EE 123 HW 02

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$$a^n u[n] \iff \frac{1}{1 - ae^{-j\omega}},\ |a| < 1 \tag{1}$$

3.a

$$x[n] = -b^n u[-n-1] = \begin{cases} -b^n & n \leqslant -1 \\ 0 & n \geqslant 0 \end{cases} \tag{2}$$

Using the definition of the DTFT,

$$\begin{aligned} X(e^{j\omega}) &= \sum_{k \in \mathbb{Z}} -b^k u[-k-1] e^{-j\omega k} && (3) \\ &= -\sum_{k \leqslant -1} b^k e^{-j\omega k} && (4) \end{aligned}$$

Letting $k' = -k$,

$$\begin{aligned} X(e^{j\omega}) &= \sum_{k' \geqslant 1} -b^{-k'} e^{j\omega k'} && (5) \\ &= \sum_{k \geqslant 0} -(b^{-1} e^{j\omega})^k - 1 && (6) \\ &= 1 - \frac{1}{1 - be^{j\omega}} && (7) \\ &= \frac{\cancel{1} - be^{j\omega} - \cancel{1}}{1 - be^{j\omega}} \cdot \frac{-be^{-j\omega}}{-be^{-j\omega}} && (8) \\ &= \frac{1}{1 - be^{-j\omega}} && (9) \end{aligned}$$

where $|b^{-1}| < 1 \implies |b| > 1$.

3.b

$$Y(e^{j\omega}) = 2e^{-j\omega} \frac{1}{1 - (-2)e^{-j\omega}} \tag{10}$$

$$\begin{aligned} \overset{\mathcal{F}^{-1}}{\implies} y[n] &= 2(-(-2)^{n-1} u[-(n-1)-1]) && (11) \\ &= (-2)^n u[-n] && (12) \end{aligned}$$

4

$$H(z) = \frac{1 - z^{-1}}{1 - 0.25z^{-2}} = \frac{1 - z^{-1}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} \tag{13}$$

4.a

Using the z-transform multiplication property,

$$Y(z) = H(z)X(z) = \frac{\cancel{1 - z^{-1}}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})\cancel{(1 - z^{-1})}} = \frac{1}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} \tag{14}$$

Then, using partial fraction decomposition,

$$Y(z) = \frac{1}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} = \frac{A}{1 - 0.5z^{-1}} + \frac{B}{1 + 0.5z^{-1}} \tag{15}$$

$$= 1 = A(1 + 0.5z^{-1}) + B(1 - 0.5z^{-1}) \tag{16}$$

$$\tag{17}$$

Letting $z = -0.5$, we get $B = \frac{1}{2}$. Letting $z = 0.5$, we get $A = \frac{1}{2}$. Then,

$$Y(z) = \frac{1}{2}\frac{1}{1 - 0.5z^{-1}} + \frac{1}{2}\frac{1}{1 + 0.5z^{-1}} \tag{18}$$

$$\overset{\mathcal{Z}^{-1}}{\Longrightarrow} y[n] = \frac{1}{2}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(-\frac{1}{2}\right)^n u[n] \tag{19}$$

4.b

$$y[n] = \delta[n] - \delta[n-1] \overset{\mathcal{Z}}{\Longrightarrow} Y(z) = 1 - z^{-1} = H(z)X(z) \tag{20}$$

meaning that $X(z) = 1 - 0.25z^{-2} = (1 - 0.5z^{-1})(1 + 0.5z^{-1})$. By definition of the z-transform,

$$x[n] = \delta[n] - \frac{1}{4}\delta[n-2] \tag{21}$$

4.c

Treating the input as the real part of $z = e^{j0.5\pi}$,

$$H(e^{j0.5\pi}) = \frac{1 - e^{-j0.5\pi}}{1 - 0.25e^{-j\pi}} = 0.8\sqrt{2}e^{j\frac{\pi}{4}} \tag{22}$$

Meaning that the final output is

$$y[n] = 0.8\sqrt{2}\cos\left(0.5\pi n + \frac{\pi}{4}\right) \tag{23}$$

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$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}2^n u[-n-1] \tag{24}$$

$$Y(z) = \frac{1 - z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)} \tag{25}$$

5.a

$$X(z) = -\frac{1}{3}\frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{4}{3}\frac{1}{1 - 2z^{-1}} \tag{26}$$

5.b

$$R_y : \frac{1}{2} < |z| < 2 \tag{27}$$

5.c

Simplifiying $X(z)$,

$$X(z) = \frac{1}{3}\frac{-(1 - 2z^{-1}) + 4\left(1 - \frac{1}{2}z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)} \tag{28}$$

$$= \frac{1}{3}\frac{-1 + \cancel{2z^{-1}} + 4 - \cancel{2z^{-1}}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)} \tag{29}$$

$$= \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - 2z^{-1}\right)} \tag{30}$$

$$\implies Y(z) = (1 - z^{-2})X(z) = X(z) - z^{-2}X(z) \tag{31}$$

$$\overset{\mathcal{Z}^{-1}}{\implies} y[n] = x[n] - x[n-2] \tag{32}$$

5.d

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2} \overset{\mathcal{Z}^{-1}}{\implies} h[n] = \delta[n] - \delta[n-2] \tag{33}$$

