EECS 16B Final Review Session

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May 6, 2021

Disclaimer

This is an unofficial review session and HKN is not affiliated with this course. Although some of the presenters may be course staff, the material covered in the review session may not be an accurate representation of the topics covered in and difficulty of the exam.

Slides are also posted at @2229 on Piazza.

HKN Drop-In Tutoring

- HKN has office hours Monday, Wednesday, and Friday from **1 PM - 3 PM** and **8 PM - 10 PM** on hkn.mu/ohqueue
- The schedule of tutors can be found at hkn.mu/tutor

System ID

System ID Parameters

(Note: This is basically all of note 7B.)

We model linear systems with:

$$\vec{x_d}[t+1] = A\vec{x_d}[t] + B\vec{u_d}[t]$$

Where

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \; B = \begin{bmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nk} \end{bmatrix}$$

System ID Parameters

Suppose we are not given A and B, and wish to determine them with some number of observations. We can treat the matrices as unknown parameters, and attempt to solve for their value.

Breaking the system down into scalars, we have:

$$\begin{bmatrix} x_1[t+1] \\ \vdots \\ x_n[t+1] \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1[t] \\ \vdots \\ x_n[t] \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nk} \end{bmatrix} \begin{bmatrix} u_1[t] \\ \vdots \\ u_k[t] \end{bmatrix}$$

Converting to Least Squares

Let us examine each row of the previous equation separately. For row r,

$$x_r[t+1] = \begin{bmatrix} a_{r1} & \cdots & a_{rn} \end{bmatrix} \begin{bmatrix} x_1[t] \\ \vdots \\ x_n[t] \end{bmatrix} + \begin{bmatrix} b_{r1} & \cdots & b_{rk} \end{bmatrix} \begin{bmatrix} u_1[t] \\ \vdots \\ u_k[t] \end{bmatrix}$$

Since inner products are symmetric, we can flip this around:

$$x_r[t+1] = \begin{bmatrix} x_1[t] & \cdots & x_n[t] \end{bmatrix} \begin{bmatrix} a_{r1} \\ \vdots \\ a_{rn} \end{bmatrix} + \begin{bmatrix} u_1[t] & \cdots & u_k[t] \end{bmatrix} \begin{bmatrix} b_{r1} \\ \vdots \\ b_{rk} \end{bmatrix}$$

Converting to Least Squares

Finally, we stack our vectors to get:

$$x_r[t+1] = \begin{bmatrix} x_1[t] & \cdots & x_n[t] & u_1[t] & \cdots & u_k[t] \end{bmatrix} \begin{bmatrix} a_{r1} \\ \vdots \\ a_{rn} \\ b_{r1} \\ \vdots \\ b_{rk} \end{bmatrix}$$

Observe that the quantities in the row vector can be observed, while the the quantities in the column vectors are unknown parameters we wish to solve for.

System ID Least Squares

Unfortunately, the previous equation is not sufficient since we have 1 equation but $n + k$ parameters. Since the equation holds for all times, we can consider all timesteps $0 < t \leq T$ to have the following:

$$\begin{bmatrix} x_r[1] \\ \vdots \\ x_r[T] \end{bmatrix} = \begin{bmatrix} x_1[0] & \cdots & x_n[0] & u_1[0] & \cdots & u_k[0] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1[T-1] & \cdots & x_n[T-1] & u_1[T-1] & \cdots & u_k[T-1] \end{bmatrix} \begin{bmatrix} a_{r1} \\ \vdots \\ a_{rn} \\ b_{r1} \\ \vdots \\ b_{rk} \end{bmatrix}$$

This allows us to solve for our unknowns as long as there are at least $T \geq n + k$ observations using least squares. However, we can make the procedure more efficient if we stack these equations horizontally for each value of r.

System ID Least Squares

Let us define the following matrices:

$$P = \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{nn} \\ b_{11} & \cdots & b_{n1} \\ \vdots & \ddots & \vdots \\ b_{1k} & \cdots & b_{nk} \end{bmatrix}$$

System ID Least Squares

Let us define the following matrices:

$$D = \begin{bmatrix} \vec{x}^\top[0] & \vec{u}^\top[0] \\ \vdots & \vdots \\ \vec{x}^\top[T-1] & \vec{u}^\top[T-1] \end{bmatrix}$$

$$= \begin{bmatrix} x_1[0] & \cdots & x_n[0] & u_1[0] & \cdots & u_k[0] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_1[T-1] & \cdots & x_n[T-1] & u_1[T-1] & \cdots & u_k[T-1] \end{bmatrix}$$

System ID Least Squares

Let us define the following matrices:

$$S = \begin{bmatrix} \vec{x}^\top[0] \\ \vdots \\ \vec{x}^\top[T-1] \end{bmatrix} = \begin{bmatrix} x_1[1] & \cdots & x_n[1] \\ \vdots & \ddots & \vdots \\ x_1[T] & \cdots & x_n[T] \end{bmatrix}$$

System ID Least Squares

With these matrices, we finally have

$$DP \approx S$$

Using least squares, we solve

$$P = \left(D^\top D\right)^{-1} D^\top S$$

Discretization

Discretization: Q1

Note: this section follows hw8 q1 almost exactly.

Suppose we have a scalar system

$$\frac{d}{dt}x(t) = \alpha x(t) + \vec{\beta}^T \vec{u}(t)$$

and we apply a constant input $\vec{u}_n$ for times $t \in [nT, (n+1)T)$ for some $T > 0$. Given $x(nT)$ solve the differential equation.

Discretization: Q1 Sol

From $t = nT$ to $t = (n+1)T$, $\vec{\beta}^T \vec{u}$ is a constant scalar. Thus, we can solve this like a normal differential equation. Let $x = x' - \frac{\vec{\beta}^T \vec{u}}{\alpha}$.

$$\begin{aligned}
\frac{d}{dt}x'(t) &= \alpha(x'(t) - \frac{\vec{\beta}^T \vec{u}}{\alpha}) + \vec{\beta}^T \vec{u} \\
&= \alpha x' \\
x'(t) &= Ae^{\alpha(t-nT)} \\
x(t) + \frac{\vec{\beta}^T \vec{u}}{\alpha} &= Ae^{\alpha(t-nT)} \\
x(t) &= Ae^{\alpha(t-nT)} - \frac{\vec{\beta}^T \vec{u}}{\alpha}
\end{aligned}$$

Discretization: Q1 Sol Continued

At this point we can use our initial condition to get

$$x(nT) = A - \frac{\vec{\beta}^T \vec{u}}{\alpha}$$

$$A = x(nT) + \frac{\vec{\beta}^T \vec{u}}{\alpha}$$

$$x = \left(x(nT) + \frac{\vec{\beta}^T \vec{u}}{\alpha} \right) e^{\alpha(t-nT)} - \frac{\vec{\beta}^T \vec{u}}{\alpha}$$

Discretization: Q2

Using the differential equation derived from question 1, create a discrete-time system to model the continuous time. In other words, if $x[n] = x(nT), \vec{u}[n] = \vec{u}(nT)$, find a relation such that

$$x[n+1] = A_d x[n] + B_d \vec{u}[n]$$

Discretization: Q2 Sol

We can solve the previous solution for $x((n+1)T)$

$$x((n+1)T) = \left(x(nT) + \frac{\vec{\beta}^T \vec{u}(nT)}{\alpha}\right) e^{\alpha((n+1)T - nT)} - \frac{\vec{\beta}^T \vec{u}(nT)}{\alpha}$$

$$x[n+1] = e^{\alpha T} x[n] + \frac{e^{\alpha T} - 1}{\alpha} \vec{\beta}^T \vec{u}[n]$$

We see that $A_d = e^{\alpha T}, B_d = ((e^{\alpha T} - 1)/\alpha)\vec{\beta}^T$

Discretization: Q3

Instead of a scalar, we instead have a diagonal matrix A such that

$$\frac{d}{dt}\vec{x} = A\vec{x} + B\vec{u}$$

Discretize this system in the same was as Q2.

