EECS 16A Midterm 1 Review Session

Presented by <NAMES >(HKN)

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Slides are also posted at @# on Piazza.

HKN Drop-In Tutoring

- HKN has office hours every weekday from **11 AM - 3 PM** and **8 PM - 10 PM** on hkn.mu/ohqueue
- The schedule of tutors can be found at hkn.mu/tutor

Matrices and Linear Transformations

Matrices

- Matrices are **collections of vectors**.
- Typically represent **systems of equations**, where each row is an equation and each column is a variable.
- Notable matrices:
  - Identity matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  - Rotation matrix

$$\begin{bmatrix} \cos(\theta) & -\sin(theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Augmented Matrices

Augmented matrices are a way of represening both sides of a system of equations using one matrix:

$$\begin{aligned} x - 2y + 3z &= 7 \\ 2x + y + z &= 4 \\ -2x + 2y - 2z &= -10 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 2 & 1 & 1 & 4 \\ -2 & 2 & -2 & -10 \end{array}\right]$$

Gaussian Elimination

Gaussian Elimination

- **Gaussian elimination** is a methof for solving systems of linear equations.
- Use row operations to reduce matrix to **row echelon form** (a matrix that is all zero below the diagonal)

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 3 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

The row operations are:

1. **Row exchange**: reordering rows
2. **Row scaling**: scaling a row by a real number
3. **Superposition**: replace a row with the sum of itself and a scalar multiple of another row

Practice: Gaussian Elimination

Let's use Gaussian elimination to solve this system of equations!

$$\begin{cases} x_1 + x_2 + x_3 = 8 \\ 2x_1 - 4x_2 + 3x_3 = 9 \\ -x_1 + 5x_2 - 2x_3 = 0 \end{cases}$$

Practice: Gaussian Elimination [Solution]

First, write out the system of equations into matrix-vector form:

$$\begin{cases} x_1 + x_2 + x_3 = 8 \\ 2x_1 - 4x_2 + 3x_3 = 9 \\ -x_1 + 5x_2 - 2x_3 = 0 \end{cases} \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -4 & 3 & 9 \\ -1 & 5 & -2 & 0 \end{array}\right]$$

Practice: Gaussian Elimination [Solution]

Now, use row operations to get the system into row echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -4 & 3 & 9 \\ -1 & 5 & -2 & 0 \end{array}\right] \xrightarrow{2R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ -1 & 5 & -2 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ -1 & 5 & -2 & 0 \end{array}\right] \xrightarrow{R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 8 \\ 0 & \boxed{6} & -1 & 7 \\ 0 & 6 & -1 & 8 \end{array}\right]$$

The values on the diagonals (boxed in the matrix above) are known as **pivots** or **leading coefficients**.

Practice: Gaussian Elimination [Solution]

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 2 & -4 & 3 & 9 \\ 0 & 6 & -1 & 8 \end{array}\right] \xrightarrow{R_2 - R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 0 & 6 & -1 & 7 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

The last row says $0 = -1$, so there is **no solution** to this system of equations!

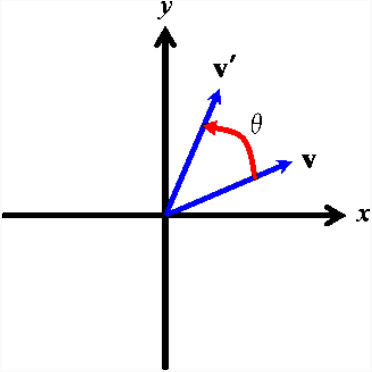
Possible results of Gaussian Elimination on $A\vec{x} = \vec{b}$

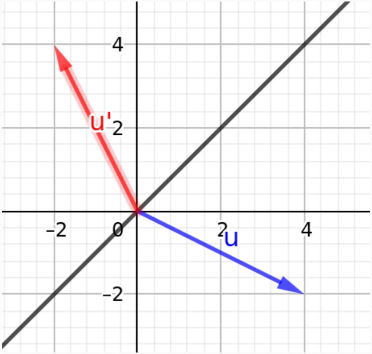
Result	Row picture	Column picture	Properties of A
Unique solution	Equations intersect at exactly one point	$\vec{b}$ is **uniquely represented** by a linear combination of the columns of A	A is **invertible**
Infinite solutions	Equations intersect along an **infinite space** (eg. line, plane, volume)	There are **multiple ways** of representing $\vec{b}$ in terms of the linear combinations of the columns of A	A has **linearly dependent columns**
No solution	Equations **do not intersect**	$\vec{b}$ is not in the span of the columns (columnspace) of A	Columnspace of A does not include $\vec{b}$. Columns of A are **linearly dependent**

Linear Transformations

Linear Transformations

- **Linear transformations** are operations that can be performed by applying a matrix to a vector.
- Some common transformations include **rotation** and **reflection**.





