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Serge Cohen
Jacques Istas

# Fractional Fields and Applications

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Serge Cohen · Jacques Istas

# Fractional Fields and Applications

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# Foreword

In May 1990, I attended the annual day of the French Mathematical Society (SMF), which, that year, had chosen the crisp new subject of wavelet analysis. The flow of wavelets had not yet spread among large areas of Science, but a few top researchers were already actively working in this field. The SMF was aware of its great potential, both inside mathematics and for other sciences, and had decided to present it to its members.

At the end of the afternoon, a short man with a brushy moustache addressed me. He explained that he was a probabilist studying Gaussian fields, had heard that I was a student of Yves Meyer (the best possible recommendation…) and wished to enroll me in his research project. I had a very light background in probability, and was rather taken aback. But, if I had learned anything from my short mathematical life as a student in the starting subject of wavelets, it certainly was that big steps forward are the consequence of chance and unexpected encounters… and I jumped in without second thoughts. The next step was a couple of weeks at Clermont-Ferrand, where Albert Benassi explained his program. Albert was not a Bourbaki-style mathematician, but he was far-sighted, and had caught sight of a rich land, which would later become a fruitful field of interactions for mathematicians, signal analysts, and image processors. When Albert was doing mathematics, he was wearing seven-league boots, and did not clear all the way between each step; so following him in this adventure was a challenge… and a very rewarding one! When I joined, Daniel Roux, then Albert's Ph.D. student, was already part of the team. Serge Cohen and Jacques Istas would join a little later, and would quickly become prominent contributors in the development of this new area of mathematics. So that the present book can certainly be described as the final, polished output of the adventure that we lived as young scientists.

One should not infer that it only is an account of 20 years of exciting mathematics. It is much more: An introduction to a subject which now is extremely active, and flourishing in many directions. The conjunction of recent high resolution data acquisition techniques and Internet has for consequence that large collections of high resolution signals and images, coming from many different areas of science, are now available. The challenge for signal and image processing is to store, transmit, treat, and classify these data, which requires the introduction of wider and wider classes of sophisticated models. Their mathematical properties have to be investigated, simulations are needed in order to confront visually models with data, and statistical tools need to be developed so that the corresponding key parameters of the models can be identified. This ambitious scientific program precisely is the purpose of this book.

A first originality is that it deals with fields, and not processes. If the literature concerning stochastic processes is extremely rich, it is much less the case for fields; this may be due to the fact that several key probabilistic tools have been developed specifically in dimension one, as a consequence of the natural interpretation of the one-dimensional variable as the time axis, and have less natural extensions in several dimensions. However, the need for a similar treatment of fields increased recently, with many motivations raising from image processing (in 2D), or for simulations of three-dimensional phenomena. A second originality is that the book encompasses the three facets of the same scientific field: A probabilistic study of classes of random fields, the development of statistical methods of identification of their parameters, and finally simulation techniques. These topics require diverse skills and usually are not met in the same book. However, each of them enriches the other two: Questions raised in one part motivate developments in another, and their conjunction will make this book an extremely valuable tool, both for mathematicians interested in understanding possible applications, and for scientists working in signal an image processing, and who want to master the mathematical background behind the models that they use; let us stress the fact that the very detailed and pedagogical chapter dealing with preliminaries make the book really accessible to scientists with a light background in probability and analysis.

This rich mixing of different aspects of the same subject certainly is in the spirit of the new way of performing scientific investigations which was initiated by Benoît Mandelbrot, half a century ago: His motivations to develop mathematical models rose from the inspection of data picked in a wide range of different sciences, and the mathematical properties of these models would often follow from observing their simulations; it is by no means a surprise that fractal analysis is a recurrent theme in this book. If the reader will allow me a bold comparison, the composition of the book with three views of the same scientific topics is reminiscent of some of the most famous Picasso portraits, where the juxtaposition of slightly different perspectives give a much deeper insight of the subject.

October 2011

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# Contents

# Notations

- a.s.: almost surely.
- $\overset{(a.s.)}{=}$: almost sure equality.
- $\#\Omega$: cardinal of the set $\Omega$.
- $\mathcal{B}(K)$is the space of bounded functions on a set $K$.
- $\mathcal{C}^H$ is the space of Hölder-continuous functions on $[0,1]^d$.
- $C^m$ is the space of $m$ times differentiable functions.
- $C^{m,n}$ is the space of $m$ times differentiable function in the first variable and $n$ times differentiable function in the second variable.
- $det(A)$: determinant of the matrix $A$.
- $\overset{(d)}{=}$: equality in distribution.
- $E_k$: let $Z^{(d)}=\mathcal{N}_1(0,1)$ (notation defined below), then $E_k = \mathbb{E}|Z|^k$.
- i.i.d.: independent and identically distributed.
- $\mathbb{E}X$: expectation of the random variable $X$.
- $\widehat{f}(\xi) = \int_{\mathbb{R}}^{d} \exp(ix.\xi)f(x)\frac{dx}{(2\pi)^{d/2}}$
- $\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(\xi)\bar{g}(\xi)\frac{d\xi}{(2\pi)^{d/2}}$.
- We denote by $\Gamma(t) = \int_0^{+\infty} x^{t-1}e^{-x}dx$ the classical Gamma function for $t > 0$.
- iff : if and only if.
- $K_X$ is the Reproducing Kernel Hilbert Space (RKHS) associated with a Gaussian field $X$.
- $\log_2(x) = \frac{\ln(x)}{\ln(2)}$.
- $\Lambda = \mathbb{Z} \times \mathbb{Z} \times \{1\}$
- $\Lambda^+ = \mathbb{N}^* \times \mathbb{Z} \times \{1\} \cup 0 \times \mathbb{Z} \times \{0\}$
- $\mathbb{N}$ the set of non-negative integers.
- $\mathbb{N}^*$ the set of positive integers.
- $\mathcal{N}_d(m, \Sigma)$: $d$-dimensional Gaussian vector of expectation $m$ and covariance matrix $\Sigma$.
- $f(x) = O(g(x))$ as $x \to x_0$ when there exists a finite constant $C$ such that $|f(x)| < C|g(x)|$ in a neighborhood of $x_0$. The point $x_0$ may be a real number, $+\infty$, or $-\infty$.

- $X_n = o_{\mathbb{P}}(Y_n)$ if $X_n,\ Y_n$ are sequences of random variables defined on the same probability space such that
$$\forall \epsilon > 0,\ \lim_{n\to\infty} \mathbb{P}(|X_n| \geq \epsilon |Y_n|) = 0.$$
  This notation can be generalized to fields $X(x),\ Y(x)$as $X(x) = o_{\mathbb{P}}(Y(x))$ in a neighborhood of some $x_0$.
- $X_n = O_{\mathbb{P}}(Y_n)$ if $X_n,\ Y_n$ are sequences of random variables defined on the same probability space such that
$$\forall \epsilon > 0,\ \exists M > 0,\ \sup_{n\in\mathbb{N}} \mathbb{P}(|X_n| > M|Y_n|) < \epsilon.$$

This notation can be generalized to fields $X(x),\ Y(x)$as $X(x) = O_{\mathbb{P}}(Y(x))$ in a neighborhood of some $x_0$.

- $f(x) = o(g(x))$ as $x \to x_0$ when $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$;$x_0$ may be a real number or $+\infty$ or $-\infty$.
- $\mathbb{Q}$ is the set of rational numbers.
- $\mathbb{R}$ the set of real numbers.
- $\Re(z),\ \Im(z)$ are respectively the real part and the complex part of a complex number $z$.
- $R(x,y) = \mathbb{E}X_x\overline{X}_y$: covariance of a second-order centered field.
- r.v.: random variable.
- $\mathcal{S}$ is the space of fast decreasing functions.
- $\mathcal{S}'$ is the space of tempered distributions.
- $S^d = \{x \in \mathbb{R}^{d+1}\ \text{ s. t. } \|x\| = 1\}$, the $d$-dimensional sphere where $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^{d+1}$.
- s.t.: such that.
- $supp f$ is the support of the function $f$.
- $\sigma(X)$ is a sigma field on the set $X$.
- ${}^tA$: transpose of the matrix $A$.
- var$X$: variance of the random variable $X$.
- $u_n \sim v_n$ means that $\lim_{n\to+\infty} \dfrac{u_n}{v_n} = 1$ .
- $u_n \asymp v_n$ means that there exists a constant $0 < C < \infty$ such that $\frac{1}{C} u_n \leq v_n \leq C u_n \quad \forall n \in \mathbb{N}$.
- $f(x) \sim g(x)$as $x \to x_0$ when $\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$;$x_0$may be a real number or $+\infty$ or $-\infty$.
- $[x]$ is the integer such that $[x] \leq x < [x] + 1$.

# Chapter 1
# Introduction

Fractals everywhere! This is the title of a bestseller, but it is also a reality: Fractals are really everywhere. What a change since the days of Charles Hermite declaring “I turn away with fright and horror of this terrible scourge of continuous functions without derivative”.[1] Historically, the first fractals are the Cantor set, and the Weierstrass function, followed by the famous Brownian motion. In these seminal examples, there were already between the lines the basic properties self-similarity and roughness, we will find throughout this book. But, what does mean this word “fractal”? Or its more or less synonymous “fractional”? There are probably as many definitions as there are people who work on the subject. We follow two tracks in this book.

- First, if an object is similar to each part of itself, i.e. the whole has the same shape as one or more of the parts.
- Second, if its Hausdorff dimension is not an integer.

A very common object is similar to each part of itself: a straight line! The self-similarity does not necessarily imply roughness. And, among the graphs of deterministic functions, the straight line is the only self-similar graph. A bit disappointing! But now replace condition “find a graph where each part is similar to itself” with the condition “find a graph where each part is *statistically* similar to itself”. We obtain an infinite number of random graphs that satisfy this statistical self-similarity. In fact, there are so many that we do not know how to classify them, even if we know how to define some broad categories. For example, if we impose the graph to be Gaussian, for each self-similarity index $0 < H \leq 1$ there exists a unique solution, up to a constant, the famous fractional Brownian motions. For $H = 1$, the graph of fractional Brownian motion is a random Gaussian straight line. For $H \neq 1$, the graph of fractional Brownian motion is statistically self-similar, rough (its Hausdorff dimension is (a.s.) equal to 2-H and its pointwise Hölder exponent is $H$). Thus fractional Brownian motions became the archetypes of random fractals, we devote to fractional Brownian motions an important section in the Chap. 3, which is concerned

[1] “Je me détourne avec horreur et effroi de cette plaie lamentable des fonctions continues qui n’ont pas de dérivées” letter of Charles Hermite to Thomas Stieltjes, 20 may 1893.

S. Cohen and J. Istas, *Fractional Fields and Applications*,
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with self-similarity in a bigger generality. A smooth curve is locally approximated by a straight line, this is a general fact. A straight line is self-similar and a smooth curve becomes locally self-similar. Similarly, parts of a random graph could locally look like a, necessarily self-similar, random graph: One then speaks of local asymptotic self-similarity (in short class). For instance, numerous random fields look locally like fractional Brownian fields and we study several within the Chap. 4. To compare a model with reality implies the statistical estimation of the parameters describing the models. The estimation of the fractional parameter of a process is an issue for which there are several approaches known. In this book we will mainly use generalized quadratic variations to estimate fractional parameters. This statistical method is treated in the Chap. 5 devoted to Statistics. Finally, the simulation of theoretical models is a necessity. Again, it is not as simple as one might think, even in the Gaussian case. Numerical problems, computations time, are still not resolved, and we dedicate the Chap. 6 to discuss these issues.

Now that the broad picture is depicted we can be more precise and describe the points we stress in this book. Our main aim is to present multifractional fields. Surprisingly enough we don't write anywhere a definition of multifractional fields. Actually we think that multifractionality is a concept that depends on the applications one is interested in. Moreover we don't write any precise definition of a fractional field either. So it seems that the study of (multi)fractional fields is not rigorous in a mathematical sense. It is wrong and we want to provide in this introduction the reader with a walkthrough to multifractional fields.

If you have a good familiarity with Probability theory, Statistics, Analysis specialized in Hausdorff dimension and some background with wavelets then you can skip the Preliminaries Chap. 2. If not or if you discover later that you are missing some basic definition then this chapter is to make the book as self contained as possible for an undergraduate student in Mathematics. Please note that one does not need to be a mathematician to read this book. We have tried to make it as friendly as possible to applied scientists and engineers. Although most of the results of Chaps. 3 and 4 are given with complete and rigorous proofs, we have tried not be too technical. A nasty consequence of this choice is the fact that this book does not make a proper account to the most recent and sophisticated results in the domain of the (multi)fractional fields. On the other side if the first three chapters are still too technical for your own background, we may suggest to start with Chaps. 5 and 6.

The beginning of the hard core of the book is in Sect. 3.2.2, where the main properties of fractional Brownian process are explained. First of all you can find a uniqueness result that shows that up to a constant it is the only Gaussian process, which is $H$-self-similar and with stationary increments. In this part we make explicit the covariance structure, the regularity of the sample paths including the computation of the Hausdorff dimension of the graph, and two integral representations of fractional Brownian motion: One which is called the moving average representation and one which is in the Fourier domain and which is called harmonizable.

So far so good but we think that restricting the word "fractional" to either Gaussian, self-similar or processes with stationary increments is too restrictive for applications. Hence we will describe various generalizations of fractional Brownian motion in

Sect. 3.3 of Chap. 3. Some are parameterized by multidimensional index, some are non-Gaussian but they all keep in some sense the $H$-self similarity property, which is the common property of most of the objects considered in the Chap. 3. At this point we would like to prevent the reader from a possible misunderstanding: This book is a treaty neither on fractional Brownian motion nor on self-similarity. There are many good references in the literature on those topics (e.g. [53, 54]) and we will avoid to talk on any properties of fractional Brownian motion related to stochastic integration because as far as we know there are not easy to generalize to other (multi)fractional fields.

The goal of the Chap. 4 is to advertise multifractional fields. At this point a reasonable question is to wonder why bother with these generalizations of fractional Brownian motion. The answer comes from the data encountered by practitioners using fractional Brownian motion. Very often they have data for which the parameter $H$ seems to vary with the time or the location where it is observed. It can be experienced because of the estimation of $H$ or by simply observing the visual roughness of the sample paths. In this case the model of fractional Brownian motion is too rigid for your data because $H$ is constant in this model or for a deeper mathematical reason: self-similarity and stationarity are global properties of the model and they cannot be adjusted to fit variations of data. A first solution to generalize the model of fractional Brownian motion is to localize the self-similarity property. This way of thinking yields a discussion carried in Sect. 4.2. In the Definition 4.2.1 of local asymptotic self-similarity the constant index $H$ is replaced by a function $x \mapsto h(x)$ which varies with the location $x \in \mathbb{R}^d$ and which is called a multifractional function. However we do not think that local asymptotic self-similarity alone is enough to call a field multifractional. Obviously one wants to be able to compute at least some pointwise Hölder exponent at each point of multifractional fields. In the Chap. 4 we provide many examples of Gaussian locally asymptotically self-similar fields: Filtered white noises, multifractional Brownian fields, step fractional Brownian processes, generalized multifractional Gaussian processes. For each of this locally asymptotically self-similar field one can compute the multifractional function and its value at a location $x$ is almost surely also the pointwise Hölder exponent. The reason why we have various models is that the assumptions on the regularity of the multifractional function $x \mapsto h(x)$ is less restrictive when a more sophisticated model is used. The previous list was ordered with growing sophistication. It is desirable because in some applications like image processing you expect discontinuous multifractional functions. Actually some data related to turbulence models have very wild "multifractional" functions for which the precise value of the pointwise Hölder exponent at a point $x$ is not interesting but only the Hausdorff dimension of the set of the points $x$ where the Hölder exponent is given is relevant. This theory yields "multifractal" processes or fields. It is not in the scope of this book since we are interested in applications where the pointwise Hölder exponent can be estimated. An introduction to multifractal processes is available in the Chap. 10 of [75]. Later in the Chap. 4 non-Gaussian multifractional fields are introduced using moving average or harmonizable integrals that extends the representations of fractional Brownian motion. Beside, new models called fractional Lévy fields show that the multifractional function obtained in the lass property is not always equal to the pointwise Hölder exponent.

At this point we think that we can propose a rule of the thumb to know if a field should be called multifractional. *Multifractional fields have three properties:*

- *Local asymptotic self-similarity with a multifractional function h.*
- *The sample paths should have at every point x a local Hölder exponent, which can be different from h(x).*
- *Almost sure efficient estimation of the multifractional function h and of the local Hölder exponent should be feasible with the observation of only one sample of the fields.*

If you accept the previous rule you can understand why we are interested in the Chap. 5 in the estimation of the parameters of multifractional models. It is the third condition to have a useful fractional model. In practice the use of various discrete generalized quadratic variations along the paths of fractional fields described in the previous chapter is our favorite tool for the estimation. It allows in some sense a unified treatment for the estimation even if other techniques exist for fractional Brownian motion. In this chapter we have to distinguish the unifractional case where $H$ is a real number and the multifractional case where we have to estimate the multifractional function $h$.

In the Chap. 6 we review some simulation techniques to get the flavor of what are the multifractional fields introduced in the previous chapters. Once again we do not focus on the simulation of fractional Brownian motion which is easily available in the literature (See [10] for instance.) but more on multifractional fields. Unfortunately many approximations have to be made to deal with computation times and this chapter should be thought more as an introduction to the problems encountered in simulating the multifractional fields than as the last words on this topic.

To finish this introduction we would like to thank many collaborators that help us to understand multifractional fields and to write this book. First of all both authors would like to express their endless gratitude to Albert Benassi. Albert has introduced both of us to the wonderful world of multifractional fields and the questions raised by their applications to real data. He was not only a collaborator for many of the articles used to write this book but also the source of most of the questions for which you have partial answers here.

Many parts of this book were taught at graduate students both in Toulouse and in Grenoble and we would like to thank the audience that helps us to improve our explanations. The universities where we are doing research and teaching leading to this book Université de Grenoble, Université de Toulouse, Université de Versailles Saint-Quentin en Yvelines and during a sabbatical semester of one of the author Cambridge University UK provided us stimulating environments. We are also indebted to many colleagues and former students. We gratefully acknowledge in particular Jean-Marc Azaïs, Jean-Marc Bardet, Alexandre Brouste, Jean-François Coeurjolly, Laure Coutin, Claire Christophe, Sébastien Déjean, Coralie Fritsch, Sébastien Gadat, Fabrice Gamboa, Stéphane Jaffard, Céline Lacaux, Sophie Lambert, Michel Ledoux, Olivier Perrin and Mario Wschebor.

# Chapter 2
# Preliminaries

In this chapter we have collected some results that will be used in the sequel of the book. We have divided these results in two parts. In the first one we recall some facts concerning stochastic processes. In the second part some results concerning fractal analysis are given.

## 2.1 Stochastic Fields

The main topic of this book is random fields. In this section we provide the reader with some theoretical background for random fields. The case of Gaussian fields, which give numerous examples, is particularly stressed. Nevertheless we assume some prerequisites for distributions of random variables and some elementary facts in probability theory. See [52] for a convenient introduction to probability theory.

### *2.1.1 Definition*

**Definition 2.1.1** *Stochastic field.*

*Let $T$ be a set and $(E, \mathcal{E})$ a measurable space. A stochastic field indexed by $T$, taking values in $(E, \mathcal{E})$, is a collection $(X_t, t \in T)$ of measurable maps $X_t$ from a probability space $(\Omega, \sigma(\Omega), \mathbb{P})$ to $(E, \mathcal{E})$.*

In this book, we are mainly concerned by real or complex valued stochastic fields indexed by $\mathbb{R}^d$, $d \geq 1$. If it is not mentioned otherwise, $T$ will be $\mathbb{R}^d$ or a Borel subset in $\mathbb{R}^d$. When $d = 1$, one usually speaks of stochastic processes rather than of stochastic fields. The measurable space $(E, \mathcal{E})$ is the space $\mathbb{R}$ or $\mathbb{C}$ endowed with its Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ or $\mathcal{B}(\mathbb{C})$. For every $t \in T$, the stochastic field $X$ yields a real or a complex valued random variable on $(\Omega, \sigma(\Omega), \mathbb{P})$. Fix now $\omega \in \Omega$. The map

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$t \to X_t(\omega)$ is the sample path of the stochastic field. We will mainly give conditions for real valued fields.

## 2.1.2 Kolmogorov's Consistency Theorem

The distribution of a stochastic field is characterized by finite dimensional distributions of the field.

**Definition 2.1.2** *Finite dimensional distributions of the field.*

*Two stochastic fields $(X_t, t \in T)$ and $(Y_t, t \in T)$ are said to be versions of each others if they have the same finite dimensional distributions. It means that for every $n \geq 1$ and $t_1, \ldots, t_n \in T$*

$$(X_{t_1}, \ldots, X_{t_n}) \overset{(d)}{=} (Y_{t_1}, \ldots, Y_{t_n}). \tag{2.1}$$

*These distributions are called the finite dimensional distributions (or margins) of the field $(X_t, t \in T)$.*

Let us stress that the one dimensional distributions are not enough to characterize distributions of a stochastic field. Actually even for $n = 2$ we can have $X_{t_1} \overset{(d)}{=} Y_{t_1}$ and $X_{t_2} \overset{(d)}{=} Y_{t_2}$, whereas $(X_{t_1}, X_{t_2})$ has a different distribution from $(Y_{t_1}, Y_{t_2})$. Let $X_{t_1} \overset{(d)}{=} Y_{t_1} \overset{(d)}{=} X_{t_2} \overset{(d)}{=} Y_{t_2}$ be standard normal variables with $X_{t_1} = X_{t_2}$ and $Y_{t_1}$ independent from $Y_{t_2}$ then the distribution $(X_{t_1}, X_{t_2})$ is clearly different from the distribution of $(Y_{t_1}, Y_{t_2})$.

Let $(X_t, t \in T)$ be a stochastic field. The distribution of the random vector $(X_{t_1}, \ldots, X_{t_n})$, $n \geq 1$, $(t_1, \ldots, t_n) \in T^n$ is characterized by its characteristic function. For $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ the characteristic function is defined by

$$\psi_{t_1,\ldots,t_n}(\lambda_1, \ldots, \lambda_n) = \mathbb{E} \exp\left( i \sum_{j=1}^{n} \lambda_j X_{t_j} \right).$$

The Kolmogorov's consistency theorem [28] ensures the existence of a stochastic field such that the finite-dimensional distributions are consistent.

**Theorem 2.1.1** *Kolmogorov's consistency theorem.*

*Let $\psi_{t_1,\ldots,t_n}(\lambda_1, \ldots, \lambda_n)$, $n \geq 1$, $(t_1, \ldots, t_n) \in T^n$ be a collection of characteristic functions. Assume that for any permutation $(\sigma(1), \ldots, \sigma(n))$ of $(1, \ldots, n)$, one has*

$$\psi_{t_{\sigma(1)},\ldots,t_{\sigma(n)}}(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}) = \psi_{t_1,\ldots,t_n}(\lambda_1, \ldots, \lambda_n), \tag{2.2}$$

*and that, for any $m \leq n$*

$$\psi_{t_1,\dots,t_m}(\lambda_1,\dots,\lambda_m) = \psi_{t_1,\dots,t_n}(\lambda_1,\dots,\lambda_m,0,\dots,0). \tag{2.3}$$

*The finite-dimensional distributions associated to these characteristic functions are called consistent. If the collection of characteristic functions is consistent, there exists a stochastic field such that, for $n \geq 1$, $(t_1,\dots,t_n) \in T^n$,*

$$\psi_{t_1,\dots,t_n}(\lambda_1,\dots,\lambda_n) = \mathbb{E}\exp\left(i\sum_{j=1}^{n}\lambda_j X_{t_j}\right). \tag{2.4}$$

Existence of the Brownian motion will be proved with the Kolmogorov's consistency theorem in the next section.

### 2.1.3 Gaussian Fields and Non-Negative Definite Functions

Many fields in this book will be Gaussian fields. First we recall some elementary facts concerning Gaussian random variables and Gaussian random vectors. The distribution of a Gaussian random variable with mean $\mathbb{E}X = m$ and variance $\sigma^2$ has a density

$$g_{m,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-m)^2}{2\sigma^2}\right)$$

with respect to Lebesgue measure. In this case the distribution of the Gaussian random variable is denoted by $\mathcal{N}(m,\sigma^2)$. Conventionally, if $X \overset{(a.s.)}{=} m \in \mathbb{R}$, the distribution of $X$ is considered as a degenerated Gaussian distribution $\mathcal{N}(m,0)$. A Gaussian random variable is call standard if $X \overset{(d)}{=} \mathcal{N}(0,1)$. The characteristic function of $X \overset{(d)}{=} \mathcal{N}(m,\sigma^2)$ is

$$\mathbb{E}\exp(i\lambda X) = \exp\left(i\lambda m - \sigma^2\lambda^2/2\right).$$

Sometime we will need complex valued Gaussian variables.

**Definition 2.1.3** *Let $X$ be a complex valued random variable. It is a standard complex Gaussian variable if $\sqrt{2}\Re X$ and $\sqrt{2}\Im X$ are two independent real standard Gaussian random variables.*

Please note that if $X$ is a standard complex Gaussian variable then

$$\mathbb{E}X = 0, \quad \mathbb{E}|X|^2 = 1,$$

this normalization explains the $\sqrt{2}$ in Definition 2.1.3.

**Definition 2.1.4** *Let $X$ be a complex valued random variable. It is a complex Gaussian variable if there is a standard complex Gaussian variable $Z$ and $a, b \in \mathbb{C}$*

*such that*

$$X = aZ + b.$$

**Definition 2.1.5** *A random vector* $X = (X_1, \ldots, X_d)$ *is called a Gaussian random vector if any finite linear combination of its coordinates* $\sum_{i=1}^{d} \lambda_i X_i$ *is a Gaussian random variable.*

One can check that if $X$ is a real valued Gaussian random vector with mean $\mathbb{E}X = m \in \mathbb{R}^d$ and covariance matrix $\Gamma_{ij} = \mathbb{E}(X_i - m_i)(X_j - m_j)$, the characteristic function of $X$ is given by

$$\mathbb{E} \exp(i\lambda.X) = \exp\left(i\lambda.m - \frac{1}{2}{}^t\lambda\Gamma\lambda\right),$$

where ${}^t\lambda$ is the transpose of the vector $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix}$. Let us recall that any covariance matrix is a non-negative symmetric matrix in the sense that $\forall \lambda \in \mathbb{R}^d,\ {}^t\lambda\Gamma\lambda \geq 0$. This classical non-negativity property for symmetric matrix is generalized to non-negative functions (Definition 2.1.9).

**Example 2.1.1** *Existence of the Brownian motion.*

Let us now prove the existence of the standard Brownian motion with the Kolmogorov's consistency theorem. One aims to construct a stochastic process $(B(t))_{t\geq 0}$ satisfying the properties of the following definition.

**Definition 2.1.6** *A process* $(B(t))_{t\geq 0}$ *satisfying*

- $B(0) = 0$,
- *for all* $s \leq t$, *the increment* $B(t) - B(s)$ *is independent of* $(B(u), u \leq s)$,
- $B(t) - B(s)$ *has a centered Gaussian distribution with variance* $|t - s|$

*is called a standard Brownian motion.*

Let $0 = t_0 \leq t_1 \leq \ldots \leq t_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Set $\Delta_j = B(t_j) - B(t_{j-1}),\ 1 \leq j \leq n$. Then

$$\sum_{j=1}^{n} \lambda_j B(t_j) = \sum_{j=1}^{n} \lambda_j \left(\sum_{k=1}^{j} \Delta_k\right).$$

Since the $\Delta_j$s have to be independent and Gaussian, the characteristic functions of the $B(t_j)$ should be given by

$$\mathbb{E}\exp\left(i\sum_{j=1}^{n}\lambda_j B(t_j)\right) = \exp\left(-1/2\sum_{j=1}^{n}\left(\sum_{k=j}^{n}\lambda_k\right)^2 (t_j - t_{j-1})\right). \quad (2.5)$$

It is now straightforward that the characteristic function (2.5) satisfies the consistency conditions (2.2) and (2.3). By Kolmogorov's consistency theorem, the standard Brownian motion $(B(t), t \geq 0)$ exists and its distribution is unique. A process $(B(t), t \geq 0)$ such that $B(t) = aB_0(t) + b$, where $a,\ b \in \mathbb{R}$, and where $B_0$ is a standard Brownian motion is called a Brownian motion. The standardization in Definition 2.1.6 amounts to fix $B(0) = 0$ almost surely and $\mathrm{var}(B(1)) = 1$.

**Definition 2.1.7** *Gaussian fields.*
*A stochastic field $X_t, t \in T$ is a Gaussian field if and only if, for all $n \geq 1$, $t_1, \ldots, t_n \in T$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian random vector.*

For Gaussian fields the distribution is characterized by the mean and the covariance functions. Let us define these functions.

**Definition 2.1.8** *Mean value and covariance function.*
*Let $(X_t, t \in T)$ be a complex valued stochastic field such that $\mathbb{E}|X_t|^2 < +\infty$, $\forall t \in T$. The mean value of $X$ is the function $t \mapsto m(t) = \mathbb{E}X_t$. A real valued stochastic field such that $\mathbb{E}X_t = 0 \ \ \forall t \in T$ is called a centered field. The covariance function of $X$ is the function*

$$(t, s) \mapsto R(t, s) = \mathbb{E}(X_t - m(t))\overline{(X_s - m(s))}.$$

If $X$ is a real valued field then the mean is also a real valued function and $R(t, s) = \mathbb{E}(X_t - m(t))(X_s - m(s))$.

We first give a characterization of covariance functions of Gaussian fields.

**Definition 2.1.9** *Non-negative definite function.*
*A complex valued function $(t, s) \to \psi(t, s), s, t \in T$ is called Hermitian if $\psi(s, t) = \overline{\psi(t, s)}$. Moreover an Hermitian function is a non-negative definite function if, for every $n \geq 1$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $t_1, \ldots, t_n \in T$*

$$\sum_{i,j=1}^{n}\lambda_i\overline{\lambda_j}\psi(t_i, t_j) \geq 0.$$

If a function $(t, s) \to \psi(t, s), s, t \in T$ is real valued, then Hermitian functions are nothing else but symmetric functions.

**Proposition 2.1.1** *Let $\psi$ be a non-negative definite function. Then, for all $t, s \in T$,*

$$0 \leq \psi(t, t), \quad (2.6)$$

$$\Re\psi(t, s)^2 \leq \psi(t, t)\psi(s, s). \quad (2.7)$$

*Proof of Proposition 2.1.1*

Apply the definition of non-negative definite function with $n = 1$, $t_1 = t$ and $\lambda_1 = 1$, the Eq. (2.6) is straightforward.

Apply now the definition of non-negative definite function with $n = 2$, $t_1 = t$, $t_2 = s$, $\lambda_1 = 1$ and $\lambda_2 = -\lambda \in \mathbb{R}$. It follows that the polynomial $\lambda \to \lambda^2\psi(s,s) - 2\lambda\Re\psi(t,s) + \psi(t,t)$ is non-negative. The discriminant $4(\Re\psi(t,s)^2 - \psi(t,t)\psi(s,s))$ is therefore non-positive and (2.7) is proved.

**Proposition 2.1.2** *The covariance function $R$ of a stochastic field is a non-negative definite function.*

*Proof of Proposition 2.1.2*

Since

$$\sum_{j,j'=1}^{n} \lambda_j \overline{\lambda}_{j'} R(t_j, t_{j'}) = \mathbb{E}\left( |\sum_{j=1}^{n} \lambda_j (X(t_j) - \mathbb{E}X(t_j))|^2 \right) \geq 0,$$

the proposition is proved.

We now prove that any non-negative definite function is a covariance function of a Gaussian field. For the sake of simplicity we suppose that the fields are real valued until the end of the section.

**Theorem 2.1.2** *Characterization of real valued Gaussian fields.*

*Let $m(t)$, $t \in T$ be a function and $R(t,s)$, $t, s \in T$ be a non-negative definite function. Then the formula, for $n \geq 1$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $t_1, \ldots, t_n \in T$,*

$$\mathbb{E}\exp\left(i\sum_{j=1}^{n}\lambda_j X_{t_j}\right) = \exp\left(i\sum_{j=1}^{n}\lambda_j m(t_j)\right) \times \exp\left(-1/2\sum_{j,j'=1}^{n}\lambda_j\lambda_{j'}R(t_j,t_{j'})\right)$$

*characterizes the distribution of an unique real valued Gaussian field $X$, whose mean value function is $m$, and whose covariance function is $R$.*

*Proof of Theorem 2.1.2*

In order to apply Kolmogorov's consistency theorem, we first need to check that the functions

$$\psi_{t_1,\ldots,t_n}(\lambda_1,\ldots,\lambda_n) = \exp\left(i\sum_{j=1}^{n}\lambda_j m(t_j)\right) \times \exp\left(-1/2\sum_{j,j'=1}^{n}\lambda_j\lambda_{j'}R(t_j,t_{j'})\right)$$

are characteristic functions. Let $\Sigma_n$ be the matrix $(R(t_j, t_{j'}))_{j,j'=1,\dots,n}$. $\Sigma_n$ is a symmetric, non-negative definite matrix. It can be written (e.g. [65]) in the form $\Sigma_n = P_n \Lambda_n P_n^{-1}$ where $P_n$ is an orthogonal matrix and $\Lambda_n$ is a diagonal matrix diag$(\lambda_1, \dots, \lambda_n)$ whose eigenvalues $\lambda_1, \dots, \lambda_n$ are non-negative. Define then $\sqrt{\Sigma_n}$ by $\sqrt{\Sigma_n} = P_n \sqrt{\Lambda_n} P_n^{-1}$ where $\sqrt{\Lambda_n} = \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})$. Let $\varepsilon_1, \dots, \varepsilon_n$ be $n$ i.i.d. standard normal variables and set $E_n = {}^t(\varepsilon_1, \dots, \varepsilon_n)$. Set $M_n = {}^t(m(t_1), \dots, m(t_n))$. Then the characteristic function of the Gaussian random vector $Z_n = M_n + \sqrt{\Sigma_n} E_n$ is $\psi_{t_1,\dots,t_n}(\lambda_1, \dots, \lambda_n)$. It is then straightforward that these characteristic functions $\psi_{t_1,\dots,t_n}(\lambda_1, \dots, \lambda_n)$ satisfy the consistency conditions (2.3). By the Kolmogorov's consistency theorem, Theorem 2.1.2 is proved.

Non-negative functions are of prime importance for Gaussian fields. We therefore recall two characterizations (see [121] for Bochner's theorem and [127] for Schoenberg's theorem). First let us extend the Definition 2.1.9 of non-negative definite functions.

**Definition 2.1.10** *A real valued function $f : T \mapsto \mathbb{R}$ is called non-negative definite if the function $(t, s) \to f(t - s)$ is non-negative definite in the sense of Definition 2.1.9 i.e. for every $n \geq 1$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, $t_1, \dots, t_n \in T$*

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j f(t_i - t_j) \geq 0 \text{ and } f(-t) = f(t)$$

**Theorem 2.1.3** *Bochner's theorem.*

*Among the continuous real valued functions, the non-negative definite functions on $T = \mathbb{R}^n$ are those functions which are the Fourier transforms of finite symmetric measures.*

In Schoenberg's theorem functions of negative types are used.

**Definition 2.1.11** *Functions of negative type.*

*A real valued symmetric function $(t, s) \to \phi(t, s), s, t \in T$ is a function of negative type if, for every $n \geq 1$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^n \lambda_i = 0$, $t_1, \dots, t_n \in T$*

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \phi(t_i, t_j) \leq 0.$$

In Proposition 2.1.2 we remark that covariance functions are non-negative functions in the sense of Definition 2.1.9. If $R(t, s) = r(t - s)$ (which means that the centered stochastic fields $X_t$ with covariance $R$ is stationary in a weak sense c.f. Definition 2.1.22) then $r$ is a non-negative function in the sense of Definition 2.1.10. In this case $\mathbb{E}(X(t) - X(s))^2 = \varphi(t, s)$ is of negative type.

**Theorem 2.1.4** *Schoenberg's theorem.*

*Let $(t, s) \to \phi(t, s), s, t \in T$ be a real valued symmetric continuous function with $\phi(t, t) = 0$.*

1. *Fix $t_0 \in T$. Define $\psi$ by*

$$\psi(t,s) = \phi(t_0,t) + \phi(t_0,s) - \phi(t,s).$$
$$\psi(s,t) = \Phi(t_0,s) + \phi(t_0,t) - \phi(s,t) = \psi(t,s).$$

   *Then $\phi$ is a function of negative type, if and only if, $\psi$ is a real valued non-negative definite function.*
2. *$\phi$ is a function of negative type, if and only if, $e^{-\lambda\phi}$ is a non-negative definite function for all $\lambda \geq 0$.*

Let us consider some consequences of these theorems.

**Corollary 2.1.1**

- *Functions $(t,s) \to ||t||^{2H} + ||s||^{2H} - ||t-s||^{2H}, t, s \in \mathbb{R}^n$ are non-negative definite functions if and only if $0 < H \leq 1$.*
- *Functions $t \to e^{-|t|^\alpha}$, $t \in \mathbb{R}$ are characteristic functions if and only if $0 < \alpha \leq 2$.*

*Proof of the Corollary 2.1.1*

Function $(t,s) \to ||t-s||^2$ is of negative type. Indeed, for $\sum_{i=1}^{n} \lambda_i = 0$

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j ||t_i - t_j||^2 = -2||\sum_{i=1}^{n} \lambda_i t_i||^2 \leq 0.$$

Using the change of variables $u = \lambda x$, one easily proves that, for $x \geq 0, 0 < H < 1$

$$x^H = \tilde{C}_H \int_0^{+\infty} \frac{e^{-\lambda x} - 1}{\lambda^{1+H}} d\lambda, \tag{2.8}$$

where $\tilde{C}_H$ is a constant depending on $H$

$$\tilde{C}_H^{-1} = \int_0^{+\infty} \frac{e^{-u} - 1}{u^{1+H}} du.$$

Then

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j ||t_i - t_j||^{2H} = \tilde{C}_H \int_0^{+\infty} \frac{\sum_{i,j=1}^{n} \lambda_i \lambda_j e^{-\lambda||t_i - t_j||^2}}{\lambda^{1+H}} d\lambda.$$

By Schoenberg's theorem, $\sum_{i,j=1}^n \lambda_i\lambda_j e^{-\lambda||t_i-t_j||^2}$ is non-negative. Since $\tilde{C}_H < 0$, functions $(t,s) \to ||t-s||^{2H}$ are of negative type for $0 < H \leq 1$. Again by Schoenberg's theorem, functions $(t,s) \to ||t||^{2H} + ||s||^{2H} - ||t-s||^{2H}$, $t,s \in \mathbb{R}^n$ are non-negative definite functions for $0 < H \leq 1$. Let us now check that there are not non-negative definite for $H > 1$. Consider now three points $t_1, t_2, t_3$ on a straight line such that $\|t_2 - t_1\| = \|t_3 - t_2\| = 1$ and $\|t_3 - t_1\| = 2$. Take $\lambda_1 = \lambda_3 = -1$ and $\lambda_2 = 2$. Then $\sum_{i,j=1}^{3} \lambda_i\lambda_j||t_i - t_j||^{2H} = -8 + 2^{2H+1}$ is strictly positive when $H > 1$. Functions $(t,s) \to ||t-s||^{2H}$ are not of negative type for $H > 1$. By Schoenberg's theorem, functions $(t,s) \to ||t||^{2H} + ||s||^{2H} - ||t-s||^{2H}$, $t,s \in \mathbb{R}^n$ are not non-negative definite functions for $H > 1$.

The part 2 of the Corollary follows then again by Schoenberg's theorem: functions $t \to e^{-|t|^\alpha}$, $t \in \mathbb{R}$ are non-negative definite functions if and only if $0 < \alpha \leq 2$. By Bochner's theorem, they are then characteristic functions. Actually, for $\alpha = 2$, it is the characteristic function of Gaussian random variable and for $0 < \alpha < 2$ it is the characteristic function of $\alpha$-symmetric random variables (See for instance [126]).

### *2.1.4 Orthonormal Expansions of Gaussian Fields*

The aim of this section is to present briefly the orthonormal expansion of Gaussian fields. One can read [1, 5, 109] for proofs and detailed results.

One can associate to every Gaussian field two Hilbert spaces. The first one is called the Gaussian space and it is a subspace of square integrable random variables. The second one is an Hilbert space of deterministic functions called Reproducing Kernel Hilbert Space. Let us first define Gaussian spaces.

**Definition 2.1.12** *A vector space $H$ of centered Gaussian random variables, which is a closed subspace of $L^2(\Omega, \mathcal{A}, P)$ is called a Gaussian space.*

Actually Gaussian spaces are generalizations of random vectors. Let us consider $(X_1, \ldots, X_d)$ a random Gaussian vector and

$$H = \{\sum_{i=1}^{d} \lambda_i X_i,\ \text{for}(\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d\}.$$

It is clear that $H$ is a finite dimensional Gaussian space.

In this section, we assume that Gaussian fields are complex valued. When the Gaussian field is real valued the following definition clearly makes sense, the reader has just to forget the conjugation bar on the second factor.

**Definition 2.1.13** *Let $X_t$, $t \in T$ be a centered Gaussian field (i.e. $\mathbb{E}X_t = 0,\ \forall t \in T$). The subspace of $L^2(\Omega, \mathcal{A}, P)$ of the linear span of margins $X_t$, $\forall t \in T$ and of their* *limits in* $L^2$ *is denoted by*

$$\mathcal{H}_X = \overline{\{Z \text{ such that } \exists n \in \mathbb{N},\ \exists \lambda_i \in \mathbb{C} \text{ for } i = 1, \dots, n \text{ and } Z = \sum_{i=1}^{n} \lambda_i X_{t_i}\}}^{L^2}\ , \tag{2.9}$$

*where* $\overline{E}^{L^2}$ *is the closure in* $L^2$ *of the set* $E$*. The space*

$$K_X = \{h_Z : T \mapsto \mathbb{C} \text{ such that } \exists Z \in \mathcal{H}_X \text{ and } h_Z(t) = \mathbb{E}\left(Z\overline{X_t}\right)\} \tag{2.10}$$

*endowed with the Hermitian form*

$$\langle h_{Z_1}, h_{Z_2} \rangle_{H_X} \stackrel{def}{=} \mathbb{E}\left(Z_1\overline{Z_2}\right) \tag{2.11}$$

*is called the Reproducing Kernel Hilbert Space (in short RKHS) of the Gaussian field* $X$*.*

Let us remark that every random variable in the Gaussian space of a Gaussian field $X$ is Gaussian since limits of Gaussian random variables are Gaussian.

**Proposition 2.1.3** *If we use the notations of Definition 2.1.13 the map* $h : \mathcal{H}_X \mapsto H_X$ *defined by* $Z \mapsto h_Z$ *is a one to one linear map such that*

$$\|h_Z\|_{K_X} = \|Z\|_{\mathcal{H}_X} \quad \forall Z \in \mathcal{H}_X. \tag{2.12}$$

*Proof of the Proposition 2.1.3*

Obviously $h$ is a linear map satisfying (2.12). Since $\mathcal{H}_X$ is the linear span of $X_t$, it is a one to one map. The isometry property is a consequence of the definition of the scalar product on the RKHS.

If $X$ is a Gaussian field, please remark that

$$h_{X_t}(s) = \mathbb{E}(X_t\overline{X_s}) = R(t, s). \tag{2.13}$$

Hence finite linear combinations of the functions $(R(t, .))_{t \in T}$ are dense in $H_X$. The name of "reproducing" Kernel Hilbert space comes from the following property:

$$\forall h_Z \in K_X \quad \langle h_Z, R(t, .)\rangle_{K_X} = \mathbb{E}(Z\overline{X_t}) = h_Z(t). \tag{2.14}$$

In particular

$$\langle R(t, .), R(s, .)\rangle_{H_X} = R(t, s) \quad \forall s,\ t \in T. \tag{2.15}$$

Let us now suppose that there exists a countable orthonormal basis $(e_n)_n \in \mathbb{N}$ of $K_X$. One will consider two orthonormal series, one in the Gaussian space, the other one in the RKHS.

**Theorem 2.1.5** *Let $X$ be a centered Gaussian field and $(e_n)_{n\in\mathbb{N}}$ an orthonormal basis of $K_X$. Let us denote by $\eta_n = h^{-1}(e_n)$ a random variable in $\mathcal{H}_X$. Then the variables $(\eta_n)_{n\in\mathbb{N}}$ are independent identically standard Gaussian random variables and constitute an orthonormal basis of $\mathcal{H}_X$. Moreover*

$$\forall t \in T, \quad X_t = \sum_{n=0}^{+\infty} \eta_n \overline{e_n(t)} \tag{2.16}$$

*where the convergence is in $L^2(\Omega)$ and*

$$\forall t \in T \quad R(t,.) = \sum_{n=0}^{+\infty} e_n(.)\overline{e_n(t)} \tag{2.17}$$

*where the convergence is in $K_X$.*

Please remark that the convergence in (2.16) can be strengthened under mild assumptions to get an almost sure convergence in a functional sense, similarly in (2.17) the convergence is very often uniform. We refer to [5] for abstract results on those questions, but many examples will be found in the next chapter.

*Proof of Theorem 2.1.5*

Since $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $K_X$, $(\eta_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_X$. Since every linear combination of the $\eta_n$'s are Gaussian random variables, orthogonality yields independence and $\|\eta_n\|_{L^2} = 1$ means that $(\eta_n)$ is a standard Gaussian random variable. Moreover $X_t \in \mathcal{H}_X$ and

$$\begin{aligned} X_t &= \sum_{n=0}^{+\infty} \mathbb{E}(X_t\overline{\eta_n})\eta_n \\ &= \sum_{n=0}^{+\infty} \langle R(t,.), e_n\rangle_{K_X}\eta_n \\ &= \sum_{n=0}^{+\infty} \eta_n\overline{e_n}(t), \end{aligned}$$

where the convergence is in $\mathcal{H}_X$. With the same arguments

$$R(t,.) = \sum_{n=0}^{+\infty} \langle R(t,.), e_n\rangle_{H_X} e_n(.)$$

where the convergence is in $K_X$.

Let us give some elementary examples of Reproducing Kernel Hilbert Space.

First let us consider $T = \{1,\dots,n\}$, then a real centered Gaussian field is nothing but a real valued centered Gaussian random vector $(X_1,\dots,X_n)$. Let us consider the covariance matrix $\mathcal{R} = (R(i, j))_{i, j \in T} = (\mathbb{E}(X_i \overline{X_j}))_{i, j \in T}$. Matrix $\mathcal{R}$ is a non-negative symmetric matrix. In this case both the Gaussian space and the RKHS are finite dimensional vector spaces. Every function $f$ in $K_X$ maps $\{1, \ldots, n\}$ onto $\mathbb{R}$ and can be viewed as a vector $(f(1), \ldots, f(n)) \in \mathbb{R}^n$. Since the linear span of $(R(i, .))_{i \in T}$ is $K_X$, there exist $\lambda_1, \ldots, \lambda_n$ in $\mathbb{R}$ such that

$$f(.) = \sum_{i=1}^{n} \lambda_i R(i, .).$$

Moreover, if $g \in K_X$ is given by

$$g(.) = \sum_{j=1}^{n} \mu_j R(j, .),$$

then

$$\langle f, g \rangle_{K_X} = \sum_{i,j=1}^{n} \lambda_i \mu_j R(i, j).$$

In this case the dimension of $K_X$ and $\mathcal{H}_X$ is the rank of the matrix $\mathcal{R}$.

Another example of Gaussian field is given by standard Brownian motion introduced in Example 2.1.1. For sake of simplicity let us take $T = [0, 1]$. One can compute the covariance of this centered Gaussian field. If $s \le t$

$$\mathbb{E}(B_s B_t) = \mathbb{E}(B_s(B_s + B_t - B_s)) = s,$$

hence,

$$R(s, t) = \min(s, t).$$

Because of (2.15) we already know that

$$\langle R(t, .), R(s, .) \rangle_{K_X} = \min(s, t).$$

Remark that the derivative of $R(s, .)$ exists for every $t \neq s$, and $\forall s \in [0, 1]$, $\frac{\partial R}{\partial u}(s, u) = \mathbf{1}_{[0,s]}(u)$. Then,

$$\langle R(t, .), R(s, .) \rangle_{K_X} = \int_0^1 \frac{\partial R}{\partial u}(s, u) \frac{\partial R}{\partial u}(t, u) du.$$

If $f(u) = \sum_{i=1}^{n} \lambda_i R(s_i, u)$ and $g(u) = \sum_{i=1}^{n'} \mu_i R(s_i, u)$ then

$$\langle f, g \rangle = \int_0^1 f'(u)g'(u)du.$$

Hence the RKHS $K_B$ of the Brownian motion $B$ is included in $H_1 = \{h : [0, 1] \mapsto \mathbb{R}$, almost everywhere derivable, such that

$$\int_0^1 (h'(u))^2 du < \infty,\ \ h(0) = 0\}.$$

Space $H_1$ endowed with the scalar product

$$\forall f, g \in H_1, \quad \langle f, g \rangle = \int_0^1 f'(u)g'(u)du$$

is classically called the Cameron Martin space of the Brownian motion and is a closed subspace of $L^2[0, 1]$. Moreover if $h \in H_1$, $h'$ is the limit in $L^2[0, 1]$ of functions of the form $\sum_{i=1}^n \lambda_i \mathbf{1}_{[0,s_i]}$ and $h(t) = \int_0^t h'(s)ds$ is the limit in $H_1$ of $\sum_{i=1}^n \lambda_i R(s_i, t)$, hence

$$K_B = H_1.$$

### 2.1.5 Orthogonality Between Gaussian Processes

In the Chap. 5 concerning statistics, we will give estimators of functionals of Gaussian processes that are using only one sample path of the process. Usually in statistics independent samples of processes are needed to identify models, but in the Gaussian setting sometimes only one sample path is enough to distinguish two Gaussian processes.

This powerful tool is related to the equivalence and orthogonality of Gaussian measures (e.g. [70, 109]). Our aim is not to present a review of general results on the equivalence and orthogonality of Gaussian measures, but only to give partial results, that will be useful later on.

Let us first recall that two probabilities $\mathbb{P}$, $\mathbb{Q}$ defined on $(\Omega, \sigma(\Omega))$ are equivalent if $\forall A \in \sigma(\Omega)$, $\mathbb{P}(A) = 0 \Leftrightarrow \mathbb{Q}(A) = 0$. The two probabilities $\mathbb{P}$, $\mathbb{Q}$ are orthogonal if $\exists A \in \sigma(\Omega)$, $\mathbb{P}(A) = 0$ and $\mathbb{Q}(A) = 1$. In the later case one can roughly speaking distinguish $\mathbb{P}$ and $\mathbb{Q}$ with only one sample $\omega$. The heuristic rule is if $\omega \in A$ then one has taken the sample $\omega$ under the probability $\mathbb{Q}$ almost surely. On the contrary if $\mathbb{P}$ and $\mathbb{Q}$ are equivalent there is no way to distinguish $\mathbb{P}$ and $\mathbb{Q}$ with only one sample.

In this book we will avoid the classical name of Gaussian measure, which is too close from the random Gaussian measures studied in the Sect. 2.1.6. In our framework a Gaussian measure is either the distribution of a random Gaussian vector, which is a probability on some finite dimensional vector space or the distribution of a Gaussian random process (or field), which is a probability on the space of functions to which belong almost surely the sample paths of the process. In this section we will use the name Gaussian process even if we are actually working with random Gaussian vectors.

We first have the fundamental theorem (e.g. [109]).

**Theorem 2.1.6** *The distribution of two Gaussian processes are either equivalent or orthogonal.*

Let $X$ and $Y$ be two Gaussian processes. A measurement of the difference between two probabilities $\mathbb{P}_X$ and $\mathbb{P}_Y$ is given by the Hellinger's distance (e.g. [44, ch. 6.4]). Let us first consider a measure $\mu$ such that both $\mathbb{P}_X$ and $\mathbb{P}_Y$ have densities $\frac{d\mathbb{P}_X}{d\mu}$ respectively $\frac{d\mathbb{P}_Y}{d\mu}$ with respect to $\mu$. Please note that one can always take $\mu = \mathbb{P}_X + \mathbb{P}_Y$. One can define

$$\rho(\mathbb{P}_X, \mathbb{P}_Y) = \int \sqrt{\frac{d\mathbb{P}_X}{d\mu}\frac{d\mathbb{P}_Y}{d\mu}} d\mu$$

This integral does not depend on $\mu$ and is usually written

$$\rho(\mathbb{P}_X, \mathbb{P}_Y) = \int \sqrt{\mathbb{P}_X \mathbb{P}_Y}.$$

The Hellinger's distance is then defined by

$$H(\mathbb{P}_X, \mathbb{P}_Y) = 1 - \rho(\mathbb{P}_X, \mathbb{P}_Y).$$

It is easy to check that two Gaussian measures are orthogonal iff $\rho(\mathbb{P}, \mathbb{Q}) = 0$.

The computation of $\rho$ is also easy for real valued random variables. For instance it is easy to check that if $X \overset{(d)}{=} \mathcal{N}(0, \sigma_X^2)$, $Y \overset{(d)}{=} \mathcal{N}(0, \sigma_Y^2)$, with $\sigma_X^2 > 0$, $\sigma_Y^2 > 0$ then

$$\begin{aligned}
\rho(\mathbb{P}_X, \mathbb{P}_Y) &= \int_{\mathbb{R}} \sqrt{\frac{1}{2\pi\sigma_X^2\sigma_Y^2} \exp\left(\frac{-x^2}{2}\left(\frac{1}{\sigma_X^2} + \frac{1}{\sigma_Y^2}\right)\right)} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma_X\sigma_Y} \int_{\mathbb{R}} \exp\left(\frac{-x^2}{4}\left(\frac{1}{\sigma_X^2} + \frac{1}{\sigma_Y^2}\right)\right) dx \\
&= \sqrt{\frac{2\sigma_X\sigma_Y}{\sigma_X^2 + \sigma_Y^2}} \qquad (2.18)
\end{aligned}$$

If both $X_1$ and $X_2$ and $Y_1$ and $Y_2$ are independent, then:

$$\rho(\mathbb{P}_{(X_1,Y_1)}, \mathbb{P}_{(X_2,Y_2)}) = \rho(\mathbb{P}_{X_1}, \mathbb{P}_{X_2}) \times \rho(\mathbb{P}_{Y_1}, \mathbb{P}_{Y_2}).$$

Another useful result is the Kakutani theorem that yields a criterion of equivalence for sequences of independent Gaussian random variables. See for instance Theorem 2.12.7 in [29]. A special instance of this theorem is when you have $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ two sequences of positive numbers. You can consider the Gaussian processes $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ where the $X_n$'s and the $Y_n$'s are sequences of independent Gaussian random variables with distributions $\mathcal{N}(0, \alpha_n^2)$ and $\mathcal{N}(0, \beta_n^2)$. Then $\mathbb{P}_X$ and $\mathbb{P}_Y$ are equivalent if and only if

$$\prod_{n\geq 0} \sqrt{\frac{2\alpha_n\beta_n}{\alpha_n^2+\beta_n^2}} \neq 0$$

or

$$\sum_{n\geq 0} \left(\frac{\alpha_n}{\beta_n} - 1\right)^2 < +\infty \tag{2.19}$$

Let $(X_t)_{t\in[0,1]}$ be a real valued centered Gaussian process parameterized by the compact interval $[0, 1]$ with continuous sample paths almost surely. Then the covariance function $R$ of $X$ is continuous on $[0, 1]^2$. There exists a classical technique to find an orthonormal basis of the RKHS of $X$. Let us first define the integral operator (still denoted by $R$)

$$R\varphi(t) = \int_0^1 R(t, s)\varphi(s)ds. \tag{2.20}$$

Since the function $R$ is bounded on $[0, 1]^2$, the integral on the right hand side of (2.20) is convergent for every function $\varphi \in L^2[0, 1]$. Moreover $R$ is a bounded linear operator from $L^2[0, 1]$ into $L^2[0, 1]$. Then the bilinear form $B$ from $L^2[0, 1] \times L^2[0, 1]$, defined by $B(\varphi, \psi) = \langle R\varphi, \psi\rangle_{L^2}$ is non-negative and symmetric. Hence there exists an orthonormal basis $(\varphi_n)_{n\geq 0}$ of $L^2[0, 1]$ such that $\varphi_n$ is the eigenfunction of the operator associated to the eigenvalue $\lambda_n^2 \geq 0$. We assume that $B(\varphi, \varphi) > 0$ for every non-vanishing function $\varphi$, or equivalently that all $\lambda_n > 0$. Moreover the functions $\varphi_n$ are continuous on $[0, 1]$ since $\varphi_n = \frac{R\varphi_n}{\lambda_n^2}$. Since for every $t \in [0, 1]$, $R(t, .) \in L^2[0, 1]$ and $(\varphi_n)_{n\geq 0}$ is an orthonormal basis of $L^2[0, 1]$ it is easy to check that

$$R(t, s) = \sum_{n\geq 0} \lambda_n^2 \varphi_n(t)\varphi_n(s), \tag{2.21}$$

where the equality is in a $L^2$ sense. Actually one can prove the Mercer theorem in this setting that claims that the convergence of the right hand series of (2.21) is indeed uniform on [0, 1] and absolute. See [117] for the proof. The alert reader may observe that the series expansion (2.21) is similar to (2.17), which is the series expansion obtained in Theorem 3.1.5 for general Gaussian fields. Actually one can extend the analogy quite completely. Let $(\eta_n)_{n\in\mathbb{N}}$ be a sequence of independent $\mathcal{N}(0,1)$ random variables, then the series

$$\tilde{X}(t) = \sum_{n\geq 0} \lambda_n \varphi_n(t)\eta_n \tag{2.22}$$

are convergent in $L^2(\Omega)$ and $\tilde{X}$ is a centered Gaussian process with covariance $R$. It is the counterpart of (2.16). Moreover one can show that $(\lambda_n\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal basis of the RKHS $K_X$ of $X$. Actually we can show the following result

**Lemma 2.1.1** *The vector space*

$$K_X = \{h \text{ such that, } \exists (a_n) \text{ with } h(t) = \sum_{n\geq 0} a_n\lambda_n\varphi_n(t), \ \sum_{n\geq 0} a_n^2 < \infty.\} \tag{2.23}$$

*endowed with the inner product* $(h,g)_{K_X} = \sum_{n\geq 0} a_n b_n$, *where* $g \in K_X$ *and* $g = \sum_{n\geq 0} b_n\lambda_n\varphi_n$ *is the RKHS of* $X$.

