EECS 16B HW00

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1 Tell Us Something You are Proud of this Semester

I am proud to have moved on to 2nd semester.

2 What are You Looking Forward to Over Winter Break?

Meeting old friends and family.

3 Where is the Sound Coming From?

3.a

We can determine that the time delay $\Delta t_1 = 9\,\mathrm{ms}$ and $\Delta t_2 = 11\,\mathrm{ms}$. Then,

$$d_1 = v_s \Delta t_1 = (300\,\mathrm{m\,s^{-1}})(9\,\mathrm{ms}) = 2.7\,\mathrm{m} \tag{1}$$

$$d_2 = v_s \Delta t_2 = (300\,\mathrm{m\,s^{-1}})(11\,\mathrm{ms}) = 3.3\,\mathrm{m} \tag{2}$$

3.b

Using trigonometry,

$$\sin(\alpha) = \frac{d}{\|\boldsymbol{p}_1\|} = \frac{1}{2} \tag{3}$$

making $\alpha = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$ rad.

3.c

Using the trilateration formula

$$2\begin{bmatrix} \boldsymbol{a}_1 - \boldsymbol{a}_2^\top \\ \boldsymbol{a}_1 - \boldsymbol{a}_3^\top \\ \boldsymbol{a}_1 - \boldsymbol{a}_4^\top \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} \|\boldsymbol{a}_1\|^2 - \|\boldsymbol{a}_2\|^2 - d_1^2 + d_2^2 \\ \|\boldsymbol{a}_1\|^2 - \|\boldsymbol{a}_3\|^2 - d_1^2 + d_3^2 \\ \|\boldsymbol{a}_1\|^2 - \|\boldsymbol{a}_4\|^2 - d_1^2 + d_4^2 \end{bmatrix} \tag{4}$$

We then plug in our values,

$$2\begin{bmatrix}\left(\begin{bmatrix}0\\4\end{bmatrix}-\begin{bmatrix}0\\2\end{bmatrix}\right)^\top\\\left(\begin{bmatrix}0\\4\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}\right)^\top\\\left(\begin{bmatrix}0\\4\end{bmatrix}-\begin{bmatrix}0\\0\end{bmatrix}\right)^\top\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}16-4-1+5\\16-1-1+10\\16-0-1+17\end{bmatrix} \tag{5}$$

$$2\begin{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}^\top\\\begin{bmatrix}0\\3\end{bmatrix}^\top\\\begin{bmatrix}0\\4\end{bmatrix}^\top\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}16\\24\\32\end{bmatrix} \tag{6}$$

$$\begin{bmatrix}0&2\\0&3\\0&4\end{bmatrix}\boldsymbol{x}=\begin{bmatrix}8\\12\\16\end{bmatrix} \tag{7}$$

Using least squares,

$$\boldsymbol{A}^\top\boldsymbol{A}=\begin{bmatrix}0&0&0\\2&3&4\end{bmatrix}\begin{bmatrix}0&2\\0&3\\0&4\end{bmatrix}=\begin{bmatrix}0&0\\0&29\end{bmatrix} \tag{8}$$

Since the matrix is non-invertible ($\det(\boldsymbol{A}^\top\boldsymbol{A})=0$), we cannot use least squares to determine the location of the transmitter. An alternative solution would be to place the microphones in non-collinear locations, so as to make the matrix linearly dependent and thus invertible.

4 Building a Classifier

4.a

We construct the least squares problem

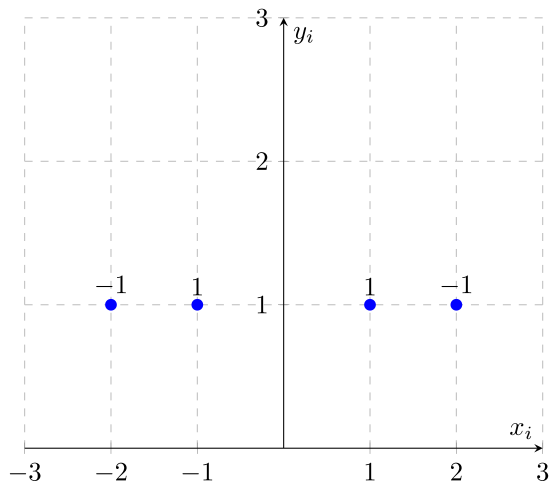
$$\begin{bmatrix}-2&1&1\\-1&1&1\\1&1&1\\2&1&1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\\\gamma\end{bmatrix}=\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix} \tag{9}$$