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$$X(z) = \frac{z^{-2}}{1 - 2.3z^{-1} + 1.6z^{-2} - 0.3z^{-3}} = \frac{2.04}{1 + 0.3z^{-1}} - \frac{3.47}{1 - z^{-1}} + \frac{1.43}{(1 - z^{-1})^2} \tag{34}$$

$$\overset{\mathcal{Z}^{-1}}{\Longrightarrow} \; x[n] = (-0.3)^n u[n] - 3.47u[n] + 1.43(n+1)u[n+1] \tag{35}$$

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a. System A, causal, stable, ROC: $|z| > 0.9$

b. System B, non-causal, stable, ROC: $|z| < 1.111$

c. System A, non-causal, unstable, ROC: $|z| < 0.9$

8

$$Z(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})\,d\theta \tag{36}$$

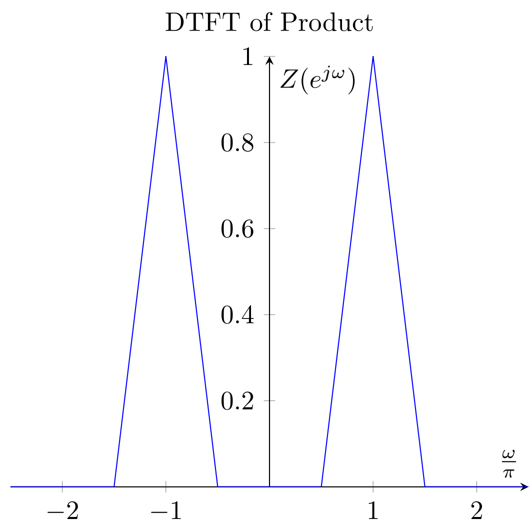
8.a

$$Y(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi + 2\pi k) + \delta(\omega + \pi + 2\pi k) \tag{37}$$

$$X(e^{j\omega}) * Y(e^{j\omega}) = \frac{1}{2}\int_{-\pi}^{\pi} X(e^{j\theta}) \sum_{k\in\mathbb{Z}} \delta((\omega-\theta) - \pi + 2\pi k) + \delta((\omega-\theta) + \pi + 2\pi k)\,d\theta \tag{38}$$

$$= \frac{1}{2}\sum_{k\in\mathbb{Z}} \int_{-\pi}^{\pi} X(e^{j\omega})(\delta(\omega - \theta - \pi + 2\pi k) + \delta(\omega + \theta + \pi + 2\pi k))\,d\theta \tag{39}$$

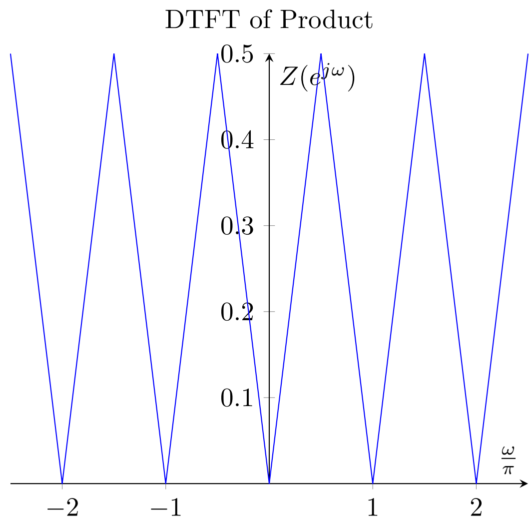
$$= \frac{1}{2}\sum_{k\in\mathbb{Z}} (X(e^{j(\omega-\pi+2\pi k)}) + X(e^{j(\omega+\pi+2\pi k)})) \tag{40}$$



8.b

$$Y(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi}{2} + 2\pi k\right) + \delta\left(\omega + \frac{\pi}{2} + 2\pi k\right) \tag{41}$$

