DATA 100 HW 01

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1 Calculus

$$\sigma(x) = \frac{1}{1 + e^{-x}} \tag{1}$$

1.a

Theorem 1. *Given the function $\sigma(x)$, $\sigma(-x) = 1 - \sigma(x)$.*

Proof.

$$\begin{aligned} \sigma(-x) &= \frac{1}{1 + e^{x}} && (2) \\ &= \frac{1 + e^{x} - e^{x}}{1 + e^{x}} && (3) \\ &= 1 - \frac{e^{x}}{1 + e^{x}} \frac{e^{-x}}{e^{-x}} && (4) \\ &= 1 - \frac{1}{1 + e^{-x}} = 1 - \sigma(x) && (5) \end{aligned}$$

□

1.b

Theorem 2. *Given the function $\sigma(x)$, $\frac{d}{dx}\sigma(x) = \sigma(x)(1 - \sigma(x))$.*

Proof.

$$\begin{aligned} \frac{d}{dx}\sigma(x) &= \frac{d}{dx}(1 + e^{-x})^{-1} = e^{-x}(1 + e^{-x})^{-2} && (6) \\ &= \frac{1}{1 + e^{-x}} \frac{e^{-x}}{1 + e^{-x}} \frac{e^{x}}{e^{x}} && (7) \\ &= \sigma(x) \frac{1}{1 + e^{x}} = \sigma(x)(1 - \sigma(x)) && (8) \end{aligned}$$

□

2 Minimization

Finding the zeroes of $f(c)$,

$$f(c) = \frac{1}{n}\sum_{i=1}^{n}(x_i - c)^2 \tag{9}$$

$$f'(c) = -\frac{2}{n}\sum_{i=1}^{n}(x_i - c) = 0 \tag{10}$$

$$\sum_{i=1}^{n}(x_i - c) = \sum_{i=1}^{n} x_i - nc = 0 \implies c = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{11}$$

Then, by the second derivative test,

$$f''(c) = -\frac{2}{n}\sum_{i=1}^{n} -1 = 2 \tag{12}$$

implying a concave up function, meaning our zero is in fact a minimum.

3 Probability & Statistics

3.a

The percent of surveyed US adults who had a great deal of confidence in both scientists and religious leaders is equal to $39\% \cdot 17\% = 6.63\%$, by the rule of multiplication.

3.b

$$\Pr(C) = 0.01 \implies \Pr(C^{\complement}) = 0.99 \tag{13}$$

$$\Pr(T \mid C) = 0.8 \tag{14}$$

$$\Pr(T \mid C^{\complement}) = 0.096 \tag{15}$$

$$\Pr(C \mid T) = \frac{\Pr(C)\Pr(T \mid C)}{\Pr(C)\Pr(T \mid C) + \Pr(C^{\complement})\Pr(T \mid C^{\complement})} \tag{16}$$

$$= \frac{0.01 \cdot 0.8}{0.01 \cdot 0.8 + 0.99 \cdot 0.096} \approx 0.078 \tag{17}$$

3.c

The standard deviation is $\sigma = 6.1$ since the inflection point of the histogram is at $\pm\sigma$, and the image visually demonstrates this.

4 Linear Algebra

4.a

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \implies \operatorname{rank}(A) = 2 \tag{18}$$

The matrix is full rank.

4.b

$$\boldsymbol{B} = \begin{bmatrix} 3 & 0 \\ -4 & 0 \end{bmatrix} \implies \text{rank}(B) = 1 \tag{19}$$

The matrix is not full rank, since $\boldsymbol{v}_1 = 0\boldsymbol{v}_2$.

4.c

$$\boldsymbol{C} = \begin{bmatrix} 0 & 5 & 10 \\ 1 & 0 & 10 \end{bmatrix} \implies \text{rank}(C) = 2 \tag{20}$$

The matrix is not full rank, since $\boldsymbol{v}_3 = 2\boldsymbol{v}_1 + 10\boldsymbol{v}_2$.

4.d

$$\boldsymbol{D} = \begin{bmatrix} 0 & -2 & 2 \\ 2 & -2 & 4 \\ 3 & 5 & -2 \end{bmatrix} \implies \text{rank}(D) = 2 \tag{21}$$

The matrix is not full rank, since $\boldsymbol{v}_3 = \boldsymbol{v}_1 - \boldsymbol{v}_2$.