Using the least squares formula,

$$\boldsymbol{A}^\top\boldsymbol{A}=\begin{bmatrix}-2&-1&1&2\\1&1&1&1\\1&1&1&1\end{bmatrix}\begin{bmatrix}-2&1&1\\-1&1&1\\1&1&1\\2&1&1\end{bmatrix}=\begin{bmatrix}10&0&0\\0&4&4\\0&4&4\end{bmatrix} \tag{10}$$

The problem is unsolvable, since columns 2 and 3 are linearly dependent. This makes $\boldsymbol{A}^\top \boldsymbol{A}$ non-invertible.

4.b



From the diagram, it is clear that it is impossible to draw a line that uniquely classifies the points by label.

4.c

We set up the least squares problem

$$\begin{bmatrix} -2 & 4 \\ -1 & 1 \\ 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \tag{11}$$

Using least squares,

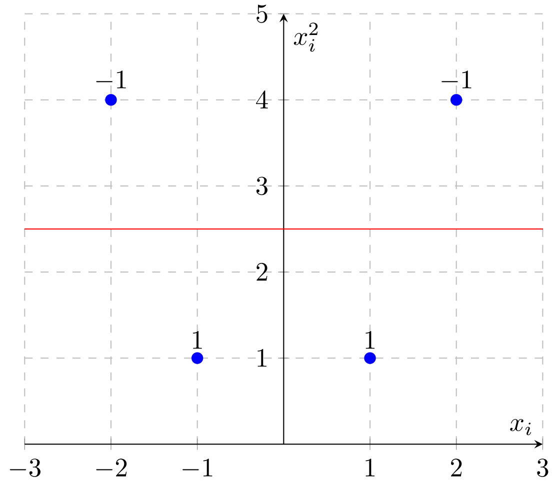
$$\boldsymbol{A}^\top \boldsymbol{A} = \begin{bmatrix} -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ -1 & 1 \\ 1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 34 \end{bmatrix} \overset{-1}{\Rightarrow} \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{34} \end{bmatrix} \tag{12}$$

$$\boldsymbol{A}^\top \boldsymbol{b} = \begin{bmatrix} -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix} \tag{13}$$

$$\hat{\boldsymbol{x}} = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{b} = \begin{bmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{34} \end{bmatrix} \begin{bmatrix} 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{6}{34} \end{bmatrix} \tag{14}$$

$$\ell \approx -\frac{3}{17} x^2 \tag{15}$$

4.d



It is possible to classify the labels using a quadratic regression.

4.e

Using the model $\ell = \alpha x + \beta x^2 + \gamma$ would create a more accurate classifier. This is due to the extra degree of freedom gained by the γ term.

5 Putting on the Pressure: Build Your Own InstantPot

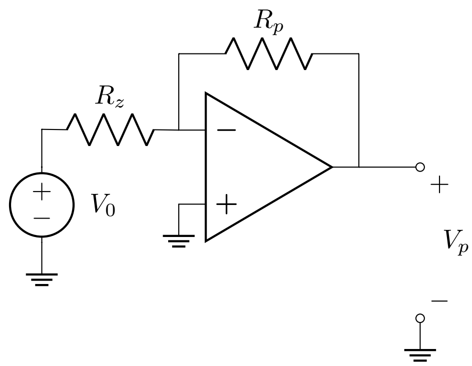
5.a

$$R_p = \frac{\rho L}{A} = \frac{0.1\,\Omega\,\mathrm{m} \cdot 0.01\,\mathrm{m}}{0.001\,\mathrm{m} \cdot 100 \times 10^{-6}\,\mathrm{m}} = 10^4\,\Omega \tag{16}$$

5.b

$$R_p(p_c) = \frac{\rho L(p_c)}{Wt} = \frac{\rho}{Wt}(L_0 + \beta p_c) \tag{17}$$