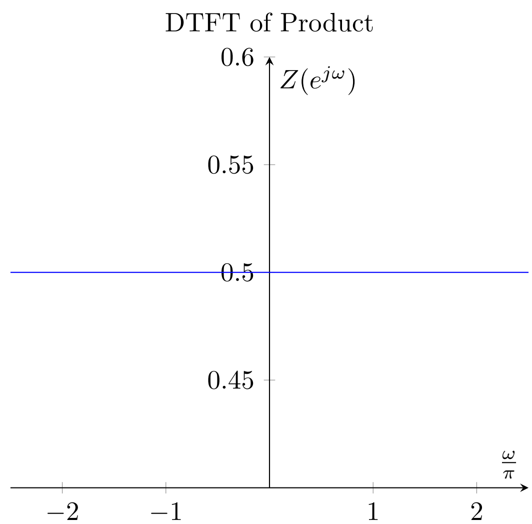
$$X(e^{j\omega}) * Y(e^{j\omega}) = \frac{1}{2}\sum_{k\in\mathbb{Z}} \left(X\left(e^{j\left(\omega - \frac{\pi}{2} + 2\pi k\right)}\right) + X\left(e^{j\left(\omega + \frac{\pi}{2} + 2\pi k\right)}\right)\right) \tag{42}$$



8.c

$$Y(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi}{4} + 2\pi k\right) + \delta\left(\omega + \frac{\pi}{4} + 2\pi k\right) + \delta\left(\omega - \frac{3\pi}{4} + 2\pi k\right) + \delta\left(\omega + \frac{3\pi}{4} + 2\pi k\right) \tag{43}$$

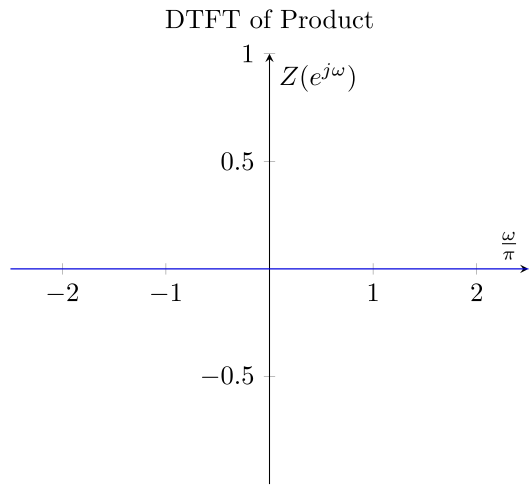
$$X(e^{j\omega}) * Y(e^{j\omega}) = \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(X\left(e^{j\left(\omega - \frac{\pi}{2} + 2\pi k\right)}\right) + X\left(e^{j\left(\omega + \frac{\pi}{2} + 2\pi k\right)}\right)\right) \tag{44}$$



8.d

$$Y(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} \cancelto{1}{e^{j\frac{\pi}{2}}}\delta(\omega - \pi + 2\pi k) + \cancelto{-1}{e^{-j\frac{\pi}{2}}}\delta(\omega + \pi + 2\pi k) \tag{45}$$

$$X(e^{j\omega}) * Y(e^{j\omega}) = \frac{1}{2}\sum_{k\in\mathbb{Z}}(X(e^{j(\omega-\pi+2\pi k)}) - X(e^{j(\omega+\pi+2\pi k)})) \tag{46}$$


DTFT of Product
$Z(e^{j\omega})$
1
0.5
−0.5
−2
−1
1
2
$\frac{\omega}{\pi}$

9

$$H(e^{j\omega}) = \begin{cases} j & \omega \in (-\pi, 0) \\ 0 & \omega = 0 \\ -j & \omega \in (0, \pi) \end{cases} \tag{47}$$

9.a

From Table 2.1 of O&S, the symmetry properties of the DTFT imply that the Hilbert filter impulse response is purely *real and odd*.

9.b

$$X(e^{j\omega}) = \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)) \tag{48}$$

$$Y(e^{j\omega}) = -j\pi(\delta(\omega - \omega_0) - \delta(\omega + \omega_0)) \tag{49}$$

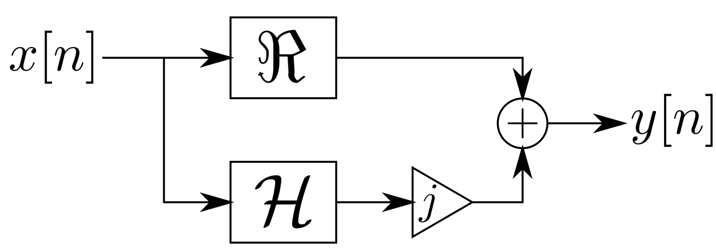
$$\xRightarrow{\mathcal{F}^{-1}} y[n] = \sin(\omega_0 n) \tag{50}$$

9.c

$$Y(e^{j\omega}) = H(e^{j\omega})^2 X(e^{j\omega}) = \begin{cases} -X(e^{j\omega}) & \omega \in (-\pi, 0) \\ 0 & \omega = 0 \\ -X(e^{j\omega}) & \omega \in (0, \pi) \end{cases} \tag{51}$$

$$= -X(e^{j\omega}) \xRightarrow{\mathcal{F}^{-1}} y[n] = -x[n] \tag{52}$$

9.d



where $\mathcal{H}$ is the Hilbert transform.