Homework 1

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2.a

Theorem 1. *The system L is not time invariant.*

Proof. We can represent a time-delayed signal as a linear combination of the other 3 signals. We can represent the 3 input signals in a matrix

$$\boldsymbol{X} = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \tag{1}$$

Then, we can find the linear combination necessary to create a time delayed signal (for example, $x_2[n-1]$) by solving the system of equations

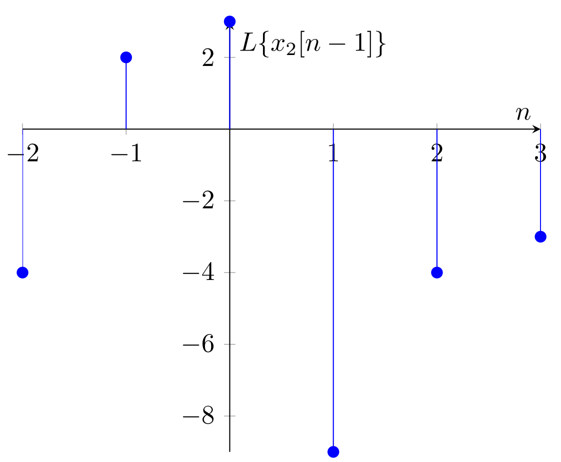
$$\boldsymbol{X}\boldsymbol{a} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \tag{2}$$

where we get $\boldsymbol{a} = \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} & -2 \end{bmatrix}^\top$. Thus, we have

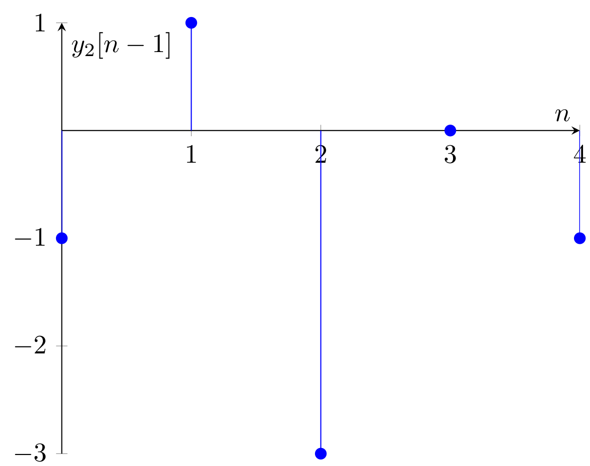
$$x_2[n-1] = -\frac{3}{2}x_1[n] + \frac{3}{2}x_2[n] - 2x_3[n] \tag{3}$$

Calculating $-\frac{3}{2}y_1[n] + \frac{3}{2}y_2[n] - 2y_3[n]$, we get the following plot:

Time invariance counterexample



Expected output signal



So the system is *not* time-invariant. ∎

2.b

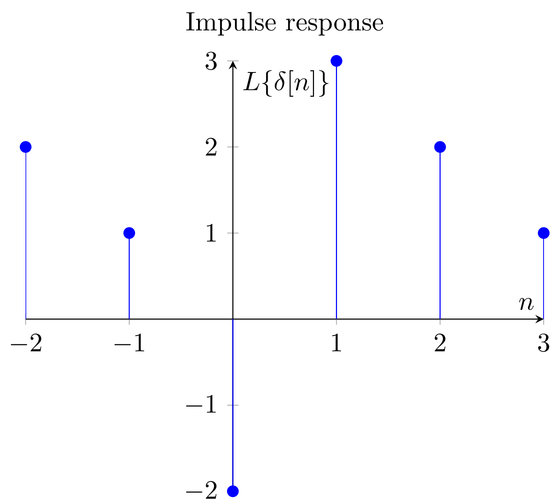
We can represent the delta function as

$$\delta[n] = \frac{1}{2}(x_1[n] - x_2[n] + 2x_3[n]) \tag{4}$$

Using the properties of linear systems, we can find the impulse response

$$h[n] = \frac{1}{2}(y_1[n] - y_2[n] + 2y_3[n]) \tag{5}$$

which gives us the following plot:



3

$$y[n] - ay[n-1] = x[n] \tag{6}$$
$$y[0] = 1 \tag{7}$$

3.a

n	$x[n]$
0	1
1	$x[1] + a = y[1]$
2	$x[2] + ax[1] + a^2 = y[2]$
3	$x[3] + ax[2] + a^2x[1] = y[3]$
$\vdots$	$\vdots$

(8)

By induction,

$$y[n] = \sum_{k=0}^{n} a^{n-k} x[k] \tag{9}$$

Finding $T\{x[n-n_0]\}$,

$$T\{x[n-n_0]\} = \sum_{k=0}^{n} a^{n-k} x[k-n_0] \tag{10}$$

Letting $k' = k - n_0$,

$$\sum_{k'=0}^{n-n_0} a^{n-n_0-k} x[k'] = y[n-n_0] \tag{11}$$

So the system is time-invariant.

