# Homework 1

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## 2

### 2.a

**Theorem 1.** *The system $L$ is not time invariant.*

*Proof.* We can represent a time-delayed signal as a linear combination of the other 3 signals. We can represent the 3 input signals in a matrix

$$\boldsymbol{X} = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \tag{1}$$

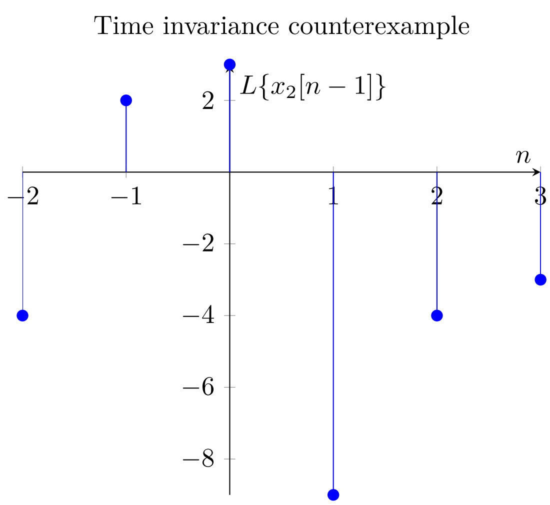
Then, we can find the linear combination necessary to create a time delayed signal (for example, $x_2[n-1]$) by solving the system of equations

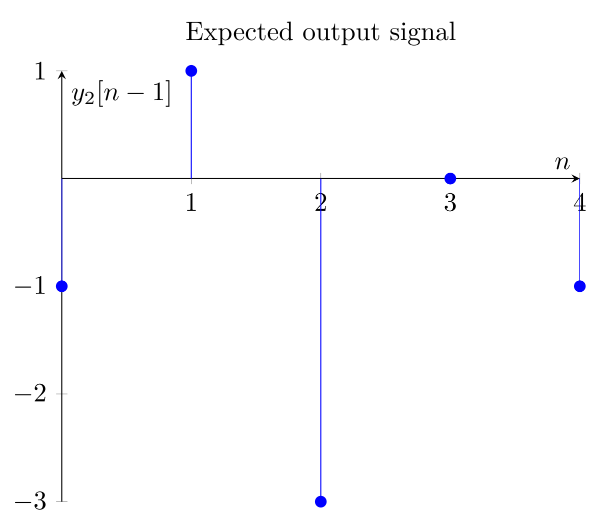
$$\boldsymbol{X}\boldsymbol{a} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \tag{2}$$

where we get $\boldsymbol{a} = \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} & -2 \end{bmatrix}^\top$. Thus, we have

$$x_2[n-1] = -\frac{3}{2}x_1[n] + \frac{3}{2}x_2[n] - 2x_3[n] \tag{3}$$

Calculating $-\frac{3}{2}y_1[n] + \frac{3}{2}y_2[n] - 2y_3[n]$, we get the following plot:




So the system is *not* time-invariant. $\square$

### 2.b

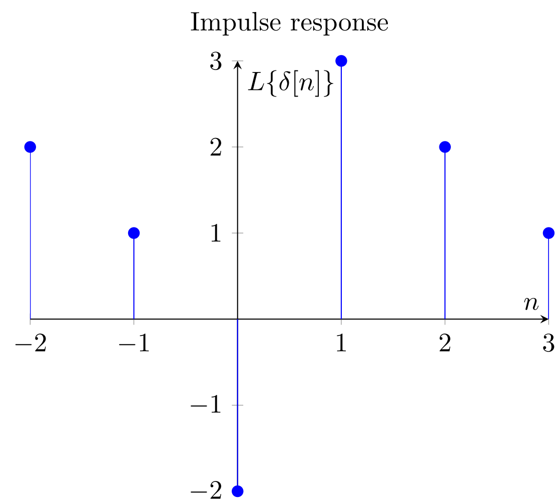
We can represent the delta function as

$$\delta[n] = \frac{1}{2}(x_1[n] - x_2[n] + 2x_3[n]) \tag{4}$$

Using the properties of linear systems, we can find the impulse response

$$h[n] = \frac{1}{2}(y_1[n] - y_2[n] + 2y_3[n]) \tag{5}$$

which gives us the following plot:




## 3

$$y[n] - ay[n-1] = x[n] \tag{6}$$
$$y[0] = 1 \tag{7}$$

### 3.a

For the homogenuous solution, assume that $y_h[n] = A\lambda^n$. Then,

$$A\lambda^n - aA\lambda^{n-1} = 0 \tag{8}$$
$$1 - a\lambda^{-1} = 0 \tag{9}$$
$$\implies \lambda = a \implies y_h[n] = Aa^n \tag{10}$$

Finding the particular solution,

| $n$ | $x[n]$ |
|---|---|
| 0 | 1 |
| 1 | $x[1] + a = y[1]$ |
| 2 | $x[2] + ax[1] + a^2 = y[2]$ |
| 3 | $x[3] + ax[2] + a^2x[1] = y[3]$ |
| $\vdots$ | $\vdots$ |

