EE 123 HW 04

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3

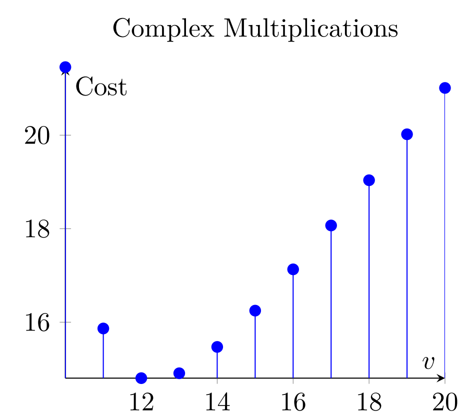
3.a

Each circular convolution in the overlap-save method will result in a length $2^v - P + 1$ signal. Then, the cost of the FFT and IFFT is $\frac{2^v}{2}\log_2(2^v) = v2^{v-1}$. Then, there is a 2^v-pointwise multiplication. The total number of multiplications is $2^v(v+1)$. Thus, the FFT for each sample will require

$$\frac{2^v(v+1)}{2^v - P + 1}. \tag{1}$$

complex multiplications per output sample.

3.b



with a minimum cost of $v = 12$. The direct evaluation would cost 500 complex multiplications per output sample, since that is the length of a given sample.

3.c

$$\lim_{v\to\infty} \frac{2^v(v+1)}{2^v - P + 1} = \lim_{v\to\infty} \frac{v+1}{1 - \left(\frac{P-1}{2^v}\right)} = v \tag{2}$$

Thus, for $P = 500$, the direct method will be more efficient for $v > 500$.

4 Fun with FFT

a. For $0 \leqslant k < N$, $H[k] = X_r[k] + W_{2N}^k X_i[k]$.

b. For $0 \leqslant k < N$, $H[k+N] = X_r[k] - W_{2N}^k X_i[k]$.

c. $X[k] = \frac{1}{2}(H[k] + H[k+N]) + \frac{j}{2} W_{2N}^{-k}(H[k] - H[k+N])$, which takes 3 multiplications and 3 additions.

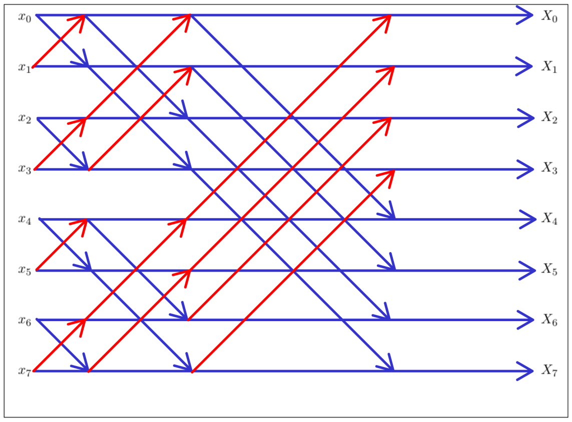
5 Hadamard Transform

5.a

$$H_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \tag{3}$$

The order that represents increasing frequency content is the sequency ordering.

5.b



6

6.a

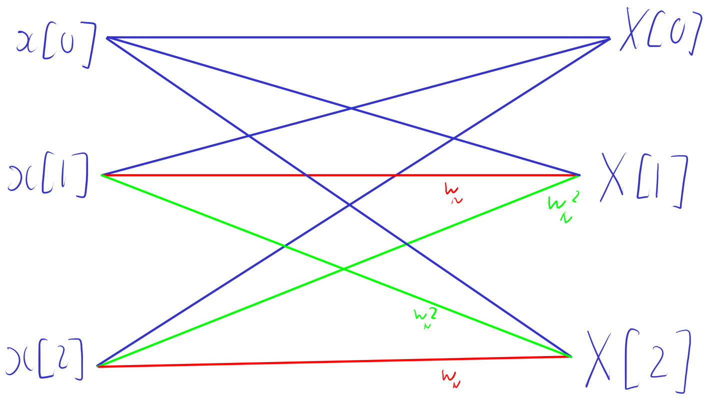
$$
\begin{aligned}
X[3k] &= \sum_{n=0}^{N-1} x[n] W_N^{3kn} && (4) \\
&= \sum_{n=0}^{\frac{N}{3}-1} x[n] W_N^{3kn} + \sum_{n=\frac{N}{3}}^{\frac{2N}{3}-1} x[n] W_N^{3kn} + \sum_{n=\frac{2N}{3}}^{N-1} x[n] W_N^{3kn} && (5) \\
&= \sum_{n=0}^{\frac{N}{3}-1} x[n] W_N^{3kn} + \sum_{n=0}^{\frac{N}{3}-1} x\left[n+\frac{N}{3}\right] W_N^{3kn} + \sum_{n=0}^{\frac{N}{3}-1} x\left[n+\frac{2N}{3}\right] W_N^{3kn} && (6) \\
&= \sum_{n=0}^{\frac{N}{3}-1} \left(x[n] + x\left[n+\frac{N}{3}\right] + x\left[n+\frac{2N}{3}\right]\right) W_N^{3kn} && (7) \\
&= \sum_{n=0}^{\frac{N}{3}-1} \underbrace{\left(x[n] + x\left[n+\frac{N}{3}\right] + x\left[n+\frac{2N}{3}\right]\right)}_{x_1[n]} W_{\frac{N}{3}}^{kn} && (8)
\end{aligned}
$$

