EE 123 HW 03

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3.a

$y[n]$ is

- $y[n \geqslant L] = 0$ since an autocorrelation implies that the range of nonzero values is $[-L-1, L-1]$.
- Conjugate-symmetric since the frequency domain is purely real.
- Odd length since the multiplication of two length L signals in the frequency domain is the convolution, which creates a length $2L - 1$.

3.b

Due to the symmetry and convolution properties of the DFT,

$$\bar{y}[n] = x[n] \, N \, x^*[-n] \tag{1}$$

where we want

$$y[n] = \bar{y}[m[n]] = x[n] * x^*[-n] \tag{2}$$

Let $N = 2L - 1$ so that we can capture the entire range of the circular autocorrelation. Since circular convolution retains the same indices as the original signals, we now have a signal with range $[0, 2N - 2]$. Thus, we must time shift by $m[n] = n + (L - 1)$ to center the signal around 0.

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k	$X[k]$
0	3
2	$0.5 - 4.5j$
4	5
5	$3.5 + 3.5j$
7	$-2.5 - 7j$

(3)

4.a

Using the symmetry properties of the DFT, we can complete the DFT as shown:

k	$X[k]$
0	3
1	$-2.5 + 7j$
2	$0.5 - 4.5j$
3	$3.5 - 3.5j$
4	5
5	$3.5 + 3.5j$
6	$0.5 + 4.5j$
7	$-2.5 - 7j$

(4)

meaning that

$$x[0] = \frac{1}{8}\sum_{k=0}^{N-1} X[k] = \frac{11}{8} \tag{5}$$

4.b

By the circular convolution and time-shift theorems of the DFT,

$$x[n] \circledast_8 \delta[n-1] = e^{-j\frac{2\pi}{8}k} X[k] \tag{6}$$

4.c

Since $W[k]$ is simply $X[k]$ evaluated at the even terms, we can represent $w[n]$ as

$$W[k] = X[2k] = \sum_{n=0}^{7} x[n] W_8^{2kn} \tag{7}$$

$$\stackrel{\mathcal{F}^{-1}}{\Longrightarrow} w[n] = \frac{1}{4}\sum_{k=0}^{3} W[k] W_4^{-kn} = \frac{1}{4}\sum_{k=0}^{3} X[2k] W_4^{-kn} \tag{8}$$

$$= \frac{1}{4}\sum_{k=0}^{7} \frac{1}{2}(1 + (-1)^k) X[k] W_8^{-kn} = \frac{1}{8}\sum_{k=0}^{7} (1 + e^{-j\pi k}) X[k] W_8^{-kn} \tag{9}$$

$$= x[n] + \sum_{k=0}^{7} e^{-j\pi k} X[k] W_8^{-kn} \tag{10}$$

$$= x[n] + \sum_{k=0}^{7} W_8^4 X[k] W_8^{-kn} = x[n] + \sum_{k=0}^{7} X[k] W_8^{-kn+4} \tag{11}$$

$$= x[n] + x[(n-4)_8] \tag{12}$$

5 Faster DFTs?

Since $f[n]$ and $g[n]$ are both purely real, then $jg[n]$ must be purely imaginary. By the linearity of the DFT, $H[k] = F[k] + jG[k]$. We can then express $F[k]$ and $G[k]$ as

$$F[k] = \Re\{H[k]\} = \frac{1}{2}(H[k] + H^*[k]) \tag{13}$$

$$G[k] = \Im\{H[k]\} = \frac{1}{2j}(H[k] - H^*[k]) \tag{14}$$

6 Diagonalizing Circulant Matrices

Theorem 1. *The DFT matrix diagonalizes all circulant matrices.*

Proof. Let $\boldsymbol{H}$ be a circulant matrix

$$\boldsymbol{H} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{N-1} & c_N \\ c_N & c_1 & \cdots & c_{N-2} & c_{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_N & c_1 \end{bmatrix} = \begin{bmatrix} c[n] \\ c[(n-1)_N] \\ \vdots \\ c[(n-(N-1))_N] \end{bmatrix} \tag{15}$$

and $\boldsymbol{D}$ be the $N \times N$ DFT matrix. Note that the DFT matrix satisfies the following relation: $\boldsymbol{D}^{-1} = \frac{1}{N}\boldsymbol{D}^*$. Then,

$$\boldsymbol{D}^{-1}\boldsymbol{H}\boldsymbol{D} = \frac{1}{N}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & \cdots & e^{j\frac{2\pi}{N}(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j\frac{2\pi}{N}(N-1)} & \cdots & e^{j\frac{2\pi}{n}(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} c[n] \\ c[(n-1)_N] \\ \vdots \\ c[(n-(N-1))_N] \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{n}} & \cdots & e^{-j\frac{2\pi}{n}(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{n}(n-1)} & \cdots & e^{-j\frac{2\pi}{n}(n-1)(n-1)} \end{bmatrix} \tag{16}$$

$$= \frac{1}{N}\begin{bmatrix} C[0] \\ W_N^{-1}C[1] \\ \vdots \\ W_N^{-(N-1)}C[N-1] \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{n}} & \cdots & e^{-j\frac{2\pi}{n}(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{n}(n-1)} & \cdots & e^{-j\frac{2\pi}{n}(n-1)(n-1)} \end{bmatrix} \tag{17}$$

$$= \frac{1}{N}\begin{bmatrix} NC[0] & 0 & \cdots & 0 \\ 0 & NC[1] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & NC[N-1] \end{bmatrix} = \begin{bmatrix} C[0] & 0 & \cdots & 0 \\ 0 & C[1] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C[N-1] \end{bmatrix} \tag{18}$$

where we use the circular shift property to rewrite the circulant matrix and use the frequency shift properties of the DFT to prove that the DFT matrix diagonalizes any circulant matrix. □