*Proof of the Lemma 2.1.1*

To check this fact, let us remark that $K_X$ is an Hilbert space and $(\lambda_n\varphi_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $K_X$. Because of Mercer theorem

$$\begin{aligned}(R(t,.),h)_{K_X} &= \left(\sum_{n\geq 0}\lambda_n\varphi_n(t)\varphi_n, \sum_{n\geq 0} a_n\lambda_n\varphi_n\right)_{K_X}\\ &= \sum_{n\geq 0}\lambda_n a_n\varphi_n(t)\\ &= h(t).\end{aligned}$$

Hence $K_X$ is the RKHS of $X$.

Please note that the partial sums

$$\tilde{X}_N = \sum_{n\leq N}\lambda_n\varphi_n\eta_n,$$

are conditional expectations of $\tilde{X}$

$$\tilde{X}_N = \mathbb{E}(\tilde{X}|\mathcal{F}_N),$$

where $\mathcal{F}_N = \sigma(\eta_n \text{ such that } n \leq N)$. Since $\tilde{X}$ has almost surely continuous sample paths, $\tilde{X}_N$ converges almost surely uniformly to $\tilde{X}$ by applying a theorem of convergence of martingale in the Banach space of continuous functions on [0, 1] (see [110], p. 104 Proposition V-2-6).

Let $X$ and $Y$ be two continuous Gaussian processes on [0, 1] with continuous mean value $m_X$ and $m_Y$ and the same continuous covariance function $R$. Let $\lambda_n^2, n \geq 0$ be the eigenvalues and $\varphi_n(t), n \geq 0$ be the $L^2$-normalized eigenfunctions of the covariance $R$. If functions $m_X$ and $m_Y$ can be expanded as uniformly convergent series on the basis $\varphi_n(t), n \geq 0$

$$\begin{aligned} m_X(t) &= \sum_{n\geq 0} m_{X,n}\varphi_n(t), \\ m_Y(t) &= \sum_{n\geq 0} m_{Y,n}\varphi_n(t), \end{aligned} \tag{2.24}$$

then we denote by $\mathbb{P}_X$ the distribution of $\sum_{n\geq 0}(\lambda_n\eta_n + m_{X,n})\varphi_n(t)$, respectively by $\mathbb{P}_Y$, the distribution of $\sum_{n\geq 0}(\lambda_n\eta_n + m_{Y,n})\varphi_n(t)$.

Let us now apply Kakutani theorem to the sequences $\tilde{X} = (\lambda_n\eta_n + m_{X,n})_{n\in\mathbb{N}}$ and $\tilde{Y} = (\lambda_n\eta_n + m_{Y,n})_{n\in\mathbb{N}}$.

The Hellinger's distance between $\mathbb{P}_{\tilde{X}}$ and $\mathbb{P}_{\tilde{Y}}$ can be computed

$$\begin{aligned} \rho(\mathbb{P}_{\tilde{X}}, \mathbb{P}_{\tilde{Y}}) &= \prod_{n\geq 0} \frac{1}{\lambda_n\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left\{-\frac{1}{4\lambda_n^2}\left[(x - m_{X,n})^2 + (x - m_{Y,n})^2\right]\right\} dx \\ &= \exp\left\{-\frac{(m_{X,n} - m_{Y,n})^2}{8\lambda_n^2}\right\}. \end{aligned}$$

It follows that $\mathbb{P}_{\tilde{X}}$ and $\mathbb{P}_{\tilde{Y}}$ are equivalent iff

$$\sum_{n\geq 0} \frac{(m_{X,n} - m_{Y,n})^2}{\lambda_n^2} < \infty. \tag{2.25}$$

Hence if the series in (2.25) is convergent $\mathbb{P}_{\tilde{X}}$ and $\mathbb{P}_{\tilde{Y}}$ are equivalent. It follows that $\mathbb{P}_X$ and $\mathbb{P}_Y$ are equivalent.

This implies in particular that the mean value function of a Gaussian process satisfying (2.24) can not be identified with the observation of a single sample path on [0, 1]. Indeed, let us give an example. Let $X$ be a Brownian motion on [0, 1] and let $Y(t) = X(t) + m(t)$. Eigenfunctions $\varphi_n(t)$ are equal to $\sqrt{2}\sin\left(\frac{(2n+1)\pi t}{2}\right)$ and eigenvalues are equal to $\lambda_n^2 = \left(\frac{2}{(2n+1)\pi}\right)^2$. Take a mean value $m(t) = \sum_{n\geq 0} m_n\sqrt{2}\sin\left(\frac{(2n+1)\pi t}{2}\right)$, with $\sum_{n\geq 0} m_n^2 n^2 < \infty$ . It follows from (2.25) that the

Gaussian measure associated with processes $X$ and $Y$ are equivalent. Therefore, the mean value cannot be estimated from the observation of a single path of $Y$.

Now let $X$ and $Y$ be two continuous centered Gaussian processes on [0, 1] with covariance functions $R_X$ and $R_Y$. Moreover, we assume that the covariance operators $R_X$ and $R_Y$ can be diagonalized on the same $L^2$-basis $\varphi_n(t), n \geq 0$

$$R_X(t, t') = \sum_{n\geq 0} \lambda_n^2 \varphi_n(t)\varphi_n(t'),$$
$$R_Y(t, t') = \sum_{n\geq 0} \mu_n^2 \varphi_n(t)\varphi_n(t').$$

Applying Kakutani theorem and the computation in (2.19) yields that the two measures $\mathbb{P}_X$ and $\mathbb{P}_Y$ are therefore equivalent if

$$\sum_{n\geq 0} \left(\frac{\lambda_n}{\mu_n} - 1\right)^2 < +\infty. \tag{2.26}$$

This implies in particular that the covariance functions $R_X,\ R_Y$ cannot be identified under the assumptions above from the observation of a single sample path on [0, 1].

### *2.1.6 Gaussian Random Measure*

Actually many Gaussian fields in this book are obtained as integrals of random measures. In this section some elementary properties of random Gaussian measures are given. Let us first give some abstract definitions.

**Definition 2.1.14** *A Gaussian random measure on a measure space* $(M, \mathcal{M}, \mu)$ *is an isometry* $\mathcal{I}$ *from the Hilbert space* $L^2(M, \mathcal{M}, \mu)$ *onto a Gaussian space included in some* $L^2(\Omega, \mathcal{A}, \mathbb{P})$.

In Definition 2.1.14 one can wonder why the isometry is called a "measure". For random Gaussian measures the $\sigma$-additivity property is replaced by the following property. If $(A_n)_{n\in\mathbb{N}}$ is a sequence of mutually disjoints measurable sets such that $\mu(\cup_{n\in\mathbb{N}} A_n) < \infty$, then the Gaussian random variables $\mathcal{I}(A_n)$ are independent because they are orthogonal by isometry. One says that random measures are independently scattered when they satisfy this property. We will also consider non-Gaussian random measure that are independently scattered. See Definition 2.1.17.

Moreover since $\sum \mathbf{1}_{A_n} = \mathbf{1}_{\cup_{n\in\mathbb{N}} A_n}$ is converging in $L^2(M, \mathcal{M}, \mu)$ we get the convergence in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ of

$$\sum_{n\in\mathbb{N}} \mathcal{I}(A_n) = \mathcal{I}(\cup_{n\in\mathbb{N}} A_n).$$

Actually one can prove that the convergence holds almost surely. Beware of the fact that in general one cannot find a negligible set $N$ such that in the complementary of $N$, $\sum_{n\in\mathbb{N}} \mathcal{I}(A_n) = \mathcal{I}(\cup_{n\in\mathbb{N}} A_n)$ holds for all sequences of mutually disjoints measurable sets such that $\mu(\cup_{n\in\mathbb{N}} A_n) < \infty$. In other words a random Gaussian measure is not in general a measure valued random variable.

Let us give two standard constructions of Gaussian random measures, that are used throughout the book.

#### 2.1.6.1 Real Valued Brownian Random Measure

**Definition 2.1.15** *Real valued Brownian random measure.*

*When $M = \mathbb{R}^d$, $\mathcal{M} = \mathcal{B}(\mathbb{R}^d)$ and $\mu(ds) = ds/(2\pi)^{d/2}$, $\mathcal{I}$ is called a Brownian random measure and will be denoted $W(ds)$.*

It is a real valued Brownian random measure, because we impose here that $L^2(ds)$ is a space of real valued functions and, in this case, the random variables in $L^2(\Omega, \mathcal{A}, \mathbb{P})$ are also real valued.

Let us now give some straightforward properties of Brownian random measures. For any $f \in L^2(\mathbb{R}^d)$, the random variable $\int_{\mathbb{R}^d} f(s)W(ds)$ is a centered Gaussian variable with variance

$$\mathbb{E}\left(\int_{\mathbb{R}^d} f(s)W(ds)\right)^2 = \int_{\mathbb{R}^d} f(s)^2 \frac{ds}{(2\pi)^{d/2}}.$$

Moreover, for any $f, g \in L^2(\mathbb{R}^d)$,

$$\mathbb{E}\left(\int_{\mathbb{R}^d} f(s)W(ds) \int_{\mathbb{R}^d} g(s)W(ds)\right) = \int_{\mathbb{R}^d} f(s)g(s) \frac{ds}{(2\pi)^{d/2}}. \tag{2.27}$$

In particular, if $f$ and $g$ are orthogonal for the inner product of $L^2(\mathbb{R}^d)$, then the random variables $\int_{\mathbb{R}^d} f(s)W(ds)$ and $\int_{\mathbb{R}^d} g(s)W(ds)$ are independent.

Let $\sum_n f_n$ be convergent series in $L^2(\mathbb{R}^d)$, then

$$\int_{\mathbb{R}^d} \sum_{n\geq 1} f_n(s)W(ds) \text{ Ł}2lim \sum_{n\geq 1} \int_{\mathbb{R}^d} f_n(s)W(ds).$$

If $(e_n)_{n\geq 1}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ and $(\varepsilon_n)_{n\geq 1}$ be a sequence of i.i.d. $\mathcal{N}(0, 1)$, one can construct formally a Brownian random measure

$$W(ds) = \sum_{n\geq 1} \varepsilon_n e_n(s) \frac{ds}{(2\pi)^{d/2}}$$

that can be defined rigorously by

$$\int_{\mathbb{R}^d} f(s) W(ds) = \sum_{n\geq 1} < f, e_n >_{L^2(\mathbb{R}^d)} \varepsilon_n, \ \forall f \in L^2(\mathbb{R}^d), \tag{2.28}$$

where

$$< f, e_n >_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(s) e_n(s) \frac{ds}{(2\pi)^{d/2}}.$$

Series expansion (2.28) can be compared to (2.16) in Theorem 2.1.5. In some sense random Gaussian random measure is reverting the construction of Theorem 2.1.5: One starts from two orthonormal bases to construct Gaussian variables instead of extracting two orthonormal bases of Hilbert spaces from a Gaussian field.

#### 2.1.6.2 Fourier Transform of a Real Valued Brownian Random Measure

Since the Fourier transform is an isometry of $L^2(\mathbb{R}^d)$ onto itself, we might expect that the Fourier transform of a Brownian random measure is still a Brownian random measure. Although this construction is very useful, we need to be more precise and to be careful.

**Definition 2.1.16** *Fourier transform of a real valued Brownian random measure.*

*If $W(ds)$ is a real valued Brownian random measure, one can define the Fourier transform of $W(ds)$, denoted by $\widehat{W}(d\xi)$, as follows. Let $\mathcal{F}$ be the space of complex valued functions $f \in L^2(\mathbb{R}^d, \mathbb{C})$ such that $f(-\xi) = \overline{f(\xi)}$. For every $f \in \mathcal{F}$, one sets*

$$\int_{\mathbb{R}^d} f(\xi) \widehat{W}(d\xi) \stackrel{def}{=} \int_{\mathbb{R}^d} \widehat{f}(s) W(ds). \tag{2.29}$$

If the real valued Brownian random measure

$$W(ds) = \sum_{n\geq 1} \varepsilon_n e_n(s) \frac{ds}{(2\pi)^{d/2}}$$

is as in (2.28), then

$$\begin{aligned}\int_{\mathbb{R}^d} f(\xi)\widehat{W}(d\xi) &= \int_{\mathbb{R}^d} \widehat{f}(s)W(ds) \\ &= \sum_{n\geq 1} < \widehat{f}, e_n >_{L^2(\mathbb{R}^d)} \varepsilon_n \\ &= \sum_{n\geq 1} < f, \widehat{e_n} >_{L^2(\mathbb{R}^d)} \varepsilon_n \\ &= \sum_{n\geq 1} \int_{\mathbb{R}}^{d} f(\xi)\overline{\widehat{e_n}(\xi)}\frac{d\xi}{(2\pi)^{d/2}}\varepsilon_n\end{aligned} \tag{2.30}$$

Hence

$$\widehat{W}(d\xi) = \sum_{n\geq 1} \varepsilon_n \overline{\widehat{e_n}(\xi)}\frac{d\xi}{(2\pi)^{d/2}}. \tag{2.31}$$

On the one hand please remark that by definition $\int_{\mathbb{R}^d} f(\xi)\widehat{W}(d\xi)$ is always a real valued Gaussian random variable. Actually a necessary and sufficient condition for a function $f$ in $L^2(\mathbb{R}^d, \mathbb{C})$ to have a real valued Fourier transform $\widehat{f}$ is $f \in \mathcal{F}$. Hence it explains why we restrict the definition of $\widehat{W}$ to $\mathcal{F}$. On the other hand $\widehat{W}$ is not defined on $L^2(\mathbb{R}, \mathbb{C})$ and it is not a Gaussian random measure in the sense of Definition 2.1.14. Nevertheless Fourier transforms of real valued Brownian random measures are very useful mainly because the isometry property with $L^2(\mathbb{R}, \mathbb{C})$ still holds. It is a straightforward consequence of Parseval equality. For every $f, g \in \mathcal{F}$

$$\mathbb{E}\left(\int_{\mathbb{R}^d} f(\xi)\widehat{W}(d\xi)\int_{\mathbb{R}^d} g(\xi)\widehat{W}(d\xi)\right) = \int_{\mathbb{R}^d} f(\xi)\overline{g(\xi)}\frac{d\xi}{(2\pi)^{d/2}} \tag{2.32}$$

$$= \int_{\mathbb{R}^d} \widehat{f}(s)\widehat{g}(s)\frac{ds}{(2\pi)^{d/2}}. \tag{2.33}$$

### *2.1.7 Poisson Random Measure*

In this book we are using Poisson random measures to build stable random measures. A Poisson random measure $N$ on a measurable state space $(S, \mathcal{S})$, endowed with deterministic measure $n$, is an independently scattered $\sigma$-additive set function defined on $\mathcal{S}_0 = \{A \in \mathcal{S} \text{ s. t. } n(A) < +\infty\}$. Thus, for $A \in \mathcal{S}_0$, the random variable $N(A)$ has a Poisson distribution with mean $n(A)$

$$\mathbf{P}(N(A) = k) = e^{-n(A)}\frac{(n(A))^k}{k!}$$

for $k = 0,\ 1,\ 2, \ldots$. The measure $n$ is called the mean measure of $N$. Moreover the Poisson random measure $N$ is independently scattered in the following sense.

**Definition 2.1.17** *A random measure $M$ is independently scattered, if for a finite number of sets $\{A_i\}_{j\in J}$ that are pairwise disjoint, the random variables $(M(A_j))_{j\in J}$ are independent.*

Furthermore the Poisson random variables have expectation and variance given by their mean i.e.

$$\mathbb{E}(N(A)) = \mathrm{var}N(A) = n(A),$$

for every $A \in \mathcal{S}_0$. Their characteristic function is easily computed

$$\mathbb{E}(e^{ivN(A)}) = \exp\left(n(A)(e^{iv} - 1)\right),$$

for every $v \in \mathbb{R}$.

Our goal is to build $\alpha$-stable (symmetric or complex isotropic) random measures to define $\alpha$-stable self-similar fields. The first step in that direction is to define a stochastic integral with respect to Poisson measures. Actually the Poisson measures used to build stable random measure are compensated. Let us define compensated Poisson measures on $\mathbb{R}^d \times \mathbb{R}$ with mean measure $n(ds, du)$

$$\widetilde{N} = N - n.$$

Then for every function $\varphi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ such that $\varphi \in L^2(\mathbb{R}^d \times \mathbb{R}, n)$ the stochastic integral

$$\int_{\mathbb{R}^d\times\mathbb{R}} \varphi(s, u)\widetilde{N}(ds, du)$$

is defined as the limit in $L^2(\Omega)$ of

$$\int_{\mathbb{R}^d\times\mathbb{R}} \varphi_k(s, u)\widetilde{N}(ds, du)$$

where $\varphi_k$ is a simple function of the form $\sum_{j\in J} a_j \mathbf{1}_{A_j}$, where $J$ is finite. Then

$$\int_{\mathbb{R}^d\times\mathbb{R}} \sum_{j\in J} a_j \mathbf{1}_{A_j} \widetilde{N}(ds, du) \overset{def}{=} \sum_{j\in J} a_j \widetilde{N}(A_j)$$

where the Poisson random variables $N(A_j)$ have intensity $n(A_j)$ and are independent since the sets $A_j$ are supposed pairwise disjoint. Consequently the characteristic function of $\sum_{j\in J} a_j \widetilde{N}(A_j)$ is

$$\exp\left[\sum_{j\in J} n(A_j)(e^{ia_j v} - 1 - ia_j v)\right],$$

and the convergence in $L^2(\Omega)$ implies that the characteristic function of the stochastic integral is $\forall v \in \mathbb{R}$

$$\mathbb{E}\exp\left(iv\int \varphi d\widetilde{N}\right) = \exp\left[\int_{\mathbb{R}^d\times\mathbb{R}}[\exp(iv\varphi) - 1 - iv\varphi]n(ds,du)\right]. \tag{2.34}$$

Moreover $\mathrm{var}N(A) = n(A)$ yields

$$\mathbb{E}\left(\int_{\mathbb{R}^d\times\mathbb{R}}\varphi(s,u)\widetilde{N}(ds,du)\right)^2 = \int_{\mathbb{R}^d\times\mathbb{R}}\varphi^2(s,u)n(ds,du), \tag{2.35}$$

first for simple functions, then for every $\varphi(s,u) \in L^2(n(ds,du))$.

Please note that as Poisson random measure, compensated Poisson random measure are independently scattered.

### 2.1.8 Lévy Random Measure

With the help of the compensated Poisson random measure, we will define Lévy random measure for which the control measure has a finite moment of order 2. It is not the case of stable Lévy measure, but the random stable measures are defined as limits of such Lévy measures, which are also useful to construct locally self-similar fields considered in the Chap. 3. Let us take a control measure $\nu$ such that

$$\int_{\mathbb{R}}|u|^p\nu(du) < \infty \quad \forall p \geq 2. \tag{2.36}$$

Then a Poisson random measure $\widetilde{N}$ is associated with the mean measure $n(ds,du) = ds\nu(du)$ and for every function $f \in L^2(\mathbb{R}^d, ds)$ one can define

$$\int_{\mathbb{R}^d} f(s)M(ds) \stackrel{def}{=} \int_{\mathbb{R}^d\times\mathbb{R}} f(s)u\widetilde{N}(ds,du), \tag{2.37}$$

since $f(s)u \in L^2(\mathbb{R}^d, ds\nu(du))$. One can easily compute the characteristic function of such a random variable and check that this variable is infinitely divisible (see for instance the Lévy Kintchine formula in [116]). Actually the name "Lévy" comes from the fact that in dimension $d = 1$, this random Lévy measure is related to the increments of processes with stationary and independent increments, which are called Lévy processes. To illustrate this point of view, let us remark that the process $X(t) = \int_{\mathbb{R}}\mathbf{1}_{[0,t]}(s)M(ds),\ t \geq 0$ is a Lévy process. (See Exercise 3.6.6.)

Let us now state an isometry property for Lévy random measure.

$$\mathbb{E}\left(\int_{\mathbb{R}^d} f(s)M(ds)\right)^2 = \int_{\mathbb{R}} u^2\nu(du)\|f\|^2_{L^2(\mathbb{R}^d)}, \tag{2.38}$$

which is a consequence of (2.35). When $A$ is a Borel set of $\mathbb{R}^d$ with finite Lebesgue measure, we will denote by $M(A)$ the random variable $M(\mathbf{1}_A)$.

Please note that Lévy random measure are independently scattered in the sense of Definition 2.1.17.

### 2.1.9 Stable Random Measure

In this book we will also consider the special instance of stable random variables (or stable measures). Let us recall characteristic functions of symmetric $\alpha$-stable random variables $X$

$$\mathbb{E}(e^{ivX}) = \exp\left(-\sigma^\alpha|v|^\alpha\right)$$

where $0 < \alpha < 2$ and where $\sigma$ is called the scale factor. Please note that the case $\alpha = 2$ formally corresponds to the Gaussian case. Because of harmonizable representations used in the sequel, we will also need complex valued stable variables. Let us introduce complex isotropic stable variables $X_1 + iX_2$, their characteristic functions are given by:

$$\mathbb{E}(e^{i(v_1X_1+v_2X_2)}) = \exp\left(-\sigma_\alpha(v_1^2 + v_2^2)^{\alpha/2}\right).$$

At this point, the $\alpha$-stable symmetric random measure $M_\alpha$ will be defined as a limit of a Lévy random measure $M_{\alpha,R}(ds)$. Let us consider $f \in L^\alpha(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, then, for any positive number $R$, the integral on the left hand side

$$\int_{\mathbb{R}^d} f(s)M_{\alpha,R}(ds) \overset{def}{=} \int_{\mathbb{R}^d\times\mathbb{R}} f(s)u\widetilde{N_R}(ds, du) \tag{2.39}$$

is defined by the integral on the right hand side, where the mean measure of $N_R$ is $n_R(ds, du) = \frac{\mathbf{1}_{|u|\le R}}{|u|^{1+\alpha}}dsdu$. Roughly speaking, $M_{\alpha,R}$ is a stable measure where the "big jumps" have been truncated. Let us now check that $\int_{\mathbb{R}^d} f(s)M_{\alpha,R}(ds)$ converges in distribution, when $R \to +\infty$, to a symmetric $\alpha$-stable random variable, which scale factor is given by $\left(\int_{\mathbb{R}^d} |f(s)|^\alpha ds\right)^{1/\alpha}$. The limit will be denoted by $\int_{\mathbb{R}^d} f(s)M_\alpha(ds)$. The random measure $M_\alpha(ds)$ is called a random stable measure and we shall only consider in this book symmetric random stable measure. Let us give a formal definition of a symmetric random stable measure (in short S$\alpha$S random measure).

**Definition 2.1.18** *Let $(E, \mathcal{E}, m)$ be a measurable space with a $\sigma$-finite measure $m$. For every $\mathcal{E}$-measurable function $f$ such that $\int_E |f|^\alpha dm < \infty$, a symmetric* *random stable measure $M_\alpha$ satisfies that $\int_E f(s)M_\alpha(ds)$ is an $\alpha$-symmetric random stable variable with scale factor $\left(\int_E |f|^\alpha dm\right)^{1/\alpha}$. Moreover $M_\alpha$ is assumed to be independently scattered.*

In this book symmetric random stable measures will be limit of Lévy random measures.

**Lemma 2.1.2** *For all functions $f \in L^\alpha(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, the limit in distribution of*

$$\lim_{R\to+\infty} \int_{\mathbb{R}^d} f(s)M_{\alpha,R}(ds)$$

*exists. Moreover the limit in distribution is the distribution of an $\alpha$-stable random variable with scale factor $\left(C(\alpha)\int_{\mathbb{R}^d} |f(s)|^\alpha ds\right)^{1/\alpha}$, where*

$$C(\alpha) = \frac{\pi}{2\alpha\Gamma(\alpha)\sin(\frac{\pi\alpha}{2})}, \tag{2.40}$$

*where $\Gamma$ is the classical Gamma function.*

*Proof of the Lemma 2.1.2*

To prove the Lemma it is enough to show the convergence of the characteristic functions of $\int_{\mathbb{R}^d} f(s)M_{\alpha,R}(ds)$ to the characteristic function of an $\alpha$-stable random variable with the scale factor given in Lemma 2.1.2.

Because of (2.34)

$$\begin{aligned} &-\log\left(\mathbb{E}\exp\left(iv\int f(s)M_{\alpha,R}(ds)\right)\right)\\ &= \int_{\mathbb{R}^d\times\mathbb{R}} [1-\exp(ivf(s)u) + ivf(s)u]ds\frac{\mathbf{1}_{|u|\le R}du}{|u|^{1+\alpha}}. \end{aligned} \tag{2.41}$$

Moreover for $0<\alpha<2$ if you change the variable $u$ into $|x|u$ you observe that the function

$$x \mapsto \int_{\mathbb{R}} [1-e^{ixu} + ixu\mathbf{1}_{|u|\le R}]\frac{du}{|u|^{1+\alpha}}$$

is homogeneous. Hence, for $0<\alpha<2$,

$$C(\alpha)|x|^\alpha = \int_{\mathbb{R}} [1-e^{ixu} + ixu\mathbf{1}_{|u|\le R}]\frac{du}{|u|^{1+\alpha}} \tag{2.42}$$

for every $x\in\mathbb{R}$, where the constant $C(\alpha)$ is given by letting $x=1$ in (2.42). Indeed $C(\alpha)$ does not depend on $R$. Hence

$$C(\alpha) = \int_{\mathbb{R}} \frac{(1-\cos(u))}{|u|^{1+\alpha}} du$$
$$= \frac{\pi}{2\alpha \Gamma(\alpha)\sin(\frac{\pi\alpha}{2})}$$

This formula for $C(\alpha)$ can be deduced for instance from (7.2.13 p. 328 in [126]). Hence,

$$C(\alpha)|v|^{\alpha} \int_{\mathbb{R}^d} |f(s)|^{\alpha} ds = \int_{\mathbb{R}^d} ds \int_{\mathbb{R}} [1 - e^{if(s)vu} + if(s)vu\mathbf{1}_{|u|\leq R}] \frac{du}{|u|^{1+\alpha}}. \quad (2.43)$$

Then,

$$-C(\alpha)|v|^{\alpha} \int_{\mathbb{R}^d} |f(s)|^{\alpha} ds - \log\left(\mathbb{E}\exp\left(iv \int f(s) M_{\alpha,R}(ds)\right)\right)$$
$$= -\int_{\mathbb{R}^d} ds \int_{\mathbb{R}} \frac{du}{|u|^{1+\alpha}} (1 - e^{if(s)vu})\mathbf{1}_{|u|>R}.$$

Please note that the last line is negative and finite for any $R > 0$. Hence, it converges to 0 by monotone convergence. Consequently the convergence in distribution is established and

$$\log\left(\mathbb{E}\exp\left(iv \int f(s) M_{\alpha}(ds)\right)\right) = -C(\alpha)|v|^{\alpha} \int_{\mathbb{R}^d} |f(s)|^{\alpha} ds. \quad (2.44)$$

**Definition 2.1.19** *For all functions $f \in L^{\alpha}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we denote by $\int_{\mathbb{R}^d} f(s)$ $M_{\alpha}$ $(ds)$*

$$\lim_{R\to+\infty} \int_{\mathbb{R}^d} f(s) M_{\alpha,R}(ds).$$

Actually $\int_{\mathbb{R}^d} f(s)M_{\alpha}(ds)$ is defined for every function $f \in L^{\alpha}(\mathbb{R}^d)$ since $L^{\alpha}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $L^{\alpha}(\mathbb{R}^d)$. At this point it is not clear that $\int_{\mathbb{R}^d} f(s)M_{\alpha}(ds)$ does define an $\alpha$-symmetric stable random measure in the sense of Definition 2.1.18. But the integration with respect to $M_{\alpha}(ds)$ could be defined as a stochastic field $(\int_{\mathbb{R}^d} f(s)M_{\alpha}(ds))_{f\in L^{\alpha}}$ parameterized by $L^{\alpha}$ as in Sect. 2.2 in [126], since the proof of Lemma 2.1.2 can be carried for any finite number of functions $f_1, \ldots, f_k \in L^{\alpha}(\mathbb{R}^d)$. Moreover the independent scattering property is a consequence of the same property for Lévy random measures, which is a direct consequence of the independent scattering for random Poisson measures. We omit the complete proof in the book but we claim that there exists an $\alpha$-symmetric stable random measure $M_{\alpha}(ds)$ in the sense of Definition 2.1.18 such that $\lim_{R\to+\infty} \int_{\mathbb{R}^d} f(s)M_{\alpha,R}(ds) \overset{(d)}{=} \int_{\mathbb{R}^d} f(s)M_{\alpha}(ds)$.

### 2.1.10 Complex Isotropic Random Measure

In Chap. 3 we need to construct a complex isotropic $\alpha$-stable random measure to define a self-similar field. Let us sketch the various steps that allow us to define this random measure starting from a complex Poisson random measure. Since this construction is parallel to the one in the real case we do not give details.

Let us consider an isotropic control measure on the complex field $\nu(dz)$. Isotropy means that if $P$ is the map $P(\rho\exp(i\theta)) = (\theta, \rho) \in [0, 2\pi) \times (0, \infty)$, then

$$P(\nu(dz)) = d\theta\nu_\rho(d\rho) \tag{2.45}$$

where $d\theta$ is the uniform measure on $(0, 2\pi)$.We still assume that

$$\int_{\mathbb{C}} |u|^p \nu(du) < \infty \quad \forall p \geq 2. \tag{2.46}$$

and we consider the associated Poisson random measure $\tilde{N} = N - n$, where $N$ is a Poisson random measure on $\mathbb{R}^d \times \mathbb{C}$ with mean measure $n(d\xi, dz) = d\xi\nu(dz)$. Similarly to the real case one can define a stochastic integral for every complex valued function $\varphi \in L^2(\mathbb{C} \times \mathbb{R}^d, d\xi\nu(dz))$ denoted by

$$\int_{\mathbb{R}^d \times \mathbb{C}} \varphi(\xi, z)\tilde{N}(d\xi, dz). \tag{2.47}$$

If $\varphi$ is real valued so is $\int \varphi d\tilde{N}$, and if $\Re(z)$ denotes the real part of a complex $z$ then $\Re(\int \varphi d\tilde{N}) = \int \Re(\varphi)d\tilde{N}$, the same property is true for the imaginary part $\Im$ of stochastic integrals. Furthermore, for every $u, \ v \in \mathbb{R}$

$$\begin{aligned}
&\mathbb{E}\exp\left(i(u\int \Re(\varphi)d\tilde{N} + v\int \Im(\varphi)d\tilde{N})\right) \\
&= \exp\left[\int_{\mathbb{R}^d \times \mathbb{C}} [\exp(i(u\Re(\varphi) + v\Im(\varphi))) - 1 - i(u\Re(\varphi) + v\Im(\varphi))]d\xi\nu(dz)\right].
\end{aligned} \tag{2.48}$$

Like real Poisson random measures, complex Poisson random measures satisfy an isometry property

$$\mathbb{E}\left|\int_{\mathbb{R}^d \times \mathbb{C}} \varphi(\xi, z)\tilde{N}(d\xi, dz)\right|^2 = \int_{\mathbb{R}^d \times \mathbb{C}} |\varphi(\xi, z)|^2 n(d\xi, dz), \tag{2.49}$$

where $|\varphi(\xi, z)|$ is the complex modulus of the complex number $\varphi(\xi, z)$. This property is obvious for simple functions of the form $\sum_{j\in J} a_j \mathbf{1}_{A_j}$ and is trivially extended to functions in $L^2$ by letting a sequence of simple functions converges to every $L^2$ function. One can now define the complex isotropic Lévy random measure

**Definition 2.1.20**

$$\int_{\mathbb{R}^d} f(\xi)M(d\xi) = \int_{\mathbb{R}^d\times\mathbb{C}} [f(\xi)z + f(-\xi)\bar{z}]\tilde{N}(d\xi, dz) \tag{2.50}$$

*for every function* $f : \mathbb{R}^d \to \mathbb{C}$ *where* $f \in L^2(\mathbb{R}^d)$.

Then, if

$$\forall \xi \in \mathbb{R}^d, \quad f(-\xi) = \overline{f(\xi)} \tag{2.51}$$

the following integral is a real number since

$$\int_{\mathbb{R}^d} f(\xi)M(d\xi) = \int_{\mathbb{R}^d\times\mathbb{C}} 2\Re(f(\xi)z)\tilde{N}(d\xi, dz) \tag{2.52}$$

$$= 2\Re\left(\int_{\mathbb{R}^d\times\mathbb{C}} f(\xi)z\tilde{N}(d\xi, dz)\right). \tag{2.53}$$

When $f$ satisfies (2.51), for every measurable odd function $a$

$$\int f(\xi)\exp(ia(\xi))M(d\xi) \overset{(d)}{=} \int f(\xi)M(d\xi), \tag{2.54}$$

is a consequence of (2.48). Moreover stochastic integrals have symmetric distributions

$$-\int f(\xi)M(d\xi) \overset{(d)}{=} \int f(\xi)M(d\xi). \tag{2.55}$$

Moreover an isometry property for the Lévy measure $M(d\xi)$ when $f$ satisfies (2.51)

$$\mathbb{E}|\int_{\mathbb{R}^d} f(\xi)M(d\xi)|^2 = 4\pi\,\|f\|^2_{L^2(\mathbb{R}^d)} \int_0^{+\infty} \rho^2\nu_\rho(d\rho) \tag{2.56}$$

holds, it is a consequence of (2.49). The properties of complex valued random Lévy measures are proved in Exercise 3.6.7.

Actually the characteristic function (2.48) allows us to compute every moments of the stochastic integrals $\int f(\xi)M(d\xi)$.

**Proposition 2.1.4** *If* $f \in L^2(\mathbb{R}^d) \bigcap L^{2p}(\mathbb{R}^d)$ *and* $f$ *satisfies* (2.51) *then* $\int f(\xi)M(d\xi)$ *is in* $L^{2p}(\Omega)$ *and*

$$\mathbb{E}\left[\left(\int f(\xi)M(d\xi)\right)^{2p}\right] = \sum_{n=1}^{p}(2\pi)^n$$
$$\sum_{P_n}\prod_{q=1}^{n}\frac{(2m_q)!\|f\|_{2m_q}^{2m_q}\int_0^{+\infty}\rho^{2m_q}\nu_\rho(d\rho)}{(m_q!)}, \qquad (2.57)$$

*where* $\sum_{P_n}$ *stands for the sum over the set of partitions* $P_n$ *of* $\{1,\ldots,2p\}$ *in* $n$ *subsets* $K_q$ *such that the cardinality of* $K_q$ *is* $2m_q$ *with* $m_q \geq 1$ *and where* $\|f\|_{2m_q}$ *is the* $L^{2m_q}(\mathbb{R}^d)$ *norm of* $f$.

*Proof of the Proposition 2.1.4*

An expansion in power series of both sides of (2.48) yields the result.

One can also define complex isotropic $\alpha$-stable random measure as a limit of complex isotropic Lévy random measures. Let us consider a complex isotropic random Lévy measures $M_{\alpha,R}$ associated to $n_R(d\xi, dz) = d\xi\frac{dz}{|z|^{1+\alpha}}\mathbf{1}_{|z|\leq R}$, one can prove a lemma similar to Lemma 2.1.2.

**Lemma 2.1.3** *For all functions* $f \in L^\alpha(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, *that satisfy (2.51), the limit in distribution of :*

$$\lim_{R\to+\infty}\int_{\mathbb{R}^d} f(\xi)M_{\alpha,R}(d\xi) \qquad (2.58)$$

*exists. Moreover the limit in distribution is a real valued* $\alpha$*-stable random variable with scale factor* $\left(C(\alpha)2^{\alpha-1}\int_0^{2\pi}|\cos(\theta)|^\alpha d\theta\int_{\mathbb{R}^d}|f(\xi)|^\alpha d\xi\right)^{1/\alpha}$, *where* $C(\alpha)$ *is defined in* (2.40).

*Proof of Lemma 2.1.3*

Let consider for $u \in \mathbb{R}$ the characteristic function

$$-\log\left(\mathbb{E}\exp\left(iu\Re\int f(\xi)M_{\alpha,R}(d\xi)\right)\right)$$
$$= \int_{\mathbb{R}^d\times[0,2\pi]\times(0,+\infty)}[\exp(iu\rho 2\Re(f(\xi)e^{i\theta})) - 1$$
$$- iu\rho 2\Re(f(\xi)e^{i\theta})]\mathbf{1}_{|\rho|\leq R}d\xi d\theta\frac{d\rho}{\rho^{1+\alpha}}.$$

As in Lemma 2.1.2, one can show that

$$\lim_{R\to\infty} -\log\left(\mathbb{E}\exp\left(iu\Re\int f(\xi)M_{\alpha,R}(d\xi)\right)\right)$$
$$= \frac{C(\alpha)}{2}|u|^\alpha\int_{\mathbb{R}^d}\int_0^{2\pi}|2\Re(f(\xi)e^{i\theta}|^\alpha d\xi d\theta,$$

which can also be written

$$C(\alpha)2^{\alpha-1}\int_0^{2\pi}|\cos(\theta)|^\alpha d\theta \int_{\mathbb{R}^d}|f(\xi)|^\alpha d\xi.$$

**Definition 2.1.21** *For all functions* $f \in L^\alpha(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, *we denote by* $\int_{\mathbb{R}^d} f(\xi)M_\alpha\ (d\xi)$

$$\lim_{R\to+\infty}\int_{\mathbb{R}^d} f(\xi)M_{\alpha,R}(d\xi).$$

## 2.1.11 Stationary Fields and Fields with Stationary Increments

In this book, stationary fields and fields with stationary increments will be considered. Let us give some definitions and representation theorems for such fields.

**Definition 2.1.22** *A field* $(X(t))_{t\in\mathbb{R}^d}$ *such that* $\mathbb{E}|X_t|^2 < +\infty, \quad \forall t \in \mathbb{R}^d$ *is called weakly stationary if the mean value is constant and if there exists a function* $r$ *such that the covariance* $R(t,s) = r(t-s)$. *It is called strictly stationary if* $\forall\delta \in \mathbb{R}^d$

$$(X(t+\delta))_{t\in\mathbb{R}^d} \overset{(d)}{=} (X(t))_{t\in\mathbb{R}^d}. \tag{2.59}$$

**Remark 2.1.1** *If* $X$ *is a strictly stationary field, then* $X$ *is weakly stationary. When* $X$ *is a Gaussian field, strict stationarity and weak stationarity are equivalent, since mean value and covariance characterize the distribution of Gaussian fields.*

**Definition 2.1.23** *If* $X$ *is a weakly stationary field, then the function* $r$ *such that the covariance* $R(t,s) = r(t-s)$ *is a non negative function in the sense of Definition 2.1.10. If* $r$ *is a continuous function, by Bochner theorem 2.1.3, there exists a finite symmetric measure* $\mu$ *on* $\mathbb{R}^d$ *such that*

$$r(t) = \int_{\mathbb{R}^d} e^{it\cdot\xi}\mu(d\xi). \tag{2.60}$$

*The measure* $\mu$ *is called the spectral measure of the field* $X$. *If* $\mu$ *admits a density with respect to the Lebesgue measure* $d\xi$, *this density is called the spectral density of* $X$.

**Example 2.1.2** *Spectral representation of Gaussian stationary field*

Let $(X_t,\ t\in\mathbb{R}^d)$ be a centered stationary Gaussian process with covariance

$$r(h) = \mathbb{E}X_{t+h}X_t.$$

Assume that $r$ is a continuous function with the corresponding spectral measure $\mu(d\xi) = f(\xi)d\xi$. Please remark that $r(h) = r(-h)$, and by Bochner's theorem $r(h) = \int_{\mathbb{R}^d} e^{ih.\xi} f(\xi)d\xi$, so $f(\xi) = f(-\xi)$ is a non negative function in $L^1(\mathbb{R}^d)$. Let us consider the field defined by

$$Y_t = (2\pi)^{\frac{d}{4}} \int_{\mathbb{R}^d} \sqrt{f(\xi)} e^{it.\xi} \widehat{W}(d\xi), \tag{2.61}$$

where $\widehat{W}(d\xi)$ is the Fourier transform of a real Brownian random measure introduced in Definition 2.1.16. Clearly, $Y$ is a centered Gaussian field with covariance function

$$\begin{aligned}\mathbb{E}Y_t Y_s &= \int_{\mathbb{R}^d} f(\xi) e^{i(t-s)\xi} d\xi \\ &= r(t-s).\end{aligned}$$

Hence $Y$ and $X$ have the same distribution and (2.61) is called the spectral representation of $X$.

Let us now consider fields with stationary increments.

**Definition 2.1.24** *A field $(X(t))_{t\in\mathbb{R}^d}$ such that*

$$(X(t+\delta) - X(s+\delta))_{t\in\mathbb{R}^d} \overset{(d)}{=} (X(t) - X(s))_{t\in\mathbb{R}^d} \tag{2.62}$$

*for every $s$ and $\delta$ in $\mathbb{R}^d$ is called a field with stationary increments.*

There exists also a spectral representation of the covariance of fields with stationary increments, which is similar to Bochner's theorem. Let us assume that $X$ is a centered field with stationary increments, such that $\mathbb{E}|X_t|^2 < +\infty, \quad \forall t \in \mathbb{R}^d$. Moreover if $X_0 = 0$ a.s. and the covariance is continuous, then there exist a sigma finite measure $\mu$ on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} \inf(1, \|\xi\|^2) d\mu(\xi) < +\infty$ and a symmetric non-negative definite matrix $\Sigma$ such that

$$R(t,s) = \int_{\mathbb{R}^d} (e^{it.\xi} - 1)(e^{-is.\xi} - 1) d\mu(\xi) +^t t\Sigma s. \tag{2.63}$$

(See a discussion concerning this result in [142] and the references therein). One can also deduce from this result an integral representation for Gaussian fields with stationary increments.

**Example 2.1.3** *Spectral representation of Gaussian stationary fields with stationary increments.*

Let us consider a centered Gaussian field $X$ with stationary increments, such that $X_0 = 0$ a.s. Let us assume that the covariance function is continuous, and that the spectral measure $\mu(d\xi) = f(\xi)d\xi$. As in stationary case, $f$ is called the spectral density of $X$. Then

$$X_t \overset{(d)}{=} (2\pi)^{\frac{d}{4}} \int_{\mathbb{R}^d} (e^{it.\xi} - 1)\sqrt{f(\xi)}\widehat{W}(d\xi) + t.N \tag{2.64}$$

where $\widehat{W}(d\xi)$ is the Fourier transform of a real Brownian random measure and where $N$ is a centered Gaussian random vector with covariance $\Sigma$. Please note that $\int_{\mathbb{R}^d} (e^{it.\xi} - 1)\sqrt{f(\xi)}\widehat{W}(d\xi)$ and $N$ are independent for all $t$. See Exercise 2.3.9.

### 2.1.12 Regularity of the Sample Paths

In this book we very often have to check whether a given field is Hölder continuous. The Kolmogorov-Chentsov's theorem (e.g. [79] for a proof) answers this question when the field have finite moments of high order. Sadly enough, the Definition 2.1.2 is not precise enough to decide if the sample paths of a field are regular or not. Classically the following definition is introduced to have a convenient definition of equality for two fields.

**Definition 2.1.25** *Modification.*
*A field $Y$ is a modification of a field $X$ if, for every $t$, $\mathbb{P}(X_t = Y_t) = 1$.*

Please note that if $Y$ is a modification of a field $X$ then $Y$ and $X$ are two versions of the same field. See Exercise 2.3.8.

One can find a modification $Y$ with jumps of a continuous process $X$. For instance let $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}[0, 1]$ and take the Lebesgue measure $\mathbb{P}$ on $\Omega$ as a probability measure. Let us consider $Y_t(\omega) = \mathbf{1}_{t\neq\omega}$, for $t \in [0, 1]$. Every sample path of $Y$ are discontinuous but $Y$ is a modification of $X_t(\omega) = 1$ for every $t \in [0, 1]$ and $\omega$. Actually $\mathbb{P}(\omega \neq t) = 0$ for every $t \in [0, 1]$.

**Theorem 2.1.7** *Kolmogorov-Chentsov's theorem.*
*Let $(X_t,\ t \in [A, B]^d)$ be a random field. If there exist three positive constants $\alpha,\ \beta,\ C$ such that, for every $t, s \in [A, B]^d$*

$$\mathbb{E}|X_t - X_s|^\alpha \le C\|t - s\|^{d+\beta},$$

*then, there exists a locally $\gamma$-Hölder continuous modification $\widetilde{X}$ of $X$ for every $\gamma < \beta/\alpha$. It means that there exist a random variable $h(\omega)$, and a constant $\delta > 0$ such that*

$$\mathbb{P}\left[\omega,\ \sup_{\|t-s\|\le h(\omega)} \frac{|\widetilde{X}_t(\omega) - \widetilde{X}_s(\omega)|}{\|t - s\|^\gamma} \le \delta\right] = 1.$$

Locally $\gamma$-Hölder functions are defined later in Definition 2.2.2. Before proving the theorem, let us give a classical application.

**Example 2.1.4** *Regularity of the Brownian motion.*

Since $B_t - B_s$ is a centered Gaussian variable of variance $|t-s|$, for all $\alpha > 0$, there exists $C_\alpha$ such that

$$\begin{aligned}\mathbb{E}|B_t - B_s|^\alpha &= C_\alpha \left(\mathbb{E}(B_t - B_s)^2\right)^{\alpha/2} \\ &= C_\alpha |t-s|^{\alpha/2}.\end{aligned}$$

We now apply Kolmogorov-Chentsov's theorem with $d = 1$, $C = C_\alpha$, and $\beta = \alpha/2 - 1$. The Brownian motion is locally $\gamma$-Hölder continuous for every $\gamma < 1/2 - 1/\alpha$. Letting $\alpha \to +\infty$, we get that the Brownian motion is locally $\gamma$-Hölder continuous for every $\gamma < 1/2 - \varepsilon$, $\varepsilon > 0$. Actually the modulus of continuity of Brownian motion has been studied in much more details. See [79] for an introduction to this study.

Let us now state a theorem that ensures the differentiability of a stochastic process (proved in [43], p. 69).

**Theorem 2.1.8** *Let $(X_t, t \in [A, B])$ be a stochastic process such that there exist three positive constants $\alpha_0,\ \beta_0,\ C_0$ such that, for every $t, t+h \in [A, B]$*

$$\mathbb{E}|X_{t+h} - X_t|^{\alpha_0} \le C_0 |h|^{1+\beta_0},$$

*and there exist three positive constants $\alpha_1,\ \beta_1 > \alpha_1,\ C_0$ such that, for every $t,\ t+h,\ t-h \in [A, B]$*

$$\mathbb{E}|X(t+h) + X(t-h) - 2X(t)|^{\alpha_1} \le C_1 |h|^{1+\beta_1}.$$

*Then there exists a modification of $X$ which has almost surely $C^1$ sample paths.*

## *2.1.13 Sequences of Continuous Fields*

In this book we will consider sequences of continuous fields that will converge in distribution. Actually continuous fields may converge in distribution in two ways. If we recall that the distribution of a field is characterized by its finite dimensional distributions (Cf. Definition 2.1.2.), the convergence of those margins is a natural prerequisite for the convergence of the distributions of fields.

**Definition 2.1.26** *Let $X^n$ be a sequence of fields. We say that $X^n$ converge in distribution to the field $X$ for all finite dimensional margins of the fields, if, for every $k \in \mathbb{N}^*$ and $t_1, \ldots, t_k \in T$*

$$\lim_{n\to+\infty} (X^n_{t_1}, \ldots, X^n_{t_k}) \overset{(d)}{=} (X_{t_1}, \ldots, X_{t_k}). \tag{2.65}$$

Please remark that we don't need that the fields are continuous to have convergence of finite dimensional margins. But this convergence might be too weak for some applications. Let us be more specific. If $X^n$ are continuous fields defined on $[A, B]^d$, sometimes we need to show that the $\sup_{t\in[A,B]^d} X^n(t)$ is converging in distribution to $\sup_{t\in[A,B]^d} X(t)$. Please see [82] (especially Sect. 1 5.3.3, p. 196) for a comprehensive study. Unfortunately the supremum of a continuous function does not depend in general on a finite sequence of points $t_k$'s and the convergence of finite dimensional margins does not imply the convergence of distributions of supremum. Let us consider the space $C^0$ of continuous functions from $[A, B]^d$ onto $\mathbb{R}$ endowed with the topology induced by the uniform metric distance

$$\rho(f, g) = \sup_{t\in[A,B]^d} |f(t) - g(t)|.$$

This topology is also the topology corresponding to the uniform convergence.

**Definition 2.1.27** *Let $X^n$ be a sequence of continuous fields. We say that $X^n$ converges in distribution to the continuous field $X$ on the space of continuous functions endowed with the topology of the uniform convergence, if, for every bounded continuous functional $F : C^0 \mapsto \mathbb{R}$*

$$\lim_{n\to+\infty} \mathbb{E}F(X^n) = \mathbb{E}F(X). \tag{2.66}$$

**Remark 2.1.2** *If the set of parameters of the fields is $\mathbb{R}^d$, we say that $X^n$ converges in distribution to the continuous field $X$ on the space of continuous functions endowed with the topology of the uniform convergence on every compact set, if for every compact $[A, B]^d$ the fields $X^n$ restricted to the set of parameters $[A, B]^d$ is converging in distribution to the continuous field $X$ restricted to $[A, B]^d$ on the space of continuous functions endowed with the topology of the uniform convergence.*

Let us prove that the convergence on $C^0$ is stronger than convergence in distribution of finite dimensional margins.

**Proposition 2.1.5** *Convergence in distribution on the space of continuous functions endowed with the topology of the uniform convergence implies the convergence of finite dimensional margins.*

*Proof of Proposition 2.1.5*
If $k \in \mathbb{N}$ and $t_1, \ldots, t_k \in [A, B]^d$ then the functional

$$F(f) = \varphi(f(t_1), \ldots, f(t_k))$$

is a bounded continuous functional on $C^0$ provided $\varphi$ is a bounded continuous function.

Moreover the supremum of a continuous function is obviously a continuous functional on $C^0$. Hence convergence in distribution on the space of continuous functions endowed with the topology of the uniform convergence implies convergence in distribution of the suprema. Actually to prove convergence in distribution on the space of continuous functions endowed with the topology of the uniform convergence we need tightness of the distributions of the continuous fields $X^n$ in addition to the convergence of finite dimensional margins.

Let $\mathcal{C}$ be the Borel sigma field on the metric set $C^0$ and $X$ be a continuous field defined on the probability space $(\Omega, \sigma(\Omega), \mathbb{P})$. The distribution of $X$ is a probability $\mathbb{P}_X$ such that for every Borel set $B$ in $\mathcal{C}$,

$$\mathbb{P}_X(B) = \mathbb{P}(\omega,\ X(\omega) \in B).$$

Then tightness of the distributions of fields $X^n$ is defined as the tightness of their distributions and is very similar to the tightness of probabilities on $\mathcal{B}(\mathbb{R}^d)$.

**Definition 2.1.28** *Tightness.*

*A sequence of probabilities* $(\mathbb{P}_n)_{n\geq 1}$ *on* $(C^0, \mathcal{C})$ *is tight, if for every* $\varepsilon > 0$, *there exists a compact set* $K_\varepsilon$ *such that* $\inf_{n\geq 1} \mathbb{P}_n(K_\varepsilon) \geq 1 - \varepsilon$

Description of the compactness in $C^0$ involves modulus of continuity of functions as stated in Ascoli's Theorem, see for instance [79].

For any function $f \in C^0$, let us introduce its modulus of continuity

$$\omega(f, \delta) = \sup_{\|t-s\|\leq\delta} |f(t) - f(s)|.$$

One can find the following theorem in [82].

**Theorem 2.1.9** *Tightness criteria.*

*Let* $(\mathbb{P}_n)_{n\geq 1}$ *be a sequence of probabilities on* $C^0$. *The sequence* $(\mathbb{P}_n)_{n\geq 1}$ *is tight if and only if*

1. *For every* $\eta > 0$, *there exist* $t_0 \in [A, B]^d$, $a > 0$ *and* $n_0$ *such that for* $n \geq n_0$

$$\mathbb{P}_n(f \in C^0,\ |f(t_0)| \geq a) \leq \eta.$$

2. *For every* $\varepsilon > 0$, $\eta > 0$, *there exist* $\delta > 0$ *and* $n_0$ *such that for* $n \geq n_0$

$$\mathbb{P}_n(f \in C^0,\ \omega(f, \delta) \geq \varepsilon) \leq \eta.$$

Actually in this book we will mainly use the following sufficient conditions to have tightness of continuous fields. One can find a more sophisticated version of this result in [86], see also [27, 82, 92] for different types of sufficient conditions.

**Corollary 2.1.2** *Tightness of random fields.*

*Let* $(X^n)_{n\geq 1}$ *be a sequence of random fields valued in* $C^0$. *The sequence of random fields* $(X^n)_{n\geq 1}$ *is tight if*

1. *there exists* $t_0 \in [A, B]^d$ *such that* $(X^n(t_0))_{n\geq 1}$ *is tight,*

2. *there exist three positive constants* $\alpha$, $\beta$ *and* $C$ *such that, for* $t, s \in [A, B]^d$

$$\sup_{n\geq 1} \mathbb{E}|X_t^n - X_s^n|^\alpha \leq C\|t-s\|^{d+\beta}.$$

**Remark 2.1.3** *It is obvious that if there exists* $\nu > 0$ *such that*

$$\sup_{n\geq 1} \mathbb{E}|X^n(t_0)|^\nu < \infty,$$

*then* $(X^n(t_0))_{n\geq 1}$ *is tight. It is often used as an alternative to assumption 1. In Corollary 2.1.2.*

**Remark 2.1.4** *Please remark that, if we consider only one field* $X$*, assumption 2, in Corollary 2.1.2, is the same as the assumption in Theorem 2.1.7. Actually one way to show that there is continuous modification of a given field is to construct a sequence converging to* $X$ *in* $C^0$*. So it is quite natural that assumption 2 is a generalization of assumption 2. In Theorem 2.1.7, the proof of the Corollary is actually also a proof of Theorem 2.1.7.*

*Proof of Corollary 2.1.2*

Let us first remark that we only have to prove that assumption 2, in Corollary 2.1.2 implies assumption 2, in Theorem 2.1.9. Without loss in generality we may assume $A = 0,\ B = 1$.

Let $D_n = \{\frac{k}{2^n},\ k \in \{0, \ldots, 2^n\}^d\}$ and $D = \cup_{n\in\mathbb{N}} D_n$. Let us fix $0 < \gamma < \frac{\beta}{\alpha}$, and define for $l \in \mathbb{N}$

$$C_l(\omega) \overset{def}{=} \sup_{\substack{n\geq 1, s\in D_n,\\ \|t-s\|=\frac{1}{2^n}}} 2^{\gamma n}|X^l(t) - X^l(s)|.$$

$\forall x > 0$,

$$\begin{aligned}
\mathbb{P}(C_l > x) &\leq \sum_{n=1}^{+\infty} \sum_{\substack{t,s\in D_n,\\ \|t-s\|=\frac{1}{2^n}}} \mathbb{P}\left(\left|X^l(t) - X^l(s)\right| > 2^{-\gamma n}x\right) \\
&\leq \frac{1}{x^\alpha} \sum_{n=1}^{+\infty} \sum_{\substack{t,s\in D_n,\\ \|t-s\|=\frac{1}{2^n}}} 2^{\gamma n\alpha}\mathbb{E}\left(|X^l(t) - X^l(s)|^\alpha\right) \\
&\leq \frac{2d}{x^\alpha} \sum_{n=1}^{+\infty} 2^{n(d+\gamma\alpha-d-\beta)} \\
&\leq \frac{d2^{\alpha\gamma-\beta+1}}{(1-2^{\alpha\gamma-\beta})x^\alpha}. \qquad (2.67)
\end{aligned}$$

For every $x > 0$, let $\Omega^l_x = \{C_l \leq x\}$. Then $\mathbb{P}(\Omega \setminus \Omega^l_x) < \frac{C_*}{x^\alpha}$, where the constant $C_* = \frac{d2^{\alpha\gamma-\beta+1}}{(1-2^{\alpha\gamma-\beta})}$ does not depend neither on $l$ neither on $x$, and $\Omega^l_x$ is such that

$$\forall n \in \mathbb{N},\ \forall t,\ s \in D_n,\ \forall \omega \in \Omega^l_x,\ \|t-s\| = \frac{1}{2^n}, \quad |X^l(t) - X^l(s)| \leq 2^{-\gamma n} x. \tag{2.68}$$

Let us show by induction that

$$\forall m \geq n+1,\ t,\ s \in D_m \quad \text{and} \quad \|t-s\| < \frac{1}{2^n}, \forall \omega \in \Omega^l_x,$$
$$|X^l(t) - X^l(s)| \leq 2dx \sum_{j=n+1}^{m} 2^{-\gamma j}. \tag{2.69}$$

For $m = n+1,\ t = k/2^{n+1},\ s = k'/2^{n+1}$ and $\|k - k'\| < 2$, hence (2.68) implies (2.69). Let us assume (2.69) for $m = n+1, \ldots, M$ and take $t,\ s \in D_{M+1}$ such that $\|t-s\| < \frac{1}{2^n}$. There exist $t',\ s' \in D_M$ such that $\|t'-s'\| < \frac{1}{2^n},\ \|t'-t\| < \frac{1}{2^{M+1}}$, $\|s'-s\| < \frac{1}{2^{M+1}}$, and such that at most $d$ coordinates of $t'$ are different from the corresponding coordinate in $t$. Similarly there will be at most $d$ coordinates of $s'$ different from the corresponding coordinate in $s$. Then, $\forall \omega \in \Omega^l_x$,

$$\begin{aligned} |X^l(t) - X^l(s)| &\leq |X^l(t) - X^l(t')| + |X^l(t') - X^l(s')| + |X^l(s') - X^l(s)| \\ &\leq dx2^{-\gamma(M+1)} + 2dx \sum_{j=n+1}^{M} 2^{-\gamma j} + dx2^{-\gamma(M+1)}. \end{aligned}$$

Hence (2.69) is proved by induction.

Let us consider $t,\ s \in D$ such that $\|t-s\| > 0$ there exists $n > 0$ such that $2^{n+1} \leq \|t-s\| < 2^n$. Then, for $m > n$ such that $t,\ s \in D_m$

$$\begin{aligned} |X^l(t) - X^l(s)| &\leq 2dx \sum_{j=n+1}^{+\infty} 2^{-\gamma j} \\ &= 2dx2^{-\gamma(n+1)} \frac{1}{1-2^{-\gamma}} \\ &\leq \frac{2dx}{1-2^{-\gamma}} \|t-s\|^\gamma, \end{aligned} \tag{2.70}$$

$\forall \omega \in \Omega^l_x$. Since the fields $X^l$ are continuous and $D$ is dense in $[0,1]^d$,

$$\mathbb{P}\left( \sup_{t,\ s \in [0,1]^d,\ t \neq s} \frac{|X^l(t) - X^l(s)|}{\|t-s\|^\gamma} > C^< x \right) \leq \frac{C_*}{x^\alpha}, \tag{2.71}$$

where $C^{<} = \frac{2d}{1-2^{-\gamma}}$. Then for $\epsilon > 0, \delta > 0,\ l \in \mathbb{N}$

$$\mathbb{P}\left(\sup_{\|t-s\|\leq\delta} |X^l(t) - X^l(s)| \geq \varepsilon\right) \leq \mathbb{P}\left(\sup_{\|t-s\|\leq\delta} \frac{|X^l(t) - X^l(s)|}{\|t-s\|^{\gamma}} > \frac{\varepsilon}{\delta^{\gamma}}\right)$$
$$\leq \frac{C_* \delta^{\alpha\gamma} C^{<\alpha}}{\varepsilon^{\alpha}} \tag{2.72}$$

Hence for all $\eta > 0$, one can take $\delta = \left(\frac{\eta\varepsilon^{\alpha}}{C_* C^{<\alpha}}\right)^{1/(\gamma\alpha)} > 0,$ which does not depend on $l$, such that $\forall l \in \mathbb{N}$,

$$\mathbb{P}(X^l \in C^0,\ \omega(X^l, \delta) \geq \varepsilon) \leq \eta.$$

For practical purposes, in order to prove the convergence in distribution of a sequence of random fields in the space of continuous functions, one uses the following result.