Common Linear Transformations: Rotation

The **rotation matrix** rotates points by a specific angle, θ:

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Use this matrix by **plugging in the desired rotation angle**, then multiply it to a vector.

$$R(\theta) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Rotation matrices also **preserve the length** of a vector. Check it! (*Think about the eigenvalues! Real? Complex? How about the magnitude of these eigenvalues?*)

Common Linear Transformations: Reflection

- The reflection matrix **reflects vectors** across a line. (Notice that such matrix also *preserves the length of a vector*.)
- Notable reflection matrices:
    - Reflection across x-axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
    - Reflection across y-axis: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
    - Reflection across line $y = x$: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Practice: Matrix Transformations

Create matrices to transform the vector $\vec{v} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$ as follows:

1. Rotate by $45\,\deg$
2. Reflect across $y = x$

Practice: Matrix Transformations

Create matrices to transform the vector $\vec{v} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$ as follows:

1. Rotate by 45 deg

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \end{bmatrix}$$

2. Reflect across $y = x$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Linear Independence

Linear Combinations

A **linear combination** of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a sum of the vectors, scaled by scalars $\{a_1, \ldots, a_n\}$:

$$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$$

- If $\{\vec{v}_1, \ldots, \vec{v}_n\}$ are the *columns of a matrix* B, the set of all linear combinations is called the **columnspace** or **range** of B.
- Note: the **columnspace** of B is a **vector space**.
  - To check a subset is a vector space, it must be **closed under addition and scalar multiplication**.

Linear Independence

- Informally speaking, a set of vectors is linearly independent if *no vector in the set can be represented as a linear combination of other vectors*.
- Formal definition:
  - Given a set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$, if some scalars $\{c_1, c_2, \ldots, c_m\}$ *neq*$\{0, \ldots, 0\}$ exist such that $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = 0$, then the vectors are **linearly dependent**.
  - If no such scalars exist, then the vectors are **linearly independent**.
- In other words, for a set of linearly independent vectors, $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_m\vec{v}_m = 0$ implies that all $c_1 = 0, c_2 = 0, \ldots, c_m = 0$ (*useful when doing proofs*).

Practice: Linear Independence

Are the columns of the following matrix linearly independent?

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

Practice: Linear Independence [Solution]

Row reduce the matrix. If any of the **pivots** (numbers on the diagonal) are 0, then the columns are **linerly dependent**.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Practice: Linear Independence [Solution]

Let's look at the row-reduced matrix as the system of equations:

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 = 0$$

Remember, that if there exist nonzero c_1, c_2, c_3 that satisfy that equation, the columns are **linearly dependent**.

$$\begin{aligned} c_2 &= -3c_3 \\ 2c_1 = -3c_2 - 5c_3 = 9c_3 - 5c_3 &= 4c_3 \\ c_1 &= 2c_3 \end{aligned}$$

There are infinitely many solutions, so the columns of A are **linearly dependent**.

Span and Rank

Span

- **Definition**: The **span** of a set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots\}$ is the set of *all possible linear combinations* of the vectors.
- The span of a collection of vectors is always a **vector space**.
- Since the column space of a matrix A is the *span of its columns*, it is a vector space.

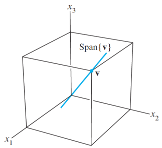


FIGURE 10 Span{**v**} as a line through the origin.

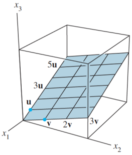
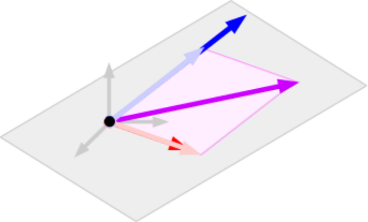


FIGURE 11 Span{**u**, **v**} as a plane through the origin.



Practice: Span

Do the following sets of vectors span $\mathbb{R}^3$?