(11)

### 3.b

### 3.c

## 4

$$y[n] = h[n] * (\cos[\omega_0 n]x[n]) = \sum_{k\geqslant 0} \frac{1}{1+k} \cos[\omega_0(n-k)]x[n-k] \tag{12}$$

### 4.a

**Theorem 2.** *The above system is not LTI.*

*Proof.* Proving linearity, suppose we are given the system responses $x_1[n] \iff y_1[n]$ and $x_2[n] \iff y_2[n]$,

$$\begin{aligned} T\{ax_1[n] + bx_2[n]\} &= \sum_{k\geqslant 0} \frac{1}{1+k} \cos^{\omega_0 n}(ax_1[n-k] + bx_2[n-k]) && (13) \\ &= a\sum_{k\geqslant 0} \frac{1}{1+k} \cos[\omega_0 n]x_1[n-k] + b\sum_{k\geqslant 0} \frac{1}{1+k} \cos[\omega_0 n]x_2[n-k] && (14) \\ &= ay_1[n] + by_2[n] && (15) \end{aligned}$$

Consider the inputs $x[n] = \delta[n]$ and $n_0 = 1$,

$$\begin{aligned} T\{\delta[n]\} = h[n] * (\cos[\omega_0 n]\delta[n]) &= \begin{cases} 0 & n < 0 \\ \frac{1}{1+n} \cos[\omega_0 n] & n \geqslant 0 \end{cases} && (16) \\ T\{\delta[n-1]\} = h[n-1] * (\cos[\omega_0 n - \omega_0]\delta[n-1]) & && (17) \\ = h[n-2] \cos[\omega_0 n - \omega_0] &= \begin{cases} 0 & n < 2 \\ \frac{1}{n-1} \cos[\omega_0 n - \omega_0] & n \geqslant 2 \end{cases} \neq y[n-1] && (18) \\ & && (19) \end{aligned}$$

□

### 4.b

The system is not BIBO stable, since if $|x[n]| < B_x$, then

$$|y[n]| = \sum_{k\geqslant 0} \frac{1}{1+k} B_x \not< \infty \tag{20}$$

due to the divergence of the harmonic series.

### 4.c

The system is causal, since the summation in the convolution with $h[n]$ only relies on past values of $x[n]$.

## 5

### 5.a

We can model the frequency response as

$$H(e^{j\omega}) = \text{rect}\left(\frac{\omega}{\pi}\right) \tag{21}$$

Finding the inverse Fourier transform,

$$
\begin{aligned}
h[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega && (22)\\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{j\omega n}\,d\omega = \frac{1}{2\pi}\left.\frac{e^{j\omega n}}{jn}\right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} && (23)\\
&= \frac{1}{2\pi jn}\left(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}\right) = \frac{\sin\left[\frac{\pi}{2}n\right]}{\pi n} && (24)
\end{aligned}
$$

### 5.b

The system is not causal, since $h[-1] = \frac{1}{\pi} \neq 0$.

### 5.c

| $n$ | $h[n]$ |
|---|---|
| 0 | $\frac{1}{\pi}$ |
| 1 | $\frac{1}{\pi}$ |
| 3 | $-\frac{1}{3\pi}$ |
| 5 | $\frac{1}{5\pi}$ |
| $\vdots$ | $\vdots$ |

(25)

The system is BIBO stable, since

$$
\sum_{k=-\infty}^{\infty} |h[k]| = \frac{1}{\pi} + \frac{2}{\pi}\underbrace{\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)}_{\frac{\pi}{4}} = \frac{1}{\pi} + \frac{1}{2} < \infty \tag{26}
$$

where we use the evenness of the sinc function and the Leibniz formula for $\pi$.

## 6

$$
y[n] = \frac{1}{6}\sum_{k=0}^{5} x[n-k] \tag{27}
$$

### 6.a

$$
\begin{aligned}
T\{e^{j\omega n}\} &= \frac{1}{6}\sum_{k=0}^{5} e^{j\omega(n-k)} && (28)\\
&= \frac{1}{6}\sum_{k=0}^{5} e^{j\omega n}e^{-j\omega k} = e^{j\omega n}\left(\frac{1}{6}\sum_{k=0}^{5} e^{-j\omega k}\right) && (29)\\
\implies H(e^{j\omega}) &= \frac{1}{6}\sum_{k=0}^{5} e^{-j\omega k} && (30)
\end{aligned}
$$

Using Equation 2.123 from Oppenheim & Schafer with $M_2 = 5$,

$$H(e^{j\omega}) = \frac{1}{6} \frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{5}{2}\omega} \tag{31}$$

$$|H(e^{j\omega})| = \frac{1}{6} \left| \frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)} \right| \tag{32}$$

$$\angle H(e^{j\omega}) = \angle \sin(3\omega) - \angle \sin\left(\frac{\omega}{2}\right) - \frac{5}{2}\omega \tag{33}$$

The zero crossings are $\omega = \frac{k\pi}{3}$ for $k \in \mathbb{Z}$.

Magnitude of Frequency Response