EECS 16B HW02

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1 Fundamental Theorem of Solutions to Differential Equations

1.a

Theorem 1. *Given* $\phi_1(x) = e^x$ *and* $\phi_2(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\phi_1(x) = \phi_2(x)$.

Proof. Proving the derivative property,

$$\frac{d}{dx}\phi_1(x) = \frac{d}{dx}e^x = e^x = \phi_1(x) \tag{1}$$

$$\frac{d}{dx}\phi_2(x) = \frac{d}{dx}\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{\cancel{n}x^{n-1}}{\cancel{n}(n-1)!} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \phi_2(x) \tag{2}$$

where we are allowed to shift the index because any singular n terms cancel to zero. Proving the initial condition,

$$\phi_1(0) = e^0 = 1 \tag{3}$$

$$\phi_2(0) = \sum_{n=0}^{\infty}\frac{0^n}{n!} = \frac{0^0}{0!} + \cancel{\frac{0^1}{1!}} + \cancel{\frac{0^2}{2!}} + \cancel{\cdots} = 1 \tag{4}$$

□

1.b

Theorem 2. *Given* $\phi_1(x) = \cos(x)$ *and* $\phi_2(x) = \cos(-x)$, $\phi_1(x) = \phi_2(x)$.

Proof. Proving the derivatives,

$$\frac{d^2}{dx^2}\phi_1(x) = \frac{d^2}{dx^2}\cos(x) = -\cos(x) = -\phi_1(x) \tag{5}$$

$$\frac{d^2}{dx^2}\phi_2(x) = \frac{d^2}{dx^2}\cos(-x) = -\cos(-x) = -\phi_2(x) \tag{6}$$

Proving the initial conditions,

$$\phi_1(x) = \phi_2(x) = \cos(0) = 1 \tag{7}$$

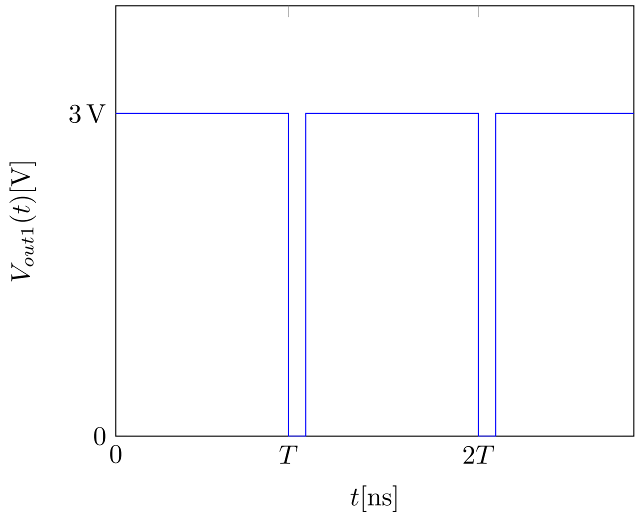
$$\frac{d}{dx}\phi_1(0) = -\sin(0) = 0 \tag{8}$$

$$\frac{d}{dx}\phi_2(0) = \sin(0) = 0 \tag{9}$$

□

2 IC Power Supply

2.a



2.b

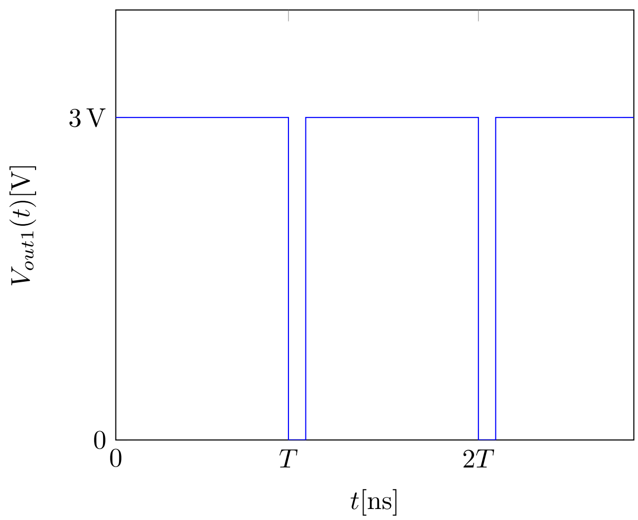
The generalized expression for V_{DD} can be represented with the differential equation