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Practice: Span [Solution]

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \implies \text{No}$$

$$\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \implies \text{Yes}$$

How do you know if a set of vectors spans $\mathbb{R}^3$?

- Are there **n pivots**, i.e. **n linearly independent vectors**?
- Use *Gaussian Elimination*!

Rank

- Definition: The **rank** of a matrix A is the *dimension of the column space of A*.
- Altenatively:
  - The number of rows with *nonzero leading coefficients*.
  - The number of *linearly independent columns*.
- A **pivot** is the first nonzero element of a row for a matrix in *row echelon form*, and a pivot column is a column that contains a pivot.
- In fact, pivots are shared by both the columns and the rows, so $\mathbf{dim(colspace(}A\mathbf{))} = \mathbf{dim\ (rowspace(}A\mathbf{))}$.
- $\mathbf{rank(}A\mathbf{)} = \mathbf{rank(}A^T\mathbf{)}$

Practice: Rank

Find the rank of B:

$$B = \begin{bmatrix} 1 & 2 & 1 & -4 \\ 0 & -1 & 1 & -2 \\ 1 & 4 & 1 & 2 \end{bmatrix}$$

Practice: Rank [Solution]

Do Gaussian Elimination!

$$\begin{bmatrix}1 & 2 & 1 & -4\\ 0 & -1 & 1 & -2\\ 1 & 4 & 1 & 2\end{bmatrix} \xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix}1 & 2 & 1 & -4\\ 0 & -1 & 1 & -2\\ 0 & 2 & 0 & 6\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 1 & -4\\ 0 & -1 & 1 & -2\\ 0 & 2 & 0 & 6\end{bmatrix} \xrightarrow[R_2 \to R_3]{R_3/2 \to R_2} \begin{bmatrix}1 & 2 & 1 & -4\\ 0 & 1 & 0 & 3\\ 0 & -1 & 1 & -2\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 1 & -4\\ 0 & 1 & 0 & 3\\ 0 & -1 & 1 & -2\end{bmatrix} \xrightarrow{R_3 + R_2 \to R_3} \begin{bmatrix}\boxed{1} & 2 & 1 & -4\\ 0 & \boxed{1} & 0 & 3\\ 0 & 0 & \boxed{1} & 1\end{bmatrix}$$

There are *3 pivot columns*, so $\mathbf{rank}(B) = \mathbf{3}$.

Matrix Inverses

Inverses

A matrix, A, is **invertible** if there exists a matrix B, such that

$$AB = BA = I_n \implies B = A^{-1}$$

Conditions for inverse to exist:

- The matrix must be **square** (n x n).
- The columns must be **linearly independent** (injective) and they must **span** $\mathbb{R}^n$ (surjective).

Inverse Properties

Here are some useful properties of matrix inverses:

- $AA^{-1} = A^{-1}A = I$
- $(A^{-1})^{-1} = A$
- $(kA)^{-1} = k^{-1}A^{-1}$ for scalar k
- $(AB)^{-1} = B^{-1}A^{-1}$ (similar to transpose)
- $(A^{-1})^T = (A^T)^{-1}$
- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I$

Invertibility

If A is a $n \times n$ **invertible** matrix, then the following are also true:

- A has n pivot positions.
- A has a trivial nullspace ($A\vec{x} = \vec{0}$ only if $\vec{x} = \vec{0}$).
- The columns and rows of A are **linearly independent**, and **span** $\mathbb{R}^n$. As a result, they form a **basis** for $\mathbb{R}^n$.
- The columnspace of A is $\mathbb{R}^n$, and is $n-$dimensional. So, A has a **rank** of n.
- For every $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a *unique solution*.
- The determinant of A is not 0.
- A does not have an eigenvalue of 0.

For a full description of the **invertible matrix theorem** (warning: parts are out of scope), look **here**.

Computing Inverses

Use Gaussian elimination! (Who would've guessed...)

- Construct an **augmented matrix** consisting of A and the identity matrix:

$$\left[A \mid I\right]$$

- Row reduce this augmented matrix until the *left side becomes the identity*, and the *right side becomes A^{-1}*:

$$\left[A \mid I\right] \longrightarrow \left[I \mid A^{-1}\right]$$

Computing Inverses: 2×2 Matrices

For a 2×2 matrix, you can **find the inverse quickly** using the following formula:

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

We can derive this from the general method of finding inverses, but this can be convenient.