5.c Pressure Sensor Circuit



Recognizing that this is an inverting op-amp, we can use the formula

$$V_p = -V_0 \frac{R_p}{R_z} \tag{18}$$

and solve for R_z. Plugging in our values of R_p,

$$V_p = -V_0 \frac{R_0 \frac{p_c}{p_{ref}}}{R_z} = -V_0 \frac{p_c}{p_{ref}} \frac{R_0}{R_z} \tag{19}$$

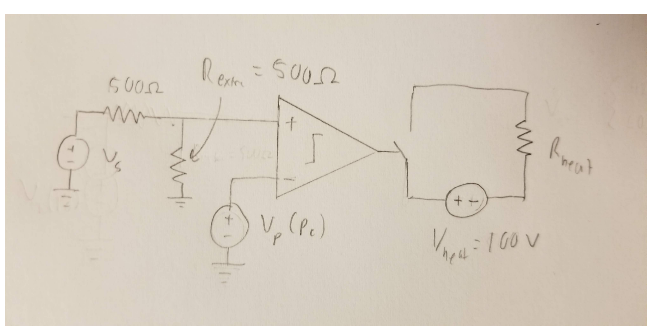
In order to satisfy our desired voltage of $V_p = -V_0 \frac{p_c}{p_{ref}}$, $\frac{R_0}{R_z} = 1$. Thus, $R_z = R_0 = 1\,\mathrm{k\Omega}$.

5.d Resistive Heating Element

$$P = \frac{V_{heat}^2}{R_{heat}} \Rightarrow 1000\,\mathrm{W} = \frac{100\,\mathrm{V}^2}{R_{heat}} \tag{20}$$

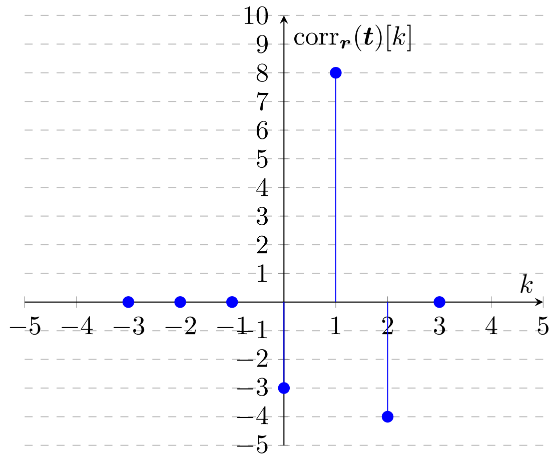
yielding $R_{heat} = 10\,\Omega$.

5.e Pressure Regulation



6 Finding Faults with PG&E

6.a



The index of the maximum correlation is $k = 1$.

6.b

Using OMP, we first find the largest magnitude of the inner product $\langle \boldsymbol{r}, \boldsymbol{u}_i \rangle$, which is $i = 1$. Finding our preliminary "weight",

$$\boldsymbol{A}^\top \boldsymbol{A} = \|\boldsymbol{u}_1\|^2 = 3 \tag{21}$$

$$\boldsymbol{A}^\top \boldsymbol{r} = \langle \boldsymbol{u}_1, \boldsymbol{r} \rangle = 5 \tag{22}$$

$$\hat{\boldsymbol{x}} = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{r} = \frac{5}{3} \tag{23}$$

Finding our new error vector,

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{A}\hat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} - \frac{5}{3} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix} \tag{24}$$

Our new maximum inner product is

i	$\langle \boldsymbol{r}', \boldsymbol{u}_i \rangle$
1	0
2	0
3	$\frac{2}{3}$
4	$\frac{4}{3}$

Then, we use least squares to find the weights,

$$\boldsymbol{A}^\top \boldsymbol{A} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & -1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \xRightarrow{-1} \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \tag{25}$$

$$\boldsymbol{A}^\top \boldsymbol{r} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \tag{26}$$

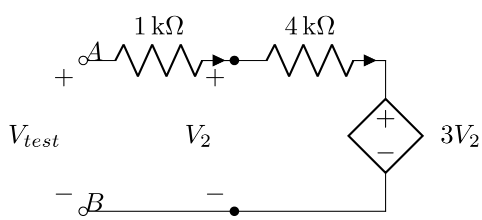
$$\hat{\boldsymbol{x}} = (\boldsymbol{A}^\top \boldsymbol{A})^{-1} \boldsymbol{A}^\top \boldsymbol{r} = \frac{1}{5} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 11 \\ 4 \end{bmatrix} \tag{27}$$

So our signal $\boldsymbol{r} = \frac{11}{5}\boldsymbol{u}_1 + \frac{4}{5}\boldsymbol{u}_4$.

