EE 105 HW 01

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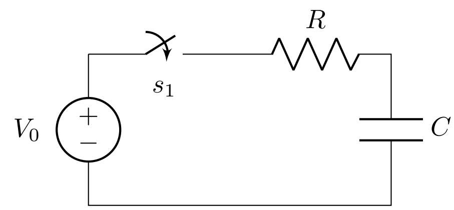


The initial condition is $V_C(t) = 0$. Analyzing the current $I_C(t)$,

$$\frac{V_0 - V_C(t)}{R} = C\frac{\mathrm{d}V_C}{\mathrm{d}t} \tag{1}$$

$$V_0 - V_C(t) = RC\frac{\mathrm{d}V_C}{\mathrm{d}t} \tag{2}$$

$$\frac{1}{V_0 - V_C(t)} = \frac{1}{RC}\frac{\mathrm{d}t}{\mathrm{d}V_C} \tag{3}$$

$$\int \frac{1}{V_0 - V_C(t)}\,\mathrm{d}V_C = \int \frac{1}{RC}\,\mathrm{d}t \tag{4}$$

$$-\ln|V_0 - V_C(t)| + C_0 = \frac{t}{RC} + C_1 \tag{5}$$

$$V_0 - V_C(t) = C_2 e^{-\frac{t}{RC}} \tag{6}$$

$$V_C(t) = V_0 - C_2 e^{-\frac{t}{RC}} \tag{7}$$

Solving for the initial condition,

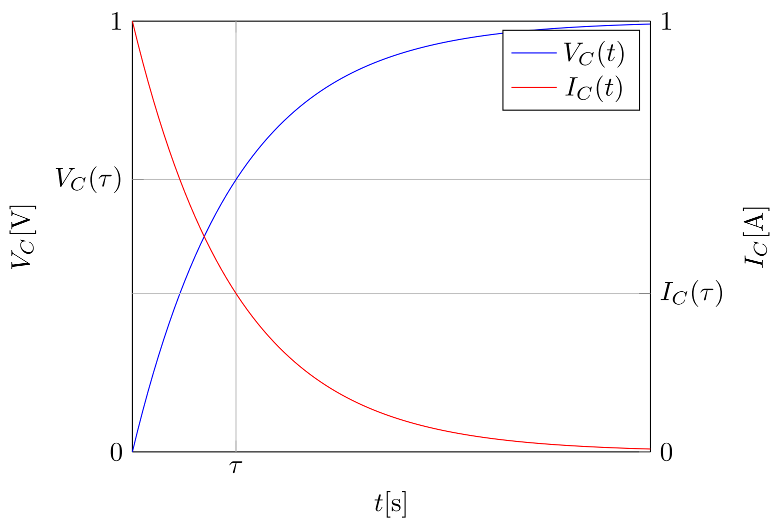
$$0 = V_0 - C_2 \implies C_2 = V_0 \tag{8}$$

Thus, we have

$$V_C(t) = V_0(1 - e^{-\frac{t}{RC}}) \tag{9}$$

$$I_C(t) = C\frac{\mathrm{d}V_C}{\mathrm{d}t} = \frac{\cancel{V_0} - \cancel{V_0} + V_0 e^{-\frac{t}{RC}}}{R} = \frac{V_0}{R}e^{-\frac{t}{RC}} \tag{10}$$

Assuming $V_0 = 1\,\mathrm{V}$, $R = 1\,\Omega$, and $C = 1\,\mathrm{F}$,



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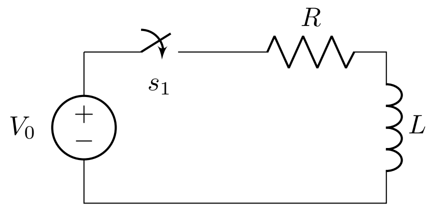