Side Note: In real numerical computation, we generally use gaussian elimination to find the inverse of a large matrix even through there is a theoretical formula deduced from Cramer's rule which has a terrible runtime.

Practice: Computing Inverses

Find the **inverse** of

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Practice: Computing Inverses [Solution]

Make an **augmented matrix** with A and the identity, and then perform **Gaussian Elimination**:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-R_1+R_2 \to R_2]{-R_1+R_3 \to R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[R_1-3R_2 \to R_1]{R_1-3R_3 \to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Vector Spaces

Vector Spaces

A set of elements closed under vector addition and scalar multiplication.

- *"No escape properties"* for addition and scalar multiplication
- Must also contain the **0 vector** (special case of closed scalar multiplication).

Let $\mathbb{V}$ be a vector space:

- If $\vec{u}, \vec{v}$ are vectors in $\mathbb{V}$, then $\vec{u} + \vec{v}$ must also be in $\mathbb{V}$
- If $\vec{u} \in \mathbb{V}$ and k is a real number, then $k\vec{u}$ must be in $\mathbb{V}$

Thus, any **linear combination** of vectors in $\mathbb{V}$ is also in $\mathbb{V}$.

Practice: Is it a Vector Space?

Are the following collections of vectors **vector spaces**?

- 2D plane (spanning $\mathbb{R}^2$)
- 5D space (spanning $\mathbb{R}^5$)
- n-D space (spanning $\mathbb{R}^n$)
- Line in $\mathbb{R}^2$ intersecting origin
- Line in $\mathbb{R}^2$ *not* intersecting origin
- First and third quadrant of $\mathbb{R}^2$
- Plane in $\mathbb{R}^3$ intersecting origin
- $\{\vec{0}\}$ (just the zero vector)
- $\{\vec{v}, \vec{v} \neq 0\}$
- $\text{span}(\vec{v})$

Practice: Is it a Vector Space? [Solution]

Are the following collections of vectors **vector spaces**?

- 2D plane (spanning $\mathbb{R}^2$) [yes]
- 5D space (spanning $\mathbb{R}^5$) [yes]
- n-D space (spanning $\mathbb{R}^n$) [yes]
- Line in $\mathbb{R}^2$ intersecting origin [yes]
- Line in $\mathbb{R}^2$ *not* intersecting origin [no]
- First and third quadrant of $\mathbb{R}^2$ [no]
- Plane in $\mathbb{R}^3$ intersecting origin [yes]
- $\{\vec{0}\}$ (just the zero vector) [yes]
- $\{\vec{v}, \vec{v} \neq 0\}$ [no]
- span$(\vec{v})$ [yes]

Subspaces

- Definition: **A subspace is a subset of a vector space that is itself a vector space**.
- Suppose we have a vector space $\mathbb{V}$. A subset $\mathbb{S}$ of $\mathbb{V}$ is *only a subspace if the following three properties are met*:
    1. The **zero vector** of $\mathbb{V}$ is in $\mathbb{S}$
    2. $\mathbb{S}$ is closed under **vector addition**
    3. $\mathbb{S}$ is closed under **scalar multiplication**

(Same rules as before!)

Basis

Definition: A **basis** of a vector space is a **linearly independent** set of vectors that **spans** the vector space.

- **Linearly independent** (*not too big*): No vectors in a basis can be written as a linear combination of the other vectors.
- **Spanning** (*not too small*): All vectors in the vector space can be represented as a linear combination of the basis vectors.

Basis

A basis does not necessarily have to span $\mathbb{R}^n$ - it can span any vector space.

- A basis is a **minimum set of vectors** required to completely span a vector space.
- eg. Any basis of $\mathbb{R}^n$ contains **exactly n vectors**. In fact, an m-dimensional vector space must have m vectors in its basis.

Bases are **not unique**! Why? How many are there?

Practice: Basis

Are the following bases for $\mathbb{R}^n$?

$$\mathbb{V}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\mathbb{V}_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Practice: Basis [Solution]

Are the following bases for $\mathbb{R}^n$?

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ Yes! The set is linearly independent and spanning.

$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ No! The vectors are negatives of each other!

