 percent are very low for the long horizon case, and much higher, with a value of about 11 percent for the all horizon case. In the long horizon world there is not a lot of risk that consumer would be interested in getting rid of. However, this risk becomes significant in the all horizon case.

The final two columns give evidence on the dispersion of consumption and wealth across agents. The first shows the mean interquartile range in both cases. There is a dramatic increase from 0.007 to 0.186. This shows that not only is consumption more volatile in the aggregate, but the inequality across agents is also greater in the all horizon case. This result is backed up by estimates of Gini coefficients for final wealth at the end of a long and all horizon run respectively.²⁴ The values of 0.008 and 0.517 for the long and all horizon cases show a dramatic increase in wealth inequality across agents. The all horizon case may be populated with a few very wealthy agents moving asset prices.

Table 3 performs some experiments on the first order conditions for consumption and risky asset returns for the all horizon case. The objective is to see how far this looks from an optimum from the individual agent perspectives. The expected value of x_t should be zero at the optimum. The row labeled “long horizon” uses a horizon length set to 500 periods for estimating each agent’s x_t . The row labeled “individual horizon”

evidence.

²⁴Gini coefficients range from zero when there is complete equality, to one. See Deaton & Muellbauer (1980) for a complete description.

uses the horizon length for each individual. The table reports distributional characteristics for x_t . The first row shows that the first order condition estimates for the individuals are all greater than their theoretical value of zero, indicating possible irrational decisions. However, in row two the distribution is more centered around zero, with only 70 percent of the estimated x_t values greater than 0. This is suggestive of the fact that individual agents may be acting rationally subject to their given short time horizons for analyzing past data.

One final test that could be performed using horizon lengths is to see how they compare to consumption and wealth measures. Initial tests have been unable to show any connection between consumption and horizon length in the evolving market. This indicates that the changing market environment does not tend to favor agents of any particular horizon which is consistent with the continuing instability in overall horizon length in the population. Obviously, this question remains an interesting one for the future.

3.4 Evolutionary Stability

The differences between the long and all horizon experiments suggest some tests of evolutionary stability. Since the all horizon experiments contain types with many different horizons it is interesting that the long horizon agents do not drive out the short horizon ones. If the latter are less rational than the former they should prove to be less fit in terms of wealth evolution. The fact that these agents cannot be driven out does not suggest that the actual equilibrium is unstable, but it would be interesting to know what some of the local properties are in terms of mutations of differing agent types.

Three experiments are tested. In the first experiment agent evolution is stopped. The market runs with a fixed population of 225 agents drawn from the long horizon distribution of $[75, 100]$, and 25 agents drawn from a special short distribution of $[5, 50]$. The objective is to see who will win in this simple static battle which has been significantly tilted to favor the long horizon types. Figure 8 shows a fast convergence to the equilibrium pricing function. The wealth weighted horizon plot shows that the longer horizon agents do eventually take control of the market. The final panel reports the ratio of the average wealth level for the different types of agents. This shows an interesting initial shift toward the shorter horizon types, but it quickly favors the long horizon agents.

In the second experiment agent evolution is reactivated for the market, and the initial population is seeded with long horizon agents only. Agents are selected for removal as in the previous experiments. However, new agent's horizons are drawn from a short horizon distribution that is uniform $[5, 50]$. It is clear from the price and horizon plots in figure 9 that this market is not converging to the equilibrium. The horizon lengths are

shorter due to the shorter horizons of the invading agents. The invading short horizon agents have kept the market from the homogeneous equilibrium, and have taken over the market. This test may not be exactly fair to the long horizon agents since they weren't given a chance to learn the equilibrium strategies and comfortably settle into trading in that world. Figure 10 shows the same experiment, but this time 1000 periods are allowed to go by before the short horizon agents are allowed to invade. The market fluctuates briefly and then settles into the equilibrium. Without agent evolution the agent horizon length is flat in this region. Once the invasion starts at period 1000 the horizon length drops a small amount and becomes much noisier, but the invasion is not successful in changing the market. The price dynamics continue in equilibrium, and the horizon length stays well above the average for the invaders of 27.5.

These results show that the equilibrium once obtained is stable under some local perturbations. However, the path to the equilibrium may be hard to find. Simply having a large fraction of long horizon agents around at the start is not enough. They must be given some amount of time to learn without the interference of the short horizon types.

3.5 Switching Costs

These runs suggest a general difficulty in evolving to the REE. This section explores parameter modifications that appear to allow convergence to occur. These modifications are not small, nor economically insignificant. They demonstrate which modifications may move to stabilize the market.

In Arthur et al. (1997) it was demonstrated that if the rate of learning were slowed down then convergence to the equilibrium occurred. It left open the question of what this rate of learning was, and how it might be affected by economic parameters in the trading environment. In this market an indirect economic mechanism will be used to slow traders down. Agents examine their rules every period, but replace the current strategy only when a new one beats it by a certain threshold. This is in the spirit of either a transaction cost of some kind, or a possible statistical hurdle that one might require. The threshold is fixed at 20 percent. A rule with a return of 10 percent would only be replaced with one with a return of 12 percent or higher. Figure 11 again shows a fast convergence to the REE. The cost of changing rules has slowed down agent adaption enough so that the market can find the homogeneous REE equilibrium. Obviously, more needs to be done before this policy recommendation can be made, but this is an interesting case where a type of market friction improves welfare for the traders.

4 Summary and Conclusions

This set of computer experiments reveals three main results. First, the selection pressure for longer horizon agents appears to be weak. Their ability to push the market toward the simple equilibrium cannot overcome the high frequency fluctuations of shorter horizon agents. Second, these same shorter horizon agents act as volatility generators keeping the longer horizon agents from ever getting a foothold in the market. In many ways this story is reminiscent of the noise trader results in DeLong et al. (1991) where the presence of the noise traders affects the ability of the rational traders to move the price. In this model they increase the amount of variability in the price far beyond that driven by fundamental variability. Some of this variability appears to be driven by heterogeneity in the consumption patterns of consumers as opposed to movements in the aggregate consumption series. Finally, this market replicates many of the time series features found in actual markets such as volatility and volume persistence, and the presence of large price jumps. Although this is a long way from looking like actual financial data, it is a possible first step in uncovering a mechanism that generates realistic market dynamics.

There are a few weaknesses that will need to be addressed in future versions of the model. The return series appears too spikey relative to actual returns. In other words, the market is generating return distributions that appear farther from Gaussianity than actual financial market returns. Some tests need to be done to find out why this is so, and if this model can reduce these extremes. Beyond this, the key empirical issue that needs to be tackled is one of calibration. This market is still not aligned with any series, and even the time horizon is kept somewhat vague. At this point it should be clear from the format of preferences and discounting that this is a longer horizon model. It eventually should be directed at time scales between 1 week and 1 quarter depending on how it is set up.

