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Research Note

A Comparison of Numerical Methods for the Study of Star Cluster Dynamics

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Summary. We compare the results of three different numerical methods for computing the evolution of a spherical star cluster from a given initial state, under the influence of internal relaxation: the N-body integration, the Monte Carlo method, and the fluid-dynamical approach. The general features of the evolution are very similar in all cases. The rates of evolution differ somewhat; for stars of equal masses, taking the N-body integrations as a reference, the Monte Carlo models evolve too fast by a factor 1.5, and the fluid-dynamical models by a factor 2 to 3.

Key words: stellar dynamics — star cluster — numerical simulation — relaxation

A number of methods have been devised in recent years for the numerical simulation of stellar systems. Here we consider the case of spherical star clusters, and the problem of their long-term evolution under the influence of internal encounters. Three different numerical approaches have been developed; in order of decreasing accuracy and increasing speed of computation, they are:

(a) The N-body integrations (see the reviews by Aarseth, 1973; Wielen, 1974). The cluster is represented by a system of N point masses, whose equations of motion are integrated numerically. This direct approach is the most realistic since it makes no a priori assumptions about the dynamical evolution of the system. Because of the heavy computing time requirement, the practical usefulness is limited to a few hundred particles.

(b) The Monte Carlo method (see the review by Hénon, 1973). This approach relies on the classical theory of relaxation in stellar systems, i.e. it assumes that the dynamical evolution is due to the cumulative effect of weak two-body encounters. The distribution function then evolves according to the well-known Fokker-Planck equation, which is solved by a Monte Carlo procedure. Two variants of this method have been developed: (b 1) in one of them (Spitzer and Hart, 1971a, b; Spitzer and Thuan, 1972), the orbital motion of the stars in the mean field is integrated step by step; (b 2) in the other (Hénon, 1971a, b, 1973), new positions of the stars are randomly chosen on their orbits at each step.

(c) The fluid-dynamical approach (Larson, 1970a, b), which also relies on the theory of encounters in the form of the Fokker-Planck equation. Additional approximations are made in order to eliminate the velocity variables; there results a set of partial differential equations in r (distance to centre) and t (time), which is solved by conventional methods.

In the present note, we compare results of the three methods applied to the same problem. With respect to earlier comparisons of the same kind (Hénon, 1971a; Spitzer and Thuan, 1972; Hénon, 1973; Wielen, 1974), the following improvements should be noted: (i) some deficiencies of the Monte Carlo method have been corrected (see Hénon, 1973); (ii) more models have been computed with methods (a) and (b); this provides an estimate of the statistical dispersion in the results, as well as better mean values; (iii) a detailed comparison of all available methods is presented for the first time.

We consider a convenient "model problem" defined as follows. At time $t=0$ the space density $\varrho(r,t)$ of the cluster conforms to Plummer's model, i.e. a polytrope of index 5:

$$\varrho(r,0)=(3/4\pi)\,MR^{-3}\,[1+(r/R)^2]^{-5/2}\,, \qquad (1)$$

where M is the total mass of the cluster, and R is a parameter which determines the dimensions of the cluster. The gravitational potential is then

$$U(r,0) = -GMR^{-1}\,[1+(r/R)^2]^{-1/2} \tag{2}$$

and the potential energy of the cluster is

$$W = -(3\pi/32)\,GM^2R^{-1}\,. \tag{3}$$

Initially the system is assumed to be in a steady state, with a velocity distribution everywhere isotropic. This implies that the initial distribution function is given by

$$f(r,V,0) = \begin{cases} (24\sqrt{2}/7\pi^3)\,G^{-5}M^{-4}R^2(-E)^{7/2} & \text{for } E<0\,, \\ 0 & \text{for } E>0\,. \end{cases} \tag{4}$$

