EECS 16B HW01

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1 Existence and Uniqueness of Solutions to Differential Equations

1.a

$$\frac{d}{dt}x_d(t) = \alpha x_0 e^{\alpha t} = \alpha x_d(t) \tag{1}$$

$$x_d(0) = x_0 e^0 = x_0 \tag{2}$$

1.b

$$z(0) = \frac{y(0)}{x(0)} = \frac{x_0}{x_0} = 1 \tag{3}$$

1.c

$$\frac{d}{dt}z(t) = \frac{y'(t)x(t) - x'(t)y(t)}{x(t)^2} \tag{4}$$

$$= \frac{\cancel{\alpha y(t)x(t)} - \cancel{\alpha x(t)y(t)}}{x(t)^2} = 0 \tag{5}$$

Since $\frac{d}{dt}z(t) = 0$, this implies that $z(t)$ is a constant. Specifically, $z(t) = 1$ due to the ratio of their initial conditions as we have shown above.

1.d

Applying the transformation $t = t_0 - \tau$,

$$\frac{d}{d\tau}\tilde{x}(\tau) = \frac{d}{d\tau}x(t_0 - \tau) = -\alpha x(t_0 - \tau) = -\alpha \tilde{x}(\tau) \tag{6}$$

1.e

Using the equation from **1.d**,

$$\frac{d}{d\tau}\tilde{y}(\tau) = -\alpha\tilde{y}(\tau) \tag{7}$$
$$\Rightarrow \tilde{y}(\tau) = x_0 e^{-\alpha(t_0-\tau)} \tag{8}$$

1.f

$$\tilde{y}(0) = x_0 e^{-\alpha(t_0)} \neq x_0 \tag{9}$$

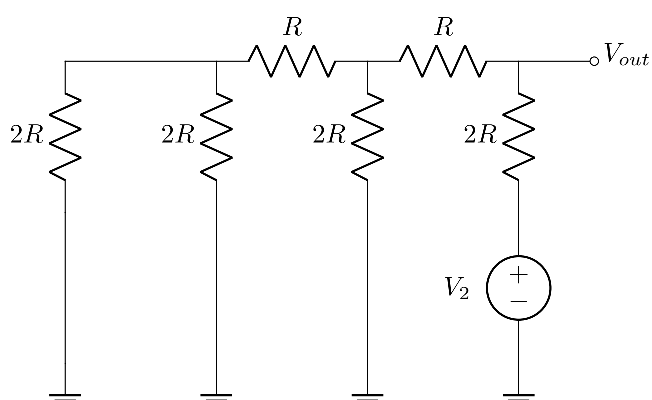
By contradiction, we establish uniqueness for all $x_0 \in \mathbb{R}$.

1.g

It is important to establish unique solutions to differential equations so that we can derive important information. If there were non-unique solutions, then multiple equations could describe behavior.

2 Digital-Analog Converter

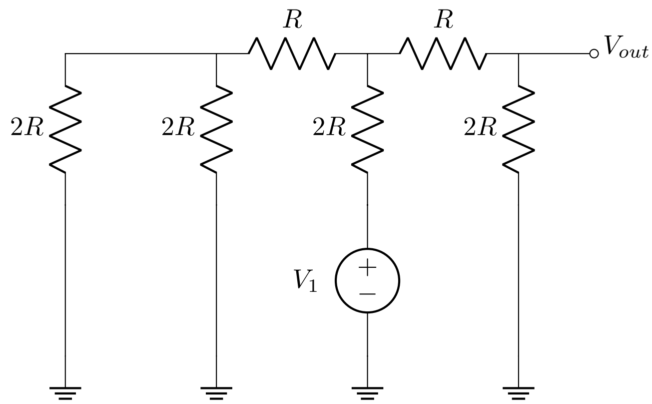
2.a



The left nest of resistors ends up having an equivalent resistance of $2R$. We can treat the resulting circuit as a voltage divider, with voltage