**Theorem 2.1.10** *Let $(X^n)_{n\geq 1}$ and $X$ be random fields valued in $C^0$ such that for all $k \in \mathbb{N}$ and for all $t_1, \ldots, t_k \in [A, B]^d$ the finite dimensional distributions of $(X^n(t_1), \ldots, X^n(t_k))$ converge to $(X(t_1), \ldots, X(t_k))$ . If there exist three positive constants $\alpha$, $\beta$ and $C$ such that, for $t, s \in [A, B]^d$*

$$\sup_{n\geq 1} \mathbb{E}|X_t^n - X_s^n|^{\alpha} \leq C\|t-s\|^{d+\beta},$$

*then $X^n$ converges to the continuous field $X$ in distribution on the space of continuous functions endowed with the topology of the uniform convergence.*

*Proof of Theorem 2.1.10*

First $(X^n)_{n\geq 1}$ is tight since convergence of finite dimensional distributions implies tightness of the distribution of $(X^n(t_0))_{n\geq 1}$ for any $t_0 \in [A, B]^d$, and we use Corollary 2.1.2. Hence there exist converging subsequences of $P_{X^n}$ in the space of continuous functions endowed with the topology of the uniform convergence. Moreover their limit $P$ is unique because of the convergence of the finite dimensional distribution of $(X^n)_{n\geq 1}$.

## 2.2 Fractal Analysis

### *2.2.1 Hölder Continuity, and Exponents*

In this book we will often discuss the regularity of the sample paths of the random fields. We choose not to be too technical in a domain where many refinements are possible. For sake of simplicity we will mainly focus our attention on the Hölder continuity and the Theorem 2.1.7 is a good example of results that can be expected. Therefore some definitions are introduced.

**Definition 2.2.1** *Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a function such that there exist $C > 0$ and $0 < H < 1$ so that*

$$|f(t) - f(u)| \leq C\|t - u\|^H \quad \forall t,\ u \in \mathbb{R}^d.$$

*The function $f$ is called $H$-Hölder continuous. The set of $H$-Hölder continuous functions on $[0, 1]^d$ is denoted by $\mathcal{C}^H$.*

Unfortunately the conclusion of many theorems like Theorem 2.1.7 is only that the sample paths are locally Hölder continuous, which means that their restrictions to every compact sets is $H$-Hölder continuous.

**Definition 2.2.2** *Let $f : \mathbb{R}^d \mapsto \mathbb{R}$ be a function such that on every compact set $K$ there exist a constant $C(K) > 0$ depending only on $K$ and $0 < H < 1$ such that*

$$|f(t) - f(u)| \leq C(K)\|t - u\|^H \quad \forall t,\ u \in K.$$

*The function $f$ is called locally $H$-Hölder continuous.*

Please note that if $H' < H$, $H$-Hölder continuous functions are $H'$-Hölder continuous. The same property is true locally. Hence if we know that a function is $H$-Hölder continuous, one can wonder what is the best $H$. These considerations lead to the definition of Hölder exponents that can have different natures : global, local or pointwise. In this book we will only be concerned with pointwise Hölder exponents.

**Definition 2.2.3** *A real valued function $f$ defined in a neighborhood of $t$, has a pointwise Hölder exponent $H$ if*

$$H(x) = \sup\left\{H', \quad \lim_{\|\epsilon\|\to 0} \frac{f(t+\epsilon) - f(t)}{\|\epsilon\|^{H'}} =\right\}. \tag{2.73}$$

## 2.2.2 Fractional Derivative and Integration

The aim of this section is to give rough ideas on fractional derivative and integration. Precise and detailed statements can be for instance found in [108, 125] for the Riemann-Liouville approach and in [135] for the Fourier approach.

### 2.2.2.1 Riemann-Liouville Approach

Let us start with heuristic computations. For an integer $n$, let $D^n$ be the nth derivative and $D^{-n}$ be the n-fold integral. Then

$$D^{-1}f(t) = \int_0^t f(s)ds.$$

Operator $D^{-1}$ is define up to a constant: we choose it such that $D^{-1}f(0) = 0$. More generally, we can check by induction that

$$D^{-n}f(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}f(s)ds.$$

Let $\nu > 0$. Let us define the fractional integration operator $D^{-\nu}$, sometimes called the Riemann-Liouville operator:

$$D^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-s)^{\nu-1}f(s)ds,$$

where $\Gamma$ is the gamma function.

For $\nu = 0$, operator $D^0$ is equal to identity by convention. It follows that $D^0 f(0)$ should be different of 0.

Let $\nu > 0$. Let $[\nu]$ be the integer part of $\nu$ and put $\{\nu\} = \nu - [\nu]$. The fractional derivative operator $D^\nu$ is defined by

$$D^\nu f(t) = D^{[\nu]+1}\left[D^{1-\{\nu\}}f(t)\right].$$

The Riemann-Liouville approach has been used for differential calculus associated to fractional Brownian motion since it is a non-anticipative way of defining fractional derivative and integration.

#### 2.2.2.2 Fourier Approach

A different approach of fractional calculus, that will be mostly used in this book, is based on the Fourier approach. Let us start too with heuristic computations. Let $\widehat{f}$ be the Fourier transform of a function $f$. It is well-known that Fourier transforms a derivative into a multiplication and an integration by a division:

$$\widehat{f^{(n)}(\xi)} = (i\xi)^n \widehat{f}(\xi).$$

The idea is to replace the integer $n$ by any real $\nu$. There is clearly a problem for defining $(i\xi)^\nu$. For $\xi \neq 0$, let us write $(i\xi)^n = \left(i\frac{\xi}{|\xi|}\right)^n |\xi|^n$. In the Fourier domain, multiplying by $i$ is a translation and multiplying by $\frac{\xi}{|\xi|}$ is an application of Riesz transform. We only keep the multiplication by the modulus of $\xi$. We define a pseudo fractional derivative/integration operator of order $\nu$ of $f$ by taking the inverse Fourier transform of $|\xi|^\nu \widehat{f}(\xi)$

$$f \to \int_{\mathbb{R}} \hat{f}(\xi) e^{-it\xi} |\xi|^\nu \frac{d\xi}{(2\pi)^{1/2}}.$$

As for the Riemann-Liouville approach, we want the $\nu$ fold integral of a function $f$ to be null at $t = 0$. This leads to the operator

$$f \to \int_{\mathbb{R}} \hat{f}(\xi)(e^{-it\xi} - 1)|\xi|^\nu \frac{d\xi}{(2\pi)^{1/2}}. \tag{2.74}$$

Now we need to make some precise statements and especially to know if the operator (2.74) should have a real meaning. In this book, we will focus on the case $-3/2 < \nu < -1/2$. Let us put $H = -\nu - 1/2$ so $0 < H < 1$. Define the so-called harmonizable fractional operator

$$\tilde{\mathcal{I}}_H f(t) = \int_{\mathbb{R}} \overline{\hat{f}(\xi)} \frac{e^{-it\xi} - 1}{|\xi|^{H+1/2}} \frac{d\xi}{(2\pi)^{1/2}}.$$

Because function $\xi \to \frac{e^{it\xi} - 1}{|\xi|^{H+1/2}}$ belongs then to $L^2$: the operator $\tilde{\mathcal{I}}^H$ is defined on $L^2$. Since

$$\frac{\widehat{e^{-it\xi} - 1}}{|\xi|^{H+1/2}} = C(|t-s|^{H-1/2} - |s|^{H-1/2}),$$

Plancherel's formula leads to the so-called moving-average fractional operator:

$$\tilde{\mathcal{I}}_H f(t) = C' \int_{\mathbb{R}} f(s)(|t-s|^{H-1/2} - |s|^{H-1/2}) \frac{ds}{(2\pi)^{1/2}}.$$

These formal computations are made precise in (3.29).

This can be done in arbitrary dimension $d \geq 1$. The $d$-dimensional harmonizable fractional operator is then

$$\tilde{\mathcal{I}}_H f(t) = \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)} \frac{e^{-it\xi} - 1}{||\xi||^{H+d/2}} \frac{d\xi}{(2\pi)^{d/2}},$$

and is defined on $L^2(\mathbb{R}^d)$. The moving-average fractional operator is obtained via Plancherel's formula

$$\tilde{\mathcal{I}}_H f(t) = C' \int_{\mathbb{R}^d} f(s)(||t-s||^{H-d/2} - ||s||^{H-d/2}) \frac{ds}{(2\pi)^{d/2}}.$$

See (3.27) for a probabilistic counterpart.

## 2.2.3 *Fractional Dimensions*

The aim of this section is to summarize basic notions of "fractal dimension", or "fractional dimension". All results are given without proofs, and we refer for instance to [54] for more details.

### 2.2.3.1 Box-Counting Dimension

If $F$ is a non-empty set of $d$-dimensional Euclidean space $\mathbb{R}^d$ the diameter of $F$ is given by $|F| = \sup\{\|x-y\|;\ x,\ y \in F\}$.

For $F$ a non-empty bounded set of $\mathbb{R}^d$, let $N_\varepsilon$ be the smallest number of sets of diameter $\varepsilon$ that can cover $F$. Basically, if this number $N_\varepsilon$ behaves, as $\varepsilon \to 0$, as a power $\varepsilon^{-H}$, the box-dimension of $F$ will be $H$. Let us be more precise. The lower and upper box-dimension of $F$ are defined by

$$\underline{\dim}_B F = \liminf_{\varepsilon \to 0^+} \frac{\log N_\varepsilon}{-\log \varepsilon},$$

and

$$\overline{\dim}_B F = \limsup_{\varepsilon \to 0^+} \frac{\log N_\varepsilon}{-\log \varepsilon}.$$

If these lower and upper box-counting dimensions are equal, we refer to the common value as the box-counting dimension of $F$

$$\dim_B F = \lim_{\varepsilon \to 0^+} \frac{\log N_\varepsilon}{-\log \varepsilon}.$$

Box-counting dimension, despite its simplicity, has several drawbacks:

- When the lower and upper box-counting dimensions are different, the box-counting dimension is not defined.

- When $F$ is unbounded, the box-counting dimension is infinite. For instance, the box-counting dimension of a straight line is infinite!
- A set $F$ and its closure $\overline{F}$ have the same box-counting dimension. It follows that the set of rational of [0, 1] has a higher box-counting dimension than the Cantor's set.
- The box-counting dimension does not generate a measure as the Hausdorff dimension that is defined in the next section.

#### 2.2.3.2 Hausdorff Measure and Dimension

Hausdorff dimension [68] has been introduced to avoid the drawbacks of box-counting dimension.[1] As opposed to box-counting dimension, we first define the Hausdorff measure and then the associated dimension.

If $\{U_i\}$ is a countable (or finite) collection of sets of diameter at most $\delta$ that cover $F$, i.e. $F \subset \bigcup_i^\infty U_i$ with $0 < |U_i| \le \delta$ for each $i$, we say that $\{U_i\}$ is a $\delta$-cover of $F$. Suppose $F$ is a subset of $\mathbb{R}^d$ and $s$ is a non-negative number. For any $\delta > 0$ we define

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_i^\infty |U_i|^s, \quad \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}. \tag{2.75}$$

As $\delta$ decreases, the class of permissible covers of $F$ is reduced, therefore $\mathcal{H}_\delta^s(F)$ increases as $\delta \to 0$. The following limit exists

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F) \tag{2.76}$$

and is called the $s$-dimensional Hausdorff measure of $F$. It can be shown that $\mathcal{H}^s$ is an outer measure on Borel sets. Hausdorff measures generalize Lebesgue measures: for integer $i$, $\mathcal{H}^i$ is, up to a constant, the usual Lebesgue measure.

Actually the limit in (2.76) is for all $s$, except eventually one value, null or infinite. First $s \to \mathcal{H}_\delta^s(F)$ is clearly a non increasing function, so is $s \to \mathcal{H}^s(F)$. Moreover if $t > s$ and $\{U_i\}$ is a $\delta$-cover of $F$ we have

$$\sum_i^\infty |U_i|^t \le \delta^{t-s} \sum_i^\infty |U_i|^s$$

so $\mathcal{H}_\delta^t(F) \le \delta^{t-s}\mathcal{H}_\delta^s(F)$. Letting $\delta \to 0$, we see that if $\mathcal{H}_\delta^s(F) < \infty$ then $\mathcal{H}_\delta^t(F) = 0$. The Hausdorff dimension of $F$ is the only possible $s$ where $s \to \mathcal{H}^s(F)$ jumps from $+\infty$ to 0, more precisely

$$\dim_H F = \inf\{s \text{ s. t. } \mathcal{H}^s(F) = 0\} = \sup\{s \text{ s. t. } \mathcal{H}^s(F) = \infty\}. \tag{2.77}$$

[1] An alternative dimension is the so-called packing dimension, or Tricot's dimension, cf. [136, 137].

#### 2.2.3.3 Some Techniques for Calculating Hausdorff Dimension

We first start with a basic inequality between dimensions for any bounded set $U$:

$$\dim_H U \leq \underline{\dim}_B U \leq \overline{\dim}_B U.$$

In this book we are mainly interested in the Hausdorff dimension of the graph of functions or random fields. In this case an upper bound of the Hausdorff dimension is usually obtained by knowing the Hölder continuity of the function. Let us state a precise result.

**Lemma 2.2.1** *For $0 < H < 1$ let $f$ be an $H$-Hölder continuous function (See Definition 2.2.1) Then* $\dim_H\{(t, f(t))\ 0 \leq t \leq 1\} \leq 2 - H$.

*Proof of Lemma 2.2.1*

The number of squares of the $1/p$-mesh that intersect the graph of $f$ is at most $p\left(\frac{Cp}{p^H} + 2\right)$, where $C$ is the constant in Definition 2.2.1. Let $F = \{(t, f(t))\ 0 \leq t \leq 1\}$,

$$\lim_{p\to+\infty} \mathcal{H}^s_{1/p}(F) = 0$$

when $s > 2 - H$. So if $s > 2 - H$, $\mathcal{H}^s(F) < \infty$ and

$$\dim_H\{(t, f(t))\ 0 \leq t \leq 1\} \leq 2 - H.$$

Actually the hard part to compute Hausdorff dimension is usually to get a lower bound. Sometimes we will use the following Lemma due to Frostmann [61].

**Lemma 2.2.2** *Let $F$ be a Borel set in $\mathbb{R}^d$. Let us define the $s$-energy of $F$ associated with a given probability measure $\mu$*

$$I_s(\mu) = \iint_{F\times F} \frac{d\mu(x)d\mu(y)}{|x-y|^s}.$$

*If there exists a probability measure $\mu$ on $F$ with*

$$I_s(\mu) < \infty$$

*then* $\dim_H F \geq s$. *If* $\mathcal{H}^s(F) > 0$ *then there exists a probability measure $\mu$ on $F$ with* $I_t(\mu) < \infty$ *for all $t < s$.*

We refer to Theorem 2.4.13 in [54] for the Proof of the Lemma 2.2.2.

**Remark 2.2.1** *Graph of a function.*

Let $F$ be the graph of a function $f$ on [0, 1]: $F = \{(t, f(t)),\quad t \in [0, 1]\}$. Then the $s$-energy of $F$ associated with a given probability measure $\mu$ is

$$I_s(\mu) = \iint_{[0,1]^2} \left( |x-y|^2 + |f(x) - f(y)|^2 \right)^{-s/2} d\mu(x) d\mu(y).$$

### 2.2.4 Lemarié-Meyer Basis

In this section we summarize some classical facts that will be used in the Sect. 3.2.2 concerning an orthonormal basis of $L^2(\mathbb{R})$, whose functions have a convenient property, called later in this section localization. This basis is actually an example of a multiresolution analysis and is related to wavelets. See [75] for a broad introduction to the topic. Since we will not use other tools from wavelets we restrict the presentation to Lemarié-Meyer basis.

There exists a function $\psi^{(1)} \in L^2(\mathbb{R})$ such that the support of $\widehat{\psi^{(1)}}$ is included in $\left\{\frac{2\pi}{3} < |\xi| < \frac{8\pi}{3}\right\}$ and such that the functions

$$\psi^{(1)}_{j,k}(x) = 2^{j/2} \psi^{(1)}(2^j x - k)$$

where $j \in \mathbb{Z},\ k \in \mathbb{Z}$, constitute an orthonormal basis of $L^2(\mathbb{R})$. See [105] for the proof and for a precise definition of $\psi^{(1)}$ that we don't need here. The set $\mathbb{Z} \times \mathbb{Z} \times \{1\}$ is denoted by $\Lambda$ and the notation $\lambda = (j,k,l)$ is used henceforth. With this notation a basis of $L^2(\mathbb{R})$ is $(\psi^{(1)}_{j,k})_{\lambda \in \Lambda}$. Each function in the basis is obtained by applying a dilation with scaling factor $2^j$ and translation of $k$ to $\psi^{(1)}$. If $j$ is negative and $|j|$ is large we say that $\psi^{(1)}_{j,k}$ corresponds to a low scale factor, where as when $j$ is positive and $|j|$ large, we speak of high scale factor. Actually one can deduce from this first basis, a second basis, where all the functions corresponding to non-positive $j$ are in some sense aggregated. Let us denote by $V_0$ the closed linear vector space spanned by $(\psi^{(1)}_{j,k})_{j \le 0, k \in \mathbb{Z}}$. There exists a function $\psi^{(0)}$ in $L^2(\mathbb{R})$ such that $(\psi^{(0)}(x-k))_{k \in \mathbb{Z}}$ is an orthonormal basis of the space $V_0$. We get another orthonormal basis of $L^2$ $(\psi^{(l)}_{j,k})_{\lambda \in \Lambda^+}$ where $\Lambda^+ = \mathbb{N}^* \times \mathbb{Z} \times \{1\} \bigcup \{0\} \times \mathbb{Z} \times \{0\}$ where $\psi^{(0)}_{0,k}(x) = \psi^{(0)}(x-k)$.

Let us now remind the localization property of the function in the Lemarié- Meyer basis. As a consequence of Fourier inverse transform we have

$$\psi^{(1)}(x) = \int_{\left\{\frac{2\pi}{3} < |\xi| < \frac{8\pi}{3}\right\}} e^{-ix\xi} \widehat{\psi^{(1)}}(\xi) \frac{d\xi}{(2\pi)^{1/2}}.$$

Hence $\psi^{(1)}$ is a $C^\infty$ function and, by integration by parts, one can show that for every $K \in \mathbb{N}$

$$|\psi^{(1)}(x)| \le \frac{C(K)}{1 + |x|^K}.$$

Hence $\psi^{(1)}$ belongs to the space $\mathcal{S}$ of fast decreasing functions. Moreover

$$|\psi_\lambda(x)| \leq \frac{C(K)2^{j/2}}{1+|2^j x - k|^K} \tag{2.78}$$

for every $K \in \mathbb{N}$. This last equation can be interpreted as $\psi_\lambda \in \mathcal{S}$ but we shall use the fact that the right hand side of (2.78) is maximum at the point $k/2^j$ as the localization of $\psi_\lambda$ in a neighborhood of $\lambda$ which is considered by an abuse of notation equal to $k/2^j$.

## 2.3 Exercises

### *2.3.1 Inequality for Anti-Correlated Gaussian Random Variables*

Let $(X, Y)$ be a centered Gaussian random vector in $\mathbb{R}^2$ such that $\mathbb{E}(XY) \leq 0$. Prove that

$$\mathbb{P}(X \geq a,\ Y \geq b) \leq \mathbb{P}(X \geq a)\mathbb{P}(Y \geq b)$$

where $a,\ b \geq 0$.

### *2.3.2 Tail of Standard Gaussian Random Variables*

1. Let $X$ be a standard Gaussian random variable. For every $r > 0$, show that

$$\sqrt{\frac{2}{\pi}}(r^{-1} - r^{-3})\exp(\frac{-r^2}{2}) \leq \mathbb{P}(|X| > r) \leq \sqrt{\frac{2}{\pi}} r^{-1}\exp(\frac{-r^2}{2}). \tag{2.79}$$

2. Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of centered Gaussian random variables such that

$$\mathbb{E}X_j^2 \leq 1.$$

Show that $\forall n \geq 2\ \forall\lambda > 1\ \exists\gamma > 0$ such that

$$\mathbb{P}(\sup_{1\leq j\leq n} |X_j| \geq \sqrt{2\lambda\log(n)}) \leq \gamma n^{1-\lambda}. \tag{2.80}$$

3. Let $(X_j)_{j\in\mathbb{N}}$ be as in the previous question. Show that there exists an almost surely positive random $C(\omega)$ variable such that

$$\sup_{1\leq j\leq n} |X_j(\omega)| \leq C(\omega)\sqrt{\log(n)}\ \forall n \geq 2 \quad \text{a.s.} \tag{2.81}$$

4. Let $(\eta_{j,k})_{(j,k)\in\mathbb{N}^*\times\mathbb{Z}}$ be a sequence of Gaussian random variables such that $\mathbb{E}\eta_{j,k} = 0$ and $\mathbb{E}\eta_{j,k}^2 \leq 1$. Show that there exists a positive random variable

$C$, which is almost surely finite, such that

$$|\eta_{j,k}| \leq C(\log(j+1)^{1/2} + \log(|k|+1)^{1/2}) \quad \text{a.s.} \tag{2.82}$$

### 2.3.3 Conditional Independence

Let $K$, $H_i$ for $i \in I$ be closed vector spaces included in a Gaussian space $H$ of centered random variables such that $K \subset H_i$ for $i \in I$. The sigma fields $\sigma(H_i)$ are said to be conditionally independent with respect to $\sigma(K)$ if

$$\mathbb{E}(Y_i\overline{Y_j}|\sigma(K)) = \mathbb{E}(Y_i|\sigma(K))\overline{\mathbb{E}(Y_j|\sigma(K))} \tag{2.83}$$

for every $i \neq j \in I$ and $\sigma(H_i)$-random variables $Y_i$ that are complex valued. Let us say that $H_i$ for $i \in I$ are orthogonally secant with respect to $K$ if

$$\mathbb{E}(X_i\overline{X_j}) = \mathbb{E}(P_K(X_i)\overline{P_K(X_j)}) \tag{2.84}$$

for every $i \neq j \in I$ where $X_i$ is in $H_i$ and $P_K$ denotes the orthogonal projection onto $K$.

1. Show that if $K = \{0\}$, conditional independence of $\sigma(H_i)$ with respect to $\sigma(K)$ is equivalent to $H_i$ for $i \in I$ orthogonally secant with respect to $K$. In this case conditional independence is nothing but independence of the sigma fields $\sigma(H_i)$.
2. Prove that (2.83) implies (2.84) when $K$ is any closed vector space included in $H_i$.
3. Show that if $H_i$ for $i \in I$ are orthogonally secant with respect to $K$ then $\sigma(H_i)$ are conditionally independent with respect to $\sigma(K)$.

The results of this exercise will be used in the next chapter see exercise 3.6.3.

### 2.3.4 Some Properties of Covariance Functions

Let $R$ and $Q$ be two covariance functions defined on the same index set $T$. Show that

1. $\lambda R$ is a covariance function for all $\lambda \geq 0$.
2. $R + Q$ is a covariance function.
3. $RQ$ is a covariance function.

### 2.3.5 Examples and Counter-Examples of Covariance Functions

1. Prove that the following functions $t \to r(t)$ are covariance functions

   a. $r(t) = (1 - |t|^\alpha)_+, \ t \in \mathbb{R}, \ 0 < \alpha \le 1$. (One can show first that $r$ is a characteristic function. Let us recall Pólya's criterion: $r$ is a characteristic function if $r$ is real valued, non-negative, $r(0) = 1$, $r(-t) = r(t)$ and convex.)
   b. $r(t) = e^{-|t|^\alpha} \cos(t)$, $0 < \alpha \le 2$.
   c. $r(t) = 1/(1+t^2)^n, \ n \in \mathbb{N}^\star$.
   d. $r(t) = K_\nu(t)$ where $K_\nu$ is the modified Bessel function of the second kind. Let us recall

$$K_\nu(t) = \frac{\Gamma(\nu + 1/2)2^\nu}{\sqrt{\pi}} \int_0^{+\infty} \frac{\cos(ut)}{(1+u^2)^{\nu+1/2}} du.$$

2. Let $f(z) = \sum_{n \ge 0} f_n z^n$ be an analytic function such that $f_n \ge 0$. Let $r(t)$ be a covariance function. Shows that $f(r(t))$ is a covariance function.
3. Is always the function $(t, s) \to R(t, s) = |t|^{2H(t)} + |s|^{2H(s)} - |t - s|^{H(t)+H(s)}$ a covariance function?

### 2.3.6 Covariance Functions on the Sphere

Let $S^2 = \{t, \ ||t||_{\mathbb{R}^3} = 1\}$ be the unit sphere of $\mathbb{R}^3$ and let $d$ be the geodesic distance on $S^2$, which can be defined by

$$\cos(d(t, t')) = t.t'$$

where $t, \ t' \in S^2 \subset \mathbb{R}^3$ and where the inner product is the Euclidean scalar product in $\mathbb{R}^3$. Fix $O$ on $\mathbb{S}^2$ and define, for $t, s \in S^2$,

$$R_H(t, s) = 1/2(d^{2H}(0, t) + d^{2H}(0, s) - d^{2H}(t, s)).$$

1. Show that $R_{1/2}$ is a covariance function. One can use the following property of the distance on the sphere. Let $H_t$ be the half-sphere $H_t = \{s \in S^2, \ d(t, s) \le \pi/2\}$. Let $ds$ be the uniform measure on $\mathbb{S}_2$ and let $d\mu(s) = 1/4ds$. Then

$$d(t, s) = \mu(H_t \Delta H_s),$$

   where $\Delta$ denotes the symmetric difference of sets.
2. Show that $R_H$ is a covariance function for $0 \le H \le 1/2$.
3. Show that $R_H$ is not a covariance function when $H > 1/2$.

4. Show that $(t, s) \mapsto \exp(-d^{2H}(t, s))$ is a covariance function iff $0 < H \leq 1/2$.
5. Show that $(t, s) \mapsto 1 - 1/2\|t - s\|^{2H}$ is a covariance function iff $0 < H \leq 1$.

### 2.3.7 Gaussian Bridges

Let $X(t), t \in [0, 1]$ be a centered Gaussian continuous process with covariance function $R(t, s)$ satisfying $R(1, 1) = 1$. The Gaussian bridge $B_X(t), t \in [0, 1]$ is by definition the stochastic process $X$ conditioned by $X(1) = 0$. Show that:

$$B_X(t) = X(t) - R(1, t)X(1).$$

### 2.3.8 Version Versus Modification

Show that if $Y$ is a modification of a field $X$ then $Y$ and $X$ are two versions of the same field.

### 2.3.9 Sum of Fields with Stationary Increments

Show that the sum of two centered Gaussian fields vanishing at $0 \in \mathbb{R}^d$ with stationary increments is with stationary increments if the fields are independent.

### 2.3.10 Equivalence of the Distributions of Gaussian Processes

Let $(X_t)_{t\in[0,1]}$ be a continuous centered Gaussian process and let $K_X$ be its reproducing kernel Hilbert space. Let $m$ be a deterministic function. Show that the measures associated with the processes $X$ and $X + m$ are equivalent iff $m \in K_X$.

# Chapter 3
# Self-Similarity

## 3.1 Introduction

Self-similarity is a major part of the mathematics. One can refer to [53] for a general reference. The self-similarity literature is quite confusing for beginners since the statement of very elementary facts may look very similar to deep theorems. On the one hand if you assume that you observe a self-similar phenomenon, then the self-similarity is an invariance property and you expect your phenomenon to be easier to study than general phenomena with no structure. On the other hand if you want to have a complete classification of self-similar fields then we can find in the literature a lot of counter-examples that prevent to draw even an heuristic picture of what is true for every self-similar fields. Following [138] a classical tool to simplify the study of the self-similarity in the stochastic case is to assume that the fields have stationary increments.

In the first section of this chapter we consider objects parameterized by real numbers: self-similarity in dimension 1. In the deterministic case self-similar functions are homogeneous functions. Homogeneous functions are therefore very useful in the study of processes which are statistically self-similar. The brief Sect. 3.2.1 states some classical facts related to the Fourier transform of the homogeneous functions. Then the self-similarity for processes i.e. fields parameterized by $d = 1$ dimensional spaces is introduced in the Sect. 3.2.2. This quite long and important section is also devoted to the celebrated fractional Brownian motions introduced in [83] and in [101], which are the only Gaussian processes which have stationary increments and are self-similar. To finish with the one dimensional case we relax the definition of the self-similarity to get another invariance property called here semi self-similarity. In the paragraph 3.2.3 Weierstrass functions both deterministic and randomized are introduced and we recall that asymptotically random Weiertstrass processes converge to fractional Brownian motion.

The self-similarity property for fields (when the space of parameters is $d$ dimensional with $d > 1$) is still more involved. In this case the stationarity of increments

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may take various forms, and it yields different models that are described in the Sect. 3.3.

In the Sect. 3.4 some non-Gaussian fields are introduced. Again we did not try to be exhaustive in this part. We chose stable fractional fields because they are limits of fractional fields that appear in the next chapter.

At last, since we expect fractional fields to be Hölder continuous, we conclude this chapter by a section where the relationship between the self-similarity and the regularity of sample paths is studied.

## 3.2 Self-Similarity and Fractional Brownian Motion

### 3.2.1 Deterministic Case

If we come back to the heuristic we gave in the introduction a self-similar function should be similar to each part of itself. So at each scale $\varepsilon > 0$ $x \mapsto f(\varepsilon x)$ should be similar to $x \mapsto f(x)$ itself. Let us be a bit formal for a while. If we denote the similarity operator by $T_\varepsilon$ then

$$f(\varepsilon x) = T_\varepsilon(f)(x) \quad \forall x \in \mathbb{R}^d. \tag{3.1}$$

One consequence of (3.1) is that $T_\varepsilon$ is a semigroup in $\varepsilon > 0$ i.e.

$$T_{\varepsilon\varepsilon'} = T_\varepsilon \circ T_{\varepsilon'}$$

for every $\varepsilon,\ \varepsilon' > 0$. If one assumes moreover that $\varepsilon \mapsto T_\varepsilon$ is smooth then it turns out that formally $T_\varepsilon(f) = \varepsilon^\alpha f$. Hence a function $f : \mathbb{R}^d \to \mathbb{R}$ is self-similar if for every $\varepsilon > 0$

$$f(\varepsilon x) = \varepsilon^\alpha f(x). \tag{3.2}$$

Obviously self-similar functions are nothing but homogeneous functions. Then if the dimension is 1 and if the function is assumed to be symmetric

$$f(x) = C|x|^\alpha.$$

But even if $d > 1$ it is classical to reduce a multivariate model to one dimensional one by assuming additional invariance. In the setting of homogeneous function it is usual to assume rotational invariance and then the only self-similar function which is rotationally invariant is

$$f(x) = C\|x\|^\alpha$$

where $\|x\|$ stands for the Euclidean norm in $\mathbb{R}^d$. The next step is to define stochastic self-similarity which is the main aim of this chapter. In the stochastic setting a spectral analysis is often carried to study self-similar fields. Returning to the deterministic case the spectral analysis consists in the computation of the Fourier transform of homogeneous functions. It is a classical result that Fourier transform of homogeneous "function" of order $\alpha$ are homogeneous "function" of order $-d-\alpha$. We recall that the Fourier transform of a function is defined in this book by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} \exp(ix.\xi) f(x) \frac{dx}{(2\pi)^{d/2}}.$$

It is a consequence of the formal computation

$$\int_{\mathbb{R}} \exp(ix.\varepsilon\xi) f(x) \frac{dx}{(2\pi)^{d/2}} = \varepsilon^{-d-\alpha} \widehat{f}(\xi) \tag{3.3}$$

given by the change of variable $x' = \varepsilon x$. Actually the computation above is formal since the integrals are not defined for $f(x) = \|x\|^\alpha$. To give a rigorous mathematical sense to the previous computation one can use Fourier transform for distributions and pseudo-functions and we refer the reader to [128] (VII,7) example 5 for precise definitions.

Since harmonic analysis of homogeneous functions is a very important tool for the study of self-similar processes with stationary increments one will present another way to make rigorous (3.3) for $0 < \alpha < 1$. With the same change of variable as in (3.3) we get

$$\int_{\mathbb{R}^d} \frac{e^{ix.\xi} - 1}{\|\xi\|^{d+\alpha}} \frac{d\xi}{(2\pi)^{d/2}} = C_\alpha \|x\|^\alpha \tag{3.4}$$

where $C_\alpha$ is a positive constant. Since homogeneous functions in dimension 1 is an uninteresting model for modeling, we will relax the condition of self-similarity in two directions in the following sections. First we present the stochastic self-similarity, where $f$ is a random field, and where the equality in the counterpart of (3.2) is only equality of distributions. Let us stress a little bit more this point that may lead to some confusion. Usually the sample paths of self-similar stochastic fields are not homogeneous functions in the sense of (3.2). Another way to relax (3.2) is to require the equality not for all $\varepsilon > 0$ but only for $\varepsilon$ in a discrete subgroup of $\mathbb{R}^+$, it is what is done usually for fractals in the deterministic setting. We will call this property semi-self similarity.

## 3.2.2 Fractional Brownian Motion

### 3.2.2.1 Covariance Structure of Fractional Brownian Motion

A $H$-self-similar field satisfies

$$(X(\varepsilon t))_{t\in\mathbb{R}} \stackrel{(d)}{=} \varepsilon^H (X(t))_{t\in\mathbb{R}} \tag{3.5}$$

for every $\varepsilon > 0$ where we recall (Definition 2.1.2) that $\overset{(d)}{=}$ means that for every positive $\varepsilon$ and for every positive integer $n$ and $(t_1, \ldots, t_n) \in \mathbb{R}^n$

$$(X(\varepsilon t_1), \ldots, X(\varepsilon t_n)) \overset{(d)}{=} \varepsilon^H (X(t_1), \ldots, X(t_n)).$$

In the last section we have insisted on the fact that a stationary hypothesis was needed to study such fields, but the naive assumption of stationary fields

$$X(x) \overset{(d)}{=} X(y). \tag{3.6}$$

for every $x$ and $y$ in $\mathbb{R}^d$, is incompatible with (3.5) unless $H = 1$ in which case it is the same property.

Actually the fields are supposed to have stationary increments. Let us recall Definition 2.1.24, when $d = 1$

$$(X(s+\delta) - X(t+\delta))_{s\in\mathbb{R}} \overset{(d)}{=} (X(s) - X(t))_{s\in\mathbb{R}}. \tag{3.7}$$

for every $t$ and $\delta$ in $\mathbb{R}$. Beside the tool of spectral analysis given by this assumption (c.f. the first chapter Sect. 2.1.11) it yields an uniqueness result for self-similar processes with stationary increments.

**Proposition 3.2.1** *Let $(X(t))_{t\in\mathbb{R}}$ be a non-constant real valued $H$-self-similar process with stationary increments. Let us also assume that $\mathbb{E}X(t)^2 < \infty$ for every $t$ and that $\lim_{\varepsilon\to 0} X(\varepsilon) \overset{(d)}{=} X(0)$. Then $0 \le H \le 1$, and $X(0) = 0$ a.s.. Moreover the covariance of the second order process $X$ is determined up to a multiplicative constant:*

$$R(s,t) = \mathbb{E}(X(s)X(t)) = \frac{\mathbb{E}(X(1)^2)}{2}\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\}. \tag{3.8}$$

*If $0 < H < 1$, $\mathbb{E}X(t) = 0$ for all $t$, and if $H = 1$ $X(t) = tX(1)$.*

*Proof of Proposition 3.2.1*:

The self-similarity yields $X(0) \overset{(d)}{=} \varepsilon^H X(0)$ and we get $X(0) \overset{(d)}{=} 0$ which is the same as $X(0) \overset{(a.s.)}{=} 0$. Then the assumption $\lim_{\varepsilon\to 0} X(\varepsilon) \overset{(d)}{=} X(0)$ is one way to deduce $0 \le H$. Since the increments of $X$ are stationary

$$X(1) - X(0) \overset{(d)}{=} X(0) - X(-1)$$

which yields $X(1) \overset{(d)}{=} -X(-1)$. Moreover

$$\mathbb{E}(|X(2)|) \le \mathbb{E}(|X(2) - X(1)|) + \mathbb{E}(|X(1)|),$$

which yields

$$2^H \mathbb{E}(|X(1)|) \leq 2\mathbb{E}(|X(1)|)$$

by using self-similarity and stationarity of the increments. Please note that the expectations considered above are finite since $X$ is a second order process. Let us prove that $\mathbb{E}(|X(1)|) \neq 0$ so that the last inequality implies $H \leq 1$. Let us suppose that $X(1) \overset{(a.s.)}{=} 0$ then $X(-1) \overset{(a.s.)}{=} 0$ and by self-similarity $X$ is identically vanishing. Let us now compute the covariance of $X$

$$\begin{aligned}\mathbb{E}(X(s)X(t)) &= \frac{1}{2}\{\mathbb{E}(X(s)^2) + \mathbb{E}(X(t)^2) - \mathbb{E}((X(s) - X(t))^2)\} \\ &= \frac{\mathbb{E}(X(1)^2)}{2}\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\}.\end{aligned}$$

Suppose that $0 < H < 1$

$$\mathbb{E}(X(1)) = \mathbb{E}(X(2) - X(1)) = (2^H - 1)\mathbb{E}(X(1))$$

which yields that $\mathbb{E}(X(1)) = 0$ and by self-similarity $X$ is a centered process and (3.8) is proved. If $H = 1$ $\mathbb{E}(X(s)X(t)) = st$ and $\mathbb{E}((X(t) - tX(1))^2) = 0$ which shows the last claim of the proposition.

Since the covariance characterizes the law of centered Gaussian processes the last proposition shows that up to multiplicative constant fractional Brownian motion is the only self-similar process with stationary increments. Let us recall this celebrated fact in the following corollary.

**Corollary 3.2.1** *Let $0 < H < 1$ and $V > 0$. There exists only one Gaussian process $(B_H(t))_{t\in\mathbb{R}}$ which is $H$-self-similar with stationary increments and such that $var(B_H(1)) = V$. Fractional Brownian motion is a centered Gaussian process and its covariance is given by:*

$$R(s,t) = \frac{V}{2}\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\}. \tag{3.9}$$

*The spectral measure of $R$ in the sense of (2.62) is $\frac{V d\xi}{C_H |\xi|^{2H+1}(2\pi)^{1/2}}$, where $C_H = \frac{\sqrt{\pi}}{H\Gamma(2H)\sin(H\pi)\sqrt{2}}$. Moreover*

$$\mathbb{E}(B_H(t) - B_H(s))^2 = V|t-s|^{2H}. \tag{3.10}$$

*A fractional Brownian motion is called a standard fractional Brownian motion if $V = 1$.*

*Proof of Corollary 3.2.1*:

The proof of uniqueness is a simple consequence of the preceding proposition. To show that there exists a Gaussian process with such a covariance one has to show that $R$ is non-negative definite (Definition 2.1.9) in the sense that for every $n \in \mathbb{N}^*$, every $(s_1, \ldots, s_n) \in \mathbb{R}^n$, and every $(a_1, \ldots, a_n) \in \mathbb{C}^n$

$$\sum_{j,k=1..n} a_j \overline{a_k} R(s_j, s_k) \geq 0. \tag{3.11}$$

We have proved (cf. Corollary 2.1.1) that $R$ is of non-negative type using Schoenberg's theorem. Let us give another proof. One can prove as in (3.4) that

$$C_H |s|^{2H} = \int_{\mathbb{R}} \frac{|e^{is.\xi} - 1|^2}{|\xi|^{2H+1}} \frac{d\xi}{(2\pi)^{1/2}}$$

with

$$C_H = \int_{\mathbb{R}} \frac{2(1-\cos(\xi))}{|\xi|^{2H+1}} \frac{d\xi}{(2\pi)^{1/2}} \tag{3.12}$$

$$= \frac{\sqrt{\pi}}{H\Gamma(2H)\sin(H\pi)\sqrt{2}}. \tag{3.13}$$

This formula for $C_H$ has already been used in (2.39). Then, since

$$R(s,t) = \frac{1}{2}\{R(s,s) + R(t,t) - R(s-t, s-t)\}$$

we get

$$R(s,t) = \int_{\mathbb{R}} \frac{(e^{is.\xi} - 1)(e^{-it.\xi} - 1)}{C_H |\xi|^{2H+1}} \frac{V d\xi}{(2\pi)^{1/2}}. \tag{3.14}$$

Please note that this equation shows that

$$\frac{V d\xi}{C_H |\xi|^{2H+1} (2\pi)^{1/2}} \tag{3.15}$$

is the spectral measure of the process with stationary increments $B_H$ in the sense defined in Sect. 2.1.11. Then

$$\sum_{j,k=1...n} a_j \overline{a_k} R(s_j, s_k) = \int_{\mathbb{R}} \left| \sum_{j=1}^{n} a_j \frac{e^{is_j.\xi} - 1}{|\xi|^{1/2+H}} \right|^2 \frac{V d\xi}{C_H (2\pi)^{1/2}} \geq 0$$

This last fact concludes the proof.

Let us remark that if $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$ and $s_j \neq s_k$ for $j \neq k$ then

$$\sum_{j,k=1...n} a_j \overline{a_k} R(s_j, s_k) > 0$$

since the functions $\frac{e^{is_j.\xi}-1}{|\xi|^{H+1/2}}$ are linearly independent in $L^2(\mathbb{R})$. It has the following consequences for the finite dimensional margins of fractional Brownian motion.

**Corollary 3.2.2** *If $n \in \mathbb{N}^*$ and $(s_1, \ldots, s_n) \in \mathbb{R}^n$ are such that $s_j \neq s_k$ for $j \neq k$ then the matrix $(R(s_j, s_k))$ is a $n$ by $n$ symmetric matrix which is positive definite and almost surely $(B_H(s_1), \ldots, B_H(s_n))$ are linearly independent.*

*Proof of Corollary 3.2.2*

The last claim is the consequence of the fact that the Gaussian vector $(B_H(s_1), \ldots, B_H(s_n))$ has a density with respect of the Lebesgue measure in $\mathbb{R}^n$ and that proper vector spaces included in $\mathbb{R}^n$ have vanishing Lebesgue measures.

The end of this section is devoted to recall some properties of fractional Brownian motion.

#### 3.2.2.2 Negative Properties of $B_H$ for $H \neq 1/2$

The first remark is that $H = \frac{1}{2}$ is an exceptional case for fractional Brownian motion. Actually it is obvious from (3.10) that the restriction on $\{t \geq 0\}$ of $B_{1/2}$ is a Brownian Motion. See Definition 2.1.6. Hence, when $H = \frac{1}{2}$, fractional Brownian motion has independent increments and is a semi-martingale. See for instance [116] for the definition of semi-martingales and the proof that Brownian motion is a semi-martingale. However it is a very bad guideline to fractional Brownian motions when $H \neq \frac{1}{2}$ for which these properties are not true. For instance it is proved in [119] that when $H \neq \frac{1}{2}$ fractional Brownian motion is not a semi-martingale. One can outline the main reason. Let us first recall the definition of quadratic variations on $[0, 1]$ of a process $X$ at scale $1/N$

$$V_N = \sum_{k=0}^{N-1} \left( X\left(\frac{k+1}{N}\right) - X\left(\frac{k}{N}\right)\right)^2. \tag{3.16}$$

For semi-martingales the limit of $V_N$ exists when $N \to \infty$ and is positive if the semi-martingale has not finite variations on compact sets. Since (3.10), the expectation of the quadratic variations of fractional Brownian motion converges to a non-vanishing real number only when $H = \frac{1}{2}$. Actually this fact prevents us from using the classical stochastic integration theory for semi-martingales to study fractional Brownian motion. Although some stochastic calculus have been recently developed for processes including fractional Brownian motion we will not consider this calculus in this book.

The covariance structure of fractional Brownian motion implies that the increments of fractional Brownian motion are independent only when $H = \frac{1}{2}$. Moreover, when $H \neq \frac{1}{2}$ fractional Brownian motion is not even a Markov process, it can be deduce from the covariance. See Exercise 3.6.3. It has important consequences on the structure of the Reproducing Kernel Hilbert Space (RKHS) of fractional Brownian motion.

#### 3.2.2.3 Reproducing Kernel Hilbert Space of fractional Brownian motion

Let us apply the result of the Sect. 2.1.4 to have a convenient description of the RKHS for fractional Brownian motion.

**Proposition 3.2.2** *Let us define a one to one correspondence $\mathcal{I}_H$ from $L^2(\mathbb{R})$ onto the RKHS of fractional Brownian motion denoted by $K_{B_H}$ as*

$$\mathcal{I}_H(\psi)(y) = \int_{\mathbb{R}} \frac{e^{-iy\xi}-1}{C_H^{1/2}|\xi|^{H+1/2}} \overline{\widehat{\psi}(\xi)} \frac{d\xi}{(2\pi)^{1/2}} \tag{3.17}$$

*for every $\psi \in L^2(\mathbb{R})$, where $C_H$ is defined in (3.12). The RKHS of fractional Brownian motion can be written as*

$$K_{B_H} = \{\phi \text{ s.t. } \exists\psi \in L^2(\mathbb{R}) \text{ such that } \phi = \mathcal{I}_H(\psi)\}. \tag{3.18}$$

*Moreover $\mathcal{I}_H$ is an isometry*

$$\langle \mathcal{I}_H(\psi_1), \mathcal{I}_{\mathcal{H}}(\psi_2)\rangle_{K_{B_H}} = \langle \psi_1, \psi_2\rangle_{L^2}. \tag{3.19}$$

*Proof of Proposition 3.2.2*

Let us start from the integral representation of the covariance of the standard fractional Brownian motion (3.14) which can be viewed as a scalar product in $L^2(\mathbb{R})$

$$R(s,t) = \langle k_s, k_t\rangle_{L^2} = \langle \widehat{k_s}, \widehat{k_t}\rangle_{L^2}$$

where

$$k_s(\xi) = \frac{\widehat{e^{-is\xi}-1}}{C_H^{1/2}|\xi|^{H+1/2}}.$$

Please note that the choice of the function $k_s$ or of its Fourier transform is arbitrary because of the Parseval identity. One can check easily that

$$\mathcal{I}_H(k_s)(t) = R(s,t).$$

Hence $\mathcal{I}_H(k_s)(.) = R(s,.)$ and

$$\langle \mathcal{I}_H(k_s), \mathcal{I}_{\mathcal{H}}(k_t)\rangle_{K_{B_H}} = \langle k_s, k_t\rangle_{L^2}.$$

The previous equation can be extended to $K_{B_H}$ first by linearity then by taking the closure. Hence $\mathcal{I}_H$ is an isometry from the space $\overline{\langle k_s,\ s \in \mathbb{R}\rangle}^{L^2}$ onto $K_{B_H}$. Then the proof will be complete if $\overline{\langle k_s,\ s \in \mathbb{R}\rangle}^{L^2} = L^2$. Actually it is equivalent to show that

$\overline{\langle \frac{e^{-is\xi}-1}{C_H^{1/2}|\xi|^{H+1/2}},\ s \in \mathbb{R}\rangle}^{L^2} = L^2$ because the Fourier transform is itself an isometry. Since $C^\infty$ functions with compact support in $\mathbb{R} \setminus \{0\}$ are dense in $L^2$ it is enough to show that if $\phi$ is a $C^\infty$ function with compact support in $\mathbb{R} \setminus \{0\}$ and if for all $s \in \mathbb{R}$

$$\int_{\mathbb{R}} \frac{e^{-is\xi}-1}{C_H^{1/2}|\xi|^{H+1/2}} \overline{\phi(\xi)} \frac{d\xi}{(2\pi)^{1/2}} = 0 \tag{3.20}$$

then $\phi = 0$. Since the support of $\phi$ does not include 0 the following integral is defined

$$\int_{\mathbb{R}} \frac{e^{-is\xi}}{C_H^{1/2}|\xi|^{H+1/2}} \overline{\phi(\xi)} \frac{d\xi}{(2\pi)^{1/2}}.$$

And because of (3.20) it does not depend on $s$. Moreover $\frac{\phi(\xi)}{C_H^{1/2}|\xi|^{H+1/2}}$ is in $L^1 \bigcap L^2$ and so is its Fourier transform which is constant and consequently vanishing. This implies $\frac{\phi(\xi)}{C_H^{1/2}|\xi|^{H+1/2}} = 0$ for every $\xi$ and then $\phi = 0$.

Before using this result we feel like commenting the meaning of the operator $\mathcal{I}_H$. It can be thought of like some kind of fractional integration as seen in Sect. 2.2.2.2. To be more precise let us consider the particular case $H = 1/2$. Then it is well known that for $f$ such that $\int_{\mathbb{R}}(1+|\xi|^2) f(\xi)^2 d\xi < \infty$,

$$\widehat{f'}(\xi) = (i\xi)\widehat{f}(\xi).$$

Hence the integration of functions $\psi \in L^2$ such that $\frac{\widehat{\psi}(\xi)}{-i\xi} \in L^2$ can be given by some inverse Fourier transform

$$\int_{\mathbb{R}} \frac{e^{-iy\xi}}{(i\xi)} \overline{\widehat{\psi}(\xi)} \frac{d\xi}{(2\pi)^{1/2}}.$$

Let us consider the primitive that vanishes at $y = 0$

$$\tilde{\mathcal{I}}_{1/2}(y) = \int_{\mathbb{R}} \frac{e^{-iy\xi}-1}{(i\xi)} \overline{\widehat{\psi}(\xi)} \frac{d\xi}{(2\pi)^{1/2}}.$$

When we compare $\tilde{\mathcal{I}}_{1/2}$ with $\mathcal{I}_H$ when $H = 1/2$ we remark that the power of the denominator $i\xi$ is given by the general formula $H + 1/2$. But in $\mathcal{I}_H$ a module was considered instead. If we concentrate on the power of this denominator it is clear that formally $\mathcal{I}_H$ is a fractional integration of order $H$ since we divide the function by a polynomial of fractional degree $H$.

#### 3.2.2.4 Series and Integral Representation of Fractional Brownian Motion

As explained in the Sect. 2.1.4 one needs an orthonormal basis of the RKHS of fractional Brownian motion to have a series expansion of this process. Since $\mathcal{I}_H$ is an isometry between $K_{B_H}$ and $L^2$ the problem is reduced to choosing an orthonormal basis for $L^2$. Any $L^2$ basis is convenient at this level of generality but since we are preparing the tool to study regularity of sample paths in Chap. 4 we need a localization property for the corresponding functions in the basis of the RKHS. In Chap. 2, Sect. 2.2.4 we have recalled the construction of the Lemarié-Meyer basis that will be used here. Actually this choice is convenient since the localization property can be carried on the orthonormal basis of the RKHS of fractional Brownian motion. But we will prove this fact only in the Chap. 4 in Sect. 4.3.2.

Now we have all the technical tools to present a series expansion of fractional Brownian motion which can also be viewed as the harmonizable representation of fractional Brownian motion.

**Theorem 3.2.1** *Let us denote by $\varphi_\lambda = \mathcal{I}_H(\psi_\lambda)$ for $\lambda \in \Lambda^+$ an orthonormal basis of the RKHS $K_{B_H}$ and let $(\eta_\lambda)_{\lambda\in\Lambda^+}$ be the corresponding sequence of i.i.d. standard Gaussian variables, one gets the following series representation of fractional Brownian motion*

$$B_H(t) = \sum_{\lambda\in\Lambda^+} \varphi_\lambda(t)\eta_\lambda. \tag{3.21}$$

*Please note that the convergence is both in $L^2$ and in almost sure sense for the uniform convergence on compact interval. Moreover if one considers the Brownian random measure (c.f. Sect. 2.1.6.2) associated to the Fourier transform of the Lemarié Meyer basis*

$$\widehat{W^+}(d\xi) = \sum_{\lambda\in\Lambda^+} \overline{\widehat{\psi_\lambda}(\xi)}\eta_\lambda d\xi \tag{3.22}$$

*one gets the harmonizable representation of fractional Brownian motion:*

$$B_H(t) = \int_{\mathbb{R}} \frac{e^{-it\xi}-1}{C_H^{1/2}|\xi|^{H+1/2}}\widehat{W^+}(d\xi) \tag{3.23}$$

*where $C_H$ is defined in (3.12).*

**Remark 3.2.1** *There exists an alternative orthonormal basis of $L^2(\mathbb{R})$ indexed by $\Lambda$, and an alternative series representation of fractional Brownian motion is deduced from this basis:*

$$B_H(t) = \sum_{\lambda\in\Lambda} \varphi_\lambda(t)\eta_\lambda. \tag{3.24}$$

*One can compare (3.21) and (3.24) by remarking that the $\lambda = (j, k, l)$ for which $j \le 0$ are synthesized in the second expansion by the coefficients where $j = 0$.*

*In the same vein we have an alternative harmonizable representation of fractional Brownian motion*

$$B_H(t) = \int_{\mathbb{R}} \frac{e^{-it\xi} - 1}{C_H^{1/2}|\xi|^{H+1/2}} \widehat{W}(d\xi) \tag{3.25}$$

*where*

$$\widehat{W}(d\xi) = \sum_{\lambda \in \Lambda} \overline{\widehat{\psi}_\lambda(\xi)} \eta_\lambda d\xi. \tag{3.26}$$

**Remark 3.2.2** *Since $B_H$ is a real valued process, one get equivalent representations*

$$B_H(t) = \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{C_H^{1/2}|\xi|^{H+1/2}} W(d\xi)$$

*where $W(d\xi)$ is defined by*

$$W(d\xi) = \sum_{\lambda \in \Lambda^+} \widehat{\psi}_\lambda(\xi) \eta_\lambda d\xi.$$

*Indeed it is a consequence of*

$$\int_{\mathbb{R}} \frac{e^{it\xi} - 1}{C_H^{1/2}|\xi|^{H+1/2}} \widehat{\psi}_\lambda(\xi) d\xi = \int_{\mathbb{R}} \frac{e^{-it\xi} - 1}{C_H^{1/2}|\xi|^{H+1/2}} \overline{\widehat{\psi}_\lambda(\xi)} d\xi.$$

*A similar remark is true for representation (3.25). In this book we will try to keep harmonizable representations consistent with the choice made for (3.23).*

*Proof of Theorem 3.2.1*

Both series expansions (3.21) and (3.24) are convergent in $L^2$ sense, it is a straightforward consequence of Theorem 2.1.5 and of the description of the RKHS of fractional Brownian motion given in Proposition 3.2.2. The harmonizable representations of fractional Brownian motion are a consequence of the Definition 2.1.16 of the Brownian random measures. To prove the almost convergence of (3.21) and of (3.24), let us first announce a regularity result for the sample paths of fractional Brownian motion which will be proved in Theorem 3.2.3: Almost surely fractional Brownian motion belongs to the Banach space of continuous functions on a compact interval endowed with the supremum norm. A classical fact for Gaussian processes (See e.g. [94]) shows that whenever a Gaussian process belongs to a Banach space almost surely the norm of the Gaussian process has a finite expectation. Hence

$$\mathbb{E}(\|B_H\|_\infty) < \infty,$$

where $\|f\|_\infty = \sup_{x \in I} |f(x)|$ and where $I$ is a compact interval. Since $\Lambda$ and $\Lambda^+$ are countable, the series can be rewritten

$$B_H = \sum_{n \in \mathbb{N}} \varphi_n \eta_n,$$

and the partial sums

$$B_{H,N} = \sum_{n \leq N} \varphi_n \eta_n,$$

are conditional expectations of $B_H$

$$B_{H,N} = \mathbb{E}(B_H | \mathcal{F}_N),$$

where $\mathcal{F}_N = \sigma(\eta_n$ such that $n \leq N)$. Hence $B_{H,N}$ converges almost surely to $B_H$ by applying a theorem of convergence of martingale in Banach spaces (see [110] p. 104 Proposition V-2-6).

**Remark 3.2.3** *In the previous theorem we refer to* $\|B_H\|_\infty = \sup_{x \in I} |B_H(x)|$ *for* $I$ *a compact interval. The distribution of* $\|B_{\frac{1}{2}}\|_\infty$ *is well known. See for instance [28] where it is shown that*

$$\mathbb{P}(\|B_{\frac{1}{2}}\|_\infty \leq \alpha) = \frac{2}{\sqrt{2\pi}} \int_0^\alpha e^{\frac{-u^2}{2}} du,$$

*when* $I = [0, 1]$. *As far as we know, when* $H \neq \frac{1}{2}$ *it is still an open problem to have a closed form formula for this distribution.*

In the previous theorem an integral representation of fractional Brownian motion is given, but there exists other representations. In the following theorem we introduce the so-called moving average representation of the fractional Brownian motion. If one needs a guideline to know what representation is useful for what problem, we can roughly think that the harmonizable representation is more adapted to a spectral description of fractional Brownian motion whereas the moving-average is the starting point of most anticipative calculus with respect to fractional Brownian motion.