Here $f(r,V,t)\,dr\,dV$ is the total mass of the stars with position r and velocity V, at time t, and E is the energy per unit mass of a star:

$$E = U + V^2/2\,. \tag{5}$$

The total energy of the system is then

$$\mathscr{E} = W/2 = -(3\pi/64)\,GM^2R^{-1}\,. \tag{6}$$

The number of stars is N. Two different mass distributions will be considered: (i) all stars have the same mass; (ii) the stars are distributed according to Wielen's prescription (1967) in four groups with relative individual masses 8, 4, 2, 1 and relative numbers 4, 10, 24, 62. Initially these proportions are the same everywhere, i.e. there is no mass segregation. The cluster is assumed to be isolated. A practical scheme for the generation of the initial positions and velocities is described in the Appendix.

The problem is to compute the evolution of the system with time. More specifically, we shall compare results for the evolution of the spatial structure of the cluster, by plotting the radii of the spheres which contain definite fractions of the total mass: 0.1 M, 0.5 M, 0.9 M, versus time. In the N-body calculations, the centre of the spheres is taken at the density centre, as defined by von Hoerner (1963); the other methods have built-in spherical symmetry. Since different people use different units, the results must be reduced to a common system. We therefore define non-dimensional variables.

$$[r] = r/r_0\,, \qquad [t] = t/t_0\,. \tag{7}$$

For the unit of length r_0, we adopt the frequently used value

$$r_0 = GM^2(-4\mathscr{E})^{-1}\,, \tag{8}$$

which gives here

$$r_0 = (16/3\pi)\,R\,. \tag{9}$$

The unit of time should be proportional to the relaxation time, rather than to the crossing time, since we are interested in the long-term evolution of the system, and we wish to compare computations for different values of N. Theoretical expressions of the relaxation time are of the form

$$K\,GM^{5/2}(-4\mathscr{E})^{-3/2}\,N/\ln(\gamma N)\,, \tag{10}$$

where K is a numerical constant which depends on the particular definition of the relaxation time, and γ is another constant, whose value varies somewhat depending on the approximations made in the theory. Here we shall adopt Spitzer and Hart's value (1971a): $\gamma = 0.4$, and we choose as unit of time

$$t_0 = GM^{5/2}(-4\mathscr{E})^{-3/2}\,N/\ln(0.4\,N)\,. \tag{11}$$

With this unit, the relaxation time defined by Spitzer and Hart (1971a) is, for Plummer's model: $[t_{rh}] = 0.0931$.

The results, expressed in the dimensionless variables $[r]$ and $[t]$, should be the same whatever the values of G, M, R, N used in the actual computation. Therefore we use $[r]$ and $[t]$ in what follows, and we drop the brackets for convenience.

Figure 1 represents the results of 9 different calculations for the equal-mass case, made with 5 different programs and 3 basically different methods. Open triangles and squares represent N-body integrations with $N = 100$ and $N = 250$ (Wielen, 1974, Models E and DE); filled triangles and squares represent two N-body integrations with $N = 250$ (Aarseth, 1974, Models VI and VII). Full lines represent three Monte Carlo computations by Hénon; one of them has been already published (Hénon, 1973); the two others are new, and based on the same program. Dotted lines represent a computation by Shull and Spitzer (1974), using essentially the procedure described by Spitzer and Thuan (1972). Dashed lines represent a fluid-dynamical computation by Larson (1970b)[1]).

The general shape of the curves is quite similar in all models. The Monte Carlo models appear to evolve somewhat faster than the N-body models, by a factor of about 1.5; the agreement can be considered satisfactory in view of the approximations made in the theory. The fluid-dynamical model evolves even faster; the ratio to N-body models is about 2 in the core and 3 in the halo. Here again, the agreement is probably as good as

[1]) Larson's models are not strictly isolated; there is an outer absorbing wall which simulates the tidal field. However, his two galactic cluster models with the wall at 100 pc and 1000 pc respectively (Larson, 1970b, pp. 103–105), or about 30 and 300 in our units, are almost identical; this indicates that the effect of the absorbing wall is negligible in these models. On Fig. 1, the case with a wall at 100 pc has been represented; the data presented here, which do not appear in the original paper (Larson, 1970b), have been kindly supplied to us by the author. The factor ln (0.4 N) in (11) has been taken equal to 3, the value used in Larson's models.