$$V_{out} = V_2\frac{2\cancel{R}}{4\cancel{R}} = \frac{1}{2}V_{DD} \tag{10}$$

2.b



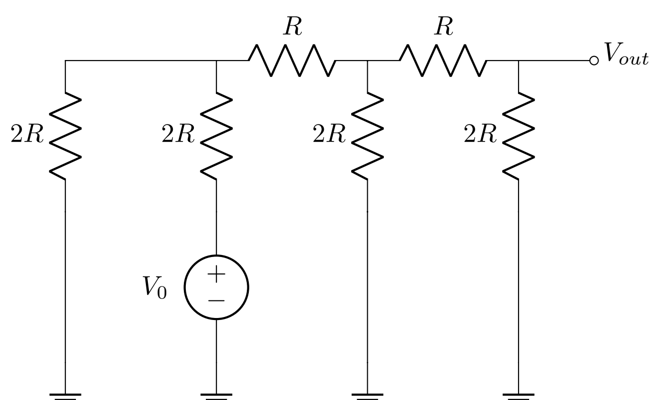
Simplifying and using NVA, we construct the matrix equation

$$\left[\begin{array}{cc|c} 2 & -1 & \frac{V_1}{2} \\ -1 & \frac{3}{2} & 0 \end{array}\right] \tag{11}$$

$$\implies \left[\begin{array}{cc|c} 1 & 0 & \frac{3V_1}{8} \\ 0 & 1 & \frac{V_1}{4} \end{array}\right] \tag{12}$$

where $V_{out} = \frac{1}{4}V_{DD}$.

2.c



Simplifying and using NVA, we construct the matrix equation

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & \frac{V_1}{2} \\ -1 & \frac{5}{2} & -1 & 0 \\ 0 & -1 & \frac{3}{2} & 0 \end{array}\right] \tag{13}$$

$$\implies \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{11V_1}{32} \\ 0 & 1 & 0 & \frac{3V_1}{16} \\ 0 & 0 & 1 & \frac{V_1}{8} \end{array}\right] \tag{14}$$

where $V_{out} = \frac{1}{8}V_{DD}$.

2.d

By the principle of superposition, the DAC's output voltage when all three bits are on is simply the addition of each individual bit's voltage:

$$V_{out} = \frac{7}{8}V_{DD} \tag{15}$$

2.e

$$V_{out} = V_{DD}\left(\frac{1}{2}b_2 + \frac{1}{4}b_1 + \frac{1}{8}b_0\right) \tag{16}$$

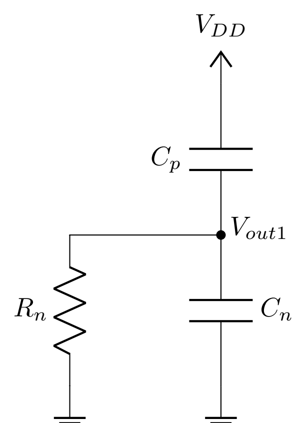
2.f

The DAC computes an analog voltage by turning the value of each bit into a weighted sum that is then applied to the supply voltage. Each bit corresponds to a power of two, and is thus given an appropriate weight.

3 Transistor Switch Model

3.a

We can model the node V_{out1} as



Writing out the node equation for V_{out1} yields

$$\frac{V_{out1}}{R} + C_n\frac{d}{dt}V_{out1} - C_p\frac{d}{dt}(V_{DD} - V_{out1}) = 0 \tag{17}$$

