# Homework 01

Bryan Ngo

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# 2

## 2.a

**Theorem 1.** *The system $L$ is not time invariant.*

*Proof.* We can represent a time-delayed signal as a linear combination of the other 3 signals. We can represent the 3 input signals in a matrix

$$\boldsymbol{X} = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \tag{1}$$

Then, we can find the linear combination necessary to create a time delayed signal (for example, $x_2[n-1]$) by solving the system of equations

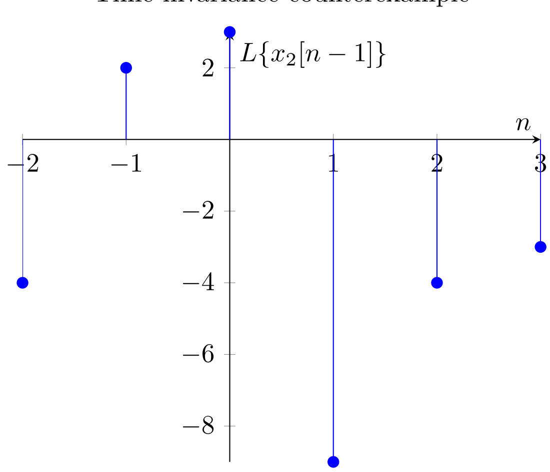
$$\boldsymbol{X}\boldsymbol{a} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \tag{2}$$

where we get $\boldsymbol{a} = \begin{bmatrix} -\frac{3}{2} & \frac{3}{2} & -2 \end{bmatrix}^\top$. Thus, we have

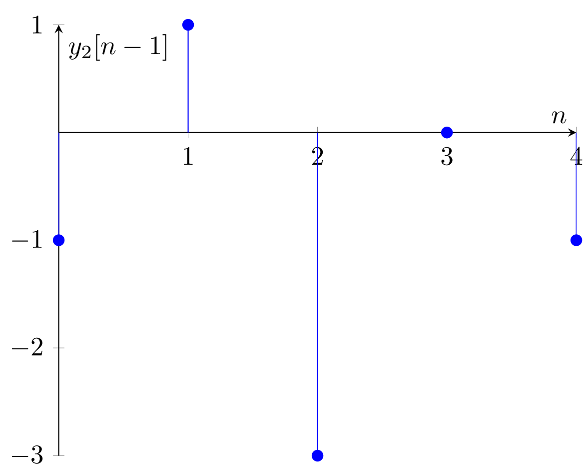
$$x_2[n-1] = -\frac{3}{2}x_1[n] + \frac{3}{2}x_2[n] - 2x_3[n] \tag{3}$$

Calculating $-\frac{3}{2}y_1[n] + \frac{3}{2}y_2[n] - 2y_3[n]$, we get the following plot:

Time invariance counterexample




Expected output signal




So the system is *not* time-invariant. $\square$

## 2.b

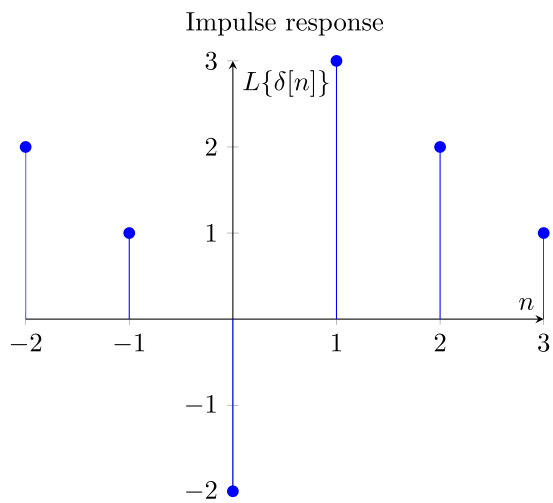
We can represent the delta function as

$$\delta[n] = \frac{1}{2}(x_1[n] - x_2[n] + 2x_3[n]) \tag{4}$$

Using the properties of linear systems, we can find the impulse response

$$h[n] = \frac{1}{2}(y_1[n] - y_2[n] + 2y_3[n]) \tag{5}$$

which gives us the following plot:



# 3

$$y[n] - ay[n-1] = x[n] \tag{6}$$
$$y[0] = 1 \tag{7}$$

## 3.a

By counterexample, consider $x[n] = \delta[n]$ and $n = 1$,

$$y[1] = a \tag{8}$$

Then, consider $x[n-1] = \delta[n-1]$ and $n = 1$,

$$y[1] - ay[0] = \delta[0] \implies y[1] = 1 + a \tag{9}$$

whereas $y[n-1] = y[0] = 1$ by the initial condition. So the system is not time-invariant.