It is difficult to separate calibration problems from a second design issue, the pricing and trading of the risk free security. Future versions need to endogenize risk free security pricing and trading, and allow some amount of borrowing to take place. In addition to being more realistic this could allow for comparison with benchmark cases that are interior solutions to the portfolio problem. At the moment both economic and computational barriers exist, but at some point they will be solved. It is best to use the simpler version now, and add multiple markets as this technology becomes better understood.

A first attempt is made to understand the possible impact of policies that might hinder trade, such as a transactions tax. In this setting this appears to help by slowing the rule interactions, and leading to a convergence to the rational expectations equilibrium. However, this result needs to be much better

understood and aligned with actual markets before it should be used as a policy tool. This market also has the characteristic that in the equilibrium portfolio holdings were constant, so the tax offered no welfare loss from reduction in trade when it was really needed.

One of the goals of this market has been to streamline some of the complexities of the Santa Fe Artificial Market.²⁵ Although the parameters have been greatly reduced, there are still many to explore. Among these are the types of information to be used by agents. Future versions may incorporate volume as well as price information, and explore both by extending, or reducing the number of indicators. It is possible that with fewer things to look at the dynamics of the market might actually settle down. Several critical parameters were not changed during the runs. Among these are, the length of time before a rule is deleted (10), the frequency of rule and agent evolution (every period), and finally the number of rules and agents (250).²⁶

This paper makes contributions along two different aspects of financial research. The first contribution is to point out that the selection pressure for long horizon agents may not be strong enough to successfully take over in a dynamically changing market environment with a large amount of agent heterogeneity.²⁷ This casts doubt on arguments for less than perfectly rational agents being driven out of a market as in Friedman (1953). Second, it was demonstrated that markets containing many different trader types generate time series with features that are also seen in many financial time series. They do this in a fashion in which these features were not prewired in the system. In this sense the macro time series features can be viewed as emergent properties of the agent based market. Finally, this research is part of the larger attempts at understanding agent based models in both economics and finance, and to divulge certain basic principles which may have applications in many different market settings.

²⁵The author has commented before that its key problem was in being too complex to interpret (LeBaron 2000).

²⁶An interesting recent paper, Egenter, Lux & Staufer (1999), explores the limiting behavior of multi-agent systems. It is crucial to understand which effects might die away as the sample size gets large. It is also interesting to note that it is not clear that small sample issues might not loom large in socio-economic systems.

²⁷This important result aligns with with several other findings. For example, Blume & Easley (1990) in finance, and Blume & Easley (1998), and Dutta & Radner (1996) on firm evolution. In general these papers suggest that various common selection criteria such as wealth or profitability, may not be closely aligned with usual economic measures of rationality.

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Table 1: *Return Summary Statistics*

Experiment	Mean	Std	Skewness	Kurtosis	Interquartile Range	Turnover
Benchmark	0.010	0.017	0.02	8.65	0.007	
Long Horizon	0.010	0.011	0.25	18.52	0.006	8.52e-12
All Horizon	0.030	0.216	6.62	93.24	0.054	0.038

Summary statistics for risky asset returns. Benchmark is from the process that would drive prices in the homogeneous rational expectations equilibrium. “Long” is for a set of agents with horizons that are drawn uniform $[75, 100]$, and “All” is for a set of agents with horizons that are drawn uniform $[5, 100]$. All simulated returns are taken from the last 1000 periods of a 10000 period run.

Table 2: *Consumption Comparisons*

Experiment	$mean(c_t)$	$std(c_t)$	$CE(c_{t,i})$	$Mean(q_{75} - q_{25})$	Gini
Benchmark	0.411	0.010			
Long Horizon	0.412	0.010	0.027	0.007	0.008
All Horizon	0.428	0.166	11.21	0.186	0.517

Mean and std. are per capita, multiplied by 100. Benchmark is from the process that would drive prices in the homogeneous rational expectations equilibrium. “Long” is for a set of agents with horizons that are drawn uniform $[75, 100]$, and “All” is for a set of agents with horizons that are drawn uniform $[5, 100]$. All simulated returns are taken from the last 1000 periods of a 10000 period run. CE reports the percentage decrease in consumption from the mean level to the certainty equivalent level. In other words, the percentage of expected consumption that consumers would be willing to give up to eliminate all uncertainty. The table reports the mean taken across the 250 individual agents. $Mean(q_{75} - q_{25})$ is the mean of the interquartile range of the individual consumption series measured in each period. Gini is the Gini coefficient for wealth in the last period.

Table 3: *First Order Conditions: $x_{t+1} = \beta(1 + r_{t+1})\frac{c_t}{c_{t+1}} - 1$*

Horizon	Mean	25% Quantile	Median	75% Quantile	Fraction > 0
Long Horizon	0.0065	0.0032	0.0045	0.0094	1.000
Individual Horizon	0.0007	-0.0002	0.0007	0.0021	0.697

Distribution properties for consumption, risky asset, first order conditions. “Long Horizon” estimates the consumption first order condition over the last 500 periods from a 10000 period run. “Individual Horizon” uses the actual horizon length for each agent to estimate the first order condition. Fraction greater than 0, reports the fraction of agents with values greater than 0.