**Theorem 3.2.2** *If* $\widehat{W}(d\xi) = \sum_{\lambda \in \Lambda} \overline{\widehat{\psi_\lambda}(\xi)} \eta_\lambda d\xi$ *is the Brownian random measure associated to the Fourier transform of the Lemarié Meyer basis, let us denote by* $W(ds) = \sum_{\lambda \in \Lambda} \psi_\lambda(s) \eta_\lambda ds$ *the Brownian random measure. Moreover for all* $0 < H < 1$

$$\int_{\mathbb{R}} \frac{e^{-it\xi} - 1}{C_H^{1/2} |\xi|^{1/2+H}} \widehat{W}(d\xi) =$$
$$\frac{2^{-H} \Gamma(5/4 - H/2)}{C_H^{1/2} \Gamma(H/2 + 1/4) |1/4 - H/2|} \int_{\mathbb{R}} \left[ |t - s|^{H-1/2} - |s|^{H-1/2} \right] W(ds) \quad a.s. \tag{3.27}$$

**Remark 3.2.4** *When $H = 1/2$ the meaning of $|s|^0$ is conventionally given by*

$$|s|^0 = \ln(1/|s|).$$

*In this case the constant before the integral in the right hand side of (3.27) is given by*

$$\sqrt{\frac{2}{\pi C_H}}.$$

**Remark 3.2.5** *Please note that the correspondence is stated in a very strong sense. It is a consequence of the fact that we have used the same sequence $(\eta_\lambda)_{\lambda\in\Lambda}$ to define the two Brownian random measures. Then one gets an almost sure identity instead of an identity in distribution as in Proposition 7.2.8 in [126]. Moreover the negligible set does not depend on $t \in \mathbb{R}$ as it can be checked in the proof.*

*Proof of Theorem 3.2.2*

Since the integral representations in (3.27) are nothing but series representations the theorem is proved if $\forall t \in \mathbb{R}$

$$\langle \frac{e^{-it\xi} - 1}{C_H^{1/2}|\xi|^{H+1/2}}, \widehat{\psi_\lambda}(\xi)\rangle_{L^2} = \tilde{D}(H)\langle [|t-s|^{H-1/2} - |s|^{H-1/2}], \psi_\lambda(s)\rangle_{L^2}, \quad (3.28)$$

where we recall that the $L^2$ space is endowed with the measure $\frac{d\xi}{\sqrt{2\pi}}$. Thanks to Parseval identity it is enough to show that

$$\frac{\widehat{e^{-it\xi} - 1}}{C_H^{1/2}|\xi|^{H+1/2}}(s) = D(H)\left(|t-s|^{H-1/2} - |s|^{H-1/2}\right) \quad (3.29)$$

for another constant $D(H)$. Let us first consider the case where $H < 1/2$. Then $|\xi|^{-(H+1/2)}$ is locally integrable and it defines an even homogeneous tempered distribution $T \in \mathcal{S}'$. See [128] for an introduction to distributions and tempered distributions. The homogeneous index of $T$ is $-(H+1/2)$. Then its Fourier transform is an even distribution of homogeneous index $-1+H+1/2$. But there exists only one even homogeneous distribution of index $H - 1/2$ which is $|s|^{H-1/2}$ up to multiplicative scalar $\tilde{D}(H)$. Hence in $\mathcal{S}'$

$$\widehat{|\xi|^{-H-1/2}}(s) = \tilde{D}(H)|s|^{H-1/2}. \quad (3.30)$$

Let us compute $\tilde{D}(H)$ by using $\widehat{\exp(-\frac{\xi^2}{2})}(s) = \exp(-\frac{s^2}{2})$. Because of Parseval identity and of Eq. (3.30)

$$\int_{\mathbb{R}} |\xi|^{-(H+1/2)} \exp\left(-\frac{\xi^2}{2}\right) \frac{d\xi}{(2\pi)^{1/2}} = \tilde{D}(H) \int_{\mathbb{R}} |s|^{H-1/2} \exp\left(-\frac{s^2}{2}\right) \frac{ds}{(2\pi)^{1/2}}. \quad (3.31)$$

Let us remark that the same identity holds when the integrals are computed on $(0, +\infty)$. When one further changes the variables so that $u = \frac{\xi^2}{2} = \frac{s^2}{2}$, one gets

$$2^{-(H/2+1/4+1/2)} \int_0^{+\infty} u^{-(H/2+3/4)} \exp(-u) du = \tilde{D}(H) 2^{(H/2-1/4-1/2)} \int_0^{+\infty} u^{H/2-3/4} \exp(-u) du. \quad (3.32)$$

And $\tilde{D}(H)$ can be expressed with the classical Gamma function:

$$\tilde{D}(H) = \frac{2^{-H}\Gamma(1/4 - H/2)}{\Gamma(H/2 + 1/4)} \quad (3.33)$$

$$= \frac{2^{-H}\Gamma(5/4 - H/2)}{\Gamma(H/2 + 1/4)(1/4 - H/2)}. \quad (3.34)$$

Since the support of $\widehat{\psi_\lambda}(\xi)$ does not contain $0 \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{1}{C_H^{1/2}|\xi|^{H+1/2}} \widehat{\psi_\lambda}(\xi) d\xi = \langle \frac{1}{C_H^{1/2}|\xi|^{H+1/2}}, \widehat{\psi_\lambda}(\xi) \rangle_{\mathcal{S}',\mathcal{S}} \quad (3.35)$$

$$= \langle \frac{\tilde{D}(H)|s|^{H-1/2}}{C_H^{1/2}}, \psi_\lambda(s) \rangle_{\mathcal{S}',\mathcal{S}}, \quad (3.36)$$

where $\langle\ ,\ \rangle_{\mathcal{S}',\mathcal{S}}$ is the notation for the evaluation of a distribution in $\mathcal{S}'$ on a function of $\mathcal{S}$. Furthermore

$$\frac{\widehat{e^{-it\xi}}}{C_H^{1/2}|\xi|^{H+1/2}}(s) = \frac{\tilde{D}(H)|s-t|^{H-1/2}}{C_H^{1/2}}$$

and

$$\langle \frac{e^{-it\xi}-1}{C_H^{1/2}|\xi|^{H+1/2}}, \widehat{\psi}(\xi)_\lambda \rangle_{L^2} = \frac{\tilde{D}(H)}{C_H^{1/2}} \langle \left(|s-t|^{H-1/2} - |s|^{H-1/2}\right), \psi_\lambda(s) \rangle_{\mathcal{S}',\mathcal{S}}$$

$$= \frac{\tilde{D}(H)}{C_H^{1/2}} \langle \left(|s-t|^{H-1/2} - |s|^{H-1/2}\right), \psi_\lambda(s) \rangle_{L^2}.$$

When $H \geq 1/2$, $|\xi|^{-(H+1/2)}$ is not a tempered distribution but the same argument as above is true by using $Pf(|\xi|^{-(H+1/2)})$ the finite part of $|\xi|^{-(H+1/2)}$ as defined in [128]. Moreover $\widehat{Pf(|\xi|^{-1})} = C \log(1/|s|) + D$ and this justifies the convention in Remark 3.2.4.

**Remark 3.2.6** *In [101] the non-anticipating moving average representation of fractional Brownian motion is given by*

$$B_H(t) = \tilde{C}_H \int_{\mathbb{R}} \left[(t-s)_+^{H-1/2} - ((-s)_+)^{H-1/2}\right] W(ds) \tag{3.37}$$

*where the multiplicative constant $\tilde{C}_H$ is given in Proposition 7.2.6 of [126] and where $(s)_+$ stands for the positive part of the real $s$. There is a corresponding harmonizable representation of the non-anticipating moving average which is given by*

$$\int_{\mathbb{R}} \frac{e^{-it\xi}-1}{i\xi|\xi|^{H-1/2}} \widehat{W}(d\xi).$$

*The proof of this correspondence for non-anticipating moving average of fractional Brownian motion follows from:*

$$\widehat{\left[(t-s)_+^{H-1/2} - ((-s)_+)^{H-1/2}\right]}(\xi) = \tilde{D}(H) \frac{e^{-it\xi}-1}{i\xi|\xi|^{H-1/2}}.$$

*Why do we prefer the so-called well-balanced moving average of fractional Brownian motion in the right hand side of (3.27)? Because it leads to straightforward generalizations in higher dimensions where the absolute value in (3.27) is replaced by the Euclidean norm. Nevertheless the non-anticipating moving average representation is more practical when one is concerned with the stochastic integration against a fractional Brownian motion.*

#### 3.2.2.5 Regularity of Fractional Brownian Motion Sample Paths

To finish this paragraph on fractional Brownian motion, let us state some results that describes the regularity of the sample paths of fractional Brownian motion. The first theorem is a rough description of the Hölder regularity of fractional Brownian motion. The next result is much more precise, it is the so-called law of the iterated logarithm but the proof of the result is no more elementary. Last the Hausdorff dimension of the graph of fractional Brownian motion is also given since it is another classical way to quantify the roughness of the sample paths of fractional Brownian motion.

**Theorem 3.2.3** *For every $H' < H$ there exists a modification of $B_H$ such that*

$$\mathbb{P}\left(\sup_{\substack{|s-t|<\epsilon(\omega)\\ |s|\le 1,\ |t|\le 1}} \left(\frac{B_H(s)-B_H(t)}{|s-t|^{H'}}\right) \le \delta\right) = 1 \tag{3.38}$$

*where $\epsilon$ is positive random variable and $\delta > 0$. Moreover the pointwise Hölder exponent for every $t \in \mathbb{R}$*

$$\sup\{H',\ \lim_{\epsilon\to 0} \frac{B_H(t+\epsilon)-B_H(t)}{|\epsilon|^{H'}} = 0\} = H \tag{3.39}$$

*almost surely.*

**Remark 3.2.7** *The first part of the theorem claims that the sample paths of fractional Brownian motion are almost surely Hölder continuous for every $H' < H$ and (3.39) express the sample paths are not $H-$ Hölder continuous.*

*Proof of Theorem 3.2.3*

The first part of the proof is a simple application of the Kolmogorov-Chentsov theorem (c.f. Theorem 2.1.7). At first sight we only have the following bound on the second order moment: $\mathbb{E}(B_H(s) - B_H(t)^2 \le C|t-s|^{2H}$ that yields only the Hölder continuity for $H' < H - 1/2$. But since fractional Brownian motion is a Gaussian process one can easily deduce from the previous bound that $\mathbb{E}(B_H(s)-B_H(t))^{2n} \le C_n|t-s|^{2Hn}$ for every positive integer $n$. Hence there exists a modification of fractional Brownian motion such that the sample paths are Hölder continuous for every $H' < H - \frac{1}{2n}$. Then by taking $n$ big enough the first part of the theorem is proved. To prove the second part let us remark that

$$\frac{B_H(t+\epsilon)-B_H(t)}{|\epsilon|^H} \overset{(d)}{=} B_H\left(\frac{\epsilon}{|\epsilon|}\right), \tag{3.40}$$

and is a standard Gaussian variable. Hence for every $H' > H$

$$\lim_{\epsilon\to 0} \frac{|\epsilon|^{H'}}{B_H(t+\epsilon)-B_H(t)} \overset{(d)}{=} 0. \tag{3.41}$$

Then the limit is also true in a convergence in probability sense. One can find a sequence $\epsilon_n \to 0$ such that

$$\lim_{n\to\infty} \frac{|\epsilon_n|^{H'}}{B_H(t+\epsilon_n)-B_H(t)} \overset{(a.s.)}{=} 0. \tag{3.42}$$

This yields that

$$\lim_{n\to\infty} \frac{B_H(t+\epsilon_n)-B_H(t)}{|\epsilon_n|^{H'}} \overset{(a.s.)}{=} +\infty$$

and that the pointwise Hölder exponent is lower than every $H' > H$ almost surely. One can conclude since the first part of the proof shows that this exponent is bigger than $H' < H$ almost surely.

At this point of the study, we may wonder why $B_H$ is not $H$-Hölder continuous. For this, one needs a precise study of the modulus of continuity of fractional Brownian motion which is given by the following result classically called the law of iterated logarithm.

**Theorem 3.2.4** *Let $B_H$ be a fractional Brownian motion, then $\forall t \in \mathbb{R}$*

$$\limsup_{\epsilon \to 0^+} \frac{B_H(t+\epsilon) - B_H(t)}{\sqrt{2\epsilon^{2H} \log(\log(1/\epsilon))}} = 1 \quad a.s. \tag{3.43}$$

**Remark 3.2.8** *The Eq. (3.43) yields the local modulus of continuity of the sample paths. When (3.43) is compared to (3.2.3) the factor $\epsilon^{2H}$ is related to the pointwise Hölder exponent whereas the factor $\log(\log(1/\epsilon))$ explains why the sample paths are not $H$-Hölder continuous. When $H = 1/2$ this result is classical and a proof using martingale properties of the standard Brownian Motion can be found in [79]. In the case of $H \neq 1/2$ one can deduce the preceding result from Theorem 1.3 in [22]. One can also find useful information on modulus of continuity of some other Gaussian processes in [62, 103, 104]. The reference [81] describes the exceptional points where the law of the iterated logarithm fails for fractional Brownian motion. Quite surprisingly we do not know elementary and easy proof for this law for fractional Brownian motion , it is the reason why we have used the ideas of [103] to prove (3.43) when $H < 1/2$.*

*Proof of the law of iterated logarithm when $H < 1/2$*

The proof of the theorem is in two parts. First the lim sup is shown to be less than 1 using a concentration inequality for Gaussian processes. Then the lim sup is shown to be greater than 1 when $H < 1/2$ since the increments of fractional Brownian motion are anticorrelated in such instance.

Upper bound for the lim sup:

Let us recall a concentration result that can be found in Theorem 12.2 [97] p 142. Let $X$ be a bounded Gaussian process, and $m$ a median of the supremum $\sup_{t\in T} X(t)$ of the process $X(t)$ defined on a set $T$ that is a real number such that $\forall \epsilon > 0$

$$\mathbb{P}(\sup_{t\in T} X(t) \le m + \epsilon) \ge 1/2; \quad \mathbb{P}(\sup_{t\in T} X(t) \le m - \epsilon) \le 1/2$$

then for any $\tau > 0$

$$\mathbb{P}(\sup_{t\in T} X(t) \le m + \tau) \ge \int_{-\infty}^{\tau/\sigma} \exp\left(\frac{-u^2}{2}\right) du \tag{3.44}$$

where $\sigma^2 = \sup_{t\in T} \mathrm{var} X(t)$. Since fractional Brownian motion is a symmetric process, 0 is a median of the supremum of $B_H$. Moreover one can easily deduce from (3.44) that for every $\tau > 0$

$$\mathbb{P}(\sup_{t\in T} |B_H(t)| \geq \tau) \leq 2\int_{\tau/\sigma}^{+\infty} \exp\left(\frac{-u^2}{2}\right) du. \tag{3.45}$$

Let us fix $0 < \theta < 1$ and $t_k = \theta^k$ for every integer $k$. For every $C_0 > 0$, let us denote $B_k = \{\sup_{0\leq s\leq t_k} |B_H(s+t) - B_H(t)| \geq C_0\sqrt{2t_k^{2H}\log(\log(1/t_k))}\}$. By (3.45)

$$\mathbb{P}(B_k) \leq 2\int_{C_0\sqrt{2\log(\log(1/t_k))}}^{+\infty} \exp\left(\frac{-u^2}{2}\right) du.$$

By using a weak form of the upper bound of (2.78) we get

$$\mathbb{P}(B_k) \leq C\exp(-C_0 \log(\log(1/\theta^k))).$$

Then

$$\log(\log(1/\theta^k))) = \log(k) + \log(\log(1/\theta)))$$

and

$$\mathbb{P}(B_k) \leq \frac{C}{k^{C_0}},$$

consequently the Borel Cantelli lemma yields that for every $C_0 > 1$ $\exists k(\omega)$ such that for all $k \geq k(\omega)$

$$\sup_{0\leq s\leq t_k} \frac{|B_H(s+t) - B_H(t)|}{\sqrt{2t_k^{2H}\log(\log(1/t_k))}} < C_0.$$

Hence

$$\sup_{t_{k+1}\leq s\leq t_k} \frac{|B_H(s+t) - B_H(t)|}{\sqrt{2t_{k+1}^{2H}\log(\log(1/t_k))}} < \frac{C_0}{\theta^{2H}}$$

and

$$\sup_{t_{k+1}\leq s\leq t_k} \frac{|B_H(s+t) - B_H(t)|}{\sqrt{2s^{2H}\log(\log(1/s))}} < \frac{C_0}{\theta^{2H}}.$$

Since the last inequality is true for every $C_0 > 1$ and $\theta < 1$ this yields the upper bound

$$\limsup_{\epsilon\to 0^+} \frac{B_H(t+\epsilon) - B_H(t)}{\sqrt{2\epsilon^{2H}\log(\log(1/\epsilon))}} \leq 1 \quad \text{a.s.} \tag{3.46}$$

Please remark that the assumption $H < 1/2$ has not been used.

Lower bound for the lim sup:

The first step is an extension of the Borel Cantelli Lemma which can be found in [103].

**Lemma 3.2.1** *Let $B_k$ be an infinite collection of events such that*

$$\mathbb{P}(B_j \cap B_k) \leq \mathbb{P}(B_j)\mathbb{P}(B_k)$$

*if $j \neq k$. If $\sum_{j=1}^{+\infty} \mathbb{P}(B_j) = +\infty$ then $\mathbb{P}(B_j \text{ infinitely often}) = 1$. (The event $B_j$ infinitely often can be also written as $\{\sum_{j=1}^{+\infty} \mathbf{1}_{B_j} = +\infty\}$.)*

*Proof of the Lemma 3.2.1*

By Cauchy Schwarz inequality

$$\left[\sum_{j=n}^{m} \mathbb{P}(B_j)\right]^2 = \left[\mathbb{E}\left(\sum_{j=n}^{m} \mathbf{1}_{B_j}\right)\right]^2 \leq \mathbb{E}\left\{\left(\sum_{j=n}^{m} \mathbf{1}_{B_j}\right)^2\right\} \mathbb{P}\left(\cup_{j=n}^{m} B_j\right),$$

then

$$\begin{aligned}\left[\sum_{j=n}^{m} \mathbb{P}(B_j)\right]^2 &\leq \left[\sum_{j=n}^{m} \mathbb{P}(B_j) + \sum_{\substack{j \neq k,\\ n \leq j \leq k \leq m}} \mathbb{P}(B_j \cap B_k)\right] \mathbb{P}\left(\cup_{j=n}^{m} B_j\right),\\ &\leq \left[\sum_{j=n}^{m} \mathbb{P}(B_j) + \sum_{\substack{j \neq k,\\ n \leq j \leq k \leq m}} \mathbb{P}(B_j)\mathbb{P}(B_k)\right] \mathbb{P}\left(\cup_{j=n}^{m} B_j\right)\end{aligned}$$

and

$$\frac{1}{\frac{1}{\sum_{j=n}^{m} \mathbb{P}(B_j)} + 1} \leq \mathbb{P}\left(\cup_{j=n}^{m} B_j\right).$$

Since $\sum_{j=1}^{+\infty} \mathbb{P}(B_j) = +\infty$, for every $\epsilon$ and $n$ one can show that

$$\mathbb{P}\left(\cup_{j \geq n} B_j\right) \geq 1 - \epsilon.$$

Hence

$$\mathbb{P}\left(\cap_{n \geq N} \cup_{j \geq n} B_j\right) = 1$$

which is the claimed result.

For a fixed $t$ the previous lemma will be applied to the events

$$B_k = \left\{ \frac{|B_H(t_k + t) - B_H(t_{k+1} + t)|}{\sqrt{2\theta^k \log(\log(\theta^{\frac{-k}{2H}}))}} \geq 1 - \sqrt{\theta} \right\},$$

where in this case we take $t_k = \theta^{\frac{k}{2H}}$ and $\theta < 1$. Since $H < 1/2$

$$\mathbb{E}(B_H(t_k + t) - B_H(t_{k+1} + t))(B_H(t_j + t) - B_H(t_{j+1} + t)) < 0$$

for $j \neq k$ and we rely on the following result to show that

$$\mathbb{P}(B_j \cap B_k) \leq \mathbb{P}(B_j)\mathbb{P}(B_k)$$

**Lemma 3.2.2** *Let $(X, Y)$ be a centered Gaussian random vector in $\mathbb{R}^2$ such that $\mathbb{E}(XY) \leq 0$ Then*

$$\mathbb{P}(X \geq a,\ Y \geq b) \leq \mathbb{P}(X \geq a)\mathbb{P}(Y \geq b)$$

*where $a,\ b \geq 0$.*

See the proof of this lemma in the solution of Exercise 2.3.1.

To prove the lower bound for the lim sup we have first to show the divergence of the series $\sum \mathbb{P}(B_k)$. By using the lower bound of (2.78) one gets

$$\mathbb{P}(B_k) \geq \frac{2}{\sqrt{2\pi}} \left( \frac{\sigma_k}{1 - \sqrt{\theta}} - \left( \frac{\sigma_k}{1 - \sqrt{\theta}} \right)^3 \right) \exp\left( \frac{-(1 - \sqrt{\theta})^2}{\sigma_k^2} \right) \tag{3.47}$$

where $\sigma_k$ is the standard deviation of $\dfrac{B_H(t_k + t) - B_H(t_{k+1} + t)}{\sqrt{2\theta^k \log(\log(\theta^{\frac{-k}{2H}}))}}$

$$\sigma_k = \frac{(1 - \theta^{\frac{1}{2H}})^H}{\sqrt{2 \log(\log(\theta^{\frac{-k}{2H}}))}}.$$

First one can remark that

$$\log(\log(\theta^{\frac{-k}{2H}})) = \log(k) - \log(2H) + \log(\log(1/\theta)) \tag{3.48}$$

hence it goes to infinity and is asymptotically equivalent to $\log(k)$. Then

$$\lim_{k \to +\infty} \frac{\sigma_k}{1 - \sqrt{\theta}} = 0 \tag{3.49}$$

and one can weaken (3.47) into:

$$\mathbb{P}(B_k) \geq C \frac{\sigma_k}{1-\sqrt{\theta}} \exp\left( \frac{-(1-\sqrt{\theta})^2}{\sigma_k^2} \right) \tag{3.50}$$

for some positive constant C. Since

$$(1-\theta^{\frac{1}{2H}})^{2H} \geq (1-\sqrt{\theta})^2$$

for $0 \leq H \leq 1$ and for every $\theta \in (0, 1)$.

$$\mathbb{P}(B_k) \geq \frac{C \ \exp\left( \frac{-(1-\sqrt{\theta})^2}{(1-\theta^{\frac{1}{2H}})^{2H}} \log(k) \right)}{(\log(k))^{1/2}}. \tag{3.51}$$

Hence $\sum \mathbb{P}(B_k) = +\infty$ and because of Lemma 3.2.1 we know that almost surely there exists an infinite number of $k$ such that:

$$|B_H(t_k+t) - B_H(t_{k+1}+t)| \geq \sqrt{2\theta^k \log(\log(\theta^{\frac{-k}{2H}}))}(1-\sqrt{\theta}).$$

Then

$$\begin{aligned}|B_H(t_k+t) - B_H(t)| \geq \sqrt{2\theta^k \log(\log(\theta^{\frac{-k}{2H}}))}(1-\sqrt{\theta}) \\ - |B_H(t_{k+1}+t) - B_H(t)|\end{aligned}$$

and because of (3.46) for every $\epsilon > 0$

$$\begin{aligned}|B_H(t_k+t) - B_H(t)| \geq (1-\sqrt{\theta})\sqrt{2\theta^k \log(\log(\theta^{\frac{-k}{2H}}))} \\ - (1+\epsilon)\sqrt{2\theta^{k+1} \log(\log(\theta^{\frac{-k+1}{2H}}))}.\end{aligned}$$

Because of the asymptotic (3.48) there exists $C$ such that

$$|B_H(t_k+t) - B_H(t)| \geq (1-C\sqrt{\theta})\sqrt{2t_k \log(\log(1/t_k))}$$

for an infinite number of $k$. By taking $\theta$ small enough, one can show that

$$\limsup_{\epsilon \to 0^+} \frac{B_H(t+\epsilon) - B_H(t)}{\sqrt{2\epsilon^{2H} \log(\log(1/\epsilon))}} \geq C'$$

for every $C' < 1$.

**Theorem 3.2.5** *Almost surely the Hausdorff dimension of the graph*

$$\{(s, B_H(s)),\ 0 \le s \le 1\}$$

*of fractional Brownian motion with Hurst exponent $H$ is* $2 - H$.

*Proof of the Theorem 3.2.5*

The upper bound is immediately given by Lemma 2.2.1 and Theorem 3.2.3. To have a lower bound the Frostmann Lemma 2.2.2 is applied. Before this, let us recall an elementary computation for Gaussian random variables.

**Lemma 3.2.3** *Let $X$ be a centered Gaussian random variable such that* $\mathrm{var} X = \sigma^2,\ h > 0,\ s > 1$

$$\mathbb{E}(X^2 + h^2)^{-s/2} \le C \frac{h^{1-s}}{\sigma}. \tag{3.52}$$

*Proof of the Lemma 3.2.3*

Let us compute the expectation

$$\mathbb{E}(X^2 + h^2)^{-s/2} = \frac{2}{\sigma\sqrt{2\pi}} \int_0^{+\infty} (x^2 + h^2)^{-s/2} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx.$$

The change of variable $\sigma^2 v = x^2$ is then applied:

$$\mathbb{E}(X^2 + h^2)^{-s/2} = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \frac{1}{\sqrt{v}(\sigma^2 v + h^2)^{s/2}} e^{-v/2} dv.$$

Since

$$\begin{aligned} \int_0^{h^2/\sigma^2} \frac{1}{\sqrt{v}(\sigma^2 v + h^2)^{s/2}} e^{-v/2} dv &\le \int_0^{h^2/\sigma^2} \frac{1}{\sqrt{v} h^s} dv \\ &\le \frac{2h^{1-s}}{\sigma}, \end{aligned}$$

$$\begin{aligned} \int_{h^2/\sigma^2}^{\infty} \frac{1}{\sqrt{v}(\sigma^2 v + h^2)^{s/2}} e^{-v/2} dv &\le \int_{h^2/\sigma^2}^{\infty} \frac{1}{\sqrt{v}\sigma^s v^{s/2}} dv \\ &\le C \frac{h^{1-s}}{\sigma} \end{aligned}$$

for some positive constant C, and (3.52) is proved.

To get the lower bound we will show that

$$I_s = \int_0^1 \int_0^1 \left(\mathbb{E}(B_H(t) - B_H(u))^2 + |t-u|^2\right)^{-s/2} dt du$$

is finite for $1 < s < 2 - H$. Please note that the measure $\mu$ that appears in the Frostmann lemma is here the Lebesgue measure. Since

$$\mathbb{E}(B_H(t) - B_H(u))^2 = |t-u|^{2H},$$

Lemma 3.2.3 is applied and one gets

$$I_s \le C \int_0^1 \int_0^1 \frac{|t-u|^{1-s}}{|t-u|^H} dt du.$$

Hence it is finite when $s < 2 - H$.

### 3.2.3 Semi Self-Similarity

In this section, the deterministic point of view of self-similarity is considered and compared to the stochastic case. As pointed out in the first section, the self-similarity has to be relaxed not to lead to trivial examples in the deterministic case. Let us start with the Weierstrass function which is a famous example in this domain. Back to Weierstrass's seminal work [141] the following function

$$\tilde{\mathcal{W}}(t) = \sum_{n=0}^{+\infty} a^n e^{ib^n t} \tag{3.53}$$

where $t \in \mathbb{R}$, $0 < a < 1$ and $ab > 1$ is a classical example of a continuous but nowhere differentiable function. In (3.53) the complex valued Weierstrass function is introduced but one can take the real or the imaginary part of $\tilde{\mathcal{W}}(t)$ to have real valued function exhibiting similar properties as to $\tilde{\mathcal{W}}(t)$. Since we aim to compare the Weierstrass function to fractional Brownian motion let us define a modified Weierstrass function originally introduced by Mandelbrot [102]

$$\mathcal{W}_r(t) = \sum_{n=-\infty}^{+\infty} \left(e^{ir^n t} - 1\right) r^{-Hn} \tag{3.54}$$

for $r > 1$ and $0 < H < 1$. This formula is to be compared to the real harmonizable representations of fractional Brownian motion (3.23) and (3.25). First the series in (3.53) and in (3.54) are convergent. Then the series (3.54) can be thought of as an harmonizable representation of the modified Weierstrass function where the stochastic measure $\frac{\widehat{W}(d\xi)}{|\xi|^{1/2}}$ has been replaced by a deterministic measure $M(d\xi) = \sum_{n=-\infty}^{+\infty} \delta_{r^n}(\xi)$. Let us now comment the differences between (3.53) and (3.54). The summation in (3.54) is for $n \in \mathbb{Z}$ to have a self-similarity type property namely

$$\mathcal{W}_r(\varepsilon t) = \varepsilon^H \mathcal{W}_r(t) \tag{3.55}$$

where $\varepsilon = r^n$ for every integer $n \in \mathbb{Z}$, the functions that satisfy (3.55) are called $r$-semi-self-similar. Then replacing $e^{ir^n t}$ by $e^{ir^n t} - 1$ is necessary to ensure the convergence of (3.54) when $n \to -\infty$. Please note that even if the Weierstrass functions are easy to simulate they provide very difficult model from a mathematical point of view. For instance the Hausdorff dimension of the graph of a Weierstrass function is still an open problem whereas the counterpart for fractional Brownian motion is elementary. Let us finish this comparison by recalling a randomization of the Weierstrass function

$$\mathcal{W}_r^{al}(t) = \sum_{n=-\infty}^{+\infty} \left(e^{ir^n t} - 1\right) r^{-Hn}(\xi_n + i\eta_n) \tag{3.56}$$

where $\xi_n$ and $\eta_n$ are real valued random variables. Many i.i.d. sequences of $(\xi_n, \eta_n)$ have been proposed in the literature (See for instance [26, 133, 114, 115]). In this section one assumes that $\xi_n$ and $\eta_n$ are independent identically distributed variables with second order moments. Then

$$\mathbb{E}\left(\Re\mathcal{W}_r^{al}(t) - \Re\mathcal{W}_r^{al}(s)\right)^2 = \frac{1}{2}\mathbb{E}\left|\mathcal{W}_r^{al}(t) - \mathcal{W}_r^{al}(s)\right|^2 \tag{3.57}$$

$$= C \sum_{n=-\infty}^{+\infty} \left(1 - \cos(r^n(t-s))\right) r^{-2Hn} \tag{3.58}$$

$$\sim \frac{C|t-s|^{2H}}{\ln r} \tag{3.59}$$

when $r \to 1^+$. This suggests to renormalize the random Weierstrass function to have convergence to fractional Brownian motion.

**Theorem 3.2.6** *Suppose that the random vector $(\xi_n, \eta_n)_{n\in\mathbb{Z}}$ are independent identically distributed random vectors such that $\mathbb{E}\xi_n = \mathbb{E}\eta_n = 0$, $\mathbb{E}\xi_n^2 = \mathbb{E}\eta_n^2 = 1$, and $\xi_n$ and $\eta_n$ are independent. Then as $r \to 1^+$, the normalized random Weierstrass function $\sqrt{\ln r}\,\Re\mathcal{W}_r^{al}$ converges in distribution to the law of fractional Brownian motion.*

**Remark 3.2.9** *In the previous theorem the convergence in distribution is for finite dimensional margins. If we assume that $\mathbb{E}(|\xi_n|^{2k} + |\eta_n|^{2k}) < +\infty$ for some integer $k$ such that $2kH > 1$, then the convergence is in the space of the sample path $C([0,1])$ endowed with the topology of the uniform convergence.*

*Sketch of the proof of the Theorem 3.2.6*

The complete proof is in [114]. We only explain here how this theorem is a consequence of a Functional Central Limit Theorem. Let us first rewrite $\sqrt{\ln r}\,\mathcal{W}_r^{al}$ as

$$S(m) = \sum_{n=-\infty}^{+\infty} f_t\left(\frac{n}{m}\right) \frac{(\xi_n + i\eta_n)}{\sqrt{m}}$$

where we have set $m = \frac{1}{\ln r}$ and

$$f_t(u) = \left(\exp(ie^u t) - 1\right) e^{-Hu}.$$

To simplify the problem, let us suppose that $m$ is an integer and consider the following process:

$$B_m(u) = \begin{cases} \frac{1}{\sqrt{m}} \sum_{j=1}^{[mu]} (\xi_j + i\eta_j), \ u \geq 0 \\ \frac{-1}{\sqrt{m}} \sum_{j=[mu]+1}^{0} (\xi_j + i\eta_j), \ u < 0. \end{cases}$$

Some central limit theorem shows that as $m \to +\infty$ the process $B_m$ converges in distribution to a complex Brownian motion $B$. Then one can write

$$\sum_{n=-\infty}^{+\infty} f_t\left(\frac{n}{m}\right) \frac{(\xi_n + i\eta_n)}{\sqrt{m}} = \int_{\mathbb{R}} f_t(u) dB_m(u) = -\int_{\mathbb{R}} B_m(u) \frac{df_t(u)}{du} du$$

and one applies the functional central limit theorem to get

$$-\int_{\mathbb{R}} B_m(u) \frac{df_t(u)}{du} du \overset{m \to +\infty}{\to} -\int_{\mathbb{R}} B(u) \frac{df_t(u)}{du} du = \int_{\mathbb{R}} f_t(u) dB(u).$$

With this presentation the limit of $S(m)$ has the following integral representation

$$\int_{\mathbb{R}} \left(\exp(ie^{iut}) - 1\right) e^{-Hu} (dB^1(u) + idB^2(u))$$

where $B^1, \ B^2$ are independent real-valued Brownian motions. Then one can check that the real part of the preceding integral has the covariance of a real valued fractional Brownian motion.

## 3.3 Self-Similarity for Multidimensional Fields

Self-similar fields parameterized by $d$-dimensional spaces, with $d > 1$ are more difficult to study than self-similar processes. For instance there is no uniqueness result like Proposition 3.2.1 for fields. In the previous sections stationary increments were needed to study spectral properties of self-similar processes. For fields, there exist at least two ways to define increments and to each one are associated self-similar fields.

### 3.3.1 Self-Similarity with Linear Stationary Increments

Let us recall Definition 2.1.24, which is a natural extension of the stationarity of the increments (3.7)

$$(X(x+\delta) - X(y+\delta))_{x\in\mathbb{R}^d} \overset{(d)}{=} (X(x) - X(y))_{x\in\mathbb{R}^d},$$

for every $y$ and $\delta$ in $\mathbb{R}^d$. Starting from the harmonizable representation in (3.25) one can easily imagine multivariate generalizations of fractional Brownian motion

$$X(x) = \int_{\mathbb{R}^d} \frac{e^{-ix.\xi} - 1}{\|\xi\|^{H+d/2}} S\left(\frac{\xi}{\|\xi\|}\right) \widehat{W}(d\xi) \tag{3.60}$$

where $x.\xi$ is the canonical scalar product in $\mathbb{R}^d$, the function $S$ is any real valued measurable function such that $S(\sigma) = S(-\sigma)$ on the sphere $S^{d-1} = \{\sigma \text{ s.t. } \|\sigma\| = 1\}$ with $0 < C_1 < S(\sigma) < C_2 < +\infty$ and $\widehat{W}(d\xi)$ is a Brownian random measure on $\mathbb{R}^d$. Because of the isometry property of Fourier transforms of real valued Brownian random measures (2.31), we have for every $x_1,\ x_2,\ y,\ \delta \in \mathbb{R}^d$:

$$\begin{aligned}
&\mathbb{E}(X(x_1+\delta) - X(y+\delta))(X(x_2+\delta) - X(y+\delta)) \\
&= \int_{\mathbb{R}^d} \frac{\left(e^{-i(x_1+\delta).\xi} - e^{-i(y+\delta).\xi}\right)\left(e^{i(x_2+\delta).\xi} - e^{i(y+\delta).\xi}\right)}{\|\xi\|^{2H+d}} S^2\left(\frac{\xi}{\|\xi\|}\right) \frac{d\xi}{(2\pi)^{d/2}}.
\end{aligned}$$

Hence this covariance does not depend on $\delta$. This shows the stationarity of the increments (3.61). Beside this invariance, these fields clearly satisfy the counterpart of the self-similar property in dimension $d$

$$(X(\varepsilon x))_{x\in\mathbb{R}^d} \overset{(d)}{=} \varepsilon^H (X(x))_{x\in\mathbb{R}^d} \tag{3.61}$$

for every $\varepsilon > 0$. If $S$ is a constant function then the fields $X$ are denoted by $B_H$ and enjoy an isotropy property

$$(X(Ox))_{x\in\mathbb{R}^d} \overset{(d)}{=} (X(x))_{x\in\mathbb{R}^d} \tag{3.62}$$

for every isometry $O$ in the orthogonal group $O(d)$. It leads to the following definition.

**Definition 3.3.1** *The standard Fractional Brownian field is a centered Gaussian field and its covariance is given by*

$$R(x,y) = \frac{1}{2}\{\|x\|^{2H} + \|y\|^{2H} - \|x-y\|^{2H}\}. \tag{3.63}$$

*It admits the harmonizable representation*

$$X(x) \overset{(d)}{=} \int_{\mathbb{R}^d} \frac{e^{-ix.\xi} - 1}{C_H^{1/2} \|\xi\|^{H+d/2}} \widehat{W}(d\xi). \tag{3.64}$$

*Moreover*

$$\mathbb{E}(B_H(x) - B_H(y))^2 = \|x - y\|^{2H}. \tag{3.65}$$

Non-standard Fractional Brownian fields have covariances given by

$$R(x, y) = \frac{V}{2}\{\|x\|^{2H} + \|y\|^{2H} - \|x - y\|^{2H}\},$$

where $V > 0$.

Please note that the constant $C_H$ is in dimension $d$

$$C_H = \int_{\mathbb{R}^d} \frac{2(1 - \cos(\xi_1))}{\|\xi\|^{d+2H}} \frac{d\xi}{(2\pi)^{d/2}} \tag{3.66}$$

$$= \frac{\pi^{1/2}\Gamma(H + 1/2)}{2^{d/2} H\Gamma(2H)\sin(\pi H)\Gamma(H + d/2)} \tag{3.67}$$

where $\xi_1$ is the first coordinate of the $d$-dimensional vector $\xi$. The formula for $C_H$ in dimension $d$ can be found in [87].

Actually most of the formula in dimension 1 can be trivially extended to fractional Brownian fields by replacing absolute value by Euclidean norm. For instance the description of the RKHS of fractional Brownian field becomes in this setting

$$\mathcal{I}_H(\psi)(y) = \int_{\mathbb{R}^d} \frac{e^{-iy.\xi} - 1}{C_H^{1/2} \|\xi\|^{H+d/2}} \overline{\widehat{\psi}(\xi)} \frac{d\xi}{(2\pi)^{d/2}}, \tag{3.68}$$

and

$$K_{B_H} = \{\phi \text{ s.t. } \exists \psi \in L^2(\mathbb{R}^d) \text{ such that } \phi = \mathcal{I}_H(\psi)\}. \tag{3.69}$$

We may wonder if the examples (3.60) are far from exhausting all the Gaussian self-similar fields with stationary increments. Actually it is shown in [51] that all such fields have a covariance of the following form

$$\mathbb{E}(X(x)X(y)) = \int_{\mathbb{R}^d} \left(e^{-ix.\xi} - 1\right)\left(e^{iy.\xi} - 1\right)\mu(d\xi) \tag{3.70}$$

where $\mu$ is a measure that can be split in polar coordinates $(\sigma, r) \in S^{d-1} \times (0, +\infty)$:

$$\mu(d\xi) = S(d\sigma) \otimes \frac{dr}{r^{2H+d}}. \tag{3.71}$$

Hence in (3.60) the only restriction is that the measure on the sphere $S^{d-1}$ is equivalent to the Lebesgue measure on $S^{d-1}$.

### 3.3.2 Self-Similarity for Sheets

One can define rectangular increments for 2-dimensional fields. Let us denote by:

$$\left(\Delta_{\delta_1,\delta_2}X\right)(x,y) = X(\delta_1+x,\delta_2+y) - X(\delta_1+x,\delta_2) - X(\delta_1,\delta_2+y) + X(\delta_1,\delta_2). \tag{3.72}$$

A field parameterized by $\mathbb{R}^2$ has stationary rectangular increments if and only if for every $(\delta_1,\delta_2)\in\mathbb{R}^2$

$$\left(\left(\Delta_{\delta_1,\delta_2}X\right)(x,y)\right)_{(x,y)\in\mathbb{R}^2} \overset{(d)}{=} \left(\left(\Delta_{0,0}X\right)(x,y)\right)_{(x,y)\in\mathbb{R}^2}. \tag{3.73}$$

"Rectangular increments" can be defined in higher dimension (cf. [82]) but the presentation is restricted here to $d=2$ to simplify notations. To produce examples of 2-dimensional self-similar fields with stationary rectangular increments one can use either moving average or harmonizable representations

$$D(x,y) = \int_{\mathbb{R}^2} \frac{e^{-ix\xi_1}-1}{(C_{H_1})^{1/2}|\xi_1|^{H_1+1/2}} \frac{e^{-iy\xi_2}-1}{(C_{H_2})^{1/2}|\xi_2|^{H_2+1/2}} \widehat{W}(d(\xi_1,\xi_2)) \tag{3.74}$$

$$= \frac{2^{-H_1}\Gamma(5/4-H_1/2)}{\Gamma(H_1/2+1/4)|1/4-H_1/2|} \frac{2^{-H_2}\Gamma(5/4-H_2/2)}{\Gamma(H_2/2+1/4)|1/4-H_2/2|} \times \int_{\mathbb{R}^2}\left[|x-s_1|^{H_1-1/2}-|s_1|^{H_1-1/2}\right]\left[|y-s_2|^{H_2-1/2}-|s_2|^{H_2-1/2}\right] \times W(d(s_1,s_2)) \tag{3.75}$$

where $0<H_1,\ H_2<1$. The proof of the almost sure equality in the previous definition is as the proof of the Theorem 3.2.2. This field called Fractional Brownian Sheet has been introduced in [78] and studied in [8, 58]. When $H_1=H_2=1/2$ fractional Brownian Sheet is a classical extension of the Brownian motion to dimension 2: the Brownian sheet. As can be easily deduced from (3.74), the covariance of the Fractional Brownian Sheet is a product :

$$\mathbb{E}D(x_1,y_1)D(x_2,y_2) = \frac{C_{H_1}C_{H_2}}{4}\{|x_1|^{2H_1}+|x_2|^{2H_1}-|x_1-x_2|^{2H_1}\} \tag{3.76}$$

$$\{|y_1|^{2H_2}+|y_2|^{2H_2}-|y_1-y_2|^{2H_2}\}. \tag{3.77}$$

With the help of this last formula one can check a rectangular self-similar property for fractional Brownian Sheet

$$(D(\varepsilon_1 x, \varepsilon_2 y))_{(x,y)\in\mathbb{R}^2} \overset{(d)}{=} \varepsilon_1^{H_1}\varepsilon_2^{H_2}(D(x, y))_{(x,y)\in\mathbb{R}^2} \tag{3.78}$$

for every $\varepsilon_1, \ \varepsilon_2 > 0$. Please note that rectangular self-similarity implies 2-dimensional self-similarity with an index $H_1 + H_2$.

## 3.4 Stable Self-Similar Fields

So far we have only considered Gaussian processes. In this section we will introduce stable self-similar fields. Through these examples we would like to recall that self-similar processes need not to be Gaussian or even to have finite expectation and variance. The interested reader can complete this quick introduction with [126].

### Moving Average Fractional Stable Fields

To construct a moving average type of self-similar fields, let us recall the moving average representation of Fractional Brownian Fields

$$B_H(t) \overset{(d)}{=} \int_{\mathbb{R}^d} \left[\|t-s\|^{H-d/2} - \|s\|^{H-d/2}\right] W(ds). \tag{3.79}$$

The previous formula is an obvious generalization of the one dimensional case (3.27), where the multiplicative constant has been dropped. It will be the starting point of moving average fractional stable field (in short mafsf).

**Definition 3.4.1** *A field $(X_H(t))_{t\in\mathbb{R}^d}$ is called a moving average fractional stable field if it admits the representation*

$$X_H(t) \overset{(d)}{=} \int_{\mathbb{R}^d} \left[\|t-s\|^{H-d/\alpha} - \|s\|^{H-d/\alpha}\right] M_\alpha(ds). \tag{3.80}$$

*where $M_\alpha$ is a real symmetric random stable measure defined in Definition 2.1.18.*

When we compare (3.79) and (3.80), we remark that the fractional power $H - d/2$ has been replaced by $H - d/\alpha$, which is consistent with the rule of the thumb that claims that Gaussian variables correspond to $\alpha = 2$.

**Proposition 3.4.1** *For $0 < H < 1$ the moving average fractional stable motion is well defined by the formula (3.80) and it is an $H$-self-similar field with stationary increments.*

*Proof of the Proposition 3.4.1*

To show that the integral in (3.80) is well defined we have to check that:

$$\int_{\mathbb{R}^d} \left| \|t-s\|^{H-d/\alpha} - \|s\|^{H-d/\alpha} \right|^{\alpha} ds < \infty. \tag{3.81}$$

The integral may diverge when $\|s\| \to \infty,\ \|s\| \to 0$ or $s \to t$. Let us rewrite the integral in (3.81)

$$\int_{S^{d-1}} \int_0^{\infty} \left| \|t-\rho u\|^{H-d/\alpha} - \|\rho u\|^{H-d/\alpha} \right|^{\alpha} \rho^{d-1} d\rho d\sigma(u) \tag{3.82}$$

where $\sigma$ is the surface measure on the unit sphere $S^{d-1}$. When $\|s\| \to \infty$ or $\rho \to \infty$

$$\left| \|t-\rho u\|^{H-d/\alpha} - \|\rho u\|^{H-d/\alpha} \right|^{\alpha} = |H-d/\alpha|^{\alpha} \rho^{H\alpha-d-\alpha} t.u + o(\rho^{H\alpha-d-\alpha}).$$

The integral in (3.82) is convergent when $\rho \to \infty$ since $H < 1$.

When $\|s\| \to 0$, if $H > d/\alpha$ the integrand is bounded, else

$$\left| \|t-\rho u\|^{H-d/\alpha} - \|\rho u\|^{H-d/\alpha} \right|^{\alpha} \sim \rho^{(H\alpha-d)}$$

and the integral in (3.82) is convergent when $\rho \to 0$ since $0 < H$.

The convergence when $s \to t$ is obtained as in the case $\|s\| \to 0$.

Let us check now the stationarity of the increments. Let $\theta = (\theta_1, \ldots, \theta_n)$ and $\mathbf{t} = (t_1, \ldots, t_n) \in (\mathbb{R}^d)^n$, the logarithm of characteristic function of the increments of the mafsf is:

$$-\log\left(\mathbb{E}\exp\left(i\sum_{j=2}^{n}\theta_j(X_H(t_j)-X_H(t_1))\right)\right) = \int_{\mathbb{R}^d}\left|\sum_{j=2}^{n}\theta_j(\|t_j-s\|^{H-d/\alpha}-\|t_1-s\|^{H-d/\alpha})\right|^{\alpha} ds.$$

Hence, it is clear that for every $\delta \in \mathbb{R}^d$

$$(X_H(t_2)-X_H(t_1), \ldots, X_H(t_n)-X_H(t_1)) \overset{(d)}{=} (X_H(t_2+\delta)-X_H(t_1+\delta), \ldots, X_H(t_n+\delta)-X_H(t_1+\delta)).$$

Since for every $\epsilon > 0$

$$-\log\left(\mathbb{E}\exp\left(i\sum_{j=1}^{n}\theta_j(X_H(\epsilon t_j)\right)\right)$$
$$= \int_{\mathbb{R}^d}\left|\sum_{j=1}^{n}\theta_j(\|\epsilon t_j - s\|^{H-d/\alpha} - \|s\|^{H-d/\alpha})\right|^{\alpha} ds.$$

By letting $s' = \epsilon s$ we get

$$-\log\left(\mathbb{E}\exp\left(i\sum_{j=1}^{n}\theta_j(\epsilon^H X_H(t_j)\right)\right) = -\log\left(\mathbb{E}\exp\left(i\sum_{j=1}^{n}\theta_j(X_H(\epsilon t_j)\right)\right),$$

which yields the self-similarity property.

**Remark 3.4.1** *We will not study the regularity of the mafsf, but we would like to stress here that when $H < d/\alpha$ the samples are nowhere bounded. See [99] for a proof when $d = 1$. So following the rules of the thumb we stated in the introduction, we are reluctant to consider mafsf as a fractional field when $H < d/\alpha$. The regularity of mafsf when $d = 1$ and $H > 1/\alpha$ is considered in the paragraph 3.5.*

### Real Harmonizable Fractional Stable Fields

Starting from (3.64) one can easily construct a stable counterpart of the harmonizable representation of fractional Brownian motion. In this book we focus on real valued harmonizable process. We recall that the limit (2.57)

$$\lim_{R\to+\infty}\int_{\mathbb{R}^d} f(\xi)M_{\alpha,R}(d\xi)$$

is our definition of $\int_{\mathbb{R}^d} f(\xi)M_\alpha(d\xi)$ for every function $f \in L^\alpha(\mathbb{R}^d)$ that satisfies (2.50). Up to the normalization constant it is consistent with Theorem 6.3.1 in [126].

Then, one can define the real harmonizable fractional stable field (in short rhfsf).

**Definition 3.4.2** *A field $(X_H(x))_{x\in\mathbb{R}^d}$ is called a real harmonizable fractional stable field if it admits the representation*

$$X_H(x) \stackrel{(d)}{=} \int_{\mathbb{R}^d}\frac{e^{-ix.\xi}-1}{\|\xi\|^{H+d/\alpha}}M_\alpha(d\xi). \tag{3.83}$$

This last definition is the stable counterpart of the harmonizable representation of fractional Brownian motion (3.25) where $H + 1/2$ has been replaced by $H + d/\alpha$, and where the normalizing constant $C_H$ has been dropped. One can also prove the following proposition.

**Proposition 3.4.2** *The real harmonizable fractional stable field is well defined by the formula (3.83) and it is a $H$-self-similar field with stationary increments.*

*Proof of the Proposition 3.4.2*
To show that the integral in (3.83) is well defined we have to check that

$$\int_{\mathbb{R}^d} \left| \frac{e^{-ix.\xi} - 1}{\|\xi\|^{H+d/\alpha}} \right|^\alpha d\xi < \infty. \tag{3.84}$$

The integral may diverge when $\|\xi\| \to \infty,\ \|\xi\| \to 0$.
The integral in (3.84) is convergent when $\|\xi\| \to \infty$ since $H > 0$.
When $\|\xi\| \to 0$

$$\left| \frac{e^{-ix.\xi} - 1}{\|\xi\|^{H+d/\alpha}} \right|^\alpha \le C\|\xi\|^{\alpha - H\alpha - d}$$

and the integral in (3.84) is convergent when $\|\xi\| \to 0$ since $H < 1$.

Let us check now the stationarity of the increments. Let $\theta = (\theta_1, \dots, \theta_n)$ and $\mathbf{x} = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$, the logarithm of the characteristic function of the increments of the rhfsf is

$$-\log\left(\mathbb{E}\exp\left(i\sum_{j=2}^n \theta_j (X_H(x_j) - X_H(x_1))\right)\right)$$
$$= \int_{\mathbb{R}^d} \left| \sum_{j=2}^n \theta_j \frac{e^{-ix_j.\xi} - e^{-ix_1.\xi}}{\|\xi\|^{H+d/\alpha}} \right|^\alpha d\xi$$
$$= \int_{\mathbb{R}^d} \left| \sum_{j=2}^n \theta_j \frac{e^{-ix_j.\xi} - 1}{\|\xi\|^{H+d/\alpha}} \right|^\alpha d\xi.$$

Hence, it is clear that for every $\delta \in \mathbb{R}^d$

$$(X_H(x_2) - X_H(x_1), \dots, X_H(x_n) - X_H(x_1)) \stackrel{(d)}{=}$$
$$(X_H(x_2 + \delta) - X_H(x_1 + \delta), \dots, X_H(x_n + \delta) - X_H(x_1 + \delta)).$$

Since for every $\epsilon > 0$

$$-\log\left(\mathbb{E}\exp\left(i\sum_{j=1}^n \theta_j (X_H(\epsilon x_j)\right)\right) = \int_{\mathbb{R}^d} \left| \sum_{j=1}^n \theta_j \frac{e^{-i\epsilon x_j.\xi} - 1}{\|\xi\|^{H+d/\alpha}} \right|^\alpha ds.$$

By letting $\xi' = \epsilon\xi$ we get

$$-\log\left(\mathbb{E}\exp\left(i\sum_{j=1}^{n}\theta_j(\epsilon^H X_H(x_j)\right)\right) = -\log\left(\mathbb{E}\exp\left(i\sum_{j=1}^{n}\theta_j(X_H(\epsilon x_j)\right)\right)$$

which yields the self-similarity property.

## 3.5 Self-Similarity and Regularity of the Sample Paths

In this section we will shortly investigate the relationship of self-similarity with regularity of the sample paths. Actually we will present some counter-examples to illustrate that the results true for fractional Brownian motion cannot be generalized to all self-similar processes.

Theorem 3.2.3 and $H$-self-similarity of fractional Brownian motion might make believe that more generally all $H$-self-similar processes are $H$-Hölder continuous. Actually this way of thinking is very often used by engineers that simulate self-similar processes with the hope to have generic $H$-Hölder continuous sample paths. Since fractional Brownian motion is the only Gaussian $H$-self-similar process with stationary increments up to a multiplicative constant (c.f. Corollary 3.2.1), one can wonder if processes that satisfy only some of these properties are still $H$-Hölder continuous. The answer is no in at least two situations:

- Gaussian processes that are with non-stationary increments,
- Non-Gaussian processes.

We are now presenting counter-examples for each case.

### Gaussian Processes with Non-Stationary Increments

If $W(ds)$ is a real Brownian random measure one can define for every real-valued function $\varphi$ such that $\varphi(0) = 0$, and

$$\int_{\mathbb{R}} \frac{|\varphi(s)|^2}{|s|^{2H+1}} ds < \infty,$$

an $H$-self-similar process

$$X_t = \int_{\mathbb{R}} \frac{\varphi(ts)}{|s|^{H+1/2}} W(ds).$$

Actually the self-similarity is a consequence of an elementary change of variable and of the fact that the function $\varphi$ is applied to the product $ts$. One can show that the increments of the process $X$ are not stationary for a convenient choice of $\varphi$. Moreover if $\varphi$ is a $C^2$ non-vanishing function with compact support included in $(0, \infty)$. The process is almost surely $C^1$ with a possible exception for $t = 0$, hence it is not $H$- Hölder continuous. Let also see Exercises 3.6.11 and 4.5.1 for other properties of the process.

## Non-Gaussian Processes

In the previous section the moving average fractional stable fields have been defined, and we have shown that there are $H$ self-similar stable processes. Let us recall the definition of the process in dimension $d = 1$

$$X_H(t) = \int_{\mathbb{R}} \left[ |t-s|^{H-1/\alpha} - |s|^{H-1/\alpha} \right] M_\alpha(ds).$$

The construction of this process is related to a Poisson random measure, and the following integral representation also holds for the moving average fractional stable motion

$$X_H(t) = \int_{\mathbb{R}} \left[ |t-s|^{H-1/\alpha} - |s|^{H-1/\alpha} \right] uN(ds, du).$$

The mean measure of $N$ is $ds \frac{du}{|u|^{1+\alpha}}$ and does not satisfy the condition (2.35), but one can see in [126] a rigorous way to define the previous integral. In [120] a rule of the thumb is given to guess the regularity of the sample paths of processes defined with Poisson random measure: It is the same as the regularity of the integrand that is, in our case, the regularity of

$$t \to \left[ |t-s|^{H-1/\alpha} - |s|^{H-1/\alpha} \right] u,$$

which is clearly Hölder continuous with parameter $H - 1/\alpha$ if $H > 1/\alpha$. Hence we claim the following result:

**Proposition 3.5.1** *If $H > 1/\alpha$ almost surely the sample paths of the moving average fractional stable motion are Hölder continuous with parameter $H - 1/\alpha$*

*Proof of the Proposition 3.5.1*

One can find a rigorous proof of this result in [134].

With this example we see that the self-similarity exponent and the Hölder continuity of the sample paths are not necessarily governed by the same parameter in a non-Gaussian setting.

## 3.6 Exercises

### 3.6.1 *Composition of Self-Similar Processes*

Let X be a $H$-self-similar process.

1. Let us suppose that $S(t) = \sup_{0<s\leq t,\ s\in\mathbb{Q}} X(s)$ is almost surely finite. Show that $S$ is a $H$-self-similar process and non-decreasing.
2. Let $X$ be a non-decreasing process and let us define $I(t) = \inf\{s \in \mathbb{Q}$ s.t. $X(s) \geq t\}$. Let us assume that $I(t)$ is almost surely finite. Prove that $I$ is a $1/H$-self-similar process, non-decreasing and right-continuous.
3. Let X be measurable and $Y$ be a $H'$-self-similar process independent of $X$. Let us assume that $I$ is almost surely finite. Prove then that $X \circ Y = (X(Y(t))$ is a $HH'$-self-similar process.

### 3.6.2 *Example*

Let $W(ds)$ be a real valued Brownian random measure. Recognize the process $X$ defined by

$$X(t) = \int_{\mathbb{R}} (\log|t-x| - \log|x|)dW(x).$$

### 3.6.3 *Markov Property for Gaussian Processes*

1. Let $(X_t)_{t\in I}$ be a Gaussian centered process indexed by an open interval $I$ in $\mathbb{R}$ with the covariance function $K$. On says that $X$ is Markov if and only if $\sigma(X_s,\ s \leq t)$ is conditionally independent to $\sigma(X_s,\ s \geq t)$ with respect to $\sigma(X_t)$. The definition of conditional independence is given in Exercise 2.3.3. Prove that $(X_t)_{t\in I}$ is Markov if and only if for every $s \leq t \leq u$
$$K(s,u) = \frac{K(s,t)K(t,u)}{K(t,t)} \tag{3.85}$$
whenever $K(t,t) \neq 0$ and $K(s,u) = 0$ otherwise.
2. Prove that fractional Brownian motion is Markov if and only if $H = 1/2$.
3. Let us consider a function $K : I \times I \to \mathbb{R}$ where $I$ is an open interval in $\mathbb{R}$ such that
$$K(s,t) = \begin{cases} \phi(s)\overline{\psi}(t) & \text{if } s \leq t \\ \psi(s)\overline{\phi}(t) & \text{if } s \geq t \end{cases} \tag{3.86}$$

where $\psi$ and $\phi$ are two continuous non-vanishing functions. They may be real or complex valued functions. Prove that if $K$ is a positive definite function (cf. Definition 2.1.9) then $\phi/\psi$ is a real valued non-negative non-decreasing function on $I$.

4. Let us consider $(B(t))_{t\geq 0}$ a Brownian motion. Then show that the covariance of

$$X_t = \psi(t)B(\phi(t)/\psi(t))$$

for $t \in I$, for $\psi,\ \phi$ two continuous non-vanishing functions such that $\phi/\psi$ is a real valued non-negative non-decreasing function on $I$ satisfies (3.86).

5. Prove that a Gaussian centered process $(X_t)_{t\in I}$ indexed by an open interval $I$ in $\mathbb{R}$ has a covariance satisfying (3.86) if and only if $X$ is continuous in $L^2$, $X_t \neq 0$ for every $t \in I$ and it is a Markov process.

### 3.6.4 Ornstein-Ühlenbeck Process

Determine all centered stationary Markov Gaussian processes $(X_t)_{t\in\mathbb{R}}$ such that their covariance is a continuous function.

### 3.6.5 Bifractional Brownian Motion

This exercise is inspired by [69].

