EECS 16B HW03

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1 Complex Numbers

1.a Length of z

$$|z| = \sqrt{x^2 + y^2} \tag{1}$$

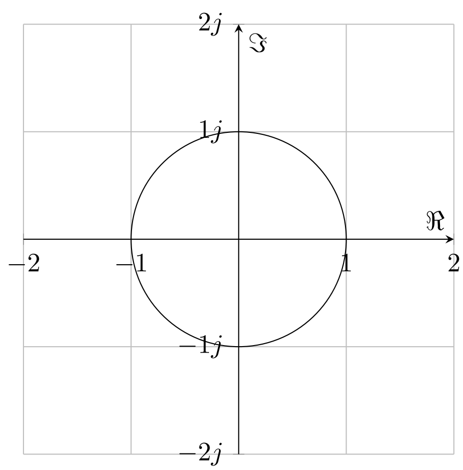
1.b Polar Representation

$$\Re(z) = x = r\cos(\theta) \tag{2}$$
$$\Im(z) = y = r\sin(\theta) \tag{3}$$

1.c Euler's Formula

$$z = x + jy = r\cos(\theta) + jr\sin(\theta) = r(\cos(\theta) + j\sin(\theta)) = re^{j\theta} \tag{4}$$

1.d Unit Complex Circle



z intersects the real axis at $z = \pm 1$. z intersects the imaginary axis at $z = \pm j$.

1.e

Proof.

$$
\begin{aligned}
z &= re^{j\theta} && (5)\\
re^{-j\theta} &= r\cos(-\theta) + jr\sin(-\theta) && (6)\\
&= r\cos(\theta) - jr\sin(\theta) && (7)\\
&= x - jy = \bar{z} && (8)
\end{aligned}
$$

□

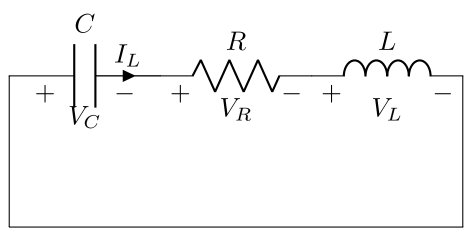
1.f

Proof.

$$
\begin{aligned}
z\bar{z} &= (x + jy)(x - jy) && \text{definition of conjugate multiplication} && (9)\\
&= x^2 - (jy)^2 && \text{difference of squares} && (10)\\
&= x^2 + y^2 = r^2 && \text{definition of } j && (11)
\end{aligned}
$$

□

2 RLC Responses: Initial Part



2.a

Using KVL,

$$V_C + V_R + L\frac{d}{dt}I_L = 0 \tag{12}$$

$$C\frac{d}{dt}V_C = I_L \tag{13}$$

Using simple algebra, we can rearrange to

$$\frac{d}{dt}I_L = -\frac{1}{L}(RI_L + V_C) \tag{14}$$

$$\frac{d}{dt}V_C = \frac{1}{C}I_L \tag{15}$$

2.b

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{16}$$

2.c

$$\begin{vmatrix} -\frac{R}{L}-\lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{vmatrix} = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0 \tag{17}$$

$$\lambda = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{18}$$

2.d

$$\lambda \in \mathbb{R} \iff \frac{R^2}{4L^2} - \frac{1}{LC} \geqslant 0 \tag{19}$$

2.e

$$\lambda \in \{jk | k \in \mathbb{R}, j^2 = -1\} \iff R = 0 \land \frac{1}{LC} \geqslant 0 \tag{20}$$

2.f

Finding the eigenvectors,

$$\begin{bmatrix} -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} & -\frac{1}{L} \\ \frac{1}{C} & \frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = \left(-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \right) \begin{bmatrix} 1 \\ y \end{bmatrix} \tag{21}$$

$$\Rightarrow y_1 = -2L\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{22}$$

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -2L\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \end{bmatrix} \tag{23}$$