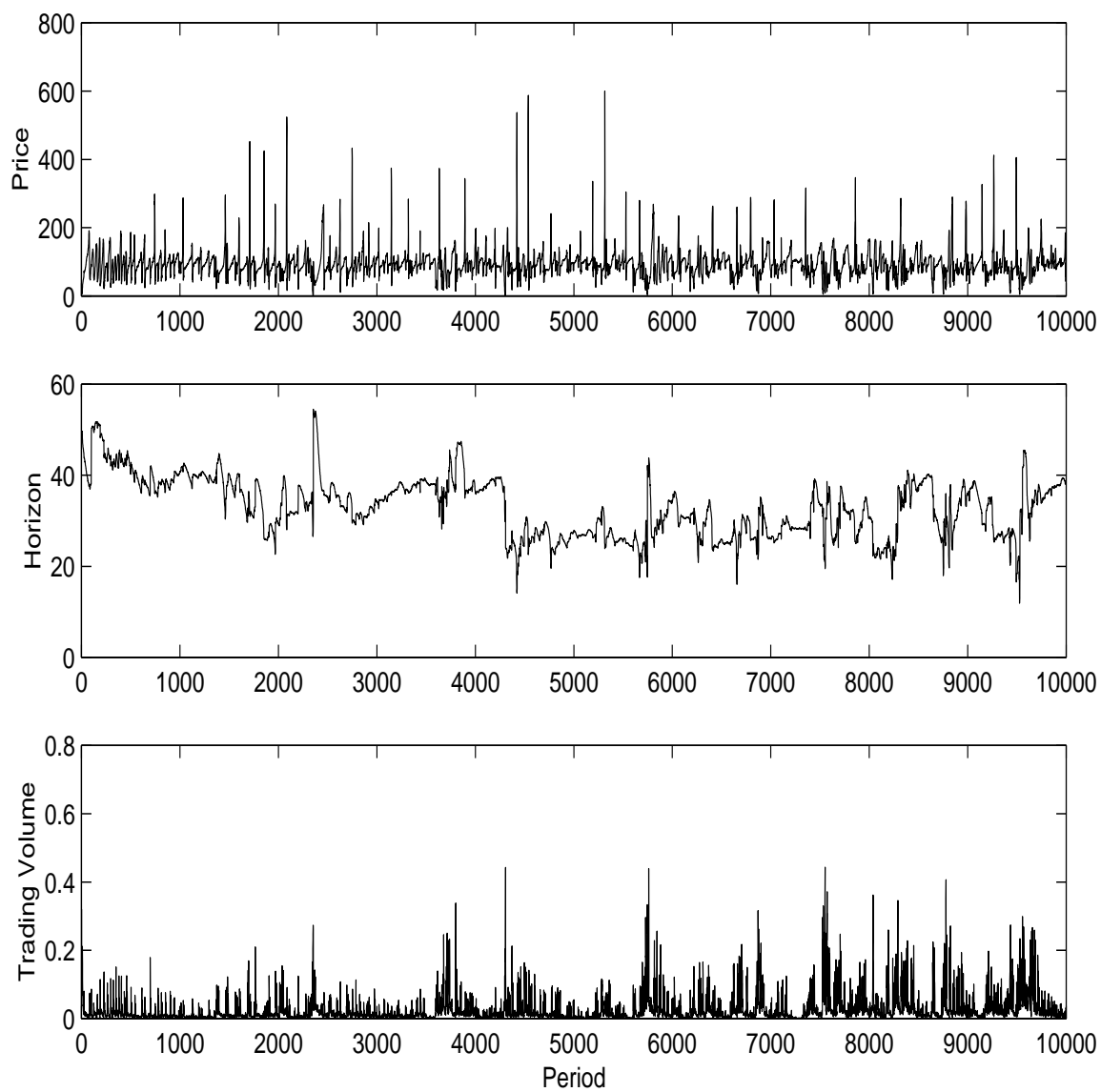


Figure 1: *All Horizon: Uniform* [5, 100]

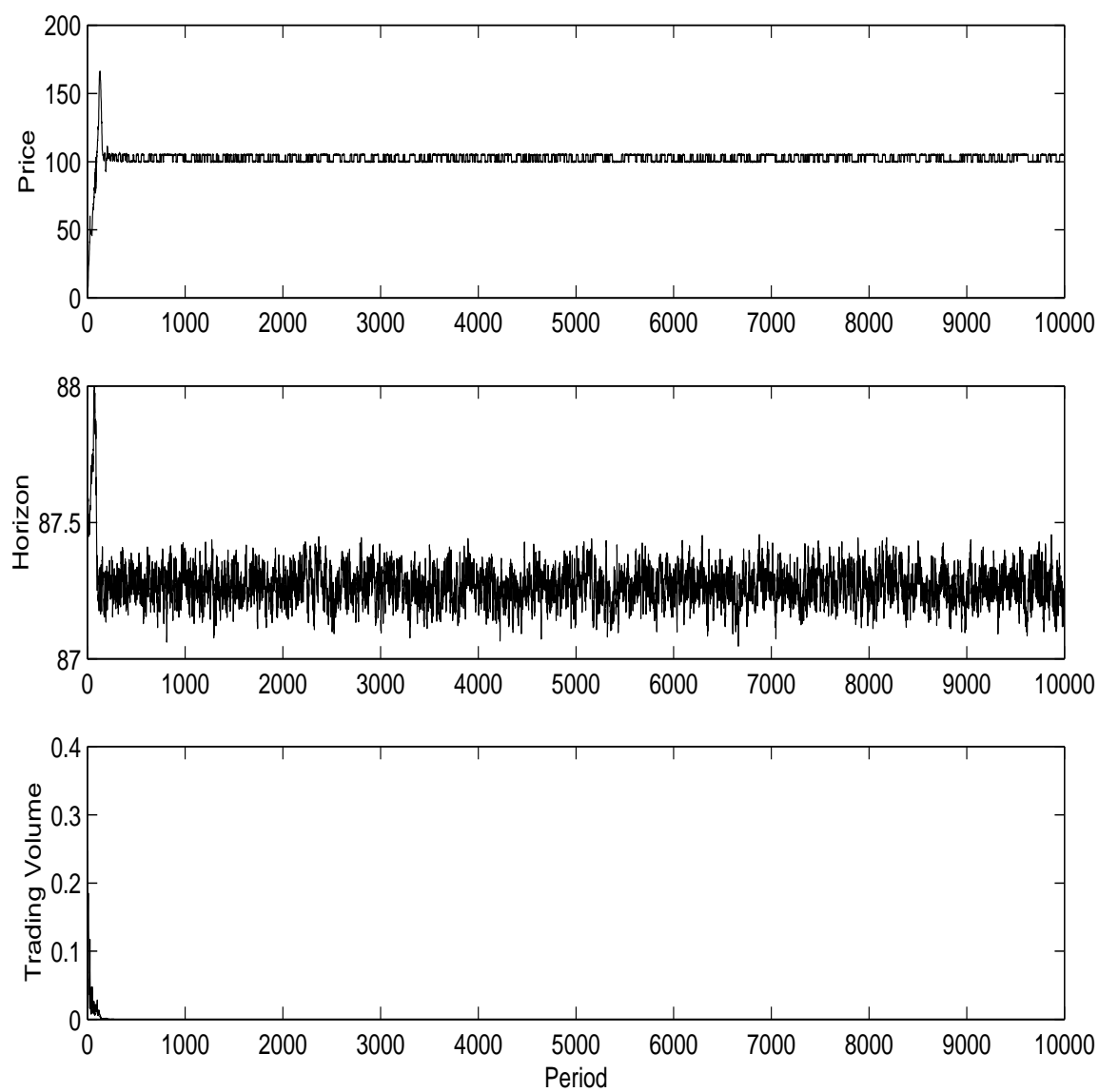


Figure 2: *Long Horizon: Uniform* [75, 100]