1. Show that for $0 < K < 1$

$$C(K) = \int_0^{+\infty} \frac{1-e^{-z}}{z^{K+1}} dz < +\infty.$$

2. Show that for $0 < K < 1$ and $y > 0$

$$C(K)y^K = \int_0^{+\infty} \frac{1-e^{-xy}}{x^{K+1}} dx. \tag{3.87}$$

3. For $0 < H < 1$ and for $0 < K < 1$ let us set

$$R(t,s) = (|t|^{2H} + |s|^{2H})^K - |t-s|^{2HK}.$$

Show that

$$R(t,s) = \frac{1}{C(K)} \int_0^{+\infty} e^{-x(|t|^{2H}+|s|^{2H})} \frac{e^{x(|t|^{2H}+|s|^{2H}-|t-s|^{2H})}-1}{x^{K+1}} dx. \tag{3.88}$$

4. Show that $R$ is of non-negative type.
5. Prove that there exists a unique centered Gaussian process $X$ such that its covariance function is $R(t, s)$.
6. Prove that $X$ is self-similar with index $HK$.
7. Has $X$ stationary increments?
8. Compute
$$\mathbf{E}\left((X(t) - X(s))^2\right).$$
Show that $\mathbf{E}\left((X(t) - X(s))^2\right) \leq 2|t - s|^{2HK}$.
9. Show that $X$ is locally Hölder continuous for every exponent $0 < \gamma < HK$.
10. Show that at every point $t$ the pointwise Hölder exponent of $X$ is $HK$.

### *3.6.6 Random Measure and Lévy Processes*

Let us consider a random Lévy measure or a real random symmetric stable measure, show that $X(t) = \int_{\mathbb{R}} \mathbf{1}_{[0,t]}(s)M(ds),\ t \geq 0$ is a Lévy process.

### *3.6.7 Properties of Complex Lévy Random Measure*

Let $M(d\xi)$ be a complex isotropic random Lévy measure.

1. Show that for every function $f \in L^2(d\xi)$ $\int f(\xi)M(d\xi)$ has a symmetric distribution i.e.
$$-\int f(\xi)M(d\xi) \overset{(d)}{=} \int f(\xi)M(d\xi).$$
2. Show that for a function $f \in L^2(d\xi)$ such that $\forall\xi \in \mathbb{R}^d, \quad f(-\xi) = \overline{f(\xi)}$ and for every measurable odd function $a$:
$$\int f(\xi)\exp(ia(\xi))M(d\xi) \overset{(d)}{=} \int f(\xi)M(d\xi),$$
3. Show that for a function $f$ such that $\forall\xi \in \mathbb{R}^d, \quad f(-\xi) = \overline{f(\xi)}$
$$\mathbb{E}|\int_{\mathbb{R}^d} f(\xi)M(d\xi)|^2 = 4\pi\,\|f\|^2_{L^2(\mathbb{R}^d)} \int_0^{+\infty} \rho^2 \nu_\rho(d\rho),$$
where $\nu_\rho$ is defined in (2.44).

### 3.6.8 Hausdorff Dimension of Graphs of Self-Similar Processes with Stationary Increments

This exercise is inspired by [19].

1. Prove that, for $1 < s < 2$, there exists a continuous function such that $f_s(\omega) \to 0$ when $|\omega| \to \infty$ and such that:

$$(x^2 + 1)^{-s/2} = \int_{\mathbb{R}} e^{ix\omega} f_s(\omega) d\omega.$$

2. Let $X$ be a $H$-self-similar process with stationary increments such that the function $\lambda \to \mathbb{E}(e^{i\lambda X(1)})$ belongs to $L^1$. Let $m$ be the Lebesgue measure on $(0, 1)$. Let us recall the Frostmann integral for the graph of the random path $\{(x, X(x)),\ x \in [0, 1])\}$ in Remark 2.2.1

$$I_s(m) = \iint_{[0,1]^2} \left(|x - y|^2 + |X(x) - X(y)|^2\right)^{-s/2} dxdy.$$

Show that

$$\mathbb{E}I_s(m) \le \sup_{\omega \in \mathbb{R}} |f_s(\omega)| \int_{\mathbb{R}} |\mathbb{E}(e^{i\lambda X(1)})| d\lambda \iint_{[0,1]^2} |x - y|^{1-H-s} dxdy.$$

3. Prove that, a.s., $\dim_H\{(x, X(x)),\ x \in [0, 1]\} \ge 2 - H$.
4. Let us assume that $X$ has almost surely $H$-Hölder continuous paths. Show that, a.s., $\dim_H\{(x, X(x)),\ x \in [0, 1]\} = 2 - H$.

### 3.6.9 Fractional Brownian Motion and Cantor Set

Recall that any real number $x \in (0, 1]$ can be written

$$x = \sum_{n=1}^{+\infty} \frac{x_n}{3^n},$$

where $x_n \in \{0, 1, 2\}$ and where we assume that there exists no $n_0$ such that for $n \ge n_0\, x_n = 0$. Let $F = \{\sum_{n=1}^{+\infty} \frac{x_n}{3^n}$ such that $x_n \in \{0, 2\}\}$ be the middle third Cantor set. Let $X$ be a fractional Brownian motion of parameter $H$. Compute the Hausdorff dimension of $\{(x, X(x)),\ x \in F\}$.

### 3.6.10 Fourier Expansion of Fractional Brownian Motion When $0 < H \leq 1/2$

Let $0 < H \leq 1/2$.

1. Show that $c_n = \int_0^1 x^{2H} \cos(n\pi x)dx \leq 0 \; \forall n \geq 1$.
2. Show that $\forall x \in [-1, 1]$ $\sum_{n=1}^{+\infty} c_n \cos(n\pi x)$ is convergent and that there exists $c_0 > 0$ such that

$$|x|^{2H} = c_0 + \sum_{n=1}^{+\infty} c_n \cos(n\pi x) \quad \forall x \in [-1, 1].$$

3. $(\xi_n, \eta_n)$ be a sequence of i.i.d. centered Gaussian random vectors in $\mathbb{R}^2$ with covariance matrix equal to identity. Show that the series

$$\sum_{n=1}^{+\infty} (\xi_n + i\eta_n)\sqrt{-c_n/2}(e^{in\pi t} - 1)$$

is convergent in $L^2$. Let us denote by $X(t)$ its sum, show that $\Re(X(t))$ is a $2\pi$ periodic Gaussian process with stationary increments such that $\mathbb{E}(\Re X(t) - \Re X(s))^2 = |t - s|^{2H}$ for all $t, s$ such that $-1 < t - s < 1$. We usually refer to $\Re(X(t))$ as fractional Brownian motion indexed by the circle $S^1$.

### 3.6.11 Exercise: Self-Similar Process with Smooth Sample Paths

Let $W(ds)$ be a real Brownian random measure (cf. Sect. 2.1.6.1), $0 < H < 1$ and let $\varphi$ be a Borel real-valued function.

1. Show that the process

$$X_t = \int_{\mathbb{R}} \frac{\varphi(ts)}{|s|^{H+1/2}} W(ds) \tag{3.89}$$

is defined if and only if $\varphi(0) = 0$, and

$$\int_{\mathbb{R}} \frac{|\varphi(s)|^2}{|s|^{2H+1}} ds < \infty. \tag{3.90}$$

This condition is always supposed to be satisfied by $\varphi$ until the end of the exercise.

2. Show that the process defined in (3.89) is $H$ self-similar.

3. Let us suppose that $\varphi$ is a $C^2$ non-vanishing function with compact support included in $(0, \infty)$. Show that $X$ has a modification with $C^1$ sample paths except maybe for $t = 0$.
4. For this question we assume that $\varphi(\lambda) = \min(|\lambda|, 1)$. Show that the condition (3.90) is satisfied and that the process defined by (3.89) is not with stationary increments.

# Chapter 4
# Asymptotic Self-Similarity

## 4.1 Introduction

Self-similarity, as described in the previous chapter, is a global property, and as such, may be to rigid for some applications. Actually, in many situations the self-similarity parameter $H$ is expected to change with time, and in spatial models, with position. For these applications, one would like to replace $H$ by a function $t \to h(t)$. Let us give a toy example to illustrate the fact that, sometimes, it is the variation of the function $t \to h(t)$, which is important for the applications. One dimensional fractional Brownian motion may be used to model the profile of a mountain, and in this model bigger is $H$, harder is the underground structure. It comes from the well known fact that, profiles of mountains, that contain hard stones is rougher, than the ones, that contain soft stones. Hence, one can hope to use $H$ to detect hard stones in the underground. But for this problem fractional Brownian motion is useless, since in this case, $H$ is constant. One way to circumvent this problem is to plug in a function $t \to h(t)$ instead of $H$ in some integral representation of fractional Brownian motion. This model is called multifractional Brownian motion and was introduced in [22] and independently in [113]. Please note that more recently Volterra type representations of fractional Brownian motion have been used to obtain other multifractional Gaussian processes in [31]. Since we did not introduced Volterra type representations of fractional Brownian motion, we will not describe their multifractional extensions. But we will show that the regularity of the sample paths of multifractional Brownian motion is governed by $h(t)$ at time $t$. Hence the function $h(t)$ can help to model the structure of the underground in our toy model. Nevertheless the dependence of $h$ with respect to $t$ destroys all the invariance properties that we had for fractional Brownian motion. In particular, multifractional Brownian motion is no more self-similar nor with stationary increments. These properties are only true locally and lead in the next section to the notion of local self-similarity property. In some sense, to be precised later, at every point $t$, there exists a tangent fractional Brownian motion. For some models, the self-similarity is lost but the parameter $H$ that describes the local asymptotic self-similarity property is constant, it is the case of filtered white

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noises. One of the problem, for applications of multifractional Brownian motion, is the regularity of the function $t \to h(t)$. We would like that this function is sufficiently irregular to allow a multifractal behavior of the sample paths. Sadly, it is not possible in the original model because of a low frequency problem, that will be pointed in the chapter. Various extensions of multifractional Brownian motion will be introduced in this chapter. The aim of this model is to allow irregular function $t \to h(t)$.

All models presented in the first section of this chapter are Gaussian, but we know that there exist non-Gaussian self-similar processes and fields. One can also construct locally self-similar non Gaussian processes and fields that are introduced in Sect. 4.4.

## 4.2 Definitions

Let us start this section with the definition of a locally asymptotically self-similar field (in short lass). This definition was first introduced in [50] and has been independently rediscovered in [15].

**Definition 4.2.1** *A field $(Y(x))_{x\in\mathbb{R}^d}$ is locally asymptotically self-similar (lass) at point $x$ if*

$$\lim_{\varepsilon\to 0^+}\left(\frac{Y(x+\varepsilon u)-Y(x)}{\varepsilon^{h(x)}}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} (T_x(u))_{u\in\mathbb{R}^d} \tag{4.1}$$

*where the non-degenerate field $(T_x(u))_{u\in\mathbb{R}^d}$ is called the tangent field at point $x$ of $Y$ and the limit is in distribution for all finite dimensional margins of the fields. Furthermore, the field is lass with multifractional function $h$ if for every $x \in \mathbb{R}^d$, it is lass at point $x$ with index $h(x)$.*

Let us make some comments about this definition, which is central in this book. First, a non-degenerate field means that the tangent field is not the null function almost surely. Then, let us remark that a field with stationary increments, which is $H$-self-similar is lass with itself as tangent field and the constant $H$ as multifractional function.

**Proposition 4.2.1** *An $H$-self-similar field $Y$ with stationary increments is lass. More precisely for every $x \in \mathbb{R}^d$*

$$\lim_{\varepsilon\to 0^+}\left(\frac{Y(x+\varepsilon u)-Y(x)}{\varepsilon^{H}}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} (Y(u))_{u\in\mathbb{R}^d}. \tag{4.2}$$

*Proof of Proposition 4.2.1*:

Since $Y$ has stationary increments, $(Y(x+\varepsilon u)-Y(x))_{u\in\mathbb{R}^d} \overset{(d)}{=} (Y(\varepsilon u))_{u\in\mathbb{R}^d}$, the self-similarity property yields $(Y(\varepsilon u))_{u\in\mathbb{R}^d} \overset{(d)}{=} \varepsilon^H (Y(u))_{u\in\mathbb{R}^d}$ and consequently the result.

In some cases, one can achieve better convergence than the limit of the finite dimensional margins. In this case one speaks of strongly locally asymptotically self-similar field.

**Definition 4.2.2** *A field $(Y(x))_{x\in\mathbb{R}^d}$ is strongly locally asymptotically self-similar (slass) at point $x$ if*

$$\lim_{\varepsilon\to 0^+}\left(\frac{Y(x+\varepsilon u)-Y(x)}{\varepsilon^{h(x)}}\right)_{u\in\mathbb{R}^d}\overset{(d)}{=}(T_x(u))_{u\in\mathbb{R}^d},\tag{4.3}$$

*where the non-degenerate field $(T_x(u))_{u\in\mathbb{R}^d}$ is called the tangent field at point $x$ of $Y$ and the limit is in distribution on the space of continuous functions endowed with the topology of the uniform convergence on every compact. Furthermore, the field is strongly lass with multifractional function $h$ if for every $x\in\mathbb{R}^d$, it is strongly lass at point $x$ with index $h(x)$.*

When we compare the Definitions 4.2.1 and 4.2.2, the only difference is that the tightness is required in the later case. Please remark that in Definition 4.2.2 the tangent field is continuous. In the examples, we will have tangent fields that are not continuous.

One can find in [56] a related definition of tangent process which are not continuous. In this case, the convergence is in the sense of the Skorokhod topology. One can also wonder, why the convergence on compact set is used in the previous definition. The aim is to have a true local property in the following sense. If $Y=Y'$ in a neighborhood of $x$ then if $Y$ is slass at $x$ then the same is true for $Y'$. If we assume uniform convergence instead of uniform convergence on compact in Definition 4.2.2, the slass property is no more local.

Please also note, that the structure of tangent processes have been studied in [55, 56]. The main result is that the tangent fields $(T_x(u))_{u\in\mathbb{R}^d}$ are $h(x)$ self-similar with stationary increments for almost every $x$ for the Lebesgue measure, but the setting in these articles is slightly different. One can also refer to [33, 90] for the theoretical study of the tangent fields.

One can also define the asymptotic self similarity at infinity, which is the counterpart of the lass property when $\epsilon\to+\infty$. Since in this case the base point $x$ is not relevant, it leads to the following definition.

**Definition 4.2.3** *A field $(Y(x))_{x\in\mathbb{R}^d}$ is H-asymptotically self-similar ($\infty$-ass) at infinity if*

$$\lim_{R\to+\infty}\left(\frac{Y(Ru)}{R^H}\right)_{u\in\mathbb{R}^d}\overset{(d)}{=}(T(u))_{u\in\mathbb{R}^d},\tag{4.4}$$

*where the non-degenerate field $(T(u))_{u\in\mathbb{R}^d}$ is called the tangent field at infinity of $Y$ and the limit is in distribution for all finite dimensional margins of the fields.*

## 4.3 Gaussian Fields

In this section we will review various Gaussian fields that are generalizations of fractional Brownian motion and we will prove that there are locally self-similar.

### *4.3.1 Filtered White Noises*

The first attempt to capture locally self-similar fields, which have non-stationary increments is the filtered white noise (in short fwn) introduced in [21]. The starting point of this generalization is the harmonizable representation of the fractional Brownian field. Recall (3.60)

$$B_H(x) \stackrel{(d)}{=} \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi}-1}{C_H^{1/2}\|\xi\|^{H+d/2}} \widehat{W}(d\xi).$$

From a signal processing point of view, the fractional Brownian field is a white noise $\widehat{W}(d\xi)$ which is observed through the "filter"

$$g(x,\xi) = \frac{1}{C_H^{1/2}\|\xi\|^{H+d/2}}.$$

If we consider the harmonizable representation of the sum

$$a(x)B_{H_1}(x) + b(x)B_{H_2}(x),$$

where $B_{H_1}$, $B_{H_2}$ are two fractional Brownian fields driven by the same white noise, it corresponds to the filter

$$g(x,\xi) = \frac{a(x)}{C_{H_1}^{1/2}\|\xi\|^{H_1+d/2}} + \frac{b(x)}{C_{H_2}^{1/2}\|\xi\|^{H_2+d/2}} \tag{4.5}$$

In this case, the two fractional Brownian fields are not independent. If we assume that $0 < H_1 < H_2 < 1$, and that the functions $a$ and $b$ are smooth, $B_{H_1}$ is preponderant. Actually we will see in Proposition 4.3.1 that fwn's are slass with a constant multifractional function $H_1$ and that the tangent field is $a(x_0)B_{H_1}$. The same properties are true if the filter

$$g(x,\xi) \sim \frac{a(x)}{C_{H_1}^{1/2}\|\xi\|^{H_1+d/2}} + \frac{b(x)}{C_{H_2}^{1/2}\|\xi\|^{H_2+d/2}}, \tag{4.6}$$

when $\|\xi\| \to +\infty$. This leads to the definition of the filtered white noises.

**Definition 4.3.1** *A field $(X(x))_{x\in\mathbb{R}^d}$ is called a filtered white noise (in short fwn), if it admits the harmonizable representation*

$$X(x) \overset{(d)}{=} \int_{\mathbb{R}^d} (e^{-ix\cdot\xi} - 1) g(x,\xi) \widehat{W}(d\xi), \tag{4.7}$$

*where the filter $g(x,\xi)$ satisfies*

$$\int_{\mathbb{R}^d} |(e^{-ix\cdot\xi} - 1) g(x,\xi)|^2 d\xi < +\infty \tag{4.8}$$

*for every $x \in \mathbb{R}^d$ and*

$$g(x,\xi) = \frac{a(x)}{\|\xi\|^{H+d/2}} + r(x,\xi), \tag{4.9}$$

*where $0 < H < 1$. The following assumptions are enforced: $a$ is a $C^2$ real valued function such that $\forall x \in \mathbb{R}^d a(x) \neq 0$. The function $r$ is $C^2$ and $\overline{r(x,\xi)} = r(x,-\xi)$. For $0 \le m,\ n \le 2,\ \exists C > 0,\ \exists \eta > H$*

$$\left|\frac{\partial^{m+n} r}{\partial x^m \partial \xi^n}(x,\xi)\right| \le \frac{C}{\|\xi\|^{\eta+n+\frac{d}{2}}}\ \forall \xi \in \mathbb{R}^d, \forall x \in \mathbb{R}^d, \tag{4.10}$$

*where $\eta > H$, and $C$ denotes a generic constant.*

**Remark 4.3.1** *The Eq. (4.9) is an asymptotic development of the filter $g$ when $\|\xi\| \to +\infty$. If $g$ satisfies (4.5) then it fulfills (4.9) for $H = H_1$ and $\eta = H_2$.*

**Remark 4.3.2** *In this section we consider fields parametrized by $x \in \mathbb{R}^d$. In Sects. 5.1.3 and 5.1.4 the parameters $H_1$ and $a$ are identified, and various alternative definitions of fwn are given when $d = 1$.*

Let us now show that a filtered white noise is locally self-similar.

**Proposition 4.3.1** *A filtered white noise $(X(x))_{x\in\mathbb{R}^d}$ is strongly locally self-similar with a multifractional function constantly equal to $H$. More precisely $\forall x_0 \in \mathbb{R}^d$*

$$\lim_{\varepsilon\to 0^+} \left(\frac{X(x_0+\varepsilon u) - X(x_0)}{\varepsilon^H}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} C_H^{1/2} a(x_0) (B_H(u))_{u\in\mathbb{R}^d} \tag{4.11}$$

*where the limit is in distribution on the space of continuous functions endowed with the topology of the uniform convergence on every compact.*

*Proof of Proposition 4.3.1*:
Let us first consider the increments of the fwn:

$$X(x)-X(x')=a(x)\int_{\mathbb{R}^d}\frac{(e^{-ix\cdot\xi}-e^{-ix'\cdot\xi})}{\|\xi\|^{H+d/2}}\widehat{W}(d\xi) \quad (4.12)$$
$$+(a(x)-a(x'))\int_{\mathbb{R}^d}\frac{e^{-ix'\cdot\xi}-1}{\|\xi\|^{H+d/2}}\widehat{W}(d\xi)$$
$$+\int_{\mathbb{R}^d}(e^{-ix\cdot\xi}-e^{-ix'\cdot\xi})r(x,\xi)\widehat{W}(d\xi)$$
$$+\int_{\mathbb{R}^d}(e^{-ix'\cdot\xi}-1)(r(x,\xi)-r(x',\xi))\widehat{W}(d\xi).$$

Then the variance of the increments is given by:

$$\mathbb{E}(X(x)-X(x'))^2=a^2(x)\|x-x'\|^{2H}$$
$$+2a(x)(a(x)-a(x'))\Re\int_{\mathbb{R}^d}\frac{(e^{-ix\cdot\xi}-e^{-ix'\cdot\xi})}{\|\xi\|^{2H+d}}(e^{ix'\cdot\xi}-1)\frac{d\xi}{(2\pi)^{d/2}} \quad (4.13)$$
$$+2a(x)\Re\int_{\mathbb{R}^d}\frac{|e^{-ix\cdot\xi}-e^{-ix'\cdot\xi}|^2}{\|\xi\|^{H+d/2}}\overline{r(x,\xi)}\frac{d\xi}{(2\pi)^{d/2}}$$
$$+2a(x)\Re\int_{\mathbb{R}^d}\frac{(e^{-ix\cdot\xi}-e^{-ix'\cdot\xi})}{\|\xi\|^{H+d/2}}(e^{ix'\cdot\xi}-1)(\overline{r(x,\xi)-r(x',\xi)})\frac{d\xi}{(2\pi)^{d/2}} \quad (4.14)$$
$$+(a(x)-a(x'))^2\|x'\|^{2H}$$
$$+2(a(x)-a(x'))\Re\int_{\mathbb{R}^d}\frac{e^{-ix'\cdot\xi}-1}{\|\xi\|^{H+d/2}}(e^{ix\cdot\xi}-e^{ix'\cdot\xi})\overline{r(x,\xi)}\frac{d\xi}{(2\pi)^{d/2}}$$
$$+2(a(x)-a(x'))\Re\int_{\mathbb{R}^d}\frac{|e^{-ix'\cdot\xi}-1|^2}{\|\xi\|^{H+d/2}}(\overline{r(x,\xi)-r(x',\xi)})\frac{d\xi}{(2\pi)^{d/2}}$$
$$+\int_{\mathbb{R}^d}|e^{-ix\cdot\xi}-e^{-ix'\cdot\xi}|^2|r(x,\xi)|^2\frac{d\xi}{(2\pi)^{d/2}}$$
$$+2\Re\int_{\mathbb{R}^d}(e^{-ix\cdot\xi}-e^{-ix'\cdot\xi})(e^{ix'\cdot\xi}-1)r(x,\xi)(\overline{r(x,\xi)-r(x',\xi)})\frac{d\xi}{(2\pi)^{d/2}}$$
$$+\int_{\mathbb{R}^d}|e^{-ix'\cdot\xi}-1|^2|r(x,\xi)-r(x',\xi)|^2\frac{d\xi}{(2\pi)^{d/2}}.$$

Let us assume $\|x\|, \|x'\| \leq R$ for some positive number $R$. Since $a$ is smooth and $r$ satisfies (4.10) one can assume

$$\sup_{\|x\|\leq R}|a(x)|\leq M,$$
$$|a(x)-a(x')|\leq K\|x-x'\|,$$
$$\sup_{x\in\mathbb{R}^d}|r(x,\xi)|\leq\frac{C_1}{\|\xi\|^{\eta+\frac{d}{2}}}$$

and $\exists C_2 > 0$

$$|r(x,\xi) - r(x',\xi)| \le \frac{C_2\|x-x'\|}{\|\xi\|^{\eta+\frac{d}{2}}}.$$

Each integral in the previous sum can be bounded using Cauchy Schwarz inequality. For instance in (4.13), if

$$I = \left| \int_{\mathbb{R}^d} \frac{(e^{-ix\cdot\xi} - e^{-ix'\cdot\xi})}{\|\xi\|^{2H+d}} (e^{-ix'\cdot\xi} - 1) \frac{d\xi}{(2\pi)^{d/2}} \right|,$$

then

$$I \le \left( \int_{\mathbb{R}^d} \frac{|e^{-ix\cdot\xi} - e^{-ix'\cdot\xi}|^2}{\|\xi\|^{2H+d}} \frac{d\xi}{(2\pi)^{d/2}} \right)^{1/2} \left( \int_{\mathbb{R}^d} \frac{|e^{-ix'\cdot\xi} - 1|^2}{\|\xi\|^{2H+d}} \frac{d\xi}{(2\pi)^{d/2}} \right)^{1/2}$$

and $I \le C_3\|x-x'\|^H\|x'\|^H$. For (4.14) one can use

$$\begin{aligned}
&\left| \Re \int_{\mathbb{R}^d} \frac{(e^{-ix\cdot\xi} - e^{-ix'\cdot\xi})}{\|\xi\|^{H+d/2}} (e^{-ix'\cdot\xi} - 1) \overline{(r(x,\xi) - r(x',\xi))} \frac{d\xi}{(2\pi)^{d/2}} \right| \\
&\le \left( \int_{\mathbb{R}^d} \frac{|e^{-ix\cdot\xi} - e^{-ix'\cdot\xi}|^2}{\|\xi\|^{2H+d}} \frac{d\xi}{(2\pi)^{d/2}} \right)^{1/2} \\
&\times \left( \int_{\mathbb{R}^d} |e^{-ix'\cdot\xi} - 1|^2 |r(x,\xi) - r(x',\xi)|^2 \frac{d\xi}{(2\pi)^{d/2}} \right)^{1/2}
\end{aligned}$$

and we get

$$\begin{aligned}
&\left| \Re \int_{\mathbb{R}^d} \frac{(e^{-ix\cdot\xi} - e^{-ix'\cdot\xi})}{\|\xi\|^{H+d/2}} (e^{-ix'\cdot\xi} - 1) \overline{(r(x,\xi) - r(x',\xi))} \frac{d\xi}{(2\pi)^{d/2}} \right| \\
&\le C_3\|x-x'\|^H C_2\|x-x'\| R^\eta.
\end{aligned}$$

By using similar arguments for the others lines, one gets

$$\begin{aligned}
\mathbb{E}(X(x) - X(x'))^2 \le\ & M^2\|x-x'\|^{2H} + 2MKC_3R^H\|x-x'\|^{H+1} \\
&+ 2MC_1\|x-x'\|^{H+\eta} + 2MC_2C_3R^\eta\|x-x'\|^{H+1} \\
&+ K^2R^{2H}\|x-x'\|^2 + 2C_1KR^H\|x-x'\|^{1+\eta} \\
&+ 2KC_2R^H\|x-x'\|^2 + C_1\|x-x'\|^{2\eta} \\
&+ 2C_2C_1R^\eta\|x-x'\|^{1+\eta} + C_2R^{2\eta}\|x-x'\|^2,
\end{aligned} \tag{4.15}$$

which implies

$$\mathbb{E}(X(x) - X(x'))^2 \leq C\|x - x'\|^{2H} \tag{4.16}$$

for $\|x\|, \|x'\| \leq R$, where the constant $C$ depends only on $R,\ K,\ M, C_1,\ C_2, C_3$.

Let us denote

$$Y_\varepsilon(u) = \frac{X(x_0 + \varepsilon u) - X(x_0)}{\varepsilon^H}.$$

To prove the tightness of the distributions of $Y_\varepsilon$ in the space of continuous functions endowed with the topology of the uniform convergence on every compact, we use Corollary 2.1.2. Since $Y_\varepsilon(0) = 0$, we just have to find $\gamma > 0$ and $\alpha > d$ such that

$$\mathbb{E}(|Y_\varepsilon(u_1) - Y_\varepsilon(u_2)|^\gamma) < C\|u_1 - u_2\|^\alpha.$$

In our case, one first checks that

$$\mathbb{E}(|Y_\varepsilon(u_1) - Y_\varepsilon(u_2)|^2) = \mathbb{E}\left(\frac{|X(x + \varepsilon u_1) - X(x + \varepsilon u_2)|^2}{\varepsilon^{2H}}\right)$$

and because of (4.16)

$$\mathbb{E}(|Y_\varepsilon(u_1) - Y_\varepsilon(u_2)|^2) \leq C\|u_1 - u_2\|^{2H}.$$

Since $Y_\varepsilon$ is a Gaussian field, we also have for every $k \in \mathbb{N}$

$$\mathbb{E}(|Y_\varepsilon(u_1) - Y_\varepsilon(u_2)|^{2k}) \leq C\|u_1 - u_2\|^{2Hk}$$

and we take $k$ such that $2Hk > d$.

To prove that filtered white noise is slass, we have to show the convergence in distribution of $(Y_\varepsilon(u_1), \ldots, Y_\varepsilon(u_n))$ toward $C_H^{1/2} a(x)(B_H(u_1), \ldots, B_H(u_n))$. Since $(Y_\varepsilon(u))_{u \in \mathbb{R}^d}$ is a Gaussian field, it is enough to show convergence of the covariance to have the first point. Because of (4.12)

$$\left| \mathbb{E}(Y_\varepsilon(u_1)Y_\varepsilon(u_2)) - \frac{1}{\varepsilon^{2H}} a(x_0 + \varepsilon u_1) a(x_0 + \varepsilon u_2) \right.$$
$$\left. \times \int_{\mathbb{R}^d} \frac{(e^{-ix_0\cdot\xi} - e^{-i(x_0+\varepsilon u_1)\cdot\xi})(e^{ix_0\cdot\xi} - e^{i(x_0+\varepsilon u_2)\cdot\xi})}{\|\xi\|^{2H+d}} \frac{d\xi}{(2\pi)^{d/2}} \right| \to 0, \tag{4.17}$$

when $\varepsilon \to 0$, and

$$\int_{\mathbb{R}^d} \frac{(e^{-ix_0.\xi} - e^{-i(x_0+\varepsilon u_1).\xi})(e^{-ix_0.\xi} - e^{-i(x_0+\varepsilon u_2).\xi})}{\|\xi\|^{2H+d}} \frac{d\xi}{(2\pi)^{d/2}}$$
$$= C_H \mathbb{E} B_H(\varepsilon u_1) B_H(\varepsilon u_2).$$

Because of self-similarity of the fractional Brownian field

$$\lim_{\varepsilon \to 0} \mathbb{E}(Y_\varepsilon(u_1) Y_\varepsilon(u_2)) = C_H a^2(x_0) \mathbb{E} B_H(u_1) B_H(u_2).$$

### *4.3.2 Multifractional Brownian Field*

In the filtered white noise model the increments are no more homogeneous as in fractional Brownian field case. It is obvious when we consider the tangent field associated with a fwn where the function $a(x)$ appears. Still the multifractional function in the previous model is constant and it is not convenient for many applications. In this section multifractional Brownian motion is introduced. This model is truly multifractional.

**Definition 4.3.2** *Let $h : \mathbb{R}^d \to (0, 1)$ be a measurable function. A real valued field is called a multifractional Brownian field (in short mBf) with multifractional function $h$, if it admits the harmonizable representation*

$$B_h(x) \overset{(d)}{=} \frac{1}{(C(h(x)))^{1/2}} \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{d/2+h(x)}} \widehat{W}(d\xi), \tag{4.18}$$

*where the normalization function is*

$$C(s) = \int_{\mathbb{R}^d} \frac{2(1-\cos(\xi_1))}{\|\xi\|^{d+2s}} \frac{d\xi}{(2\pi)^{d/2}}, \tag{4.19}$$

$$= \frac{\pi^{1/2} \Gamma(s+1/2)}{2^{d/2} \Gamma(2s) \sin(\pi s) \Gamma(s+d/2)} \tag{4.20}$$

*where $\xi_1$ is the first coordinate of the $d$-dimensional vector $\xi$.*

**Remark 4.3.3** *If the multifractional function $h(x) = H$ is a constant then multifractional Brownian field is a fractional Brownian field. The multiplicative deterministic function in (4.18) is such that*

$$\mathbb{E} B_h^2(x) = 1$$

*for every $x \in S^{d-1}$.*

**Remark 4.3.4** *If $d = 1$ multifractional Brownian fields (mBf) are often called multifractional Brownian motions (mBm).*

Our first aim is to obtain explicit expression for the covariance of mBm.

**Proposition 4.3.2** *Let $B_h$ be an mBm with multifractional function $h$. Then,*

$$\mathbb{E}(B_h(x)B_h(y)) = D(h(x), h(y))(\|x\|^{h(x)+h(y)} + \|y\|^{h(x)+h(y)} - \|x-y\|^{h(x)+h(y)}), \tag{4.21}$$

*where*

$$D(s,t) = \frac{\sqrt{\Gamma(2s+1)\Gamma(2t+1)\sin(\pi s)\sin(\pi t)}\,\Gamma(\frac{s+t+1}{2})}{2\Gamma(s+t+1)\sin(\pi(s+t)/2)\Gamma(\frac{s+t+d}{2})}, \ \forall s,\ t \in (0,1). \tag{4.22}$$

*Proof of Proposition 4.3.2*:
By definition,

$$\begin{aligned}
\mathbb{E}(B_h(x)B_h(y)) &= \mathbb{E}\Bigg(\frac{1}{C(h(x))^{1/2}}\int_{\mathbb{R}^d}\frac{e^{-ix\cdot\xi}-1}{\|\xi\|^{h(x)+\frac{d}{2}}}\widehat{W}(d\xi) \\
&\qquad \frac{1}{C(h(y))^{1/2}}\int_{\mathbb{R}^d}\frac{e^{iy\cdot\xi}-1}{|\xi|^{h(y)+\frac{d}{2}}}\widehat{W}(d\xi)\Bigg) \\
&= \frac{1}{C(h(x))^{1/2}C(h(y))^{1/2}}\int_{\mathbb{R}^d}\frac{(e^{-ix\cdot\xi}-1)(e^{iy\cdot\xi}-1)}{\|\xi\|^{h(x)+h(y)+d}}\frac{d\xi}{(2\pi)^{d/2}}
\end{aligned} \tag{4.23}$$

where the value of $C(h(x)) = \frac{\pi^{1/2}\Gamma(h(x)+1/2)}{2^{d/2}h(x)\Gamma(2h(x))\sin(\pi h(x))\Gamma(h(x)+d/2)}$ is deduced from (3.66).

Fix $x, y \in \mathbb{R}^d$, and let $B_H$ be a standard fractional Brownian field with fractional parameter $H = \frac{h(x)+h(y)}{2}$. Because of the harmonizable representation of fractional Brownian field we know (3.64)

$$\begin{aligned}
\mathbb{E}(B_H(x)B_H(y)) &= \frac{1}{C_H}\int_{\mathbb{R}^d}\frac{(e^{-ix\cdot\xi}-1)(e^{iy\cdot\xi}-1)}{\|\xi\|^{2H+d}}\frac{d\xi}{(2\pi)^{d/2}} \\
&= \frac{1}{2}(\|x\|^{2H} + \|y\|^{2H} - \|x-y\|^{2H})
\end{aligned}$$

where $C_s = \frac{\pi^{1/2}\Gamma(s+1/2)}{2^{d/2}s\Gamma(2s)\sin(\pi s)\Gamma(s+d/2)}$.

Thus

$$\begin{aligned}
\int_{\mathbb{R}^d}\frac{(e^{-ix\cdot\xi}-1)(e^{iy\cdot\xi}-1)}{\|\xi\|^{h(x)+h(y)+d}}\frac{d\xi}{(2\pi)^{d/2}} &= C_{\frac{h(x)+h(y)}{2}}\mathbb{E}(B_H(x)B_H(y)) \\
&= \frac{C_{\frac{h(x)+h(y)}{2}}}{2}\left(\|x\|^{h(x)+h(y)} + \|y\|^{h(x)+h(y)} - \|x-y\|^{h(x)+h(y)}\right).
\end{aligned}$$

Substituting in (4.23):

$$\mathbb{E}(B_h(x)B_h(y)) = \frac{C_{\frac{h(x)+h(y)}{2}}}{2C(h(x))^{1/2}C(h(y))^{1/2}}(\|x\|^{h(x)+h(y)} + \|y\|^{h(x)+h(y)} - \|x-y\|^{h(x)+h(y)}). \tag{4.24}$$

Replacing $s$ by $\dfrac{h(x)+h(y)}{2}$ in $C_s$ and using the identity $x\Gamma(x) = \Gamma(x+1)$ yields the announced equality.

One has a series expansion for the mBm similar to the one we get for fractional Brownian motion. We state the results only when $d = 1$.

In Chap. 2, Sect. 2.2.4, the set $\mathbb{Z} \times \mathbb{Z} \times \{1\}$ is denoted by $\Lambda$ and the functions $\psi_\lambda$ are defined. Using these conventions a new function $\chi_\lambda(x, y)$ is now introduced.

**Definition 4.3.3** *For $\lambda \in \Lambda$, $x \in \mathbb{R}$, $y \in (0,1)$ let us define*

$$\chi_\lambda(x, y) = \int_{\mathbb{R}} \frac{e^{-ix\xi} - 1}{|\xi|^{y+1/2}} \overline{\widehat{\psi_\lambda}(\xi)} \frac{d\xi}{(2\pi)^{1/2}}. \tag{4.25}$$

**Remark 4.3.5** *Let us remark that the functions $\chi_\lambda(t, H) = \mathcal{I}_H(\psi_\lambda)$. They are used in series expansion of fractional Brownian motion in Theorem 3.2.1. Moreover $\chi$ is depending on two variables to allow the Hurst exponent to vary for multifractional Brownian motion.*

**Theorem 4.3.1** *Let*

$$\widehat{W^+}(d\xi) = \sum_{\lambda \in \Lambda^+} \overline{\widehat{\psi_\lambda}(\xi)} \eta_\lambda d\xi$$

*be a Brownian random measure as in (3.22) and the functions $(\chi_\lambda(x, y)_{\lambda \in \Lambda^+})$ defined in (4.25) one gets the following series representation of the mBm*

$$B_h(t) = \frac{1}{C(h(t))^{1/2}} \sum_{\lambda \in \Lambda^+} \chi_\lambda(t, h(t))\eta_\lambda. \tag{4.26}$$

*Please note that the convergence is in $L^2$. Moreover, if $h$ is locally Hölder continuous with exponent $\beta$, the series (4.26) converge almost surely for the uniform convergence on compact interval.*

*Proof of Theorem 4.3.1*:

One can use similar arguments to those used in Theorem 3.2.1 to prove the theorem. Please note that the continuity of the sample paths of mBm is a consequence of Theorem 4.3.2. The uniform convergence can also be proved directly. See Exercise 4.5.4.

**Remark 4.3.6** *One also has a series expansion*

$$B_h(t) = \frac{1}{C(h(t))^{1/2}} \sum_{\lambda \in \Lambda} \chi_\lambda(t, h(t))\eta_\lambda. \tag{4.27}$$

We have the same correspondence between the harmonizable representation of the mBm and a moving average representation that the one we get for the fractional Brownian field.

**Proposition 4.3.3** *If $\widehat{W}(d\xi) = \sum_{\lambda\in\Lambda} \overline{\widehat{\psi_\lambda}(\xi)}\eta_\lambda d\xi$ is the Brownian random measure associated to the Fourier transform of the Lemarié Meyer basis, and $W(ds) = \sum_{\lambda\in\Lambda} \psi_\lambda(s)\eta_\lambda ds$ its Fourier transform, then $\forall x \in \mathbb{R}^d$*

$$\begin{aligned}\frac{1}{C(h(x))^{1/2}} \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{d/2+h(x)}} \widehat{W}(d\xi) = \\ \frac{2^{-h(x)}\Gamma(1 + d/4 - h(x)/2)}{C(h(x))^{1/2}\Gamma(h(x)/2 + d/4)|d/4 - h(x)/2|} \\ \times \int_{\mathbb{R}^d} \left[\|x - s\|^{h(x)-d/2} - \|s\|^{h(x)-d/2}\right] W(ds) \quad a.s.\end{aligned} \tag{4.28}$$

**Remark 4.3.7** *When $d = 1$ and $h(x) = 1/2$ the meaning of $|s|^0$ is the same as in the Remark 3.2.4 and it is conventionally given by*

$$|s|^0 = \ln(1/|s|).$$

*In this case the constant before the integral in the right hand side of* (4.28) *is given by*

$$\sqrt{\frac{2}{\pi C_{1/2}}}.$$

**Remark 4.3.8** *One can wonder if the Remark 3.2.6 still holds in the multifractional case. Surprisingly the answer is no. Actually a process also called multifractional Brownian motion in the literature, is defined in* [113] *as*

$$\tilde{B}_h(t) = \tilde{C}_H \int_{\mathbb{R}} \left[(t - s)_+^{h(t)-1/2} - (s)_+^{h(t)-1/2}\right] W(ds). \tag{4.29}$$

*It is shown in* [131] *that its harmonizable representation is not*

$$\int_{\mathbb{R}} \frac{e^{-it\xi} - 1}{i\xi|\xi|^{h(t)-1/2}} \widehat{W}(d\xi)$$

*in general, when the multifractional function is not constant. Even if $\tilde{B}_h$ and $B_h$ enjoy the same properties for local self-similarity or smoothness of the sample paths, there is no simple transformation that maps the sample paths of $B_h$ into those of $\tilde{B}_h$.*

*Proof of Proposition 4.3.3*:

Since $x$ is fixed, the proof of Proposition 4.3.3 is the same as the proof of Theorem 3.2.2, when $d = 1$. When $d \neq 1$, we only have to check the normalizing function that appears in (4.28). In dimension 1 the constant is obtained in Eq. (3.31) thanks to Parseval identity. In dimension $d$ this equation becomes

$$\int_{\mathbb{R}^d} \|\xi\|^{-(H+d/2)} \exp\left(-\frac{\|\xi\|^2}{2}\right) \frac{d\xi}{(2\pi)^{d/2}} \\ = \tilde{D}(H) \int_{\mathbb{R}^d} \|s\|^{H-d/2} \exp\left(-\frac{\|s\|^2}{2}\right) \frac{ds}{(2\pi)^{d/2}}. \tag{4.30}$$

where

$$\widehat{\|\xi\|^{-H-d/2}}(s) = \tilde{D}(H)\|s\|^{H-d/2}.$$

If one uses the changes of variable $u = \|\xi\|^2/2 = \|s\|^2/2$, one gets

$$2^{-H/2+d/4-1} \int_0^{+\infty} u^{-H/2+d/4-1} \exp(-u)du \\ = \tilde{D}(H) 2^{H/2+d/4-1} \int_0^{+\infty} u^{H/2+d/4-1} \exp(-u)du, \tag{4.31}$$

which yields the value of $\tilde{D}(H)$.

Let us show that the mBf is strongly locally asymptotically self-similar. Since mBf is a Gaussian field, we first show that the renormalized increments converge in $L^2$ in the following Proposition.

**Proposition 4.3.4** *Let $h : \mathbb{R}^d \mapsto (0, 1)$ be a $\beta$-Hölder continuous multifractional function and $B_h$ be the corresponding multifractional Brownian field. Let us assume $\beta > \sup_{x\in\mathbb{R}^d} h(x)$, then*

$$\lim_{\varepsilon\to 0^+} \frac{\mathbb{E}(B_h(x+\varepsilon u_1) - B_h(x+\varepsilon u_2))^2}{\varepsilon^{2h(x)}} = \|u_1 - u_2\|^{2h(x)} \tag{4.32}$$

*$\forall u_1, u_2 \in \mathbb{R}^d$. Moreover the limit is uniform, when $u_1, u_2 \in K$, where $K$ is a given compact.*

*Proof of Proposition 4.3.4*:

By the isometry property of (2.31) we get

$$\mathbb{E}(B_h(x+\varepsilon u_1) - B_h(x+\varepsilon u_2))^2 = \frac{1}{C(h(x+\varepsilon u_1))C(h(x+\varepsilon u_2))}$$

$$\times \int_{\mathbb{R}^d} \frac{|T(x,\xi,\varepsilon,u_1,u_2)|^2}{\|\xi\|^{2(h(x+\varepsilon u_1)+h(x+\varepsilon u_2))+d}} \frac{d\xi}{(2\pi)^{d/2}},$$

where we will drop the variables for the function $T$, which is

$$\begin{aligned} T = &(e^{-i(x+\varepsilon u_1)\xi} - 1)C^{1/2}(h(x+\varepsilon u_2))\|\xi\|^{h(x+\varepsilon u_2)} \\ &- (e^{-i(x+\varepsilon u_2)\xi} - 1)C^{1/2}(h(x+\varepsilon u_1))\|\xi\|^{h(x+\varepsilon u_1)}. \end{aligned} \tag{4.33}$$

Let us define $w(x) = C^{1/2}(h(x))$ and let us split $T$ in three parts corresponding to the variation of one single factor $T = T_1 + T_2 + T_3$, where

$$\begin{aligned} T_1 &= w(x+\varepsilon u_2)\|\xi\|^{h(x+\varepsilon u_2)} e^{-i(x+\varepsilon u_2)\xi}(e^{i(\varepsilon(u_2-u_1)\xi} - 1) \\ T_2 &= w(x+\varepsilon u_2)(e^{-i(x+\varepsilon u_2)\xi} - 1)(\|\xi\|^{h(x+\varepsilon u_2)} - \|\xi\|^{h(x+\varepsilon u_1)}) \\ T_3 &= (w(x+\varepsilon u_2) - w(x+\varepsilon u_1))(e^{-i(x+\varepsilon u_2)\xi} - 1)\|\xi\|^{h(x+\varepsilon u_1)}. \end{aligned}$$

Then, define

$$I_{ij} = \frac{1}{w^2(x+\varepsilon u_1)w^2(x+\varepsilon u_2)} \int_{\mathbb{R}^d} \frac{T_i \bar{T}_j}{\|\xi\|^{2(h(x+\varepsilon u_1)+h(x+\varepsilon u_2))+d}} \frac{d\xi}{(2\pi)^{d/2}},$$

for $1 \le i,\ j \le 3$. Hence,

$$\mathbb{E}(B_h(x+\varepsilon u_1) - B_h(x+\varepsilon u_2))^2 = \sum_{1\le i,\ j\le 3} I_{ij}. \tag{4.34}$$

Let us show that $I_{11}$ is preponderant in the preceding sum. First

$$I_{11} = \frac{1}{w^2(x+\varepsilon u_1)} \int_{\mathbb{R}^d} \frac{|e^{i(\varepsilon(u_2-u_1)\xi} - 1)|^2}{\|\xi\|^{2h(x+\varepsilon u_1)+d}} \frac{d\xi}{(2\pi)^{d/2}}.$$

Using the change of variable $\lambda = \varepsilon\|u_2 - u_1\|\xi$, we get

$$I_{11} = \varepsilon^{2h(x+\varepsilon u_1)}\|u_2 - u_1\|^{2h(x+\varepsilon u_1)}, \tag{4.35}$$

then

$$\lim_{\varepsilon\to 0^+} \frac{I_{11}}{\varepsilon^{2h(x)}} = \|u_2 - u_1\|^{2h(x)}.$$

Let us show that the other terms in the sum (4.34) are negligible when $\varepsilon \to 0^+$. Let us consider for instance $I_{22}$, we need an upper bound for $|\|\xi\|^{h(x+\varepsilon u_2)} - \|\xi\|^{h(x+\varepsilon u_1)}|$. This will be a consequence of the Hölder continuity of the multifractional function. More precisely, if we write $\forall \xi \ne 0$

$$
\begin{aligned}
&|\|\xi\|^{h(x+\varepsilon u_2)} - \|\xi\|^{h(x+\varepsilon u_1)}| \\
&\quad = |1 - \exp(\log(\|\xi\|)(h(x+\varepsilon u_1) - h(x+\varepsilon u_2)))|\|\xi\|^{h(x+\varepsilon u_2)}
\end{aligned} \tag{4.36}
$$

and we use the existence of $C > 0$ such that $|1 - e^y| \le C|y|$ for every $y < 1$, one gets $\forall \xi \neq 0$

$$
\begin{aligned}
|\|\xi\|^{h(x+\varepsilon u_2)} - \|\xi\|^{h(x+\varepsilon u_1)}| \le C[&\mathbf{1}(C|\log(\|\xi\|)|\varepsilon^\beta \|u_2 - u_1\|^\beta \le 1)\varepsilon^\beta \\
&\|u_2 - u_1\|^\beta \|\xi\|^{h(x+\varepsilon u_2)} |\log(\|\xi\|)| \\
&+ \mathbf{1}(\|\xi\| > \exp(C\varepsilon^{-\beta}\|u_2 - u_1\|^{-\beta}))\|\xi\|^{M_\varepsilon} \\
&+ \mathbf{1}(\|\xi\| < \exp(-C\varepsilon^{-\beta}\|u_2 - u_1\|^{-\beta}))\|\xi\|^{m_\varepsilon}].
\end{aligned}
$$

In the previous formula, $M_\varepsilon = \max(h(x+\varepsilon u_1), h(x+\varepsilon u_2))$, $m_\varepsilon = \min(h(x+\varepsilon u_1), h(x+\varepsilon u_2))$ and because of the Hölder continuity of $h$ there exists $C > 0$ such that $M_\varepsilon > h(x) - C\varepsilon^\beta$ and $h(x) + C\varepsilon^\beta > m_\varepsilon > 0$. Then one gets

$$
I_{22} \le C\varepsilon^{2\beta} \int \mathbf{1}_{C|\log(\|\xi\|)|\varepsilon^\beta \|u_2 - u_1\|^\beta \le 1} \frac{|e^{-i(x+\varepsilon u_2)\xi} - 1|^2 \log^2(\|\xi\|)}{\|\xi\|^{2h(x+\varepsilon u_1)+d}} \frac{d\xi}{(2\pi)^{d/2}} \tag{4.37}
$$

$$
+ C \int \mathbf{1}_{\|\xi\| > \exp C\varepsilon^{-\beta}\|u_2 - u_1\|^{-\beta}} \frac{|e^{-i(x+\varepsilon u_2)\xi} - 1|^2}{\|\xi\|^{2(h(x)-C\varepsilon^\beta)+d}} \frac{d\xi}{(2\pi)^{d/2}} \tag{4.38}
$$

$$
+ C \int \mathbf{1}_{\|\xi\| < \exp -C\varepsilon^{-\beta}\|u_2 - u_1\|^{-\beta}} \frac{|e^{-i(x+\varepsilon u_2)\xi} - 1|^2}{\|\xi\|^{2(h(x)+C\varepsilon^\beta)+d}} \frac{d\xi}{(2\pi)^{d/2}}. \tag{4.39}
$$

The integral in the right hand side of (4.37) is bounded, since

$$
\int_{\mathbb{R}^d} \frac{|e^{-iy\xi} - 1|^2 \log^2(\|\xi\|)}{\|\xi\|^{2H+d}} \frac{d\xi}{(2\pi)^{d/2}}
$$

is a continuous function of $(y, H)$ on any compact included in $\mathbb{R}^d \times (0, 1)$. Then we deduce

$$
\begin{aligned}
I_{22} \le C(&\varepsilon^{2\beta} + \exp\left(-C(h(x) - C\varepsilon^\beta)\varepsilon^{-\beta}\|u_2 - u_1\|^{-\beta}\right) \\
&+ \exp\left(C(h(x) + C\varepsilon^\beta - 1)\varepsilon^{-\beta}\|u_2 - u_1\|^{-\beta}\right),
\end{aligned} \tag{4.40}
$$

which yields

$$
\lim_{\varepsilon \to 0^+} \frac{I_{22}}{\varepsilon^{2h(x)}} = 0. \tag{4.41}
$$

Using the Hölder continuity of $w$, one gets that

$$
I_{33} < C\varepsilon^{2\beta}\|u_2 - u_1\|^{2\beta} \tag{4.42}
$$

and since $\beta > \sup_{x\in\mathbb{R}^d} h(x)$, hence

$$\lim_{\varepsilon\to 0^+} \frac{I_{33}}{\varepsilon^{2h(x)}} = 0. \tag{4.43}$$

When $i \neq j$, we have by Cauchy Schwarz inequality

$$\begin{aligned}
|I_{ij}| &\leq C \int_{\mathbb{R}^d} \frac{T_i \bar{T_j}}{\|\xi\|^{2(h(x+\varepsilon u_1)+h(x+\varepsilon u_2))+d}} \frac{d\xi}{(2\pi)^{d/2}} \\
&\leq C \left( \int_{\mathbb{R}^d} \frac{|T_i|^2}{\|\xi\|^{2(h(x+\varepsilon u_1)+h(x+\varepsilon u_2))+d}} \frac{d\xi}{(2\pi)^{d/2}} \right)^{1/2} \\
&\quad \times \left( \int_{\mathbb{R}^d} \frac{|T_j|^2}{\|\xi\|^{2(h(x+\varepsilon u_1)+h(x+\varepsilon u_2))+d}} \frac{d\xi}{(2\pi)^{d/2}} \right)^{1/2} \\
&\leq C (I_{ii})^{1/2} (I_{jj})^{1/2}.
\end{aligned} \tag{4.44}$$

Hence when $i \neq j$,

$$\lim_{\varepsilon\to 0^+} \frac{I_{ij}}{\varepsilon^{2h(x)}} = 0,$$

and the proposition is proved.

One can deduce from the preceding proposition that the mBf is strongly locally asymptotically self-similar, with arguments similar to those used to show the same property for fwn.

**Proposition 4.3.5** *Let $h : \mathbb{R}^d \mapsto (0, 1)$ be a $\beta$-Hölder continuous multifractional function and $B_h$ be the corresponding multifractional Brownian field. Let us assume $\beta > \sup_{x\in\mathbb{R}^d} h(x)$, then, for every $x \in \mathbb{R}^d$ the mBf is strongly locally asymptotically self-similar, with tangent field a fractional Brownian field with Hurst exponent $H = h(x)$. More precisely*

$$\lim_{\varepsilon\to 0^+} \left( \frac{B_h(x+\varepsilon u) - B_h(x)}{\varepsilon^{h(x)}} \right)_{u\in\mathbb{R}^d} \stackrel{(d)}{=} (B_H(u))_{u\in\mathbb{R}^d} \tag{4.45}$$

*where $H = h(x)$ and $B_H$ is a fractional Brownian field with Hurst exponent $H$.*

**Remark 4.3.9** *One may wonder what happens if $\beta < \sup_{x\in\mathbb{R}^d} h(x)$. Actually in this case, the pointwise Hölder exponent of the mBf is $\beta$, hence there is no limit for the ratio in (4.45). This discussion is continued in the Sect. 4.3.4.*

*Proof of Proposition 4.3.5*:

To prove the convergence of finite dimensional margins, we use Proposition 4.3.4. Let us fix $x \in \mathbb{R}^d$, for every $u_1, u_2 \in \mathbb{R}^d$

$$\frac{\mathbb{E}(B_h(x+\varepsilon u_1)-B_h(x))(B_h(x+\varepsilon u_2)-B_h(x))}{\varepsilon^{2h(x)}} = \frac{1}{2\varepsilon^{2h(x)}}$$
$$\times\,(\mathbb{E}(B_h(x+\varepsilon u_1)-B_h(x))^2+\mathbb{E}(B_h(x+\varepsilon u_2)-B_h(x))^2$$
$$-\,\mathbb{E}(B_h(x+\varepsilon u_2)-B_h(x+\varepsilon u_1))^2).$$

Hence,

$$\lim_{\varepsilon\to 0^+}\left(\frac{\mathbb{E}(B_h(x+\varepsilon u_1)-B_h(x))(B_h(x+\varepsilon u_2)-B_h(x))}{\varepsilon^{2h(x)}}\right)$$
$$=\frac{1}{2}\{\|u_1\|^{2h(x)}+\|u_2\|^{2h(x)}-\|u_1-u_2\|^{2h(x)}\},$$

which yields the convergence of finite dimensional margins.

Then, to have the convergence in distribution in the space of continuous functions endowed with the topology of the uniform convergence on every compact, we have to prove the tightness of the renormalized increments

$$Y_\varepsilon(u)=\frac{B_h(x+\varepsilon u)-B_h(x)}{\varepsilon^{h(x)}}.$$

Let us fix a compact $K$ then because of (4.34), (4.35), (4.40), (4.42), (4.44)

$$\lim_{\varepsilon\to 0}\frac{\mathbb{E}(Y_\varepsilon(u_1)-Y_\varepsilon(u_2))^2}{\|u_1-u_2\|^{2h(x)}}=1$$

uniformly when $u_1, u_2$ are in the compact $K$. Hence,

$$\sup_{\varepsilon>0}\mathbb{E}(Y_\varepsilon(u_1)-Y_\varepsilon(u_2))^2\le C\|u_1-u_2\|^{2h(x)}.$$

Then, one can choose $p\in\mathbb{N}$ such that $ph(x)>d$ and apply Corollary 2.1.2, since

$$\mathbb{E}(Y_\varepsilon(u_1)-Y_\varepsilon(u_2))^{2p}\le C\|u_1-u_2\|^{ph(x)}.$$

Next, we are studying the regularity of the trajectories of the mBm and we are starting with a technical proposition that shows that the localization property of the Lemarié-Meyer basis recalled in Sect. 2.2.4 is also true for the functions $\chi_\lambda$.

**Proposition 4.3.6** *Let us recall that for* $x\in\mathbb{R}$, $y\in(0,1)$, $\lambda\in\Lambda^+$ *and* $j>0$

$$\chi_\lambda(x,y)=\int_{\mathbb{R}}\frac{e^{-ix\xi}-1}{|\xi|^{y+1/2}}\overline{\widehat{\psi_\lambda}(\xi)}\frac{d\xi}{(2\pi)^{1/2}} \tag{4.46}$$

*and let us define for* $\lambda=(0,k,0)$

$$\tilde{\chi}_\lambda(x, y) = \int_{\mathbb{R}} \frac{e^{-ix\xi} - 1 + ix\xi - 1/2x^2\xi^2}{|\xi|^{y+1/2}} \overline{\widehat{\psi_\lambda}(\xi)} \frac{d\xi}{(2\pi)^{1/2}}. \tag{4.47}$$

*Then for $j > 0 \chi_\lambda$ is an analytic function and satisfies for every $K \in \mathbb{N}^*$*

$$|\chi_\lambda(x, y)| \le C(K) 2^{-yj} \left( \frac{1}{1 + |2^j x - k|^K} + \frac{1}{1 + |k|^K} \right) \tag{4.48}$$

*and for $x,\ x' \in \mathbb{R}$, and $y,\ y' \in (0, 1)$*

$$|\chi_\lambda(x, y) - \chi_\lambda(x', y')| \le C(K) 2^{-\min(y,y')j} \left[ \frac{2^j |x - x'| + j|y - y'|}{1 + |2^j x - k|^K} + \frac{j|y - y'|}{1 + |k|^K} \right]. \tag{4.49}$$

*Moreover $\tilde{\chi}_\lambda(x, y)$ is a $C^2$ function and satisfies for $K = 0,\ 1,\ 2$*

$$|\tilde{\chi}_\lambda(x, y)| \le C(K) \left( \frac{1}{1 + |x - k|^K} + \frac{1}{1 + |k|^K} \right) \tag{4.50}$$

*and*

$$|\tilde{\chi}_\lambda(x, y) - \tilde{\chi}_\lambda(x', y')| \le C(K) \left[ \frac{|x - x'|}{1 + |x - k|^K} + |y - y'| \left( \frac{1}{1 + |x - k|^K} + \frac{1}{1 + |k|^K} \right) \right]. \tag{4.51}$$

The proof of the Proposition 4.3.6 is postponed to the Appendix.

