EECS 16B MT2 Redo

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1

$$\frac{d}{dt}S = -\beta\frac{IS}{N} \tag{1}$$
$$\frac{d}{dt}I = \beta\frac{IS}{N} - \gamma I \tag{2}$$
$$\frac{d}{dt}R = \gamma I \tag{3}$$

The Jacobian is

$$\begin{bmatrix} -\beta\frac{I}{N} & -\beta\frac{S}{N} & 0 \\ \beta\frac{I}{N} & \beta\frac{S}{N} - \gamma & 0 \\ 0 & \gamma & 0 \end{bmatrix} \implies \begin{bmatrix} 0 & -\beta & 0 \\ 0 & \beta - \gamma & 0 \\ 0 & \gamma & 0 \end{bmatrix} \tag{4}$$

2

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{5}$$
$$\boldsymbol{z}(t) = \boldsymbol{T}\boldsymbol{x}(t) \tag{6}$$
$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} \tag{7}$$

In terms of $\boldsymbol{z}$, the system is

$$\frac{d}{dt}\boldsymbol{z}(t) = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}\boldsymbol{z}(t) + \boldsymbol{T}\boldsymbol{B}\boldsymbol{u}(t) \tag{8}$$

The controllability matrix is

$$\boldsymbol{\mathcal{C}}' = \begin{bmatrix} \boldsymbol{T}\boldsymbol{B} & \boldsymbol{T}\boldsymbol{A}\boldsymbol{B} & \cdots & \boldsymbol{T}\boldsymbol{A}^{n-1}\boldsymbol{B} \end{bmatrix} = \boldsymbol{T}\boldsymbol{\mathcal{C}} \tag{9}$$

3

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$\boldsymbol{A}\left[\boldsymbol{A}\right]^\top = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

The SVD of $\boldsymbol{A}$ is

$$\boldsymbol{U} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

$$[\boldsymbol{V}]^\top = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

1. By inspection, none of $\boldsymbol{A}\,[\boldsymbol{A}]^\top$'s eigenvalues are similar to each other.
2. $\boldsymbol{\Sigma}$ clearly shows the nonzero singular values of $3, 2, 1$.
3. If we remove the last row, the last row of $\boldsymbol{\Sigma}$ is removed, so the nonzero singular values remain unchanged.

4

$$\boldsymbol{A} = \sigma_1 \boldsymbol{u}_1 \,[\boldsymbol{v}_1]^\top + \sigma_2 \boldsymbol{u}_2 \,[\boldsymbol{v}_2]^\top + \cdots \tag{15}$$

A is true by definition of SVD. **B** is not true since in general $\sigma_i = \sqrt{\lambda_i} \in \mathbb{C}$. If $\lambda_i \in \mathbb{R}$, $\sigma_i > 0$. **C** is not true by definition of the compact SVD, any matrix with rank r will have r nonzero eigenvalues. **D** is not true since all values can be multiplied by constant factors.

5

$$\frac{d}{dt}S = -\beta\frac{IS}{N} \tag{16}$$

$$\frac{d}{dt}I = \beta\frac{IS}{N} - \gamma I \tag{17}$$

$$\frac{d}{dt}R = \gamma I \tag{18}$$

The system has infinitely many equilibrium points. Since I will be zero, S can be any number since it will be multiplied to zero. R can be any number since it doesn't appear in the state model.

6

Since $\boldsymbol{A}$ is doubled, its eigevalues are also doubled. Given the general solution to the vector case and a diagonalizable $\boldsymbol{A}$,

$$\boldsymbol{x}_d[k+1] = \boldsymbol{V} e^{\boldsymbol{\Lambda} T} \boldsymbol{V}^{-1} \boldsymbol{x}_d[k] \tag{19}$$

where $\boldsymbol{\Lambda}, \boldsymbol{V}$ is a matrix containing the eigenvalues and eigenvectors, respectively. Since we are effectively doubling the term in the exponential, this is equivalent to squaring the entire discrete system.

7

$$\frac{d}{dt}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}_1 u \tag{20}$$

$$\frac{d}{dt}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \tag{21}$$

$$\boldsymbol{B} = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \tag{22}$$

The system is already controllable, so by the Cayley-Hamilton theorem, adding new vectors does not change the span.

