

EE 123 HW 02

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$$a^n u[n] \iff \frac{1}{1 - ae^{-j\omega}}, \quad |a| < 1 \quad (1)$$

3.a

$$x[n] = -b^n u[-n - 1] = \begin{cases} -b^n & n \leq -1 \\ 0 & n \geq 0 \end{cases} \quad (2)$$

Using the definition of the DTFT,

$$X(e^{j\omega}) = \sum_{k \in \mathbb{Z}} -b^k u[-k - 1] e^{-j\omega k} \quad (3)$$

$$= - \sum_{k \leq -1} b^k e^{-j\omega k} \quad (4)$$

Letting $k' = -k$,

$$X(e^{j\omega}) = \sum_{k' \geq 1} -b^{-k'} e^{j\omega k'} \quad (5)$$

$$= \sum_{k \geq 0} -(b^{-1} e^{j\omega})^k - 1 \quad (6)$$

$$= 1 - \frac{1}{1 - be^{j\omega}} \quad (7)$$

$$= \frac{1 - be^{j\omega} - 1}{1 - be^{j\omega}} \cdot \frac{-be^{-j\omega}}{-be^{-j\omega}} \quad (8)$$

$$= \frac{1}{1 - be^{-j\omega}} \quad (9)$$

where $|b^{-1}| < 1 \implies |b| > 1$.

3.b

$$Y(e^{j\omega}) = 2e^{-j\omega} \frac{1}{1 - (-2)e^{-j\omega}} \quad (10)$$

$$\xrightarrow{\mathcal{F}^{-1}} y[n] = 2(-(-2)^{n-1} u[-(n-1) - 1]) \quad (11)$$

$$= (-2)^n u[-n] \quad (12)$$

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$$H(z) = \frac{1 - z^{-1}}{1 - 0.25z^{-2}} = \frac{1 - z^{-1}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} \quad (13)$$

4.a

Using the Z-transform multiplication property,

$$Y(z) = H(z)X(z) = \frac{\cancel{1} \cancel{z^{-1}}}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})\cancel{(1 - z^{-1})}} = \frac{1}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} \quad (14)$$

Then, using partial fraction decomposition,

$$Y(z) = \frac{1}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})} = \frac{A}{1 - 0.5z^{-1}} + \frac{B}{1 + 0.5z^{-1}} \quad (15)$$

$$= 1 = A(1 + 0.5z^{-1}) + B(1 - 0.5z^{-1}) \quad (16)$$

$$(17)$$

Letting $z = -0.5$, we get $B = \frac{1}{2}$. Letting $z = 0.5$, we get $A = \frac{1}{2}$. Then,

$$Y(z) = \frac{1}{2} \frac{1}{1 - 0.5z^{-1}} + \frac{1}{2} \frac{1}{1 + 0.5z^{-1}} \quad (18)$$

$$\xRightarrow{\mathcal{Z}^{-1}} y[n] = \frac{1}{2} \left(\frac{1}{2}\right)^n u[n] + \frac{1}{2} \left(-\frac{1}{2}\right)^n u[n] \quad (19)$$

4.b

$$y[n] = \delta[n] - \delta[n - 1] \xRightarrow{\mathcal{Z}} Y(z) = 1 - z^{-1} = H(z)X(z) \quad (20)$$

meaning that $X(z) = 1 - 0.25z^{-2} = (1 - 0.5z^{-1})(1 + 0.5z^{-1})$. By the convolution property of the Z-transform,

$$x[n] = \left(\left(\frac{1}{2}\right)^n u[n]\right) * \left(\left(-\frac{1}{2}\right)^n u[n]\right) \quad (21)$$

4.c

Treating the input as the real part of $z = e^{j0.5\pi}$,

$$H(e^{j0.5\pi}) = \frac{1 - e^{-j0.5\pi}}{1 - 0.25e^{-j\pi}} = 0.8\sqrt{2}e^{j\frac{\pi}{4}} \quad (22)$$

Meaning that the final output is

$$y[n] = 0.8\sqrt{2} \cos\left(0.5\pi n + \frac{\pi}{4}\right) \quad (23)$$

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$$x[n] = -\frac{1}{3} \left(\frac{1}{2}\right)^n u[n] - \frac{4}{3} 2^n u[-n-1] \quad (24)$$

$$Y(z) = \frac{1 - z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})} \quad (25)$$

5.a

$$X(z) = -\frac{1}{3} \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{4}{3} \frac{1}{1 - 2z^{-1}} \quad (26)$$