$$C\frac{d}{dt}V_{DD} + i(t) - \frac{V_s - V_{DD}}{R} = 0 \tag{10}$$

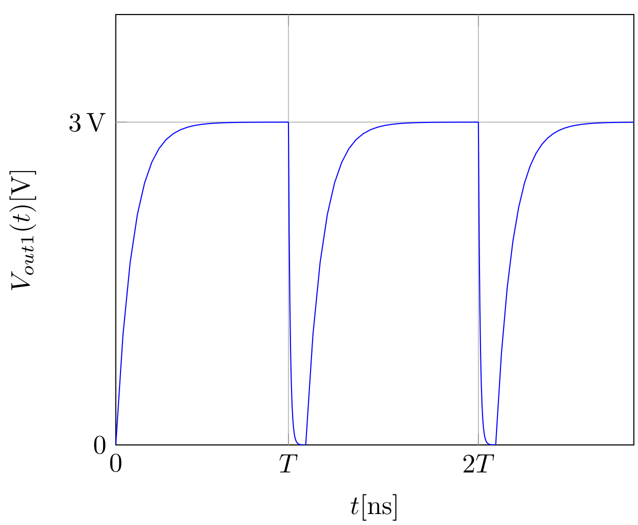
$$\frac{d}{dt}V_{DD} + \frac{V_{DD}}{RC} = \frac{V_s}{RC} - \frac{i(t)}{C} \tag{11}$$

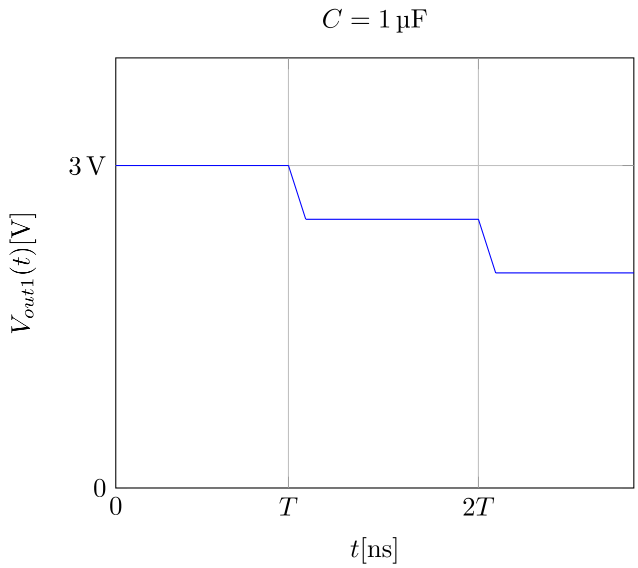
with $i(0) = 0$.

$C = 1\,\mathrm{pF}$



$C = 1\,\mathrm{nF}$





2.c

It is better to have a higher C than a higher R. This is because a higher capacitance means that the capacitor will take longer to discharge completely, resisting larger changes in voltage, possibly outlasting t_p.

3 Simple Scalar DEs Driven by an Input

3.a

Proof. We will prove that $\delta(t) = x_g(t) - y(t) = 0$, thuss proving uniqueness. First proving the initial conditions cancel,

$$\delta(t) = x_g(0) - y(0) = x_0 - x_0 = 0 \tag{12}$$

Next, we prove uniqueness of the derivative:

$$\begin{aligned} \frac{d}{dt}\delta(t) &= \frac{d}{dt}(x_g - y) & (13) \\ &= \lambda x_g + \cancel{u(t)} - \lambda y - \cancel{u(t)} & (14) \\ &= \lambda(x_g - y) & (15) \end{aligned}$$

$\square$

3.b

Proof.

$$x_c(t) = x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)}\, d\tau \tag{16}$$

$$x_c(0) = x_0 \cancelto{1}{e^{\lambda 0}} + \cancel{\int_0^0 u(\tau) e^{\lambda(t-\tau)}}\, d\tau = x_0 \tag{17}$$

$$\frac{d}{dt} x_c(t) = \lambda x_0 e^{\lambda t} + \underbrace{\int_0^t \frac{\partial}{\partial t} u(\tau) e^{\lambda(t-\tau)}\, d\tau}_{\text{Leibniz integral rule}} + \underbrace{u(t) \cancelto{1}{e^{\lambda(t-t)}}}_{\text{FTC}} \tag{18}$$

$$= \lambda x_c(t) + \lambda \int_0^t u(\tau) e^{\lambda(t-\tau)}\, d\tau + u(t) \tag{19}$$

$$= \lambda x_c(t) + u(t) \tag{20}$$

□

3.c

Plugging in,

$$x_c(t) = x_0 e^{\lambda t} + \int_0^t e^{s\tau} e^{\lambda(t-\tau)}\, d\tau = x_0 e^{\lambda t} + \int_0^t e^{(s-\lambda)\tau + \lambda t}\, d\tau \tag{21}$$

