EECS 16A Midterm 2 Review Session

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Disclaimer

This is an unofficial review session and HKN is not affiliated with this course. Although some of the presenters may be course staff, the material covered in the review session may not be an accurate representation of the topics covered in and difficulty of the exam.

Slides are also posted at @836 on Piazza.

HKN Drop-In Tutoring

- HKN has office hours Monday, Wednesday, and Friday from **1 PM - 3 PM** and **8 PM - 10 PM** on hkn.mu/ohqueue
- The schedule of tutors can be found at hkn.mu/tutor

Resistors

Resistance

- A **resistor** is a circuit element used to:
  - *Dissipate/consume* energy
  - Lowers voltage / forces a *voltage drop* across its terminals
  - Restricts (resists) the *flow of current*
- **Ohm's Law**: $\Delta V = IR \leftarrow$ voltage-current relationship
- Unit of resistance: **Ohm** (Ω)
  - Also (from Ohm's law), we see that $1\Omega = (1V)/(1A)$. Why does that make sense?
- Good way to think about them is a *"bumpy road"* that prevents the electrical current from travelling smoothly

Resistivity (ρ)

- *How much resistance* a material naturally has.
- For example, metal has a much lower resistivity than plastic.
- **Physical equation** for resistance:

$$R = \rho \frac{L}{A}$$

  - **R** is the *resistance*
  - **L** is the material's *length*
  - **A** is the material's *cross-sectional area*
  - ρ is the *constant of resistivity* for that material
- Units of resistivity are in Ω-m

Quick Resistivity Question

We have a rectangular wire made out of copper (Cu) whose *cross-sectional area* is $1 \times 10^{-9}\,\mathrm{m}^2$ and whose *length* is $0.2\,\mathrm{m}$.
What is its resistance?

$(\rho_{Cu} = 1.68 \times 10^{-8}\,\Omega\mathrm{m})$

Quick Resistivity Question [Solution]

We have a rectangular wire made out of copper (Cu) whose *cross-sectional area* is $1 \times 10^{-9}\,\text{m}^2$ and whose *length* is $0.2\,\text{m}$.
What is its resistance?

($\rho_{Cu} = 1.68 \times 10^{-8}\,\Omega\text{m}$)

$$R = \rho\frac{L}{A} = 1.68 \times 10^{-8}\,\Omega\text{m} \cdot \frac{(0.2\,\text{m})}{1^{-9}\,\text{m}^2} = 3.36\,\Omega$$

Resistivity: Sanity Check

What is the resistance of a material when $L \to 0$? ∞?

What is the resistance of a material when $A \to 0$? ∞?

Does this make sense intuitively?

Resistivity: Sanity Check [Solution]

What is the resistance of a material when $L \to 0$? ∞?

When $L \to 0$, $R \to 0$. When $L \to \infty$, $R \to \infty$.

What is the resistance of a material when $A \to 0$? ∞?

When $A \to 0$, $R \to \infty$. When $A \to \infty$, $R \to 0$.

Does this make sense intuitively?

Yes! As the *length of the resistor increases*, it's **more difficult for electrons to move** across the resistor and vice versa.

For the area, as the *cross sectional area increases*m there is a **wider gap for electrons to flow**, so resistance decreases and vice versa.

It's useful to think of the analogy of *water flowing through a pipe.*

Resistivity: Sanity Check

What happens when we **double the length** of a resistor?

What happens when we **double the area** of a resistor?

Resistivity: Sanity Check [Solution]

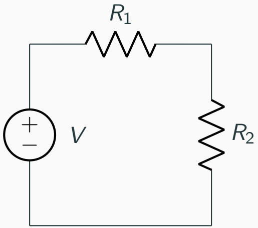
What happens when we double the length of a resistor?

Doubling the length **doubles the resistance**.

What happens when we double the area of a resistor?

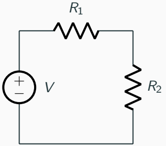
Doubling the area **halves the resistance**.

Resistors in Series



- **Series**: Every element is on the same path
- **Current** through R_1 is **the same as** the **current** through R_2.

Resistors in Series



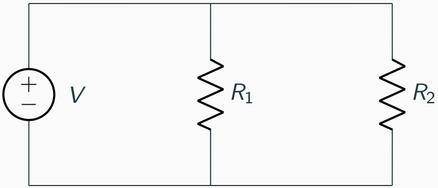
Want to model $V_{total} = IR_{total}$ (the **equivalent total resistance** in the circuit):

1. $V_1 = IR_1$
2. $V_2 = IR_2$
3. $V_{total} = V_1 + V_2$
4. $V_{total} = IR_1 + IR_2$
5. $V_{total} = I(R_1 + R_2)$
6. $R_{total} = R_1 + R_2$

TLDR: Just add them!

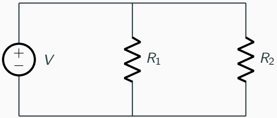
$$R_{total} = \sum_n R_n$$

Resistors in Parallel



- **Parallel**: When you have *multiple paths between two nodes*.
- **Voltage** across R_1 is **the same as** the **voltage** across R_2.

Resistors in Parallel



As with series resistors, we want to model $V_{total} = IR_{total}$:

1. $V_1 = V_2 = V_{total}$
2. $V_{total} = I_1 R_1$
3. $V_{total} = I_2 R_2$
4. $I_{total} = I_1 + I_2$
5. $V_{total}/R_{total} = V_1/R_1 + V_2/R_2$
6. $1/R_{total} = 1/R_1 + 1/R_2$

Resistors in Series

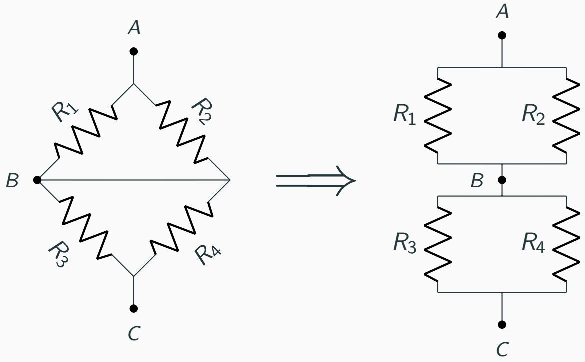
TLDR:

$$1/R_{total} = \sum_n 1/R_n$$

Equivalent Resistance: Steps to Solve

- **Decide** *what two nodes* you're finding your resistance over—normally, it will be the resistance between the *terminals of a voltage or current source, or between two open terminals*.
- **Break the problem down**: which resistors are in **parallel**? Which resistors are in **series**?
- **Use equivalent resistance equations** to *simplify resistors one "group" at a time* until you are left with a single resistor.
- **Redraw, redraw, redraw!** Sometimes, a circuit will be drawn to confuse you; in these cases, *carefully redraw the circuit in a more manageable way.*

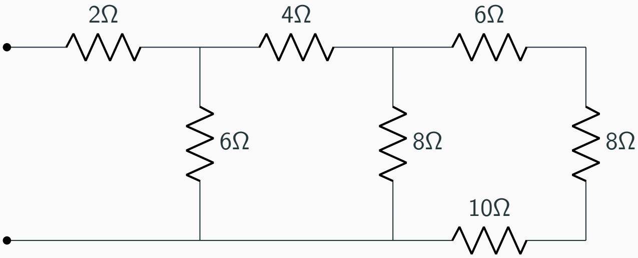
Equivalent Resistance



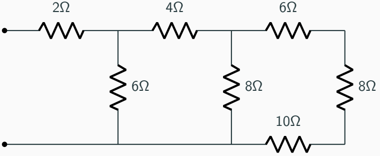
Careful! Make sure to *preserve all the nodes*. Remember that a single node consists of *all wires connected to a junction* (for example, node B in this circuit).