The initial condition is $I_L(t) = 0$. Analyzing the voltage loop including $V_L(t)$,

$$V_0 - I_L(t)R - L\frac{\mathrm{d}I_L}{\mathrm{d}t} = 0 \tag{11}$$

$$\frac{V_0}{R} - I_L(t) = \frac{L}{R}\frac{\mathrm{d}I_L}{\mathrm{d}t} \tag{12}$$

$$\int \frac{1}{\frac{V_0}{R} - I_L(t)}\,\mathrm{d}I_L = \int \frac{R}{L}\,\mathrm{d}t \tag{13}$$

$$-\ln\left|\frac{V_0}{R} - I_L(t)\right| + C_0 = \frac{R}{L}t + C_1 \tag{14}$$

$$\frac{V_0}{R} - I_L(t) = C_2 e^{-\frac{R}{L}t} \tag{15}$$

$$I_L(t) = \frac{V_0}{R} - C_2 e^{-\frac{R}{L}t} \tag{16}$$

Solving for the initial condition,

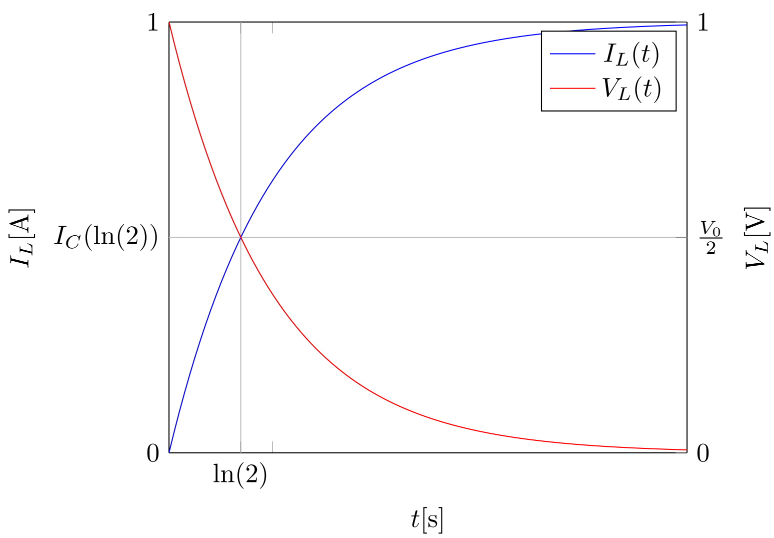
$$0 = \frac{V_0}{R} - C_2 \implies C_2 = \frac{V_0}{R} \tag{17}$$

Thus, we have

$$I_L(t) = \frac{V_0}{R}(1 - e^{-\frac{R}{L}t}) \tag{18}$$

$$V_L(t) = \cancel{V_0} - \cancel{V_0} + V_0 e^{-\frac{R}{L}t} = V_0 e^{-\frac{R}{L}t} \tag{19}$$

Assuming $V_0 = 1\,\mathrm{V}$, $R = 1\,\Omega$, and $L = 1\,\mathrm{H}$,



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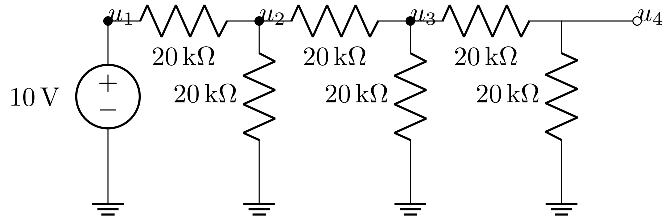
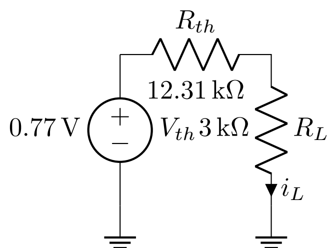


Using nodal voltage analysis, we obtain the following equations:

$$u_1 = 10\,\mathrm{V} \tag{20}$$
$$u_1 = 3u_2 - u_3 \tag{21}$$
$$u_2 = 3u_3 - u_4 \tag{22}$$
$$u_3 = 2u_4 \tag{23}$$

Solving for these gives us $u_4 = \frac{1}{13}10\,\mathrm{V} \approx 0.77\,\mathrm{V}$. The Thévenin equivalent resistance is $\frac{8}{13}20\,\mathrm{k\Omega} \approx 12.31\,\mathrm{k\Omega}$. The equivalent circuit is thus



We find that $i_L = \frac{V_{th}}{R_{th}+R_L} \approx 5.03 \times 10^{-5}\,\mathrm{A}$.