8

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{23}$$

1. By definition of equilibrium, $\boldsymbol{x}^* = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{u}^*$. Thus, for $\boldsymbol{u} = \boldsymbol{0}$, $\boldsymbol{x} = \boldsymbol{0}$.
2. Using the previous definition, this is false if $\boldsymbol{A}$ is uninvertible.
3. By definition of an equilibrium point, $\frac{d}{dt}\boldsymbol{x}(t) = 0 \ \forall t \geqslant 0$, so $\boldsymbol{x}(t) = C \in \mathbb{R}$.
4. Using the very first definition, if $\boldsymbol{A}$ is invertible, then there exists a unique mapping from $\boldsymbol{u}^*$ to $\boldsymbol{x}^*$.
5. This is trivial due to the linearity of the system.

9

Since the system is defined in $\mathbb{R}^3$, by Cayley-Hamilton the system is controllable in 3 steps. Thus, from any $\boldsymbol{x}[0]$, we can reach any other $\boldsymbol{x}[2]$ using a bounded set of inputs.

10

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 3 & 6 \\ 4 & 4 & 8 \\ 5 & 1 & 2 \end{bmatrix} \tag{24}$$

A matrix has as many nonzero singular values as its rank, so since this matrix has linearly dependent rows up until the last one, it has $r = 2$, and thus two nonzero singular values.

11

$$\boldsymbol{y} = \boldsymbol{D}\boldsymbol{p} + \boldsymbol{e} \tag{25}$$

Given the equation for least squares

$$\hat{\boldsymbol{p}} = ([\boldsymbol{D}]^\top \boldsymbol{D})^{-1} [\boldsymbol{D}]^\top \boldsymbol{y} \tag{26}$$

it is clear that $[\boldsymbol{D}]^\top \boldsymbol{D}$ must be invertible to ensure a unique solution.

12

$$x[t+1] = bu[t] + e[t] \tag{27}$$
$$u[0] = u[1] = u[2] = u[3] = 1 \tag{28}$$

We simply set up the least squares problem

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} b = \begin{bmatrix} x[1] \\ x[2] \\ x[3] \\ x[4] \end{bmatrix} \tag{29}$$

Thus, we simply plug in various values of $x[t]$ and find the least squares solution, arriving at

$$\boldsymbol{x} = \begin{bmatrix} 0.1 \\ 1.1 \\ 1.9 \\ 0.9 \end{bmatrix} \tag{30}$$

13

1. If $\boldsymbol{Q}\,[\boldsymbol{Q}]^\top = \boldsymbol{I}$, then $\lambda = 1$, so $\sigma = \sqrt{\lambda} = 1$.
2. The rank determines how many linearly independent eigenvectors exist. All linearly dependent eigenvectors are in the null space of $\boldsymbol{A}$, and therefore have a zero singular value.
3. There are no restrictions on the dimensions or rank of $\boldsymbol{A}$ in order to obtain its SVD.

14

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{31}$$

1. The dimensions of $\boldsymbol{x}, \boldsymbol{u}$ do not uniquely determine controllability, only $\boldsymbol{A}, \boldsymbol{B}$.
2. We can uniquely determine the controllability matrix by knowing $\boldsymbol{A}, \boldsymbol{B}$.
3. If $m = n$, then we can uniquely determine the input sequence by inverting $\boldsymbol{B}$.
4. If $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{0}$, then all subsequent multiplications will also be $\boldsymbol{0}$, so the rank will not change.
5. Rank does not determine controllability.

15

$$\begin{bmatrix} u[1] & u[0] \\ u[2] & u[1] \\ \vdots & \vdots \\ u[N] & u[N-1] \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} y[2] \\ y[3] \\ \vdots \\ y[N+1] \end{bmatrix} \tag{32}$$

We can replace the left column of the first matrix with the function definition,

$$\begin{bmatrix} \lambda u[0] & u[0] \\ \lambda u[1] & u[1] \\ \vdots & \vdots \\ \lambda u[N-1] & u[N-1] \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} y[2] \\ y[3] \\ \vdots \\ y[N+1] \end{bmatrix} \tag{33}$$

Thus the system is linearly dependent and unsolvable.

