Homework 01

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2

2.a

Theorem 1. *The system L is not time invariant.*

Proof. We can represent a time-delayed signal as a linear combination of the other 3 signals. We can represent the 3 input signals in a matrix

$$\boldsymbol{X} = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \tag{1}$$

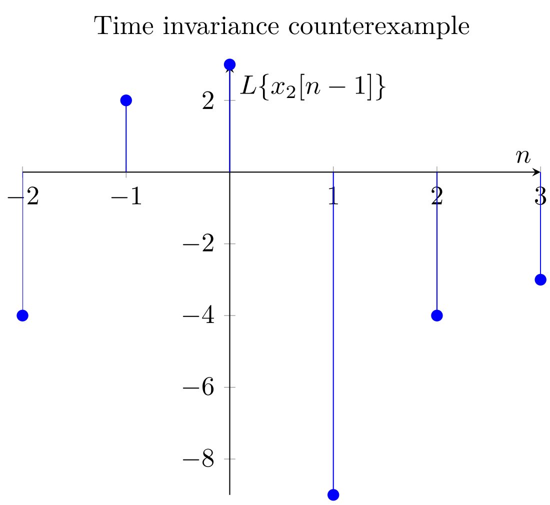
Then, we can find the linear combination necessary to create a time delayed signal (for example, $x_2[n-1]$) by solving the system of equations

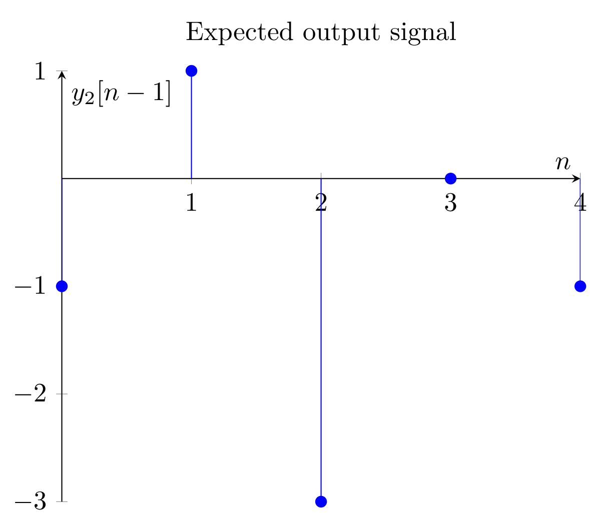
$$\boldsymbol{X}\boldsymbol{a} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \tag{2}$$

where we get $\boldsymbol{a} = \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} & -2 \end{bmatrix}^\top$. Thus, we have

$$x_2[n-1] = -\frac{3}{2}x_1[n] + \frac{3}{2}x_2[n] - 2x_3[n] \tag{3}$$

Calculating $-\frac{3}{2}y_1[n] + \frac{3}{2}y_2[n] - 2y_3[n]$, we get the following plot:





So the system is *not* time-invariant. □

2.b

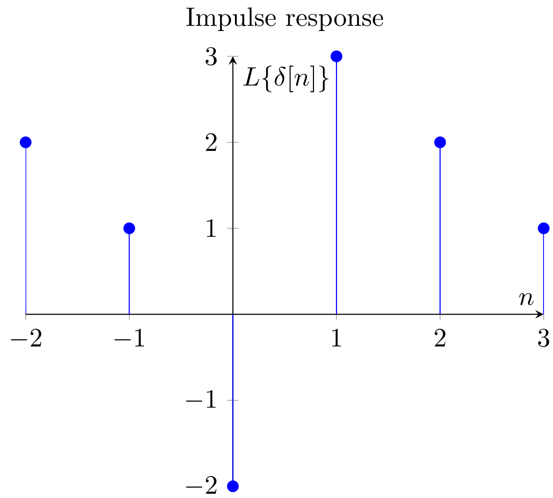
We can represent the delta function as

$$\delta[n] = \frac{1}{2}(x_1[n] - x_2[n] + 2x_3[n]) \tag{4}$$

Using the properties of linear systems, we can find the impulse response

$$h[n] = \frac{1}{2}(y_1[n] - y_2[n] + 2y_3[n]) \tag{5}$$

which gives us the following plot:



3

$$y[n] - ay[n-1] = x[n] \tag{6}$$
$$y[0] = 1 \tag{7}$$

3.a

By counterexample, consider $x[n] = \delta[n]$ and $n = 1$,

$$y[1] = a \tag{8}$$

Then, consider $x[n-1] = \delta[n-1]$ and $n = 1$,

$$y[1] - ay[0] = \delta[0] \implies y[1] = 1 + a \tag{9}$$

So the system is not time-invariant.