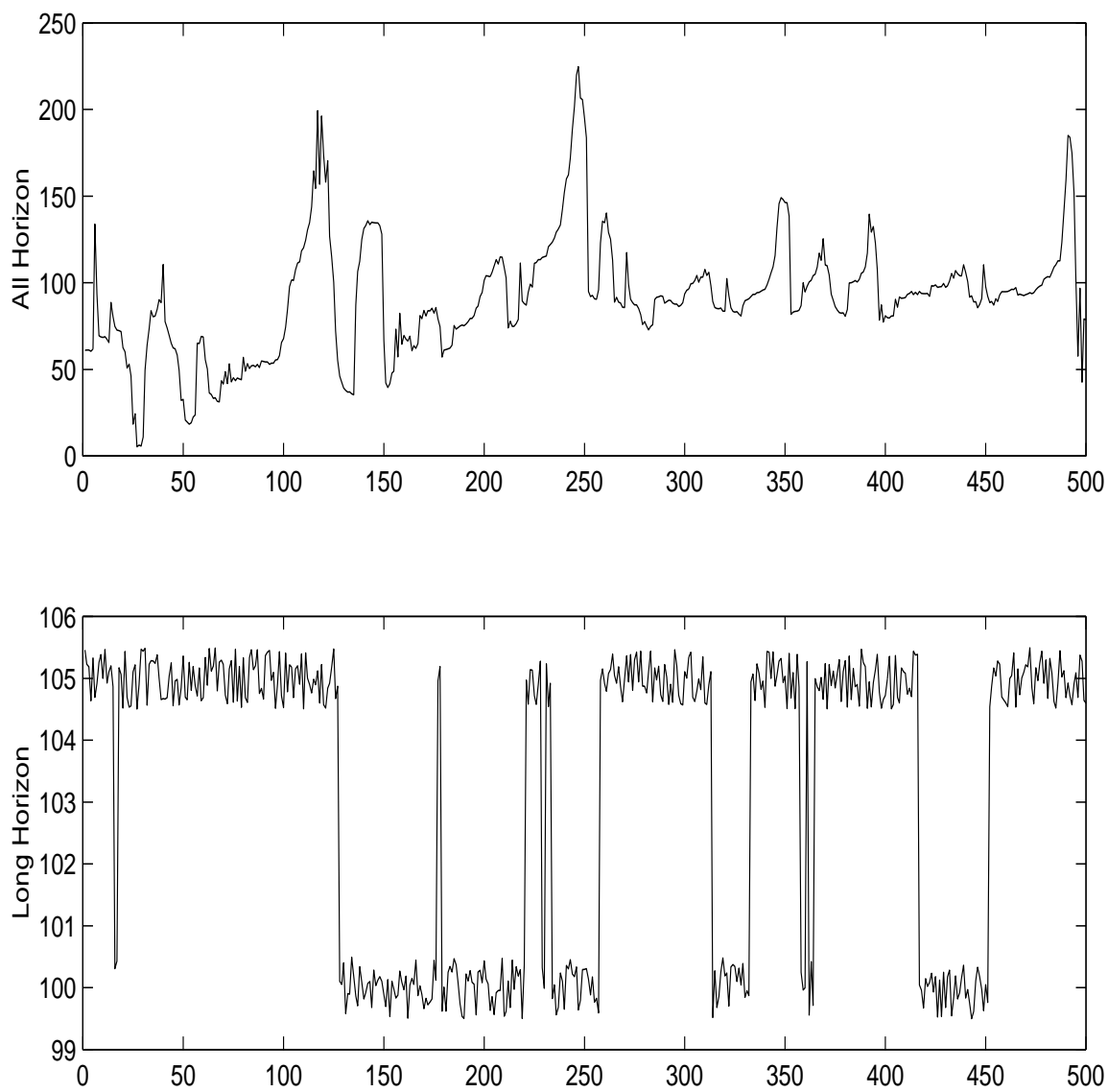


Figure 3: *Price Comparison: All and Long Horizons*

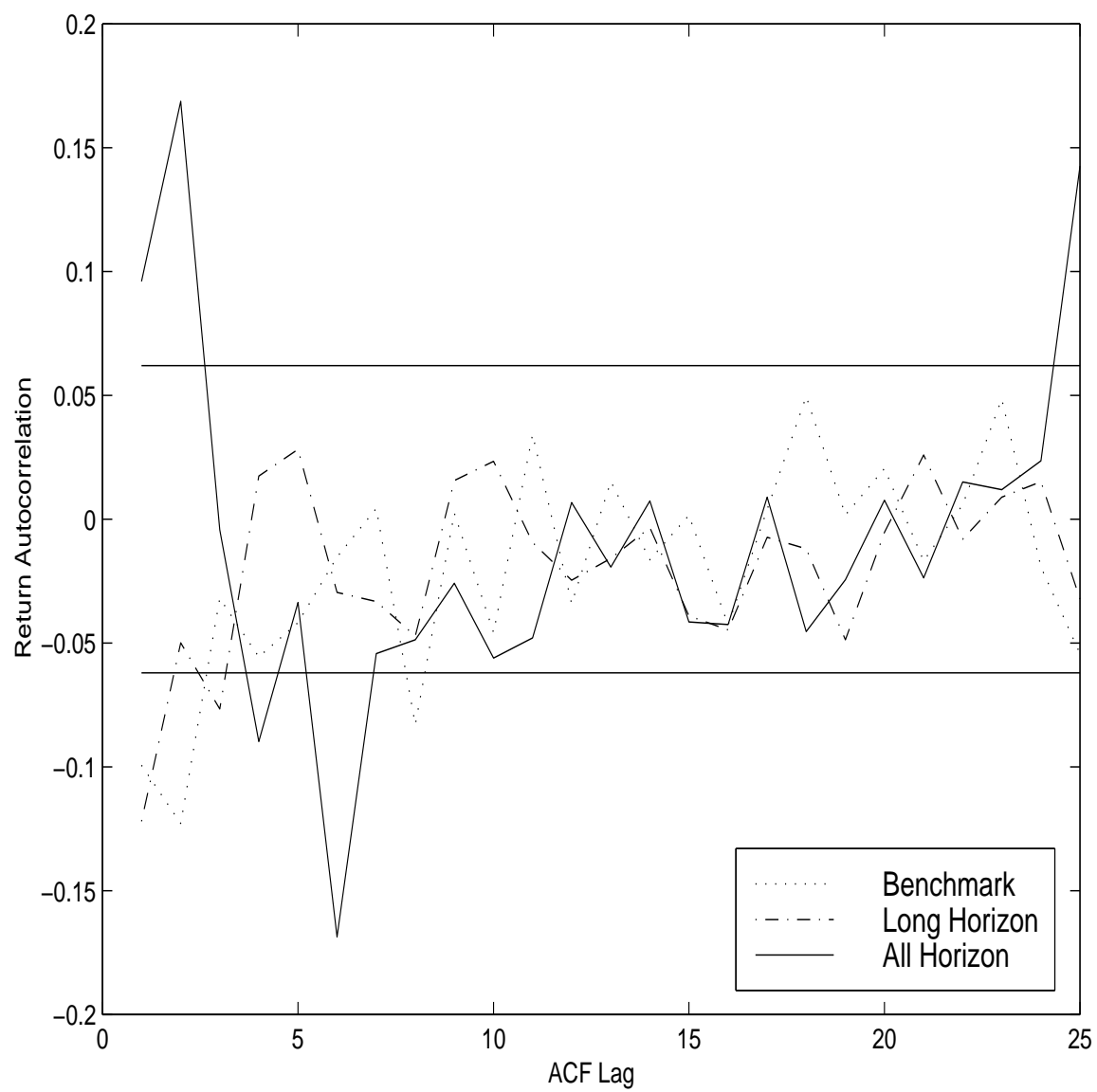


Figure 4: *Return Autocorrelations*

