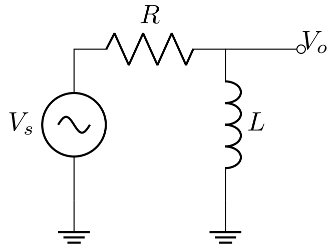
EE 105 HW 02

1



(a) The differential equation governing the circuit is

$$V_s(t) - I_L(t)R - L\frac{\mathrm{d}I_L}{\mathrm{d}t} = 0 \tag{1}$$

$$\implies \frac{\mathrm{d}I_L}{\mathrm{d}t} + \frac{R}{L}I_L(t) = \frac{1}{L}V_s(t) \tag{2}$$

The homogeneous solution to the differential equation is

$$I_{L,h}(t) = A_h e^{-\frac{R}{L}t} \tag{3}$$

Assuming the particular solution is of the form $A_p e^{bt}$, we have

$$A_p b e^{bt} + \frac{R}{L}A_p e^{bt} = \frac{1}{L}u(t) \tag{4}$$

Pattern matching, we find that $b = 0$, and $A_p = \frac{1}{R}u(t)$, so our particular solution is

$$I_{L,p}(t) = \frac{1}{R}u(t) \tag{5}$$

The final solution is

$$I_L(t) = I_{L,h}(t) + I_{L,p}(t) = \frac{1}{R}u(t) + A_h e^{-\frac{R}{L}t} \tag{6}$$

$$I_L(0) = 0 = \frac{1}{R} + A_h \implies A_h = -\frac{1}{R}u(t) \tag{7}$$

The final equation is

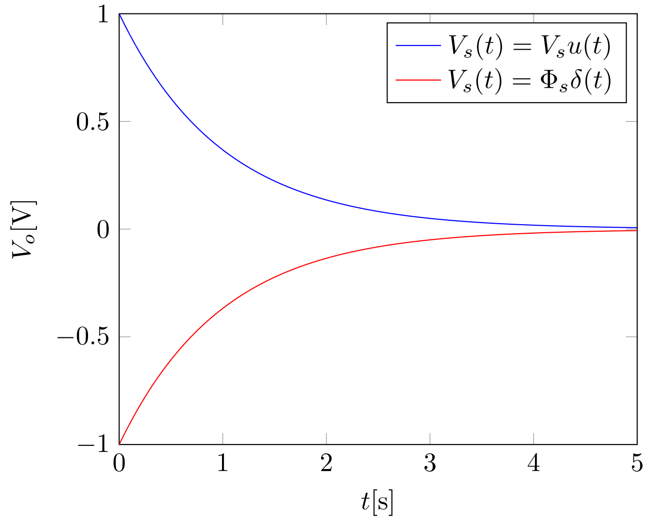
$$I_L(t) = \frac{1}{R}(1 - e^{-\frac{R}{L}t})u(t) \tag{8}$$

$$V_o(t) = L\frac{\mathrm{d}I_L}{\mathrm{d}t} = \cancelto{0}{\frac{L}{R}(1 - e^{-\frac{R}{L}t})}\delta(t) + e^{-\frac{R}{L}t}u(t) \tag{9}$$

The impulse response, by definition, is

$$h(t) = \frac{\mathrm{d}\Phi_o}{\mathrm{d}t} = \cancelto{1}{e^{-\frac{R}{L}t}}\delta(t) - \frac{R}{L}e^{-\frac{R}{L}t}u(t) = \delta(t) - \frac{R}{L}e^{-\frac{R}{L}t}u(t) \tag{10}$$

Assuming $V_s = 1\,\mathrm{V}$, $R = 1\,\Omega$, $L = 1\,\mathrm{H}$,



(b) Finding the particular solution for the frequency response, we have

$$Abe^{bt} + \frac{R}{L}Ae^{bt} = \frac{1}{L}V_s e^{j\omega t} \tag{11}$$

Pattern matching, we find that $b = j\omega$, and $A = \frac{V_s}{R+j\omega L}$. The final equation is

$$I_L(t) = \frac{V_s}{R + j\omega L}e^{j\omega t} \tag{12}$$

$$V_o(t) = L\frac{\mathrm{d}I_L}{\mathrm{d}t} = \frac{V_s L}{R + j\omega L} j\omega e^{j\omega t} \tag{13}$$

$$= \frac{V_s e^{j\omega t}}{R + j\omega L}(j\omega L) \implies \frac{V_o(t)}{V_s(t)} = \frac{j\omega L}{R + j\omega L} \tag{14}$$

$$\left|\frac{V_o(t)}{V_s(t)}\right| = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}} \tag{15}$$

$$\angle\frac{V_o(t)}{V_s(t)} = \tan^{-1}\left(\frac{R}{\omega L}\right) \tag{16}$$

This is a high-pass filter since $\lim_{\omega\to 0}|H(\omega)| = 0$ and $\lim_{\omega\to\infty}|H(\omega)| = 1$.