Discretiziation: Q3 Sol

Expanding the original system out line-by-line gives

$$\frac{d}{dt}x_i = a_i x_i + b_i \vec{u}_i$$

where x_i is the ith variable of $\vec{x}$, a_i is the diagonal entry of A, and b_i is the row of B.

Discretization: Generic Matrix

Math not shown, but we can perform a change of basis from our original space to our diagonal space, and then apply the results of the previous part.

Controls

Reviewing State Space

Discrete Time State Space Model:

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

Where $\vec{x}[\cdot]$ as the state vector, $u[\cdot]$ as the input vector.

Controllability

Goal: Modify $x[n]$ to be in any state we desire.

$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

Expand out $x[t]$ in terms of the initial state and all inputs,

$$\vec{x}[t] = Bu[t-1] + ABu[t-2] + A^2Bu[t-3] + \cdots + A^{t-1}Bu[0] + A^t\vec{x}[0]$$

$$\vec{x}[t] - A^t\vec{x}[0] = \underbrace{\begin{bmatrix} B & AB & A^2B & \cdots & A^{t-1}B \end{bmatrix}}_{\triangleq \mathcal{C}} \begin{bmatrix} u[t-1] \\ u[t-2] \\ \vdots \\ u[0] \end{bmatrix}$$

Controllability

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$$\vec{x}[t+1] = A\vec{x}[t] + B\vec{u}[t]$$

Expand out $x[t]$ in terms of the initial state and all inputs,

$$\vec{x}[t] = Bu[t-1] + ABu[t-2] + A^2Bu[t-3] + \cdots + A^{t-1}Bu[0] + A^t\vec{x}[0]$$

$$\vec{x}[t] - A^t\vec{x}[0] = \underbrace{\begin{bmatrix} B & AB & A^2B & \cdots & A^{t-1}B \end{bmatrix}}_{\triangleq \mathcal{C}} \begin{bmatrix} u[t-1] \\ u[t-2] \\ \vdots \\ u[0] \end{bmatrix}$$

Given the initial condition, $x[0]$ the output of the system can be expressed in terms of the solely our inputs!

What states can we change $x[n]$ to?

$$\vec{x}[t] - A^t\vec{x}[0] = \underbrace{\begin{bmatrix} B & AB & A^2B & \cdots & A^{t-1}B \end{bmatrix}}_{\triangleq \mathcal{C}} \begin{bmatrix} u[t-1] \\ u[t-2] \\ \vdots \\ u[0] \end{bmatrix}$$

The $\text{Col}(\mathcal{C})$ determines the subspace $\vec{u}(t)$ can map to.

In order to control the state to any vector in $\mathbb{R}^n$, $\text{Col}(\mathcal{C}) = R^n$, or it must be full rank.

i.e. The system is Controllable if and only if

$$\text{rank}\,\mathcal{C} = \text{rank}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = n$$

Minimum Energy Control

Controllability Recap

Recall that any system with state equation

$$\vec{x}[n+1] = A\vec{x}[n] + B\vec{u}[n]$$

is controllable if and only if the controllability matrix

$$C = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

is full rank. For simplicity, we shall only consider the case where $u[t]$ is a scalar, so B is a vector and C is a full rank $n \times n$ matrix.

Controllability Recap

If we wanted to arrive at our desired state $\vec{x}^*$ in n timesteps, then we can simply choose

$$\vec{u} = \begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[0] \end{bmatrix} = C^{-1}\vec{x}^*$$

Minimum Energy Control

If we want to arrive in $t > n$ timestamps, then there may be multiple solutions. The corresponding controllability matrix becomes

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{t-1}B \end{bmatrix}$$

One natural way to define the "best" solution is to minimize the norm of $\vec{u}$. This is known as minimum energy control.

Minimum Energy Control Solution

Notice that $\mathcal{C}$ has a nontrivial nullspace. If we have a particular solution $\vec{u}$ satisfying $\mathcal{C}\vec{u} = \vec{x}^*$ and any vector $\vec{a} \in \mathrm{Null}(\mathcal{C})$, then

$$\mathcal{C}(\vec{u} + \vec{a}) = \mathcal{C}\vec{u} + \mathcal{C}\vec{a} = \mathcal{C}\vec{u} + \vec{0} = \vec{x}^*$$

It turns out that the optimal solution is $\vec{u_0} \perp \mathrm{Null}(\mathcal{C})$.

Minimum Energy Control Proof

Let $\vec{u}_0$ be defined as before, and consider a solution vector $\vec{u} = \vec{u}_0 + \vec{a}$ where $\vec{a}$ is in the nullspace of $\mathcal{C}$. We calculate the norm:

$$\begin{aligned} |\vec{u}| &= \langle \vec{u}, \vec{u} \rangle \\ &= \langle \vec{u}_0 + \vec{a}, \vec{u}_0 + \vec{a} \rangle \\ &= \langle \vec{u}_0, \vec{u}_0 \rangle + 2 \langle \vec{u}_0, \vec{a} \rangle + \langle \vec{a}, \vec{a} \rangle \\ &= \langle \vec{u}_0, \vec{u}_0 \rangle + \langle \vec{a}, \vec{a} \rangle \\ &= |\vec{u}_0| + |\vec{a}| \end{aligned}$$

Thus, to minimize the norm of $\vec{u}$ we must minimize the norm of $\vec{a}$, so the optimal solution is $\vec{u} = \vec{u}_0$.

Stability

Stability in Discrete Time

A discrete system is stable iff all eigenvalues have magnitude less than 1. If any eigenvalue has magnitude greater than 1, then any state vector with a nonzero corresponding eigenvector component will have that component repeatedly magnified.

For example: $x[t+1] = 2x[t]$

Stability in Discrete Time

A discrete system is stable iff

$$\forall x \in eig(A) : |x| < 1$$

The eigenvectors form a basis (called the eigenbasis) which spans the entire space if A is full rank. (can you prove this?)

If any eigenvalue has magnitude greater than 1, then any state vector with a nonzero corresponding eigenvector component will have that component repeatedly magnified.

Stability in Discrete Time

How do the eigenvalues govern system dynamics?

If initial state is x(0), and there's no control input, the nth state is

$$x(n) = A^n x(0)$$

If any eigenvalue of A is larger in magnitude than 1, it "blows up" through repeated exponentiation - the system destabilizes!

