EECS 16B HW06

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1 Inverted Pendulum on a Rolling Cart

1.a

$$\frac{d}{dt}\begin{bmatrix} x_1(t) = \theta(t) \\ x_2(t) = \dot{\theta}(t) \\ x_3(t) = \dot{y}(t) \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{\ell(\frac{M}{m}+\sin^2(x_1))}\left(-\frac{u}{m}\cos(x_1) - x_2^2\ell\cos(x_1)\sin(x_1) + \frac{M+m}{m}g\sin(x_1)\right) \\ \frac{1}{(\frac{M}{m}+\sin^2(x_1))}\left(\frac{u}{m} + x_2^2\ell\sin(x_1) - g\sin(x_1)\cos(x_1)\right) \end{bmatrix} \tag{1}$$

1.b

$$\begin{bmatrix} x_2 \\ \frac{1}{\ell(\frac{M}{m}+\sin^2(x_1))}\left(-\frac{u}{m}\cos(x_1) - x_2^2\ell\cos(x_1)\sin(x_1) + \frac{M+m}{m}g\sin(x_1)\right) \\ \frac{1}{(\frac{M}{m}+\sin^2(x_1))}\left(\frac{u}{m} + x_2^2\ell\sin(x_1) - g\sin(x_1)\cos(x_1)\right) \end{bmatrix}_{x_1=x_2=x_3=u=0} \tag{2}$$

$$\begin{bmatrix} 0 \\ \frac{1}{\ell(\frac{M}{m}+\sin^2(0))}\left(-\frac{0}{m}\cos(0) - 0^2\ell\cos(0)\sin(0) + \frac{M+m}{m}g\sin(0)\right) \\ \frac{1}{(\frac{M}{m}+\sin^2(0))}\left(\frac{0}{m} + 0^2\ell\sin(0) - g\sin(0)\cos(0)\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

1.c

Given the linearization equation

$$f(\boldsymbol{x}, \boldsymbol{u}) \approx f(\boldsymbol{x}^*, \boldsymbol{u}^*) + \nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*, \boldsymbol{u}^*)(\boldsymbol{x} - \boldsymbol{x}^*) + \nabla_{\boldsymbol{u}} f(\boldsymbol{x}^*, \boldsymbol{u}^*)(\boldsymbol{u} - \boldsymbol{u}^*) \tag{4}$$

we first find the Jacobians

$$\nabla_{\boldsymbol{x}} f(\boldsymbol{x}^*, \boldsymbol{u}^*) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{\ell m}g & 0 & 0 \\ \frac{m}{M}g & 0 & 0 \end{bmatrix} \tag{5}$$

$$\nabla_{\boldsymbol{u}} f(\boldsymbol{x}^*, \boldsymbol{u}^*) = \begin{bmatrix} 0 \\ -\frac{1}{\ell M} \\ \frac{1}{M} \end{bmatrix} \tag{6}$$

So our final linearization is

$$f(\boldsymbol{x}, \boldsymbol{u}) \approx \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ \frac{M+m}{\ell m}g & 0 & 0 \\ \frac{m}{M}g & 0 & 0 \end{bmatrix}}_{\boldsymbol{A}} \boldsymbol{x} + \underbrace{\begin{bmatrix} 0 \\ -\frac{1}{\ell M} \\ \frac{1}{M} \end{bmatrix}}_{\boldsymbol{B}} \boldsymbol{u} \tag{7}$$

2 Single Dimension Linearization

$$\frac{d}{dt}x(t) = \sin(x(t)) + u(t) \tag{8}$$

2.a

$$x^* = \sin^{-1}(-u^*) = \{k\pi | k \in \mathbb{Z}^{[-4,4]}\} \tag{9}$$

2.b

$$\partial_x f(x^*, u^*) = \cos(0) = 1 \tag{10}$$
$$\partial_u f(x^*, u^*) = 1 \tag{11}$$
$$\Rightarrow f(x, u) \approx x + u \tag{12}$$

2.c

Finding the solution to the continuous-time solution and assuming a zero-hold interpolation,

$$x(t) = x(nT)e^{t-nT} + \int_{kT}^{t} e^{t-\tau} u(\tau)\, d\tau \tag{13}$$
$$x((k+1)T) = x(kT)e^{T} + u(kT) \int_{kT}^{(k+1)T} e^{(k+1)T-\tau} d\tau \tag{14}$$
$$x[k+1] = e^{T} x[k] + (e^{T} - 1)u[k] \tag{15}$$

3 Linearizing for Understanding Amplification

3.a

By Ohm's Law,

$$I_C = \frac{V_{DD} - V_{out}}{R} \Longrightarrow V_{out} = V_{DD} - I_C R \tag{16}$$

3.b

$$m = \lim_{V_{in} \to V_{in}^*} \frac{I(V_{in}) - I_C^*}{V_{in} - V_{in}^*} \tag{17}$$

Note that this is simply the definition of the derivative of I_C with respect to V_{in} at the point V_{in}^*,

$$m = \frac{d}{dV_{in}} I_C(V_{in}^*) = \frac{1}{V_{TH}} I_S e^{\frac{V_{in}^*}{V_{TH}}} = \frac{I_C^*}{V_{TH}} \tag{18}$$

3.c

Substituting I_C into **3.a**,

$$V_{out} = V_{DD} - (I_C^* + m\, \delta V_{in)}) R \tag{19}$$
$$= \underbrace{(V_{DD} - I_C^* R)}_{V_{out}^*} - \frac{I_C^*}{V_{TH}}\, \delta V_{in} R \tag{20}$$
$$\Rightarrow \frac{\delta V_{out}}{\delta V_{in}} = -\frac{I_C^* R}{V_{TH}} \tag{21}$$

3.d

$$\frac{\delta V_{out}}{\delta V_{in}} = -\frac{I_C^* R}{V_{TH}} = -\frac{1\,\mathrm{mA} \cdot 1\,\mathrm{k\Omega}}{26\,\mathrm{mV}} \approx -38.5 \tag{22}$$

3.e

$$\frac{\delta V_{out}}{\delta V_{in}} = -\frac{I_C^* R}{V_{TH}} = -\frac{9\,\mathrm{mA} \cdot 1\,\mathrm{k\Omega}}{26\,\mathrm{mV}} \approx -346.1 \tag{23}$$

5 Homework Process and Study Group

1. I referred to lecture notes
2. I worked on this homework by myself.
3. I worked on this homework on Sunday, then completed working on it on Thursday.
4. Some of the derivatives in 1.b were quite unwieldy.
5. 3 hours.