$$= x_0 e^{\lambda t} + \frac{1}{s-\lambda} e^{(s-\lambda)\tau+\lambda t} \Big|_0^t \tag{22}$$

$$= x_0 e^{\lambda t} + \left(\frac{1}{s-\lambda} e^{st-\lambda t+\lambda t}\right) - \left(\frac{1}{s-\lambda} e^{\lambda t}\right) \tag{23}$$

$$= x_0 e^{\lambda t} + \frac{1}{s-\lambda}(e^{st} - e^{\lambda t}) \tag{24}$$

3.d

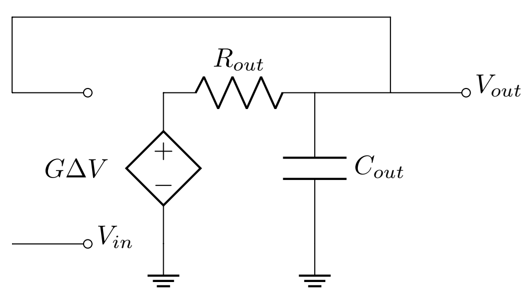
Plugging in,

$$x_c(t) = x_0 e^{\lambda t} + \int_0^t e^{\lambda\tau} e^{\lambda(t-\tau)}\, d\tau = x_0 e^{\lambda t} + \int_0^t e^{\lambda t}\, d\tau \tag{25}$$

$$= x_0 e^{\lambda t} + t e^{\lambda t}\, d\tau \tag{26}$$

4 Op-Amp Stability

4.a



4.b

The KCL equation for V_{out} is

$$
\begin{aligned}
C_{out}\frac{d}{dt}V_{out} - \frac{G(V_{in} - V_{out}) - V_{out}}{R_{out}} &= 0 \qquad (27)\\
\frac{d}{dt}V_{out} = \frac{GV_{in} - (G+1)V_{out}}{R_{out}C_{out}} &= \frac{G+1}{R_{out}C_{out}}\left(\frac{G}{G+1}V_{in} - V_{out}\right) \qquad (28)\\
\int \frac{1}{\frac{G}{G+1}V_{in} - V_{out}}\, dV_{out} &= \int \frac{G+1}{R_{out}C_{out}}\, dt \qquad (29)\\
-\ln\left|\frac{G}{G+1}V_{in} - V_{out}\right| &= \frac{G+1}{R_{out}C_{out}}t + K \qquad (30)\\
V_{out}(t) &= \frac{G}{G+1}V_{in} - V_0 \exp\left(-\frac{G+1}{R_{out}C_{out}}t\right) \qquad (31)\\
\lim_{t\to\infty} V_{out}(t) &= \frac{G}{G+1}V_{in} \qquad (32)
\end{aligned}
$$

4.c

Using similar techniques,

$$
\begin{aligned}
C_{out}\frac{d}{dt}V_{out} - \frac{G(V_{out} - V_{in}) - V_{out}}{R_{out}} &= 0 \qquad (33)\\
\frac{d}{dt}V_{out} = \frac{(G-1)V_{out} - GV_{in}}{R_{out}C_{out}} &= \frac{G-1}{R_{out}C_{out}}\left(V_{out} - \frac{G}{G-1}V_{in}\right) \qquad (34)\\
\int \frac{1}{V_{out} - \frac{G}{G-1}V_{in}}\, dV_{out} &= \int \frac{G-1}{R_{out}C_{out}}\, dt \qquad (35)\\
\ln\left|V_{out} - \frac{G}{G-1}V_{in}\right| &= \frac{G-1}{R_{out}C_{out}}t + K \qquad (36)\\
V_{out}(t) &= \frac{G}{G-1}V_{in} + V_0 \exp\left(\frac{G-1}{R_{out}C_{out}}t\right) \qquad (37)\\
\lim_{t\to\infty} V_{out}(t) &= \pm\infty \qquad (38)
\end{aligned}
$$

Depending on the initial condition, the op-amp will rail either to the positive or negative. In fact, for $V_{in} > 0$, the op-amp will rail negative due to the fact that when we solve for the initial condition, $V_0 = -\frac{G}{G-1}V_{in} < 0$.

4.d

Given the $V_{out}(t)$ from **4.b**,

$$
\lim_{G\to\infty} V_{out}(t) = V_{in} - \cancel{V_0 \exp\left(\lim_{G\to\infty} -\frac{G+1}{R_{out}C_{out}}t\right)} = V_{in} \qquad (39)
$$

6 Homework Process and Study Group

1. I referred to my lecture notes.
2. I did this homework by myself.
3. I worked on this homework in one sitting, and revised it the day after.
4. 4 hours.