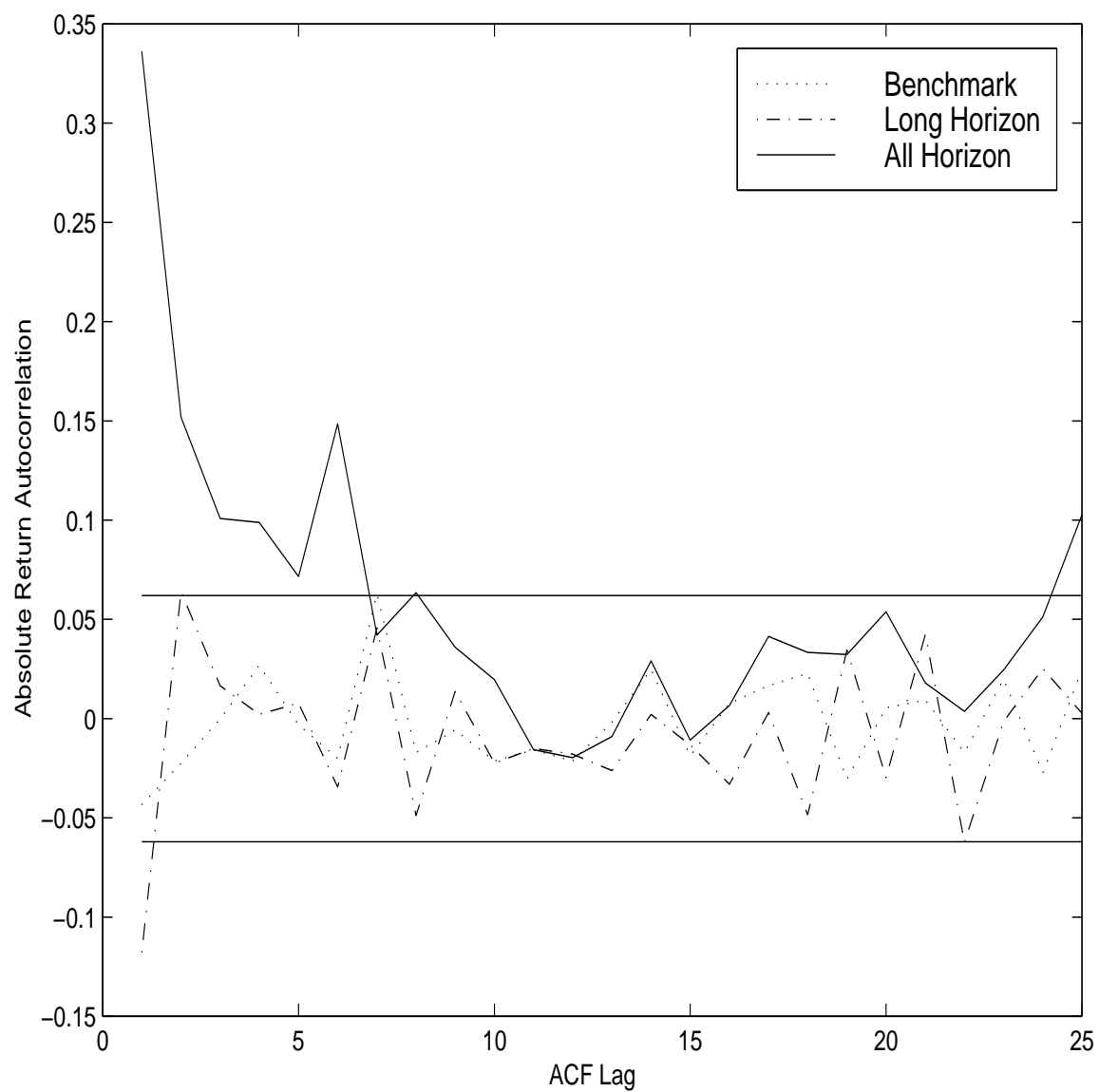


Figure 5: *Absolute Return Autocorrelations*

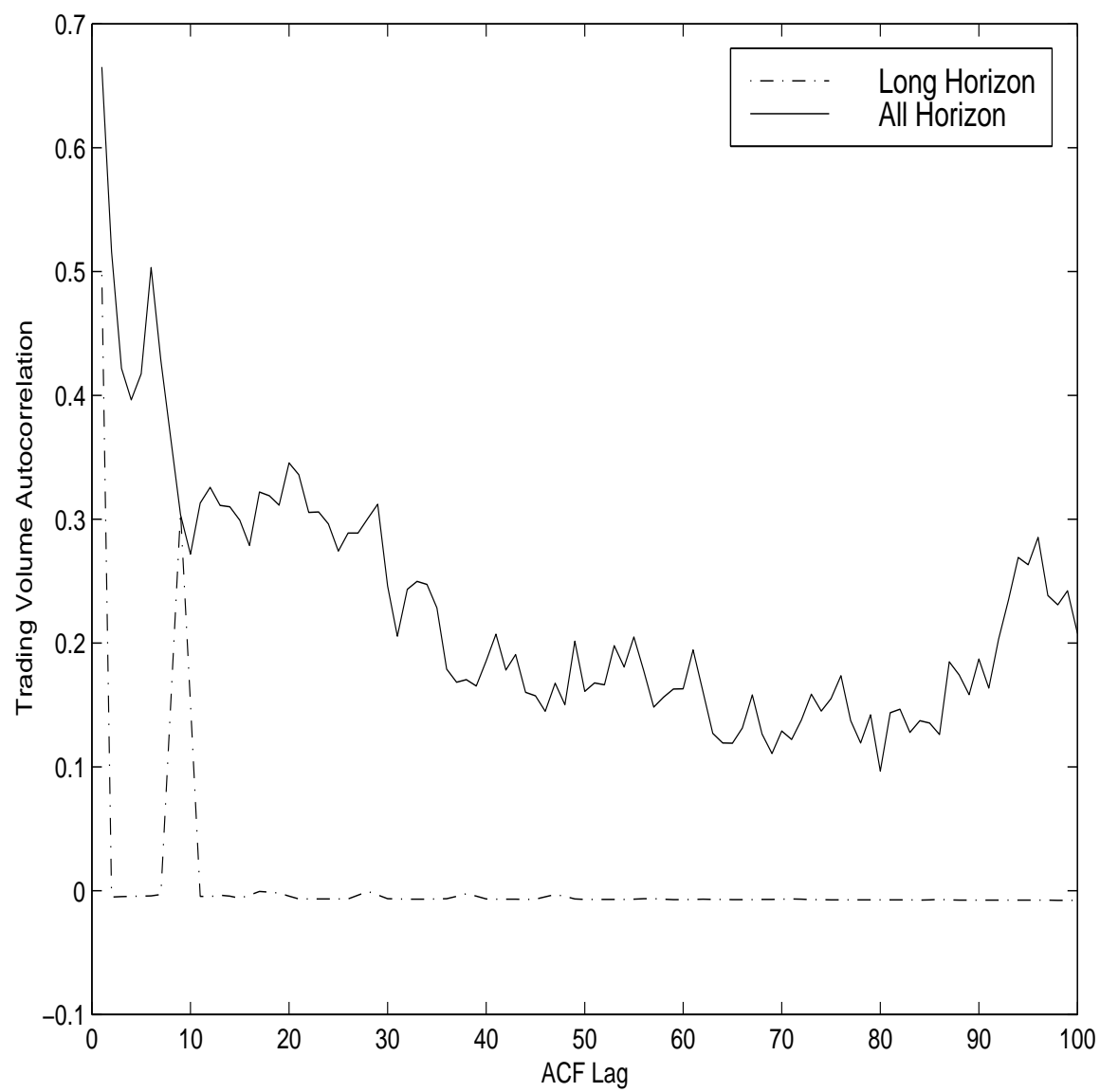