Stability in Discrete Time

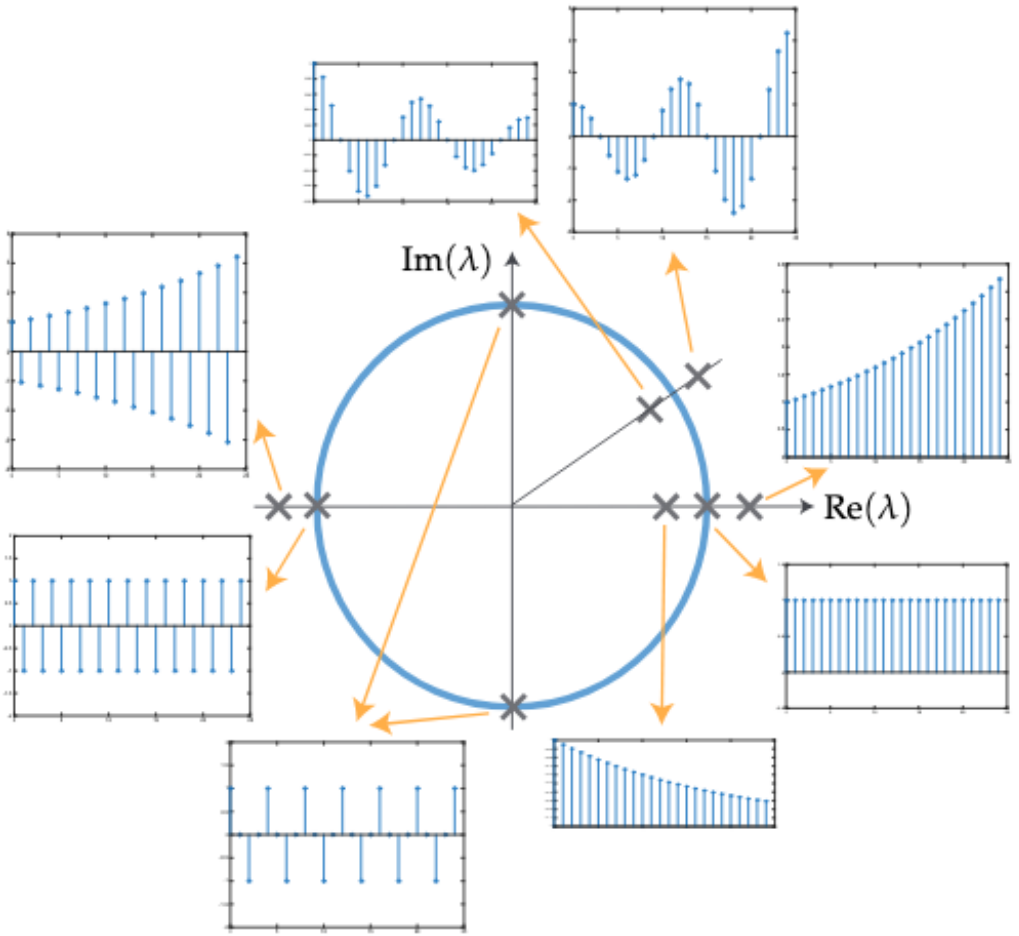


Figure 1: The real part of λ^t for various values of λ in the complex plane. It grows unbounded when $|\lambda| > 1$, decays to zero when $|\lambda| < 1$, and has constant amplitude when λ is on the unit circle ($|\lambda| = 1$).

Stability in Continuous Time

A continuous system is stable iff the real parts of all eigenvalues are negative. If any eigenvalue is positive, then any state vector with a nonzero corresponding eigenvector component will have that component grow exponentially to infinity.

For example: $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = 2\,x(t)$

Stability in Continuous Time

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = ax(t) + bu(t)$$

$$x(t) = e^{at}x(0) + b\int_0^t e^{a(t-s)}u(s)\,\mathrm{d}s$$

For scalar case, system is stable if $\mathrm{Re}\{a\} < 0$ and not stable if $\mathrm{Re}\{a\} > 0$.

By careful application of diagonalization, we get the same result for the eigenvalues of A in the matrix case.

Stability in Continuous Time

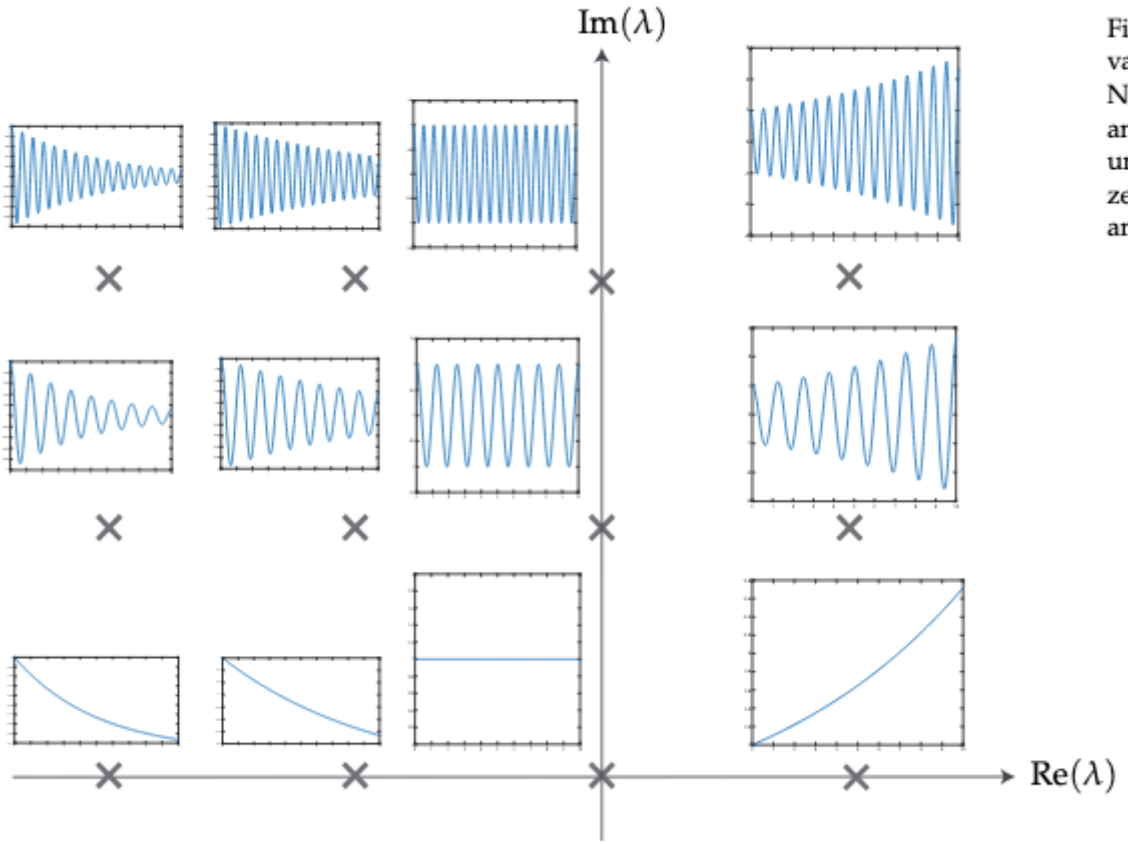


Figure 2: The real part of $e^{\lambda t}$ for various values of λ in the complex plane. Note that $e^{\lambda t}$ is oscillatory when λ has an imaginary component. It grows unbounded when $\text{Re}\{\lambda\} > 0$, decays to zero when $\text{Re}\{\lambda\} < 0$, and has constant amplitude when $\text{Re}\{\lambda\} = 0$.

Stability in Continuous Time

How do the eigenvalues govern system dynamics?

