EECS 16B HW11

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1 Eigenvalues of an Upper Triangular Matrix

1.a

Theorem 1. *Given a matrix $\boldsymbol{A} \in \mathbb{C}^{n\times n}$ in the form*

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \tag{1}$$

$\boldsymbol{A} - a_{kk}\boldsymbol{I}$ does not have full rank for some $k \in [1, n]$.

Proof.

$$\boldsymbol{A} - a_{kk}\boldsymbol{I} = \begin{bmatrix} a_{11} - a_{kk} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ 0 & a_{22} - a_{kk} & a_{23} & a_{24} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{kk} - a_{kk} & \cdots & a_{kn} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} - a_{kk} \end{bmatrix} = \begin{bmatrix} a_{11} - a_{kk} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} - a_{kk} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & a_{nn} - a_{kk} \end{bmatrix} \tag{2}$$

$\boldsymbol{A} - a_{kk}\boldsymbol{I}$ nulls one of the entries of the matrix. that means that when we back-substitute the values of the matrix from $a_{nn} - a_{kk}$ upward, the k-th row does not have an unknown variable since it is nulled. In other words, you effectively lose a pivot when subtracting a diagonal entry. Therefore, the solution to the matrix will always be inconsistent. □

1.b

Theorem 2. *If $\boldsymbol{A} - \lambda\boldsymbol{I}$ does not have full rank, then λ is a diagonal value of $\boldsymbol{A}$.*

Proof. If $\lambda \neq a_{kk}$, then all the diagonal entries will be nonzero, and thus there is a pivot in every column and has full rank. Inversely, since $\boldsymbol{A} - \lambda\boldsymbol{I}$ is not full rank, there must be a zero along the diagonal given the result of **Theorem 1**. □

2 SVD

2.a

Theorem 3. *Given some $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and nonzero $\boldsymbol{x} \in \mathbb{R}^n$, $\|\boldsymbol{A}\boldsymbol{x}\| \geqslant \sigma_{min}\|\boldsymbol{x}\|$.*

Proof.

$$\begin{aligned}
\|\boldsymbol{A}\boldsymbol{x}\|^2 &= \|\boldsymbol{U}\boldsymbol{\Sigma}\,[\boldsymbol{V}]^\top\boldsymbol{x}\|^2 && (3)\\
&= \|\boldsymbol{\Sigma}\boldsymbol{x}\|^2 && (4)\\
&= \sum_{i=1}^{n}(\sigma_i x_i)^2 && (5)\\
&= (\sigma_1 x_1)^2 + (\sigma_2 x_2)^2 + \cdots + (\sigma_{min} x_{min})^2 && (6)\\
&\geqslant \sigma_{min}^2 x_1^2 + \sigma_{min}^2 x_2^2 + \cdots \sigma_{min}^2 x_{min}^2 && (7)
\end{aligned}$$

The above is true since singular values are, by definition, positive. Since the square root function is monotonically increasing, the inequality holds for the magnitude. □

2.b

$$\boldsymbol{x} = \begin{bmatrix}\sin(\theta)\\ \cos(\theta)\end{bmatrix} \tag{8}$$

The SVD of $\boldsymbol{A}$ cannot be determined due to the fact that an inconsistency appears. Using the fact that σ_1, σ_2 can be determined at the points were $\boldsymbol{x} = \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, respectively, we find $\sigma_1 = \sigma_2 = 3.6$. However, when evaluating the magnitude at $\theta = \frac{\pi}{4}$, we get the equation

$$\left\|\frac{1}{\sqrt{2}}\begin{bmatrix}3.6\\ 3.6\end{bmatrix}\right\| \stackrel{?}{=} 5 \tag{9}$$

which is simply not true.