7 Fun with Circuits

7.a Equivalent Resistance

If we apply a test voltage V_{test} at V_1,



The NVA equations are

$$I_{R_2} - I_{R_1} = 0 \tag{28}$$

$$\frac{V_2 - 3V_2}{4\,\mathrm{k\Omega}} - \frac{V_{test} - V_2}{1\,\mathrm{k\Omega}} = 0 \tag{29}$$

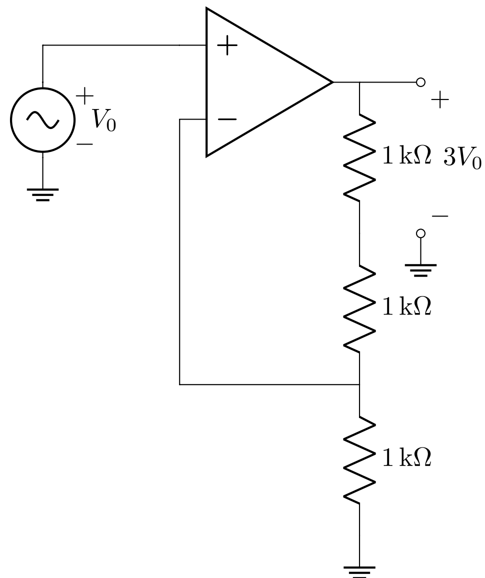
$$-\frac{V_2}{2\,\mathrm{k\Omega}} - \frac{V_{test}}{1\,\mathrm{k\Omega}} + \frac{V_2}{1\,\mathrm{k\Omega}} = 0 \tag{30}$$

$$V_2 = 2V_{test} \tag{31}$$

$$I_{R_1} = \frac{V_{test} - 2V_{test}}{1\,\mathrm{k\Omega}} = -\frac{V_{test}}{1\,\mathrm{k\Omega}} \tag{32}$$

$$R_{eq} = \frac{V_{test}}{I_{R_1}} = -1\,\mathrm{k\Omega} \tag{33}$$

7.b Amplifier Design



As a non-inverting amplifier,

$$V_{out} = V_{in}\left(1 + \frac{R_t}{R_b}\right) \tag{34}$$

$$= V_{in}\left(1 + \frac{2\,\mathrm{k\Omega}}{1\,\mathrm{k\Omega}}\right) = 3V_{in} \tag{35}$$

8 Projections and Eigenvectors

8.a

The matrix $\boldsymbol{M} = \boldsymbol{x}\boldsymbol{y}^\top \in \mathbb{R}^{n\times n}$.

8.b

$$\boldsymbol{M} = \begin{bmatrix}1\\2\end{bmatrix}\begin{bmatrix}4 & 5\end{bmatrix} = \begin{bmatrix}4 & 5\\8 & 10\end{bmatrix} \tag{36}$$

$$\Rightarrow \begin{vmatrix}4-\lambda & 5\\8 & 10-\lambda\end{vmatrix} = 0 \tag{37}$$

$$(4-\lambda)(10-\lambda) - 40 = 0 \tag{38}$$

$$40 - 4\lambda - 10\lambda + \lambda^2 - 40 = 0 \tag{39}$$

$$\lambda^2 - 14\lambda = 0 \tag{40}$$

$$\lambda = 0, 14 \tag{41}$$

Finding the eigenvectors,

$$\left[\begin{array}{cc|c} 4 & 5 & 0 \\ 8 & 10 & 0 \end{array}\right] \tag{42}$$

$$\overset{r_2-2r_1\to r_2}{\Longrightarrow} \left[\begin{array}{cc|c} 4 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{43}$$