5.b

$$R_y : \frac{1}{2} < |z| < 2 \quad (27)$$

5.c

Simplifying $X(z)$,

$$X(z) = \frac{1 - (1 - 2z^{-1}) + 4\left(1 - \frac{1}{2}z^{-1}\right)}{3 \left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})} \quad (28)$$

$$= \frac{1 - 1 + \cancel{2z^{-1}} + 4 - \cancel{2z^{-1}}}{3 \left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})} \quad (29)$$

$$= \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})} \implies Y(z) = (1 - z^{-2})X(z) = X(z) - z^{-2}X(z) \quad (30)$$

$$\xRightarrow{\mathcal{Z}^{-1}} y[n] = x[n] - x[n-2] \quad (31)$$

5.d

$$H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-2} \xRightarrow{\mathcal{Z}^{-1}} h[n] = \delta[n] - \delta[n-2] \quad (32)$$

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$$X(z) = \frac{z^{-2}}{1 - 2.3z^{-1} + 1.6z^{-2} - 0.3z^{-3}} = \frac{2.04}{1 + 0.3z^{-1}} - \frac{3.47}{1 - z^{-1}} + \frac{1.43}{(1 - z^{-1})^2} \quad (33)$$

$$\stackrel{\mathcal{Z}^{-1}}{\Longrightarrow} x[n] = (-0.3)^n u[n] - 3.47u[n] + (n+1)u[n+1] \quad (34)$$

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- a. System A, causal, stable, ROC: $|z| > 0.9$
- b. System B, non-causal, stable, ROC: $|z| < 1.111$
- c. System A, non-causal, unstable, ROC: $|z| < 0.9$

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$$Z(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) d\theta \quad (35)$$

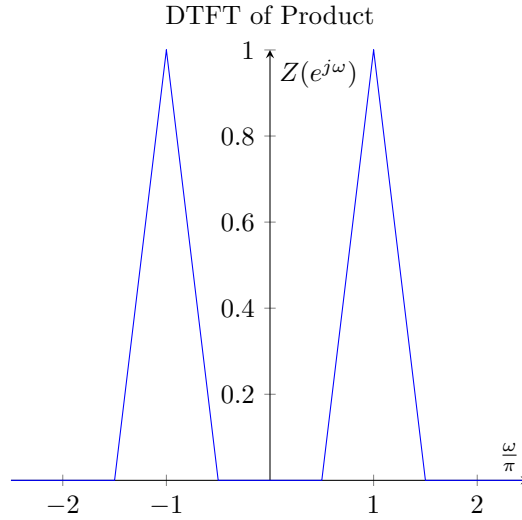
8.a

$$Y(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - \pi + 2\pi k) + \delta(\omega + \pi + 2\pi k) \quad (36)$$

$$X(e^{j\omega}) * Y(e^{j\omega}) = \frac{1}{2} \int_{-\pi}^{\pi} X(e^{j\theta}) \sum_{k \in \mathbb{Z}} \delta((\omega - \theta) - \pi + 2\pi k) + \delta((\omega - \theta) + \pi + 2\pi k) d\theta \quad (37)$$

$$= \frac{1}{2} \sum_{k \in \mathbb{Z}} \int_{-\pi}^{\pi} X(e^{j\omega}) (\delta(\omega - \theta - \pi + 2\pi k) + \delta(\omega + \theta + \pi + 2\pi k)) d\theta \quad (38)$$

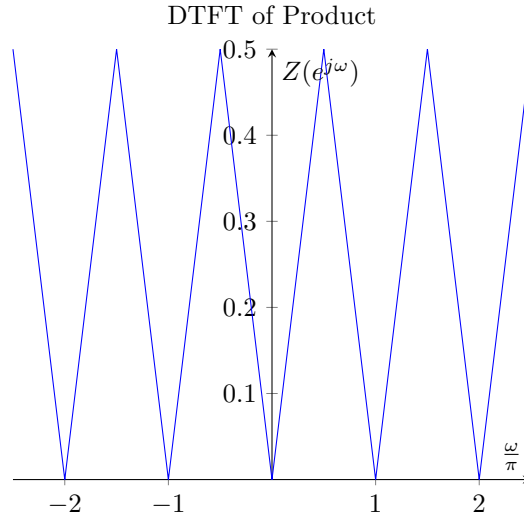
$$= \frac{1}{2} \sum_{k \in \mathbb{Z}} (X(e^{j(\omega - \pi + 2\pi k)}) + X(e^{j(\omega + \pi + 2\pi k)})) \quad (39)$$



8.b

$$Y(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi}{2} + 2\pi k\right) + \delta\left(\omega + \frac{\pi}{2} + 2\pi k\right) \quad (40)$$