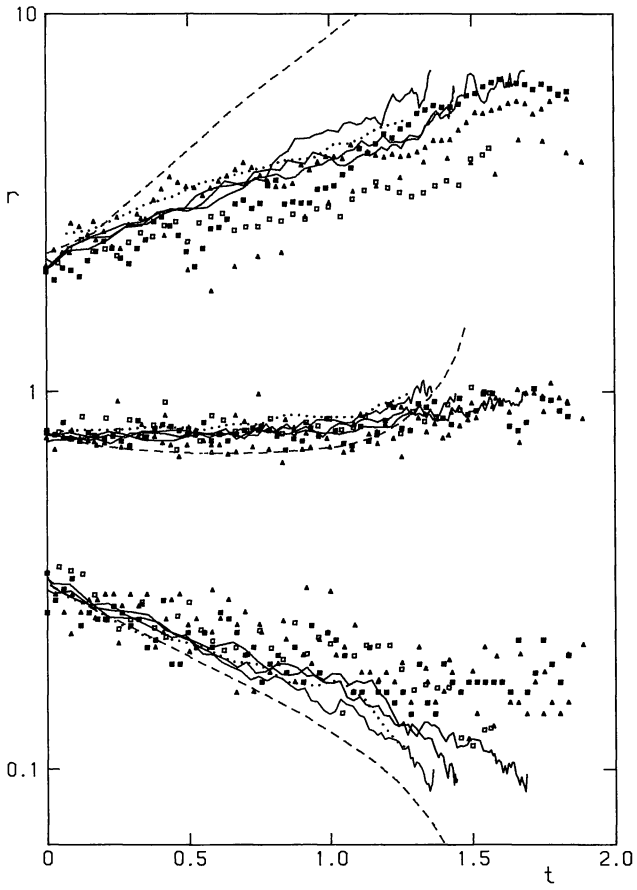

Fig. 1. Radii containing 10%, 50%, 90% of the mass, plotted versus time for a cluster with stars of equal masses. Open triangles and squares: N-body integrations with $N=100$ and $N=250$ (Wielen). Filled triangles and squares: N-body integrations with $N=250$ (Aarseth). Full lines: Monte Carlo models (Hénon). Dotted lines: Monte Carlo model (Shull and Spitzer). Dashed lines: fluid-dynamical model (Larson)