In the case of fractional Brownian motion, we had precise results stated as a law of the iterated logarithm. In the case of the mBm, we will only consider the Hölder exponent and the one dimensional case. To obtain the Hölder exponent, we use the series expansion. To be more precise, since we know the Hölder property of fractional Brownian motion, we will consider the mBm as a fractional Brownian motion with a varying Hurst exponent by introducing a process index by $(t, H)$, where $H$ is the Hurst exponent. This leads to the definition of the following process.

**Definition 4.3.4** *A real valued field is called a Hurst process $(X(x, y))_{(x,y) \in \mathbb{R} \times (0,1)}$ if it admits the harmonizable representation*

$$X(x, y) \stackrel{(d)}{=} \int_{\mathbb{R}} \frac{e^{-ix\xi} - 1}{|\xi|^{1/2+y}} \widehat{W}(d\xi). \tag{4.52}$$

**Remark 4.3.10** *Please remark that if* $h : \mathbb{R} \to (0, 1)$ *then*

$$\frac{1}{(C(h(x)))^{1/2}} X(x, h(x))$$

*is a mBm.*

First we need to prove that the Hurst process is uniformly Lipschitz continuous with respect to the second variable.

**Lemma 4.3.1** *Let X be a Hurst process and K be a compact set in* $\mathbb{R}$. *There exist* $\varepsilon$ *such that* $0 < \varepsilon < 1/2$ *and a positive random variable* $C(\omega)$ *such that* $\forall y,\ y' \in (\varepsilon, 1-\varepsilon)$

$$\sup_{x \in K} |X(x, y) - X(x, y')| \le C(\omega)|y - y'| \tag{4.53}$$

*Proof of Lemma 4.3.1*:

One uses a series expansion of the Hurst process where the coefficients are the functions $\chi$ defined in Proposition 4.3.6. It will be convenient to use the Brownian random measure $\widehat{W^+}(d\xi)$ for representing the Hurst process

$$X(x, y) = \int_{\mathbb{R}} \frac{e^{-ix\xi} - 1}{|\xi|^{1/2+y}} \widehat{W^+}(d\xi).$$

Let us split the Hurst process in three parts. Let

$$X^{\sharp}(x, y) = \int_{|\xi| \ge \frac{4\pi}{3}} \frac{e^{-ix\xi} - 1}{|\xi|^{1/2+y}} \widehat{W^+}(d\xi). \tag{4.54}$$

Similarly, let us define the low frequency part of $X$ by

$$X^{\flat}(x, y) = \int_{|\xi| \le \frac{4\pi}{3}} \frac{e^{-ix\xi} - 1 + ix\xi - 1/2x^2\xi^2}{|\xi|^{1/2+y}} \widehat{W^+}(d\xi), \tag{4.55}$$

$$X^{\flat\flat}(x, y) = \int_{|\xi| \le \frac{4\pi}{3}} \frac{-ix\xi + 1/2x^2\xi^2}{|\xi|^{y+1/2}} \widehat{W^+}(d\xi). \tag{4.56}$$

Clearly

$$X = X^{\sharp} + X^{\flat} + X^{\flat\flat}.$$

Let us recall that the support of $\widehat{\psi^{(1)}_{j,k}}$ is such that

$$supp(\widehat{\psi}^{(1)}_{j,k}) \subset \left\{ \frac{2^{j+1}\pi}{3} \le |\xi| \le \frac{2^{j+3}\pi}{3} \right\}.$$

Hence,

$$X^{\sharp}(x,y) = \sum_{(j,k,1)\in\Lambda^+} \chi_{j,k,1}(x,y)\eta_{j,k}, \tag{4.57}$$

where $\eta_{j,k}$s are defined with the Brownian random measure $\widehat{W^+}(d\xi)$. Because the support of $\widehat{\psi^{(0)}}$ is included in $\{|\xi| < \frac{4\pi}{3}\}$

$$X^{\flat}(x,y) = \sum_{k\in\mathbb{Z}} \tilde{\chi}_{0,k,0}(x,y)\eta_{0,k,0}. \tag{4.58}$$

Please note that the series are converging almost surely uniformly on compact set. The proof of this fact is the same as the study of the increments of the processes, which is done in the following lines, but in a simpler instance. We omit it to avoid redundancy.

Let us first prove that (4.53) holds for $X^{\sharp}$. To have an upper bound for the increments of $X^{\sharp}$, we use (4.49), we get

$$\begin{aligned}
|X^{\sharp}(x,y) - X^{\sharp}(x,y')| &\leq \sum_{(j,k,1)\in\Lambda^+} |\chi_{j,k,1}(x,y) - \chi_{j,k,1}(x,y')||\eta_{j,k}| \\
&\leq C|y-y'| \\
&\quad \times \sum_{(j,k,1)\in\Lambda^+} 2^{-\min(y,y')j} j \left(\frac{1}{1+|2^j x-k|^2} + \frac{1}{1+|k|^2}\right)|\eta_{j,k}|.
\end{aligned}$$

Then we will show the uniform convergence of the last series. For this aim one uses a classical inequality for sequences of standard Gaussian random variables. Let us recall that (2.81) yields a positive random variable $C$ such that

$$|\eta_{j,k}| \leq C(\log(j+1)^{1/2} + \log(|k|+1)^{1/2}).$$

Then the uniform convergence of the series can be deduced.

Let us now consider the increments of $X^{\flat}$. In this case, we get

$$\begin{aligned}
|X^{\flat}(x,y) - X^{\flat}(x,y')| &\leq \sum_{k\in\mathbb{Z}} |\tilde{\chi}_{0,k,0}(x,y) - \tilde{\chi}_{0,k,0}(x,y')||\eta_{0,k,0}| \\
&\leq |y-y'| \sum_{k\in\mathbb{Z}} C\left(\frac{1}{1+|k|^2}\right)|\eta_{0,k,0}|.
\end{aligned}$$

Then, one can conclude with the same arguments used for $X^{\sharp}$ because of (2.81).

Finally let us consider $X^{\flat\flat}$. Let us rewrite

$$X^{\flat\flat}(x,y) = -ixY_1(y) + 1/2x^2Y_2(y), \tag{4.59}$$

where

$$Y_\alpha(y) = \int_{|\xi|\le\frac{4\pi}{3}} \frac{\xi^\alpha}{|\xi|^{y+1/2}} \widehat{W^+}(d\xi) \tag{4.60}$$

for $\alpha \in \{1, 2\}$. One shows that each field $Y_\alpha$ admits a modification which has almost surely $C^1$-sample paths on $(0, 1)$. Then, since $X^{\flat\flat}$ is a polynomial in the variable $x$ which belongs to a compact $K$, (4.53) is satisfied by $X^{\flat\flat}$.

One can prove with a Taylor expansion the existence of a constant $C > 0$ such that for $1/2 > \varepsilon > 0$

- $\mathbb{E}(Y_\alpha(y) - Y_\alpha(y'))^2 \le C(y - y')^2$, for every $y, y' \in [\varepsilon, 1-\varepsilon]$,
- $\mathbb{E}(Y_\alpha(y+\delta) + Y_\alpha(y-\delta) - 2Y_\alpha(y))^2 \le C\delta^4$, for every $y \in [\varepsilon, 1-\varepsilon]$ and $\delta$ such that $(y-\delta, y+\delta) \in [\varepsilon, 1-\varepsilon]^2$.

According to the Theorem 2.1.8, these statements imply the existence of a modification of $Y_\alpha$ which has almost surely $C^1$-sample paths. The proof of the following Lemma is a way to use series expansions to prove Hölder continuity of fractional Brownian motion already proved in Theorem 3.2.3. Actually, it yields an uniformity with respect to $H$, which will be useful to study the regularity of the mBm.

**Lemma 4.3.2** *Let $X$ be a Hurst process and $K$ be a compact set of $\mathbb{R}$. There exist $\varepsilon$ such that $0 < \varepsilon < 1/2$ and a positive random variable $C(\omega)$ such that $\forall y \in (\varepsilon, 1-\varepsilon)$, $\forall x,\ x' \in K$ and $\forall y' < y$*

$$|X(x, y) - X(x', y)| \le C(\omega)|x - x'|^{y'}. \tag{4.61}$$

*Proof of Lemma 4.3.2*:

As in Lemma 4.3.1, let us first prove (4.61) for $X^\sharp$. Let $x, x' \in K$ be such that $|x - x'| < 1/2$ and define $j_0 > 0$ so that $2^{-(j_0+1)} \le |x - x'| < 2^{-j_0}$. Then, $\forall y \in (0, 1)$

$$\begin{aligned} X^\sharp(x, y) - X^\sharp(x', y) &= \sum_{j=1}^{j_0-1} \sum_{k\in\mathbb{Z}} (\chi_{j,k,1}(x, y) - \chi_{j,k,1}(x', y))\eta_{j,k} \\ &\quad + \sum_{j=j_0-1}^{+\infty} \sum_{k\in\mathbb{Z}} (\chi_{j,k,1}(x, y) - \chi_{j,k,1}(x', y))\eta_{j,k}. \end{aligned}$$

Because of (4.49) and of (2.81)

$$\begin{aligned} \left|\sum_{j=1}^{j_0-1} \sum_{k\in\mathbb{Z}} (\chi_{j,k,1}(x, y) - \chi_{j,k,1}(x', y))\eta_{j,k}\right| &\le C|x - x'| \sum_{j=1}^{j_0-1} 2^{j(1-y)} \\ \times \sum_{k\in\mathbb{Z}} \left( \frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1 + |2^j x - k|^2} \right. &+ \left. \frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1+|k|^2} \right). \end{aligned}$$

Then, remark that the function

$$F(j,x)=\sum_{k\in\mathbb{Z}}\frac{\log(1+|k|)^{1/2}}{1+|2^jx-k|^2}$$

is bounded on $\mathbb{N}^*\times K$. Hence,

$$\Big|\sum_{j=1}^{j_0}\sum_{k\in\mathbb{Z}}(\chi_{j,k,1}(x,y)-\chi_{j,k,1}(x',y))\eta_{j,k}\Big|\leq C|x-x'|2^{j_0(1-y')} \tag{4.62}$$

for every $y'<y$. Then, because of (4.48) and of (2.81)

$$\begin{aligned}&\Big|\sum_{j=j_0}^{+\infty}\sum_{k\in\mathbb{Z}}(\chi_{j,k,1}(x,y)-\chi_{j,k,1}(x',y))\eta_{j,k}\Big|\\&\quad\leq C\sum_{j=j_0}^{+\infty}2^{-yj}\left(\frac{\log(1+j)^{1/2}+\log(1+|k|)^{1/2}}{1+|2^jx-k|^2}+\frac{\log(1+j)^{1/2}+\log(1+|k|)^{1/2}}{1+|k|^2}\right).\end{aligned}$$

It yields

$$\Big|\sum_{j=j_0}^{+\infty}\sum_{k\in\mathbb{Z}}(\chi_{j,k,1}(x,y)-\chi_{j,k,1}(x',y))\eta_{j,k}\Big|\leq C2^{-j_0y'}. \tag{4.63}$$

Since $|x-x'|<2^{-j_0}$, and because of (4.62), (4.63)

$$\begin{aligned}|X^\sharp(x,y)-X^\sharp(x',y)|&\leq C2^{-j_0y'}\\&\leq C|x-x'|^{y'},\end{aligned}$$

because $2^{-(j_0+1)}\leq|x-x'|$.

Please remark that the set of probability 1, where the previous inequality holds, depends only on $(\eta_{j,k})_{j,\,k\in\mathbb{N}^*\times\mathbb{Z}}$. Let us now prove (4.61) for $X^\flat$. One can show that

$$\begin{aligned}|X^\flat(x,y)-X^\flat(x',y)|&\leq\sum_{k\in\mathbb{Z}}|\tilde{\chi}_{0,k,0}(x,y)-\tilde{\chi}_{0,k,0}(x',y)||\eta_{0,k,0}|\\&\leq|x-x'|\sum_{k\in\mathbb{Z}}C\left(\frac{1}{1+|k|^2}\right)|\eta_{0,k,0}|.\end{aligned} \tag{4.64}$$

Then (4.61) holds for $X^\flat$. The same fact is true for $X^{\flat\flat}$ because of (4.59).

We can now show that the sample paths of mBm are almost surely Hölder continuous with an exponent less than the infimum of the multifractional function, and the regularity of the multifractional function. Let us state the result more precisely in the next theorem.

**Theorem 4.3.2** *Let $I$ be a compact interval and $h$ be function $h : \mathbb{R} \mapsto (0, 1)$ locally Hölder with exponent $\beta$. Let $B_h$ be the corresponding multifractional Brownian motion and $m = \inf\{h(x),\ \ x \in I\}$. For every $H < \min(m, \beta)$ there exists a modification of $B_h$ such that almost surely the sample paths of $B_h$ are $H$-Hölder continuous on $I$.*

*Proof of Theorem 4.3.2*:
Since $t \mapsto \frac{1}{(C(h(t)))^{1/2}}$ is $\beta$-Hölder continuous we only have to prove that $t \mapsto X(t, h(t))$ is $H$-Hölder continuous on $I$ for every $H < \min(m, \beta)$. Let us split the renormalized increments in two parts for $t, t' \in I$

$$\frac{X(t,h(t)) - X(t',h(t'))}{|t-t'|^H} = \frac{X(t,h(t)) - X(t',h(t))}{|t-t'|^H} + \frac{X(t',h(t)) - X(t',h(t'))}{|t-t'|^H}.$$

Because of Lemma 4.3.2 the first renormalized increments on the right hand side of the previous equation is bounded and so is the second because of Lemma 4.3.1. Hence the theorem is proved.

As a consequence one can compute the pointwise Hölder exponent for an mBm.

**Theorem 4.3.3** *Let $h$ be function $h : \mathbb{R} \mapsto (0, 1)$ locally Hölder with exponent $\beta$. Let $B_h$ be the corresponding multifractional Brownian motion and fix $t$ such that $h(t) < \beta$, the pointwise Hölder exponent of $B_h$ at $t$ is*

$$\sup\left\{H',\ \lim_{\epsilon\to 0} \frac{B_h(t+\epsilon) - B_h(t)}{|\epsilon|^{H'}} = 0\right\} = h(t) \tag{4.65}$$

*almost surely.*

*Proof of Theorem 4.3.3*:
Let $H' < h(t)$. Since $h$ is continuous, there exists a compact interval $I$, which is a neighborhood of $t$ such that $\min(h(s), s \in I) > H'$. Then, because of Theorem 4.3.2, $B_h$ is $H'$-Hölder continuous on $I$. Hence, $\sup\{H'',\ \lim_{\epsilon\to 0} \frac{B_h(t+\epsilon)-B_h(t)}{|\epsilon|^{H''}} = 0\} \geq h(t)$ almost surely.

Let $H' > h(t)$. Because of Proposition 4.3.5, we have

$$\lim_{\epsilon\to 0} \frac{|\epsilon|^{H'}}{B_h(t+\epsilon) - B_h(t)} \stackrel{(d)}{=} 0.$$

Then, we can conclude as in Theorem 3.2.3. However let us repeat this important argument here. The limit above is also true in a convergence in probability sense. Hence, one can find a sequence $\epsilon_n \to 0$ such that

$$\lim_{n\to\infty} \frac{|\epsilon_n|^{H'}}{B_h(t+\epsilon_n) - B_h(t)} \stackrel{(a.s.)}{=} 0.$$

This yields

$$\lim_{n\to\infty} \frac{B_h(t+\epsilon_n) - B_h(t)}{|\epsilon_n|^{H'}} \overset{(a.s.)}{=} +\infty,$$

and that the pointwise Hölder exponent is lower than every $H' > h(t)$. One can conclude since the first part of the proof shows that the pointwise exponent is bigger than $H' < h(t)\, a.s.$

### 4.3.3 Step Fractional Brownian Motion

For mBm the Hurst exponent $H$ is replaced by a scaling function $t \to h(t)$. To show the regularity of the sample paths and the local self similarity we had to assume that the multifractional function is itself Hölder continuous. Actually concerning the regularity of the sample paths one can show that if the multifractional function jumps at one point, almost surely the sample paths have a jump at the same point (cf. Exercise 4.5.3). In some applications it is desirable to have an abrupt change of the multifractional functions and hence of the statistic properties of the process and no discontinuity of the sample paths. The step fractional Brownian motion (in short sfBm) is a generalization of the mBms for piecewise constant multifractional functions. We will study the regularity of the sample paths and the local self similarity of the sfBm. Identification of the multifractional function is postponed to Sect. 5.3.2. First the sfBm is constructed starting from the series expansion of the mBm. Let us recall that an mBm can be derived from the Hurst process as we did in Remark 4.3.10

$$B_h(t) = \frac{1}{(C(h(t)))^{1/2}} X(t, h(t)).$$

Since the normalizing function $y \mapsto C(y)$ is smooth, a jump for the multifractional function will generally result in a jump for $x \mapsto C(h(x))$. Let also recall the series expansion of the mBm

$$B_h(t) = \frac{1}{(C(h(t)))^{1/2}} \sum_{\lambda\in\Lambda^+} \chi_\lambda(t, h(t))\eta_\lambda.$$

Since we want to avoid discontinuities to produce the sfBm we will replace the variable $h(t)$ in the series above by $h(\lambda)$. Because of the properties of the functions $\chi$ it will not change very much the process in a neighborhood of $\lambda$ and the perturbation will be negligible when $t$ is away from $\lambda$. It leads to the following model.

**Definition 4.3.5** *Let h be a multifractional function such that*

***Hyp 4.3.1*** *There exist $N \in \mathbb{N}^*$, an increasing sequence of real numbers $a_1, \dots, a_N$ and $H_0, \dots, H_N \in (0,1)$ such that $h(t) = \Sigma_{i=0}^{N} 1_{[a_i, a_{i+1})}(t)\, H_i$, with $a_0 = -\infty$ and $a_{N+1} = +\infty$.*

*Let $\eta_\lambda$, $\lambda \in \Lambda^+$ be i.i.d. standard Gaussian variables, the step fractional Brownian Motion (in short sfBm) associated to the multifractional function $h$ is defined by*

$$Q_h(t) = \sum_{\lambda \in \Lambda^+} \chi_\lambda(t, h(\lambda))\eta_\lambda. \tag{4.66}$$

Next we will study the regularity of the sample paths of the sfBm. For every interval $I \subset \mathbb{R}$, we denote by $H^*(I) = \inf\{h(u), \text{ for } u \in I\}$.

We have the following continuity result:

**Theorem 4.3.4** *Let $Q_h$ be a sfBm with multifractional function $h$ satisfying **Hyp 4.3.1**. For each open interval $I_0$ in $\mathbb{R}$ and for every compact interval $J \subset I_0$, $Q_h$ is $\alpha$ Hölder continuous for $\alpha \in [0, H^*(I_0))$.*

*Proof of Theorem 4.3.4*:

Please note that the series (4.66) converge almost surely uniformly on compact sets. It can be proved as in Theorem 4.3.1.

To study the Hölder continuity of $Q_h$ on $J$, one can only consider $\lambda \in I_0$ in the series (4.66). Let us define for every interval $I$

$$Q_h^I(t) = \sum_{\lambda \in \Lambda^+,\ \lambda \in I} \chi_\lambda(t, h(\lambda))\eta_\lambda.$$

Then, one can split $Q_h$ into two processes $Q_h = Q_h^{I_0} + Q_h^{I_0^c}$, where $I_0^c$ is the complement set of $I_0$. Let us show that $Q_h^{I_0^c}$ is Lipschitz on $J$. To study the regularity of $Q_h^{I_0^c}$ we use the decomposition

$$Q_h^{I_0^c} = Q_h^{I_0^c,\sharp} + Q_h^{I_0^c,\flat} + Q_h^{I_0^c,\flat\flat},$$

which has been already used for the Hurst process in Lemma 4.3.1.

$$\begin{aligned}
Q_h^{I_0^c,\sharp}(t) &= \sum_{(j,k,1) \in \Lambda^+,\ k/2^j \notin I_0} \chi_{(j,k,1)}(t, h(k/2^j))\eta_{(j,k,1)}, \\
Q_h^{I_0^c,\flat}(t) &= \sum_{(0,k,0) \in \Lambda^+,\ k \notin I_0} \tilde{\chi}_{(0,k,0)}(t, h(k/2^j))\eta_{(0,k,0)} \\
Q_h^{I_0^c,\flat\flat}(t) &= \sum_{(0,k,0) \in \Lambda^+,\ k \notin I_0} P_k(t)\eta_{(0,k,0)},
\end{aligned}$$

where $P_k(t) = \int_{\mathbb{R}} \frac{-it\xi + 1/2t^2\xi^2}{|\xi|^{h(k)+1/2}} \overline{\widehat{\psi_{(0,k,0)}}(\xi)} \frac{d\xi}{(2\pi)^{1/2}}$. Please remark that

$$Q_h^{I_0^c,\flat\flat}(t) = -itY_1 + 1/2t^2Y_2, \tag{4.67}$$

where $Y_1,\ Y_2$ are Gaussian random variables defined by series. Since $J \subset I_0$ and is compact, there exists $\epsilon > 0$ such that $|t - t'| > \epsilon$ for every $t \in J$ and $t' \in I_0^c$. Then,

$$Q_h^{I_0^c,\sharp}(t) = \sum_{(j,k,1)\in\Lambda^+,\ k/2^j \notin I_0} \chi_{(j,k,1)}(t, h(k/2^j))\eta_{(j,k,1)}.$$

Hence,

$$\begin{aligned} |Q_h^{I_0^c,\sharp}(t) - Q_h^{I_0^c,\sharp}(t')| &\leq \sum_{\substack{(j,k,1)\in\Lambda^+,\\ k/2^j \notin I_0}} |\chi_{(j,k,1)}(t, h(k/2^j)) - \chi_{(j,k,1)}(t', h(k/2^j))||\eta_{(j,k,1)}| \\ &\leq C \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}, |\frac{k}{2^j} - t| > \epsilon} 2^{-h(k/2^j)j} \frac{2^j|t - t'|}{1 + |2^j t - k|^3} |\eta_{(j,k,1)}|, \end{aligned}$$

because of (4.49). Since $|2^j t - k| > C2^j\epsilon$, we have $1 + |2^j t - k|^3 > C2^j\epsilon(1 + |2^j t - k|^2)$ and

$$|Q_h^{I_0^c,\sharp}(t) - Q_h^{I_0^c,\sharp}(t')| \leq C \left( \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}, |\frac{k}{2^j} - t| > \epsilon} \frac{2^{-H^*(I_0^c)j}|\eta_{(j,k,1)}|}{1 + |2^j t - k|^2} \right) |t - t'|.$$

Because (2.81), the series on the right hand side of the inequality is almost surely finite. Hence we have shown the Lipschitz continuity of $Q_h^{I_0^c,\sharp}$ on $J$. We can also prove the Lipschitz continuity of $Q_h^{I_0^c,\flat}$ as we did in (4.64). Lipschitz continuity of $Q_h^{I_0^c,\flat\flat}$ is a consequence of (4.67).

Let us now study the Hölder continuity of $Q_h^{I_0}$. Since the proof is very similar to the proof of Lemma 4.3.2, let us only consider $Q_h^{I_0,\sharp}$. Let $t, t' \in J$ be such that $|t - t'| < 1/2$ and define $j_0 > 0$ so that $2^{-(j_0+1)} \leq |t - t'| < 2^{-j_0}$. Then,

$$\begin{aligned} Q_h^{I_0,\sharp}(t) - Q_h^{I_0,\sharp}(t') = &\sum_{j=1}^{j_0-1} \sum_{k\in\mathbb{Z},\ k/2^j \in I_0} (\chi_{j,k,1}(t, h(k/2^j)) - \chi_{j,k,1}(t', h(k/2^j)))\eta_{j,k} \\ &+ \sum_{j=j_0}^{+\infty} \sum_{k\in\mathbb{Z},\ k/2^j \in I_0} (\chi_{j,k,1}(t, h(k/2^j)) - \chi_{j,k,1}(t', h(k/2^j)))\eta_{j,k}. \end{aligned}$$

Because of (4.49) and of (2.81)

$$\left|\sum_{j=1}^{j_0-1} \sum_{k\in\mathbb{Z},\ k/2^j\in I_0} (\chi_{j,k,1}(t, h(k/2^j)) - \chi_{j,k,1}(t', h(k/2^j))\eta_{j,k}\right| \leq C|t-t'|$$
$$\times \sum_{j=1}^{j_0-1} \sum_{k\in\mathbb{Z},\ k/2^j\in I_0} 2^{j(1-h(k/2^j))} \left(\frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1+|2^j t - k|^2}\right.$$
$$\left. + \frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1+|k|^2}\right).$$

Since $k/2^j \in I_0$, we have $2^{j(1-h(k/2^j))} < 2^{j(1-H^*(I_0))}$ and

$$\left|\sum_{j=1}^{j_0-1} \sum_{k\in\mathbb{Z},\ k/2^j\in I_0} (\chi_{j,k,1}(t, h(k/2^j)) - \chi_{j,k,1}(t', h(k/2^j))\eta_{j,k}\right| \leq C|t-t'|$$
$$\times \sum_{j=1}^{j_0-1} 2^{j(1-H^*(I_0))} \sum_{k\in\mathbb{Z},\ k/2^j\in I_0} \left(\frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1+|2^j t - k|^2}\right.$$
$$\left. + \frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1+|k|^2}\right). \qquad (4.68)$$

Then,

$$\left|\sum_{j=1}^{j_0-1} \sum_{k\in\mathbb{Z},\ k/2^j\in I_0} (\chi_{j,k,1}(t, h(k/2^j)) - \chi_{j,k,1}(t', h(k/2^j)))\eta_{j,k}\right|$$
$$\leq C 2^{j_0(1-H')}|t-t'| \qquad (4.69)$$

for every $H' < H^*(I_0)$. Because of (4.48)

$$\left|\sum_{j=j_0}^{+\infty} \sum_{k\in\mathbb{Z},\ k/2^j\in I_0} (\chi_{j,k,1}(t, h(k/2^j)) - \chi_{j,k,1}(t', h(k/2^j)))\eta_{j,k}\right|$$
$$\leq C \sum_{j=j_0}^{+\infty} 2^{-H'j} \left(\frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1+|2^j t - k|^2}\right.$$
$$\left. + \frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1+|k|^2}\right) |\eta_{j,k}|.$$

It yields

$$\left| \sum_{j=j_0}^{+\infty} \sum_{k\in\mathbb{Z},\ k/2^j\in I_0} (\chi_{j,k,1}(t, h(k/2^j)) - \chi_{j,k,1}(t', h(k/2^j)))\eta_{j,k} \right| \le C 2^{-H' j_0}. \quad (4.70)$$

Since $|t - t'| < 2^{-j_0}$, and because of (4.69), (4.70)

$$\begin{aligned} |Q_h^{I_0,\sharp}(t) - Q_h^{I_0,\sharp}(t')| &\le C 2^{-j_0 H'} \\ &\le C|t - t'|^{H'}, \end{aligned}$$

because $2^{-(j_0+1)} \le |t - t'|$. It is straightforward to show that $Q_h^{I_0,\flat}$, $Q_h^{I_0,\flat\flat}$ are Lipschitz. The proof of the theorem is finished.

One can also prove the slass property for the sfBm.

**Proposition 4.3.7** *Let $Q_h$ be a sfBm with multifractional function $h$ satisfying **Hyp 4.3.1**. For all $t$ fixed in $(a_i, a_{i+1})$ it is strongly locally self-similar*

$$\lim_{\epsilon\to 0^+} \left( \frac{Q_h(t+\epsilon u) - Q_h(t)}{\epsilon^{H_i}} \right)_{u\in\mathbb{R}} \overset{(d)}{=} C(H_i)^{1/2} \left(B_{H_i}(u)\right)_{u\in\mathbb{R}}$$

*where $B_{H_i}$ is a fractional Brownian motion with Hurst exponent $H_i$. The convergence is a convergence in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets.*

*Proof of Proposition 4.3.7*:

The strong local self-similarity is obtained by using the same technique as for fwn and mBm. Hence, we first give an asymptotic result for the variance of the increments of the sfBm.

**Lemma 4.3.3** *Let $Q_h$ be a sfBm with multifractional function $h$ satisfying **Hyp 4.3.1**. For all $t$ fixed in $(a_i, a_{i+1})$,*

$$\lim_{\varepsilon\to 0^+} \frac{\mathbb{E}(Q_h(t+\varepsilon u_1) - Q_h(t+\varepsilon u_2))^2}{\varepsilon^{2H_i}} = C(H_i)|u_1 - u_2|^{2H_i} \quad (4.71)$$

$\forall u_1,\ u_2 \in \mathbb{R}$.

*Proof of Lemma 4.3.3*:

Let us fix $t \in (a_i, a_{i+1})$, where the index $i$ is uniquely determined. Since we do not have a convenient integral representation of the sfBm, we will compare $\mathbb{E}(Q_h(t + \varepsilon u_1) - Q_h(t + \varepsilon u_2))^2$ with the same quantity for $B_{H_i}$. Because of the series expansion of fractional Brownian motion we have

$$
\begin{aligned}
\mathbb{E}(Q_h(t+\varepsilon u_1)-Q_h(t+\varepsilon u_2))^2 &= \sum_{\lambda\in\Lambda^+}(\chi_\lambda(t+\varepsilon u_1,h(\lambda))-\chi_\lambda(t+\varepsilon u_2,h(\lambda)))^2 \\
&= \sum_{j=1}^{n}\sum_{\substack{\lambda\in\Lambda^+,\\ \lambda\in[a_j,a_{j+1})}}(\chi_\lambda(t+\varepsilon u_1,H_j)-\chi_\lambda(t+\varepsilon u_2,H_j))^2 \\
&= \sum_{\lambda\in\Lambda^+}(\chi_\lambda(t+\varepsilon u_1,H_i)-\chi_\lambda(t+\varepsilon u_2,H_i))^2 \qquad (4.72)\\
&\quad -\sum_{\substack{\lambda\in\Lambda^+,\\ \lambda\notin[a_i,a_{i+1})}}(\chi_\lambda(t+\varepsilon u_1,H_i)-\chi_\lambda(t+\varepsilon u_2,H_i))^2 \qquad (4.73)\\
&\quad +\sum_{\substack{j\neq i,\\ \lambda\in\Lambda^+,\\ \lambda\in[a_j,a_{j+1})}}(\chi_\lambda(t+\varepsilon u_1,H_j)-\chi_\lambda(t+\varepsilon u_2,H_j))^2. \qquad (4.74)
\end{aligned}
$$

Since fractional Brownian motion can be written

$$
B_{H_i}(t)=\frac{1}{C(H_i)^{1/2}}\sum_{\lambda\in\Lambda^+}\chi_\lambda(t,H_i)\eta_\lambda,
$$

the right hand side of (4.72) is equal to

$$
C(H_i)\mathbb{E}(B_{H_i}(t+\varepsilon u_1)-B_{H_i}(t+\varepsilon u_2))^2=C(H_i)\varepsilon^{2H_i}|u_1-u_2|^{2H_i}.
$$

Let us show that the right hand side of (4.73) is negligible with respect to $\varepsilon^{2H_i}$, when $\varepsilon\to 0^+$. Since $u_1,\ u_2$ are fixed, one can assume that $\varepsilon$ is small enough so that $t+\varepsilon u_1,\ t+\varepsilon u_2\in(a_i,a_{i+1})$. Moreover $\lambda\notin[a_i,a_{i+1})$ in (4.73), and there exists $\delta>0$ such that $\min(|t+\varepsilon u_1-\lambda|,|t+\varepsilon u_2-\lambda|)>\delta$. Then,

$$
\sum_{\lambda\in\Lambda^+,\ \lambda\notin[a_i,a_{i+1})}(\chi_\lambda(t+\varepsilon u_1,H_i)-\chi_\lambda(t+\varepsilon u_2,H_i))^2\le C\varepsilon^2|u_1-u_2|^2,
$$

where $C$ is a generic positive constant. The proof of this fact is similar to the proof of the Lipschitz continuity of $Q_h^{I_0^c}$ and is omitted .

Similar arguments show that the right hand side of (4.74) is negligible with respect to $\varepsilon^{2H_i}$, when $\varepsilon\to 0^+$. Hence, the proof of Lemma 4.3.3 is finished.

The proof of Proposition 4.3.7 using Lemma 4.3.3 is similar to the proof of Proposition 4.3.5 and left to the reader.

### *4.3.4 Generalized Multifractional Gaussian Process*

As stated in the section devoted to the sfBm an important drawback of the mBm is the regularity assumption, we need to assume on the multifractional function, to compute the Hölder exponent of the sample path. Actually if we consider the series expansion (4.26) of the mBm and consider only a finite number of terms in

$$\sum_{\lambda \in \Lambda} \chi_\lambda(t, h(t))\eta_\lambda,$$

the sum process cannot be in general more regular than $h$. On the other hand the Hölder exponent of the sample paths of the mBm is related to the behavior of $\chi_\lambda(x, y)$, when the frequency part $j$ of $\lambda = (j, k)$, tends to infinity. So, in some sense, the obstruction to have an irregular multifractional function is a "low" frequency problem (i.e. $j \leq j_0$ for $j_0$ fixed). These considerations lead to propose a generalized model of the mBm, where the multifractional function $h(t)$ may depend on the frequency and, thus, becomes a function $(t, \xi) \mapsto H(t, \xi)$. Let us discuss more precisely assumptions required for this function.

***Hyp 4.3.2*** *A function* $H : \mathbb{R} \times \mathbb{R} \mapsto [a, b] \subset (0, 1)$ *is called a frequency lift of a multifractional function h if*

$$h(t) = \lim_{\xi \to +\infty} H(t, \xi), \tag{4.75}$$

$$H(t, -\xi) = H(t, \xi), \tag{4.76}$$

*H is twice differentiable with respect to the variable* $\xi$ *and satisfies*

- *for* $0 < |\xi| \leq \frac{4\pi}{3}$,
$$H(t, \xi) = b;$$
- *for* $\frac{4\pi}{3} < |\xi|$, $\exists g$ *such that* $0 < g < \min(a/2, 1/4)$ *and* $\exists \beta$ *such that* $b < \beta \leq 1$
$$\left| \frac{\partial^k H}{\partial \xi^k}(t, \xi) - \frac{\partial^k H}{\partial \xi^k}(t', \xi) \right| \leq C|\xi|^{g-k}|t - t'|^\beta, \tag{4.77}$$
*for* $t,\ t' \in [0, 1]$, *and* $k = 0, 1, 2$,
$$\left| \frac{\partial^k H}{\partial \xi^k}(t, \xi) \right| \leq C|\xi|^{g-k}, \tag{4.78}$$
*for* $t \in [0, 1]$, *and* $k = 0, 1, 2$.

These assumptions, that are needed to construct an Hölder continuous model, are quite technical and we will give some comments before introducing the model. First if one wishes a model similar to mBm but for an irregular multifractional function $h$, we have to construct a frequency lift of the desired multifractional function $h$. Examples of such construction are given later in the section, but it is obvious that there are in general many frequency lifts of a multifractional function. In ***Hyp 4.3.2*** the cut-off at $\xi = \frac{4\pi}{3}$ is arbitrary and can be replaced by any non-negative constant. The inequality $0 < g < \min(a/2, 1/4)$ has a deeper meaning: it expresses the fact that the Hölder constant of $H(t, \xi)$ cannot grow too fast when $\xi \to +\infty$. Finally, the inequality $b < \beta$ is similar to the condition $\beta > \sup_{x \in \mathbb{R}^d} h(x)$ in Theorem 4.3.3. Let us now introduce the model.

**Definition 4.3.6** *Let $H : \mathbb{R} \times \mathbb{R} \to (a, b) \subset (0, 1)$ be a frequency lift function of a multifractional function h that satisfies* ***Hyp 4.3.2****. A real valued field $X_H$ is called a generalized multifractional Gaussian process (in short gmGp) with function $H$, if it admits the harmonizable representation*

$$X_H(t) \overset{(d)}{=} \int_{\mathbb{R}} \frac{e^{-it\cdot\xi} - 1}{|\xi|^{1/2+H(t,\xi)}} \widehat{W}(d\xi). \tag{4.79}$$

**Remark 4.3.11** *One can remark that if $H_1$ and $H_2$ are two frequency lifts of the same multifractional function, then the gmGps $X_{H_1}$ and $X_{H_2}$ have different distributions. Nevertheless they share many local properties, since the frequency lifts have the same asymptotic behavior when $\xi \to +\infty$. The following result on the regularity of the sample path of gmGp is an example of a local property, which is the same for all frequency lifts of a multifractional function.*

Let us state the regularity result for gmGp.

**Theorem 1** *Under* ***Hyp 4.3.2****, the pointwise Hölder exponent of a gmGp associated to a frequency lift H and a multifractional function h at each point t satisfies almost surely,*

$$\sup\left\{H',\ \lim_{\epsilon\to 0} \frac{X_H(t+\epsilon) - X_H(t)}{|\epsilon|^{H'}} = 0\right\} = h(t). \tag{4.80}$$

*Proof of Theorem 1*:

The proof is admitted and can be found in [7].

Let us now prove the slass property for the gmGp. As we did for the mBm, let us start with a proposition that yields the asymptotic behavior of the $L^2$ norm of the increments. The slass requires that $H$ converges to $h$ fast enough. Technically it leads to the ***Hyp 4.3.3***, which is written in the following proposition.

**Proposition 4.3.8** *Let $H : \mathbb{R} \times \mathbb{R} \to (a, b) \subset (0, 1)$ be a frequency lift function of a multifractional function h, which satisfies* ***Hyp 4.3.2****, with $\beta = 1$. For a fixed $t$, let us assume that $\frac{2+3a}{4-a} > h(t)$ and that*

***Hyp 4.3.3*** *There exist $C, R > 0$ and $c > \frac{h(t)-a}{1-h(t)}$*

$$|H(t, \xi) - h(t)| < C/|\xi|^{2c}, \tag{4.81}$$

*for $|\xi| > R$. Then, the associated gmGp $X_H$ satisfies*

$$\lim_{\varepsilon \to 0^+} \frac{\mathbb{E}(X_H(t+\varepsilon u_1) - X_H(t+\varepsilon u_2))^2}{\varepsilon^{2h(t)}} = C(h(t))|u_1 - u_2|^{2h(t)}. \tag{4.82}$$

*Moreover the limit is uniform, when $u_1, u_2 \in K$, where $K$ is a fixed compact.*

**Remark 4.3.12** *If $a \geq 1/2$ then $\frac{2+3a}{4-a} > h(t), \ \forall t \in \mathbb{R}$.*

*Proof of Proposition 4.3.8*:
By the isometry

$$\mathbb{E}(X_H(t+\varepsilon u_1) - X_H(t+\varepsilon u_2))^2 = \int_{\mathbb{R}} \frac{|T|^2(x, \xi, \varepsilon, u_1, u_2)}{|\xi|^{2(H(t+\varepsilon u_2,\xi)+H(t+\varepsilon u_1,\xi))+1}} \frac{d\xi}{\sqrt{2\pi}}$$

where the variables are dropped in what follows,

$$T = (e^{-i(t+\varepsilon u_1)\xi} - 1)|\xi|^{H(t+\varepsilon u_2,\xi)} - (e^{-i(t+\varepsilon u_2)\xi} - 1)|\xi|^{H(t+\varepsilon u_1,\xi)}.$$

Let us split $T = T_1 + T_2$, where

$$\begin{aligned} T_1 &= |\xi|^{H(t+\varepsilon u_2,\xi)+1/2} e^{-i(t+\varepsilon u_2)\xi} (e^{i\varepsilon(u_2-u_1)\xi} - 1) \\ T_2 &= (e^{-i(t+\varepsilon u_2)\xi} - 1) \left( |\xi|^{H(t+\varepsilon u_2,\xi)+1/2} - |\xi|^{H(t+\varepsilon u_1,\xi)+1/2} \right). \end{aligned}$$

Let us define

$$I_{ij} = \int_{\mathbb{R}} \frac{T_i \bar{T}_j}{|\xi|^{2(H(t+\varepsilon u_1,\xi)+H(t+\varepsilon u_2,\xi)+1)}} \frac{d\xi}{(2\pi)^{1/2}},$$

for $1 \leq i, \ j \leq 2$. Hence,

$$\mathbb{E}(X_H(t+\varepsilon u_1) - X_H(t+\varepsilon u_2))^2 = \sum_{1 \leq i, \ j \leq 2} I_{ij}. \tag{4.83}$$

Let us compare

$$I_{11} = \int_{\mathbb{R}} \frac{|e^{i\varepsilon(u_2-u_1)\xi} - 1|^2}{|\xi|^{2H(t+\varepsilon u_1,\xi)+1}} \frac{d\xi}{(2\pi)^{1/2}} \tag{4.84}$$

with

$$\tilde{I}_{11} = \int_{\mathbb{R}} \frac{|e^{i\varepsilon(u_2-u_1)\xi} - 1|^2}{|\xi|^{2H(t,\xi)+1}} \frac{d\xi}{(2\pi)^{1/2}}. \tag{4.85}$$

Let us denote

$$R_{11} = I_{11} - \tilde{I}_{11}$$
$$= \int_{\mathbb{R}} |e^{i\varepsilon(u_2-u_1)\xi} - 1|^2 \left( \frac{1}{|\xi|^{2H(t+\varepsilon u_1,\xi)+1}} - \frac{1}{|\xi|^{2H(t,\xi)+1}} \right) \frac{d\xi}{(2\pi)^{1/2}}.$$

Let us bound the parenthesis in $R_{11}$ by using (4.77), it yields

$$\left( \frac{1}{|\xi|^{2H(t+\varepsilon u_1,\xi)+1}} - \frac{1}{|\xi|^{2H(t,\xi)+1}} \right) \leq C \frac{|H(t,\xi) - H(t+\varepsilon u_1,\xi)| \log|\xi|}{|\xi|^{2a+1}}$$
$$\leq C \frac{\varepsilon |u_1| \log|\xi| |\xi|^g}{|\xi|^{2a+1}}$$

for $4\pi/3 \leq |\xi| \leq C\varepsilon^{-1/g}$. Let $u_1 \neq u_2, 0 < H' < \frac{1-h(t)}{1-a}$, and $g < g' < \min(a/2, 1/4)$. Since $H(t,\xi) = H(t+\varepsilon u_1,\xi)$ for $|\xi| \leq 4\pi/3$, we get

$$R_{11} \leq C \int_{4\pi/3 \leq |\xi| \leq (\varepsilon|u_1-u_2|)^{-H'}} \frac{(\varepsilon|u_1-u_2|)^2}{|\xi|^{2a-1}} d\xi \tag{4.86}$$
$$+ C \int_{(\varepsilon|u_1-u_2|)^{-H'} \leq |\xi| \leq \varepsilon^{-1/g'}} \left( \frac{1}{|\xi|^{2H(t+\varepsilon u_1,\xi)+1}} - \frac{1}{|\xi|^{2H(t,\xi)+1}} \right) d\xi \tag{4.87}$$
$$+ C \int_{\varepsilon^{-1/g'} \leq |\xi|} \frac{d\xi}{|\xi|^{2a+1}}. \tag{4.88}$$

Please note that the term in (4.87) disappears if $\{(\varepsilon|u_1-u_2|)^{2H'-2} \leq |\xi| \leq \varepsilon^{-1/g'}\}$ is an empty set. Then,

$$R_{11} \leq C \left( (\varepsilon|u_1-u_2|)^{2-(2-2a)H'} + \varepsilon|u_1| \int_{(\varepsilon|u_1-u_2|)^{-H'} \leq |\xi|} |\xi|^{g'-2a-1} d\xi + \varepsilon^{2a/g'} \right)$$

Hence,

$$R_{11} \leq C \Big( (\varepsilon|u_1-u_2|)^{2-(2-2a)H'}$$
$$+ |u_1| \; |u_1-u_2|^{-H'(g'-2a)} \varepsilon^{-H'(g'-2a)+1} + \varepsilon^{2a/g'} \Big).$$

Since $\frac{2+3a}{4-a} > h(t)$ and $g < a/2$, $\frac{1+a-g}{2-g} > h(t)$ and it can be rewritten

$$\frac{1-h(t)}{1-a}(2a-g) + 1 > 2h(t).$$

Hence, there exists $H' < \frac{1-h(t)}{1-a}$ such that $-H'(g'-2a)+1 > 2h(t)$. For such a choice of $H'$, $R_{11} = o(\varepsilon^{2h(t)})$. Let us now consider the asymptotic of $\tilde{I}_{11}$ when $\varepsilon \to 0$. First,

$$\begin{aligned}
&\tilde{I}_{11} - \varepsilon^{2h(t)}|u_1-u_2|^{2h(t)}C(h(t)) \\
&\quad = \int_{\mathbb{R}} |e^{i\varepsilon(u_2-u_1)\xi}-1|^2 \left(\frac{1}{|\xi|^{2H(t+\varepsilon u_1,\xi)+1}} - \frac{1}{|\xi|^{2h(t)+1}}\right)\frac{d\xi}{(2\pi)^{1/2}}.
\end{aligned} \tag{4.89}$$

Hence,

$$\tilde{I}_{11} - \varepsilon^{2h(t)}|u_1-u_2|^{2h(t)}C(h(t)) \le C\int_{|\xi|\le 4\pi/3} \frac{(\varepsilon|u_1-u_2|)^2}{|\xi|^{2b-1}}d\xi \tag{4.90}$$

$$+ C\int_{4\pi/3<|\xi|\le(\varepsilon|u_1-u_2|)^{-H'}} \frac{(\varepsilon|u_1-u_2|)^2}{|\xi|^{2a-1}}d\xi \tag{4.91}$$

$$+ C\int_{(\varepsilon|u_1-u_2|)^{-H'}<|\xi|} \frac{|H(t,\xi)-h(t)|\log|\xi|}{|\xi|^{2a+1}}d\xi. \tag{4.92}$$

One easily gets that the term in (4.90) is a $o(\varepsilon^{2h(t)})$. If we take $H' < \frac{1-h(t)}{1-a}$ we already proved that the term in (4.91), which is similar to (4.87), is a $o(\varepsilon^{2h(t)})$. Actually

$$\int_{(\varepsilon|u_1-u_2|)^{-H'}<|\xi|} \frac{|H(t,\xi)-h(t)|\log|\xi|}{|\xi|^{2a+1}}d\xi < C(\varepsilon|u_1-u_2|)^{2H'(a+c')},$$

for $c' < c$, because of assumption ***Hyp 4.3.3***. Since $c > \frac{h(t)-a}{1-h(t)}$, one can choose $c' < c$, $H' < \frac{1-h(t)}{1-a}$ such that $2H'(a+c) > 2h(t)$. Hence, (4.92) is also a $o(\varepsilon^{2h(t)})$. Let us now consider

$$\begin{aligned}
&I_{22} \\
&\quad = \int_{4\pi/3\le|\xi|} |e^{-i(t+\varepsilon u_2)\xi}-1|^2 \left(|\xi|^{-H(t+\varepsilon u_1,\xi)-1/2} - |\xi|^{-H(t+\varepsilon u_2,\xi)-1/2}\right)^2 \frac{d\xi}{(2\pi)^{1/2}}.
\end{aligned}$$

If $g < g' < \min(a/2, 1/4)$, then

$$I_{22} \le C\varepsilon^2|u_2-u_1|^2 \int_{4\pi/3<|\xi|\le(\varepsilon|u_1-u_2|)^{-1/g'}} \frac{|e^{-i(t+\varepsilon u_2)\xi}-1|^2\log^2\xi|\xi|^g}{|\xi|^{2H(t+\varepsilon u_1,\xi)}}d\xi \tag{4.93}$$

$$+ C\int_{(\varepsilon|u_1-u_2|)^{-1/g'}\le|\xi|} \frac{|e^{-i(t+\varepsilon u_2)\xi}-1|^2}{|\xi|^{2a+1}}d\xi. \tag{4.94}$$

The integral in (4.93) is bounded by $\int_{4\pi/3<|\xi|\leq\infty}\log^2\xi|\xi|^{g-2a-1}<+\infty$. The term in (4.94) is bounded by $C(\varepsilon|u_1-u_2|)^4$. Hence the proposition is proved.

One can now prove the slass property for the gmGp.

**Proposition 4.3.9** *Let $H:\mathbb{R}\times\mathbb{R}\to(a,b)\subset(0,1)$ be a frequency lift function of a multifractional function $h$, which satisfies **Hyp 4.3.2**, with $\beta=1$. For a fixed $t$, let us assume that $\frac{2+3a}{4-a}>h(t)$ and that **Hyp 4.3.3** is fulfilled. The corresponding gmGp $X_H$ is strongly locally asymptotically self-similar at point $t$, with tangent field a fractional Brownian motion with Hurst exponent $H=h(t)$. More precisely*

$$\lim_{\varepsilon\to 0^+}\left(\frac{X_H(t+\varepsilon u)-X_H(t)}{\varepsilon^{h(t)}}\right)_{u\in\mathbb{R}^d}\overset{(d)}{=}C(H)(B_H(u))_{u\in\mathbb{R}^d}\tag{4.95}$$

*where $H=h(t)$ and $B_H$ is a fractional Brownian motion with Hurst exponent $H$.*

*Proof of Proposition 4.3.9*:

To prove the convergence of finite dimensional margins, we use Proposition 4.3.8. Let us fix $t\in\mathbb{R}$, for every $u_1,\ u_2\in\mathbb{R}$

$$\frac{\mathbb{E}(X_H(t+\varepsilon u_1)-X_H(t))(X_H(t+\varepsilon u_2)-X_H(t))}{\varepsilon^{2h(t)}}=\frac{1}{2\varepsilon^{2h(t)}}$$
$$(\mathbb{E}(X_H(t+\varepsilon u_1)-X_H(t))^2+\mathbb{E}(X_H(t+\varepsilon u_2)-X_H(t))^2$$
$$-\mathbb{E}(X_H(t+\varepsilon u_2)-X_H(t+\varepsilon u_1))^2).$$

Hence,

$$\lim_{\varepsilon\to 0^+}\left(\frac{\mathbb{E}(X_H(t+\varepsilon u_1)-X_H(t))(X_H(t+\varepsilon u_2)-X_H(t))}{\varepsilon^{2h(t)}}\right)$$
$$=\frac{C(h(t))}{2}\{|u_1|^{2h(t)}+|u_2|^{2h(t)}-|u_1-u_2|^{2h(t)}\},$$

which yields the convergence of finite dimensional margins.

Then, to have the convergence in distribution in the space of continuous functions endowed with the topology of the uniform convergence on every compact, we have to prove the tightness of the renormalized increments

$$Y_\varepsilon(u)=\frac{X_H(t+\varepsilon u)-X_H(t)}{\varepsilon^{h(t)}}.$$

Let us fix a compact $K$ and let us suppose that $u_1,\ u_2\in K$, then

$$\lim_{\varepsilon\to 0^+}\mathbb{E}(Y_\varepsilon(u_1)-Y_\varepsilon(u_2))^2=C(h(t))|u_1-u_2|^{2h(t)}$$

uniformly on $K$. Hence,

$$\sup_{\varepsilon>0} \mathbb{E}(Y_\varepsilon(u_1) - Y_\varepsilon(u_2))^2 < C|u_1 - u_2|^{2h(t)}.$$

Then, one can choose $p \in \mathbb{N}$ such that $ph(t) > 1$ and apply Corollary 2.1.2, since

$$\mathbb{E}(Y_\varepsilon(u_1) - Y_\varepsilon(u_2))^{2p} \le C|u_1 - u_2|^{ph(t)}.$$

#### 4.3.4.1 Examples

In this section we present three examples of application of Theorem 1, that illustrate the way this theorem can be used. In each of these examples, one assumes given a precise multifractional function. Then, one constructs a gmGp by exhibiting a frequency lift $H(t, \xi)$ such that the assumption ***Hyp 4.3.2*** is fulfilled.

Hölder continuous multifractional functions

Remark first that if the multifractional function $h$ is $\beta$-Hölder continuous with $\beta > b$, one may choose a frequency lift function $H(t, \xi)$ that interpolates a constant for $0 \le \xi \le \frac{4\pi}{3}$ and the multifractional function $h(t)$ when $\xi \ge \frac{8\pi}{3}$. One can take for instance

$$H(t, \xi) = b \quad \text{if} \quad 0 \le \xi \le \frac{4\pi}{3}$$

and

$$H(t, \xi) = b + (h(t) - b)\sin^4\left(\frac{3\xi}{8} - \frac{\pi}{2}\right) \quad \text{if} \quad \frac{4\pi}{3} \le \xi \le \frac{8\pi}{3},$$

and $H(t, \xi) = h(t)$ if $\xi \ge \frac{8\pi}{3}$. The frequency lift $H(t, \xi)$ is defined by (4.76), when $\xi < 0$. In this case the gmGp exhibits the same properties as a multifractional Brownian motion.

Multifractional functions with one jump

In this example we consider a multifractional function that has only one discontinuity and that is piecewise constant. This very simple example can be easily extended to the case of piecewise constant functions with a finite number of jumps. The gmGp associated to this kind of multifractional function are comparable to sfBm for the applications. Let us suppose that

$$h(t) = b\mathbf{1}_{[0,t_0]} + a\mathbf{1}_{(t_0,1]} \tag{4.96}$$

where $t_0 \in (0, 1)$ and $0 < a < b < 1$. Then one can construct a frequency lift $H$ of $h$ as follows:

- For $0 < \xi \le \frac{4\pi}{3}$,

$$H(t, \xi) = b \quad \forall t \in [0, 1].$$

- for $\frac{4\pi}{3} < \xi$ let us choose $0 < f < \min(a/6, 1/12)$

$$\begin{aligned} H(t, \xi) &= b && \text{if} \quad t \leq t_0 \\ &= a + (b - a) \sin^4 \left( \frac{\pi \xi^f}{2} \left( t - t_0 - \frac{1}{\xi^f} \right) \right) && \text{if} \quad t \in \left[ t_0, t_0 + \frac{1}{\xi^f} \right) \\ &= a && \text{if} \quad t \geq t_0 + \frac{1}{\xi^f}. \end{aligned}$$

The frequency lift $H(t, \xi)$ is defined by (4.76), when $\xi < 0$. An elementary computation shows that ***Hyp 4.3.2*** is fulfilled with $g = 3f$ and $\beta = 1$. Moreover this gmGp is slass at every point $t$ such that $\frac{2+3a}{4-a} > h(t)$. Actually ***Hyp 4.3.3*** is satisfied since $H(t, \xi) = h(t)$ for $\xi$ large enough.

Multifractional functions with an accumulation of jumps

Let us address the more interesting case of piecewise constant multifractional functions which have an infinite number of jumps. In this example let us suppose the multifractional function is defined by

$$\begin{aligned} h(t) &= 1/2 && \text{if } t \leq 1/2 \\ &= \frac{1}{2} + \frac{1}{k+1} && \text{if } \frac{1}{2} + \frac{1}{k+1} < t \leq \frac{1}{2} + \frac{1}{k} \\ &= \frac{1}{2} + \frac{1}{3} && \text{if } 1 \leq t \end{aligned} \tag{4.97}$$

where $k \geq 2$. The multifractional function has an accumulation of jumps at time $t = 1/2$. Let us now exhibit a frequency lift of the multifractional function defined in (4.97) by using the construction of the previous example, which is valid when the multifractional function has only a finite number of jumps. Let us define:

$$\begin{aligned} h_{k_0}(t) &= 1/2 && \text{if } t \leq 1/2 \\ &= \frac{1}{2} + \frac{1}{k_0+1} && \text{if } 1/2 < t \leq \frac{1}{2} + \frac{1}{k_0+1} \\ &= \frac{1}{2} + \frac{1}{k+1} && \text{if } \frac{1}{2} + \frac{1}{k+1} < t \leq \frac{1}{2} + \frac{1}{k}, \text{ for } 2 \leq k \leq k_0, \\ &= \frac{1}{2} + \frac{1}{3} && \text{if } 1 \leq t \end{aligned} \tag{4.98}$$

when only $k_0$ jumps are considered. Then one can construct, with the same technique as in the example of the previous paragraph, a frequency lift of $h_{k_0}$ which is denoted by

$H_{k_0}(t,\xi)$ and is defined for $(k_0(k_0+1))^{1/f} \le \xi$. If we fix $f < \min(a/6, 1/12)$ in the previous example, then this last condition simply means that $\left[\frac{1}{2}+\frac{1}{k}, \frac{1}{2}+\frac{1}{k}+\frac{1}{\xi^f}\right]$ is included in the intervals where $h_{k_0}$ is constant. Hence this condition allows us to manage the jumps of $h_{k_0}$ separately. Let us now introduce a frequency lift of (4.97) by taking more and more jumps into account when $\xi \to +\infty$. Define

$$K(\xi) = \sup\{k \in \mathbb{N}^* \text{ such that } (k(k+1))^{1/f} \le \xi\} \tag{4.99}$$

and define the frequency lift function with the $H_{k_0}$'s

$$H(t,\xi) = H_{K(\xi)}(t,\xi). \tag{4.100}$$

It is not difficult to check that this frequency lift satisfies ***Hyp 4.3.2*** since

$$\frac{\partial^m H}{\partial \xi^m}(t,\xi) = \frac{\partial^m H_{k_0}}{\partial \xi^m}(t,\xi)$$

when $(k_0(k_0+1))^{1/f} < \xi < ((k_0+1)(k_0+2))^{1/f}$. Moreover, in this example the gmGp is everywhere slass. First $a = 1/2$ implies that $\frac{2+3a}{4-a} > h(t),\ \forall t \in \mathbb{R}$. Second if $t \le 1/2$, $H(t,\xi) = h(t)$. If $t > 1/2$, let $k$ be such that $\frac{1}{2}+\frac{1}{k+1} < t \le \frac{1}{2}+\frac{1}{k}$. Then for $\xi$ big enough

$$H(t,\xi) = H_k(t,\xi) = h_k(t) = h(t)$$

and ***Hyp 4.3.3*** is satisfied.

### *4.3.5 Gaussian Random Weierstrass Function*

In Sect. 3.2.3, we have introduced the real random Weierstrass function

$$\Re \mathcal{W}_r^{al} = \Re \sum_{n=-\infty}^{+\infty} \big(\exp(ir^n t) - 1\big)\, r^{-Hn}(\xi_n + i\eta_n)$$

with $(\xi_n, \eta_n)_{n\in\mathbb{Z}}$ independent identically distributed random vectors such that $\mathbb{E}\xi_n = \mathbb{E}\eta_n = 0$, $\mathbb{E}\xi_n^2 = \mathbb{E}\eta_n^2 = 1$, and $\xi_n$ and $\eta_n$ are independent. In this section, we will show that the real random Weierstrass function is not lass. From the point of view of this chapter, this section should be considered as a counter-example. For the sake of simplicity, we assume that $(\xi_n, \eta_n)_{n\in\mathbb{Z}}$ are Gaussian random vectors. First of all, we show that if the real random Weierstrass function was lass, the only possible index would be $H$. This fact is not surprising, if we recall that real random Weierstrass functions are $H$-semi-self similar, but we can illustrate this fact with the following Proposition.