6.b

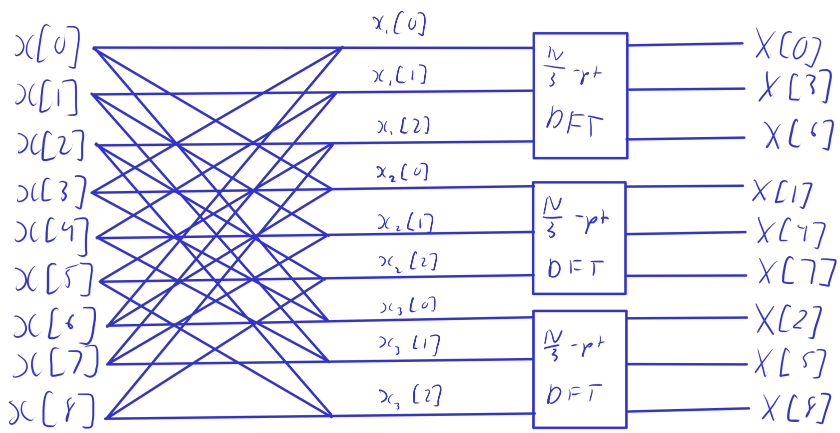
$$
\begin{aligned}
X[3k+1] &= \sum_{n=0}^{N-1} x[n] W_N^{n(3k+1)} && (9)\\
&= \sum_{n=0}^{\frac{N}{3}-1} x[n] W_N^{n(3k+1)} + \sum_{n=\frac{N}{3}}^{\frac{2N}{3}-1} x[n] W_N^{n(3k+1)} + \sum_{n=\frac{2N}{3}}^{N-1} x[n] W_N^{n(3k+1)} && (10)\\
&= \sum_{n=0}^{\frac{N}{3}-1} x[n] W_N^{n(3k+1)} + \sum_{n=0}^{\frac{N}{3}-1} x\left[n+\frac{N}{3}\right] W_N^{\frac{N}{3}} W_N^{n(3k+1)} + \sum_{n=0}^{\frac{N}{3}-1} x\left[n+\frac{2N}{3}\right] W_N^{\frac{2N}{3}} W_N^{n(3k+1)} && (11)\\
&= \sum_{n=0}^{\frac{N}{3}-1} \left(x[n] + x\left[n+\frac{N}{3}\right] W_N^{\frac{N}{3}} + x\left[n+\frac{2N}{3}\right] W_N^{\frac{2N}{3}}\right) W_N^{n(3k+1)} && (12)\\
&= \sum_{n=0}^{\frac{N}{3}-1} \underbrace{\left(x[n] + x\left[n+\frac{N}{3}\right] W_N^{\frac{N}{3}} + x\left[n+\frac{2N}{3}\right] W_N^{\frac{2N}{3}}\right) W_N^{n}}_{x_2[n]} W_{\frac{N}{3}}^{kn} && (13)
\end{aligned}
$$

$$
\begin{aligned}
X[3k+2] &= \sum_{n=0}^{N-1} x[n] W_N^{n(3k+2)} && (14)\\
&= \sum_{n=0}^{\frac{N}{3}-1} x[n] W_N^{n(3k+2)} + \sum_{n=\frac{N}{3}}^{\frac{2N}{3}-1} x[n] W_N^{n(3k+2)} + \sum_{n=\frac{2N}{3}}^{N-1} x[n] W_N^{n(3k+2)} && (15)\\
&= \sum_{n=0}^{\frac{N}{3}-1} x[n] W_N^{n(3k+2)} + \sum_{n=0}^{\frac{N}{3}-1} x\left[n+\frac{N}{3}\right] W_N^{\frac{2N}{3}} W_N^{n(3k+2)} + \sum_{n=0}^{\frac{N}{3}-1} x\left[n+\frac{2N}{3}\right] W_N^{\frac{4N}{3}} W_N^{n(3k+2)} && (16)\\
&= \sum_{n=0}^{\frac{N}{3}-1} \left(x[n] + x\left[n+\frac{N}{3}\right] W_N^{\frac{2N}{3}} + x\left[n+\frac{2N}{3}\right] W_N^{\frac{4N}{3}}\right) W_N^{n(3k+2)} && (17)\\
&= \sum_{n=0}^{\frac{N}{3}-1} \underbrace{\left(x[n] + x\left[n+\frac{N}{3}\right] W_N^{\frac{N}{3}} + x\left[n+\frac{2N}{3}\right] W_N^{\frac{2N}{3}}\right) W_N^{2n}}_{x_3[n]} W_{\frac{N}{3}}^{kn} && (18)
\end{aligned}
$$