Problem: Resistor Equivalence

Find the equivalent resistance for this circuit:



Problem: Resistor Equivalence [Solution]

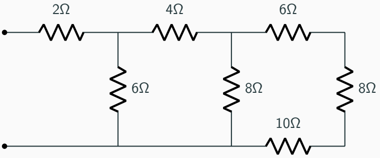


First, add up the three resistors on the right:
$R_{right} = (6 + 8 + 10)\Omega = 24\Omega$

That resistor group is in parallel with the 8Ω resistor to the left:
$1/R_{pt2} = 1/8 + 1/24 = 4/24 \Rightarrow R_{pt2} = 6\Omega$

Now, R_{pt2} is in series with the 4Ω resistor:
$R_{pt3} = 6\Omega + 4\Omega = 10\Omega$

Problem: Resistor Equivalence [Solution]



The resistor from the previous part is in parallel with the 6Ω resistor to its left:

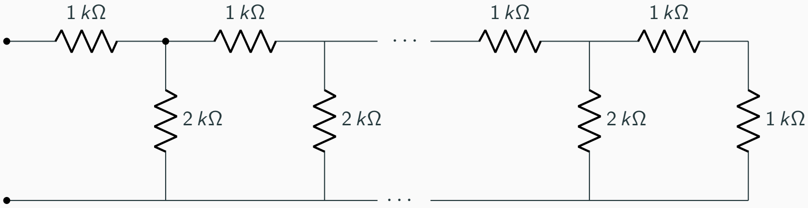
$1/R_{pt4} = 1/10 + 1/6 = 4/15 \Rightarrow R_{pt4} = 3.75\Omega$

Finally, the 2Ω resistor is in series with the rest of the circuit:

$R_{total} = 2\Omega + 3.75\Omega = 5.75\Omega$

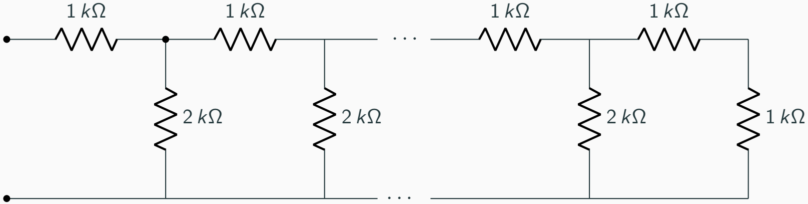
Problem: Infinite Series

Find the effective resistance between the two nodes (*Hint: start at the end and see if you can find a pattern*)



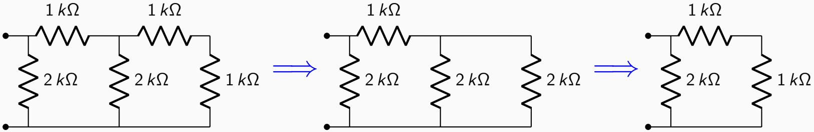
Problem: Infinite Series [Solution]

Find the effective resistance between the two nodes: $2\ k\Omega$

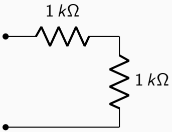


Problem: Infinite Series [Work]

First, look at the end:



Notice that the equivalent circuit is *the same as the three end resistors in the original circuit*. This pattern continues, until the whole circuit is reduced to:



So, the **equivalent resistance** of the whole circuit is $2\,k\Omega$.

Kirchhoff’s Laws

Kirchhoff's Current Law

$$\sum I_{node} = 0$$

- Comes from *conservation of charge*.
- Equivalent restatement: the **sum of current entering a node = sum of current leaving a node**
- Use KCL to *write an equation for each node* when solving circuits

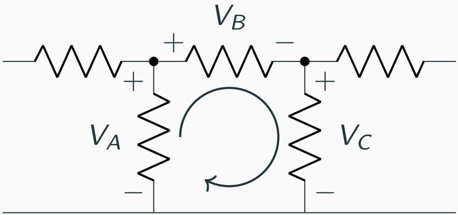
Kirchhoff's Voltage Law

$$\sum V_{loop} = 0$$

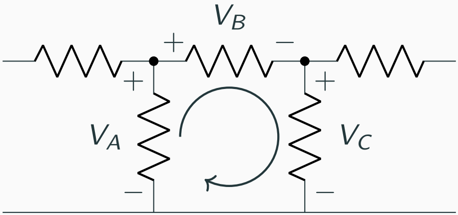
- **Net potential around any loop in a circuit is zero**
- *Careful!*
  - First, **define the + and - ends of each circuit element** in the circuit (use *passive sign convention*)
  - As you traverse the loop and pass by circuit elements, **ADD** the voltage if the loop goes from **the - to the + end**; **SUBTRACT** the voltage otherwise!

Practice: KVL

Try writing the KVL equation for this loop:



Practice: KVL [Solution]

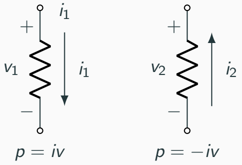


Remember to **add** voltages that go from - **to +** and **subtract** otherwise! So, the KVL equation for the center loop is:

$$\sum V = V_a = V_b + V_c$$

Passive Sign Convention

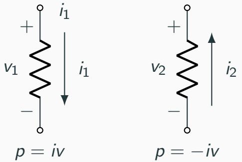
- Used to define **sign of power** in a circuit
- Electrical component *consumes power* $\rightarrow$ *positive power*
  - Current runs from $+$ *to* - *end* of component
- Electrical component *produces power* $\rightarrow$ *negative power*
  - Current runs from - *to* $+$ *end* of component
- Voltage drop across element with reference to current direction is **voltage at the base of the arrow minus the voltage at the tip of the arrow**.



In the figure, what are V_1 and V_2 in terms of V_+ and V_- if you use the passive sign convention?

Passive Sign Convention [Solution]

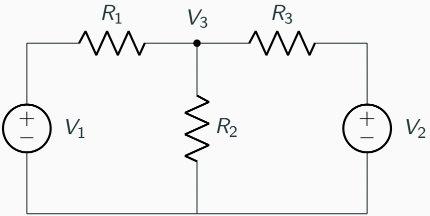
- Used to define **sign of power** in a circuit
- Electrical component *consumes power* $\rightarrow$ *positive power*
  - Current runs from $+$ *to* - *end* of component
- Electrical component *produces power* $\rightarrow$ *negative power*
  - Current runs from - *to* $+$ *end* of component
- Voltage drop across element with reference to current direction is **voltage at the base of the arrow minus the voltage at the tip of the arrow**.



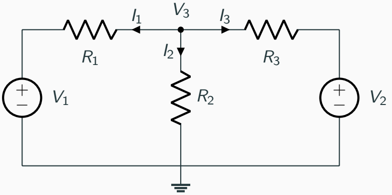
$V_1 = V_+ - V_-$ and $V_2 = V_- - V_+$

Practice: Solving for a Node Voltage

Solve for V_3 in terms of R_1, R_2, R_3, V_1, and V_2.
(Hint: Use KCL to relate each current coming out of node V_3 and Ohm's Law to express these currents in terms of voltages).



Practice: Solving for a Node Voltage [Solution]

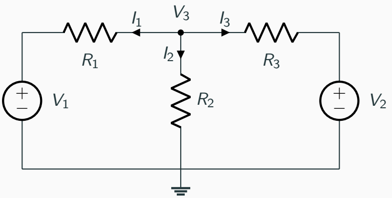


By doing **KCL** at V_3, we get: $I_1 + I_2 + I_3 = 0$

We can apply **Ohm's Law** to all three currents to get:

$$I_1 = \frac{V_3 - V_1}{R_1},\ I_2 = \frac{V_3}{R_2},\ I_3 = \frac{V_3 - V_2}{R_3}$$

Practice: Solving for a Node Voltage [Solution]



Plugging into the **KCL equation**, we have:

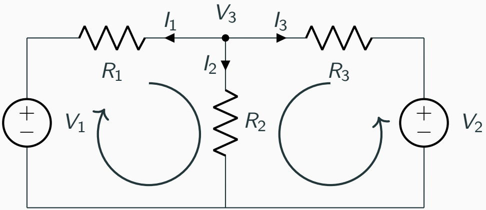
$$\frac{V_3 - V_1}{R_1} + \frac{V_3}{R_2} + \frac{V_3 - V_2}{R_3} = 0$$

$$V_3\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = \frac{V_1}{R_1} + \frac{V_2}{R_2}$$

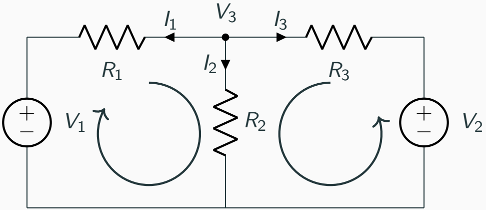
$$V_3 = \frac{\frac{V_1}{R_1} + \frac{V_2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}}$$