3.b

$$\frac{V_{out1}}{R} + C_n \frac{d}{dt} V_{out1} - \cancel{C_p \frac{d}{dt} V_{DD}} + C_p \frac{d}{dt} V_{out1} = 0 \tag{18}$$

$$\frac{d}{dt} V_{out1} = -\frac{V_{out1}}{\underbrace{R(C_p + C_n)}_{\tau}} \tag{19}$$

$$\int \frac{1}{V_{out1}}\, dV_{out1} = \int -\frac{1}{\tau}\, dt \tag{20}$$

$$\ln |V_{out1}| = -\frac{1}{\tau} t + C \tag{21}$$

$$V_{out1} = V_0 e^{-\frac{1}{\tau} t} \tag{22}$$

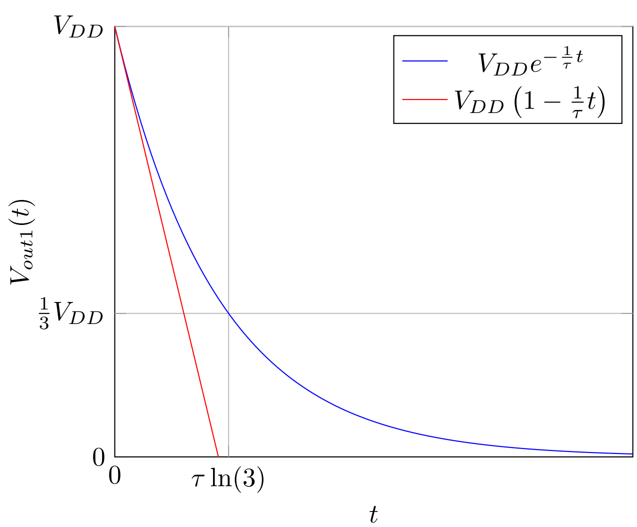
Plugging in our initial condition $V_{out1}(0) = V_{DD}$,

$$V_{out1}(0) = V_0 = V_{DD} \tag{23}$$

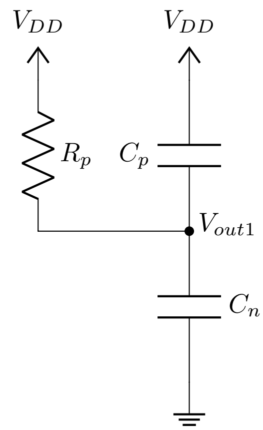
So our solution is

$$V_{out1}(t) = V_{DD} e^{-\frac{1}{\tau} t} \tag{24}$$

3.c



3.d



The node equation is

$$C_n \frac{d}{dt} V_{out1} - C_p \frac{d}{dt} (\cancel{V_{DD}} - V_{out1}) - \frac{V_{DD} - V_{out1}}{R_p} = 0 \tag{25}$$

Solving,

$$C_n \frac{d}{dt} V_{out1} + C_p \frac{d}{dt} V_{out1} - \frac{V_{DD} - V_{out1}}{R_p} = 0 \tag{26}$$

$$\frac{d}{dt} V_{out1} = \frac{V_{DD} - V_{out1}}{\underbrace{R_p(C_p + C_n)}_{\tau}} \tag{27}$$

$$\int \frac{1}{V_{DD} - V_{out1}}\, dV_{out1} = \int \frac{1}{\tau}\, d\tau \tag{28}$$

$$-\ln |V_{DD} - V_{out1}| = \frac{1}{\tau} + C \tag{29}$$

$$V_{DD} - V_{out1} = V_0 e^{-\frac{1}{\tau} t} \tag{30}$$

$$V_{out1}(t) = V_{DD} - V_0 e^{-\frac{1}{\tau} t} \tag{31}$$

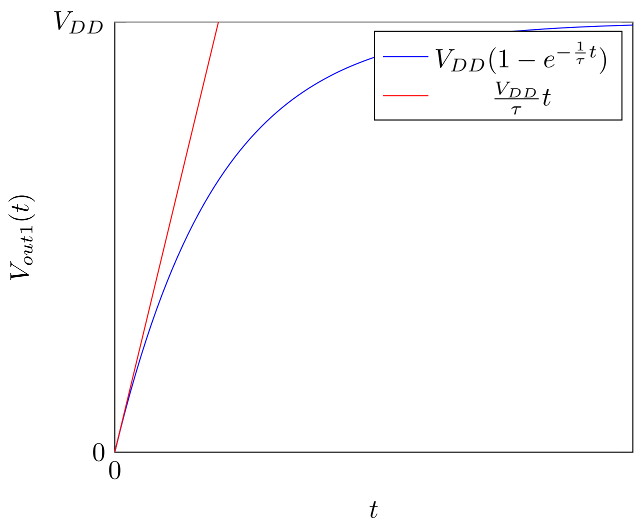
Plugging in the initial condition $V_{out1}(0) = 0$,