2

(a)

$$F(s) = \int_{\mathbb{R}^+} 7t^2 e^{-st}\ \mathrm{d}t \tag{17}$$

By tabular integration, we have

$$\begin{array}{c|c} u & v \\ \hline 7t^2 & e^{-st} \\ 14t & -\frac{1}{s}e^{-st} \\ 14 & \frac{1}{s^2}e^{-st} \\ 0 & -\frac{1}{s^3}e^{-st} \end{array} \tag{18}$$

So the final integral is

$$F(s) = -\frac{7t^2e^{-st}}{s} - \frac{14te^{-st}}{s^2} - \frac{14e^{-st}}{s^3}\bigg|_0^\infty = \frac{14}{s^3} \tag{19}$$

where the integral converges when $\mathrm{Re}\{s\} > 0$.

(b)

$$\begin{aligned}
F(s) &= \int_{\mathbb{R}+} 3e^{-2t}\cos(5t)e^{-st}\ \mathrm{d}t && (20)\\
&= \int_{\mathbb{R}+} 3e^{-2t}\left(\frac{e^{j5t}+e^{-j5t}}{2}\right)e^{-st}\ \mathrm{d}t && (21)\\
&= \frac{3}{2}\int_{\mathbb{R}+} e^{(-2-s+5j)t} + e^{(-2-s-5j)t}\ \mathrm{d}t && (22)\\
&= \frac{3}{2}\left(\frac{1}{-2-s+5j}e^{(-2-s+5j)t} + \frac{1}{-2-s-5j}e^{(-2-s-5j)t}\bigg|_0^\infty\right) && (23)\\
&= \frac{3}{2}\left(\frac{1}{2+s+5j} + \frac{1}{2+s-5j}\right) && (24)\\
&= \frac{3}{2}\left(\frac{2+s-5j+2+s+5j}{(2+s)^2+25}\right) = 3\left(\frac{2+s}{(2+s)^2+25}\right) && (25)
\end{aligned}$$

where the integral converges when $\mathrm{Re}\{s\} > 2$.

3

(a)

$$\begin{aligned}
F(s) &= \frac{1}{s-1} + \frac{4}{(s-3)^2} + \frac{7}{(s-5)^3} && (26)\\
&= \frac{0!}{(s-1)^1} + \frac{4}{1!}\frac{1!}{(s-3)^2} + \frac{7}{2!}\frac{2!}{(s-5)^3} && (27)
\end{aligned}$$

Using the Laplace transform pair $t^n e^{-\alpha t} \iff \frac{n!}{(s+\alpha)^{n+1}}$, we get

$$f(t) = e^t + 4te^{3t} + \frac{7}{2}t^2e^{5t} \tag{28}$$

(b)

$$F(s) = \frac{(s+3)+8}{(s+3)^2+4} = \frac{s+3}{(s+3)^2+4} + 4\frac{2}{(s+3)^2+4} \tag{29}$$

Using the Laplace transform pairs $e^{-\alpha t}\sin(\omega t) \iff \frac{\omega}{(s+\alpha)^2+\omega^2}$ and $e^{-\alpha t}\cos(\omega t) \iff \frac{s+\alpha}{(s+\alpha)^2+\omega^2}$, we get

$$f(t) = e^{-3t}\cos(2t) + 4e^{-3t}\sin(2t) \tag{30}$$

(c)

$$\begin{aligned}
F(s) &= \frac{1}{(s+2)(s+4)} = \frac{A}{s+2} + \frac{B}{s+4} && (31)\\
\implies 1 &= A(s+4) + B(s+2) && (32)
\end{aligned}$$