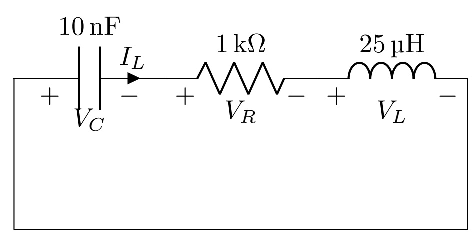
$$\Rightarrow y_2 = 2L\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{24}$$

$$\boldsymbol{v_2} = \begin{bmatrix} 1 \\ 2L\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \end{bmatrix} \tag{25}$$

2.g

$$\frac{d}{dt}\boldsymbol{z}(t) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \boldsymbol{z}(t) \tag{26}$$

3 RLC Responses: Overdamped Case



3.a

$$\boldsymbol{V} \approx \begin{bmatrix} 1 & 1 \\ -995 & 995 \end{bmatrix} \tag{27}$$

$$\boldsymbol{z}(0) = \boldsymbol{V}^{-1} \begin{bmatrix} 0\,\mathrm{A} \\ 1\,\mathrm{V} \end{bmatrix} \approx \begin{bmatrix} -502 \times 10^{-6} \\ 502 \times 10^{-6} \end{bmatrix} \tag{28}$$

3.b

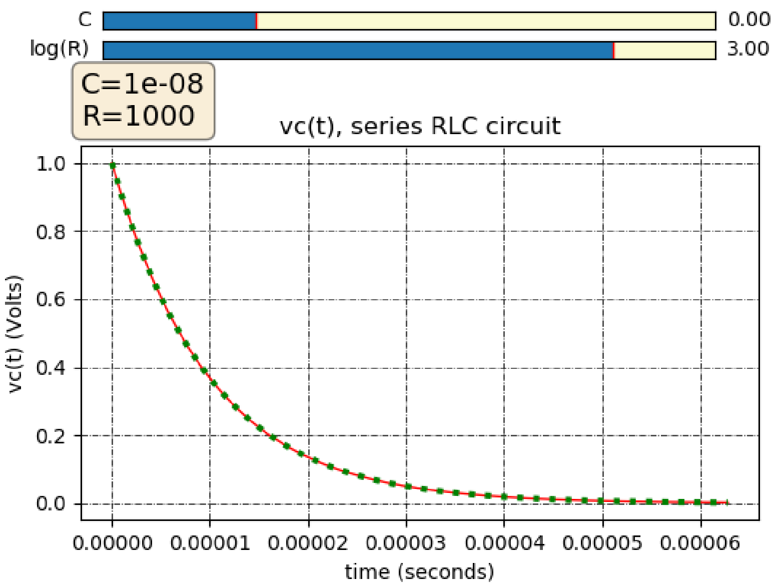
$$\lambda_1 \approx -100\,\mathrm{ks}^{-1} \tag{29}$$

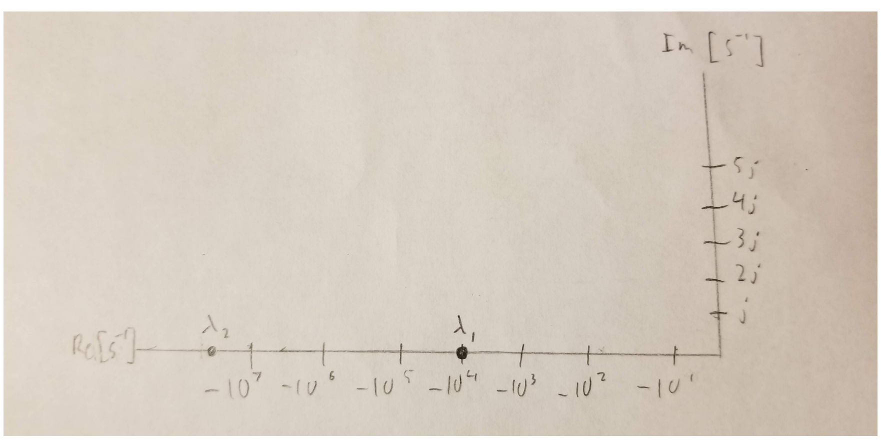
$$\lambda_2 \approx -40\,\mathrm{Ms}^{-1} \tag{30}$$

$$\boldsymbol{z}(t) = \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix} \tag{31}$$

$$\boldsymbol{x}(t) = \boldsymbol{V}\boldsymbol{z}(t) \approx \begin{bmatrix} 1 & 1 \\ -995 & 995 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} + e^{\lambda_2 t} \\ -995 e^{\lambda_1 t} + 995 e^{\lambda_2 t} \end{bmatrix} \tag{32}$$