3.b

Given the initial condition $y[0] = 1$, our system is affine, but not linear, since all linear systems cross the origin.

3.c

$$\begin{array}{||c|c||} \hline n & y[n] \\ \hline 0 & 0 \\ 1 & x[1] \\ 2 & x[2] + ax[1] \\ 3 & x[3] + ax[2] + a^2x[1] \\ \vdots & \vdots \\ \hline \end{array} \tag{10}$$

By induction,

$$y[n] = \sum_{k=1}^{n} a^{n-k}x[k] = \sum_{k=0}^{n} a^{n-k}x[k] - a^n x[0] \tag{11}$$

The system is still not time-invariant, but it is now linear due to the initial condition $y[0] = 0$.

4

$$y[n] = h[n] * (\cos[\omega_0 n] x[n]) = \sum_{k \geqslant 0} \frac{1}{1+k} \cos[\omega_0 (n-k)] x[n-k] \tag{12}$$

4.a

Proving linearity, suppose we are given the system responses $x_1[n] \iff y_1[n]$ and $x_2[n] \iff y_2[n]$,

$$\begin{aligned} T\{ax_1[n] + bx_2[n]\} &= h[n] * (\cos[\omega_0 n](ax_1[n] + bx_2[n])) & (13)\\ &= h[n] * (\cos[\omega_0 n] a x_1[n] + \cos[\omega_0 n] b x_2[n]) & (14)\\ &= a(h[n] * \cos[\omega_0 n] x_1[n]) + b(h[n] * \cos[\omega_0 n] x_2[n]) & (15)\\ &= a y_1[n] + b y_2[n] & (16) \end{aligned}$$

For time invariance,

$$\begin{aligned} T\{x[n-n_0]\} &= h[n] * (\cos[\omega_0 n] x[n-n_0]) & (17)\\ y[n-n_0] &= h[n-n_0] * (\cos[\omega_0 (n-n_0)] x[n-n_0]) & (18) \end{aligned}$$

The two are clearly not the same, so our system is not time invariant and thus not LTI.

4.b

We can bound the cosine function with 1. Similar to Example 2.18 in Oppenheim & Schafer, if $|x[n]| < B_x$,

$$|y[n]| = \sum_{k \geqslant 0} \frac{1}{1+k} B_x = B_x \sum_{k \geqslant 1} \frac{1}{k} \tag{19}$$

which is a divergent harmonic series.

4.c

The system is causal since the $y[n]$ consists of a memoryless multiplication by the cosine function and convolution with a causal LTI system, which is overall causal.

5

5.a

We can model the frequency response as

$$H(e^{j\omega}) = \begin{cases} 1 & \omega < \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{20}$$

Finding the inverse Fourier transform, we can use Table 2.3 from Oppenheim & Schafer to find that the impulse response is

$$h[n] = \frac{\sin\left[\frac{\pi}{2}n\right]}{\pi n} \tag{21}$$

5.b

The system is not causal, since $h[-1] = \frac{1}{\pi} \neq 0$.

5.c

This is a specific case of Example 2.18 in Oppenheim & Schafer. Each term of the absolute sum of the sinc function only decreases on the order of $\frac{1}{n}$, so we can treat the sum as the harmonic sum, which diverges. Thus, the low pass filter is not BIBO stable.