$$\boldsymbol{e}_1 = \operatorname{span}\left\{ \begin{bmatrix} -\frac{5}{4} \\ 1 \end{bmatrix} \right\} \tag{44}$$

$$\left[\begin{array}{cc|c} -10 & 5 & 0 \\ 8 & -4 & 0 \end{array}\right] \tag{45}$$

$$\overset{r_1/-5\to r_1, r_2/4\to r_2}{\Longrightarrow} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \tag{46}$$

$$\overset{r_2-r_1\to r_2}{\Longrightarrow} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \tag{47}$$

$$\boldsymbol{e}_2 = \operatorname{span}\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\} \tag{48}$$

8.c

$$\begin{aligned} \operatorname{proj}_{\boldsymbol{M}}(\boldsymbol{x}) &= \boldsymbol{M}(\boldsymbol{M}^\top\boldsymbol{M})^{-1}\boldsymbol{M}^\top\boldsymbol{x} && (49) \\ &= \boldsymbol{M}((\boldsymbol{x}\boldsymbol{y}^\top)^\top\boldsymbol{x}\boldsymbol{y}^\top)^{-1}(\boldsymbol{x}\boldsymbol{y}^\top)^\top\boldsymbol{x} && (50) \\ &= \boldsymbol{M}(\boldsymbol{y}\boldsymbol{x}^\top\boldsymbol{x}\boldsymbol{y}^\top)^{-1}\boldsymbol{y}\boldsymbol{x}^\top\boldsymbol{x} && (51) \\ &= \boldsymbol{M}(\cancel{\|\boldsymbol{x}\|^2}\boldsymbol{y}\boldsymbol{y}^\top)^{-1}\boldsymbol{y}\cancel{\|\boldsymbol{x}\|^2} && (52) \\ &= \boldsymbol{x}\boldsymbol{y}^\top(\boldsymbol{y}\boldsymbol{y}^\top)^{-1}\boldsymbol{y} && (53) \\ &= \boldsymbol{x} && (54) \end{aligned}$$

8.d

Theorem 1. *Suppose $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\boldsymbol{M} = \boldsymbol{x}\boldsymbol{y}^\top$. Then, for some $\boldsymbol{z} \in \mathbb{R}^n$ such that $\langle \boldsymbol{z}, \boldsymbol{y}\rangle = 0$, $\boldsymbol{z} \in \operatorname{null}(\boldsymbol{M})$.*

Proof.

$$\begin{aligned} \langle \boldsymbol{z}, \boldsymbol{y}\rangle = \langle \boldsymbol{y}, \boldsymbol{z}\rangle &= 0 && \text{commutativity of inner product} && (55) \\ \boldsymbol{y}^\top\boldsymbol{z} &= 0 && \text{definition of inner product} && (56) \\ \boldsymbol{x}\boldsymbol{y}^\top\boldsymbol{z} &= 0 && \text{multiply by } \boldsymbol{x} \text{ on both sides} && (57) \\ \boldsymbol{M}\boldsymbol{z} &= 0 && \text{definition of } \boldsymbol{M} && (58) \\ \boldsymbol{z} &\in \operatorname{null}(\boldsymbol{M}) && \text{definition of nullspace} && (59) \end{aligned}$$

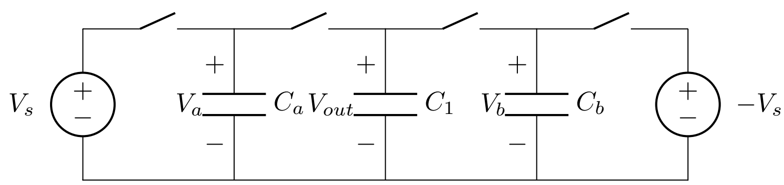
□

8.e

$$
\begin{aligned}
\boldsymbol{M} &= \begin{bmatrix} y_1\boldsymbol{x} & y_2\boldsymbol{x} & \cdots & y_n\boldsymbol{x} \end{bmatrix} && (60)\\
\boldsymbol{M}\boldsymbol{x} &= \begin{bmatrix} y_1\boldsymbol{x} & y_2\boldsymbol{x} & \cdots & y_n\boldsymbol{x} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} && (61)\\
&= x_1y_1\boldsymbol{x} + x_2y_2\boldsymbol{x} + \cdots + x_ny_n\boldsymbol{x} && (62)\\
&= \sum_{i=1}^{n} x_iy_i\boldsymbol{x} && (63)\\
&= \langle \boldsymbol{x}, \boldsymbol{y} \rangle \, \boldsymbol{x} && (64)
\end{aligned}
$$