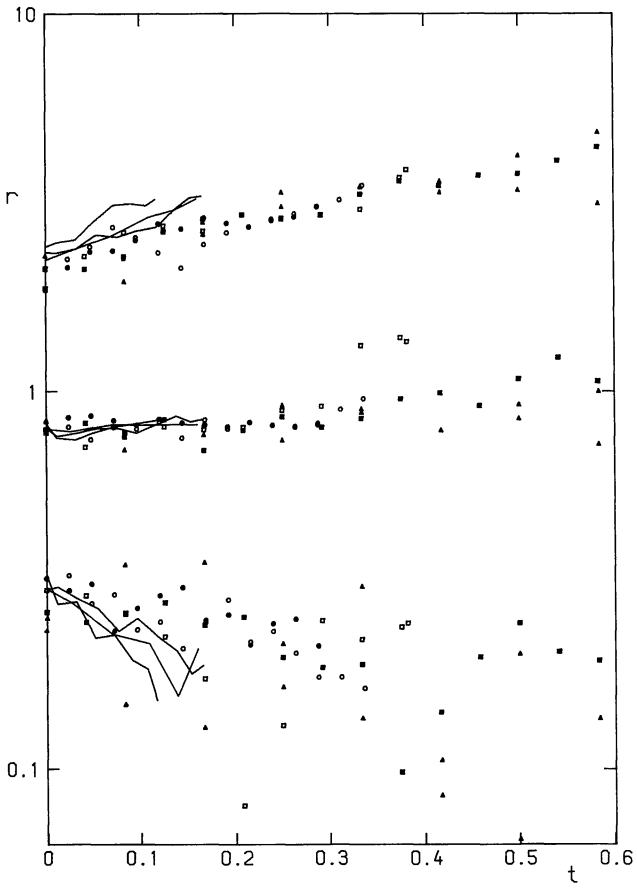

Fig. 2. Radii containing 10%, 50%, 90% of the mass, plotted versus time for a cluster with stars of different masses. Open and filled symbols: N-body integrations with $N=100$ (triangles), $N=250$ (squares), $N=500$ (circles) (Wielen). Full lines: Monte Carlo models (Hénon)

could be expected since this method involves further approximations.

In all cases but one, the contraction of the core ends in a singular event. In all three of the N-body integrations with $N=250$, a close binary forms near the centre, at $t=1.41$ and 1.61 (S.J.A.) and $t=1.52$ (R.W.), and starts absorbing more and more of the negative binding energy of the cluster; eventually it becomes so close that the computation has to be stopped. In the N-body integration with $N=100$ (R.W.), no such event is observed during the period covered by the integration ($0<t<1.88$). In the Monte Carlo computations (M.H.), the innermost shell collapses rather abruptly at $t=1.36$, 1.44 and 1.69, respectively. At present the computations cannot be reliably continued beyond this point. In Shull and Spitzer's model, an extrapolation of the curve corresponding to 2% of the mass (see Spitzer and Thuan, 1972) indicates that the centre of the system collapses at $t=1.34$. In the fluid-dynamical model, the central density becomes infinite at $t=1.49$.

This comparison appears to confirm the essential correctness of the classical theory of relaxation, based on the consideration of binary encounters, at least for predicting the evolution of the spatial structure. The good agreement between the N-body integrations with $N=100$ or 250 and the Monte Carlo models, which correspond essentially to the limit $N\to\infty$, also suggests that one can now interpolate with some confidence and apply the results to globular clusters, with N in the range 10^4-10^6.

Figure 1 shows that the two independent Monte Carlo methods are now in full agreement, as already reported (Hénon, 1973); the serious discrepancies found in earlier comparisons (Spitzer and Hart, 1971b; Spitzer and Thuan, 1972) have been eliminated by the improvements made in both programs.

Figure 1 is also instructive in that it gives an idea of the intrinsic dispersion of the results in the N-body integrations and in the Monte Carlo computations. This dispersion is partly due to the fact that random numbers are used for the selection of initial coordinates,

and also during the course of the computation in the Monte Carlo methods, and partly to the build-up of errors for individual orbits (Miller, 1964; Lecar, 1968). The dispersion should be proportional to $N^{-1/2}$ in the N-body integrations, and to $n^{-1/2}$ in the Monte Carlo models, n being the number of "superstars" or "shells" used to represent the distribution function (here $n=1000$ in all cases). As is shown by the Figure, an isolated computation can deviate appreciably from the mean, and should not be trusted beyond a certain point for quantitative results. As an illustration, we remark that an earlier comparison between Monte Carlo and N-body models (Hénon, 1973), based on a smaller number of models, indicated a discrepancy by a factor 2 in the rates of evolution; with more models available, this factor is now reduced to about 1.5.

We compare now the computing times, using the IBM 7090 computer as a reference. The computing time for an N-body integration is roughly proportional to $N^3/\ln(0.4\,N)$ and is about 20 hrs for $N=100$, 200 hrs for $N=250$. The computing time in the Monte Carlo methods is approximately proportional to n, and is of the order of 5 hrs for Hénon's models, 30 hrs for Shull and Spitzer's model, with $n=1000$. Finally, Larson's fluid-dynamical computation takes about 7 min.