## 3.b

Given the initial condition $y[0] = 1$, scaling the input does not change this fact, so the system *cannot* be linear.

## 3.c

The system is still not time-invariant due to the same counterexample as above. The system is now linear as we can see because we no longer have an affine transformation at $y[0]$.

# 4

$$y[n] = h[n] * (\cos[\omega_0 n]x[n]) = \sum_{k \geqslant 0} \frac{1}{1+k} \cos[\omega_0 (n-k)]x[n-k] \tag{10}$$

## 4.a

Proving linearity, suppose we are given the system responses $x_1[n] \iff y_1[n]$ and $x_2[n] \iff y_2[n]$,

$$\begin{aligned} T\{ax_1[n] + bx_2[n]\} &= h[n] * (\cos[\omega_0 n](ax_1[n] + bx_2[n])) & (11)\\ &= h[n] * (\cos[\omega_0 n]ax_1[n] + \cos[\omega_0 n]bx_2[n]) & (12)\\ &= a(h[n] * \cos[\omega_0 n]x_1[n]) + b(h[n] * \cos[\omega_0 n]x_2[n]) & (13)\\ &= ay_1[n] + by_2[n] & (14) \end{aligned}$$

For time invariance,

$$\begin{aligned} T\{x[n-n_0]\} &= h[n] * (\cos[\omega_0 n]x[n-n_0]) & (15)\\ y[n-n_0] &= h[n-n_0] * (\cos[\omega_0 (n-n_0)]x[n-n_0]) & (16) \end{aligned}$$

The two are clearly not the same, so our system is not time invariant and thus not LTI.

## 4.b

We can bound the cosine function with 1. Similar to Example 2.18 in Oppenheim & Schafer, if $|x[n]| < B_x$,

$$|y[n]| = \sum_{k \geqslant 0} \frac{1}{1+k} B_x = B_x \sum_{k \geqslant 1} \frac{1}{k} \tag{17}$$

which is a divergent harmonic series.

## 4.c

The system is causal since the $y[n]$ consists of a memoryless multiplication by the cosine function and convolution with a causal LTI system, which is overall causal.

# 5

## 5.a

We can model the frequency response as

$$H(e^{j\omega}) = \begin{cases} 1 & \omega < \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{18}$$

Finding the inverse Fourier transform, we can use Table 2.3 from Oppenheim & Schafer to find that the impulse response is

$$h[n] = \frac{\sin\left[\frac{\pi}{2}n\right]}{\pi n} \tag{19}$$

## 5.b

The system is not causal, since $h[-1] = \frac{1}{\pi} \neq 0$.

## 5.c

This is a specific case of Example 2.18 in Oppenheim & Schafer. Each term of the absolute sum of the sinc function only decreases on the order of $\frac{1}{n}$, so we can treat the sum as the harmonic series, which diverges. Thus, the low pass filter is not BIBO stable.

## 6

$$y[n] = \frac{1}{6}\sum_{k=0}^{5} x[n-k] \tag{20}$$

### 6.a

Using Equation 2.123 from Oppenheim & Schafer with $M_2 = 5$,

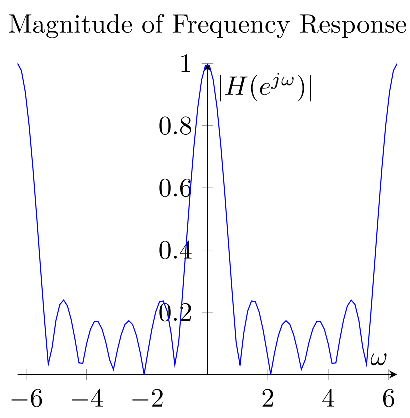
$$H(e^{j\omega}) = \frac{1}{6}\frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\frac{5}{2}\omega} \tag{21}$$