Practice: KVL

Write **KVL equations** around the left loop and the right loop.
(*Remember to label the* $+$ *and - terminals of all circuit elements!*)



Practice: KVL [Solution]

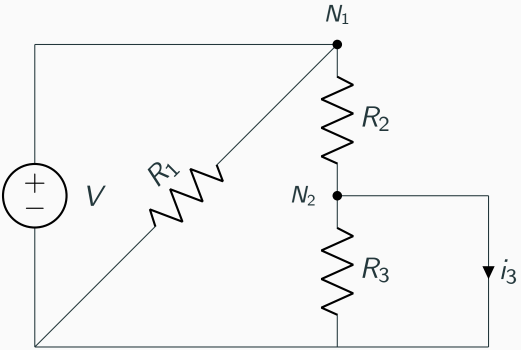


Left Loop: $\sum V = V_1 + I_1 R_1 - I_2 R_2$

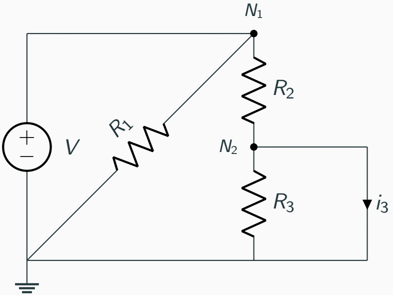
Right Loop: $\sum V = V_2 + I_3 R_3 - I_2 R_2$

Practice: Find i_3

Find i_3 in the following circuit:



Practice: Find i_3 [Solution]



$$i_3 = V/R_2$$

Notice that i_3 is *equivalent to the current through resistor* R_2 because no current goes through R_3. And since the voltage at node N_1 is V and the voltage at N_2 is 0, the current $i_3 = V/R_2$.

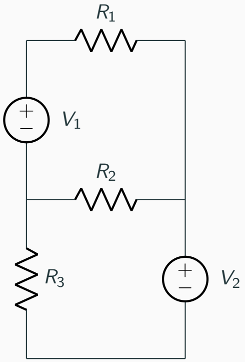
Nodal Analysis

Nodal Analysis Procedure

1. Pick a **ground** (reference) node.
2. **Label nodes** with voltage set by voltage sources.
3. Label remaining nodes.
4. Label **element voltages and currents** (*passive sign convention!*)
5. Set up *KCL equations*.
6. Find expressions for *element currents* (in EE16A, just use **Ohm's Law**).
7. Substitute element currents in KCL Equations.
8. Solve the system.

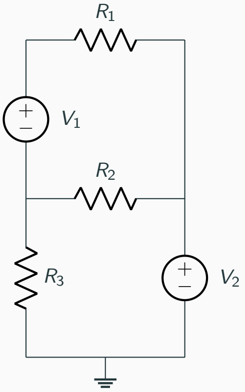
Nodal Analysis Walkthrough

Let's solve this circuit!



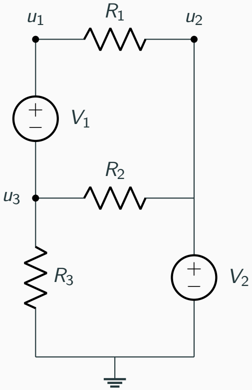
Nodal Analysis Walkthrough

1. Label a ground



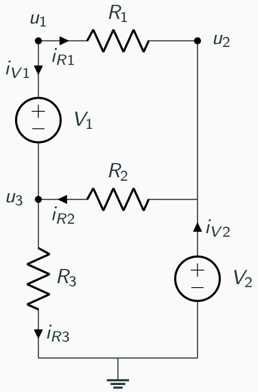
Nodal Analysis Walkthrough

1. Label a ground
2. Label all the nodes



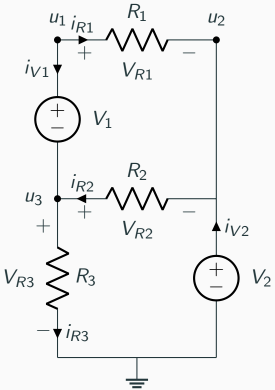
Nodal Analysis Walkthrough

1. Label a ground
2. Label all the nodes
3. Label all the currents



Nodal Analysis Walkthrough

1. Label a ground
2. Label all the nodes
3. Label all the currents
4. Label all the voltages (passive sign convention!)



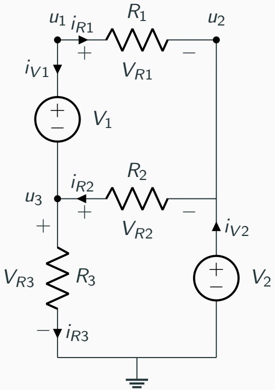
Nodal Analysis Walkthrough

Write **KCL equations** at each node:

At u_1: $i_{V1} + i_{R1} = 0$

At u_2: $i_{R1} + i_{V2} - i_{R2} = 0$

At u_3: $i_{V1} + i_{R2} - i_{R3} = 0$



Nodal Analysis Walkthrough

Write **I-V equations** (Ohm's Law):

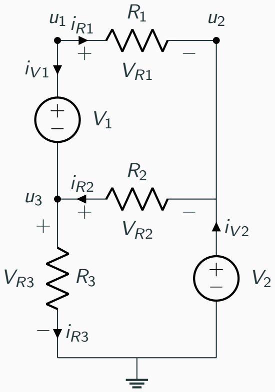
$V_{R1} = i_{R1} R_1 = u_1 - u_2$

$V_{R2} = i_{R2} R_2 = V_2 - u_3$

$V_{R3} = i_{R3} R_3 = u_3 - 0$

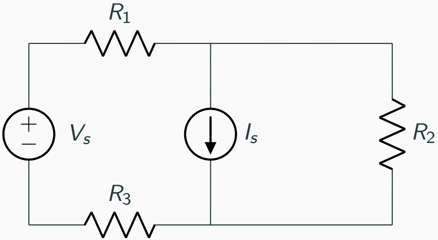
$V_1 = u_1 - u_3$

$V_2 = u_2 - 0$



Practice: Nodal Analysis

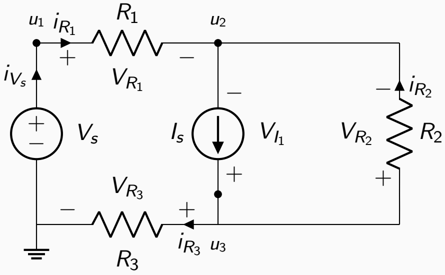
Solve the circuit! (find voltages at all nodes, currents through all circuit elements)



Where $V_s = 6\,V$, $I_s = 2\,A$, $R_1 = 2\,\Omega$, $R_2 = 4\,\Omega$, and $R_3 = 8\,\Omega$.

Practice: Nodal Analysis [Solution]

First, label **nodes**, **voltages**, and **currents**.



Practice: Nodal Analysis [Solution]

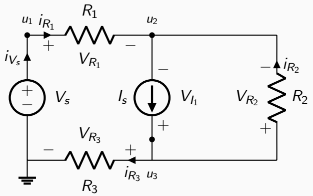
Long way: write out a system of equations in terms of currents and node voltages, plug into a matrix, and solve.