Figure 6: *Trading volume Autocorrelations*

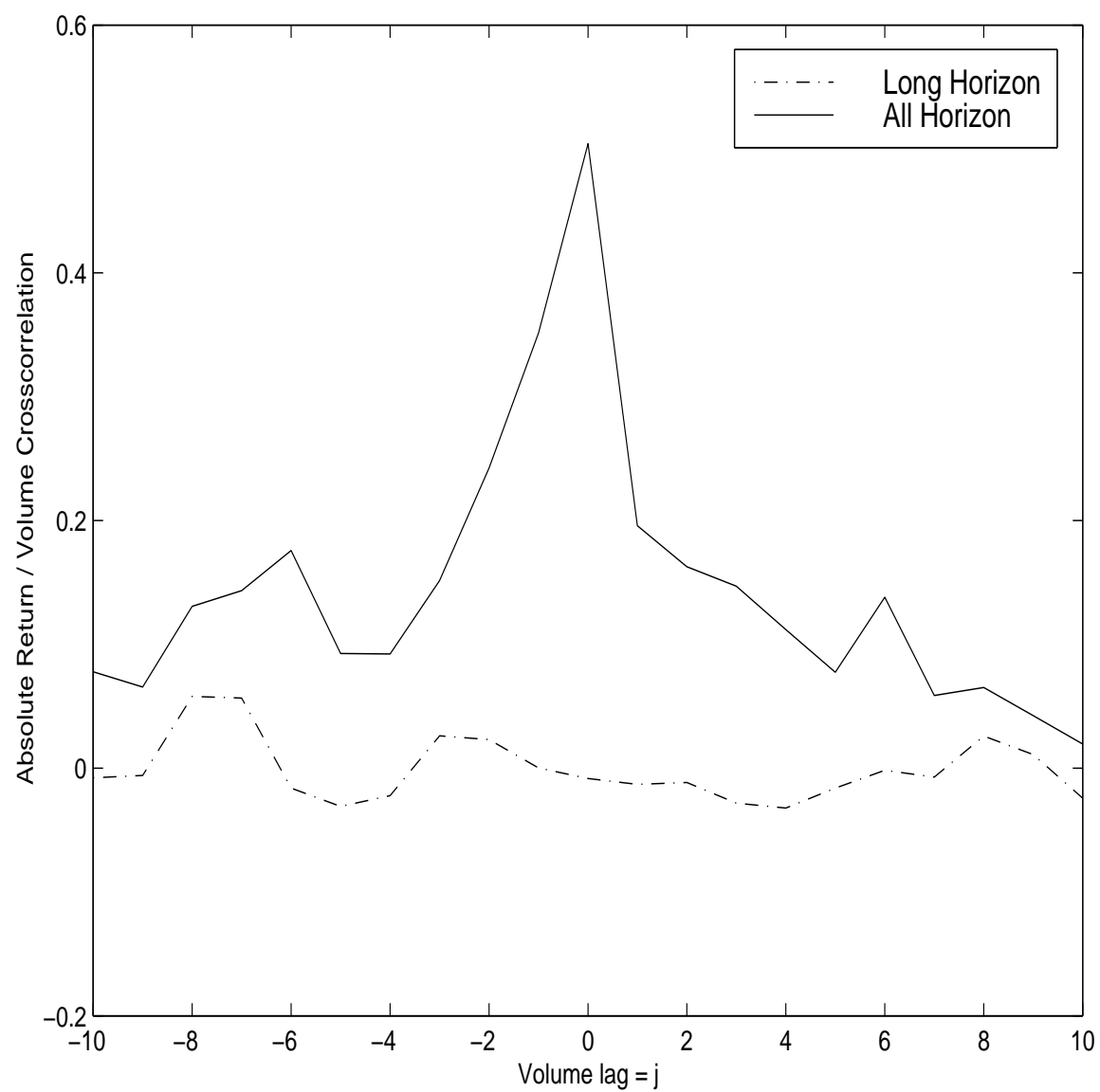


Figure 7: *Absolute Return/Volume Crosscorrelation*

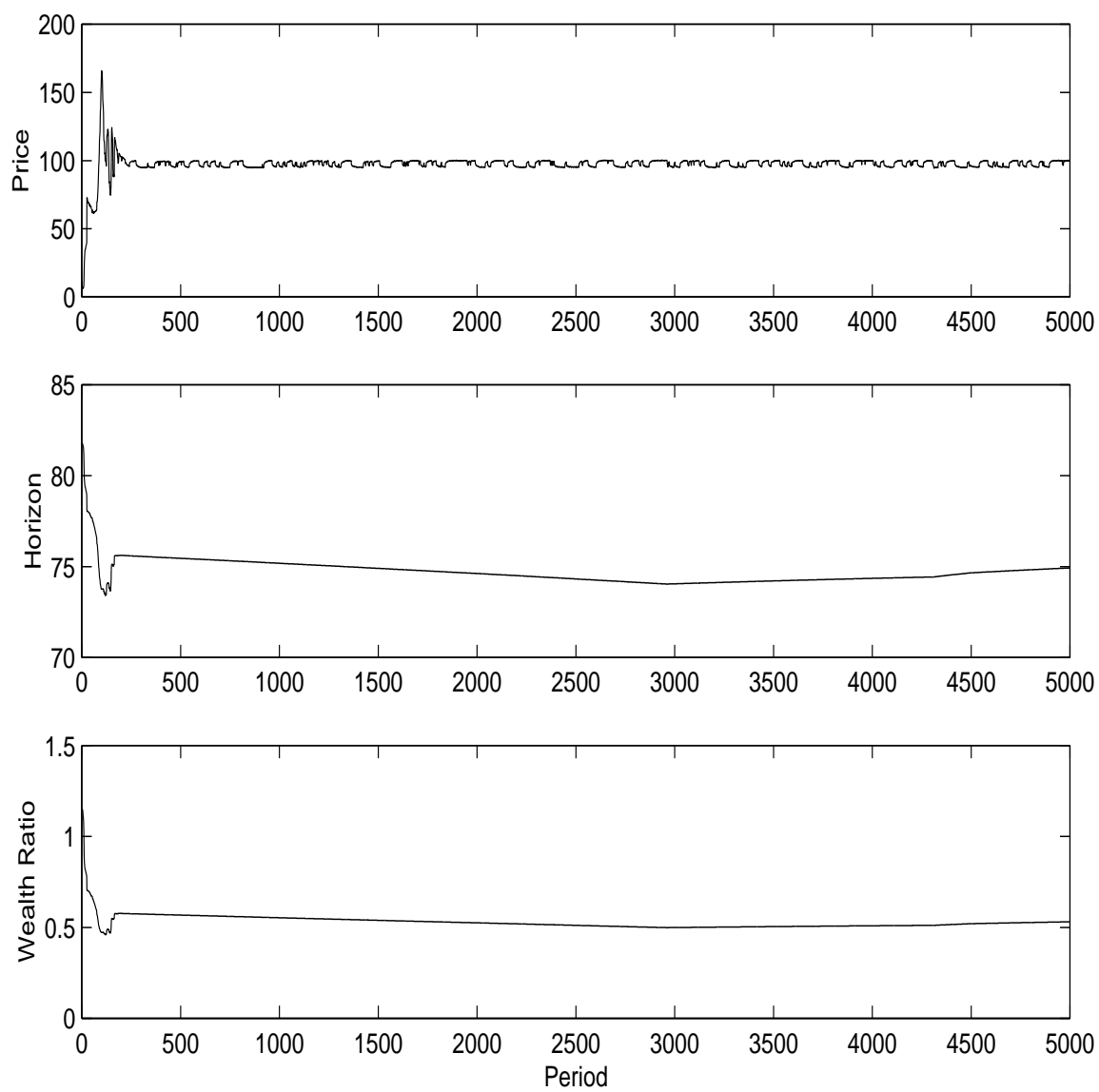