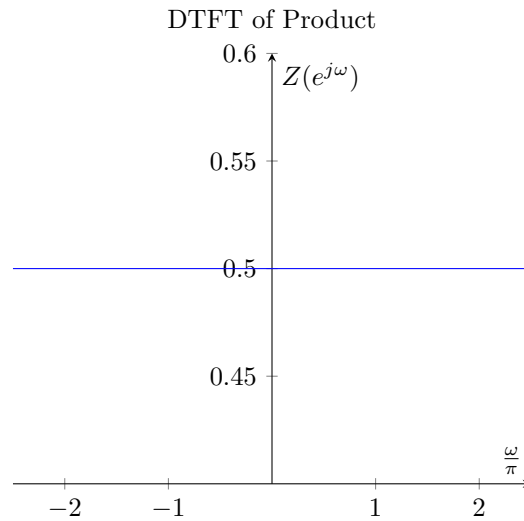
$$X(e^{j\omega}) * Y(e^{j\omega}) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \left(X\left(e^{j\left(\omega - \frac{\pi}{2} + 2\pi k\right)}\right) + X\left(e^{j\left(\omega + \frac{\pi}{2} + 2\pi k\right)}\right) \right) \quad (41)$$



8.c

$$Y(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{\pi}{4} + 2\pi k\right) + \delta\left(\omega + \frac{\pi}{4} + 2\pi k\right) + \delta\left(\omega - \frac{3\pi}{4} + 2\pi k\right) \delta\left(\omega + \frac{3\pi}{4} + 2\pi k\right) \quad (42)$$

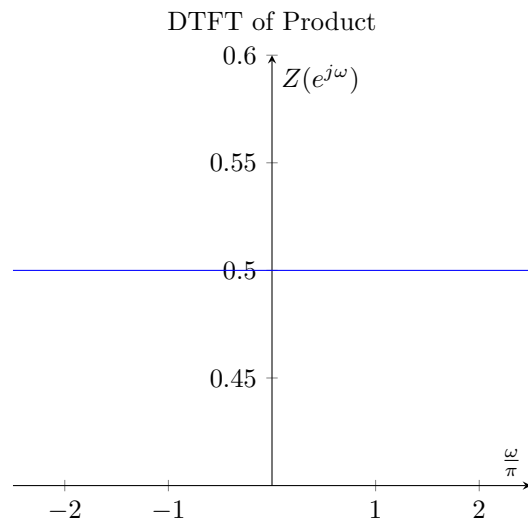
$$X(e^{j\omega}) * Y(e^{j\omega}) = \frac{1}{2} \sum_{k \in \mathbb{Z}} \left(X\left(e^{j\left(\omega - \frac{\pi}{2} + 2\pi k\right)}\right) + X\left(e^{j\left(\omega + \frac{\pi}{2} + 2\pi k\right)}\right) \right) \quad (43)$$



8.d

$$Y(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} e^{j\frac{\pi}{2}} \delta(\omega - \pi + 2\pi k) + e^{-j\frac{\pi}{2}} \delta(\omega + \pi + 2\pi k) \quad (44)$$

$$X(e^{j\omega}) * Y(e^{j\omega}) = \frac{1}{2} \sum_{k \in \mathbb{Z}} (X(e^{j(\omega - \pi + 2\pi k)}) - X(e^{j(\omega + \pi + 2\pi k)})) \quad (45)$$



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$$H(e^{j\omega}) = \begin{cases} j & \omega \in (-\pi, 0) \\ 0 & \omega = 0 \\ -j & \omega \in (0, \pi) \end{cases} \quad (46)$$

9.a

The Hilbert filter is odd-symmetric, Hermitian, and complex.

9.b

$$X(e^{j\omega}) = \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)) \quad (47)$$

$$Y(e^{j\omega}) = j\pi(-\delta(\omega - \omega_0) + \delta(\omega_0)) \quad (48)$$

$$\xRightarrow{\mathcal{F}^{-1}} y[n] = 2 \sin(\omega_0 n) \quad (49)$$

9.c

$$Y(e^{j\omega}) = H(e^{j\omega})^2 X(e^{j\omega}) = \begin{cases} -X(e^{j\omega}) & \omega \in (-\pi, 0) \\ 0 & \omega = 0 \\ X(e^{j\omega}) & \omega \in (0, \pi) \end{cases} = X(e^{j\omega})(u(\omega) - u(\omega)) \quad (50)$$

$$\xRightarrow{\mathcal{F}^{-1}} y[n] = x[n] * \quad (51)$$