2.c

$$\begin{aligned}
\|\boldsymbol{y}\| &= \|\boldsymbol{A}\boldsymbol{B}\boldsymbol{x}\| && (10)\\
&= \|\boldsymbol{U_A}\boldsymbol{\Sigma_A}\,[\boldsymbol{V_A}]^\top \boldsymbol{U_B}\boldsymbol{\Sigma_B}\,[\boldsymbol{V_B}]^\top\boldsymbol{x}\| && (11)\\
&= \|\boldsymbol{\Sigma_A}\boldsymbol{\Sigma_B}\boldsymbol{x}\| \leqslant \sigma_1^{(\boldsymbol{A})}\sigma_1^{(\boldsymbol{B})} && (12)
\end{aligned}$$

3 Otto the Pilot

$$\lambda = v \pm j\omega \tag{13}$$

3.a

The oscillatory transient can be described with the equation

$$y = e^{vt}\sin(\omega t) \tag{14}$$

Since the altitude at 5 min is tangent to $1 + e^{vt}$, we can simply solve for v by plugging known values from the graph,

$$\begin{aligned}
1 + e^{v\cdot 5} &= 1.4843 && (15)\\
\Re(\lambda) = v &= \frac{\ln(1.4853 - 1)}{5} \approx -0.15 && (16)
\end{aligned}$$

Then, to find ω, we simply find the wavelength, which is $T = 10\,\mathrm{min}$, so $\Im(\lambda) = \omega = \frac{\pi}{T} \approx 0.31\,\mathrm{rad\,min^{-1}}$.

3.b

$$\frac{d}{dt}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \tag{17}$$

$$\det\left(\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} - \lambda \boldsymbol{I}\right) = \lambda^2 - a_2\lambda - a_1 = 0 \tag{18}$$

Plugging in our values of λ,

$$\begin{bmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 \\ \lambda_2^2 \end{bmatrix} \tag{19}$$

$$\Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \approx \begin{bmatrix} -0.12 \\ -0.29 \end{bmatrix} \tag{20}$$

3.c

$$\det\left(\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} - \lambda \boldsymbol{I}\right) = \lambda^2 - a_2\lambda - a_1 \tag{21}$$

$$\Rightarrow \lambda = \frac{a_2}{2} \pm \sqrt{\frac{a_2^2}{4} - a_1} \tag{22}$$

If the system is to be critically damped, then the term inside the square root is to be zero, i.e.

$$\frac{a_2^2}{4} = a_1 \Longrightarrow a_2 = \pm 2\sqrt{a_1} \tag{23}$$

4 Balance – Linearizing a Vector System

$$(I_1 + (m_1 + m_2)L^2)\frac{d^2\theta_2(t)}{dt^2} = -K_t u(t) + (m_1 + m_2)Lg\sin(\theta_1(t)) \tag{24}$$

$$I_2\frac{d^2\theta_2(t)}{dt^2} = K_t u(t) \tag{25}$$

$$0.001\frac{d^2\theta_2(t)}{dt^2} = -0.025u(t) + 0.1\sin(\theta_1(t)) \tag{26}$$

$$(5 \times 10^{-5})\frac{d^2\theta_2(t)}{dt^2} = 0.025u(t) \tag{27}$$

4.a

The state model is

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ 100\sin(x_1) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -25 \\ 500 \end{bmatrix} u(t) \tag{28}$$

Our Jacobians are

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{29}$$

$$\boldsymbol{B} = \begin{bmatrix} 0 \\ -25 \\ 500 \end{bmatrix} \tag{30}$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \approx \begin{bmatrix} 0 & 1 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -25 \\ 500 \end{bmatrix} u(t) \tag{31}$$

4.b

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} 0 & -25 & 0 \\ -25 & 0 & -2500 \\ 500 & 0 & 0 \end{bmatrix} \implies \operatorname{span}(\boldsymbol{\mathcal{C}}) = \mathbb{R}^3 \tag{32}$$

Since the system is controllable, we are able to assign arbitrary closed-loop eigenvalues.

4.c

Using `numpy`,

```python
import numpy as np
import math


A = np.array([
    [0, 1, 0],
    [100, 0, 0],
    [0, 0, 0]
])
B = np.array([0, -25, 500])
K = np.array([20, 5, 0.01])


closed_loop = A + np.outer(B, K)
print(np.linalg.eigvals(closed_loop))
```

We get $\lambda \approx -116.60, -1.70 + 1.20\text{j}, -1.70 - 1.20\text{j}$. Since $\Re(\lambda) < 0$, the system is indeed stable.