6.c



6.d



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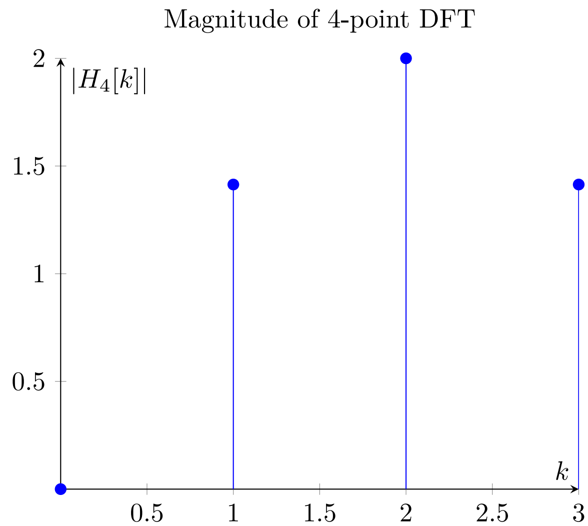
a. $|X[k]| \leqslant N$ for $k = 0$.

b. We want $x[n]$ to be a constant under the DFT, so we can cancel out the complex exponential terms to obtain $x[n] = e^{j\theta} W_N^{-kn}$ for all $\theta \in \mathbb{R}$, and $k, n \in \mathbb{Z}$.

8

8.a

$$H_4[k] = \sum_{k=0}^{3} h[n]W_4^{kn} = 1 - W_4^k = 1 - (-j)^k = \{0, 1+j, 2, 1-j\} \tag{19}$$



The DFT is not even, not odd, is conjugate symmetric. The DFT is a high-pass filter since it lets in $\omega = \pi$, which is the highest frequency, and blocks out $\omega = 0$, the DC frequency.

8.b

It is not possible to uniquely identify $x[n]$ since the expression

$$X[k] = \frac{Y[k]}{H[k]} \tag{20}$$

involves $H[0] = 0$, so the expression is undefined at $k = 0$, meaning that

$$X[k] = \begin{cases} C & k \equiv 0 \pmod 4 \\ \frac{Y[k]}{1-(-j)^k} & k \in [1,3] \pmod 4 \end{cases} \tag{21}$$

for some $C \in \mathbb{C}$.

8.c

Using Parseval's theorem for the DFT,

$$\sum_{n=0}^{3} |x[n]|^2 = \frac{1}{4} \sum_{k=0}^{3} |X[k]|^2 = D \tag{22}$$

$$= \frac{1}{4} \left(|X[0]|^2 + |X[1]|^2 + |X[2]|^2 + |X[3]|^2 \right) \tag{23}$$

$$\implies X[k] = \begin{cases} \pm\sqrt{4D - \frac{1}{\sqrt{2}}|Y[1]|^2 - \frac{1}{2}|Y[2]|^2 - \frac{1}{\sqrt{2}}|Y[3]|^2} & k \equiv 0 \pmod 4 \\ \frac{Y[k]}{1-(-j)^k} & k \in [1,3] \pmod 4 \end{cases} \tag{24}$$

8.d

Using the frequency shift property of the DFT,

$$\tilde{Y}[k] = X[k]\tilde{H}[(k+1)_N] \tag{25}$$

Assuming nothing else about $x[n]$,

$$X[k] = \begin{cases} \frac{Y[k]}{1-(-j)^{k+1}} & k \in [0,2] \pmod 4 \\ C & k \equiv 3 \pmod 4 \end{cases} \tag{26}$$

for $C \in \mathbb{C}$. Assuming the sum holds from the previous part,

$$X[k] = \begin{cases} \frac{Y[k]}{1-(-j)^{k+1}} & k \in [0,2] \pmod 4 \\ \pm\sqrt{4D - \frac{1}{\sqrt{2}}|Y[0]|^2 - \frac{1}{2}|Y[1]|^2 - \frac{1}{\sqrt{2}}|Y[2]|^2} & k \equiv 3 \pmod 4 \end{cases} \tag{27}$$