EECS 16B HW07

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1 LED Strip

1.a

We can use a vector $\boldsymbol{x}$ that represents the brightness of all the LEDs,

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \tag{1}$$

where our input is the brightness of the left-most LED. We will assume leftmost to be the brightness x_1.

1.b

The system can be written is

$$\boldsymbol{x}[t+1] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \tag{2}$$

1.c

Proof. To prove controllability, we must prove that

$$\text{span}\{\boldsymbol{u}, \boldsymbol{A}\boldsymbol{u}, \boldsymbol{A}^2\boldsymbol{u}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{u}\} = \mathbb{R}^n \tag{3}$$

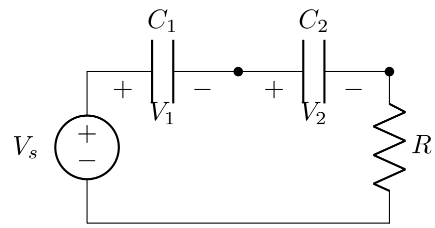
The system is controllable because every value of the state vector can be influenced by the input. This is because we have a sort of "daisy chain" between the LEDs, which means every brightness can be traced back to the input, with a certain delay. That is, we can change the brightness of any given LED at any time. ☐

1.d

$$x[0] = \begin{bmatrix} 0 \\ 127 \\ 0 \\ 255 \\ 0 \end{bmatrix} \tag{4}$$

We can simply maintain this brightness by looping through the state vector's entries and pushing the next brightness. By $t = 5$, the brightness will be the same. We can display any pattern by simply pushing the brightness one at a time.

2 Controllability in Circuits



2.a

Performing KCL at the node between the capacitors and across the resistor and using KVL,

$$V_s - V_1 - V_2 - V_R = 0 \tag{5}$$

$$C_2 \frac{d}{dt} V_2 - C_1 \frac{d}{dt} V_1 = 0 \tag{6}$$

$$\frac{V_s - V_1 - V_2}{R} - C_2 \frac{d}{dt} V_2 = 0 \tag{7}$$

In state space form,

$$\frac{d}{dt}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} \frac{C_2}{C_1}\frac{d}{dt}V_2 \\ \frac{V_s - V_1 - V_2}{RC_2} \end{bmatrix} = \begin{bmatrix} \frac{V_s - V_1 - V_2}{RC_1} \\ \frac{V_s - V_1 - V_2}{RC_2} \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} -\frac{1}{RC_1} & -\frac{1}{RC_1} \\ -\frac{1}{RC_2} & -\frac{1}{RC_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{RC_2} \end{bmatrix} V_s \tag{9}$$

2.b

Proof. At $t = 2$, the span is

$$\text{span}\left\{ \begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{RC_2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{R^2C_1^2} - \frac{1}{R^2C_1C_2} \\ -\frac{1}{R^2C_2C_1} - \frac{1}{R^2C_2^2} \end{bmatrix} \right\} = \mathbb{R} \tag{10}$$

If we focus on the $\boldsymbol{Ab}$ vector,

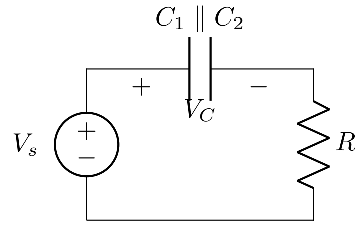
$$\begin{bmatrix} -\frac{1}{R^2C_1^2} - \frac{1}{R^2C_1C_2} \\ -\frac{1}{R^2C_2C_1} - \frac{1}{R^2C_2^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{RC_1}\left(-\frac{1}{RC_1} - \frac{1}{RC_2}\right) \\ \frac{1}{RC_2}\left(-\frac{1}{RC_1} - \frac{1}{RC_2}\right) \end{bmatrix} = \left(-\frac{1}{RC_1} - \frac{1}{RC_2}\right)\begin{bmatrix} \frac{1}{RC_1} \\ \frac{1}{RC_2} \end{bmatrix} \tag{11}$$

Since $\boldsymbol{Ab}$ is simply a scaled version of $\boldsymbol{b}$, this means that the span does not increase from $t = 1$ to $t = 2$. Thus, the span will never increase and remain at $\mathbb{R}$ and is uncontrollable. $\square$

2.c

The system does not appear to be controllable because V_1 is dependent on V_2, as is seen in **Equation 8**. This means that V_1 cannot be independently controlled by V_s, our input.