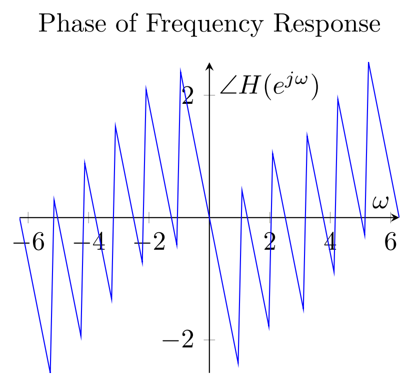
$$|H(e^{j\omega})| = \frac{1}{6}\left|\frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)}\right| \tag{22}$$

$$\angle H(e^{j\omega}) = -\frac{5}{2}\omega \tag{23}$$

$$\tag{24}$$

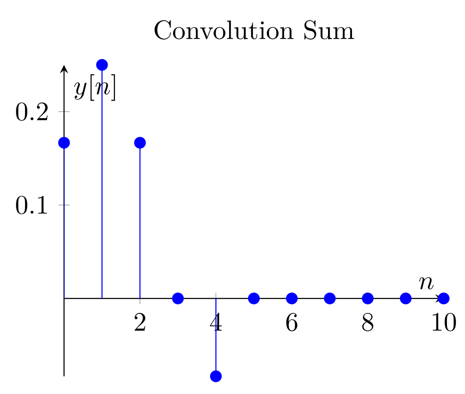
The zero crossings are $\omega = \frac{k\pi}{3}$ for nonzero $k \in \mathbb{Z}$, by solving for the equation $\sin(3\omega) = 0$.




### 6.b

Let $\omega_1 = \frac{\pi}{3}$. Then, the output $y[n]$ is as shown:




The transient at the start of the convolution sum comes from at the beginning of the MAF, where the positive terms from the first 2 inputs of the cosine function. The settling of the convolution sum comes from the fact that the sum of a sinusoid without a vertical shift over 1 period is 0. Since the MAF has a window of size 6, and the cosine has a period of 6 as well, once the window is filled by the cosine, it we get 0. For any value of the convolution sum afterwards, it remains 0 due to the MAF now being saturated with a constant 1 period of the cosine, which has a sum of 0.

# 7

$$y[n] = \mathcal{P}\{x[n-2], x[n-1], x[n], x[n+1], x[n+2]\} \tag{25}$$

## 7.a

We can transform the problem into a least-squares problem:

$$\begin{bmatrix} x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \tag{26}$$

By the least squares formula $\hat{\boldsymbol{a}} = (\boldsymbol{P}^\top \boldsymbol{P})^{-1} \boldsymbol{P}^\top \boldsymbol{x}$, we get that

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} -3 & 12 & 17 & 12 & -3 \\ -7 & -3.5 & 0 & 3.5 & 7 \\ 5 & -2.5 & -5 & -2.5 & 5 \end{bmatrix} \begin{bmatrix} x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \end{bmatrix} \tag{27}$$

$$\implies y[0] = a_0 = \frac{1}{35}(-3x[-2] + 12x[-1] + 17x[0] + 12x[1] - 3x[2]) \tag{28}$$

## 7.b

$$\begin{bmatrix} x[-1] \\ x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \tag{29}$$

which gives us the solution

$$y[1] = a_0 = \frac{1}{35}(-3x[-1] + 12x[0] + 17x[1] + 12x[2] - 3x[3]) \tag{30}$$

## 7.c

The system is linear since it entirely consists of additions and scalar multiplications. The system is time-invariant since for any shift $n_0$, the corresponding $x[n-n_0]$ is simply shifted and put into the solution, which is what is expected from a time-shifted output. The system is stable since if $x[n]$ is bounded by $B_x$, $y[n]$ is bounded by $B_x$. Since the system is LTI, the system has a frequency response. We do not actually need to perform a regression for every $n$, since the coefficient matrix multiplying the set of $x[n-k]$ is constant.

## 7.d

We can see that points further from the operating point have the largest weights, while weights further away have smaller and even negative weights. This means that it better represents a distribution at a given range.