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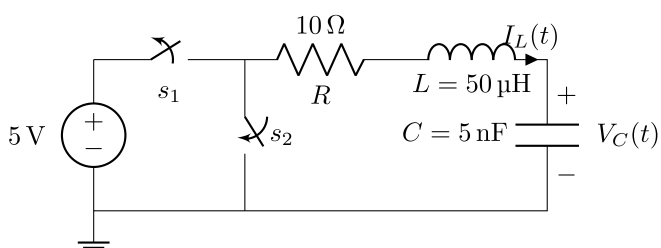

The initial conditions are $V_C(0) = 5\,\mathrm{V}$ and $I_L(0) = 0\,\mathrm{A}$. Performing nodal voltage analysis at $t \geqslant 0$, we obtain the equations

$$I_L(t) = C\frac{\mathrm{d}V_C}{\mathrm{d}t} \tag{24}$$
$$-I_L(t)R - V_C(t) = L\frac{\mathrm{d}I_L}{\mathrm{d}t} \tag{25}$$

Let $x_1(t) = V_C(t)$ and $x_2(t) = I_L(t)$. In state space form, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \boldsymbol{x}(t) = \underbrace{\begin{bmatrix} 0 & \frac{1}{5\times10^{-9}} \\ -\frac{1}{50\times10^{-6}} & -\frac{10}{50\times10^{-6}} \end{bmatrix}}_{\boldsymbol{A}} \boldsymbol{x}(t) \tag{26}$$

The eigenvalues are

$$\begin{vmatrix} -\lambda & 2\times 10^8 \\ -2\times 10^4 & -2\times 10^5-\lambda \end{vmatrix} = \lambda^2 + (2\times 10^5)\lambda + (4\times 10^{12}) = 0 \tag{27}$$

$$\implies \lambda = -10^5 \pm \sqrt{10^{10} - 4\times 10^{12}} \tag{28}$$

$$= -10^5 \pm 10^5\sqrt{1-4\times 10^2} = -10^5 \pm 10^5\sqrt{399}j \tag{29}$$

$$\approx -10^5 \pm (2\times 10^6)j \tag{30}$$

The eigenvectors are

$$\boldsymbol{A}-\lambda_1\boldsymbol{I} = \begin{bmatrix} 10^5 - 10^5\sqrt{399}j & 2\times 10^8 \\ -2\times 10^4 & -2\times 10^5 + 10^5 - 10^5\sqrt{399}j \end{bmatrix} \tag{31}$$

$$= \begin{bmatrix} 10^5 - 10^5\sqrt{399}j & 2\times 10^8 \\ -2\times 10^4 & -10^5 - 10^5\sqrt{399}j \end{bmatrix} \tag{32}$$

$$R_1 \leftarrow \frac{R_1}{-\lambda_1} \implies \begin{bmatrix} 1 & 5+5\sqrt{399}j \\ -2\times 10^4 & -10^5 - 10^5\sqrt{399}j \end{bmatrix} \tag{33}$$

$$R_2 \leftarrow R_2 - (2\times 10^4)R_1 \implies \begin{bmatrix} 1 & 5+5\sqrt{399}j \\ 0 & 0 \end{bmatrix} \implies \boldsymbol{v}_1 = \begin{bmatrix} -5-5\sqrt{399}j \\ 1 \end{bmatrix} \approx \begin{bmatrix} -5-100j \\ 1 \end{bmatrix} \tag{34}$$