3.b

Letting $T\{x_1[n]\} = y_1[n]$ and $T\{x_1[n]\} = y_2[n]$,

$$\begin{aligned} T\{\alpha x_1[n] + \beta x_2[n]\} &= \sum_{k=0}^{n} a^{n-k} (\alpha x_1[k] + \beta x_2[k]) && (12) \\ &= \sum_{k=0}^{n} a^{n-k} \alpha x_1[k] + \sum_{k=0}^{n} a^{n-k} \beta x_2[k] && (13) \\ &= \alpha \sum_{k=0}^{n} a^{n-k} x_1[n] + \beta \sum_{k=0}^{n} x_2[k] = \alpha y_1[n] + \beta y_2[n] && (14) \end{aligned}$$

So the system is linear.

3.c

n	$x[n]$
0	1
1	$x[1] = y[1]$
2	$x[2] + ax[1] = y[2]$
3	$x[3] + ax[2] + a^2x[1] = y[3]$
$\vdots$	$\vdots$

(15)

By induction,

$$y[n] = \sum_{k=1}^{n} a^{n-k} x[k] = \sum_{k=0}^{n} a^{n-k} x[k] - a^n x[0] \tag{16}$$

which no longer makes the system time-invariant.

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$$y[n] = h[n] * (\cos[\omega_0 n]x[n]) = \sum_{k\geqslant 0} \frac{1}{1+k} \cos[\omega_0(n-k)]x[n-k] \tag{17}$$

4.a

Consider the inputs $x[n] = \delta[n]$ and $n_0 = 1$,

$$T\{\delta[n]\} = h[n] * (\cos[\omega_0 n]\delta[n]) = \begin{cases} 0 & n < 0 \\ \frac{1}{1+n}\cos[\omega_0 n] & n \geqslant 0 \end{cases} \tag{18}$$

$$T\{\delta[n-1]\} = h[n-1] * (\cos[\omega_0 n - \omega_0]\delta[n-1]) \tag{19}$$

$$= h[n-2]\cos[\omega_0 n - \omega_0] = \begin{cases} 0 & n < 2 \\ \frac{1}{n-1}\cos[\omega_0 n - \omega_0] & n \geqslant 2 \end{cases} \neq y[n-1] \tag{20}$$

So the system is *not* time-invariant.

4.b

The system is not BIBO stable, since if $|x[n]| < B_x$, then

$$|y[n]| = \sum_{k\geqslant 0} \frac{1}{1+k} B_x \not< \infty \tag{21}$$

due to the divergence of the harmonic series.

4.c

The system is causal, since the summation in the convolution with $h[n]$ only relies on past values of $x[n]$.

5

5.a

We can model the frequency response as

$$H(e^{j\omega}) = \text{rect}\left(\frac{\omega}{\pi}\right) \tag{22}$$

Finding the inverse Fourier transform,

$$\begin{aligned} h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega & (23)\\ &= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{j\omega n}\,d\omega = \frac{1}{2\pi}\left.\frac{e^{j\omega n}}{jn}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} & (24)\\ &= \frac{1}{2\pi jn}\left(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right) = \frac{\sin\left[\frac{\pi}{2}n\right]}{\pi n} & (25) \end{aligned}$$

5.b

The system is not causal, since $h[-1] = \frac{1}{\pi} \neq 0$.

5.c

n	$h[n]$
0	$\frac{1}{\pi}$
1	$\frac{1}{\pi}$
3	$-\frac{1}{3\pi}$
5	$\frac{1}{5\pi}$
$\vdots$	$\vdots$

(26)

The system is BIBO stable, since

$$\sum_{k=-\infty}^{\infty} |h[k]| = \frac{1}{\pi} + \frac{2}{\pi}\underbrace{\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)}_{\frac{\pi}{4}} = \frac{1}{\pi} + \frac{1}{2} < \infty \tag{27}$$

where we use the evenness of the sinc function and the Leibniz formula for π.

6

$$y[n] = \frac{1}{6}\sum_{k=0}^{5} x[n-k] \tag{28}$$

6.a

$$T\{e^{j\omega n}\} = \frac{1}{6}\sum_{k=0}^{5} e^{j\omega(n-k)} \tag{29}$$

$$= \frac{1}{6}\sum_{k=0}^{5} e^{j\omega n}e^{-j\omega k} = e^{j\omega n}\left(\frac{1}{6}\sum_{k=0}^{5} e^{-j\omega k}\right) \tag{30}$$

$$\implies H(e^{j\omega}) = \frac{1}{6}\sum_{k=0}^{5} e^{-j\omega k} \tag{31}$$

Using Equation 2.123 from Oppenheim & Schafer with $M_2 = 5$,

$$H(e^{j\omega}) = \frac{1}{6}\frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)}e^{-j\frac{5}{2}\omega} \tag{32}$$

$$|H(e^{j\omega})| = \frac{1}{6}\left|\frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)}\right| \tag{33}$$

$$\angle H(e^{j\omega}) = \angle\sin(3\omega) - \angle\sin\left(\frac{\omega}{2}\right) - \frac{5}{2}\omega \tag{34}$$

The zero crossings are $\omega = \frac{k\pi}{3}$ for $k \in \mathbb{Z}$.

Magnitude of Frequency Response