If initial state is $\vec{x}(0)$, and there's no control input, state at time t is

$$\vec{x}(t) = e^{At}\vec{x}(0)$$

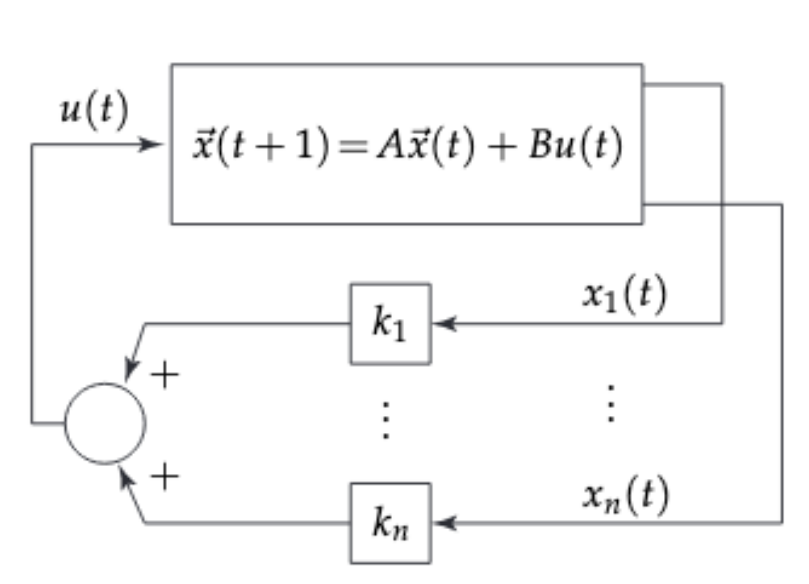
If any eigenvalue of A is larger in magnitude than 1, it "blows up" through repeated exponentiation — the system destabilizes!

Stability Through State Feedback

- If we add a feedback path (modifying the input values with the state) our state update equation changes

$$\vec{x}(t+1) = (A + BK)\vec{x}(t)$$

- What determines the stability of this new system?



State Feedback

- By designing K, we can give our system specific dynamic properties
    - Can analyze and design the way its state changes over time
- If our "open-loop" system is unstable, choosing the right values of K can make it stable!
- Is this always possible?

Example: Controllability and Stability

$$\vec{x}[t+1] = \begin{bmatrix} -5 & 0 \\ 7 & 6 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u[t]$$

$$\vec{y}[t] = \begin{bmatrix} 1 & 1 \end{bmatrix} \vec{x}[t]$$

Controllable?

Stable for $u[t] = 0$?

Example: Controllability and Stability

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Controllable? **Yes**

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Example: Controllability and Stability

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Controllable? **Yes**

Stable for $u[t] = 0$? **No**

Upper Triangularization

Upper Triangularization

- Recall that not all square matrices are diagonalizable
  - An $n \times n$ matrix is diagonalizable iff has n linearly independent eigenvectors
- However, all square matrices can be brought into upper triangular form
- I'll walk through the proof from the notes
- (But I'm not sure how useful this will be / how they would ask questions about this on the test)
- (So if people want to I can instead start taking questions on SVD, time- and frequency-domain analysis of RLC circuits, and phasors)

Upper Triangularization Proof

- What are we trying to prove?
  - Remember that if M is diagonalizable, this means that there exists a matrix P such that PMP^{-1} was diagonal
  - In our case, we want to prove that for any square matrix A, there exists a matrix T such that TAT^{-1} is upper triangular
- We will proceed by induction
- First prove a base case (a 1×1 matrix must be upper triangular)
- Prove that if there exists such a matrix T_0 for a $k \times k$ matrix, then there exists the matrix T for a size $(k+1) \times (k+1)$ matrix

Upper Triangularization Proof

- Clearly a 1×1 matrix is upper triangular
- First we choose one arbitrary eigenvalue / eigenvector pair, choose an orthonormal basis for $\mathbb{R}^n$ (with Gram-Schmidt), then define V formed with those vectors.

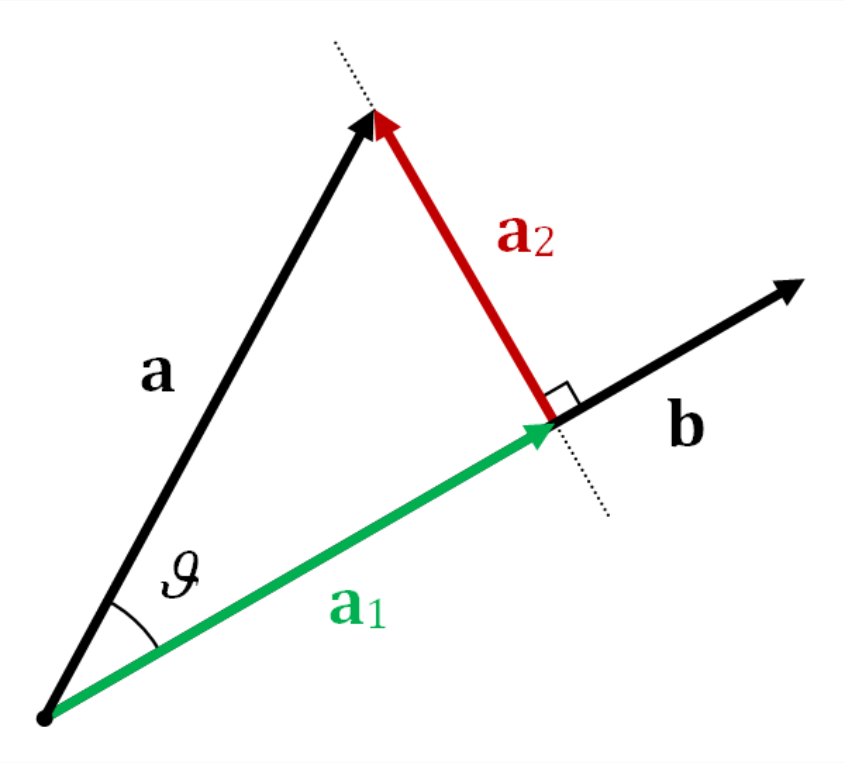
We can upper triangularize $(k+1) \times (k+1)$ matrices if we assume that $k \times k$ matrices can be upper triangularized. To show this, let A be an arbitrary $(k+1) \times (k+1)$ matrix and let $\lambda, \vec{v}$ by an eigenvalue/vector pair: $A\vec{v} = \lambda\vec{v}$. Normalize $\vec{v}$ so that $\|\vec{v}\| = 1$ and choose k other vectors $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^{k+1}$ such that $\{\vec{v}, \vec{v}_1, \ldots, \vec{v}_k\}$ is an orthonormal basis for $\mathbb{R}^{k+1}$. Then the $(k+1) \times (k+1)$ matrix $V = \begin{bmatrix} \vec{v} & \vec{v}_1 & \cdots & \vec{v}_k \end{bmatrix}$ is orthogonal, *i.e.* $V^{-1} = V^T$.

Gram-Schmidt

Gram-Schmidt

Given a set of basis vectors $\vec{s_1}, \vec{s_2}, \ldots, \vec{s_n}$, Gram-Schmidt converts this into an orthonormal set of basis vectors with the same span and rank $\vec{q_1}, \vec{q_2}, \ldots, \vec{q_n}$.

Gram-Schmidt



$$\vec{a_1} = \frac{\langle \vec{a}, \vec{b} \rangle}{\|b\|^2} \vec{b} \tag{1}$$

Gram-Schmidt

$$\vec{q}_1 = \frac{\vec{s}_1}{\|\vec{s}_1\|} \tag{2}$$

$$\vec{e}_i = \vec{s}_i - \sum_{k=1}^{i-1} \langle s_i, q_k \rangle q_k \tag{3}$$

$$\vec{q}_i = \frac{\vec{e}_i}{\|\vec{e}_i\|} \tag{4}$$

GS True or False

- You can perform GS with the $\vec{s}$ vectors in any order.

GS True or False

- You can perform GS with the $\vec{s}$ vectors in any order. **True**
- You *must* normalize the vector after subtracting the projections.

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- Gram-Schmidt changes the span of the basis vectors.