6

$$y[n] = \frac{1}{6}\sum_{k=0}^{5} x[n-k] \tag{22}$$

6.a

Using Equation 2.123 from Oppenheim & Schafer with $M_2 = 5$,

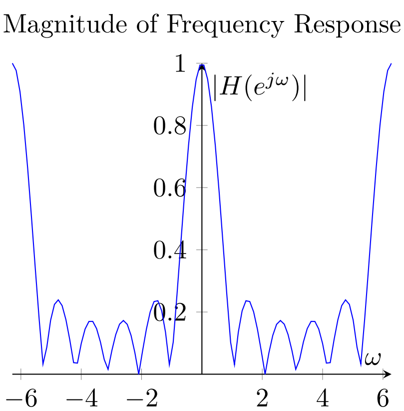
$$H(e^{j\omega}) = \frac{1}{6}\frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)}e^{-j\frac{5}{2}\omega} \tag{23}$$

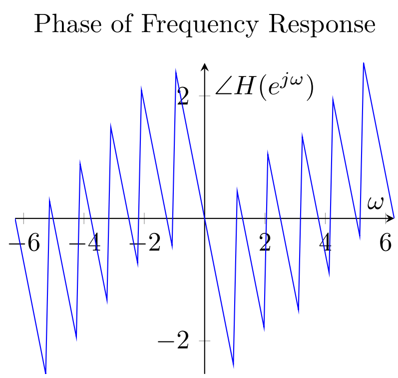
$$|H(e^{j\omega})| = \frac{1}{6}\left|\frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)}\right| \tag{24}$$

$$\angle H(e^{j\omega}) = -\frac{5}{2}\omega \tag{25}$$

$$\tag{26}$$

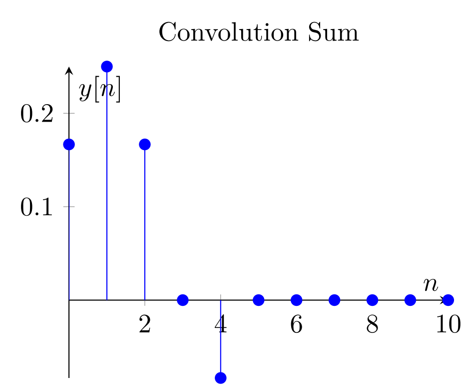
The zero crossings are $\omega = \frac{k\pi}{3}$ for $k \in \mathbb{Z}$, by solving for the equation $\sin(3\omega) = 0$.





6.b

We let $\omega_1 = \frac{\pi}{3}$. Then, the output $y[n]$ is as shown:



The explanation for the transient comes from the initial spike that dominates the MAF.

7

$$y[n] = \mathcal{P}\{x[n-2], x[n-1], x[n], x[n+1], x[n+2]\} \tag{27}$$

7.a

We can transform the problem into a least-squares problem:

$$\begin{bmatrix} x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \tag{28}$$

By the least squares formula $\hat{\boldsymbol{a}} = (\boldsymbol{P}^\top \boldsymbol{P})^{-1} \boldsymbol{P}^\top \boldsymbol{x}$, we get that

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} -3 & 12 & 17 & 12 & -3 \\ -7 & -3.5 & 0 & 3.5 & 7 \\ 5 & -2.5 & -5 & -2.5 & 5 \end{bmatrix} \begin{bmatrix} x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \end{bmatrix} \tag{29}$$

$$\implies y[0] = a_0 = \frac{1}{35}(-3x[-2] + 12x[-1] + 17x[0] + 12x[1] - 3x[2]) \tag{30}$$

7.b

$$\begin{bmatrix} x[-1] \\ x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \tag{31}$$

which gives us the solution

$$y[1] = a_0 = \frac{1}{35}(-3x[-1] + 12x[0] + 17x[1] + 12x[2] - 3x[3]) \tag{32}$$

7.c

The system is linear since it entirely consists of additions and scalar multiplications. The system is time-invariant since for any shift n_0, the corresponding $x[n-n_0]$ is simply shifted and put into the solution, which is what is expected from a time-shifted output. The system is stable since if $x[n]$ is bounded by B_x, $y[n]$ is bounded by B_x. Since the system is LTI, the system has a frequency response. We do not actually need to perform a regression for every n, since the coefficient matrix multiplying the set of $x[n-k]$ is constant.

7.d

We can see that points further from the operating point have the largest weights, while weights further away have smaller and even negative weights. This means that it better represents a distribution at a given range.