$$0 = V_{DD} - V_0 \tag{32}$$

$$V_0 = V_{DD} \tag{33}$$

Yielding the final equation

$$V_{out1}(t) = V_{DD}(1 - e^{-\frac{1}{\tau} t}) \tag{34}$$



3.e

Since the circuit goes through a voltage cycle of V_{DD}, then the total charge is $V_{DD}(C_p + C_n)$ by definition of a capacitor. Plugging in, this gives us 2 fC.

4 RC Circuit

4.a

$$I(0) = \frac{V_s}{R} \tag{35}$$

$$\lim_{t\to\infty} I(t) = 0 \tag{36}$$

4.b

$$V_s - V_R(t) - V_C(t) = 0 \tag{37}$$

$$V_s - I_C(t)R - \frac{1}{C}Q_C(t) = 0 \tag{38}$$

$$R\frac{d}{dt}I_C(t) + \frac{1}{C}I_C(t) = 0 \tag{39}$$

4.c

The eigenvalue $\lambda = -\frac{1}{RC}$.

4.d

$$\frac{d}{dt}I_C(t) = -\frac{1}{RC}I_C(t) \tag{40}$$

$$\int \frac{1}{I_C}\,dI_C = \int -\frac{1}{RC}\,dt \tag{41}$$

$$\ln|I_C| = -\frac{1}{RC}t + K \tag{42}$$

$$I_C(t) = I_0 e^{-\frac{1}{RC}t} \tag{43}$$

Plugging in the initial values,

$$I_C(0) = I_0 = \frac{V_s}{R} \tag{44}$$

$$I_C(t) = \frac{V_s}{R}e^{-\frac{1}{RC}t} \tag{45}$$

4.e

Solving for $0.05\frac{V_s}{R}$,

$$0.05 = e^{-\frac{1}{RC}t} \tag{46}$$

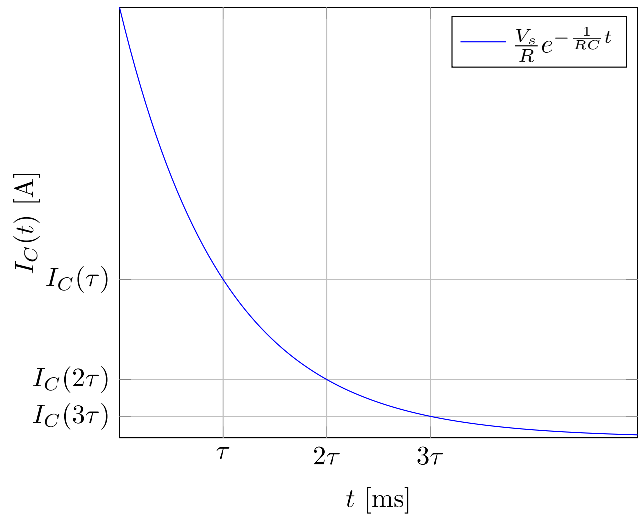
$$\ln(0.05) = -\frac{1}{RC}t \tag{47}$$

$$t = -RC \cdot \ln(0.05) \rightsquigarrow 2.99\,\mathrm{ms} \tag{48}$$

4.f

1. Make R smaller.
2. Make C smaller.

4.g



6 Homework Process and Study Group

a. I used Note 3 from 28 January.

b. I worked on this homework by myself.

c. I worked on this homework in one sitting.

d. I spend 4 hours on this homework.