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Singular Value Decomposition

SVD Theorem

Any matrix $A \in \mathbb{R}^{m \times n}$ can be decomposed into the product of three matrices

$$A = U \Sigma V^T$$

$$U : m \times m$$

$$\Sigma : m \times n$$

$$V^T : n \times n$$

Such that U, V are unitary matrices and Σ only has nonnegative values along its main diagonal.

SVD: Compact Form

We can also express the SVD as

$$A = \mathcal{U} S \mathcal{V}^T$$

$$\mathcal{U} : m \times r$$

$$S : r \times r$$

$$\mathcal{V}^T : r \times n$$

where r is the rank of A. The compact form matrices maintain properties of the original matrices, but have entries removed whenever they correspond to zero singular values.

SVD: Outer Product Form

Lastly, we can express

$$A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$$

where $\vec{u}_i, \vec{v}_i$ are the columns of U, V, respectively, and σ_i are corresponding diagonal entry of the matrix Σ

Computing SVD with A^TA

$$A^TA = U\Sigma V^T V\Sigma^T U^T$$
$$= U\Sigma^2 U^T$$

This is an eigen decomposition since Σ^2 is diagonal and $U^{-1} = U^T$. Thus solving for the eigenvalues and eigenvectors of A^TA give $\lambda_i = \sigma_i^2$ with eigenvectors which correspond to the right singular vectors. We need to sort by decreasing σ_i.

Side note: $\Sigma^T\Sigma$ is not actually equal to Σ^2, but the former product yields a matrix with singular values squared on the diagonal entries, hence we call it Σ^2

Computing SVD with $A^T A$

Given a right singular vector $\vec{v}_i$ which we found from the previous part, we can apply it

$$
\begin{aligned}
A\vec{v}_i &= \left(\sum_{k=1}^{r} \sigma_k \vec{u}_k \vec{v}_k^T \right) \vec{v}_i \\
&= \sum_{k=1}^{r} \sigma_k \vec{u}_k \vec{v}_k^T \vec{i} \\
&= \sigma_i \vec{u}_i \\
\vec{u}_i &= \frac{1}{\sigma_i} A\vec{v}_i
\end{aligned}
$$

Computing SVD with AA^T

Similar calculations yield $\sigma_i = \sqrt{\lambda_i}$ of AA^T with eigenvectors as left singular vectors, and $\vec{v}_i = \frac{1}{\sigma_i} A^T \vec{u}_i$

Intepretation of SVD

- Unitary matrices act as rotation in a given space. A diagonal matrix stretches in a given coordinate space.
- SVD visualization (open in browser)

Intepretation of SVD

For a product $A\vec{x}$, we can decompose every vector $\vec{x}$ into a linear combination of right singular vectors

$$\vec{x} = \sum_{i=1}^{n} \alpha_i \vec{v}_i$$

Thus, we can see exactly which parts of $\vec{x}$ affect the output.

Compression of Low-Rank Matrices

- Suppose I had a matrix $A \in \mathbb{R}^{m \times n}$ with $m, n >> rank(A)$. How could I more efficiently store A and compute products like $A\vec{x}$?

Compression of Low-Rank Matrices

- Suppose I had a matrix $A \in \mathbb{R}^{m \times n}$ with $m, n >> rank(A)$. How could I more efficiently store A and compute products like $A\vec{x}$?

- With the SVD, we only have to save r set of two vectors and a scalar, which saves us a lot of space if the rank is small with respect to the matrix. Also, less computation is carried out if we represent the matrix as the outer product form.

Principle Component Analysis

PCA

PCA is a linear dimensionality reduction tool. Given data $\vec{x_i} \in \mathbb{R}^d$, we can create a mapping $T : \mathbb{R}^d \to \mathbb{R}^{d'}, d' < d$ such that the variance in the dataset is still captured

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5. To project data into the representative subspace: $T(x) := V_{d'}^T x$

PCA: Computation

The mapping T can then be expressed as

$$T(\vec{x}) = V_k^T \vec{x}$$

If we apply this transformation onto the entire dataset (which has row vectors), we can say

$$T(A) = B = AV_k$$

where $B \in \mathbb{R}^{n \times k}$

PCA: computation

If we were to show the projected vectors in the original space, we can multiply back with the projection vectors

$$A' = BV_k^T$$

Classification

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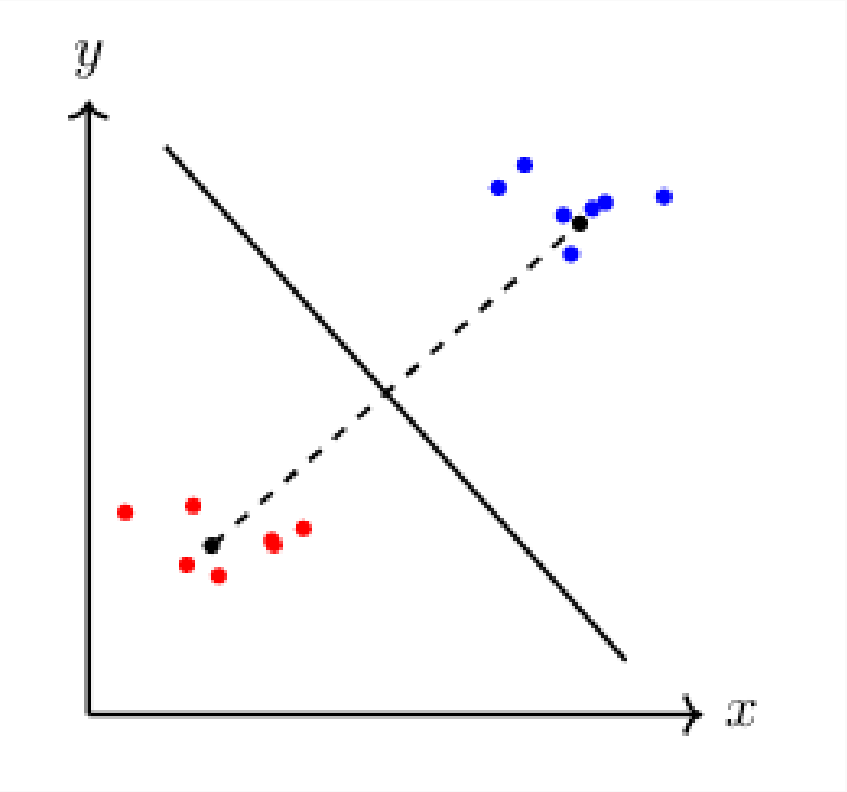
Given data points $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$ and labels $\ell_1, \ell_2, \ldots, \ell_m$, and $\ell_i = \pm 1$, we want to find a vector $\vec{w}$ such that all points with label 1 have a positive inner product and all points with label -1 have a negative inner product. We accomplish this by assigning a *cost function* $c(\vec{x}^T \vec{w}, \ell_i)$and try to minimize this cost function. Least squares is one type of cost function, but we can define new ones.



Quadratic Approximation

We can linearize a quadratic cost function in the form

$$c(\vec{w}^* + \delta\vec{w}) \approx c(\vec{w}^*) + \begin{bmatrix} \frac{\partial c}{\partial w_1} & \frac{\partial c}{\partial w_2} & \cdots & \frac{\partial c}{\partial w_n} \end{bmatrix} (\delta\vec{w}) \tag{5}$$

$$+\frac{1}{2}(\delta\vec{w})^T \underbrace{\begin{bmatrix} \frac{\partial^2 c}{\partial w_1 \partial w_1} & \cdots & \frac{\partial^2 c}{\partial w_1 \partial w_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 c}{\partial w_n \partial w_1} & \cdots & \frac{\partial^2 c}{\partial w_n \partial w_n} \end{bmatrix}}_{\text{Hessian}} (\delta\vec{w}) \tag{6}$$

Linearization

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- Recall that if we have $\frac{dx}{dt} = \lambda x(t) + bu(t)$ we know a solution to this is:

$$x(t) = x(0)e^{\lambda t} + \int_0^t e^{\lambda(t-\tau)} u(\tau)\ d\tau$$

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- Big Picture: linearize f around an operating point and then treat it as a linear function in a small neighborhood of that point.
- Why linearization?
  It allows you to treat the system as a linear one, which is very helpful as linear ODE are (usually) much easier to solve!