Figure 2 is a more restricted comparison for the case of unequal masses. Isolated symbols represent the results of 6 N-body integrations (Wielen, 1974; Models P, P2, DP, DP2, FP, FP2): open and filled triangles are two models with $N=100$, open and filled squares are two models with $N=250$, and open and filled circles are two models with $N=500$. Curves represent three new Monte Carlo computations (Hénon). Here again the Monte Carlo models evolve faster than the N-body integrations. They end when the innermost shell collapses, at $t=0.12$, 0.17, 0.17 respectively. In the N-body integrations, a close central binary forms at about $t=0.5$, on the average, ranging from 0.3 to 1.0 for the individual models. The latter method is seen here to have a definite advantage in that it allows one to follow the evolution over a much longer time.

Appendix: Generation of Initial Coordinates

Plummer's model was found to be convenient for a comparison of methods, and might be adopted as a standard model for such comparisons. Therefore we think it useful to give here a detailed prescription for the construction of initial positions and velocities.

There must be available a subroutine which generates normalized random numbers X, with uniform probability distribution between 0 and 1. We consider the system as defined by (1), (2) and (4), taking $G=1$, $M=1$, $R=1$ for convenience. In the equal-mass case, each star then has a mass $m=1/N$. From (1), the mass inside a sphere of radius r is

$$M(r)=r^3(1+r^2)^{-3/2}. \tag{A1}$$

In order to select a value of r for a star, we simply generate a random number X_1 and equate $M(r)$ to X_1, so that r is given by

$$r=(X_1^{-2/3}-1)^{-1/2}. \tag{A2}$$

The actual position (x, y, z) of the star should now be selected on the sphere of radius r, with uniform probability. This is done by the usual trick: we generate two normalized random numbers X_2 and X_3 and compute

$$\begin{aligned} &z=(1-2X_2)\,r, \quad x=(r^2-z^2)^{1/2}\cos 2\pi X_3, \\ &y=(r^2-z^2)^{1/2}\sin 2\pi X_3. \end{aligned} \tag{A3}$$

Next, we compute the velocity modulus for the same star. The maximum value of V at distance r from the centre is the escape velocity

$$V_e=(-2U)^{1/2}=2^{1/2}(1+r^2)^{-1/4}. \tag{A4}$$

We write $V/V_e=q$. Then (4) shows that the probability distribution of q is proportional to

$$g(q)=q^2(1-q^2)^{7/2}. \tag{A5}$$

A convenient way to sample q according to this distribution is provided by von Neumann's rejection technique. Possible values of q range from 0 to 1, and $g(q)$ is always less than 0.1. Therefore we generate two normalized random numbers X_4 and X_5; if $0.1\,X_5<g(X_4)$, we adopt $q=X_4$; if not, a new pair of random numbers is tried, until one is found which satisfies the inequality. The velocity modulus is then obtained, using (A4). Since the velocity distribution is isotropic, the three velocity coordinates u, v, w are computed from V in the same way as the three space coordinates from r, using two new random numbers X_6 and X_7:

$$\begin{aligned} &w=(1-2X_6)\,V, \quad u=(V^2-w^2)^{1/2}\cos 2\pi X_7, \\ &v=(V^2-w^2)^{1/2}\sin 2\pi X_7. \end{aligned} \tag{A6}$$

The whole procedure is repeated for each of the N stars. Finally, the values of m, x, y, z, u, v, w may be scaled to suit the numerical scheme used. If a cluster with mass M and energy $\mathscr{E}$ is desired, while keeping $G=1$, then masses should be multiplied by M, lengths by $(3\pi/64)\,M^2|\mathscr{E}|^{-1}$, and velocities by $(64/3\pi)\,|\mathscr{E}|^{1/2}\,M^{-1/2}$.

Postscript. Monte Carlo models have recently been extended past the formation of the central singularity. In addition, the agreement between all three methods has been improved by a new evaluation of the constant γ (Hénon, 1974).

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