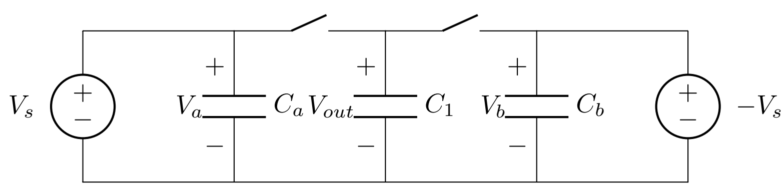
We find that $\boldsymbol{M}$ as eigenvector $\boldsymbol{x}$ with eigenvalue $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$.

9 Electronic Level



9.a

During phase 1,



which leaves the capacitors with charge

$$
\begin{aligned}
Q_a &= C_aV_s && (65)\\
Q_b &= -C_bV_s && (66)\\
Q_{tot}(0) = Q_{tot}(1) &= V_s(C_a - C_b) && (67)
\end{aligned}
$$

Plugging in our values of C_a and C_b,

$$\begin{aligned}
V_s(C_a - C_b) &= V_s\left(C_0\left(100 + \frac{\alpha}{\alpha_{ref}}\right) - C_0\left(100 - \frac{\alpha}{\alpha_{ref}}\right)\right) && (68)\\
&= V_s\left(\cancel{100C_0} + C_0\frac{\alpha}{\alpha_{ref}} - \cancel{100C_0} + C_0\frac{\alpha}{\alpha_{ref}}\right) && (69)\\
&= 2C_0V_s\frac{\alpha}{\alpha_{ref}} && (70)
\end{aligned}$$

9.b

$$\begin{aligned}
V_{out}(1) &= \frac{Q_{tot}(1)}{\underbrace{C_a + C_b + C_1}_{\text{in parallel}}} && (71)\\
&= \frac{Q_{tot}(1)}{C_0\left(100 + \frac{\alpha}{\alpha_{ref}}\right) + C_0\left(100 - \frac{\alpha}{\alpha_{ref}}\right)} && (72)\\
&= \frac{Q_{tot}(1)}{200C_0 + C_1} && (73)
\end{aligned}$$

9.c

We can construct the equation for the total charge on the capacitor as

$$Q_{tot}(k) = Q_{tot}(k-1) + V_s(C_a - C_b) \qquad (74)$$

Then, plugging into the equation in **7.b**, we get

$$\begin{aligned}
V_{tot}(k) &= \frac{Q_{tot}(k-1) + 2C_0V_s\frac{\alpha}{\alpha_{ref}}}{200C_0 + C_1} && (75)\\
&= \frac{C_1V_{out}(k-1) + 2C_0V_s\frac{\alpha}{\alpha_{ref}}}{200C_0 + C_1} && (76)
\end{aligned}$$

9.d

By induction,

$$\begin{aligned}
V_{out}(1) &= \gamma V_{out}(0) + \beta = \beta && (77)\\
V_{out}(2) &= \gamma\beta + \beta && (78)\\
V_{out}(3) &= \gamma(\gamma\beta + \beta) + \beta = \gamma^2\beta + \gamma\beta + \beta && (79)\\
&\vdots && (80)\\
\lim_{k\to\infty} V_{out}(k) &= \sum_{i=0}^{\infty}\beta\gamma^i = \frac{\beta}{1-\gamma} && (81)
\end{aligned}$$

10 Homework Process and Study Group

10.a

I used notes from EECS 16A to assist me with the homework.

10.b

I did this homework by myself.

10.c

I spent 3 hours doing the homework, then went to eat dinner. Afterwards, I finished the homework in 2 hours.

10.d

It took me 5 hours to do this homework.