Linearizing a Single-Variable Function

- Suppose we have $f(x)$ that is a non linear function. We can use a first order Taylor polynomial to linearize the function, this is equivalent to finding the slope of the tangent line of $f(x)$ at a particular point.
- From calculus: $f(x) \approx f(x^*) + f'(x^*)(x - x^*)$.
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- Example: Linearize $f(x) = 3e^{x^2+2}$ around x^*
- Solution:
  $f(x^*) = 3e^{(x^*)^2+2}$
  $f'(x) = 3e^{x^2+2}(2x) = 6xe^{x^2+2}$
  $f'(x^*) = 6x^*e^{(x^*)^2+2}$
  Therefore : $f(x) \approx 3e^{(x^*)^2+2} + 6x^*e^{(x^*)^2+2}(x - x^*)$

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(iii) Define $x_l(t) = x(t) - x^*$ and $u_l(t) = u(t) - u^*$, and re-write the ODE in terms of $x_l(t)$ and $u_l(t)$. By plugging in you get: $\frac{dx_l(t)}{dt} = f(x_l(t) + x^*) + b(u_l(t) + u^*)$

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(v) Linearize the ODE: $f(x_l(t) + x^*) \approx f(x^*) + f'(x^*)x_l(t)$. Here we assume that $x_l(t)$ is also small. This is something that we will need to verify in the next step!

Linearizing Steps for $\frac{dx(t)}{dt} = f(x(t)) + bu(t)$ (Continued)

(vi) Plug (vi) back into (iii) and we obtain :
$\frac{dx_l(t)}{dt} \approx f'(x^*)f(x_l(t)) + \cancel{f(x^*)} + bu_l(t) + \cancel{bu^*} =$
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How do we know if the linearization is valid? Well, if we have $\frac{dx_l(t)}{dt} = \lambda x_l(t) + bu(t)$ we know the solution doesn't blow up if $\lambda < 0$ as we will have a term $e^{\lambda t}$.

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Hint: $\cos(x^*) = 0$ has multiple solutions, which means that we can find numerous DC operating points, can you guess which one would result in a stable system ?

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What if we had chosen a different DC Operating point, say $-\frac{\pi}{2}$? When we linearize the system we see that the solution will "explode" around that particular DC operating point.

Linearization of Vector Functions

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For example:

$$f_1 \approx f_1(\vec{x}^*) + \frac{\partial f_1}{\partial x_1}(\vec{x}^*)(x_1 - x_1^*) + ... + \frac{\partial f_n}{\partial x_1}(\vec{x}^*)(x_n - x_n^*)$$

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Repeating this for all n functions in $\vec{f}$ we see we get a system of scalar, linearized, multivariate functions which makes you think, wouldn't it be nice to express it in a shorthand matrix notation?

Jacobian Matrix

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Continuing from the previous slide we have:

$$\begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{bmatrix} \approx \begin{bmatrix} f_1(\vec{x}^*) \\ \vdots \\ f_n(\vec{x}^*) \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}^*) & \dots & \frac{\partial f_1}{\partial x_n}(\vec{x}^*) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{x}^*) & \dots & \frac{\partial f_n}{\partial x_n}(\vec{x}^*) \end{bmatrix} (\vec{x} - \vec{x}^*)$$

Linearization with Jacobians Example

$$\text{Linearize } \vec{f}(\vec{x}(t)) = \begin{bmatrix} \sin(x_1(t) \times x_2(t)) + 2x_1(t)x_3^2(t) \\ x_3(t)\cos(x_2(t)) + \frac{x_1(t)}{x_3(t)} \\ x_1(t) + 2x_3(t)x_2^3(t) \end{bmatrix} \text{ about } \vec{x}^* = \begin{bmatrix} 0 \\ 2\pi \\ \frac{2\pi}{3} \end{bmatrix}$$

Solutions

Find the Jacobian:

$$\begin{bmatrix} x_2(t)\cos(x_1(t)\times x_2(t)) + 2x_3^2(t) & x_1(t)\cos(x_1(t)\times x_2(t)) & 4x_1(t)x_3(t) \\ \frac{1}{x_3(t)} & -x_3(t)\sin(x_2(t)) & \cos(x_2(t)) - \frac{x_1(t)}{x_3^2(t)} \\ 1 & 6x_3(t)x_2^2(t) & 2x_2^3(t) \end{bmatrix}$$

Evaluate the Jacobian about $\vec{x}^*$:

$$\begin{bmatrix} 5\pi & 0 & 0 \\ \frac{2\pi}{3} & 0 & 1 \\ 1 & 36\pi^3 & 16\pi^3 \end{bmatrix}$$

Linearize:

$$\vec{f}(\vec{x}(t)) \approx \begin{bmatrix} 0 \\ \frac{3\pi}{4} \\ 24\pi^4 \end{bmatrix} + \begin{bmatrix} 5\pi & 0 & 0 \\ \frac{2\pi}{3} & 0 & 1 \\ 1 & 36\pi^3 & 16\pi^3 \end{bmatrix} \begin{bmatrix} x_1(t) - 0 \\ x_2(t) - \frac{3\pi}{4} \\ x_3(t) - 24\pi^4 \end{bmatrix}$$

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$\vec{f}(\vec{x}(t), \vec{u}(t)) \approx \vec{f}(\vec{x}^*, \vec{u}^*) + J_{\vec{x}}\vec{x}_l(t) + J_{\vec{u}}\vec{u}_l(t)$

(iv) Plug (iv) back into the ODE:
$\frac{d\vec{x}}{dt} \approx \cancel{\vec{f}(\vec{x}^*, \vec{u}^*)} + J_{\vec{x}}\vec{x}_l(t) + J_{\vec{u}}\vec{u}_l(t)$

Linearizing Vector ODE Systems Example

Given a DC input $u^* = 1$, linearize:

$$\frac{d\vec{x}(t)}{dt} = \begin{bmatrix} x_1^2(t) - x_2(t)u(t) \\ x_2^2(t)(1 + x_1(t)) + \sin(\pi x_1(t)u(t)) \end{bmatrix}$$

Solutions

Again, we will do this in steps:

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$$x_2^{*2}(x_1^* + 1) + \sin(\pi x_1^* u^*) = 0 \tag{8}$$

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(iv) Linearize,

$$\vec{f}(\vec{x}(t), u(t)) \approx \vec{f}(\vec{x}^*, 1) + \begin{bmatrix} -2 & -1 \\ 1 - \pi & 0 \end{bmatrix} \vec{x}_l(t) + \begin{bmatrix} -1 \\ \pi \end{bmatrix} u_l(t)$$

Solutions Continued

(v) Substitute linear approximation back into the system,

$$\frac{d\vec{x}(t)}{dt} \approx \cancel{\vec{f}(\vec{x}^*, 1)} + \begin{bmatrix} -2 & -1 \\ 1-\pi & 0 \end{bmatrix} \vec{x}_l(t) + \begin{bmatrix} -1 \\ \pi \end{bmatrix} u_l(t)$$

