EE 120 Midterm 1 Review Session

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If you want to follow along: head to <Slides location and/or link >

Slides are also posted at @# on Piazza.

HKN Drop-In Tutoring

- HKN has office hours every weekday from **11 AM - 3 PM** and **8 PM - 10 PM** on hkn.mu/ohqueue
- The schedule of tutors can be found at hkn.mu/tutor
  <Itemize the presenter hours here >

LCCDE

Linear Constant-Coefficient Difference Equations

- Discrete LTI systems can be described by LCCDEs in the following form

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{L=0}^{M} b_L x(n-L)$$

- Taking the Fourier transform on both sides and applying linearity and time-shifting properties results in

$$Y(e^{jw})[\sum_{k=0}^{N} a_k e^{-j\omega k}] = X(e^{jw})[\sum_{L=0}^{N} b_L e^{-j\omega L}]$$

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{\sum_{L=0}^{N} b_L e^{-j\omega L}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

Linear Constant-Coefficient Difference Equations

- LCCDEs for CT LTI systems have the following equations

$$\sum_{n=0}^{N} a_n \frac{d^n}{dt^n} y(t) = \sum_{m=0}^{M} b_m \frac{d^m}{dt^m} x(t)$$

$$H(\omega) = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{n=0}^{N} a_n (j\omega)^n}$$

DT LTI Systems

Discrete Signals and Systems

- A discrete signal is described by a function

$$x : \mathbb{Z} \to \mathbb{R}$$

- A delta function is the unit impulse and any signal can be decomposed into a sum of deltas

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \forall x$$

- A discrete system, H, takes an input signal, x, and yields an output signal, y

Linear and Time-Invariant Systems

Linearity

- **Scaling** an input by some amount scales the output by the same amount

$$\alpha x_1[n] \xrightarrow{H} \alpha y_1[n]$$

- **Superposition** of two inputs generates a superposition of two outputs

$$x_1[n] + x_2[n] \xrightarrow{H} y_1[n] + y_2[n]$$

Time-Invariance

- A time invariant system means that a shift in time in the input results in a shift in time in the output

$$x_1[n-N] \xrightarrow{H} y_1[n-N] \quad \forall N \in \mathbb{Z}$$

Impulse Response

- The impulse response is the output of a system for a $\delta[n]$ input

$$x[n] = \delta[n] \xrightarrow{H} y[n] = h[n]$$

Verify that knowing $h[n]$ allows you to determine the output $y[n]$ for any input in an LTI system. *Hint: decompose* $x[n]$

Proof and the Convolutional Sum

Proof Decompose $x[n]$

$$x[n] = \sum_k x[k]\delta[n-k]$$

Recall the definition of the impulse response

$$\delta[n] \xrightarrow{H} h[n]$$

Apply linearity and time-invariance to the definition of the impulse response

$$\sum_k x[k]\delta[n-k] \xrightarrow{H} \sum_k x[k]h[n-k]$$

Convolution

- This is the convolutional sum, and denoted by $(x * h)[n]$
- Note: $(x * h)[n] = (h * x)[n]$

LTI Example Problems

Determine if the system described by the input-output relationship is 1) linear and 2) time-invariant. *Credit: Discussion #1 EE120 Sp21*

1. $y[n] = 7x[n+1]$
2. $y[n] = x[n]x[n-1]$
3. $y[n] = e^{x[n]}$
4. $y[n] = x[-n]$
5. $y[n] = v[n]x[n]$, where v is some fixed signal

LTI Example Solutions

1. $y[n] = 7x[n+1]$
    - Linear: **Yes**
    - Time-invariant: **Yes**
2. $y[n] = x[n]x[n-1]$
    - Linear: **No**, any scaling factor becomes squared at the output
    - Time-invariant: **Yes**
3. $y[n] = e^{x[n]}$
    - Linear: **No**, exponentials are non-linear (verify at $n = 0$)
    - Time-invariant: **Yes**
4. $y[n] = x[-n]$
    - Linear: **Yes**
    - Time-invariant: **No**, $\delta[n-1]$ input results in $\delta[-n-1]$
5. $y[n] = v[n]x[n]$, where v is some fixed signal
    - Linear: **Yes**
    - Time-invariant: **No**, consider when $v[n] = \delta[n]$ for $x[n]$ and $\tilde{x}[n] = x[n-1]$; $y[0] = x[0]$ and $\tilde{y[0]} = x[-1]$

DT System Example Problem

Consider the *linear* DT system H

$$\delta[n-k] \xrightarrow{H} h_k[n] = \alpha^{|k|}u[n-k] \quad \forall|\alpha| < 1$$

Show that the output, $y[n]$, to a general input, $x[n]$ is

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

Credit: Midterm #1 EE120 Sp20

DT System Example Solution

Decompose $x[n]$

$$x[n] = \sum_{m=-\infty}^{+\infty} x[m]\delta[n-m]$$

Recall the definition of the system H

$$\delta[n-m] \xrightarrow{H} h_m[n]$$

Apply linearity

$$x[n] = \sum_{m=-\infty}^{+\infty} x[m]\delta[n-m] \xrightarrow{H} \sum_{m=-\infty}^{+\infty} x[m]h_m[n] = y[n]$$

Notice the differences in this sum from the usual convolutional sum.

