EE 123 HW 02

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$$a^n u[n] \iff \frac{1}{1 - ae^{-j\omega}},\ |a| < 1 \tag{1}$$

3.a

$$x[n] = -b^n u[-n-1] = \begin{cases} -b^n & n \leqslant -1 \\ 0 & n \geqslant 0 \end{cases} \tag{2}$$

Using the definition of the DTFT,

$$\begin{aligned} X(e^{j\omega}) &= \sum_{k \in \mathbb{Z}} -b^k u[-k-1] e^{-j\omega k} &(3)\\ &= -\sum_{k \leqslant -1} b^k e^{-j\omega k} &(4) \end{aligned}$$

Letting $k' = -k$,

$$\begin{aligned} X(e^{j\omega}) &= \sum_{k' \geqslant 1} -b^{-k'} e^{j\omega k'} &(5)\\ &= \sum_{k \geqslant 0} -(b^{-1} e^{j\omega})^k - 1 &(6)\\ &= 1 - \frac{1}{1 - be^{j\omega}} &(7)\\ &= \frac{\cancel{1} - be^{j\omega} - \cancel{1}}{1 - be^{j\omega}} \cdot \frac{-be^{-j\omega}}{-be^{-j\omega}} &(8)\\ &= \frac{1}{1 - be^{-j\omega}} &(9) \end{aligned}$$

where $|b^{-1}| < 1 \implies |b| > 1$.

3.b

$$\begin{aligned} Y(e^{j\omega}) &= 2e^{-j\omega} \frac{1}{1 - (-2)e^{-j\omega}} &(10)\\ \overset{\mathcal{F}^{-1}}{\implies} y[n] &= 2(-(-2)^{n-1} u[-(n-1)-1]) &(11)\\ &= (-2)^n u[-n] &(12) \end{aligned}$$

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$$H(z) = \frac{1-z^{-1}}{1-0.25z^{-2}} = \frac{1-z^{-1}}{(1-0.5z^{-1})(1+0.5z^{-1})} \tag{13}$$

4.a

Using the Z-transform multiplication property,

$$Y(z) = H(z)X(z) = \frac{\cancel{1-z^{-1}}}{(1-0.5z^{-1})(1+0.5z^{-1})\cancel{(1-z^{-1})}} = \frac{1}{(1-0.5z^{-1})(1+0.5z^{-1})} \tag{14}$$

Then, using partial fraction decomposition,

$$Y(z) = \frac{1}{(1-0.5z^{-1})(1+0.5z^{-1})} = \frac{A}{1-0.5z^{-1}} + \frac{B}{1+0.5z^{-1}} \tag{15}$$

$$= 1 = A(1+0.5z^{-1}) + B(1-0.5z^{-1}) \tag{16}$$

$$\tag{17}$$

Letting $z=-0.5$, we get $B=\frac{1}{2}$. Letting $z=0.5$, we get $A=\frac{1}{2}$. Then,

$$Y(z) = \frac{1}{2}\frac{1}{1-0.5z^{-1}} + \frac{1}{2}\frac{1}{1+0.5z^{-1}} \tag{18}$$

$$\overset{\mathcal{Z}^{-1}}{\Longrightarrow} y[n] = \frac{1}{2}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(-\frac{1}{2}\right)^n u[n] \tag{19}$$

4.b

$$y[n] = \delta[n] - \delta[n-1] \overset{\mathcal{Z}}{\Longrightarrow} Y(z) = 1 - z^{-1} = H(z)X(z) \tag{20}$$

meaning that $X(z) = 1-0.25z^{-2} = (1-0.5z^{-1})(1+0.5z^{-1})$. By the convolution property of the Z-transform,

$$x[n] = \left(\left(\frac{1}{2}\right)^n u[n]\right) * \left(\left(-\frac{1}{2}\right)^n u[n]\right) \tag{21}$$

4.c

Treating the input as the real part of $z = e^{j0.5\pi}$,

$$H(e^{j0.5\pi}) = \frac{1-e^{-j0.5\pi}}{1-0.25e^{-j\pi}} = 0.8\sqrt{2}e^{j\frac{\pi}{4}} \tag{22}$$

Meaning that the final output is

$$y[n] = 0.8\sqrt{2}\cos\left(0.5\pi n + \frac{\pi}{4}\right) \tag{23}$$

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$$x[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{4}{3}2^n u[-n-1] \tag{24}$$

$$Y(z) = \frac{1-z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})} \tag{25}$$

5.a

$$X(z) = -\frac{1}{3}\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{4}{3}\frac{1}{1-2z^{-1}} \tag{26}$$

5.b

$$R_y : \frac{1}{2} < |z| < 2 \tag{27}$$

5.c

Simplifiying $X(z)$,

$$X(z) = \frac{1}{3}\frac{-(1-2z^{-1}) + 4\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})} \tag{28}$$

$$= \frac{1}{3}\frac{-1+\cancel{2z^{-1}}+4-\cancel{2z^{-1}}}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})} \tag{29}$$

$$= \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)(1-2z^{-1})} \implies Y(z) \qquad = (1-z^{-2})X(z) = X(z) - z^{-2}X(z) \tag{30}$$

$$\overset{\mathcal{Z}^{-1}}{\implies} y[n] = x[n] - x[n-2] \tag{31}$$

5.d

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2} \overset{\mathcal{Z}^{-1}}{\implies} h[n] = \delta[n] - \delta[n-2] \tag{32}$$

5.e

$$X(z) = \frac{z^{-2}}{1-2.3z^{-1}+1.6z^{-2}-0.3z^{-3}} = \frac{2.04}{1+0.3z^{-1}} - \frac{3.47}{1-z^{-1}} + \frac{1.43}{(1-z^{-1})^2} \tag{33}$$

$$\overset{\mathcal{Z}^{-1}}{\implies} x[n] = (-0.3)^n u[n] - 3.47u[n] + (n+1)u[n+1] \tag{34}$$

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a.