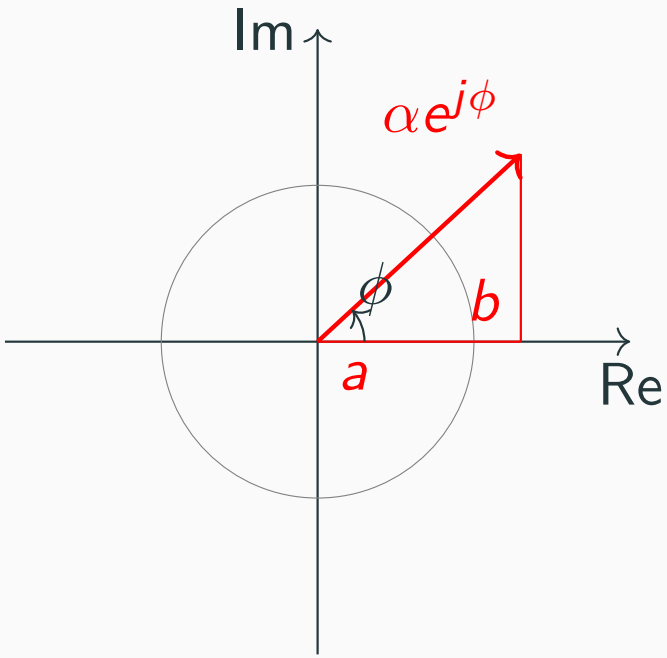
Discrete Fourier Transform

Complex Numbers Review

- The *complex numbers* are pairs of real numbers
- with *Real* and Complex Parts
- Visualize on the *Complex Plane*
- and analogize to a 2-d vector with the L-2 norm
- Write in two forms:
  - $a + bj$ (Cartesian)
  - $\alpha \exp(j\phi)$ (polar)

  and relate using Euler's formula

$$\alpha \exp(j\phi) = \alpha \cos(\phi) + j\alpha \sin(\phi) \quad (9)$$



More Useful Formulae

- Sinusoids to complex exponentials:

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \tag{10}$$

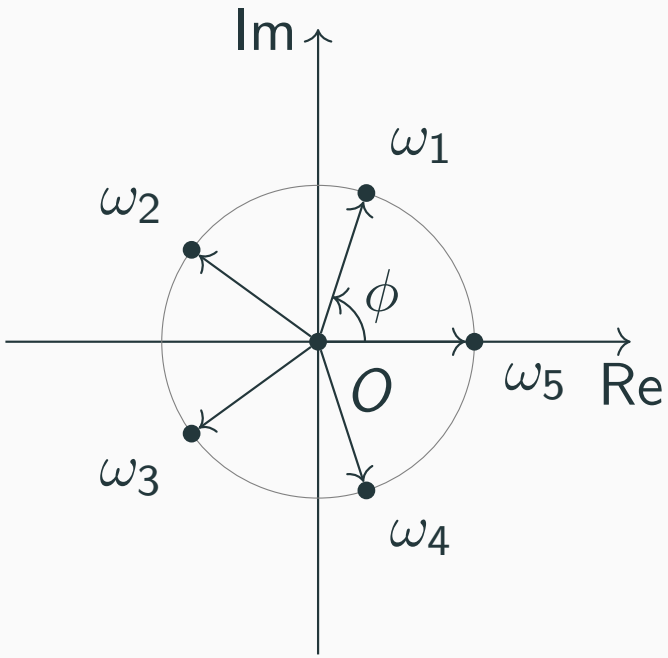
$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{11}$$

- Complex Inner Product:

$$\langle x|y\rangle = y^* x$$

$$\|x\|_2^2 = \langle x|x\rangle$$

Roots of Unity



Roots of Unity

- **N$^{\text{th}}$ root of unity** ω_N is defined to satisfy

$$(\omega_N)^N = 1$$

- N numbers sweeping around the complex unit circle satisfy this:

$$\omega_{k,N} = e^{j\frac{2\pi k}{N}}$$

- Noting that the k can be pulled out of the exponent, this can also be written

$$\omega_N^k$$

DFT (Discrete Fourier Transform) Basics

- Transforms between time- and frequency-domains
- N-dimensional space to N-dimensional space
- Orthonormal basis: $U^*U = UU^* = I$
- DFT is a basis of **harmonics**: $u_n[k] = \frac{1}{\sqrt{N}}\omega_N^{nk}$ which oscillate around the complex circle

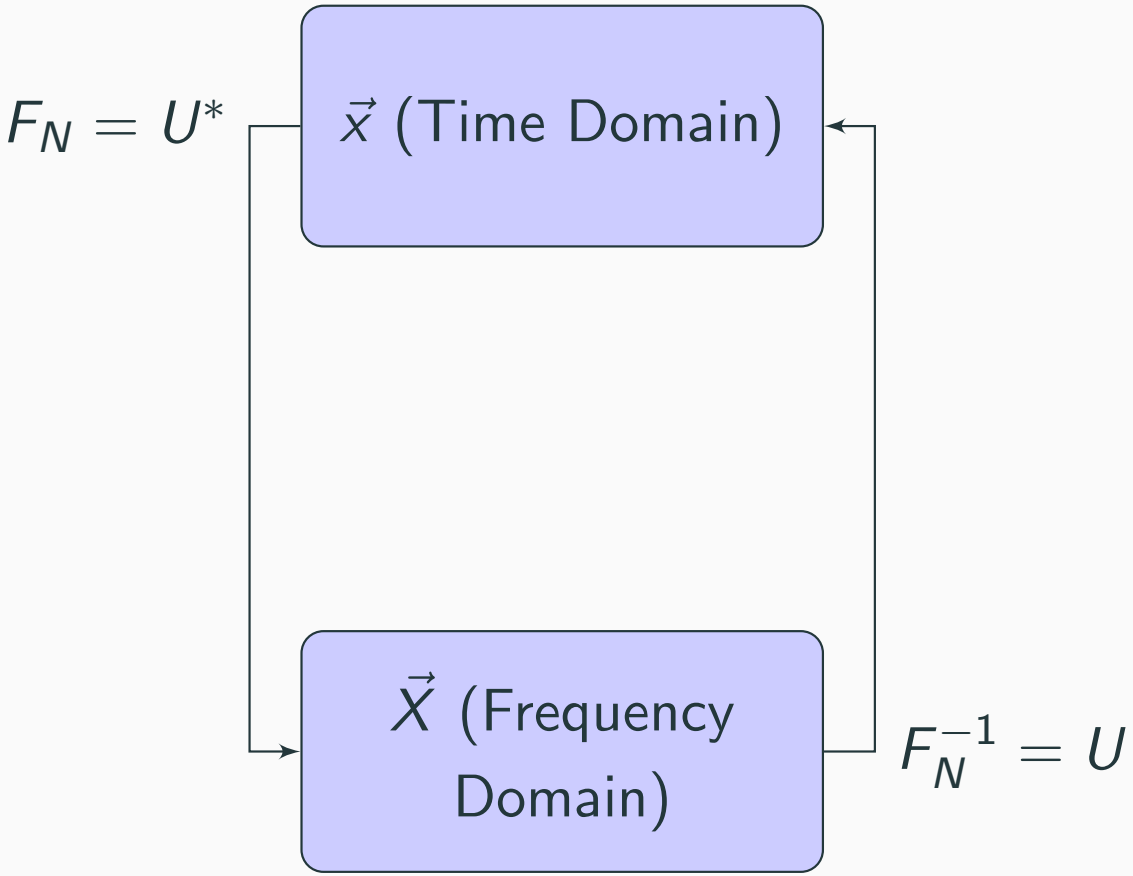
$$U = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{pmatrix}$$