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$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t)x_2(t) \\ \cos\left(\frac{\pi}{2}x_1(t)\right) \end{bmatrix} + \begin{bmatrix} x_1^2(t) \\ 0 \end{bmatrix} u(t) \tag{34}$$

Our linearizations are

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ -\frac{\pi}{2} & 0 \end{bmatrix} \tag{35}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{36}$$

17

By process of elimination, **A** has negative singular values, **B** is not full rank, **C** is wrong because the multiplication is incorrect, **D** has incorrect singular values.

18

By KVL,

$$u - i_1 R_1 - L_1 \frac{d}{dt} i_1 = 0 \tag{37}$$

$$u - i_2 R_2 - L_2 \frac{d}{dt} i_2 = 0 \tag{38}$$

Our state model is

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ \frac{1}{L_2} \end{bmatrix} u \tag{39}$$

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} \frac{1}{L_1} & -\frac{R_1}{L_1^2} \\ \frac{1}{L_2} & -\frac{R_2}{L_2^2} \end{bmatrix} \tag{40}$$

In order to have an uncontrollable system, $\frac{R_1}{L_1} = \frac{R_2}{L_2}$. Therefore, $R_2 = 2\,\mathrm{m}\Omega$.

19

$$\boldsymbol{A} = \sigma_1 \boldsymbol{u}_1 [\boldsymbol{v}_1]^\top + \sigma_2 \boldsymbol{u}_2 [\boldsymbol{v}_2]^\top + \sigma_3 \boldsymbol{u}_3 [\boldsymbol{v}_3]^\top \tag{41}$$

$\boldsymbol{v}\,[\boldsymbol{v}]^\top \neq 1$ because $\boldsymbol{v}\,[\boldsymbol{v}]^\top$ is the identity matrix, not a scalar.

20

$$\frac{d}{dt}x(t) = (a - by(t))x(t) \tag{42}$$
$$\frac{d}{dt}y(t) = (cx(t) - d)y(t) \tag{43}$$

Let $x^* = y^* = 0$. Then,

$$\boldsymbol{A} = \begin{bmatrix} a & 0 \\ 0 & -d \end{bmatrix} \implies \lambda = a, -d \tag{44}$$

Similarly, let $x^* = \frac{d}{c}, y^* = \frac{a}{b}$. Then,

$$\boldsymbol{A} = \begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix} \implies \lambda = \pm j\sqrt{ad} \tag{45}$$

21

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \tag{46}$$

It is not possible to have two distinct equilibrium points for a linear system since that implies that the line defining the system in the state space has two zeros, which is impossible.

22

In the case of $\boldsymbol{A}\left[\boldsymbol{A}\right]^\top$, every eigenvector $\boldsymbol{v}$ can also be replaced with $-\boldsymbol{v}$. The argument works the same with $\left[\boldsymbol{A}\right]^\top\boldsymbol{A}$ and $\boldsymbol{u}$. Thus, there are two choices for every vector in the SVD, leading to 2^n SVDs.

23

$$\boldsymbol{B} = \begin{bmatrix} 1 & 5 & 1 & 1 & 2 \\ 2 & 7 & 2 & 9 & 4 \\ 3 & 3 & 3 & 4 & 6 \end{bmatrix} \tag{47}$$

By definition, a singular value $\sigma \in \mathbb{R}^+$, so it has to be 4.04.

24

$$x[t+1] = ax[t] + bu(t) + e[t] \tag{48}$$

The least squares equation is

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \tag{49}$$

By least squares, we obtain $a = \frac{1}{2}, b = -\frac{1}{2}$.

25

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix} \tag{50}$$

Finding $x_2(t)$,

$$x_2(t+T) - x_2(t) = \int_t^{t+T} \frac{d}{d\tau} x_2(\tau)\, d\tau = Tu(t) \tag{51}$$

Since the component of $\boldsymbol{B}$ in the discretization in $x_2[t+1]$ is one, $T = 1$.