KCL equations:

At u_1: $i_{V_s} - i_{R_1} = 0$

At u_2: $i_{R_1} + i_{R_2} - I_s = 0$

At u_3: $I_s - i_{R_2} - i_{R_3} = 0$



I-V Relationships:

$V_{R_1} = i_{R_1} R_1 = u_1 - u_2$

$V_{R_2} = i_{R_2} R_2 = u_3 - u_2$

$V_{R_3} = i_{R_3} R_3 = u_3 - 0$

$V_s = u_1 - 0$

$V_{I_s} = u_3 - u_2$

Solving for known values:

$i_{V_s} - i_{R_1} = 0 \qquad i_{R_2} R_2 - u_3 + u_2 = 0$

$i_{R_1} + i_{R_2} = I_2 \qquad i_{R_3} R_3 - u_3 = 0$

$i_{R_2} + i_{R_3} = I_s \qquad u_1 = V_s$

$i_{R_1} R_1 - u_1 + u_2 = 0$

$V_{I_s} - u_3 + u_2 = 0$

Practice: Nodal Analysis [Solution]

You can now *put the equations into a matrix and solve* (Gaussian elimination not shown):

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ R_1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & R_2 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & R_3 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_{R_1} \\ i_{R_2} \\ i_{R_3} \\ i_{V_s} \\ V_{I_s} \\ u_1 \\ u_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ I_s \\ I_s \\ 0 \\ 0 \\ 0 \\ V_s \\ 0 \end{bmatrix}$$

Solving and *plugging in given values* gives:

$$\boxed{\begin{array}{llll} i_{R_1} = 1\,A & i_{R_2} = 1\,A & i_{R_3} = 1\,A & i_{V_s} = 1\,A \\ V_{I_s} = 4\,V & u_1 = 6\,V & u_2 = 4\,V & u_3 = 8\,V \end{array}}$$

Practice: Nodal Analysis [Solution]

Potentially shorter solution:

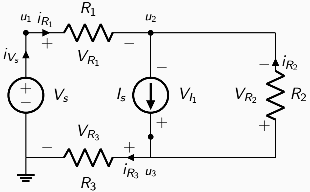
V_s is between u_1 and ground, so

$\boxed{u_1 = V_s = 6V}$

KCL at u_2 gives: $i_{R_1} + i_{R_2} = I_s \quad (1)$

KVL on left loop clockwise gives:

$V_s - i_{R_1} R_1 + V_{I_s} - i_{R_3} R_3 = 0 \qquad (2)$



Solving (1), (2) for i_{R1}, noticing that $i_{R_3} = i_{R_1}$:

$I_s - i_{R_2} = i_{R_1} = \frac{V_s}{R_1+R_3} + \frac{V_{I_s}}{R_1+R_3}$ but $V_{I_s} = V_{R_2} = i_{R_2} R_2$ (parallel)

$I_s - \frac{V_{I_s}}{R_2} = \frac{V_s}{R_1+R_3} + \frac{V_{I_s}}{R_1+R_3} \Rightarrow \boxed{V_{I_s} = 4\ V}$

Practice: Nodal Analysis [Soltuion]

Potentially shorter solution (continued):

Plugging V_{I_s} into (2) to solve for $i_{R_1} = i_{R_3}$:

$i_{R_1} = \frac{V_s + V_{I_s}}{R_1 + R_2} = 1\,A$ so $\boxed{i_{R_1} = i_{R_3} = 1\,A}$

Ohm's Law for R_3 gives: $u_3 = i_{R_3} R_3 \Rightarrow \boxed{u_3 = 8\;V}$

Looking at the voltage across the current source: $V_{I_s} = u_3 - u_2$

$u_2 = u_3 - V_{I_s} = 4\;V \Rightarrow \boxed{u_2 = 5\;V}$

Ohm's Law for R_2: $u_3 - u_2 = i_{R_2} R_2$

$i_{R_2} = \frac{u_3 - u_2}{R_2} = 1\,A \Rightarrow \boxed{i_{R_2} = 1\,A}$

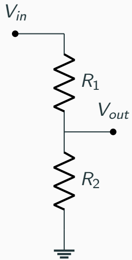
Voltage and Current Dividers

Voltage Divider

Super useful, a way to *lower the voltage by a desired amount by using resistors*:

$$\boxed{V_{out} = V_{in}\frac{R_2}{R_1 + R_2}}$$

Intuition: If R_1 is much greater, then most of the voltage is lost across R_1 , so V_{out} is small.



Voltage across resistor is **proportional to its resistance**

Can also be used to find *voltage across top resistor*.

Example: Voltage Dividers

If we want to have our output voltage be **half of our input voltage**, how can we use a **voltage divider** to do so?

Remember, V_{out} of a voltage divider is $V_{in}\frac{R_2}{R_1+R_2}$.

Example: Voltage Divider [Solution]

If we want to **halve our input voltage** how can we use a **voltage divider** to do so?

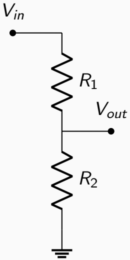
Solution: If we look at the equation

$V_{out} = V_{in}\frac{R_2}{R_1+R_2}$

We take $R_1 = R_2$ and then the fraction

$\frac{R_2}{R_1+R_2} = 1/2$

So, we halve our voltage!

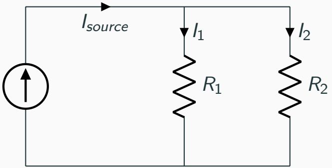


Current Divider

Similar to the voltage divider, a way to get a **fraction of the input current** based on the resistors of our circuit.

$I_1 = I_{source}\frac{R_2}{R_1+R_2}$

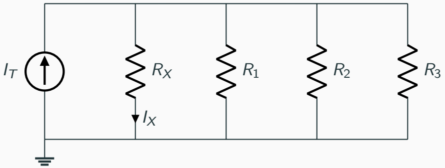
Intuitively: The more resistance in our R_2 value means that there will be **more current going through the first branch** (I_1).



Current Divider

Note in our formula

$I_X = I_T \frac{R_T}{R_X + R_T}$



The R_T corresponds to the **equivalent resistance** of the rest of the circuit. (Not including R_X)

Superposition

Superposition

If a circuit has linear elements (like resistors), we can analyze voltages and currents through it by **considering only one voltage/current source at a time**.

To do this, we **"zero" out** the other independent sources.

Zeroing out:

- **Voltage sources** become **ideal wires** ($V = 0$)
- **Current sources** become **open circuits** ($I = 0$)

Note: Superposition is a *long* process, but mostly mechanical.

Superposition: Steps to Solve

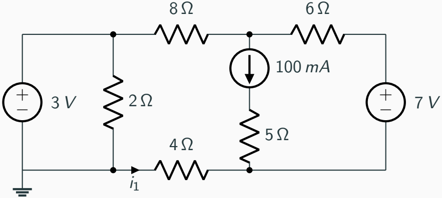
- **Note** what *goal* you want to achieve: are you finding a *voltage across two nodes*? A *current through a resistor*? Keep this in mind.
- **Identify** each **independent** voltage and current source you have in your circuit. You will have *one subproblem to solve per source*.
- **For each subproblem**:
  - For each source, **zero out the other independent sources** and redraw the circuit.
  - **Solve for your subgoal** once per subproblem.
- Once you have each subgoal, **add them all together**. That's your answer!

Practice: Superposition

Goal: Find i_1.

• How many subproblems will you have to solve?

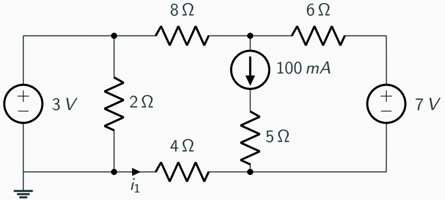
• Draw out your equivalent circuits for each subproblem.



Practice: Superposition [Solution]

Goal: Find i_1.

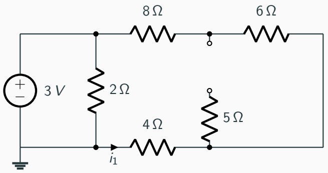
- How many subproblems will you have to solve? 3: one for each independent source!
- Draw out your equivalent circuits for each subproblem. (See next slide)

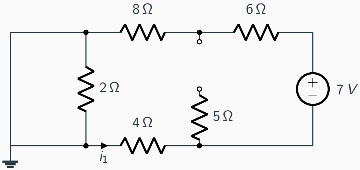


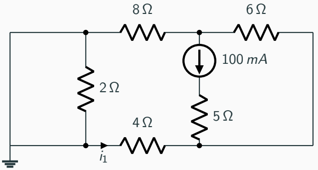
Practice: Superposition [Solution]

Goal: Find i_1.