DT System Example Problem Continued

Derive a closed-form expression (no summations) for the output $y[n]$ when the input $x[n] = u[n]$, the unit step

Recall:

$$h_k[n] = \alpha^{|k|} u[n-k] \quad \forall |\alpha| < 1$$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h_k[n]$$

$$\sum_{k=M}^{N} \alpha^k = \begin{cases} \frac{\alpha^{N+1} - \alpha^M}{\alpha - 1} & \alpha \neq 1 \\ N - M + 1 & \alpha = 1 \end{cases}$$

DT System Example Solution

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] h_k[n]$$

Change the bounds and plug in for $x[k]$ and $h_k[n]$

$$y[n] = \sum_{k=0}^{+\infty} 1 \cdot \alpha^{|k|} u[n-k]$$

$$y[n] = u[n] + \alpha u[n-1] + \alpha^2 u[n-2] + \dots$$

$$y[0] = 1$$

$$y[1] = 1 + \alpha$$

$$y[2] = 1 + \alpha + \alpha^2$$

$$y[n] = \sum_{p=0}^{n} \alpha^p = \frac{\alpha^{n+1} - 1}{\alpha - 1}$$

Fourier Transforms

Continuous Time Fourier Transform

- Given a continuous-time signal $x(t)$, you can apply the **Continuous Time Fourier Transform** (CTFT) to find its **spectrum**, $X(\omega)$.
- $X(\omega)$ is also known as the **frequency domain representation** of $x(t)$.
- The CTFT (sometimes known as the **CTFT analysis equation**) can be written as follows:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\, dt.$$

CTFT (Continued)

- Given the spectrum of a signal, you can find the time-domain representation using the inverse CTFT (also known as the **CTFT synthesis equation**):

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{i\omega t} \, d\omega.$$

- If $h(t)$ is the impulse response of LTI system $\mathcal{H}$, the CTFT of $h(t)$, $H(\omega)$ is known as the **frequency response** of $\mathcal{H}$.
    - For any input signal of the form $x(t) = e^{i\omega t}$, the output of the system will be $y(t) = H(\omega)x(t)$.
    - We sometimes call such an $x(t)$ an **eigenfunction** of $\mathcal{H}$.

CTFT (Continued)

- You can only use the CTFT analysis equation if $x(t)$ is absolutely integrable, i.e.

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty.$$

- For signals that are not absolutely integrable but have a CTFT representation, you can find the spectrum using other methods.
  - For periodic signals, you can calculate the CTFS and convert from the CTFS to the CTFT.
  - Later on, you will learn about properties of Fourier Transforms you can utilize.

Practice: CTFT

1. Find the CTFT of the signal $x(t) = e^t(u(t+3) - u(t-3))$.
2. Given that the CTFT of $y(t)$ is $Y(\omega) = \delta(\omega - \pi)$, find $y(t)$.

Practice: CTFT (Solutions)

1. $X(\omega) = \frac{2i}{1-i\omega} sin(3 - 3i\omega)$.

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt & &= \int_{-3}^{3} e^t e^{-i\omega t}\,dt \\ &= \int_{-3}^{3} e^{(1-i\omega)t}\,dt & &= \frac{1}{1-i\omega} e^{1-i\omega t}\Big|_{-3}^{3} \\ &= \frac{1}{1-i\omega}\left(e^{3-3i\omega} - e^{-(3-3i\omega)}\right) &&= \frac{2i}{1-i\omega}\sin(3-3i\omega). \end{aligned}$$

2. $y(t) = e^{i\pi t}$.

Practice: CTFT (Solutions)

1. $X(\omega) = \frac{2i}{1-i\omega} sin(3 - 3i\omega)$.
2. $y(t) = e^{i\pi t}$.

$$
\begin{aligned}
y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)e^{i\omega t}\, d\omega \qquad &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\pi)e^{i\omega t}\, d\omega \\
= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\pi)e^{i\pi t}\, d\omega &= \frac{1}{2\pi}e^{i\pi t}.
\end{aligned}
$$

Continuous Time Fourier Series

- For periodic continuous-time signals (where $x(t) = x(t+T), \forall t$), we can represent $x(t)$ as a sum of complex exponentials at multiples of the fundamental frequency $\omega_0 = \frac{2\pi}{T}$.
- The CTFS synthesis equation is as follows:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{i\frac{2\pi}{T}kt},$$

where X_k is the k^{th} **Fourier coefficient** of $x(t)$.