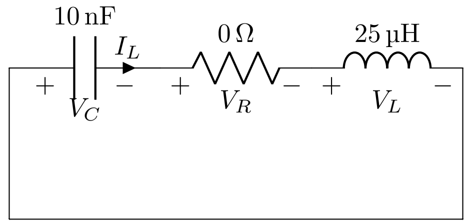
3.c





The graph seems to be a plain exponential curve, due to the heavy damping of the resistor.

4 RLC Responses: Undamped Case



4.a

$$\boldsymbol{V} \approx \begin{bmatrix} 1 & 1 \\ -100j & 100j \end{bmatrix} \tag{33}$$

$$\boldsymbol{z}(0) = \boldsymbol{V}^{-1} \begin{bmatrix} 0\,\mathrm{A} \\ 1\,\mathrm{V} \end{bmatrix} \approx \frac{1}{200} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{34}$$

4.b

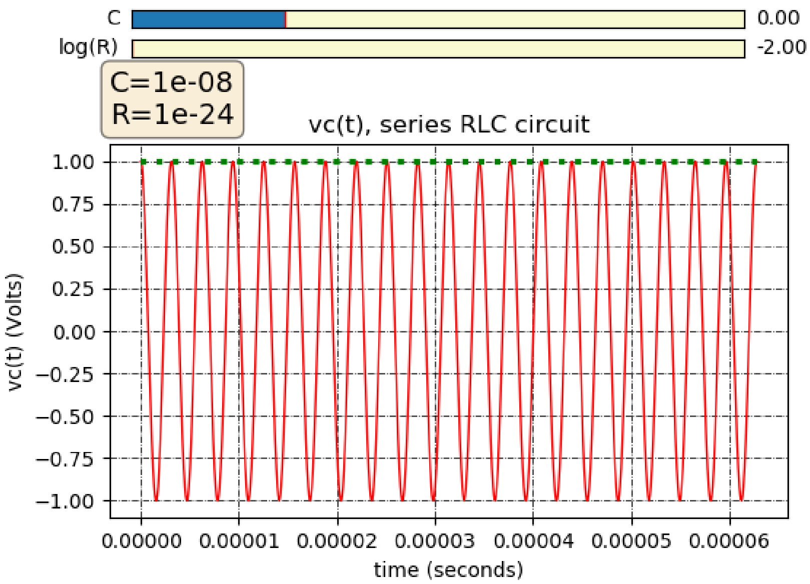
$$\lambda_1 = 2\mathrm{j}\,\mathrm{Ms}^{-1} \tag{35}$$

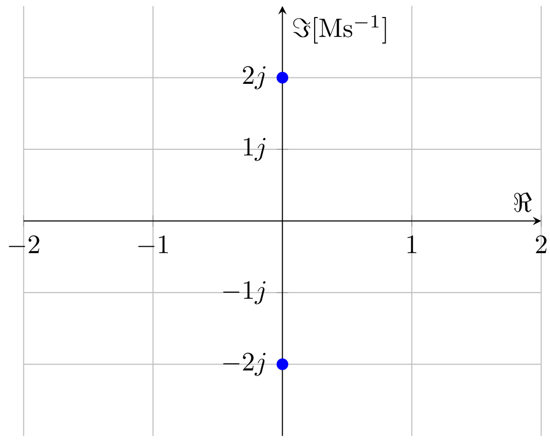
$$\lambda_2 = -2\mathrm{j}\,\mathrm{Ms}^{-1} \tag{36}$$

$$\boldsymbol{z}(t) = \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix} \tag{37}$$

$$\boldsymbol{x}(t) = \boldsymbol{V}\,\boldsymbol{z}(t) = \begin{bmatrix} 1 & 1 \\ -100j & 100j \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} + e^{\lambda_2 t} \\ -100je^{\lambda_1 t} + 100je^{\lambda_2 t} \end{bmatrix} \tag{38}$$

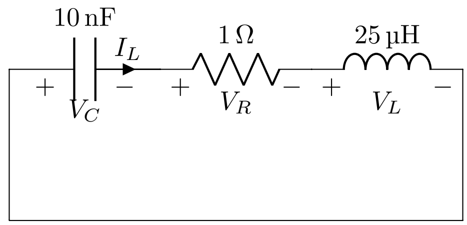
4.c





The case here shows that the circuit is *not* transient.