- Draw out your equivalent circuits for each subproblem $\rightarrow$

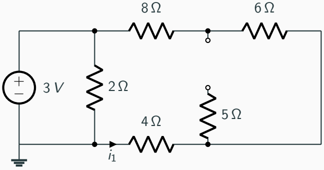


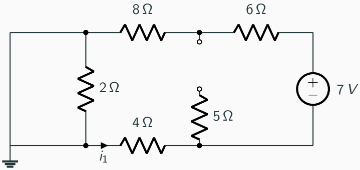


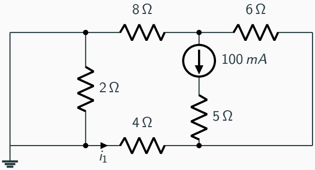


Practice: Superposition

Subgoals: Find i_1 for each subcircuit.

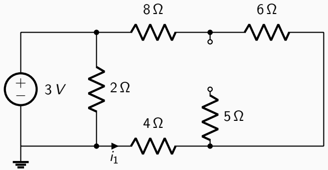






Practice: Superposition [Solution]

Subgoal: Find $i_{1,part1}$ for the $3\,V$ circuit.



Let V_{eq} be the voltage drop across the branch that includes the $4\,\Omega$ resistor. *Remember to follow passive sign convention!*

$V_{eq} = 0 - 3\,V = -3\,V$

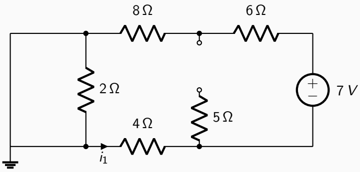
Let R_{eq} be the series combination of resistors on that branch.

$R_{eq} = 4\,\Omega + 6\,\Omega + 8\,\Omega = 18\,\Omega$

$$i_{1,part1} = \frac{V_{eq}}{R_{eq}} = \frac{-3\,V}{18\,\Omega} = -167\,mA$$

Practice: Superposition [Solution]

Subgoal: Find $i_{1,part2}$ for the $7\,V$ circuit.



Let V_{eq} be the voltage drop across the branch that includes the $4\,\Omega$ resistor.

$V_{eq} = 7\,V - 0 = 7\,V$

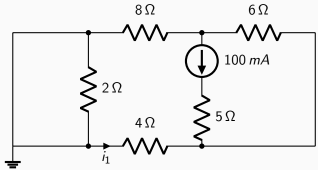
Let R_{eq} be the series combination of resistors on that branch. Note that the $2\,\Omega$ resistor is in parallel with a *short*, so we disregard it.

$R_{eq} = 6\,\Omega + 8\,\Omega + 4\,\Omega = 18\,\Omega$

$$i_{1,part2} == \frac{V_{eq}}{R_{eq}} = \frac{7\,V}{18\,\Omega} = 389\,mA$$

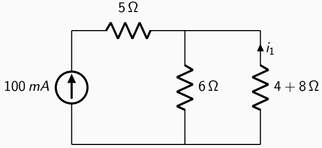
Practice: Superposition [Solution]

Subgoal: Find $i_{1,part3}$ for the 100 mA circuit.



Use a **current divider** and **equivalent resistances** to redraw the circuit!

Note: disregard the 2 Ω resistor because (in parallel with a short).



Use our *curent divider formula* (with a negation because i_1 is going in the opposite direction of the current source).

$$i_{1,part3} = -100\,mA \frac{6\,\Omega}{6\,\Omega + 12\,\Omega} = -33\,mA$$

Practice: Superposition [Solution]

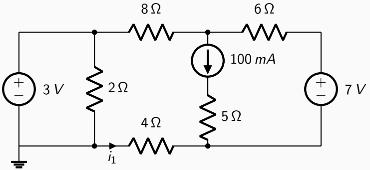
Goal: Find i_1

Add the result of each subpart:

$i_1 = i_{1,part1} + i_{1,part2} + i_{1,part3}$

$i_1 = -167\,mA - 33\,mA + 389\,mA$

$i_1 = 189\,mA$



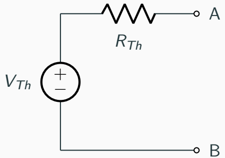
Note: you can also add up voltages this way using superposition.

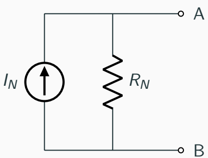
Thévenin and Norton Equivalent Circuits

Equivalent Circuits

Any **linear electrical network** with only voltage sources, current sources, and resistances can be replaced by either of the following:

- **Thévenin Equivalent**: An equivalent voltage source V_{Th} in series with an equivalent resistance R_{Th}.
- **Norton Equivalent**: An equivalent current source I_N in parallel with an equivalent resistance R_N





Equivalent Circuits: Steps to Solve

- Find **Thévenin voltage**, V_{Th}, by treating the output terminals as an **open circuit** and *finding the voltage across them.*
- Or, find the **Norton current**, I_N, by treating the output terminals as a **short circuit** and *finding the current through that short*.
    - Note: resistors in parallel with a short can be disregarded.

Equivalent Circuits: Steps to Solve

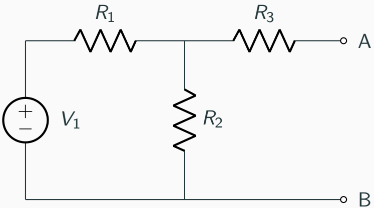
Find R_{Th} or R_N by either:

- Connecting a **test voltage source** V_{test} across the terminals and calculating I_{test}, the **current out** of the source, or
- Connecting a **test current source** I_{test} and measuring V_{test}, the **voltage across** the source.

Also, note that $R_{Th} = R_N = V_{Th}/I_N$.

Practice: Equivalent Circuits

Draw the **Thévenin and Norton equivalent circuits** for the following circuits across terminals A and B, with A as the $+$ terminal and B as the $-$ terminal.



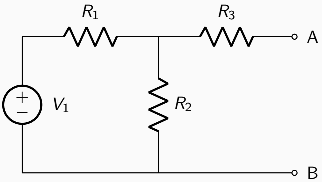
Where $V_1 = 10\,V$ and $R_1 = R_2 = R_3 = 200\,\Omega$.

Practice: Equivalent Circuits [Solution]

Find V_{Th}:

Note that R_3 does not affect V_{Th} because we're looking at the voltage across an open circuit.

So, V_{Th} is the voltage across R_2

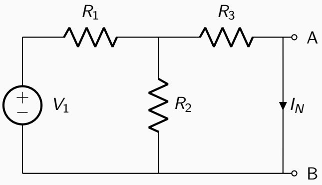


$$V_{Th} = V_1 \frac{R_2}{R_1 + R_2} = 10\,V \cdot \frac{200\,\Omega}{400\,\Omega} = 5\,V$$

Practice: Equivalent Circuits [Solution]

Find I_N:

We can use our **current divider** formula by finding I_{total} then using the relationship between R_2 and R_3 to find I_N.



$$I_{total} = \frac{V_1}{R_1 + (R_2||R_3)} = \frac{10\ V}{200(200||200)\,\Omega} = 1/30\,A$$

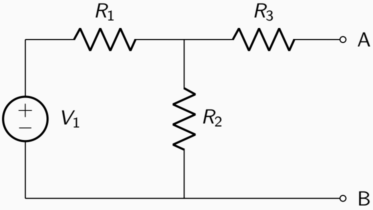
$$I_N = I_{total}\frac{R_2}{R_2 + R_3} = (1/30\,A) \cdot \frac{200\,\Omega}{400\,\Omega} = 1/60\,A$$

Practice: Equivalent Circuits [Solution]

Find $R_{Th} = R_N = R_{eq}$:

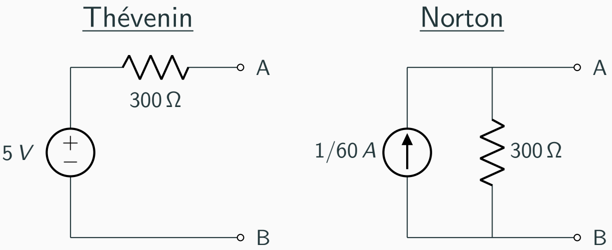
$R_{eq} = V_{Th}/I_N$

$R_{eq} = 5\,V/(1/60)\,A = 300\,\Omega$



Practice: Equivalent Circuits [Solution]

Finally, redraw the equivalent circuits:

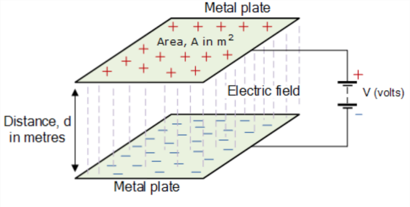


Capacitors and Capacitance

Introduction to Capacitors

Capacitor: generally two surfaces that **store charge**, with non-conductive material between plates.

