EE 123 HW 03

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3

3.a

$y[n]$ is

- $y[n \geqslant L] = 0$ since an autocorrelation implies that the range of nonzero values is $[-L-1, L-1]$.
- Conjugate-symmetric since the frequency domain is purely real.
- Odd length since the multiplication of two length L signals in the frequency domain is the convolution, which creates a length $2L-1$.

3.b

Due to the symmetry and convolution properties of the DFT,

$$\bar{y}[n] = x[n] \textcircled{N} x^*[-n] \tag{1}$$

where we want

$$y[n] = \bar{y}[m[n]] = x[n] * x^*[-n] \tag{2}$$

Let $N = 2L - 1$ so that we can capture the entire range of the circular autocorrelation. Since circular convolution retains the same indices as the original signals, we now have a signal with range $[0, 2N-2]$. Thus, we must time shift by $m[n] = n - L - 1$ to correct the range of the signal.

4

$$\begin{array}{||c|c||}
k & X[k] \\ \hline
0 & 3 \\
2 & 0.5 - 4.5j \\
4 & 5 \\
5 & 3.5 + 3.5j \\
7 & -2.5 - 7j
\end{array} \tag{3}$$

4.a

Using the symmetry properties of the DFT, we can complete the DFT as shown:

$$\begin{array}{||c|c||}
k & X[k] \\ \hline
0 & 3 \\
1 & -2.5 + 7j \\
2 & 0.5 - 4.5j \\
3 & 3.5 - 3.5j \\
4 & 5 \\
5 & 3.5 + 3.5j \\
6 & 0.5 + 4.5j \\
7 & -2.5 - 7j
\end{array} \tag{4}$$

meaning that

$$x[0] = \frac{1}{8}\sum_{k=0}^{N-1} X[k] = \frac{11}{8} \tag{5}$$

4.b

By the circular convolution and time-shift theorems of the DFT,

$$x[n] \, ⑧ \, \delta[n-1] = e^{-j\frac{2\pi}{8}k} X[k] \tag{6}$$

4.c

Since $W[k]$ is simply $X[k]$ evaluated at the even terms, we can represent $w[n]$ as

$$W[k] = X[2k] = \sum_{n=0}^{7} x[n] W_8^{2kn} \tag{7}$$

$$\overset{\mathcal{F}^{-1}}{\Longrightarrow} \; w[n] = \frac{1}{4}\sum_{k=0}^{3} W[k] W_4^{-kn} = \frac{1}{4}\sum_{k=0}^{3} X[2k] W_4^{-kn} \tag{8}$$

$$= \frac{1}{4}\sum_{k=0}^{7} \frac{1}{2}(1 + (-1)^k) X[k] W_8^{-kn} = \frac{1}{8}\sum_{k=0}^{7} (1 + e^{-j\pi k}) X[k] W_8^{-kn} \tag{9}$$

$$= x[n] + \sum_{k=0}^{7} e^{-j\pi k} X[k] W_8^{-kn} \tag{10}$$

$$= x[n] + \sum_{k=0}^{7} W_8^4 X[k] W_8^{-kn} = x[n] + \sum_{k=0}^{7} X[k] W_8^{-kn+4} \tag{11}$$

$$= x[n] + x[(n-4)_8] \tag{12}$$

5 Faster DFTs?

Since $f[n]$ and $g[n]$ are both purely real, then $jg[n]$ must be purely imaginary. By the linearity of the DFT, $H[k] = F[k] + jG[k]$. We can then express $F[k]$ and $G[k]$ as

$$F[k] = \Re\{H[k]\} = \frac{1}{2}(H[k] + H^*[k]) \tag{13}$$

$$G[k] = \Im\{H[k]\} = \frac{1}{2j}(H[k] - H^*[k]) \tag{14}$$

6 Diagonalizing Circulant Matrices

Theorem 1. *The DFT matrix diagonalizes all circulant matrices.*

Proof. Let $\boldsymbol{H}$ be a circulant matrix

$$\boldsymbol{H} = \begin{bmatrix} c_1 & c_2 & \cdots & c_{N-1} & c_N \\ c_N & c_1 & \cdots & c_{N-2} & c_{N-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_N & c_1 \end{bmatrix} = \begin{bmatrix} c[n] \\ c[(n-1)_N] \\ \vdots \\ c[(n-(N-1))_N] \end{bmatrix} \tag{15}$$

and $\boldsymbol{D}$ be the $N \times N$ DFT matrix. Note that the DFT matrix satisfies the following relation: $\boldsymbol{D}^{-1} = \frac{1}{N}\boldsymbol{D}^*$. Then,

$$\boldsymbol{D}^{-1}\boldsymbol{H}\boldsymbol{D} = \frac{1}{N}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & \cdots & e^{j\frac{2\pi}{N}(N-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{j\frac{2\pi}{N}(N-1)} & \cdots & e^{j\frac{2\pi}{n}(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} c[n] \\ c[(n-1)_N] \\ \vdots \\ c[(n-(N-1))_N] \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{n}} & \cdots & e^{-j\frac{2\pi}{n}(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{n}(n-1)} & \cdots & e^{-j\frac{2\pi}{n}(n-1)(n-1)} \end{bmatrix} \tag{16}$$

$$= \frac{1}{N}\begin{bmatrix} C[k] \\ W_N^{-1}C[k] \\ \vdots \\ W_N^{-(N-1)}C[k] \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{n}} & \cdots & e^{-j\frac{2\pi}{n}(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{-j\frac{2\pi}{n}(n-1)} & \cdots & e^{-j\frac{2\pi}{n}(n-1)(n-1)} \end{bmatrix} \tag{17}$$

$$= \frac{1}{N}\begin{bmatrix} NC[0] & 0 & \cdots & 0 \\ 0 & NC[0] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & NC[0] \end{bmatrix} \tag{18}$$

which is diagonal. $\square$