5 RLC Responses: Underdamped Case



5.a

$$\boldsymbol{V} \approx \begin{bmatrix} 1 & 1 \\ -100j & 100j \end{bmatrix} \tag{39}$$

$$\boldsymbol{z}(0) = \boldsymbol{V}^{-1} \begin{bmatrix} 0\,\mathrm{A} \\ 1\,\mathrm{V} \end{bmatrix} \approx \frac{1}{200} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{40}$$

5.b

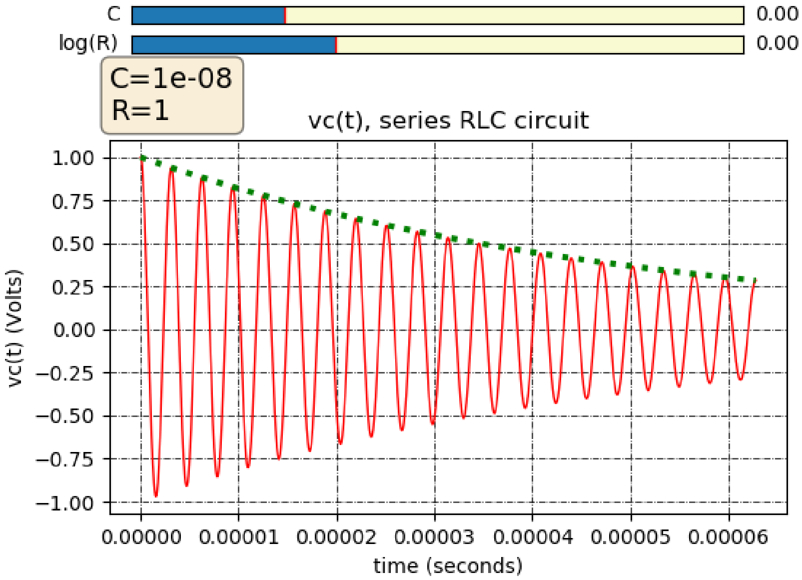
$$\lambda_1 = (-20 + 2000\mathrm{j})\,\mathrm{ks}^{-1} \tag{41}$$

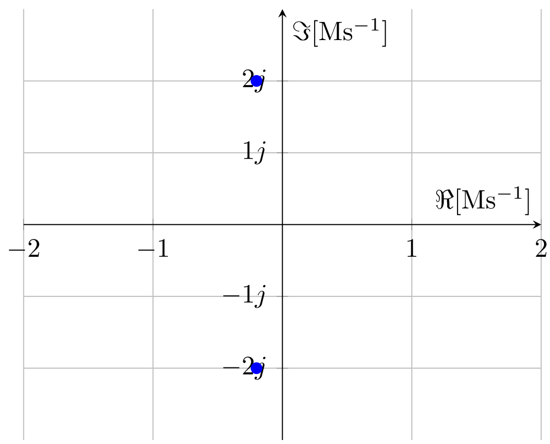
$$\lambda_2 = (-20 - 2000\mathrm{j})\,\mathrm{ks}^{-1} \tag{42}$$

$$\boldsymbol{z}(t) = \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix} \tag{43}$$

$$\boldsymbol{x}(t) = \boldsymbol{V}\boldsymbol{z}(t) = \begin{bmatrix} 1 & 1 \\ -100j & 100j \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} + e^{\lambda_2 t} \\ -100je^{\lambda_1 t} + 100je^{\lambda_2 t} \end{bmatrix} \tag{44}$$

5.c





The graph appears to exhibit transient behavior, due to the asymptotic decline.