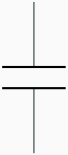
$$CV = Q$$
$$C = \epsilon A/d$$



- C: capacitance
- A: area of capacitor (one plate)
- d: distance between plates
- ϵ: "permittivity", a constant depending on the material in the space between the two plates

Circuit model of a capacitor

Unit is the **Farad** (F) $\rightarrow$ Coulombs per volt (C/V)



$C = Q/V$
capacitance = charge/voltage ($F = C/V$)

$E = \frac{1}{2}QV = \frac{1}{2}CV^2$
energy = 1/2 * capacitance * voltage squared ($J = C/V \cdot V^2 = CV$)

Sanity Check: Parallel Plate Capacitor

$$C = \epsilon \frac{A}{d}$$

What is the **capacitance** of pair of parallel plates when

- $A \to 0$?
- $A \to \infty$?
- $d \to 0$?
- $d \to \infty$?

Does this make sense intuitively?

Sanity Check: Parallel Plate Capacitor [Solution]

$$C = \epsilon \frac{A}{d}$$

What is the **capacitance** of pair of parallel plates when

- $A \to 0$? $C \to 0$
- $A \to \infty$? $C \to \infty$
- $d \to 0$? $C \to \infty$
- $d \to \infty$? $C \to 0$

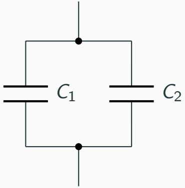
Does this make sense intuitively?

Yes, since as the area of the capacitor increases, the capacitor can hold more charge and vice versa. As the distance between the plates decreases, the charges escape to the other plate more easily.

Capacitors in Parallel

We know that the two capacitors must be at the **same voltage** but *not necessarily have the same charge*. So:

$$C_{eq} = Q/V$$
$$C_{eq} = (Q_1 + Q_2)/V$$
$$C_{eq} = Q_1/V + Q_2/V$$
$$C_{eq} = C_1 + C_2$$



TLDR: Just add them

$C_{eq} = \sum_n C_n$

Capacitors in Series

We know that both C_1 and C_2 have the **same charge** Q stored in them since the *current going through each of the capacitors must leave through the other*. On the other hand, the voltages **sum to the total voltage**.

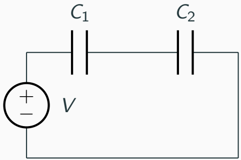
Knowing this:

$Q = C_{eq}(V_1 + V_2)$

$1/C_{eq} = (V_1 + V_2)/Q$

$V_1/Q + V_2/Q$

$1/C_{eq} = 1/C_1 + 1/C_2$



TLDR:

$$1/C_{eq} = \sum_n 1/C_n$$

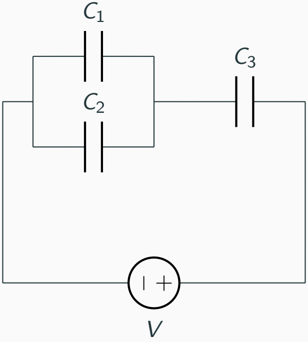
Equivalent Capacitance: Steps to Solve

- **Decide** what **two nodes** you're finding your capacitance over.
  - Normally, it will be the capacitance between the *terminals of a voltage or current source, or between two open terminals*.
- **Break the problem down**: which capacitors are in **parallel**? Which capacitors are in **series**?
- **Use these equivalent capacitance equations** to simplify capacitances one "group" at a time until you are left with a single capacitance.

Note: Capacitor equations are exactly opposite of resistor equations!

Practice: Equivalent Capacitance

Find the **total capacitance** in this circuit.



Practice: Equivalent Capacitance [Solution]

Find the **total capacitance** in this circuit.

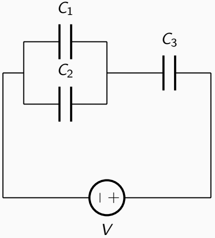
The parallel portion becomes:

$C_{par} = C_1 + C_2$

Then we use our series equation:

$1/C_{eq} = 1/C_{par} + 1/C_3 = \frac{C_{par}+C_3}{C_{par}C_3}$

$C_{eq} = \frac{C_{par}C_3}{C_{par}+C_3} = \frac{C_1C_3+C_2C_3}{C_1+C_2+C_3}$



Charging a Capacitor

- When a capacitor is supplied with current, it **charges up**.
- $V = Q/C$, so the voltage *increases with time*.
- When the capacitor **discharges**, it *loses charge and (therefore) voltage*.

Working with charges over time:

- Charge on capacitor after t seconds (constant current): $I \cdot t$.
- $Q_{final} = C(V_{final} - V_{init})$
- $I = CdV/dt = C\Delta V/\Delta t$
- $C = I\Delta t/\Delta V$
- $Q_{final} = C\Delta V = I\Delta t = I(t_{final} - t_{init})$

Note: we'll be working with discrete time in 16A.

Power

Power (Definition)

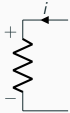
Power is the amount of energy supplied/dissipated per unit time:

- Units: Watts (Joules / second)
- General equation: $P = IV$
  - For resistors, $P = IV = V^2/R = I^2 R$

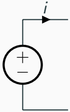
Power Conventions

We use **"passive sign convention"**

- Elements **consuming power** have current *entering the higher voltage node*:



- Elements **supplying power** have current *exiting the higher voltage node*:



Power

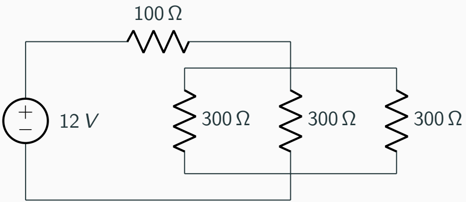
A circuit element either **supplies** or **consumes** power.

- Voltage and current sources **supply power**
- Resistors always **consume power**
  - *Convert electrical energy to heat*
- Capacitors can *either supply or consume* power.
  - Consumes energy when charging the capacitor
  - Supplies energy when discharging the capacitor

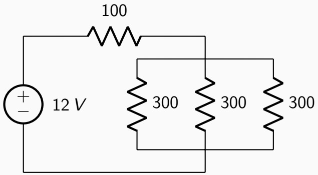
Practice: Power

A. Find the **total power consumed** by the resistors.

B. Find the **power supplied** by the battery.



Practice: Power [Solution]



A: Power dissipated by resistor:

Equivalent Resistance:

- Parallel network: $1/R_p = 1/300 + 1/300 + 1/300 = 1/100$
- $R_p = 100$, so $R_{eq} = 100 + 100 = 200$

The voltage across the resistors is 12 V, so $P = V^2/R = 12^2/200 = 0.72\ W$

B: Power supplied by battery:

By *conservation of energy*, **total power = 0**, so power consumed is the equal to power supplied.

$P = 0.72\ W$

Charge Sharing

Charge Sharing

Capacitors can be first charged up, then *reconfigured into a different circuit*, usually via switches.

States to Analyze:

1. **Initial state**: after charging up
2. **Final state**: after charges redistribute in new configuration

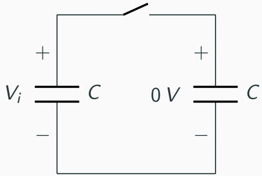
NOTE! *Charge at a floating node is *always* conserved from phase 1 to 2.*

Charge Sharing: Steps to Solve

- **Note how many states** you will have to find charges for, and **draw their equivalent circuits**. Generally, there are two: an *initial state and a final state*.
  - Sometimes, there will be intermediate stages – but you *solve those much like you will a two-state problem*.
- Find the **charges on all capacitors in your initial state** in terms of your knowns.
  - Most often, you'll be given an *initial charge on one cap, or a voltage source, and the capacitances of all caps*.
  - For 16A, this is generally taken once the charges stabilize.
- Find the **charges on all capacitors in your second state**, knowing that $Q_{total,final} = Q_{total,init}$ at the floating node(s) in your system because **charge is conserved**, in terms of your knowns.