Figure 8: *Static Horizons: 225 Long, 25 Short*

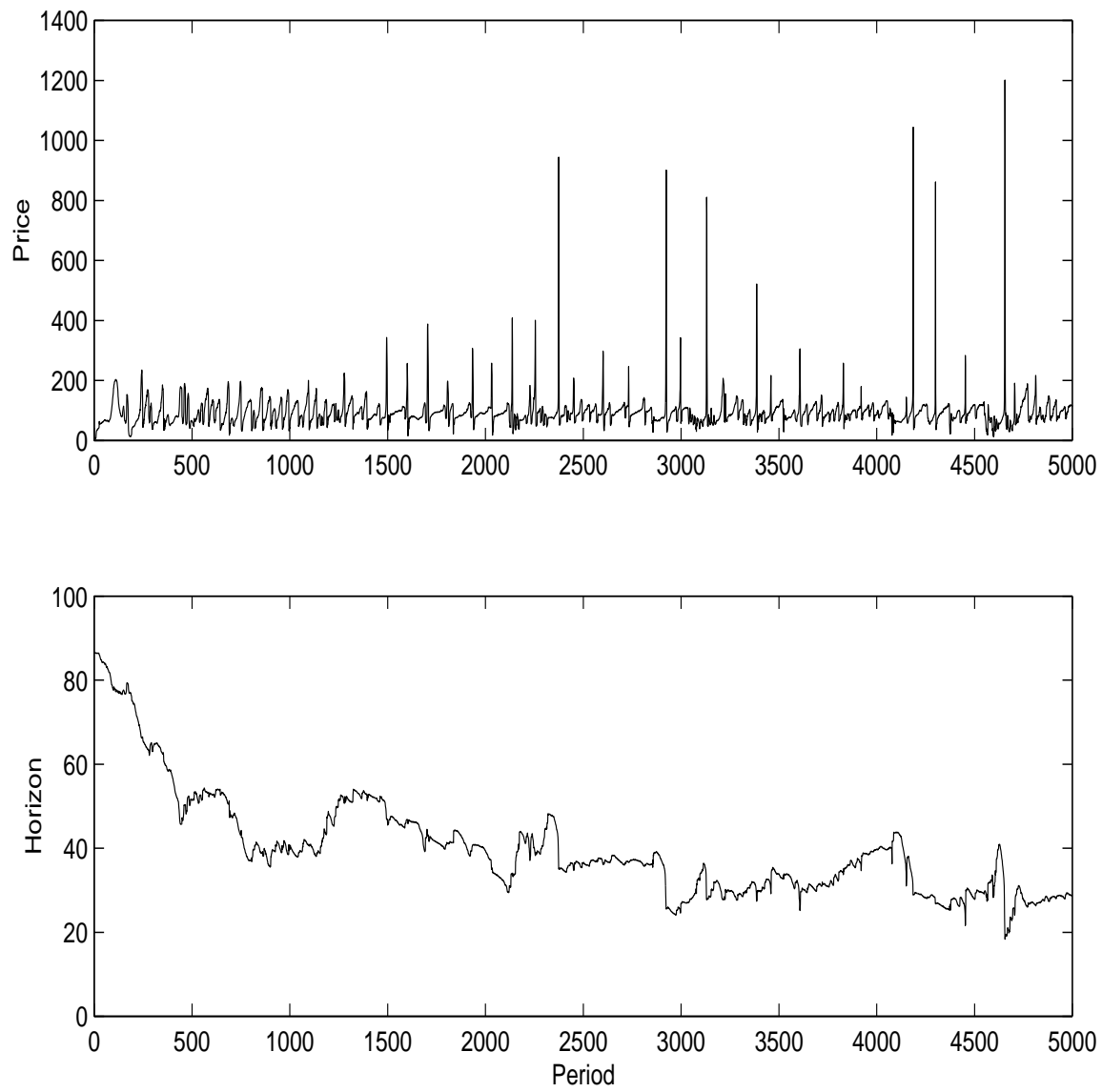


Figure 9: *Short horizon invasion*

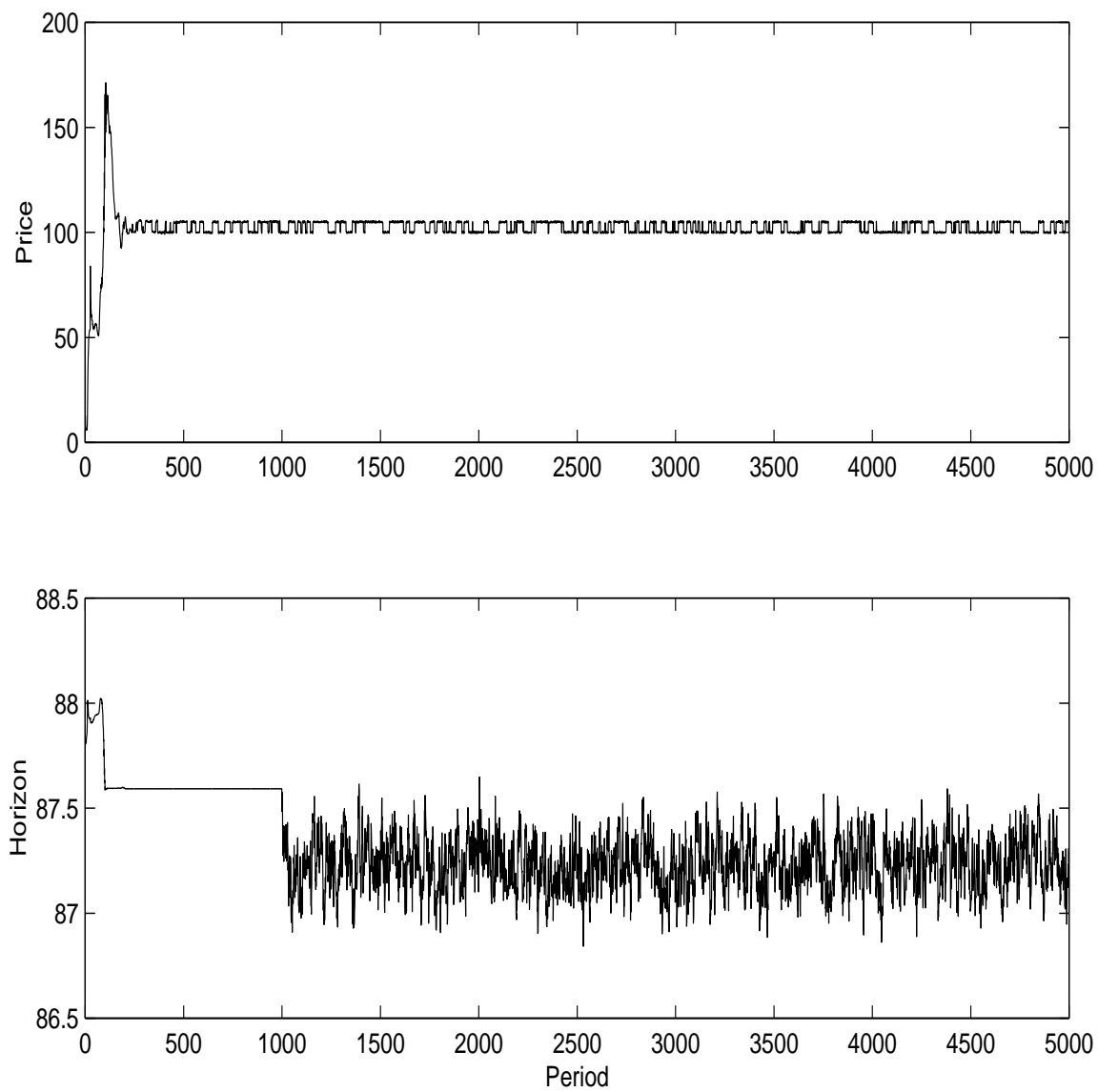