2.d



We can still control V_s in this system, but it is controllable now since every quantity is independently controllable via the input.

3 Controllability and Discretization

$$\frac{d}{dt}p(t) = v(t) \tag{12}$$
$$\frac{d}{dt}v(t) = u(t) \tag{13}$$

3.a

Converting to matrix form,

$$\frac{d}{dt}\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p(t) \\ v(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{14}$$

The span at $t = 2$ are

$$\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} = \mathbb{R}^2 \tag{15}$$

Since the two vectors are linearly independent (specifically, $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$), the system spans the required space and is thus, controllable.

3.b

Suppose we find the change in the state variables $p(t), v(t)$ according to a change T in time,

$$p(t+T) - p(t) = \int_t^{t+T} v(\tau)\, d\tau \tag{16}$$
$$v(t+T) - v(t) = \int_t^{t+T} u(\tau)\, d\tau = Tu(t) \tag{17}$$

Substituting $v(\tau) = v(t) + (\tau - t)u(t)$,

$$\begin{aligned} p(t+T) - p(t) &= \int_t^{t+T} v(t) + (\tau - t)u(t)\, d\tau && (18) \\ &= Tv(t) + \int_t^{t+T} (\tau - t)u(t)\, d\tau && (19) \\ &= Tv(t) + \frac{(\tau - t)^2}{2}u(t)\Big|_t^{t+T} && (20) \\ &= Tv(t) + \frac{T^2}{2}u(t) && (21) \end{aligned}$$

In matrix form, our discretized state system is

$$\begin{bmatrix} p[t+1] \\ v[t+t] \end{bmatrix} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p[t] \\ v[t] \end{bmatrix} + \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} u[t] \tag{22}$$

3.c

Our span at $t = 2$ is

$$\operatorname{span}\left\{ \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}, \begin{bmatrix} \frac{3T^2}{2} \\ \frac{T^2}{2} \end{bmatrix} \right\} = \mathbb{R}^2 \tag{23}$$

By inspection, the two vectors are clearly able to span $\mathbb{R}^2$, and thus the system is controllable.

4 Understanding the SIXT33N Car Control Model

$$v[k] = d[k+1] - d[k] = \theta u[k] - \beta \tag{24}$$

4.a

Setting $v[k] = v^*$,

$$u[k] = \frac{v^* + \beta}{\theta} \tag{25}$$

4.b

$$v[k] = \begin{cases} -\beta & u[k] = 0 \\ 255\theta - \beta & u[k] = 255 \end{cases} \tag{26}$$

We are able to slow the car down by reducing $u[k]$.

4.c

At $u[k] = 0$, $v[k] = -\beta$. This means that the car is supposedly going *backwards*, contrary to our intuition, which should be at $u[k] = 0$, $v[k] = 0$. The model *cannot* accurately describe this phenomenon.

4.d

We can determine $\theta_L, \theta_R, \beta_L, \beta_R$ simply by applying a series of known inputs and an expected series of velocity outputs and using statistical techniques like least squares to create θ, β.

4.e

Since the velocity curve of SIXT33N is nonlinear, we need to restrict our speeds to a very small range. This makes the curve look linear around that range. Then, since $\theta_L, \theta_R, \beta_L, \beta_R$ are separate values, we can use different parameters for each wheel, thus counteracting any deviance from the model.

6 Homework Process and Study Group

1. I used the Lecture 7B notes and Discussion 8A solutions.
2. I worked on this homework by myself.
3. I worked on this homework in one sitting.
4. Perhaps a little more explanation on the work going through **3.b** since it was not discussed in lecture.
5. 4 hours.