Practice: Charge Sharing

Suppose the left capacitor has an initial voltage of V_i.

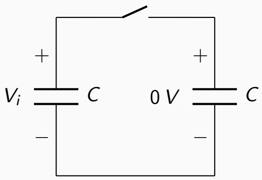


What happens when we close the switch?

Practice: Charge Sharing [Solution]

Assuming ideal switch connection, **total charge must be conserved**.

Phase 1 is when the switch is open and Phase 2 is after we *close the switch and reach steady state*. Let V_f be the voltage across the capacitors in Phase 2.



$Q_{total,\phi_1} = Q_{1,\phi_1} + Q_{2,\phi_1} = CV_i + C(0) = Q_{1,\phi_1}$

$Q_{total,\phi_2} = Q_{1,\phi_2} + Q_{2,\phi_2} = CV_f + CV_f = 2CV_f = 2Q_{1,\phi_2}$

But, $Q_{total,\phi_1} = Q_{total,\phi_2}$. Therefore, $Q_{1,\phi_2} = Q_{1,\phi_1}/2$

$E_{\phi_1} = \frac{{Q_{1,\phi_1}}^2}{2C}$

$E_{\phi_2} = \frac{{Q_{1,\phi_2}}^2}{2C} + \frac{{Q_{2,\phi_2}}^2}{2C} = 2\frac{{Q_{1,\phi_2}}^2}{2C} = \frac{1}{2}\frac{{Q_{1,\phi_1}}^2}{2C}$

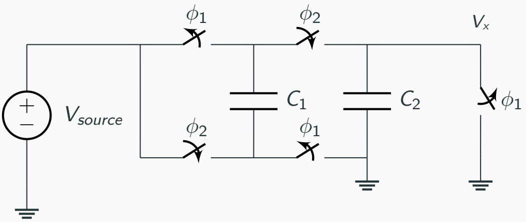
Therefore, $E_{\phi_2} = E_{\phi_1}/2$. So energy was dissipated between Phase 1 and Phase 2!

Challenge Practice: Charge Sharing

Phase 1: all ϕ_1 switches are **closed**, and all ϕ_2 switches are **open**.

Phase 2: all ϕ_1 switches are **open**, and all ϕ_2 switches are **closed**.

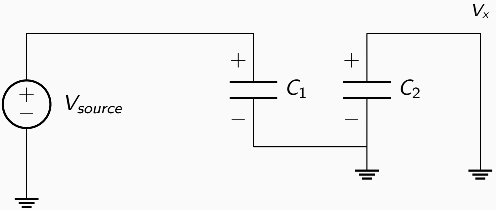
What is V_x during phase 2 if $C_1 = C$ and $C_2 = 9C$?



Challenge Practice: Charge Sharing [Solution]

Phase 1

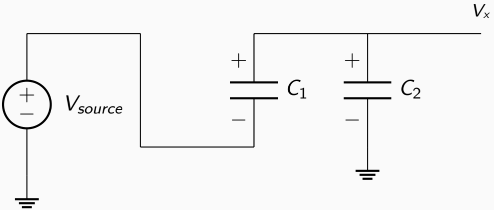
$$Q_1 = C_1 V_1 = C_1(V_S - 0) = CV_S$$
$$Q_2 = C_2 V_2 = (9C)(0 - 0) = 0$$



Challenge Practice: Charge Sharing [Solution]

Phase 2

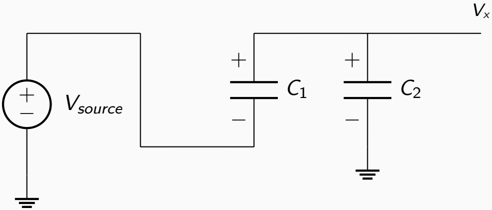
$$V_1 = V_x - V_S \;\rightarrow\; Q_1 = C_1(V_x - V_S) = C(V_X - V_S)$$
$$V_2 = V_X \;\rightarrow\; Q_2 = C_2 V_X = 9CV_X$$



Challenge Practice: Charge Sharing [Solution]

Total charge in Phase 1 = Total charge in Phase 2

$$Q_{1,\phi_1} + Q_{2,\phi_1} = Q_{1,\phi_2} + Q_{2,\phi_2}$$
$$CV_S + 0 = C(V_X - V_S) + 9CV_X$$
$$2CV_S = 10CV_X \;\rightarrow\; V_X = V_S/5$$



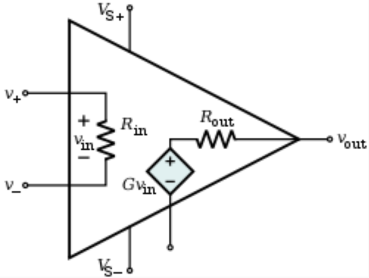
Op-Amps

Important facts: **always applicable** (in 16A):

$$V_{out} = G(V_+ - V_-)$$

$$V_{S+} \geq V_{out} \geq V_{S-}$$

The last fact says that the output voltage **"clips" if the input voltage difference is too large**.

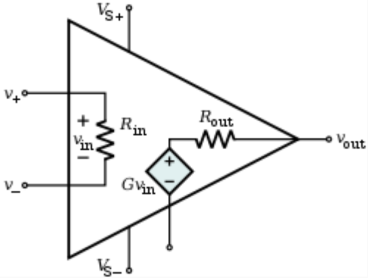


Op-Amps: Gain

Gain can be defined differently depending on the problem.

Sometimes gain is just defined as the **ratio** of V_{out} to $V_{+} - V_{-}$.

Other times, with a *single voltage source* V_{in}, it can be defined as the ratio of V_{out} to V_{in}.

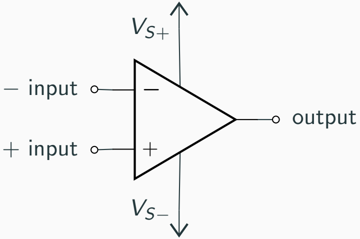


Usually, “gain” refers to **voltage gain**, and will be $G = V_{out}/V_{in}$.

Comparators

A device that **compares two voltages** and outputs a digital signal indicating which is larger.

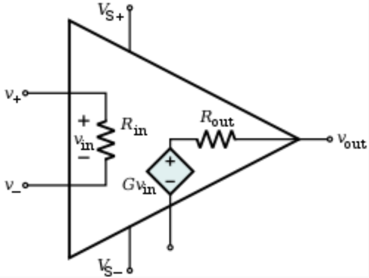
An **op-amp** can act as a comparator because, when $V_+ > V_-$, it outputs V_{S+}, and when $V_+ < V_-$, it output V_{S-}.



Golden Rules

For **ideal op-amps**, take
$G \to \infty,\ R_{in} \to \infty,\ R_{out} \to 0$

For all ideal op-amps, **input terminals draw no current**:
$I_- = I_+ = 0$

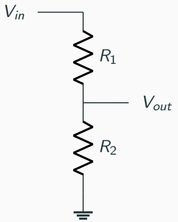


For all ideal op-amps in **negative feedback**, there is **no voltage difference** between the two input teminals: $V_- = V_+$

Loading

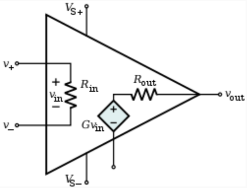
<u>Load-dependent Circuit: Voltage Divider</u>

V_{out} depends on the **load**, or the *current drawn from the V_{out} terminal*. The voltage divider formula for V_{out} can only be used if V_{out} is not loaded, i.e., no current is drawn from it.



<u>Load-independent Circuit: Ideal Op-Amp</u>

Ideal op-amps: V_{out} is **INDEPENDENT of the load**. The *internal voltage source* guarantees that V_{out} is **kept the same**. (*But the current produced from the output can be different depending on the load*.)