DFT (Discrete Fourier Transform) Basics

Use U to transform from the frequency domain to the time domain, and use $F_N = U^{-1} = U^*$ to transform from the time domain to frequency.

$$F_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \dots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \dots & \omega^{-2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{2(n-1)} & \dots & \omega^{-(n-1)(n-1)} \end{pmatrix}$$

Using the DFT (Discrete Fourier Transform)



Systems in the (Discrete) Time Domain

- Reminder: a **system** connects inputs and outputs by solving a non-homogenous differential (difference in discrete time) equation
- Equivalent to **convolving** an **impulse response** over the input

$$C_{\vec{h}}\vec{x} = \vec{y}$$

where the **Circulant Matrix** for the convolution with $\vec{h}$ is written

$$C_{\vec{h}} = \begin{pmatrix} h[0] & h[n-1] & \dots & h[1] \\ h[1] & h[0] & \dots & h[2] \\ \vdots & \vdots & \vdots & \vdots \\ h[n-1] & h[n-2] & \dots h[0] & \end{pmatrix}$$

Systems in the (Discrete) Time Domain

- Solve systems by **convolving** an **impulse response** over the input

$$C_{\vec{h}}\vec{x} = \vec{y}$$

- This is a matrix-vector multiply, $O(N^2)$.
- Is there a better way?

Systems in the Frequency Domain

- If it's hard it time domain, do it in frequency!

$$U^*\vec{x} = \vec{X} \qquad\qquad C_{\vec{h}} = U\Lambda U^*$$

$$Y = \Lambda X$$

- Matrix is diagonal in the DFT basis, find Y by simple scalar multiplication, $O(n)$.
- Still need to move data to/from frequency domain, but the FFT (see EE123) does this in $O(n \log n)$.

Amplitude and Phase Response

- How to systems respond to real signals?
- Complex exponentials are **eigenfunctions** of a system.

$$\exp\left(jk\tfrac{2\pi}{N}t\right) \longrightarrow \boxed{H} \longrightarrow \lambda_k \exp\left(jk\tfrac{2\pi}{N}t\right)$$

$$\cos\left(k\tfrac{2\pi}{N}t\right) \longrightarrow \boxed{H} \longrightarrow \lambda_k \cos\left(k\tfrac{2\pi}{N}t\right) = |\lambda_k| \cos\left(k\tfrac{2\pi}{N}t + \angle\lambda_k\right)$$

where λ_k is the k-th eigenvalue of $C_{\vec{h}}$.

$|\lambda_k|$: **magnitude response**, $\angle\lambda_k$: **phase response**

Conjugate Symmetry & Parseval's Theorem

- The DFT basis vectors are **conjugate symmetric**:

$$\vec{u}_k = \vec{u}^*_{N-k}$$

- If a time-domain signal is real, is DFT-domain representation is conjugate-symmetric

$$x[t] \in \text{Re} \implies X[k] = X[\bar{-k}] = \delta X[N-k]$$

- Parseval's Theorem: Since the DFT matrix is orthonormal (and therefore unitary), "energy" in conserved:

$$\|\vec{x}\|_2 = \left\|\vec{X}\right\|_2$$

DFT True/False

1. The columns of the DFT matrix are orthonormal

2. The DFT matrix is conjugate-symmetric

3. $\|\vec{u}_k\|_2 = \sqrt{N}$

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   True
2. The DFT matrix is conjugate-symmetric
   True
3. $\|\vec{u}_k\|_2 = \sqrt{N}$
   False. That's the points of the scaling factor in our definition of the basis elements.

Samling and DFT of Sinusoids

- If we have a sinusoidal signal in the form $A\cos\left(\frac{2\pi}{N}(kt)+\varphi\right)$, we can form a time-domain vector $\vec{x}$ by sampling it at N locations.
- If you take the DFT of this vector ($\vec{X} = F_N\vec{x}$), you will find only the k-th and $N-k$-th slots non-zero. Specifically, they'll take value $A\frac{\sqrt{N}}{2}\exp(j\varphi)$ (check Parseval's theorem with this!).
- If there is a DC offset to the signal, $\frac{X[0]}{\sqrt{N}}$ will be equal to it.

DFT Practice Problem Warmup

Find the DFT coefficients for the following $N = 8$ signal:

$$x[t] = e^{j\frac{\pi}{2}t} + e^{j\frac{3\pi}{2}t}$$

DFT Practice Problem Warmup

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$$x[t] = e^{j\frac{\pi}{2}t} + e^{j\frac{3\pi}{2}t}$$

Hint: since the DFT can be written as a matrix multiply, is is a **linear transformation**.

DFT Practice Problem Warmup Solution

$$
\begin{aligned}
x[t] &= e^{j\frac{\pi}{2}t} + e^{j\frac{3\pi}{2}t} \\
&= e^{2j\pi\frac{2}{8}t} + e^{2j\pi\frac{6}{8}t} \\
&= u_2[t] + u_6[t] \\
X[k] &= \begin{pmatrix} 0 & 0 & \sqrt{8} & 0 & 0 & 0 & \sqrt{8} & 0 & 0 \end{pmatrix}
\end{aligned}
$$

DFT Practice Problem

Find the DFT for the same signal, but for $N = 3$. (The signal is not a sum of these harmonics, so we can't use the same trick)

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$$x[t] = e^{j\frac{\pi}{2}t} + e^{j\frac{3\pi}{2}t}$$

$$\vec{x} = \begin{pmatrix} 2 & 0 & -2 \end{pmatrix}^{\mathsf{T}}$$

$$F_3\vec{x} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{-j\frac{2\pi}{3}} & e^{-j\frac{4\pi}{3}} \\ 1 & e^{-j\frac{4\pi}{3}} & e^{-j\frac{2\pi}{3}} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 2 - 2e^{j\frac{2\pi}{3}} \\ 2 - 2e^{-j\frac{2\pi}{3}} \end{pmatrix}$$

DFT Practice: Real Sinusoid

Let $x[n] = \cos\left(\frac{2\pi}{5}n\right)$ for $n \in \{0, 1, \cdots, 4\}$.

Compute the DFT coefficients $\vec{X}$ for the signal $\vec{x}$.

Something to think about: what to the coefficients represent?

DFT Practice: Real Sinusoid — Solution

Recall that $\cos\left(\frac{2\pi}{5}n\right) = \frac{1}{2}\left(e^{j\frac{2\pi}{5}n} + e^{j\frac{2\pi}{5}n}\right)$

and that $\vec{u}_1[k] = \omega_5^{-k} = e^{-j\frac{2\pi}{5}k}$.

We see that $\vec{x} = \frac{1}{2}(\vec{u}_1 + \vec{u}_{5-1})$. Therefore,

$$\vec{X} = \begin{pmatrix} \vec{u}_0^{\mathsf{T}} \\ \vec{u}_1^{\mathsf{T}} \\ \vec{u}_2^{\mathsf{T}} \\ \vec{u}_3^{\mathsf{T}} \\ \vec{u}_4^{\mathsf{T}} \end{pmatrix} \left(\frac{1}{2}(\vec{u}_1 + \vec{u}_{5-1})\right)$$

Since the vectors are orthonormal,

$$\vec{X} = \begin{pmatrix} 0 & \frac{\sqrt{5}}{2} & 0 & 0 & \frac{\sqrt{5}}{2} \end{pmatrix}^{\mathsf{T}}$$