Null Space

Null Space

- **Definition**: The **null space** of a matrix (transformation) is the set of all solutions to the homogeneous equation $A\vec{x} = \vec{0}$.
- It is a *subspace of* $\mathbb{R}^n$.
- Solving a null space:
    1. Reduce to **reduced row echelon form**.
    2. Find solution to the system of equations.
    3. Represent the solutions in *matrix form*.

Practice: Null Space

Find the **null space** of A:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 2 \end{bmatrix}$$

Practice: Null Space [Solution]

First, **row reduce**.

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Practice: Null Space [Solution]

- We saw that in the row-reduced matrix, there were **3 pivot columns**.
- Additionally, we know that there are 5 total "variables"
- Thus, we can say that there are **2 free variables**, and *obtain a basis for our null space in terms of these free variables*!
- The **pivot columns** occur at x_1, x_3, and x_5, so we can set $x_2 = r$ and $x_4 = s$
- Let's find our basis in terms of r, s!

Practice: Null Space [Solution]

Let's find our basis in terms of r, s!

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_1 = 2r + s, \;\; x_2 = r$$

$$x_3 = -2s, \;\; x_4 = s, \;\; x_5 = 0$$

In matrix form:

$$\vec{x} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

Eigenvalues and Eigenvectors

Definition: Eigenvalues and Eigenvectors

****The most important concept in linear algebra**

An scalar-vector pair $\lambda, \vec{v}$ are an **eigenvalue-eigenvector pair** of matrix A if applying A to $\vec{v}$ produces a version of $\vec{v}$ scaled by λ.

$$A\vec{v} = \lambda\vec{v}$$

Properties:

- Eigenvectors w/ distinct eigenvalues are **linearly independent**
- A matrix is **non invertible** iff 0 is an eigenvalue
- A scalar times an eigenvector is still an eigenvector
- Eigenvalues remain the **same across transposes** - not necessarily true for eigenvectors!

$$A^{-1}\vec{v} = \lambda^{-1}\vec{v} \text{ and } A^n\vec{v} = \lambda^n\vec{v}$$

Finding Eigenvalues and Eigenvectors

To find eigenvalues and eigenvectors of A:

- Find solutions to the **characteristic polynomial**, which is:

$$\det(A - \lambda I) = 0$$

Aside: the determinant of a 2×2 *matrix is*

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- The solutions for λ give the **eigenvalues** of A.
- For each eigenvalue λ, calculate the null space of

$$A - \lambda I$$

- The basis for the null space will be your **eigenvectors**.

Practice: Eigenvalues and Eigenvectors

Find the **eigenvalues and eigenvectors** of

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

Practice: Eigenvalues and Eigenvectors [Solution]

First, find the solutions to the **characteristic polynomial**.

$$\begin{aligned}
\det(A - \lambda I) &= 0 \\
(3 - \lambda)(4 - \lambda) - 2 \cdot 1 &= 0 \\
\lambda^2 - 7\lambda + 10 &= 0 \\
\lambda_1 = 2,\ \lambda_2 &= 5
\end{aligned}$$

Practice: Eigenvalues and Eigenvectors [Solution]

Then, find the **null space** of $A - \lambda I$ for each lambda:

$$\left[\begin{array}{cc|c} 3-\lambda_1 & 1 & 0 \\ 2 & 4-\lambda_1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3-\lambda_2 & 1 & 0 \\ 2 & 4-\lambda_2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_2} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

PageRank

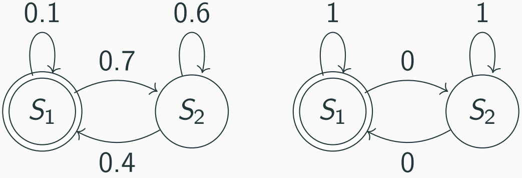
Graphs, Flow, and Transition Matrices

A **transition matrix** represents a directed graph of states and transitions.

- Edges of the graph represent *what fraction of one state moves to the next*.
- Entry **ij** in matrix means *fraction of water from node j entering node i*.
- Columns sum to ≤ 1 (*What would it mean of the sum of a column was greater than 1?*).
- Examples: Social networks, PageRank

Transition Matrix Examples

Here are some examples of **graphs** and their corresponding **transition matrices**. Notice that element ij represents the *fractional transition* from state j to state i.



$$T = \begin{bmatrix} 0.1 & 0.4 \\ 0.7 & 0.6 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Practice: Transition Matrices

- What do we know about the system if all *columns* each **sum to 1**?
- Less than 1? (Think about what physically happens if the system was water flows)
- Greater than 1?