Figure 10: *Short horizon invasion: Wait 1000 periods*

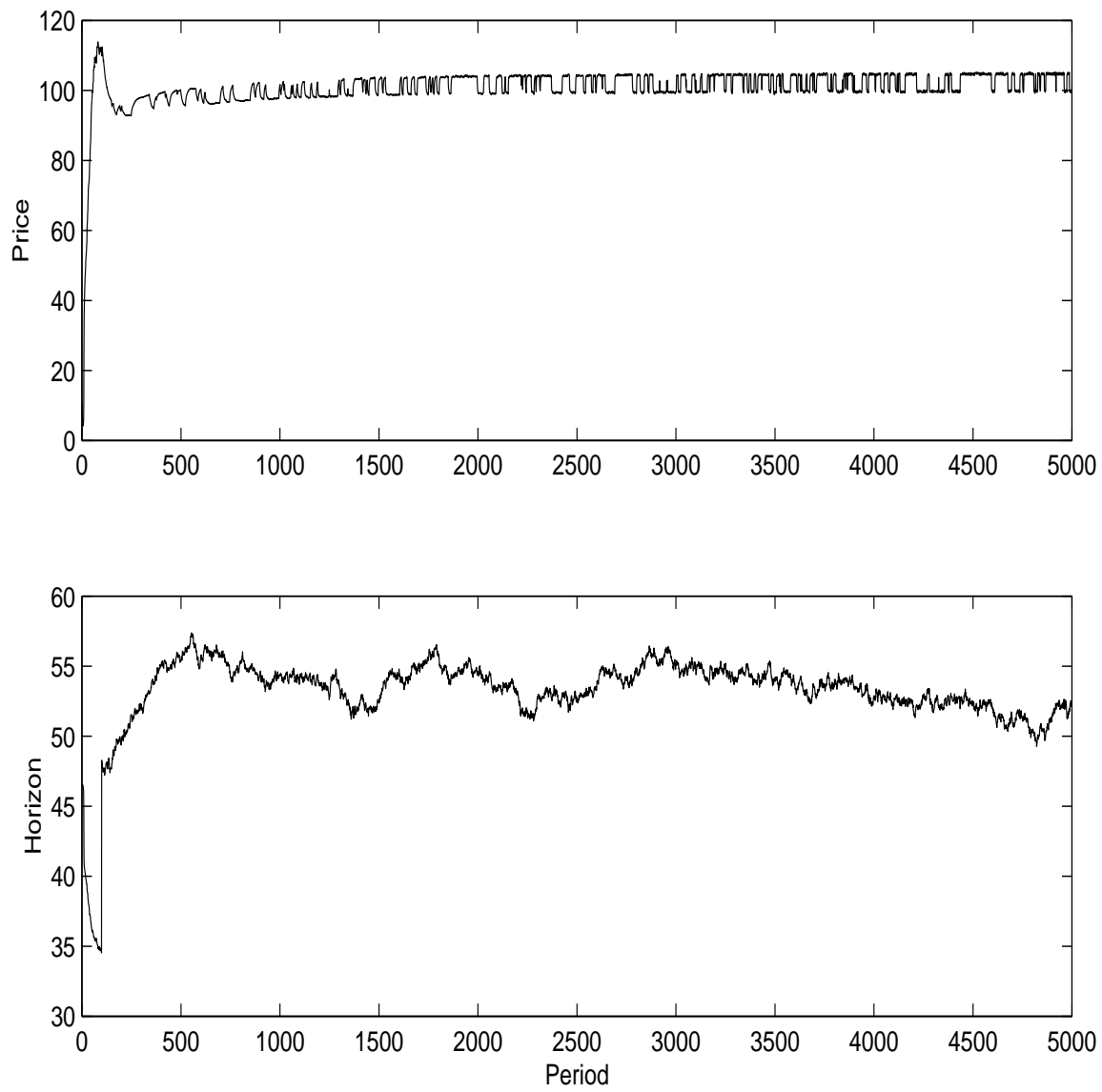


Figure 11: *Rule change cost = 20 percent*