CTFS (Continued)

- X_k is can be calculated as follows (CTFS analysis equation):

$$X_k = \frac{1}{T} \int_{<T>} x(t) e^{-i\frac{2\pi}{T}kt}\, dt.$$

  Note that $\int_{<T>}$ denotes an integral over any interval of length T (for instance, 0 to T or $-T/2$ to $T/2$).

- You can convert from the CTFS to the CTFT as follows:

$$X(\omega) = 2\pi \sum_{k=-\infty}^{\infty} X_k \delta(\omega - \frac{2\pi k}{T}).$$

Finding CTFS Coefficients

- Determine the period, T, and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ of the signal.
- If you can write the signal as a sum of complex exponentials (for example, if it's a cos or a sin), pattern-match the sum with the CTFS synthesis equation.
- Otherwise, use the CTFS analysis equation to find expressions for the Fourier coefficients.

Practice: CTFS

1. Find the fundamental frequency and nonzero CTFS coefficients of $x(t) = \sin(\frac{\pi}{2}t) + \cos(2\pi t)$.
2. Find the fundamental frequency and nonzero CTFS coefficients of $y(t) = \sum_{n=-\infty}^{\infty} \delta(t - n)$.

Practice: CTFS (Solutions)

1. $\omega_0 = \pi/2$, $X_{-4} = \frac{1}{2}$, $X_-1 = \frac{-1}{2i}$, $X_1 = \frac{1}{2i}$, $X_4 = \frac{1}{2}$.
   First find the period of $x(t)$:
   $$x(t+T) = \sin(\frac{\pi}{2}(t+T)) + \cos(2\pi(t+T)) = x(t).$$
   The fundamental period of $x(t)$ is the smallest T such that both $\frac{\pi}{2}T$ and $2\pi T$ are integer multiples of 2π, so $T = 4$.
   $$\omega_0 = \frac{2\pi}{T} = \frac{\pi}{2}.$$

Practice: CTFS (Solutions)

1. $\omega_0 = \pi/2$, $X_{-4} = \frac{1}{2}$, $X_{-}1 = \frac{-1}{2i}$, $X_1 = \frac{1}{2i}$, $X_4 = \frac{1}{2}$.
   Now, we can write out $x(t)$ using Euler's formula and pattern-match with the CTFS synthesis equation.

$$\begin{aligned} x(t) &= \frac{1}{2i}\left(e^{\frac{\pi}{2}t} - e^{-\frac{\pi}{2}t}\right) + \frac{1}{2}\left(e^{2\pi t} + e^{-2\pi t}\right) \\ &= \frac{1}{2i}\left(e^{\frac{\pi}{2}t} - e^{-\frac{\pi}{2}t}\right) + \frac{1}{2}\left(e^{\frac{\pi}{2}4t} + e^{-\frac{\pi}{2}4t}\right) \end{aligned}$$

   Pattern-matching with

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{i\frac{\pi}{2}kT},$$

   we can see that $X_{-4} = \frac{1}{2}$, $X_{-}1 = \frac{-1}{2i}$, $X_1 = \frac{1}{2i}$, $X_4 = \frac{1}{2}$, and the rest of the CTFS coefficients are 0.
2. $\omega_0 = 2\pi$, $X_k = 1$, $\forall k$.

Practice: CTFS (Solutions)

- $\omega_0 = \pi/2$, $X_{-4} = \frac{1}{2}$, $X_{-}1 = \frac{-1}{2i}$, $X_1 = \frac{1}{2i}$, $X_4 = \frac{1}{2}$.
- $\omega_0 = 2\pi$, $Y_k = 1$, $\forall k$.

  $y(t)$ repeats itself every 1 timestep, so it has a period $T = 1$ and fundamental frequency $\omega_0 = 2\pi$.

  $y(t)$ cannot be easily written as a sum of sinusoids, so let's plug $y(t)$ into the CTFS analysis equation. Note that only one Dirac delta appears every period of the signal.

$$\begin{aligned} Y_k &= \frac{1}{T}\int_{-T/2}^{T/2} y(t)e^{-i\frac{2\pi}{T}t}\,dt \\ &= \int_{-1/2}^{1/2} \delta(t)e^{-i2\pi t}\,dt = e^{-i2\pi(0)} = 1. \end{aligned}$$

Frequency Domain and Convolution

- **Convolution** in the time domain corresponds to **multiplication** in the frequency domain.
- If $y(t) = x(t) * h(t)$ in the time domain, then $Y(\omega) = X(\omega)H(\omega)$ in the CTFT domain.
- Likewise, multiplication in the time domain corresponds to convolution in the frequency domain.
- If $y(t) = x(t)h(t)$, then $Y(\omega) = X(\omega) * H(\omega)$.