5.d

Even though $\lambda_1, \lambda_2 \in \mathbb{C}$, it can be shown that the differential equation is nothing more than a linear combination of sine and cosine waves multiplied by an exponential function, due to the fact that

$$\sin(t) = \frac{e^{jt} - e^{-jt}}{2j} \tag{45}$$

$$\cos(t) = \frac{e^{jt} + e^{-jt}}{2} \tag{46}$$

6 RLC Responses: Critically Damped

6.a

There will only be one eigenvalue of A when $\frac{R^2}{4L^2} = \frac{1}{LC} \Rightarrow R = 2\sqrt{\frac{L}{C}}$.

6.b

$$\begin{vmatrix} -\frac{2}{L}\sqrt{\frac{L}{C}} - \lambda & -\frac{1}{L} \\ \frac{1}{C} & -\lambda \end{vmatrix} = \lambda^2 + \frac{2}{L}\sqrt{\frac{L}{C}}\lambda + \frac{1}{LC} = 0 \tag{47}$$

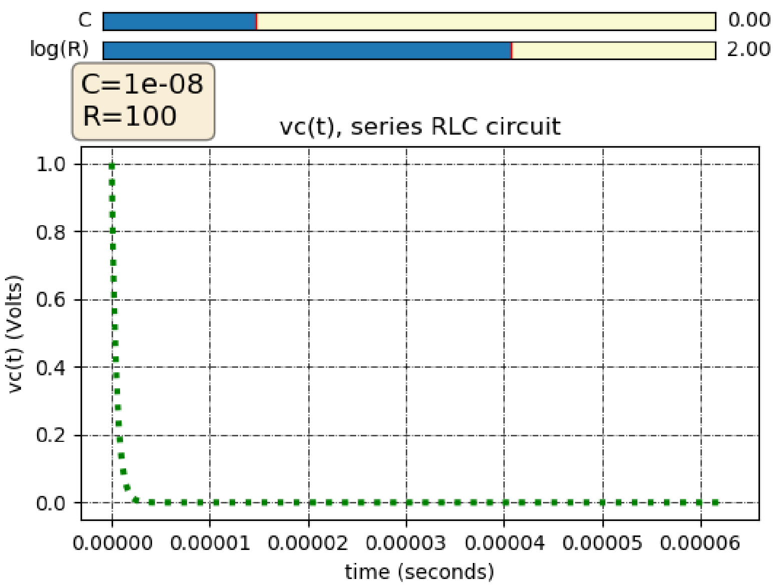
$$\lambda = -2\frac{1}{\sqrt{LC}} \tag{48}$$

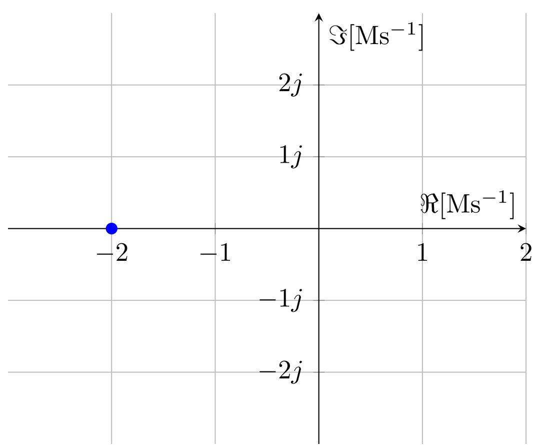
$$\begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{2}{\sqrt{LC}} \end{bmatrix} \begin{bmatrix} 1 \\ y \end{bmatrix} = -\frac{2}{\sqrt{LC}} \begin{bmatrix} 1 \\ y \end{bmatrix} \tag{49}$$

$$y = \frac{2L}{\sqrt{LC}}, -\frac{\sqrt{LC}}{4C} \tag{50}$$

Thus, we are able to find 2 eigenvectors for the eigenspace.

6.c





The graph seems to decay faster than at any other eigenvalue. If R is shifted up, then it becomes overdamped and the curve shallows. If R is shifted down, then it becomes underdamped and begins to oscillate.

8 Homework Process and Study Group

1. I referred to Note 3, Discussion 3B, and my lecture notes.
2. I worked on this homework by myself.
3. I did this homework in one sitting.
4. Around 4.5 hours.