**Proposition 4.3.10** *For $r > 1$ and $0 < H < 1$, let $\Re\mathcal{W}_r^{al}$ be a real random Weierstrass function, then there exists a positive constant $C$ such that*

$$\frac{1}{C}|t-s|^{2H} \leq \mathbb{E}\left(\Re\mathcal{W}_r^{al}(t) - \Re\mathcal{W}_r^{al}(s)\right)^2 \leq C|t-s|^{2H} \tag{4.101}$$

*for $|t-s|$ small enough.*

*Proof of Proposition 4.3.10*:

Let us first recall the series expansion for $\mathbb{E}\left(\Re\mathcal{W}_r^{al}(t) - \Re\mathcal{W}_r^{al}(s)\right)^2$ obtained in Sect. 3.2.3. First,

$$\begin{aligned}\Re\mathcal{W}_r^{al}(t) - \Re\mathcal{W}_r^{al}(s) = \sum_{n=-\infty}^{+\infty} r^{-Hn}\big([\cos(r^n t) - \cos(r^n s)]\xi_n \\ - [\sin(r^n t) - \sin(r^n s)]\eta_n\big)\,.\end{aligned} \tag{4.102}$$

Hence,

$$\begin{aligned}\mathbb{E}\left(\Re\mathcal{W}_r^{al}(t) - \Re\mathcal{W}_r^{al}(s)\right)^2 &= \sum_{n=-\infty}^{+\infty} r^{-2Hn}\Big([\cos(r^n t) - \cos(r^n s)]^2 \\ &\qquad + [\sin(r^n t) - \sin(r^n s)]^2\Big) \\ &= \sum_{n=-\infty}^{+\infty} r^{-2Hn}|e^{ir^n t} - e^{ir^n s}|^2 \\ &= 4\sum_{n=-\infty}^{+\infty} r^{-2Hn}(1 - \cos(r^n(t-s))) \\ &= 8\sum_{n=-\infty}^{+\infty} r^{-2Hn}\sin^2\left(\frac{r^n(t-s)}{2}\right)\end{aligned}$$

Let us first show the upper bound. One can assume that $s+1 > t > s, r > 1$, and let $n_0 = -[\log(t-s)/\log(r)]$. For $n \leq n_0$,

$$\begin{aligned}\sum_{n\leq n_0} r^{-2Hn}\sin^2\left(\frac{r^n(t-s)}{2}\right) &\leq C\sum_{n\leq n_0} r^{2(1-H)n}(t-s)^2 \\ &\leq Cr^{-2n_0 H} \\ &\leq C(t-s)^{2H}.\end{aligned}$$

Then

$$\sum_{n\geq n_0} r^{-2Hn}\sin^2\left(\frac{r^n(t-s)}{2}\right) \leq 4\sum_{n\geq n_0} r^{-2Hn}$$
$$\leq Cr^{-2Hn_0}$$
$$\leq C(t-s)^{2H}.$$

The upper bound for (4.101) is established.

Let us denote by $\sigma_r(t) = 8\sum_{n=-\infty}^{+\infty}(r^n t)^{-2H}\sin^2\left(\frac{r^n t}{2}\right)$ for $t > 0$. It is straightforward that

$$\sigma_r(rt) = \sigma_r(t) \tag{4.103}$$

for each $t \in \mathbb{R}$. Moreover $\forall t \neq 0,\quad \sigma_r(t) > 0$ because $\forall t \neq 0$ and $r > 1$ there exists an integer $n \in \mathbb{Z}$ such that $\{r^n t,\ n \in \mathbb{Z}\} \not\subseteq \{k\pi,\ k \in \mathbb{Z}\}$. Moreover $\sigma_r$ is a continuous function of $t$ on $[1, r]$, then $\inf_{1\leq t\leq r}\sigma_r(t) > 0$. Consequently $\inf_{t\neq 0}\sigma_r(t) > 0$. Since $\frac{\mathbb{E}\left(\Re\mathcal{W}_r^{al}(t)-\Re\mathcal{W}_r^{al}(s)\right)^2}{|t-s|^{2H}} = \sigma_r(|t-s|)$, inequalities (4.101) are now established.

We can now prove that real random Weierstrass function are not lass.

**Proposition 4.3.11** *For $r > 1$ and $0 < H < 1$, real random Weierstrass functions are not lass for $H \neq 1/2$.*

*Proof of Proposition 4.3.11*:

In this section real random Weierstrass functions are Gaussian processes, then the lass property for parameter $H$ is equivalent to the existence of the limit of

$$\frac{\mathbb{E}\left(\Re\mathcal{W}_r^{al}(t+\epsilon) - \Re\mathcal{W}_r^{al}(t)\right)^2}{|\epsilon|^{2H}},$$

when $\epsilon \to 0$. Let $\epsilon_n = \theta r^{-n}$. If real random Weierstrass functions were lass, then

$$\lim_{n\to\infty}\frac{\mathbb{E}\left(\Re\mathcal{W}_r^{al}(t+\epsilon_n) - \Re\mathcal{W}_r^{al}(t)\right)^2}{|\epsilon_n|^{2H}}$$

would exist and would not depend on $\theta$. Let us exhibit a contradiction. First, let us remark that

$$\sigma_r(\theta) = \frac{\mathbb{E}\left(\Re\mathcal{W}_r^{al}(t+\theta r^{-n}) - \Re\mathcal{W}_r^{al}(t)\right)^2}{|\theta r^{-n}|^{2H}}.$$

It would be surprising that $\sigma_r(\theta)$ would be constant, but figures in [26] show $\sigma_r$ is numerically not far from being constant for some $r$. Hence, we give a short proof of the fact that $\sigma_r$ is not constant. Let us consider $F_r(\theta) = |\theta|^{2H}\sigma_r(\theta)$. When $H < 1/2$, (4.101) shows that $\theta \mapsto F_r(\theta)$ is not differentiable and, hence, $\sigma_r(\theta)$ cannot be constant. When $H > 1/2$, $F_r$ is differentiable and

$$(F_r)'(\theta) = 8 \sum_{n=-\infty}^{+\infty} r^{(1-2H)n} \sin(r^n \theta).$$

But one can show that $(F_r)'$ is not differentiable with similar arguments. Consequently $\sigma_r$ cannot be constant.

### 4.3.6 Anisotropy

A natural idea is to define the lass property in a directional way. Let us be more precise. We say that a stochastic field $Y$ has a directional lass property at point $x \in \mathbb{R}^d$ if

$$\lim_{\varepsilon \to 0^+} \left( \frac{Y(x+\varepsilon\theta u) - Y(x)}{\varepsilon^{H(\theta)}} \right)_{u \in \mathbb{R}^d, \theta \in S^{d-1}} \stackrel{(d)}{=} (T_{x,\theta}(u))_{u \in \mathbb{R}^d, \theta \in S^{d-1}}, \tag{4.104}$$

where the convergence is for finite dimensional distributions. The answer is somewhat disappointing [30]. Anisotropy does not really exist for Gaussian fields.

**Proposition 4.3.12** *Let $Y$ be a Gaussian stochastic field with stationary increments and having a directional lass property with function $H(\theta)$. Then*

- *either $H$ is constant,*
- *either $H$ is constant, except for one direction, where it is larger.*

*Proof of Proposition 4.3.12*:
Let us choose $\theta_0$, $\theta_1$ and $\theta$. For all $\varepsilon > 0$, let

$$\varepsilon\theta = r(\varepsilon)\theta_0 + s(\varepsilon)\theta_1,$$

where $r(\varepsilon)$ and $s(\varepsilon)$ are proportional to $\varepsilon$. Let now consider the increments

$$Y(\varepsilon\theta) - Y(0) = (Y(\varepsilon\theta) - Y(r(\varepsilon)\theta_0)) + (Y(r(\varepsilon)\theta_0) - Y(0)).$$

By stationarity of the increments, $(Y(\varepsilon\theta) - Y(r(\varepsilon)\theta_0))$ is distributed as $(Y(s(\varepsilon)\theta_1) - Y(0))$. Therefore

$$\mathbb{E}(Y(\varepsilon\theta) - Y(0))^2 \le 2\mathbb{E}(Y(s(\varepsilon)\theta_1) - Y(0))^2 + 2\mathbb{E}(Y(r(\varepsilon)\theta_0) - Y(0))^2.$$

Let us divide by $\varepsilon$ and let $\varepsilon \to 0^+$. Applying (4.104) shows that the map $H$ is such that, for all $\theta, \theta_0, \theta_1$, $H(\theta) \ge \min(H(\theta_0), H(\theta_1))$, which implies Proposition 4.3.12.

## 4.4 Lévy Fields

In this part we consider fractional fields that are non-Gaussian fields with finite variance. Moreover we will show that there are locally self-similar. In many aspects these fields are intermediate between Gaussian and stable fractional fields studied in the last chapter. Actually, it is well known that in some fields of applications data do not fit Gaussian models, see for instance [100, 129, 139] for image modeling. More recently other processes that are neither Gaussian nor stable have been proposed to model Internet traffic (cf. [41, 77]).

### *4.4.1 Moving Average Fractional Lévy Fields*

First a counterpart of moving average fractional stable field is introduced when a Lévy random measure of Sect. 2.1.8 is used in the integral representation.

**Definition 4.4.1** *Let $M(ds)$ be a random Lévy measure defined by (2.36) that satisfies the finite moment assumption (2.35). Let us call a real valued field $(X_H(t))_{t\in\mathbb{R}^d}$ which admits a well-balanced moving-average representation*

$$X_H(t) \stackrel{(d)}{=} \int_{\mathbb{R}^d} \left( ||t-s||^{H-\frac{d}{2}} - ||s||^{H-\frac{d}{2}} \right) M(ds),$$

*a moving average fractional Lévy field (in short mafLf) with parameter* $0 < H < 1$.

For the sake of simplicity, we omit the case $d = 1$, $H = 1/2$: $X_{1/2}(t)$ is equal in distribution to $\int_0^t M(ds)$, which is a Lévy process.

Let us illustrate this construction with a simple example: $d = 1$ and $\nu(du) = \frac{1}{2}(\delta_{-1}(du) + \delta_1(du))$, where $\delta$'s are Dirac masses. In this case $M(ds)$ is a compound Poisson random measure and can be written as an infinite sum of random Dirac masses

$$M(ds) = \sum_{n\in\mathbb{Z}} \delta_{S_n}(ds)\varepsilon_n$$

where $S_{n+1} - S_n$ are identically independent random variables with an exponential law, and $\varepsilon_n$ are identically distributed independent Bernoulli random variables such that $\mathbf{P}(\varepsilon_n = 1) = \mathbf{P}(\varepsilon_n = -1) = 1/2$. The $\varepsilon_n$'s are independent of the $S_n$'s. Since the measure $\nu$ is finite and $\int_{\mathbb{R}} u\nu(du) = 0$, the corresponding mafLm is in this special case

$$X(t) = \sum_{n=-\infty}^{+\infty} \varepsilon_n (|t-S_n|^{H-1/2} - |S_n|^{H-1/2}). \tag{4.105}$$

Even if the previous limit is in $L^2$ sense, it suggests that the regularity of the sample paths can be governed by $H - 1/2$. In the following section we need other tools to prove this fact, but this guess happens to be true. Moreover one obtains almost sure convergence by using shot noise series representation of the Lévy measure. This idea is used in the Sect. 6.5.

Because of the isometry property of the Lévy random measure, mafLfs have finite second order moments and moreover have the same covariance structure as fractional Brownian field. But they have different distributions than fractional Brownian field, in particular they are non-Gaussian.

**Proposition 4.4.1** *The covariance structure of mafLfs is*

$$R(s,t)=\mathbb{E}(X(s)X(t))=\frac{var(X(1))}{2}\left\{\|s\|^{2H}+\|t\|^{2H}-\|s-t\|^{2H}\right\}. \quad (4.106)$$

*MafLfs have stationary increments.*

*Proof of Proposition 4.4.1*:

The first claim is consequence of the isometry property for the Lévy random measure (2.37). Actually a consequence of (2.37) is that

$$\mathbb{E}\left(\int_{\mathbb{R}^d} f(s)M(ds)\int_{\mathbb{R}^d} g(s)M(ds)\right)=\int_{\mathbb{R}} u^2\nu(du)\int_{\mathbb{R}^d} f(s)g(s)ds. \quad (4.107)$$

Then $\mathbb{E}(X(s)X(t))=\mathbb{E}B_H(s)B_H(t)$ since fractional Brownian field and mafLf have the same kernel in their integral representation. Finally (4.106) is a consequence of (3.63).

Let $\theta=(\theta_1,\dots,\theta_n)$ and $t=(t_1,\dots,t_n)\in(\mathbb{R}^d)^n$, the logarithm of the characteristic function of the increments of the mafLf is

$$\begin{aligned}
&-\log\left(\mathbb{E}\exp\left(i\sum_{j=2}^n\theta_j(X_H(t_j)-X_H(t_1))\right)\right)\\
&=\int_{\mathbb{R}^d\times\mathbb{R}}\left[\exp\left(i\sum_{j=2}^n u\theta_j(\|t_j-s\|^{H-d/2}-\|t_1-s\|^{H-d/2})\right)\right.\\
&\quad\left.-1-i\sum_{j=2}^n u\theta_j\left(\|t_j-s\|^{H-d/2}-\|t_1-s\|^{H-d/2}\right)\right]ds\nu(du).
\end{aligned}$$

Then if we set $s'=t_1+s$ and use the invariance by translation of the Lebesgue measure one gets

$$-\log\left(\mathbb{E}\exp\left(i\sum_{j=2}^n\theta_j(X_H(t_j)-X_H(t_1))\right)\right)$$

$$= -\log\left(\mathbb{E}\exp\left(i\sum_{j=2}^{n}\theta_j X_H(t_j - t_1)\right)\right)$$

and the stationarity of the increments of the mafLfs is proved.

However mafLfs are not self-similar. We are now investigating asymptotic self-similarity at infinity of the mafLf. In the sense of Definition 4.2.3, mafLfs have a fractional Brownian field as tangent field at infinity.

**Proposition 4.4.2** *The mafLfs are asymptotically self-similar at infinity with parameter $H$*

$$\lim_{R\to+\infty}\left(\frac{X_H(Rt)}{R^H}\right)_{t\in\mathbb{R}^d} \overset{(d)}{=} \int_{\mathbb{R}} u^2\nu(du) \times (B_H(t))_{t\in\mathbb{R}^d}\,, \tag{4.108}$$

*where the convergence is the convergence of the finite dimensional margins and $B_H$ is a fractional Brownian field of index $H$.*

*Proof:*
Let us consider the multivariate function

$$g_{t,v,H}(R,s,u) = iu\sum_{k=1}^{n} v_k \frac{\|Rt_k - s\|^{H-d/2} - \|s\|^{H-d/2}}{R^H} \tag{4.109}$$

where $t = (t_1, \ldots, t_n) \in (\mathbb{R}^d)^n$, and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. Then

$$\mathbb{E}\exp\left(i\sum_{k=1}^{n} v_k \frac{X_H(Rt_k)}{R^H}\right)$$
$$= \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}} [\exp(g_{t,v,H}(R,s,u)) - 1 - g_{t,v,H}(R,s,u)]ds\nu(du)\right). \tag{4.110}$$

The change of variable $s = R\sigma$ is applied to the integral of the previous right hand term to get

$$\int_{\mathbb{R}^d\times\mathbb{R}} [\exp(R^{-d/2}g_{t,v,H}(1,\sigma,u)) - 1 - R^{-d/2}g_{t,v,H}(1,\sigma,u)]R^d d\sigma\,\nu(du). \tag{4.111}$$

Then as $R \to +\infty$, a dominated convergence argument yields that

$$\lim_{R\to+\infty}\mathbb{E}\exp\left(i\sum_{k=1}^{n} v_k \frac{X_H(Rt_k)}{R^H}\right)$$
$$= \exp\left(\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}} g^2_{t,v,H}(1,\sigma,u)d\sigma\,\nu(du)\right). \tag{4.112}$$

Therefore the logarithm of the previous limit is

$$-\frac{1}{2}\int_0^{+\infty} u^2\nu(du)\int_{\mathbb{R}^d}\left(\sum_{k=1}^{n} v_k(\|t_k-\sigma\|^{H-d/2}-\|\sigma\|^{H-d/2})\right)^2 d\sigma \quad (4.113)$$

and this last integral is the variance of $\sum_{k=1}^n v_k B_H(t_k)$ which concludes the proof of the convergence of finite dimensional margins.

#### 4.4.1.1 Regularity of the Sample Paths

To investigate the regularity of the sample paths of mafLf one can use the Kolmogorov Theorem 2.1.7 to show that the sample paths are locally Hölder-continuous for every exponent $H' < H - d/2$ when $H > d/2$. It is a direct application of the isometry property. The questions are then: What happens when $H < d/2$? If $H-d/2>0$ can we show that the "true" exponent is strictly larger than $H-d/2$? If we consider the integrand: $G(t,s) = \|t-s\|^{H-d/2} - \|s\|^{H-d/2}$ it is clear that when $H - d/2 < 0$, $G(.,s)$ is not locally bounded, and when $H > d/2$, it is not $H'$-Hölderian if $H' > H - d/2$ in a neighborhood of $s$. Following Rosinski's rule of the thumb in [120], it is known that the simple paths of the integral defining $X_H(t)$ cannot be "smoother" than the integrand $G$.

Let us now make precise statements.

**Proposition 4.4.3** *If $H > d/2$, for every $H' < H - d/2$ there exists a continuous modification of the mafLf $X_H$ such that almost surely the sample paths of $X_H$ are locally $H'$ Hölder continuous i.e.*

$$\mathbb{P}\left[\omega;\ \sup_{0<\|s-t\|<\epsilon(\omega),\|s\|\le 1,\|t\|\le 1}\left(\frac{X_H(s)-X_H(t)}{\|s-t\|^{H'}}\right)\le\delta\right]=1 \quad (4.114)$$

*where $\epsilon(\omega)$ is an almost surely positive random variable and $\delta > 0$. Moreover for every $H' > H - d/2$, $\mathbb{P}(X_H \notin \mathcal{C}^{H'}) > 0$, where $\mathcal{C}^{H'}$ is the space of Hölder-continuous functions on $[0,1]^d$. Furthermore if the control measure $\nu$ of the random measure $M$ is not finite, $\mathbb{P}(X_H \notin \mathcal{C}^{H'}) = 1$.*

*Proof:*
Because of the isometry property

$$\mathbb{E}(X_H(s)-X_H(t))^2 = \mathbb{E}(B_H(s)-B_H(t))^2 = C\|t-s\|^{2H},$$

where $B_H$ is a fractional Brownian field. The property (4.114) is then a direct consequence of Kolmogorov Theorem 2.1.7. To prove the second part of the proposition Theorem 4 of [120] will be applied to $X_H$. First we take a separable modification of $X_H$ with a separable representation. The next step is to used the symmetrization argument of Sect. 4.5 in [120] if $\nu$ is not already symmetric. Then we can remark that the kernel $t \mapsto \|t-s\|^{H-d/2} - \|s\|^{H-d/2} \notin \mathcal{C}^{H'}$ for every $H' > H - d/2$, and the conclusion of Theorem 4 is applied to the measurable linear subspace $\mathcal{C}^{H'}$ to get $\mathbb{P}(X_H \notin \mathcal{C}^{H'}) > 0$. To show that this probability is actually one, we rely on a zero-one law. The process $X_H$ can be viewed as an infinitely divisible law on the Banach space $\mathcal{C}[0,1]$ of the continuous functions endowed with the supremum norm. Let us consider the map

$$\begin{aligned} \varphi : \mathbb{R} \times \mathbb{R}^d &\to \mathcal{C}[0,1] \\ (u,s) &\to \{t \mapsto u(\|t-s\|^{H-d/2} - \|s\|^{H-d/2}\|)\}\,. \end{aligned}$$

The Lévy random measure $F(df)$ of the infinitely divisible law defined by $X_H$ is now given by $\varphi(\nu^{sym}(du) \times ds) = F(df)$ where $\nu^{sym}$ is the control measure of the symmetrized process. Hence $F((\mathcal{C}[0,1] \setminus \mathcal{C}^{H'}) = +\infty$ if $\nu^{sym}(\mathbb{R}) = +\infty$. Corollary 11 of [76] and $\mathbb{P}(X_H \notin \mathcal{C}^{H'}) > 0$ yield the last result of the proposition.

Now let us go back to the case $H < d/2$.

**Proposition 4.4.4** *If $H < d/2$, for every compact interval $K \subset \mathbb{R}^d$*

$$\mathbb{P}(X_H \notin \mathcal{B}(K)) > 0,$$

*where $\mathcal{B}(K)$ is the space of bounded functions on $K$.*

In this case, we remark that $t \to \|t-s\|^{H-d/2} - \|s\|^{H-d/2} \notin \mathcal{B}(K)$ for every $s \in K$. The proposition is then proved by applying Theorem 4 of [120] to $\mathcal{B}(K)$.

#### 4.4.1.2 Local Asymptotic Self-Similarity

We will now investigate local self-similarity for mafLfs. It should be noted that mafLfs, in general, do not have a tangent field. In this section we focus on the truncated stable case. In view of Propositions 4.4.2 and 4.4.5, the truncated stable case can be viewed as a bridge between fBf and moving average stable field. Let

$$\nu_{\alpha,1}(du) = \frac{\mathbf{1}_{\{|u|\le 1\}} du}{|u|^{1+\alpha}}$$

be a control measure in the sense of Sect. 2.1.8 associated to the Lévy random measure $M_{\alpha,1}$. Denote the corresponding mafLf by $X_{H,\alpha}$

$$X_{H,\alpha}(t) = \int_{\mathbb{R}^d} \left(\|t-s\|^{H-\frac{d}{2}} - \|s\|^{H-\frac{d}{2}}\right) dM_{\alpha,1}(s).$$

**Proposition 4.4.5** *Let us assume that $\widetilde{H}$ defined by $\widetilde{H} - \frac{d}{\alpha} = H - \frac{d}{2}$ is such that $0 < \widetilde{H} < 1$. The mafLf $X_{H,\alpha}$ with control measure*

$$\nu_{\alpha,1}(du) = \frac{\mathbf{1}_{\{|u|\le 1\}} du}{|u|^{1+\alpha}}$$

*is locally self-similar with parameter $\widetilde{H}$. For every fixed $t \in \mathbb{R}^d$*

$$\lim_{\epsilon \to 0^+} \left( \frac{X_{H,\alpha}(t+\epsilon x) - X_{H,\alpha}(t)}{\epsilon^{\widetilde{H}}} \right)_{x \in \mathbb{R}^d} \stackrel{(d)}{=} \left( Y_{\widetilde{H}}(x) \right)_{x \in \mathbb{R}^d}, \tag{4.115}$$

*where the limit is in distribution for all finite dimensional margins of the field. The limit is a moving average fractional stable field that has a representation:*

$$Y_{\widetilde{H}}(x) = \int_{\mathbb{R}^d} \left( \|x - \sigma\|^{\widetilde{H} - d/\alpha} - \|\sigma\|^{\widetilde{H} - d/\alpha} \right) M_\alpha(d\sigma), \tag{4.116}$$

*where $M_\alpha(d\sigma)$ is a stable $\alpha$-symmetric random measure.*

*Proof:*

Since the mafLf has stationary increments we only have to prove the convergence for $t = 0$. We consider a multivariate function:

$$g_{t,v,H}(\epsilon, s, u) = iu \sum_{k=1}^{n} v_k \frac{\|\epsilon t_k - s\|^{H-d/2} - \|s\|^{H-d/2}}{\epsilon^{\widetilde{H}}} \tag{4.117}$$

where $t \in (\mathbb{R}^d)^n$, and $v \in \mathbb{R}^n$. Then

$$\begin{aligned} &\mathbb{E} \exp\left( i \sum_{k=1}^{n} v_k \frac{X_{\widetilde{H}}(\epsilon t_k)}{\epsilon^{\widetilde{H}}} \right) \\ &\quad = \exp\left( \int_{\mathbb{R}^d \times \mathbb{R}} [\exp(g_{t,v,H}(\epsilon, s, u)) - 1 - g_{t,v,H}(\epsilon, s, u)] ds\nu(du) \right). \end{aligned} \tag{4.118}$$

Then the change of variable $\sigma = \frac{s}{\epsilon}$ is applied and $\widetilde{H}$ has been chosen such that the integral in the previous equation is now

$$\begin{aligned} \int_{\mathbb{R}^d \times \mathbb{R}} &[\exp(g_{t,v,H}(1, \sigma, \epsilon^{-d/\alpha} u)) - 1 - g_{t,v,H}(1, \sigma, \epsilon^{-d/\alpha} u)] \\ &\qquad \mathbf{1}(|u| < 1) \epsilon^d d\sigma \frac{du}{|u|^{1+\alpha}}. \end{aligned} \tag{4.119}$$

Let us set $w = \epsilon^{-d/\alpha} u$. The integral becomes

$$\begin{aligned} I(\epsilon) = \int_{\mathbb{R}^d \times \mathbb{R}} &[\exp(g_{t,v,H}(1, \sigma, w)) - 1 - g_{t,v,H}(1, \sigma, w)] \\ &\qquad \mathbf{1}(|w| < \epsilon^{-d/\alpha}) d\sigma \frac{dw}{|w|^{1+\alpha}}. \end{aligned} \tag{4.120}$$

Let us recall that

$$- C(\alpha)|x|^{\alpha} = \int_{\mathbb{R}} [e^{ixr} - 1 - ixr\mathbf{1}(|r| \le \epsilon^{-d/\alpha})] \frac{dr}{|r|^{1+\alpha}} \tag{4.121}$$

for every $\epsilon > 0$, where $C(\alpha) = 2\int_0^{+\infty}(1-\cos(r))\frac{dr}{r^{1+\alpha}}$. Let us write

$$J_{\epsilon} = \int_{\mathbb{R}} [e^{ixr} - 1 - ixr]\mathbf{1}(|r| \le \epsilon^{-d/\alpha}) \frac{dr}{|r|^{1+\alpha}}$$

then

$$\begin{aligned} \lim_{\epsilon \to 0^+} \left(J_{\epsilon} + C(\alpha)|x|^{\alpha}\right) &= \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} [1 - e^{ixr}]\mathbf{1}(|r| > \epsilon^{-d/\alpha}) \frac{dr}{|r|^{1+\alpha}} \\ &= 0, \end{aligned}$$

hence

$$\lim_{\epsilon \to 0^+} I(\epsilon) = -C(\alpha) \int_{\mathbb{R}^d} |g_{t,v,H}(1, \sigma, 1)|^{\alpha} d\sigma. \tag{4.122}$$

Since this last expression is the logarithm of

$$\mathbb{E}\exp\left(i\sum_{k=1}^{n} v_k Y_{\widetilde{H}}(t_k)\right),$$

the proof is complete.

### 4.4.2 Real Harmonizable Fractional Lévy Fields

A counterpart of real harmonizable fractional stable field is introduced, when a random Lévy measure of Sect. 2.1.10 is used in the integral representation.

**Definition 4.4.2** *Let us call a real harmonizable fractional Lévy field (in short rhfLf) with parameter $0 < H < 1$ a real valued field $(X_H(t))_{t\in\mathbb{R}^d}$, which admits an harmonizable representation*

$$X_H(x) \stackrel{(d)}{=} \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{\frac{d}{2}+H}} M(d\xi),$$

*where $M(d\xi)$ is a complex isotropic random Lévy measure defined in Definition 2.1.20 that satisfies the finite moment assumption (2.45).*

Please note that, since

$$\xi \mapsto \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{\frac{d}{2}+H}}$$

satisfies (2.50), almost surely rhfLfs are real valued! Because of the isometry property (2.55) of the complex isotropy random Lévy measure, rhfLfs have finite second order moments. As in the case of mafLf they share the same covariance structure than fractional Brownian field. But they have different distributions, in particular they are non-Gaussian. We will see later that the regularity properties of rhfLfs are quite different from those of mafLfs.

**Proposition 4.4.6** *The covariance structure of rhfLfs is*

$$R(x, y) = \mathbb{E}(X(x)X(y)) = \frac{var(X(1))}{2}\left\{\|x\|^{2H} + \|y\|^{2H} - \|x-y\|^{2H}\right\}. \tag{4.123}$$

*RhfLfs have stationary increments.*

*Proof of Proposition 4.4.6*:

The first claim is consequence of the isometry property and of the fact that fractional Brownian fields and rhfLfs have the same kernel in their integral representation.

Let $\theta = (\theta_1, \ldots, \theta_n)$ and $\mathbf{x} = (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$, the logarithm of characteristic function of the increments of the rhfLf is:

$$\begin{aligned}
&-\log\left(\mathbb{E}\exp\left(i\sum_{j=2}^{n}\theta_j(X_H(x_j) - X_H(x_1))\right)\right) \\
&= \int_{\mathbb{R}^d\times\mathbb{C}} \exp\left(i\sum_{j=2}^{n}\theta_j 2\Re\left(z\frac{e^{-ix_j.\xi} - e^{-ix_1.\xi}}{\|\xi\|^{H+d/2}}\right)\right. \\
&\qquad\qquad \left. -1 - i\sum_{j=2}^{n}\theta_j 2\Re\left(z\frac{e^{-ix_j.\xi} - e^{-ix_1.\xi}}{\|\xi\|^{H+d/2}}\right)\right) d\xi\,\nu(dz).
\end{aligned}$$

Then, if we set $z' = e^{-ix_1.\xi}z$, and use the invariance by rotation of $\nu(dz)$ one gets

$$\begin{aligned}
&-\log\left(\mathbb{E}\exp\left(i\sum_{j=2}^{n}\theta_j(X_H(x_j) - X_H(x_1))\right)\right) \\
&= -\log\left(\mathbb{E}\exp\left(i\sum_{j=2}^{n}\theta_j X_H(x_j - x_1)\right)\right)
\end{aligned}$$

and the stationarity of the increments of the rhfLf.

Now we investigate some properties of self-similarity and regularity types that the rhfLf shares with fractional Brownian field. In the first part of this section we prove that two asymptotic self-similarity properties are true for the rhfLf. In the second part we see that almost surely the paths of the rhfLf are Hölder-continuous with a pointwise Hölder exponent $H$.

#### 4.4.2.1 Asymptotic Self-Similarity

Since we know the characteristic function of stochastic integrals of the measure $M(d\xi)$ we can prove the local self-similarity of the rhfLfs. Actually it is a consequence of the homogeneity property of $\frac{1}{\|\xi\|^{\frac{d}{2}+H}}$ and of a central limit theorem for the stochastic measure $M(d\xi)$.

**Proposition 4.4.7** *The real harmonizable fractional Lévy field is strongly locally self-similar with parameter $H$ in the sense that for every fixed $x \in \mathbb{R}^d$*

$$\lim_{\epsilon\to 0^+}\left(\frac{X_H(x+\epsilon u)-X_H(x)}{\epsilon^H}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} C(H)\left(2\pi\int_0^{+\infty}\rho^2\nu_\rho(d\rho)\right)^{1/2}(B_H(u))_{u\in\mathbb{R}^d}\,, \tag{4.124}$$

*where $B_H$ is a standard fractional Brownian field, where $C(H)$ is given in (3.66), and the limit is in distribution on the space of continuous functions endowed with the topology of the uniform convergence on every compact.*

*Proof of Proposition 4.4.7*:

The convergence of the finite dimensional margins is proved first. Since the rhfLf has stationary increments, we only have to prove the convergence for $x = 0$. Let us consider the multivariate function:

$$g_{u,v,H}(\epsilon,\xi,z) = i2\Re\left(z\sum_{k=1}^n v_k\frac{e^{-i\epsilon u_k\cdot\xi}-1}{\epsilon^H\|\xi\|^{\frac{d}{2}+H}}\right) \tag{4.125}$$

where $u = (u_1,\ldots,u_n) \in (\mathbb{R}^n)^d$ and $v = (v_1,\ldots,v_n) \in \mathbb{R}^n$. Then

$$\mathbb{E}\exp\left(i\sum_{k=1}^n v_k\frac{X_H(\epsilon u_k)}{\epsilon^H}\right) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{C}}[\exp(g_{u,v,H}(\epsilon,\xi,z))-1-g_{u,v,H}(\epsilon,\xi,z)]d\xi d\nu(z)\right). \tag{4.126}$$

The change of variable $\lambda = \epsilon\xi$ is applied to the integral of the previous right hand term to get:

$$\int_{\mathbb{R}^d \times \mathbb{C}} [\exp(\epsilon^{d/2} g_{u,v,H}(1,\lambda,z)) - 1 - \epsilon^{d/2} g_{u,v,H}(1,\lambda,z)] \frac{d\lambda}{\epsilon^d} d\nu(z). \tag{4.127}$$

Then as $\epsilon \to 0^+$ a dominated convergence argument yields that

$$\lim_{\epsilon \to 0^+} \mathbb{E} \exp\left( i \sum_{k=1}^n v_k \frac{X_H(\epsilon u_k)}{\epsilon^H} \right) = \exp\left( \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{C}} g^2_{u,v,H}(1,\lambda,z) d\lambda d\nu(z) \right). \tag{4.128}$$

Moreover (2.44) allows us to express the logarithm of the previous limit as:

$$-2\pi \int_0^{+\infty} \rho^2 \nu_\rho(d\rho) \int_{\mathbb{R}^d} \frac{|\sum_{k=1}^n v_k (e^{-iu_k \cdot \lambda} - 1)|^2}{\|\lambda\|^{d+2H}} d\lambda \tag{4.129}$$

and this last integral is the variance of $C(H) \sum_{k=1}^n v_k B_H(u_k)$ which concludes the proof of the convergence of finite dimensional margins.

Let us proceed to the proof that the distributions are tight. We need to estimate

$$\mathbb{E}(X_H(x) - X_H(y))^{2p}$$

for $p$ large enough. Unfortunately, when $H > 1 - d/2$, these moments are not finite because of the asymptotic of the integrand:

$$g_0(x,\xi) = \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{\frac{d}{2}+H}} \tag{4.130}$$

when $\|\xi\| \to 0$. In the case $H > 1 - d/2$, we thus apply a transformation to the integrand $g_0$ to analyze in two different ways its behavior at both ends of the spectrum.

Let us first consider the easy case: $H \leq 1 - d/2$. Then $g_0(x,.) \in L^{2q}(\mathbb{R}^d)$ $\forall q \in \mathbb{N}^*$ and

$$\|g_0(x,.) - g_0(y,.)\|^{2q}_{L^{2q}(\mathbb{R}^d)} = \|x-y\|^{2Hq+d(q-1)} \|g_0(e_1,.)\|^{2q}_{L^{2q}(\mathbb{R}^d)}$$

where $e_1 = (1,0,\ldots,0) \in \mathbb{R}^d$. Because of (2.56) we know that

$$\mathbb{E}(X_H(x) - X_H(y))^{2p} = \sum_{n=1}^p D(n) \|x-y\|^{2Hp+d(p-n)}$$

for some nonnegative constants $D(n)$. Hence there exists $C < +\infty$ such that

$$\mathbb{E}(X_H(x) - X_H(y))^{2p} \leq C \|x-y\|^{2Hp} \tag{4.131}$$

where $x$, $y$ are in a fixed compact. Hence if $H \leq 1 - d/2$

$$\mathbb{E}\left(\frac{(X_H(x+\epsilon u) - X_H(x+\epsilon v))^{2p}}{\epsilon^{2Hp}}\right) \leq \|u - v\|^{2Hp}$$

and one can take $p > \frac{d}{2H}$ to show the tightness with the help of Corollary 2.1.2.

When $H > 1 - d/2$ let us take $K$ an integer such that $K \geq 1 + d/2$,

$$P_K(t) = \sum_{k=1}^{K} \frac{t^k}{k!},$$

and $\varphi$ an even $C^1$-function such that $\varphi(t) = 1$ when $|t| \leq 1/2$ and $\varphi(t) = 0$ when $|t| > 1$. Then

$$g_K(x, \xi) = \frac{e^{-ix\cdot\xi} - 1 - P_K(-ix \cdot \xi)\varphi(\|x\|\|\xi\|)}{\|\xi\|^{\frac{d}{2}+H}}$$

is in $L^{2q}(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$ and $q \in \mathbb{N}^*$.

The field $X_H$ is then split into two fields $X_H = X_H^+ + X_H^-$ where

$$X_H^+(x) = \int g_K(x, \xi) M(d\xi) \tag{4.132}$$

and

$$X_H^-(x) = \int \frac{P_K(-ix \cdot \xi)}{\|\xi\|^{\frac{d}{2}+H}} \varphi(\|x\|\|\xi\|) M(d\xi). \tag{4.133}$$

A method similar to the one used for $X_H$ when $H \leq 1 - d/2$ is applied to $X_H^+$ and we check that $X_H^-$ has almost surely $C^1$ paths because of Theorem 2.1.8.

Let us start by the remark

$$\|g_K(x, .)\|_{L^{2q}(\mathbb{R}^d)}^{2q} = \|x\|^{2Hq+d(q-1)} \|g_K(e_1, .)\|_{L^{2q}(\mathbb{R}^d)}^{2q}.$$

As in the easy case we have to estimate when $\epsilon \to 0^+$

$$I_\epsilon = \int_{\mathbb{R}^d} |g_K(x, \xi) - g_K(x + \epsilon u, \xi)|^{2q} d\xi.$$

Let us split this integral into

$$I_\epsilon^+ = \int_{\epsilon\|\xi\|\geq 1} |g_K(x, \xi) - g_K(x + \epsilon u, \xi)|^{2q} d\xi$$

and

$$I_\epsilon^- = \int_{\epsilon\|\xi\|<1} |g_K(x,\xi) - g_K(x+\epsilon u,\xi)|^{2q} d\xi$$

as $I_\epsilon = I_\epsilon^+ + I_\epsilon^-$. Actually

$$|g_K(x,\xi) - g_K(y,\xi)| = |g_0(x-y,\xi)|$$

on $\{\epsilon\|\xi\| \geq 1\}$ for $\epsilon$ small enough, and we get by the change of variable $\lambda = \epsilon\xi$

$$I_\epsilon^+ = \epsilon^{2Hq+d(q-1)} \int_{\|\lambda\|\geq 1} \frac{|e^{-ie_1\cdot\lambda} - 1|^{2q}}{\|\lambda\|^{2Hq+dq}} d\lambda. \tag{4.134}$$

Then a Taylor expansion is applied to $I_\epsilon^-$

$$I_\epsilon^- = \int_{\|\xi\|<\frac{1}{\epsilon}} |dg_K(\theta(x,\epsilon u,\xi),\xi).\epsilon u|^{2q} d\xi,$$

where $dg_K(\theta(x,\epsilon u,\xi),\xi)$ is the differential of the map $g_K(.,\xi)$ and $\theta(x,\epsilon,\xi)$ is a point in the segment $(x, x+\epsilon u)$. Note that

$$\int_{\|\xi\|<C} \|dg_K(\theta(x,\epsilon u,\xi),\xi)\|^{2q} d\xi < +\infty$$

for every fixed $C$ and that

$$\|dg_K(\theta(x,\epsilon u,\xi),\xi)\|^{2q} = O(\|\xi\|^{2q(1-\frac{d}{2}-H)}) \quad \text{when} \quad \|\xi\| \to +\infty,$$

hence

$$\left(\int_{\|\xi\|<\frac{1}{\epsilon}} \|dg_K(\theta(x,\epsilon u,\xi),\xi)\|^{2q} d\xi\right) \epsilon^{2q} = 0(\epsilon^{2Hq+d(q-1)}) \quad \text{when} \quad \epsilon \to 0^+$$

and

$$|I_\epsilon^-| \leq C\epsilon^{2Hq+d(q-1)} \tag{4.135}$$

when $\epsilon \to 0^+$. Because of (4.134) and (4.135) there exists a positive constant $C$ such that

$$\int_{\mathbb{R}^d} |g_K(x,\xi) - g_K(y,\xi)|^{2q} d\xi \leq C\|x-y\|^{2Hq+d(q-1)}$$

and consequently

$$\mathbb{E}(X_H^+(x) - X_H^+(y))^{2p} \leq C\|x-y\|^{2Hp}. \tag{4.136}$$

when $\|x\| \leq 1$, $\|y\| \leq 1$, which yields that the distributions of

$$\left( \frac{X_H^+(x+\epsilon.) - X_H^+(x)}{\epsilon^H} \right)_{\epsilon>0}$$

are tight. To conclude let us write $X_H^-$ for $\|x\| < \epsilon$ as

$$\int_{\epsilon\|\xi\|\leq 1/2} \frac{P_K(-ix\cdot\xi)}{\|\xi\|^{\frac{d}{2}+H}} M(d\xi) + \int_{1/2\leq\epsilon\|\xi\|\leq 1} \frac{P_K(-ix\cdot\xi)}{\|\xi\|^{\frac{d}{2}+H}} \varphi(\|x\|\|\xi\|) M(d\xi).$$

The first integral of the previous line is actually a polynomial in the variables $(x_1, \ldots, x_d)$ with coefficients that are random variables, hence it has almost surely $C^1$ paths. Let us remark that the integrand of the second integral is bounded with compact support in $\mathbb{R}^d \times \mathbb{C}$ and is $C^1$ in the variable $x$, so is the integral which yields that $X_H^-$ is almost surely $C^1$. Then it is clear that

$$\lim_{\epsilon\to 0^+} \left( \frac{X_H^-(x+\epsilon u) - X_H^-(x)}{\epsilon^H} \right)_{u\in\mathbb{R}^d} \overset{(a.s.)}{=} 0$$

which concludes the proof.

We now want to exhibit an example of rhfLf that has asymptotic self-similarity properties when the increment is taken on large scales. Actually if the control measure $\nu_\rho(d\rho)$ is $\frac{d\rho}{|\rho|^{1+\alpha}}\mathbf{1}(|\rho| < 1)$ where $0 < \alpha < 2$, we show that at large scales the rhfLf is asymptotically self-similar with parameter $0 < \tilde{H} < 1$ such that $\tilde{H} + \frac{d}{\alpha} = H + \frac{d}{2}$.

Heuristically it means that at large scales the truncation of the Lévy measure disappears.

Moreover the limit field is a rhfsm with parameter $\tilde{H}$. This shows that at large scales the behavior of rhfLf can be very far from the Gaussian model even if the rhfLfs are fields that have moments of order 2. The rhfLf with control measure $\frac{d\rho}{|\rho|^{1+\alpha}}\mathbf{1}(|\rho| < 1)$ can be viewed roughly speaking as in between a rhfsm at large scales and a fractional Brownian field at low scales. Let us now state precisely the asymptotic self-similarity.

**Proposition 4.4.8** *Let us assume that $\tilde{H}$ defined by $\tilde{H} + \frac{d}{\alpha} = H + \frac{d}{2}$ is such that $0 < \tilde{H} < 1$. The real harmonizable fractional Lévy field with control measure $\nu_\rho(d\rho)$*

$$\frac{d\rho}{|\rho|^{1+\alpha}}\mathbf{1}(|\rho| < 1)$$

*is asymptotically self-similar at infinity with parameter $\tilde{H}$*

$$\lim_{R\to+\infty} \left( \frac{X_H(Ru)}{R^{\tilde{H}}} \right)_{u\in\mathbb{R}^d} \overset{(d)}{=} \left( Y_{\tilde{H}}(u) \right)_{u\in\mathbb{R}^d} \tag{4.137}$$

*where the limit is in distribution for all finite dimensional margins of the fields and the limit is a real harmonizable fractional stable field that has a representation*

$$Y_{\tilde{H}}(u) = \int_{\mathbb{R}^d} \frac{e^{-iu\cdot\xi} - 1}{\|\xi\|^{\frac{d}{\alpha}+\tilde{H}}} M_\alpha(d\xi), \tag{4.138}$$

*where $M_\alpha(d\xi)$ is complex isotropic $\alpha$-stable random measure defined in Definition 2.1.12.*

*Proof of Proposition 4.4.8*:
As Proposition 4.4.5 we consider a multivariate function

$$g_{u,v,H}(R,\xi,z) = i2\Re\left(z\sum_{k=1}^{n} v_k \frac{e^{-iRu_k\cdot\xi} - 1}{R^{\tilde{H}}\|\xi\|^{\frac{d}{2}+H}}\right) \tag{4.139}$$

where $u \in (\mathbb{R}^n)^d$ and $v \in \mathbb{R}^n$. And

$$\begin{aligned}&\mathbb{E}\exp\left(i\sum_{k=1}^{n} v_k \frac{X_H(Ru_k)}{R^{\tilde{H}}}\right)\\ &\quad= \exp\left(\int_{\mathbb{R}^d\times\mathbb{C}} [\exp(g_{u,v,H}(R,\xi,z)) - 1 - g_{u,v,H}(R,\xi,z)]d\xi d\nu(z)\right).\end{aligned} \tag{4.140}$$

Then the change of variable $\lambda = R\xi$ is applied and $\tilde{H}$ has been chosen such that integral in the previous equation is now

$$\begin{aligned}\int_{\mathbb{R}^d\times[0,2\pi]\times\mathbb{R}_*^+} &[\exp(g_{u,v,H}(1,\lambda,R^{d/\alpha}\rho e^{i\theta})) - 1 - g_{u,v,H}(1,\lambda,R^{d/\alpha}\rho e^{i\theta})]\\ &\mathbf{1}(|\rho|<1)R^{-d}d\lambda d\theta\frac{d\rho}{|\rho|^{1+\alpha}}.\end{aligned} \tag{4.141}$$

Let us set $r = R^{d/\alpha}\rho$ the integral becomes

$$\begin{aligned}I(R) = \int_{\mathbb{R}^d\times[0,2\pi]\times\mathbb{R}} &[\exp(g_{u,v,H}(1,\lambda,re^{i\theta})) - 1 - g_{u,v,H}(1,\lambda,re^{i\theta})]\\ &\mathbf{1}(|r|<R^{d/\alpha})d\lambda d\theta\frac{dr}{2|r|^{1+\alpha}}.\end{aligned} \tag{4.142}$$

Let us recall that (4.121)

$$-C(\alpha)|x|^\alpha = \int_{\mathbb{R}} [e^{ixr} - 1 - ixr\mathbf{1}(|r|\le R^{d/\alpha})]\frac{dr}{|r|^{1+\alpha}}$$

for every $R > 0$, where $C(\alpha) = \int_0^{+\infty}(1-\cos(r))\frac{dr}{r^{1+\alpha}}$ is given in (2.39). If we write

$$J_R = \int_{\mathbb{R}} [e^{ixr} - 1 - ixr]\mathbf{1}(|r| \leq R^{d/\alpha})\frac{dr}{2|r|^{1+\alpha}},$$

then

$$\begin{aligned}\lim_{R\to+\infty}\left(J_R + \frac{C(\alpha)}{2}|x|^\alpha\right) &= \lim_{R\to+\infty}\int_{\mathbb{R}}[e^{ixr}-1]\mathbf{1}(|r| > R^{d/\alpha})\frac{dr}{2|r|^{1+\alpha}} \\ &= 0.\end{aligned}$$

Moreover the non positive function

$$\int_{\mathbb{R}}[e^{rg_{u,v,H}(1,\lambda,re^{i\theta})} - 1]\mathbf{1}(|r| > R^{d/\alpha})\frac{dr}{2|r|^{1+\alpha}}$$

is increasing with respects to $R$, and

$$\int_{\mathbb{R}^d\times[0,2\pi]}\int_{\mathbb{R}}[e^{rg_{u,v,H}(1,\lambda,re^{i\theta})} - 1]\mathbf{1}(|r| > R^{d/\alpha})\frac{dr}{2|r|^{1+\alpha}}d\lambda d\theta < \infty.$$

Hence, by monotone convergence,

$$\lim_{R\to+\infty} I(R) = -\frac{C(\alpha)}{2}\int_{\mathbb{R}^d\times[0,2\pi]}\left|2\Re\left(e^{i\theta}\sum_{k=1}^n v_k\frac{e^{-iu_k\lambda}-1}{\|\lambda\|^{\frac{d}{\alpha}+\tilde{H}}}\right)\right|^\alpha d\lambda d\theta \quad (4.143)$$

which is also:

$$-\frac{C(\alpha))}{2}\int_0^{2\pi}|2\cos(\theta)|^\alpha d\theta\int_{\mathbb{R}^d}\left|\sum_{k=1}^n v_k\frac{e^{-iu_k\lambda}-1}{\|\lambda\|^{\frac{d}{\alpha}+\tilde{H}}}\right|^\alpha d\lambda. \quad (4.144)$$

Since this last expression is the logarithm of

$$\mathbb{E}\exp\left(i\sum_{k=1}^n v_k Y_{\tilde{H}}(u_k)\right),$$

(cf. (2.57)) the proof is complete.

#### 4.4.2.2 Regularity of the Sample Paths of the rhfLf

The Kolmogorov theorem 2.1.7 and Proposition 2.1.4 show that $H$ is the pointwise Hölder exponent of the sample paths of the rhfLfs. Let us recall the Definition 2.2.3

of the pointwise exponent $H_f(x)$ of a deterministic function $f$ at point $x$ by

$$H_f(x) = \sup\{H', \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\|\epsilon\|^{H'}} = 0\}. \tag{4.145}$$

Then the regularity of the sample paths is described by the following proposition.

**Proposition 4.4.9** *For every $H' < H$ there exists a continuous modification of the rhfLf $X_H$ such that almost surely the sample paths of $X_H$ are locally $H'$ Hölder continuous i.e.*

$$\mathbb{P}\left[\omega; \sup_{0<\|x-y\|<\epsilon(\omega), \|x\|\le 1, \|y\|\le 1} \left(\frac{X_H(x) - X_H(y)}{\|x-y\|^{H'}}\right) \le \delta\right] = 1 \tag{4.146}$$

*where $\epsilon(\omega)$ is an almost surely positive random variable and $\delta > 0$. Moreover at every point $x$ the pointwise exponent $H_{X_H}(x)$ of the rhfLf $X_H$ is almost surely equal to $H$.*

*Proof of Proposition 4.4.9*:
In the first part of the proof we will use the estimation of the moments

$$\mathbb{E}(X_H(x) - X_H(y))^{2p}$$

performed in the proof of Proposition 4.4.7 and Kolmogorov theorem. When $H \le 1 - d/2$ we already know by (4.131) that

$$\mathbb{E}(X_H(x) - X_H(y))^{2p} \le C\|x-y\|^{2Hp}$$

when $\|x\| \le 1$, $\|y\| \le 1$ and Kolmogorov theorem yields (4.146) for every $H' < H$. When $H > 1 - d/2$ we recall that $X_H$ has been split into

$$X_H = X_H^+ + X_H^-$$

where $X_H^+$ and $X_H^-$ are defined in (4.132) and (4.133). Furthermore we know that $X_H^+$ is $H'$-Hölder-continuous for every $H' < H$ by Kolmogorov theorem and the inequality (4.136), and that $X_H^-$ has almost surely $C^1$ sample paths which concludes the proof of (4.146).

Because of (4.114) at every point $x$ the Hölder exponent satisfies $H(x) \ge H$. To show $H(x) \le H$ let us use the local self-similarity (4.124). Actually if $H' > H$ we can deduce from (4.124) that

$$\lim_{\epsilon \to 0^+} \frac{\epsilon^{H'}}{|X_H(x+\epsilon u) - X_H(x)|} \stackrel{(d)}{=} 0,$$

which is also a convergence in probability. Hence we can find a sequence $(\epsilon_n)_{n\in\mathbb{N}} \to 0^+$ such that:

$$\lim_{n\to+\infty} \frac{|X_H(x+\epsilon_n u) - X_H(x)|}{\epsilon_n^{H'}} = +\infty \quad \text{almost surely.}$$

This argument concludes the proof of Proposition 4.4.3.

### 4.4.3 A Comparison of Lévy Fields

In the previous sections, we have introduced two different models of fractional fields with finite second moments that have the same covariance structure as fractional Brownian field. Since both models have very different sample paths properties, it shows that the covariance structure cannot characterize the smoothness of fractional fields. For modeling purpose a comparison of mafLf and rhfLf may be useful.

First we can remark that the kernel of the rhfLf is the Fourier transform of the kernel of the mafLf. This fact was proved in (3.29), when we showed that moving average and harmonizable representation of fractional Brownian field are equivalent. In the case of fractional Lévy fields the Fourier transform of a Lévy measure is not a Lévy measure and it explains why rhfLf and mafLf have different distributions. Nevertheless the self-similarity of fractional Lévy fields is reminiscent of this fact. Indeed in the limit in Proposition 4.4.2 for mafLf and in Proposition 4.4.7 for rhfLf is a fractional Brownian field. But in Proposition 4.4.2 the large scale behavior of mafLf is investigated, whereas Proposition 4.4.7 is concerned with the small scale behavior of rhfLf. It reminds us of the fact that the large scale behavior of a Fourier transform is related to the small scale behavior of deterministic functions. The same remark is true, when we compare Propositions 4.4.5 and 4.4.8. Even if the limits are not the same in distribution in this last case they are of the same stable type.

### 4.4.4 Real Harmonizable Multifractional Lévy Fields

In this section we introduce a class of Lévy fields proposed in [88] that yields multifractional fields related to rhfLf. These fields called real harmonizable multifractional Lévy fields are non Gaussian counterpart to multifractional Brownian field.

**Definition 4.4.3** *Let $h : \mathbb{R}^d \to (0,1)$ be a measurable function. A real valued field is called a real harmonizable multifractional Lévy field (in short rhmLf) with multifractional function $h$, if it admits the harmonizable representation*

$$X_h(x) \overset{(d)}{=} \frac{1}{(C(h(x)))^{1/2}} \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{d/2+h(x)}} M(d\xi), \tag{4.147}$$

*where $M$ is a complex isotropic random Lévy measure, and the normalization function is defined in* (4.19).

**Remark 4.4.1** *We have checked for the mBm that the kernel function in* (4.147) *is in $L^2$ and thus the integral in* (4.147) *is defined.*

**Proposition 4.4.10** *The covariance structure of rhmLfs with multifractional function $h$ is the same as the covariance structure of multifractional Brownian field $B_h$ with the same multifractional function*

$$R(x, y) = \mathbb{E}(X_h(x)X_h(y)) = \left(4\pi \int_0^{+\infty} e^2 \gamma_{\mathrm{e}}(\mathrm{d_e})\right) \mathbb{E}(B_h(x)B_h(y)). \tag{4.148}$$

*Proof of Proposition 4.4.10*

It is a consequence of the isometry property for the complex isotropic random Lévy measure $M$.

In the following we list some properties of rhmLf referring to [88] for the proofs, and we suppose that the multifractional function $h$ is $\beta$-Hölder continuous.

First the smoothness of rhmLf is the same as those of mBm.

**Proposition 4.4.11** *Let $h$ be a function $h : \mathbb{R}^d \mapsto (0, 1)$, and $X_h$ be the corresponding rhmLf. Let $K$ be a compact set, and $m = \inf\{h(x),\ x \in K\}$. For every $H < \min(m, \beta)$ there exists a modification of $X_h$ such that almost surely the sample paths of $X_h$ are $H$-Hölder continuous on $K$.*

Second rhmLf are strongly locally asymptotically self-similar.

**Proposition 4.4.12** *Let $h : \mathbb{R}^d \mapsto (0, 1)$ be a $\beta$-Hölder continuous multifractional function and $X_h$ the corresponding rhmLf. Let us assume $\beta > \sup_{x\in\mathbb{R}^d} h(x)$, then, for every $x \in \mathbb{R}^d$ the rhmLf is strongly locally asymptotically self-similar, with tangent field a fractional Brownian field with Hurst exponent $H = h(x)$. More precisely*

$$\lim_{\varepsilon\to 0^+} \left(\frac{X_h(x+\varepsilon u) - X_h(x)}{\varepsilon^{h(x)}}\right)_{u\in\mathbb{R}^d} \overset{(d)}{=} \sqrt{4\pi \int_0^{+\infty} \rho^2 \nu_\rho(d\rho)}(B_H(u))_{u\in\mathbb{R}^d} \tag{4.149}$$

*where $H = h(x)$ and $B_H$ is a fractional Brownian field with Hurst exponent $H$.*

By extending the techniques used for mBm to the random Lévy measure, one can show the following result.

**Corollary 4.4.1** *Let $h$ be function $h : \mathbb{R}^d \mapsto (0, 1)$ locally Hölder with exponent $\beta$. Let $X_h$ be the corresponding rhmLf and fix $x$ such that $h(x) < \beta$, the pointwise Hölder exponent of $X_h$ at $x$ is*

$$\sup\{H',\ \lim_{\epsilon\to 0} \frac{X_h(x+\epsilon) - X_h(x)}{|\epsilon|^{H'}} = 0\} = h(x) \tag{4.150}$$

*almost surely.*

## 4.5 Exercises

### 4.5.1 *Lass and Self-Similarity*

In this exercise we consider again the process defined in Exercise 3.6.11, for $0 < H < 1$,

$$X_t = \int_{\mathbb{R}} \frac{\varphi(ts)}{|s|^{H+1/2}} W(ds).$$

We assume $\varphi(0) = 0$, and

$$\int_{\mathbb{R}} \frac{|\varphi(s)|^2}{|s|^{2H+1}} ds < \infty.$$

1. Show that the process is $H$-locally asymptotically self-similar at $t = 0$.
2. We suppose that $\varphi$ is a $C^2$ non-vanishing function with compact support included in $(0, \infty)$. Show that the process is 1-locally asymptotically self-similar at $t \neq 0$. What is the tangent process in this case?

### 4.5.2 *Bivariate Lass*

This exercise is inspired by [87]. Let $h : \mathbb{R}^d \mapsto (0, 1)$ be a $\beta$-Hölder continuous multifractional function and

$$B_h(x) = \frac{1}{(C(h(x)))^{1/2}} \int_{\mathbb{R}^d} \frac{e^{-ix.\xi} - 1}{\|\xi\|^{d/2+h(x)}} \widehat{W}(d\xi)$$

be the corresponding multifractional Brownian field. Let us assume $\beta > \sup_{x\in\mathbb{R}^d} h(x)$. Let $(\tilde{B}(x, u))_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d}$ be the centered Gaussian field such that $\forall x \in \mathbb{R}^d$, $(\tilde{B}(x, u))_{u\in\mathbb{R}^d}$ is a standard fractional Brownian field with parameter $h(x)$. Let us also assume that $\forall n \in \mathbb{N}^*$ and $x_i$, $i = 1, \ldots, n$ pairwise distinct points in $\mathbb{R}^d$, $(\tilde{B}(x_1, u))_{u\in\mathbb{R}^d}, \ldots, (\tilde{B}(x_n, u))_{u\in\mathbb{R}^d}$ are independent fields. Let us denote by

$$X(x, y) = \frac{1}{(C(y))^{1/2}} \int_{\mathbb{R}^d} \frac{e^{-ix\cdot\xi} - 1}{\|\xi\|^{d/2+y}} \widehat{W}(d\xi)$$

a Hurst field, where $x \in \mathbb{R}^d$ and where we take the same Fourier transform of a real Brownian random measure $\widehat{W}(d\xi)$ for the mBm and the Hurst field.