Practice: Transition Matrices [Solution]

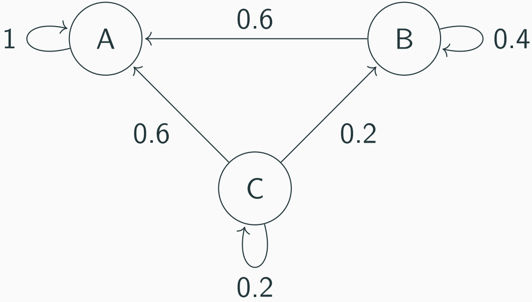
- What do we know about the system if all *columns* each **sum to 1**? **Flow is conserved: no "water" is added or lost.**
- Less than 1? (Think about what physically happens if the system was water flows) **Flow is lost.**
- Greater than 1? **Flow increases.**

PageRank

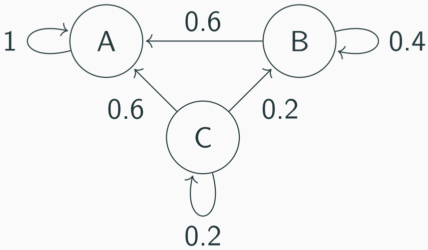
- Given a flow graph, we wish to **rank the nodes by their "importance"**, i.e. which state will hold the most "water" when the system reaches a steady state.
- Procedure:
    1. Write A, the **transition matrix** of the flow diagram.
    2. Find the eigenvector(s) of A, whose **eigenvalue is 1**. If $A\vec{v} = \vec{v}$, $\vec{v}$ is a **steady state vector**.
    3. $\vec{v}$ contains the *values the nodes will stabilize at*, and ranks the importance of each node

Practice: PageRank

Find the **transition matrix** for this flow diagram.



Practice: PageRank [Solution]



Assuming the first column represents *flow from state A*, the second represents *flow from state B*, etc. the **transition matrix** is:

$$T = \begin{bmatrix} 1 & 0.6 & 0.6 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 \end{bmatrix}$$

Practice: PageRank

Find the **steady state values** of the system.

$$T = \begin{bmatrix} 1 & 0.6 & 0.6 \\ 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 \end{bmatrix}$$

Practice: PageRank [Solution]

To find the **steady state values**, find the *eigenvectors corresponding to an eigenvalue of 1*.

$$A\vec{v} = \vec{v}$$

$$(A - I)\vec{v} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 0 & 0.6 & 0.6 & 0 \\ 0 & -0.6 & 0.2 & 0 \\ 0 & 0 & -0.8 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Diagonalization

Diagonalization

Goal:

- Find a basis such that a given matrix transformation can be expressed through **scaling by eigenvalues**. This is also called **eigendecomposition**.

$$A = P\Lambda P^{-1}$$

- Motivation: makes computing powers of A easier.

How:

- Λ is a *diagonal matrix of eigenvalues*.
- P is a matrix whose *columns are the corresponding eigenvectors*.

Practice: Diagonalization

Diagonalize the following matrix:

$$A = \begin{bmatrix} 3 & 3 \\ 1 & 5 \end{bmatrix}$$

Practice: Diagonalization [Solution]

First, **find eigenvalues**.

$$\det\left(\begin{bmatrix}3 & 3\\1 & 5\end{bmatrix} - \lambda I\right) = \det\left(\begin{bmatrix}3-\lambda & 3\\1 & 5-\lambda\end{bmatrix} = 0\right)$$

$$(3-\lambda)(5\lambda) - 3 = (\lambda-2)(\lambda-6) = 0\lambda = 2, 6$$

$$\Lambda = \begin{bmatrix}2 & 0\\0 & 6\end{bmatrix}$$

Practice: Diagonalization [Solution]

Next, **find eigenvectors**.

$$A - 2I = 0 \to \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{eigenvector}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$A - 6I = 0 \to \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{eigenvector}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}$$

Practice: Diagonalization [Solution]

Find the **inverse of eigenvector matrix**.

$$P^{-1} = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{(-3)(1) - (1)(1)} \begin{bmatrix} 1 & -1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -1/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

Practice: Diagonalization [Solution]

Plug into the diagonalization equation!

$$A = P\Lambda P^{-1}$$

$$A = \begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} -1/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$