$$\boldsymbol{A}-\lambda_2\boldsymbol{I} = \begin{bmatrix} 10^5 + 10^5\sqrt{399}j & 2\times 10^8 \\ -2\times 10^4 & -2\times 10^5 + 10^5 + 10^5\sqrt{399}j \end{bmatrix} \tag{35}$$

$$= \begin{bmatrix} 10^5 + 10^5\sqrt{399}j & 2\times 10^8 \\ -2\times 10^4 & -10^5 + 10^5\sqrt{399}j \end{bmatrix} \tag{36}$$

$$R_1 \leftarrow \frac{R_1}{-\lambda_2} \implies \begin{bmatrix} 1 & 5-5\sqrt{399}j \\ -2\times 10^4 & -10^5 + 10^5\sqrt{399}j \end{bmatrix} \tag{37}$$

$$R_2 \leftarrow R_2 + (2\times 10^4)R_1 \implies \begin{bmatrix} 1 & 5-5\sqrt{399}j \\ 0 & 0 \end{bmatrix} \implies \boldsymbol{v}_1 = \begin{bmatrix} -5+5\sqrt{399}j \\ 1 \end{bmatrix} \approx \begin{bmatrix} -5+100j \\ 1 \end{bmatrix} \tag{38}$$

$$\implies \boldsymbol{V} \approx \begin{bmatrix} -5-100j & -5+100j \\ 1 & 1 \end{bmatrix} \tag{39}$$

$$\boldsymbol{V}^{-1} \approx \frac{1}{200}\begin{bmatrix} j & 100+5j \\ -j & 100-5j \end{bmatrix} \tag{40}$$

The transformed initial condition is

$$\boldsymbol{z}(0) = \boldsymbol{V}^{-1}\begin{bmatrix} 5 \\ 0 \end{bmatrix} \approx \begin{bmatrix} \frac{j}{40} \\ -\frac{j}{40} \end{bmatrix} \tag{41}$$

Solving the diagonalized system, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{z}(t) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\boldsymbol{z}(t) \implies \boldsymbol{z}(t) \approx \begin{bmatrix} \frac{j}{40}e^{(-10^5+(2\times 10^6)j)t} \\ -\frac{j}{40}e^{(-10^5-(2\times 10^6)j)t} \end{bmatrix} \tag{42}$$

The final equation is thus

$$\boldsymbol{x}(t) = \boldsymbol{V}\boldsymbol{z}(t) \approx \frac{j}{40}\begin{bmatrix} -5-100j & -5+100j \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{(-10^5+(2\times10^6)j)t} \\ -e^{(-10^5-(2\times10^6)j)t} \end{bmatrix} \tag{43}$$

$$= \frac{j}{40}\begin{bmatrix} -5\big(e^{(-10^5+(2\times10^6)j)t} - e^{(-10^5-(2\times10^6)j)t}\big) \\ e^{(-10^5+(2\times10^6)j)t} - e^{(-10^5-(2\times10^6)j)t} \end{bmatrix} \tag{44}$$

$$= \frac{je^{-10^5 t}}{40}\begin{bmatrix} -5\big(e^{(2\times10^6)jt} - e^{(-2\times10^6)jt}\big) - 100j\big(e^{(2\times10^6)jt} + e^{(-2\times10^6)jt}\big) \\ e^{(2\times10^6)jt} - e^{(-2\times10^6)jt} \end{bmatrix} \tag{45}$$

$$= \frac{je^{-10^5 t}}{40}\begin{bmatrix} -10j\sin\big((2\times10^6)t\big) - 200j\cos\big((2\times10^6)t\big) \\ 2j\sin\big((2\times10^6)t\big) \end{bmatrix} \tag{46}$$

$$= \frac{e^{-10^5 t}}{40}\begin{bmatrix} 10\sin\big((2\times10^6)t\big) + 200\cos\big((2\times10^6)t\big) \\ -2\sin\big((2\times10^6)t\big) \end{bmatrix} \tag{47}$$