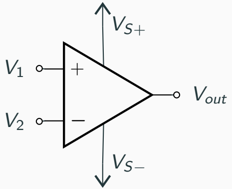


Op-Amp Configurations

Basic Op-Amp Configurations

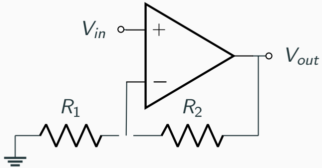
With **ideal op-amps**, you should use op-amp configurations as **building blocks** without regarding the effect of one block's R_{eq} on another block:

Voltage Comparator



$$V_{out} = \begin{cases} V_{S+} & V_1 > V_2 \\ V_{S-} & V_1 < V_2 \end{cases}$$

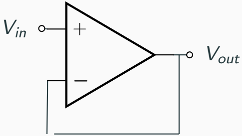
Non-Inverting Amplifier



$$V_{out} = V_{in} \cdot (1 + R_2/R_1)$$

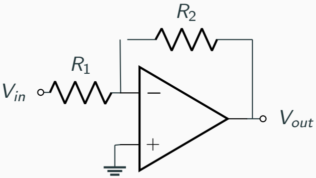
Basic Op-Amp Configurations

Unity Gain Buffer



$$V_{out} = V_{in}$$

Inverting Amplifier



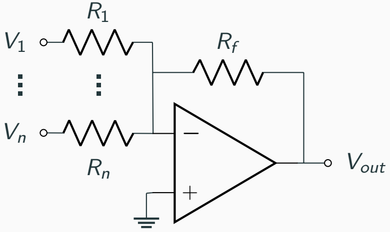
$$V_{out} = -V_{in}\frac{R_2}{R_1}$$

Also, look **here** for more useful op-amps configurations!

These might also show up on the final.

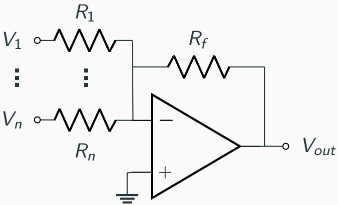
Practice: Op-Amp Analysis

Compute V_{out}:



(Hint: What basic op-amp configuration does this look like?)

Practice: Op-Amp Analysis [Solution]



This is similar to the **inverting amplifier**.

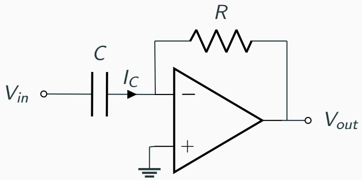
KCL at the negative terminal gives:

$$V_1/R_1 + \cdots + V_n/R_n = -V_{out}/R_f$$
$$V_{out} = -R_f(V_1/R_1 + \cdots + V_n/R_n)$$

Note that you can't use the formula for the inverting amplifier here because there are multiple voltage sources.

Practice: Calculus in Op-Amps!

Find V_{out} as a function of V_{in}:



Practice: Calculus in Op-Amps! [Solution]

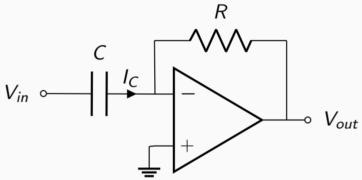
Notice that the circuit is in **negative feedback**:

By the first golden rule,

$I_c = C\frac{dV_{in}}{dt} + \frac{V_{out}}{R}$

By the second golden rule,

$V_{out} = -RC\frac{dV_{in}}{dt}$

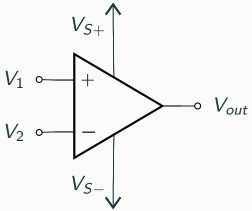


Deriving Op-Amp Configurations: Comparator

What is V_{out} when:

$V_1 > V_2$?

$V_2 > V_1$?

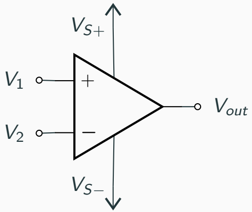


Deriving Op-Amp Configurations: Comparator [Solution]

What is V_{out} when:

$V_1 > V_2$? V_{S+}

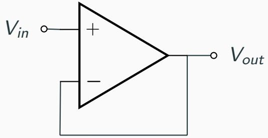
$V_2 > V_1$? V_{S-}



Reason: For *ideal op-amps*, the **gain is really large** (infinite). If there is a difference between V_1 and V_2, V_{out} will clip to the power rails.

Deriving Op-Amp Configurations: Buffer

Use the golden rules to find V_{out}.



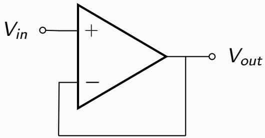
Deriving Op-Amp Configurations: Buffer [Solution]

Use the golden rules to find V_{out}.

The op-amp is in **negative feedback**,
so $V_+ = V_-$

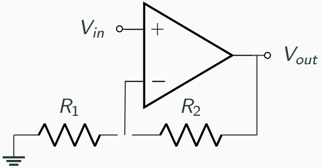
$V_{in} = V_+$ and $V_{out} = V_-$

So $V_{out} = V_{in}$.



Deriving Op-Amp Configurations: Non-Inverting Amplifier

Compute V_{out}.



Non-Inverting Amplifier [Solution]

Compute V_{out}.

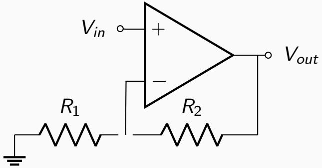
Apply **golden rules**: $V_{in} = V_+ = V_-$

Use **voltage divider**:

$V_- = V_{out}\frac{R_1}{R_1+R_2}$

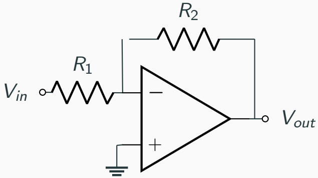
Combine and simplify:

$V_{out} = V_{in}(1 + R_2/R_1)$



Deriving Op-Amp Configurations: Inverting Amplifier

Compute V_{out}.



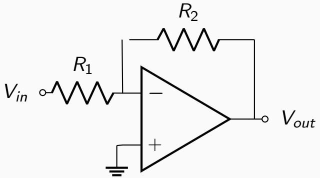
Inverting Amplifier [Solution]

Compute V_{out}.

Apply **KCL** at the negative terminal:

$V_{in}/R_1 + V_{out}/R_2 = 0$

So $V_{out} = V_{in}(-R_2/R_1)$



Circuit Cheat Sheet

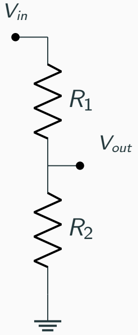
Cheat Sheet: Equations

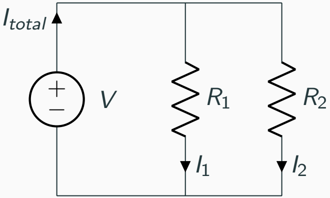
Here are some *useful equations for solving circuits*:

Ohm's Law:	$V = IR$
Current:	$I = \frac{dQ}{dt}$
Resistance:	$R = \rho L/A$
Capacitance:	$C = \epsilon A/d$
Energy:	$E = \frac{1}{2}CV^2$
Power:	$P = IV = V^2/R = I^2R$
Charge-Capacitance:	$Q = CV$
Resistors in Series:	$R_{eq} = \sum_n R_n$
Resistors in Parallel:	$1/R_{eq} = \sum_n 1/R_n$
Capacitors in Series:	$1/C_{eq} = \sum_n 1/C_n$
Capacitors in Parallel:	$C_{eq} = \sum_n C_n$

Cheat Sheet: Common Circuits

Voltage Divider	Current Divider
$V_{out} = V_{in} \frac{R_2}{R_1+R_2}$	$I_1 = I_{total} \frac{R_2}{R_1+R_2}$





Cheat Sheet: Circuit Concepts

- **Charge on a capacitor** after t seconds at constant current:

$$Q_{total} = It$$

- **Conservation of Charge**:

$$Q_{total} = Q_{initial}$$

- **Thevenin/Norton** equivalent circuits:

$$\begin{aligned} V_{Th} &= V_{OC} \\ I_N &= I_{SC} \\ R_{eq} = V_{Th}/I_N &= V_{OC}/I_{SC} \end{aligned}$$

- **Golden Rules** (apply to ideal op amps):

$$\begin{aligned} I_- = I_+ &= 0 \,\text{(always)} \\ V_- = V_+ &\,\text{(in negative feedback)} \end{aligned}$$

Cheat Sheet: Op Amp Configurations

Non-Inverting Amplifier	**Inverting Amplifier**
$V_{out}/V_{in} = \frac{R_1+R_2}{R_1}$	$V_{out}/V_{in} = -R_2/R_1$