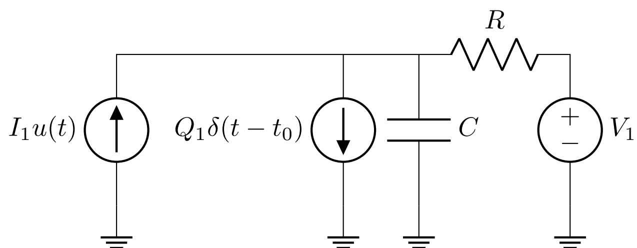
Plugging in $s = -4$ and $s = -2$, we get $A = \frac{1}{2}$ and $B = -\frac{1}{2}$, respectively. We then get

$$F(s) = \frac{1}{2}\frac{1}{s+2} - \frac{1}{2}\frac{1}{s+4} \tag{33}$$

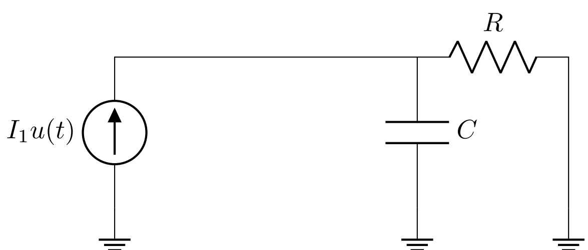
Using the Laplace transform pair $e^{-\alpha t} \iff \frac{1}{s+\alpha}$,

$$f(t) = \frac{1}{2}e^{-2t} - \frac{1}{2}e^{-4t} \tag{34}$$

4



Using superposition, we find $V_c(t)$ considering only $I_1u(t)$:



The initial condition is $V_c(0) = 0$. The homogeneous solution is $V_c(t) = A_h e^{-\frac{t}{RC}}$. The particular solution governing the circuit is

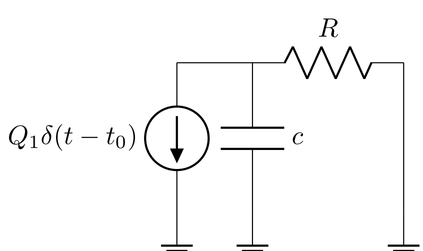
$$\frac{I_1}{C}u(t) = \frac{\mathrm{d}V_c}{\mathrm{d}t} + \frac{1}{RC}V_c(t) \tag{35}$$

$$\frac{I_1}{C}u(t) = A_p b e^{bt} + \frac{A_p}{RC}e^{bt} \tag{36}$$

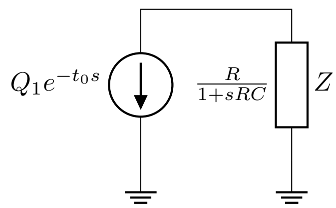
$$\implies V_c(t) = I_1 R u(t) \tag{37}$$

$$\implies V_c(t) = I_1 R(1 - e^{-\frac{t}{RC}})u(t) \tag{38}$$

Finding $V_c(t)$ considering only $Q_1\delta(t-t_0)$:



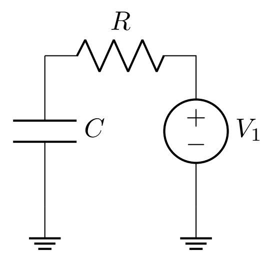
The initial condition is $V_c(t_0) = 0$. Taking the Laplace transform,



By Ohm's law, we get

$$V(s) = -\frac{QRe^{-t_0 s}}{\frac{1}{RC} + sRC} = -\frac{\frac{Q}{C}e^{-t_0 s}}{\frac{1}{RC} + s} \xRightarrow{\mathcal{L}^{-1}} V_c(t) = -\frac{Q}{C}e^{-\frac{t-t_0}{RC}}u(t-t_0) \tag{39}$$

Finding $V_c(t)$ considering only V_1:



Since V_1 is a constant, $V_c(t) = V_1$ is at steady state. The final equation for the circuit is

$$V_c(t) = I_1 R(1 - e^{-\frac{t}{RC}})u(t) - \frac{Q}{C}e^{-\frac{t-t_0}{RC}}u(t-t_0) + V_1 \tag{40}$$

Assuming $I_1 = 1\,\mathrm{A}$, $Q_1 = 1\,\mathrm{C}$, $C = 1\,\mathrm{F}$, $R = 1\,\Omega$, $V_1 = 1\,\mathrm{V}$, and $t_0 = 2\,\mathrm{s}$,