1. Let us define $Y_{\varepsilon,1}(x, u) = \frac{X(x+\varepsilon u,h(x))-X(x,h(x))}{\varepsilon^{h(x)}}$. Show that

$$\mathbb{E}(Y_{\varepsilon,1}(x, u)Y_{\varepsilon,1}(x, v)) = \mathbb{E}(\tilde{B}(x, u)\tilde{B}(x, v)).$$

2. Let $x \neq y$ show that

$$\lim_{\varepsilon \to 0^+} \mathbb{E}(Y_{\varepsilon,1}(x,u) Y_{\varepsilon,1}(y,v)) = 0.$$

3. Let us define $Y_{\varepsilon,2}(x,u) = \frac{X(x+\varepsilon u, h(x+\varepsilon u)) - X(x+\varepsilon u, h(x))}{\varepsilon^{h(x)}}$. Show that

$$\lim_{\varepsilon \to 0^+} \mathbb{E}(Y_{\varepsilon,2}(x,u))^2 = 0.$$

4. Show that

$$\lim_{\varepsilon \to 0^+} \left( \frac{B_h(x+\varepsilon u) - B_h(x)}{\varepsilon^{h(x)}} \right)_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d} \overset{(d)}{=} (\tilde{B}(x,u))_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d}.$$

### 4.5.3 Multifractional Functions with Jumps

Let us consider a Borel function $h$ such that $\forall t \in \mathbb{R},\ h(t) \in (0,1)$ and

$$\lim_{t \to 1^-} h(t) = \frac{1}{2}, \quad \lim_{t \to 1^+} h(t) = \frac{1}{4}. \tag{4.151}$$

1. Let us consider the normalization function $C$ of an mBm defined in (4.19) for $d = 1$. Show that $\lim_{t\to 1^-} C(h(t)) = \sqrt{\frac{\pi}{2}}$ and $\lim_{t\to 1^+} C(h(t)) = 1$.
2. Let us a consider an mBm $(B_h(t))_{t\in\mathbb{R}}$ associated with the multifractional function $h$ that satisfies (4.151). Show that $B_h$ has almost surely a jump for $t = 1$.

### 4.5.4 Uniform Convergence of the Series Expansion of the mBm

In this exercise we give another proof of the almost sure uniform convergence on a compact $\mathcal{K}$ of the series

$$\frac{1}{(C(h(t)))^{1/2}} \sum_{\lambda \in \Lambda^+} \chi_\lambda(t, h(t)) \eta_\lambda \tag{4.152}$$

to the mBm (4.26) $B_h$. Let $h$ be function locally Hölder continuous with exponent $\beta$.

1. Show the uniform convergence of the series on $\mathcal{K}$

$$\sum_{j \geq J, k \in \mathbb{Z}} \chi_{j,k,1}(t, h(t)) \eta_{j,k,1}$$

to 0 when $J \to +\infty$.

2. For $j > 0$ show the uniform convergence of the series on $\mathcal{K}$.

$$\sum_{|k|\geq K} \chi_{j,k,1}(t, h(t))\eta_{j,k,1}$$

to 0 when $K \to +\infty$.

3. Show the uniform convergence of the series on $\mathcal{K}$

$$\sum_{|k|\geq K} \tilde{\chi}_{0,k,0}(t, h(t))\eta_{0,k,0}.$$

to 0 when $K \to +\infty$.

4. Show the uniform convergence of the series on $\mathcal{K}$

$$\sum_{|k|\geq K} \chi_{0,k,0}(t, h(t))\eta_{0,k,0}.$$

to 0 when $K \to +\infty$.

5. Deduce from the previous questions the almost sure uniform convergence of the series (4.152) to a mBm.

# Chapter 5
# Statistics

In this chapter we would like to discuss the use of the models introduced in the previous chapters for Statistics. One of the major question is the estimation of the various parameters in those models. The common framework of this estimation is that we observe only one sample path of the field on a finite set of locations in a compact set. Most of the results will then be asymptotics, when the mesh of the grid of the locations where the fields are observed is decreasing to 0. In short we are doing fill-in statistics. Typically we are supposed to observe $\left(\frac{k_1}{N}, \ldots, \frac{k_d}{N}\right), \ 0 \le k_i \le N, \ i = 1, \ldots, d$. To study this problem we will first consider the unifractional case, where the multifractional function is constant in Sect. 5.1. We obtain very often strong consistency of our estimator and confidence intervals related to some central limit theorems. The question of the optimality of the rate of convergence for our estimator is addressed in Sect. 5.2 using the framework of efficiency and Cramer-Rao bounds. Then we generalize the estimation to the multifractional case in Sect. 5.3. Section 5.4 is devoted to the extension of our techniques to the case where the field is no more Gaussian and to the estimation of intermittency. Finally we conclude this chapter by an application to the approximation of the integrals of fractional processes in Sect. 5.5.

## 5.1 Unifractional Case

The aim of this section is to perform the identification of fractional parameters of a unifractional Gaussian process $X$. A process will be called unifractional if the multifractional function of Definition 4.2.1 is constant. As explained in the introduction of this chapter the process $X$ is assumed to be observed on a discrete sampling design $\frac{k}{N}, k = 0, \ldots, N$. Moreover the variance of the increments of process $X$ is assumed to satisfy the following expansion

S. Cohen and J. Istas, *Fractional Fields and Applications*,
Mathématiques et Applications 73, DOI: 10.1007/978-3-642-36739-7_5,

$$\mathbb{E}(X(t+h)-X(t))^2 = C(t)|h|^{2H} + r(t,h), \tag{5.1}$$

where the remainder function $r(t,h)$ is, in a sense to precise, of order $o(|h|^{2H})$ as $h \to 0$. The typical examples are filtered white noises. We will essentially focus on the estimation of the fractional index $H$ and of the singularity function $C(t)$. As usual in Statistics, three cases have to be considered.

1. Parametric case.
   This is essentially the case when $X$ is a fractional Brownian motion . The remainder term $r(t,h)$ is then equal to zero and function $C(t)$ is indeed a constant $C$.
2. Semi-parametric case.
   The remainder term is not zero anymore, and singularity function $C(t)$ is a constant or a function, but $C(t)$ is not an interesting parameter: The aim is only to perform the estimation of parameter $H$. Nevertheless, some cases can be considered. For instance, some estimators require $C$ to be constant or for the process to be centered. This is the case for instance if the estimator requires stationary increments for process $X$.
3. Non-parametric case.
   The framework is the same as in the semi-parametric case, but now function $C$ is an interesting parameter.

The outline of the section will be as follows. We will first present the parametric case: The maximum likelihood estimator, the Whittle's approximation and the variations estimators. These estimators work for fractional Brownian motion s or the standard fractional Brownian motion . We will then present the semi-parametric situation. The main idea is that the existing methods can be more or less viewed as a particular case of the so-called generalized quadratic variations estimators. Next we will focus on generalized quadratic variations estimators, giving precise statements and proofs. Afterwards, we will mention some extensions of the previous results.

### *5.1.1 Maximum Likelihood Estimator and Whittle's Approximation*

In this section we recall some classical results concerning the estimation of $H$ using time series techniques.

Let us consider the observations

$$Y_N = \{B_H(k+1) - B_H(k), k = 0, \dots, N-1\}$$

from a fractional Brownian motion $B_H$. Denote by $\Sigma_N$ the covariance matrix of the sample

$$\{B_H(k+1) - B_H(k), k = 0, \dots, N-1\}.$$

Because of Corollary 3.2.1

$$\begin{aligned}\Sigma_{k,\ell} &= \mathbb{E}\left([B_H(k+1) - B_H(k)][B_H(\ell+1) - B_H(\ell)]\right) \\ &= C\left(|\ell - k + 1|^{2H} - 2|\ell - k|^{2H} + |\ell - k - 1|^{2H}\right).\end{aligned}$$

The likelihood of the vector $Y_N$ is given by

$$L_N(H, C) = \frac{1}{(2\pi)^{\frac{N}{2}} det(\Sigma_N)} \exp\left(-\frac{1}{2} Y_N \Sigma_N^{-1t} Y_N\right).$$

Let us remark that $\Sigma_N$ is invertible as consequence of Corollary 3.2.2. The maximum likelihood estimator is obtained by minimizing $-\log L_N(H, C)$

$$(\widehat{H}_N, \widehat{C}_N) = Argmin_{H\in(\frac{1}{2},1), C\in(0,+\infty)} \log det(\Sigma_N) + \frac{1}{2} Y_N \Sigma_N^{-1t} Y_N.$$

As $N \to +\infty$, this estimator $(\widehat{H}_N, \widehat{C}_N)$ is strongly consistent and asymptotically Gaussian (e.g. [25]). Unfortunately, the maximization of the likelihood leads to computational problems. The minimization is costly in terms of CPU time. The computation of the matrix $\Sigma_N^{-1}$ is unstable. Therefore, an alternative method, known as Whittle's approximation has been introduced.

Denote by $f(\lambda, H, C)$ the spectral density of $B_H(k+1) - B_H(k)$

$$f(\lambda, H, C) = \frac{C}{\sqrt{\frac{\pi}{2}} C_H} \left\{(1 - \cos(\lambda)) + \sum_{k=-\infty}^{+\infty} |\lambda + 2k\pi|^{-1-H}\right\}$$

for $\lambda \in [-\pi, \pi]$. Here $f$ is the spectral density of a time series which is a periodicization of the spectral density of fractional Brownian motion given in 3.15. Actually the matrix $\Sigma_N = \mathcal{T}_N(f)$, where $\mathcal{T}_N$ denotes a Toeplitz matrix of dimension $N$ associated to the spectral density $f$. See [66] for a classical introduction to Toeplitz matrices.

The idea of Whittle's approximation is to replace the inverse matrix $\Sigma_N^{-1}$ by the matrix $A_N = \mathcal{T}_N(1/f)$ which turns out to be

$$A_{k,\ell} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \frac{1}{f(\lambda, H, C)} e^{i(k-\ell)} d\lambda \quad k, \ell = 0, \ldots, N-1$$

and we know that roughly speaking when $N \to \infty \quad A_N \sim \Sigma_N^{-1}$. Moreover we have

$$\lim_{N\to+\infty} \frac{1}{N} \log det \Sigma_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log f(\lambda, H, C) d\lambda \; .$$

An approximate likelihood estimator is then obtained by minimizing this pseudo-likelihood function

$$(\widehat{H}_N, \widehat{C}_N) =$$
$$Argmin_{H\in(\frac{1}{2},1),C\in(0,\infty)} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log f(\lambda, H, C) + \frac{1}{N} Y_N {A_N}^t Y_N.$$

We obtain the following theorem (e.g. [59]).

**Theorem 5.1.1** *For* $\frac{1}{2} < H < 1$,

1. *Strong consistency.*
$$\lim_{N\to+\infty} (\widehat{H}_N, \widehat{C}_N) \overset{(a.s.)}{=} (H, C).$$

2. *Asymptotic normality.*
$$\lim_{N\to+\infty} \sqrt{N}(\widehat{H}_N - H, \widehat{C}_N - C) \overset{(d)}{=} \mathcal{N}_2(0, 2D^{-1}),$$
*where the matrix* $D = (D_{i,j}),\ i, j = 1, 2$ *is*
$$D_{1,1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{\partial}{\partial H}\log f(\lambda, H, C)\right)^2 d\lambda,$$
$$D_{2,2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{\partial}{\partial C}\log f(\lambda, H, C)\right)^2 d\lambda,$$
$$D_{1,2} = D_{2,1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial}{\partial H}\log f(\lambda, H, C)\frac{\partial}{\partial C}\log f(\lambda, H, C) d\lambda.$$

## *5.1.2 Variations Estimator for the Standard Fractional Brownian Motion*

### 5.1.2.1 Quadratic Variations

The maximum likelihood method and the Whittle's approximation use observations of a fractional Brownian motion path on an unbounded domain $\{B_H(k), k = 0, \dots, N\}$ asymptotically in $N$. If we come back to our framework where the observations are fill-in statistics in a compact interval $\left(\frac{k}{N}\right)$,$0 \le k \le N$, we can still estimate the parameter $H$.

In this case it is classical to use the quadratic variations on [0, 1] of the process $X$ at scale $1/N$ defined by

$$V_N = \sum_{k=0}^{N-1} \left(X\left(\frac{k+1}{N}\right) - X\left(\frac{k}{N}\right)\right)^2.$$

A result (e.g. [67]) ensures that

$$\lim_{N\to+\infty} N^{2H-1} V_N \overset{(a.s.)}{=} 1.$$

The quadratic variations can therefore be used to identify parameter $H$

$$\widehat{H}_N = \frac{1}{2} + \frac{1}{2}\log_2 \frac{V_{N/2}}{V_N}.$$

When considering the limiting distribution of the quadratic variations, [67] found two cases.

1. $0 < H < 3/4$. The variable $\sqrt{N}\left(N^{2H-1}V_n - 1\right)$ converges in distribution, as $N \to +\infty$, to a Gaussian variable.
2. $3/4 < H < 1$. The variable $N^{2-2H}\left(N^{2H-1}V_n - 1\right)$ converges in distribution, as $N \to +\infty$, to a non-Gaussian variable.

Therefore, the rate of convergence of the estimator of $H$ based on quadratic variations dramatically fails when $3/4 < H < 1$.

#### 5.1.2.2 Extension of Quadratic Variations

To circumvent this problem, [73] introduce generalized quadratic variations associated with a finite second order derivative. Let $a = (a_k, k = 0, \ldots, K)$ be a discrete sequence of real numbers satisfying

$$\sum\nolimits_{k=0}^{K} a_k = 0, \sum\nolimits_{k=0}^{K} k a_k = 0.$$

The reason of the condition $\sum\nolimits_{k=0}^{K} k a_k = 0$ will be given later. For instance, sequence $a = (-1, 1)$ does not fulfill these conditions, but sequence $a = (-1, 2, -1)$ does. The increments of a standard $B_H$ associated with sequence $a$ are defined by

$$\Delta_p B_H = \sum_{k=0}^{K} a_k B_H\left(\frac{k+p}{N}\right).$$

Normalized $\beta$-variations associated with a sequence $(a_k)$ are then defined by

$$W_{N,\beta} = \frac{1}{N-K+1}\sum_{p=0}^{N-K} |\Delta_p B_H|^{\beta}.$$

Define the function $c(\beta, H)$

$$c(\beta, H) = \left( -\frac{1}{2} \sum_{k,k'=0}^{K} a_k a_{k'} |k - k'|^{2H} \right)^{\frac{\beta}{2}}$$

so that

$$\mathbb{E} W_{N,\beta} = \frac{1}{N^{\beta H}} \, c(\beta, H) \, E_\beta,$$

where if $Z \overset{(d)}{=} \mathcal{N}_1(0, 1)$ then $E_\beta = \mathbb{E}|Z|^\beta$. Now define

$$g_{N,\beta}(t) = \frac{1}{N^{\beta t}} \, c(\beta, H) \, E_\beta.$$

The idea of the estimation is to take the value of $t$ that fits $g_{N,\beta}(t)$ and $W_{N,\beta}$

$$\widehat{H}_N = g_{N,\beta}^{-1}(W_{N,\beta}). \tag{5.2}$$

We then obtain the following theorem ([35, 112]).

**Theorem 5.1.2**
*1. Strong consistency.*

$$\lim_{N \to +\infty} \widehat{H}_N \overset{(a.s.)}{=} H.$$

*2. Asymptotic normality.*

$$\lim_{N \to +\infty} \sqrt{N} \log(N)(\widehat{H}_N - H) \overset{(d)}{=} \mathcal{N}_1(0, \Sigma_\beta),$$

*where*

$$\Sigma_\beta = \frac{1}{\beta^2} \sum_{j \geq 1} 2^{2(j-1)}((j-1)!)^2 \sum_{i=-\infty}^{+\infty} \left( \frac{\sum_{k,k'=0}^{K} a_k a_{k'} |k - k' + i|^{2H}}{\sum_{k,k'=0}^{K} a_k a_{k'} |k - k'|^{2H}} \right)^{2j}.$$

*This variance $\Sigma_\beta$ is minimal for $\beta = 2$ the quadratic variations are optimal among the $\beta$-variations.*

**Remark 5.1.1**
*The same method of estimation can be used for non-standard fractional Brownian motion ([35]), but the rate of convergence of the estimator of $H$ is only $\sqrt{N}$ then.*

**Remark 5.1.2**
*Time-frequency methods.*

*Let $\psi$ be kernel such that $\int \psi(x)dx = 0$. $\psi$ can be a wavelet for instance. Let $j \geq 0$ and $k = 0, \ldots, 2^j - 1$. Let $X$ be a fBm then the coefficients*

$\int X(x)2^j\psi(2^jx-k)dx$ *are centered Gaussian variables with variance of order* $2^{-2jH}$. *It has been proposed (cf. [2]) to estimate parameter* $H$ *via an estimation of the variance* $\mathbb{E}\left(\int X(x)2^j\psi(2^jx-k)dx\right)^2$.

*Let* $f$ *be an arbitrary function. The estimation of* $\int f(x)2^j\psi(2^jx-k)dx$ *will be done via some quadrature formula. We refer to Sect. 5.5 for quadrature formula for stochastic processes like* $\sum_{\ell=0}^{N} c_{\ell,N} f\left(\frac{\ell}{N}\right)$, *where the* $(c_{\ell,N})$ *depend on kernel* $\psi$. *Any reasonable quadrature formula will be exact for constant function* $f$ *if* $\sum_{\ell=0}^{N} c_{\ell,N}=0$. *The estimation of*

$$\mathbb{E}\left(\int X(x)2^j\psi(2^jx-k)dx\right)^2$$

*will therefore be done with the formula* $\left(\sum_{\ell=0}^{N} c_{\ell,N} X\left(\frac{\ell}{N}\right)\right)^2$. *In other words, time-frequency methods are particular cases of generalized quadratic variations methods. Nevertheless it has been used in [64] to estimate the Hurst parameter from a discrete noisy data.*

### *5.1.3 Application to Filtered White Noises*

As in Sect. 5.1.2 let $a_k$, $k=0,\dots,K$ be a finite sequence of real numbers satisfying $\sum_{k=0}^{K} a_k=0$, $\sum_{k=0}^{K} ka_k=0$. Actually in the setting of fwn we also have to assume $\sum_{k=0}^{K} k^2a_k=0$. We can take $a=(-1,3,-3,1)$ for instance.

Generalized quadratic variations of process $X$ on $[0,1]$ at scale $1/N$ associated with sequence $(a_k)$ are defined by

$$V_N=\sum_{p=0}^{N-K}(\Delta_p X)^2,$$

where

$$\Delta_p X=\sum_{k=0}^{K} a_k X\left(\frac{k+p}{N}\right).$$

Generalized quadratic variations of various Gaussian processes have been widely studied ([21, 71, 73, 80]). We use them for the estimation problem in the framework of filtered white noises ([21]). Let us rewrite the Definition 4.3.1 of fwn when $d = 1$.

**Hyp 5.1.1**

*Filtered white noises ([21]).*

*Let $g(t, \xi)$ be the following harmonizable fractional-integral type kernel*

$$g(t, \xi) = \frac{a(t)}{|\xi|^{\frac{1}{2}+H}} + \varepsilon(t, \xi), \tag{5.3}$$

*with $a(t) \in C^2$, $\varepsilon(t, \xi) \in C^{2,2}$ satisfying, for $i, j = 0, 1, 2$:*

$$\left| \frac{\partial^{i+j}}{\partial t^i \partial \xi^j} \varepsilon(t, \xi) \right| \leq \frac{C}{|\xi|^{\frac{1}{2}+\eta+j}}\ , \tag{5.4}$$

*with $\eta > H$ and $\overline{\varepsilon(t, \xi)} = \varepsilon(t, -\xi)$. Here $C$ denotes a generic constant that can change from an occurrence to another.*

Let $\widehat{W}$ be the Fourier transform of a real valued Brownian random measure (See Definition 2.1.16.). Let us assume $\int_{\mathbb{R}} |(e^{-ix.\xi} - 1)g(x, \xi)|^2 d\xi < +\infty$. Define the process $X$ by

$$X(t) = \int_{\mathbb{R}} g(t, \xi)(e^{it\xi} - 1)\widehat{W}(d\xi).$$

The process $X$ is a filtered white noise.

**Remark 5.1.3** *Please remark that in Definition 4.3.1 we have a minus sign in front of the factor $it\xi$ and*

$$X(t) = \int_{\mathbb{R}} g(t, \xi)(e^{-it\xi} - 1)\widehat{W}(d\xi).$$

*Actually since $a$ is real valued, $\overline{\varepsilon(t, \xi)} = \varepsilon(t, -\xi)$, and $\overline{g(t, \xi)} = \varepsilon(t, -\xi)$*

$$\int_{\mathbb{R}} g(t, \xi)(e^{-it\xi} - 1)\widehat{W}(d\xi) \overset{(d)}{=} \int_{\mathbb{R}} g(t, \xi)(e^{it\xi} - 1)\widehat{W}(d\xi).$$

*So the sign does not matter.*

When $a(t) = 1$ and $\varepsilon(t, \xi) = 0$, the resulting process is a fractional Brownian motion . By construction, the value of a filtered white noise at 0 is equal to zero. This restriction does not matter: The identification results we present below are still valid if we add an arbitrary random variable at time $t = 0$ to a filtered white noise. The function $\varepsilon$ is clearly a remainder term, and conditions on this function are given to ensure that $\varepsilon$ really is a remainder term. Even if it is difficult to explain without a deep inspection of the proof, the condition $a \in C^2$ implies a decrease of the increments of orders 1 and 2 of the process.

We should wonder whether the stationary processes considered for instance by([73, 74, 80]) can be viewed as filtered white noise. Such is the case, via a minor modification of the process. Let us recall the setting of example 2.1.2 and let $f(\xi)$ be the spectral density of a Gaussian centered stationary process $Y$. By Bochner's theorem, this spectral density is a positive function. The process $Y$ can then be represented through a stochastic integral with respect to a Fourier transform of a real Brownian measure $\widehat{W}$

$$Y(t) = \int_{\mathbb{R}} \sqrt{f(\xi)} e^{it\xi} \widehat{W}(d\xi).$$

Process $X(t) = Y(t) - Y(0)$ is then represented as follows

$$X(t) = \int_{\mathbb{R}} \sqrt{f(\xi)}(e^{it\xi} - 1)\widehat{W}(d\xi)$$

If $f(\xi)$ verifies an expansion like, as $|\xi| \to +\infty$,

$$f(\xi) = \frac{a}{|\xi|^{1+2H}} + \varepsilon(\xi),$$

with

$$|\varepsilon(\xi)| \leq \frac{C}{|\xi|^{1+2H}},$$

and for some $\eta > 0$ and $j = 1, 2$

$$|\varepsilon^{(j)}(\xi)| \leq \frac{C}{|\xi|^{\eta+j}},$$

then the process $X$ is a filtered white noise.

The estimator of parameter $H$ derived from the generalized quadratic variations is then as follows

$$\widehat{H}_N = \frac{1}{2} + \frac{1}{2}\log_2 \frac{V_{N/2}}{V_N}. \tag{5.5}$$

**Theorem 5.1.3**

*Let $X$ be a process satisfying* **Hyp** *5.1.1.*

1. *Strong consistency.*

$$\lim_{N \to +\infty} \widehat{H}_N \overset{(a.s.)}{=} H.$$

2. *Asymptotic normality. If in* **Hyp** *5.1.1 $\eta > \frac{1}{2} + H$*
*As $N \to +\infty$, $\sqrt{N}(\widehat{H}_N - H)$ converges in distribution to a centered Gaussian variable.*

The proof of Theorem 5.1.3 (e.g. [21, 74]) is rather technical. We give a sketch of the proof, and the complete proof is postponed to the Appendix.

#### 5.1.3.1 Sketch of the Proof of Theorem 5.1.3

Let $R(t, t')$ be the covariance function of the process $X$ with respect to $t, t'$.

Let $W_N = N^{2H-1} V_N$. A heuristic computation of the variance of $W_N$ makes it clear why the non-Gaussian limit when $H > 3/4$ disappears with the assumption $\sum_{k=0}^{K} k a_k = 0$. Since $X$ is Gaussian,

$$\begin{aligned}
\mathrm{var} W_N &= 2N^{4H-2} \sum_{p,p'=0}^{N-K} \left(\mathbb{E}\Delta_p X \Delta_{p'} X\right)^2 \\
&= 2N^{4H-2} \sum_{p,p'=0}^{N-K} \left( \sum_{k,k'=0}^{K} a_k a_{k'} R\left(\frac{k+p}{N}, \frac{k'+p'}{N}\right)\right)^2 .
\end{aligned}$$

This sum is split into two parts.

- $|p - p'| \le K$. The condition $\sum_{k=0}^{K} a_k = 0$ implies that

$$\sum_{k,k'=0}^{K} a_k a_{k'} R\left(\frac{k+p}{N}, \frac{k'+p'}{N}\right) = O(N^{-2H}).$$

  The sum for $|p - p'| \le K$ is a $0(N^{-1})$.
- $|p - p'| > K$. A Taylor expansion up to order 2, where the remainder term has been omitted, leads to $R\left(\frac{k+p}{N}, \frac{k'+p'}{N}\right) = A + B$,

$$\begin{aligned}
A &= R\left(\frac{p}{N}, \frac{p'}{N}\right) + \frac{k}{N}\frac{\partial R}{\partial t}\left(\frac{p}{N}, \frac{p'}{N}\right) + \frac{k'}{N}\frac{\partial R}{\partial t'}\left(\frac{p}{N}, \frac{p'}{N}\right) \\
&\quad + \frac{k^2}{2N^2}\frac{\partial^2 R}{\partial t^2}\left(\frac{p}{N}, \frac{p'}{N}\right) + \frac{k'^2}{2N^2}\frac{\partial^2 R}{\partial t'^2}\left(\frac{p}{N}, \frac{p'}{N}\right), \\
B &= \frac{kk'}{N^2}\frac{\partial^2 R}{\partial t \partial t'}\left(\frac{p}{N}, \frac{p'}{N}\right).
\end{aligned}$$

– Let us now assume $\sum_{k=0}^{K} a_k = 0$ and $\sum_{k=0}^{K} ka_k \neq 0$. The term $\sum_{k,k'=0}^{K} a_k a_{k'} A$ vanishes. The term $\sum_{k,k'=0}^{K} a_k a_{k'} B$ does not vanish and is a $O(N^{-2})$. The sum for $|p - p'| > K$ is a $0(N^{4H-4})$. If we compare this term $0(N^{4H-4})$ with the previous term $0(N^{-1})$, we can understand the critical value $H = 3/4$ of [67]: For $H < 3/4$, the main term is the first one and the rate of convergence is equal to $O(N^{-1})$. For $H > 3/4$, the second term is the main one and the rate of convergence $0(N^{4H-4})$ is slower.

– Let us now assume $\sum_{k=0}^{K} a_k = 0$ and $\sum_{k=0}^{K} ka_k = 0$. The term $\sum_{k,k'=0}^{K} a_k a_{k'} B$ vanishes.

  The condition $\sum_{k=0}^{K} ka_k = 0$ allows us to take into account that the covariance function $R(t, t')$ is twice differentiable outside the diagonal $\{t = t'\}$. In other words, the increments $\Delta_p X$ et $\Delta_{p'} X$ are more decorellated, and a rate of convergence of order $O(N^{-1})$ is always obtained.

The full proof of Theorem 5.1.3 is given in Appendix A.

**Remark 5.1.4**

*Non-centered processes.*

*Let $X$ be a process satisfying* **Hyp** *5.1.1. Let $m(t)$ be $C^1$. Then part 1. of Theorem 5.1.3 holds for the process $Y(t) = X(t) + m(t)$.*

To prove this remark, it is sufficient to notice that:

$$V_N = \sum_{p=0}^{N-K} \left(\Delta_p X\right)^2 + \sum_{p=0}^{N-K} \left(\Delta_p m\right)^2 + 2 \sum_{p=0}^{N-K} \left(\Delta_p X \Delta_p m\right),$$

where $V_N$'s are the variations associated with the process $Y$. Then

$$\sum_{p=0}^{N-K} \left(\Delta_p m\right)^2 \leq C N^{-1}$$

and

$$\sum_{p=0}^{N-K} \left(\Delta_p X\right)^2 \leq C N^{1-2H}.$$

By Cauchy Schwarz inequality

$$|\sum_{p=0}^{N-K}\left(\Delta_p X \Delta_p m\right)| \le C N^{1-2\eta}$$

with $\eta > H$. In sect A.2 we prove that $\sum_{p=0}^{N-K}\left(\Delta_p X\right)^2 \sim C' N^{1-2H}$, with $C' > 0$, hence $V_N \sim C' \sum_{p=0}^{N-K}\left(\Delta_p X\right)^2$.

#### 5.1.3.2 Optimality of the Sequence *a*

The asymptotic variance of estimator (5.5) is known in some cases (E.g. [74]). The best sequence $a_k, k = 0, \dots, K$, i.e. the sequence that minimizes this asymptotics variance, is unknown at the moment. Coeurjolly [34, 35] has computed this asymptotically variance for a lot of discrete sequences $a_k, k = 0, \dots, K$. The best results are obtained for the following sequences:

1. $0 < H < 3/4$.

$$\begin{aligned} a_0 &= 1, \\ a_1 &= -1. \end{aligned}$$

2. $3/4 \le H < 1$.

$$\begin{aligned} a_0 &\sim 0.4829629, \\ a_1 &\sim -0.8365163, \\ a_2 &= 0.22414386, \\ a_3 &\sim 0.12940952. \end{aligned}$$

Notice that the two sequences are the discrete sequences associated with the Haar basis and the so-called Daubechies-4 wavelet basis ([17]).

#### 5.1.3.3 Filtered White Noises of Arbitrary Fractional Index

For the sake of simplicity, the filtered white noises have been introduced with a fractional index $0 < H < 1$. Indeed there is no reason for this restriction. We will now introduce filtered white noises of arbitrary fractional index $H > 0$. Let

$$e_n(\xi) = e^{i\xi} - \sum_{k=0}^{n} \frac{(i\xi)^k}{k!},$$

and

$$g(t,\xi) = \frac{a(t)}{|\xi|^{\frac{1}{2}+H}} + \varepsilon(t,\xi).$$

The integer $n$ is chosen such that the function $g(t,\xi)e_n(\xi)$ is square integrable for each $t$. It means that $H$ is not an integer and $n < H < n+1$. The case $n = 0$ corresponds to the previous case $0 < H < 1$. **Hyp** 5.1.1 has to be generalized.

**Hyp 5.1.2**

- $n+1 > H > n$ *and* $\varepsilon(t,\xi) \in C^{2(n+1),2}([0,1]\times\mathbb{R})$ *is a function such that*

$$\left|\frac{\partial^{i+j}}{\partial t^i \partial \xi^j}\varepsilon(t,\xi)\right| \le \frac{C}{|\xi|^{\frac{1}{2}+\eta+j}},$$

*for* $i = 0$ *to* $2(n+1)$ *and* $j = 0$ *to* $2$ *with* $\eta > 1/2 + H$.
- $a \in C^{2n+2}([0,1])$.

Now let $a_k, k = 0,\ldots,K$ be a discrete sequence of real satisfying, with $J \ge 4n+2$

$$\sum_{k=0}^{K} k^j a_k = 0,\ 0 \le j \le J,\ \sum_{k=0}^{K} k^{J+1} a_k \ne 0.$$

Generalized quadratic variations of process $X$ on $[0,1]$ at scale $1/N$ associated with sequence $(a_k)$ are still defined by

$$V_N = \sum_{p=0}^{N-K} (\Delta_p X)^2,$$

where

$$\Delta_p X = \sum_{k=0}^{K} a_k X\left(\frac{k+p}{N}\right).$$

The estimator of the parameter $H$ derived from the generalized quadratic variations is still as follows:

$$\widehat{H}_N = \frac{1}{2} + \frac{1}{2}\log_2 \frac{V_{N/2}}{V_N}.$$

**Theorem 5.1.4**

*Let $X$ be a process satisfying* **Hyp** *5.1.2.*

1. *Strong consistency.*

$$\lim_{N\to+\infty} \widehat{H}_N \overset{(a.s.)}{=} H.$$

2. *Asymptotic normality.*
   *As $N \to +\infty$, $\sqrt{N}(\widehat{H}_N - H)$ converges in distribution to a centered Gaussian variable.*

The proof of Theorem 5.1.4 is identical to the proof of Theorem 5.1.3 and is therefore omitted.

### *5.1.4 Further Fractional Parameters*

For a Gaussian process satisfying (5.1), the parameter $H$ drives the first order expansion of the covariance function. We can have a look at the second order expansion of the covariance function. The natural question then arises: Is the second order parameter identifiable? We will investigate this question in the case of filtered white noises (cf. **Hyp** 5.1.1). Various submodels of filtered white noises may be considered [21], we will focus here on the following. Let $\widehat{W}$ and $\widehat{\widehat{W}}$ be two independent Fourier transform of real Brownian random measures. Let $a(t)$ and $b(t)$ be two non-vanishing $C^2$ functions and let $0 < H_1 < H_2 < 1$. Let us then define process $X$ on [0, 1] by:

$$X(t) = \int_{\mathbb{R}} \frac{a(t)(e^{it\xi} - 1)}{|\xi|^{\frac{1}{2}+H_1}} \widehat{W}(d\xi) + \int_{\mathbb{R}} \frac{b(t)(e^{it\xi} - 1)}{|\xi|^{\frac{1}{2}+H_2}} \widehat{\widehat{W}}(d\xi).$$

Please remark that $X(t) = a(t)B_{H_1}(t) + b(t)B_{H_2}(t)$, where $B_{H_1}$ and $B_{H_2}$ are independent non-standard fractional Brownian motions. Let $V_N$ be the generalized quadratic variations associated with sequence $(a_k)$ (cf. Sect. 5.1.3). Since $H_2 > H_1$ one can consider a filtered white noise

$$\tilde{X}(t) = \int_{\mathbb{R}} g(t, \xi)(e^{it\xi} - 1)\widehat{W}(d\xi)$$

with filter $g(t, \xi) = \frac{a(t)}{|\xi|^{\frac{1}{2}+H_1}} + \varepsilon(t, \xi)$. The rest $\varepsilon(t, \xi) = \frac{b(t)}{|\xi|^{\frac{1}{2}+H_2}}$ and $\eta = H_2$ in 5.1.1. Hence for the filtered white noise $\tilde{X}$, an estimator of $H_1$ is given by

$$\widehat{H}_{1,N} = \frac{1}{2} + \frac{1}{2}\log_2 \frac{V_{N/2}}{V_N}.$$

It is still true for $X$ even if it deserves an additional proof (See [21]). The idea for estimating $H_2$ is to consider $V_{N/2} - 2^{2H_1-1}V_N$, which asymptotic behavior is of order $1 - H_1 - H_2$ and to plug in the estimator $\widehat{H}_{1,N}$ for $H_1$. Define therefore

$$\widetilde{V}_N = V_{N/2} - \frac{V_{N^2/2}}{V_{N^2}} V_N.$$

An estimator of $H_2$ is then given by:

$$\widehat{H}_{2,N} = \frac{1}{2} + \frac{1}{2}\log_2 \frac{\widetilde{V}_{N/2}}{\widetilde{V}_N}.$$

These estimators are strongly consistent [21].

**Theorem 5.1.5**

*Strong consistency.*

*Let us assume $H_2 - H_1 < 1/2$. Then:*

$$\lim_{N\to+\infty} \widehat{H}_{1,N} \overset{(a.s.)}{=} H_1,$$
$$\lim_{N\to+\infty} \widehat{H}_{2,N} \overset{(a.s.)}{=} H_2.$$

**Remark 5.1.5** *The following question arises: Should the condition $H_2 - H_1 < 1/2$ be improved? No general answer is available. Nevertheless, we can give a partial but satisfactory answer. It has been shown (cf. Theorem 3.2.1) that fractional Brownian motion can be expanded in series* (3.21)

$$B_H(t) = \sum_{\lambda\in\Lambda^+} \chi_\lambda(t, H_1)\eta_\lambda.$$

*where $\chi_\lambda(t)$ are defined in Proposition 4.3.6 and $\eta_\lambda$ are i.i.d. centered normalized Gaussian variables. Let us recall that $\Lambda^+ = \mathbb{N}^* \times \mathbb{Z} \times \{1\}$, and if $\lambda = (j, k, 1)$ $\chi_\lambda(t, H_1)$ is roughly equivalent to $2^{-jH_1}$ when $j \to +\infty$.*

*Let us then consider a process*

$$X(t) = \sum_{j>0}\sum_{k=0}^{2^j} 2^{-jH_1}\varepsilon_{j,k}\psi_{j,k}(t), \tag{5.6}$$

*where $\varepsilon_{j,k}$ are i.i.d. centered normalized Gaussian variables and the $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ is an orthonormal basis of $L^2$. The process $(X(t))_{t\in[0,1]}$ will in practice keep many qualitative properties of fractional Brownian motion restricted on* $[0, 1]$. *In particular its sample paths are $H_1$-Hölder continuous under weak assumptions on $\psi$. Let us now consider*

$$Y(t) = \sum_{j>0}\sum_{k=0}^{2^j} 2^{-jH_1}(1 + 2^{-j(H_2-H_1)})\varepsilon_{j,k}\psi_{j,k}(t),$$

*with $0 < H_1 < H_2 < 1$. The process $Y$ can be viewed as an analogous of $B_{H_1}(t) + B_{H_2}(t)$, where $B_{H_1}$ and $B_{H_2}$ are non-standard fractional Brownian motions. The condition (2.25) can be applied to processes $X$ and $Y$: The associated Gaussian measures are equivalent iff $H_2 - H_1 > 1/2$. Therefore in this setting the condition of Theorem 5.1.5 $H_2 - H_1 < 1/2$ can not be improved.*

### 5.1.5 Singularity Function: Interests and Estimations

In the unifractional case we have so far explained how to estimate the index $H$ and some related quantities. In this section we will roughly describe how to estimate the so-called singularity function $C(t)$ [cf. (5.1)]. It means that we have to estimate more delicate information on our models. The estimation problem will be presented in the setting of filtered white noises. Then several statistical problems related to the estimation of singularity function are considered.

#### 5.1.5.1 Filtered White Noise

Let $X$ be a filtered white noise satisfying **Hyp** 5.1.1. Let us recall Proposition 4.3.1 we get

$$\lim_{\varepsilon \to 0^+} \left( \frac{X(t+\varepsilon u) - X(t)}{\varepsilon^H} ) \right)_{u \in \mathbb{R}^d} \overset{(d)}{=} C_H^{1/2} a(t) (B_H(u))_{u \in \mathbb{R}^d}.$$

A comparison with (5.1) leads to the fact that $C(t)$ is up to a multiplicative constant equal to $a^2(t)$. Hence our aim is to perform the (non-parametric) estimation of function $a^2(t)$. More precisely we will only estimate the scalar product in $L^2$ of the square of this function with a given function. Actually it is the first step for the pointwise estimation of the function $a^2(t)$. The other steps consist in a series expansion of $a^2$ in an orthonormal basis $\phi_n$ of $L^2$ and the reconstruction of $a^2$ after truncation of the series expansion and estimation of the scalar products kept in the expansion.

Hence if $\phi$ is a $C^1$ function, the aim is now to estimate

$$I(\phi) = \int_0^1 a^2(t)\phi(t)dt.$$

Let us define a generalized quadratic variation at scale $1/N$ associated with $X, \phi$ and a sequence $(a_k)$ (cf. Sect. 5.1.3)

$$\begin{aligned} V_N(\phi) &= \sum_{p=0}^{N-K} \Delta_p X \Delta_p \phi X \\ &= \sum_{p=0}^{N-K} \sum_{k,k'=0}^{K} a_k a_{k'} \phi\left(\frac{k+p}{N}\right) X\left(\frac{k+p}{N}\right) X\left(\frac{k'+p}{N}\right). \end{aligned}$$

Let us define $F_\gamma$

$$F_\gamma(x) = \int_{\mathbb{R}} \sum_{k,k'=0}^{K} a_k a_{k'} \frac{e^{i(x+k-k')u}}{|u|^{\gamma+1}} du,$$

which is considered in the proof of Theorem 5.1.3. See Proposition A.2 in the Appendix. When the fractional index $H$ is unknown, the estimator of $I(\phi)$ is

$$\widehat{I}_{N,\widehat{H}_N}(\phi) = \frac{N^{2\widehat{H_N}-1}V_N(\phi)}{F_{2\widehat{H}_N}(0)} .$$

When the fractional index $H$ is known, the estimator of $I(\phi)$ is:

$$\widehat{I}_{N,H}(\phi) = \frac{N^{2H-1}V_N(\phi)}{F_{2H}(0)}.$$

**Theorem 5.1.6**

*Let $X$ be a process satisfying* **Hyp** *5.1.1 with $\eta > H + \frac{1}{2}$.*

1. *Strong consistency.*

$$\lim_{N\to+\infty} \widehat{I}_{N,\widehat{H}_N}(\phi) \overset{(a.s.)}{=} I(\phi),$$

$$\lim_{N\to+\infty} \widehat{I}_{N,H}(\phi) \overset{(a.s.)}{=} I(\phi).$$

2. *Asymptotic normality.*
   *As $N \to +\infty, \sqrt{N}(\widehat{I}_{N,H}(\phi) - I(\phi))$ converges in distribution to a centered Gaussian variable.*
   *As $N \to +\infty, \dfrac{\sqrt{N}}{\log N}(\widehat{I}_{N,\widehat{H}_N}(\phi) - I(\phi))$ converges in distribution to a centered Gaussian variable.*

*Sketch of the proof of Theorem 5.1.6*:

We will only give the proof for the estimator $\widehat{I}_{N,\widehat{H}_N}(\phi)$, the proof for $\widehat{I}_{N,H}(\phi)$ is simpler.

$\log\left(\widehat{I}_{N,\widehat{H}_N}(\phi)\right)$ can be split as follows

$$\begin{aligned}
\log\left(\widehat{I}_{N,\widehat{H}_N}(\phi)\right) &= T_1 + T_2 + T_3 + T_4 \\
T_1 &= 2\left(\widehat{H}_N - H\right)\log(N) \\
T_2 &= (2H-1)\log(N) + \log\frac{\mathbb{E}V_N(\phi)}{F_{2H}(0)} \\
T_3 &= \log\frac{F_{2H}(0)}{F_{2\widehat{H}_N}(0)} \\
T_4 &= \log\frac{V_N(\phi)}{\mathbb{E}V_N(\phi)}.
\end{aligned}$$

Following the proof of Proposition A.2.2 we obtain clearly

$$\mathbb{E}\left(\log N\left(\widehat{H}_N - H\right)\right)^4 = O\left(\frac{\log^4 N}{N^2}\right).$$

Then an application of Borel-Cantelli's Lemma implies that $T_1 \to 0$ (a.s.). Convergence of $T_2$ to $\log \int_0^1 \phi a^2$ is as in (A.30). Since the function $\gamma \to F_{2\gamma}(0)$ is continuous, $T_3 \to 0$. At last we get the almost sure convergence of $\widehat{I}_{N,\widehat{H}_N}(\phi))$ because (A.34) shows that $T_4 \to 0$.

Since $\eta > 1/2$, $\frac{\sqrt{N}}{\log N} T_1$ is the preponderant term and clearly converges to a Gaussian variable. Actually $T_2 - \log \int_0^1 \phi a^2$ is an $O(N^{-\eta})$, $\sqrt{N} T_3$ converges to a Gaussian variable because the function $\gamma \to F_{2\gamma}(0)$ is $C^1$ and $\sqrt{N} T_4$ converges to a Gaussian variable.

#### 5.1.5.2 Other Models

In the previous subsection we have observed that renormalized quadratic variations converge to integrals of the singularity functions for filtered white noises. Actually this fact is classical in other settings.

**Hyp 5.1.3**

*Let $X$ be a centered Gaussian process with a covariance function $R(t, t')$ satisfying the following conditions*

1. *$R$ is continuous on* $[0, 1]^2$,
2. *There exist two functions $c_1(s)$, $c_2(t)$ such that* $\frac{\partial^4 (R - c_1 - c_2)}{\partial t^2 \partial s^2}$ *exists and is a continuous function on* $[0, 1]^2 \backslash Diag$, *where* $Diag = \{(u, v) :$ *such that* $u = v\}$ *and there exists a constant $C_0$ and a real number* $\gamma \in (0, 2)$ *such that*

$$\left| \frac{\partial^4 (R - c_1 - c_2)(s, t)}{\partial t^2 \partial s^2} \right| \le \frac{C_0}{|s - t|^{\gamma + 2}}. \tag{5.7}$$

3. *Let us define two order increments:*

$$\delta_1^h f(s, t) = f(s + h, t) + f(s - h, t) - 2f(s, t) \tag{5.8}$$

$$\delta_2^h f(s, t) = f(s, t + h) + f(s, t - h) - 2f(s, t) \tag{5.9}$$

*and let us suppose that there exists a bounded function $g$ defined on* $(0, 1)$ *such that:*

$$\lim_{h \to 0^+} \sup_{t \in [h, 1-h]} \left| \frac{(\delta_1^h \circ \delta_2^h R)(t, t)}{h^{2-\gamma}} - g(t) \right| = 0. \tag{5.10}$$

**Theorem 5.1.7** *Let the process X satisfies* **Hyp** *5.1.3 for* $\gamma \in (0, 2)$ *then*

$$\lim_{N\to\infty} N^{1-\gamma} \sum_{k=1}^{N-1} [X_{(k+1)/N} + X_{(k-1)/N} - 2X_{k/N}]^2 = \int_0^1 g(t)dt. \qquad (5.11)$$

**Remark 5.1.6** *If assumption* **Hyp** *5.1.3.2. and assumption* **Hyp** *5.1.3.3. are satisfied for* $\gamma_0 \in (0, 2)$, *they are also satisfied for* $\gamma > \gamma_0$ *but the corresponding function* $g_\gamma$ *is vanishing. When* $\gamma_0$ *is the infimum of the real number such that assumptions* **Hyp** *5.1.3.2. is satisfied* $g_{\gamma_0}$ *can be viewed as a generalization of the singularity function introduced in [11].*

**Remark 5.1.7** *In assumption* **Hyp** *5.1.3.2 the functions* $c_1(s),\ c_2(t)$ *are introduced so that the convergence 5.11 can be applied to fractional Brownian motion with covariance*

$$R(s, t) = \frac{1}{2}\{|s|^{2H} + |t|^{2H} - |t - s|^{2H}\} \qquad (5.12)$$

*where* $0 < H < 1$. *Actually the partial derivatives of the covariance R do not exist when* $s = 0$ *but assumption (2) still holds. Assumption* **Hyp** *5.1.3.1. is clearly fulfilled for the fractional Brownian motion. Since*

$$\begin{aligned}\delta_1^h \circ \delta_2^h r(t,t) = 4r(t,t) + 2r(t-h, t+h) - 4r(t+h, t) - 4r(t-h, t)\\ + r(t+h, t+h) + r(t-h, t-h), \qquad (5.13)\end{aligned}$$

*we get* $\delta_1^h \circ \delta_2^h r(t,t) = (4 - 2^{2H})h^{2H}$. *If one makes the choice* $\gamma = 2 - 2H$ *then* $\forall t \in [h, 1-h]\ \ g(t) = (4 - 2^{2H})$, *assumption* **Hyp** *5.1.3.3. is fulfilled. The convergence of renormalized quadratic variations for fractional Brownian motion was first studied in [63]*

The estimation of the singularity function with the help of Theorem 5.1.7 is easier when the covariance $R$ is known analytically, whereas the framework of filtered white noises is more adapted to Gaussian processes for which the representation with respect to the Brownian random measure is asymptotically known. Some applications of the previous result can be found in [42]. Generalizations are also performed in [12–14].

### *5.1.6 Higher Dimension*

The previous results were given for sake of simplicity in dimension one. But they may easily be written in higher dimension.

Two directions may be chosen when modeling fractional processes in higher dimension: The isotropic and the anisotropic ones.

Let $\widehat{W}$ be the Fourier transform of a real valued Brownian random measure on $L^2(\mathbb{R}^d)$. Let us recall fractional Brownian field harmonizable representation (Definition 3.64), for $0 < H < 1$:

$$X_H(x) = \int_{\mathbb{R}^d} \frac{e^{-ix.\xi} - 1}{(C_H)^{1/2}\|\xi\|^{H+d/2}} \widehat{W}(d\xi).$$

We can easily check that the fractional parameter is $H$ in every direction, the field is therefore isotropic. The identification of $H$ may be performed either with observations on a rectangle or on a segment.

Fractional Brownian sheet [95] have been defined in Sect. 3.3 in (3.74), for $0 < H_1, H_2 < 1$

$$X_{H_1,H_2}(x) = \int_{\mathbb{R}^2} \frac{e^{-ix\xi_1} - 1}{(C_{H_1})^{1/2}|\xi_1|^{H_1+1/2}} \frac{e^{-iy\xi_2} - 1}{(C_{H_2})^{1/2}|\xi_2|^{H_2+1/2}} \widehat{W}(d(\xi_1, \xi_2)).$$

We recall that fractional Brownian sheet is an anisotropic field: There is one fractional parameter $H_j$ in each direction given by the axis. Unfortunately, in the other direction, one can easily check that the infimum of the $H_j$ dominates the fractional behavior: In almost every direction, the fractional index is given by $\inf_{j=1,2} H_j$. See Proposition 4.3.12. Other Gaussian anisotropic fields have been considered ([8, 30, 78]), the properties are similar: There is an unavoidable lack of real anisotropy.

In both cases, isotropic or anisotropic, the estimation procedure and proofs are very close to those of dimension one. We will only give the isotropic case, leaving to the reader the estimation of $\inf_{j=1,2} H_j$ as an exercise.

The field $X$ is observed at $\left(\frac{k_1}{N}, \ldots, \frac{k_d}{N}\right)$, $0 \le k_i \le N$, $i = 1, \ldots, d$.

Let $(a_\ell)$, $\ell = 0, \ldots, K$ be a real valued sequence such that

$$\sum_{\ell=0}^{K} a_\ell = 0\ , \quad \sum_{\ell=0}^{K} \ell a_\ell = 0\ , \sum_{\ell=0}^{K} \ell^2 a_\ell = 0\ . \tag{5.14}$$

For $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ define

$$a_{\mathbf{k}} = a_{k_1} \ldots a_{k_d}.$$

Define the increments of the field $X$ associated with the sequence $a$:

$$\begin{aligned}\Delta X_{\mathbf{p}} &= \sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} X_H\left(\frac{\mathbf{k}+\mathbf{p}}{N}\right)\\ &\overset{def}{=} \sum_{k_1,\ldots,k_d=0}^{K} a_{k_1}\ldots a_{k_d} X_H\left(\frac{k_1+p_1}{N}, \ldots, \frac{k_d+p_d}{N}\right),\end{aligned}$$

For instance we can take $K = 3,\ a_0 = -1,\ a_1 = 3,\ a_2 = -3,\ a_3 = 1$. Define the generalized quadratic variations associated with sequence $a$:

$$V_N = \sum_{\mathbf{p}=\mathbf{0}}^{\mathbf{N}-\mathbf{K}} (\Delta X_{\mathbf{p}})^2 \stackrel{def}{=} \sum_{p_1,\dots,p_d=0}^{N-K} (\Delta X_{(p_1,\dots,p_d)})^2.$$

The estimator of the fractional index $H$ is

$$\widehat{H}_N = \frac{1}{2} + \frac{1}{2}\log_2 \frac{V_{N/2}}{V_N}.$$

**Theorem 5.1.8** *Let $X_H$ be a fractional Brownian motion. The estimator $\widehat{H}_N$ satisfies*

- *Strong consistency*

$$\lim_{N\to+\infty} \widehat{H}_N \stackrel{(a.s.)}{=} H,$$

- *Asymptotic normality*
  *As $N \to +\infty$, $N^{\frac{d}{2}}(\widehat{H}_N - H)$ converges to a centered Gaussian variable.*

*Proof of Theorem 5.1.8*

The computation of the expectation of $V_N$ is straightforward and is left to the reader.

$$\operatorname{var} V_N = 2 \sum_{\mathbf{p},\mathbf{p}'=0}^{\mathbf{N}-\mathbf{K}} \left( \int_{\mathbb{R}} e^{-i\frac{\mathbf{p}-\mathbf{p}'}{N}\cdot\xi} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} e^{i\frac{\mathbf{k}}{N}\cdot\xi}\right|^2}{||\xi||^{d+2H}} d\xi \right)^2 ,$$

The change of variables $\lambda = \dfrac{\xi}{n}$ leads to

$$\int_{\mathbb{R}^d} e^{-i\frac{\mathbf{p}-\mathbf{p}'}{N}\cdot\xi} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} e^{-i\frac{\mathbf{k}}{N}\cdot\xi}\right|^2}{||\xi||^{d+2H}} d\xi = N^{-2H} \int_{\mathbb{R}^d} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\lambda} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\lambda}\right|^2}{||\lambda||^{d+2H}} d\lambda. \tag{5.15}$$

Let us define the operator $\mathbf{D} = \prod_{j=1}^{d} \dfrac{\partial}{\partial x_i}$. Let us suppose that $\forall j,\quad p_j \neq p'_j$, integrating by parts leads to:

$$\int_{\mathbb{R}^d} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\lambda} \frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\lambda}\right|^2}{||\lambda||^{d+2H}} d\lambda$$

$$= i^d \prod_{j=1}^{d} \frac{1}{(p_j - p'_j)} \int_{\mathbb{R}^d} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\lambda} \mathbf{D}\left[\frac{\left|\sum_{\mathbf{k}=\mathbf{0}}^{\mathbf{K}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\lambda}\right|^2}{||\lambda||^{d+2H}}\right] d\lambda. \tag{5.16}$$

The conditions 5.14 ensure the convergence of the integral. As $N \to +\infty$,

$$\left(\frac{1}{N} \sum_{m,m'=0,m\neq m'}^{N-K} \frac{1}{(m-m')^2}\right) \to C,$$

so the variance of the quadratic variations is of order $N^{-d}$. The remainder of the proof is then similar to the end of the proof in dimension one.

## 5.2 Efficiency

We can wonder if the estimates proposed previously are among the best. One classical way to address this issue is to use the so-called efficiency theory related to the Cramer Rao bound. Another way to classify estimators is to use the minimax theory, Sect. 5.2.2 is devoted to this question.

### *5.2.1 Cramer-Rao Bounds*

Let $x = (x_1, \ldots, x_N)$ be a random vector whose probability distribution depends on an unknown parameter $\theta \in \mathbb{R}^k$. Under suitable smoothness assumptions (e.g. [14], Chap. 6), we obtain that every estimators $\widehat{\theta}_N$, which are unbiased i.e. $\mathbb{E}\widehat{\theta}_N = \theta$, satisfy the following inequality

$$\mathbb{E}(\widehat{\theta}_N - \theta)^2 \geq I_N^{-1}(\theta),$$

where $I_N(\theta)$ is the Fisher information matrix defined by

$$I_N(\theta) = \left\{\mathbb{E}\left(\frac{\partial}{\partial\theta_i} \log L(x,\theta) \frac{\partial}{\partial\theta_j} \log L(x,\theta)\right)\right\}_{1\leq i,j\leq k},$$

where $L(x, \ldots)$ is the likelihood of the model.

Our statistical model is given by the data set $B_H\left(\frac{k}{N}\right)$, $k = 0, \ldots, N$, where $B_H$ is a fractional Brownian motion. Parameters $H$ and $C$ are defined as usual

$$\mathbb{E}(B_H(t) - B_H(s))^2 = C^2|t-s|^{2H}.$$

The Cramer-Rao bounds are given by the following Theorem ([36, 46]).

**Theorem 5.2.1**

1. *Standard case ($C \equiv 1$).*
   *As $N \to +\infty$*
$$I_N^{-1}(H) \sim \frac{1}{2} \frac{1}{N \log^2 N} \ .$$

2. *Non-standard case.*
   *As $N \to +\infty$*
$$I_N^{-1}(H, C^2) \asymp \begin{pmatrix} 1/N & \log(N)/N \\ \log N/N & \log^2 N/N \end{pmatrix}.$$

This means that the estimator (5.2) defined in Theorem 5.1.2 Sect. 5.1.2 is asymptotically efficient for the standard case and that the estimator (5.5) defined in Sect. 5.1.3 Theorem 5.1.3 is asymptotically efficient for the non-standard case.

### 5.2.2 *Minimax Rates*

Let us introduce the optimality of the estimator $\hat{H}_n$ in the setting of minimax theory. Let $\Theta$ be the subset of non-standard fBms with fractional index $0 < H < 1$. For a process $X \in \Theta$, denote by $H_X$ its fractional parameter. Denote by $\mathcal{H}_N$ the set of all possible estimators based on the observations $X\left(\frac{k}{N}\right)$, $k = 0, \ldots, N$. We then have the following minimax bounds [93].

**Theorem 5.2.2**

$$\liminf_{N \to \infty} \inf_{\widehat{H}_N \in \mathcal{H}_N} \sup_{X \in \Theta} N \, \mathbb{E}(\widehat{H}_N - H_X)^2 > 0.$$

This means that the estimator (5.5) defined in Theorem 5.1.3 Sect. 5.1.3 achieves the minimax rate of convergence. For generalizations to bigger subsets $\Theta$, where semi-parametric estimations of the fractional parameter $H$ are performed, we refer to [93].

## 5.3 Multifractional Case

Construction and properties of Multifractional Gaussian processes have been done in Sect. 4.3.2. Here we only recall the basic construction in the case of smooth fractional function and the case of a step wise fractional function. Please note that the estimation of the multifractional function of a generalized multifractional Gaussian process defined in Sect. 4.3.4 can be found in [9].

### 5.3.1 Smooth Multifractional Function

The idea is to come back to filtered white noises (cf. assumption 5.1.1). The parameter $H$ is replaced by a $C^1$ function $h(t)$:

$$g(t,\xi) = \frac{a(t)}{|\xi|^{\frac{1}{2}+h(t)}}. \tag{5.17}$$

Notice that a remainder term $\varepsilon(t,\xi)$ may be added to function $g(t,\xi)$ as for filtered white noises. It has been omitted here only for the sake of simplicity. Let $\widehat{W}$ be the Fourier transform of a real Brownian random measure. Define the process $X$ by

$$X(t) = \int_{\mathbb{R}} g(t,\xi)(e^{it\xi}-1)\widehat{W}(d\xi).$$

For any $t \in ]0,1[$, $\varepsilon > 0$ and $N > 0$, define the $(\varepsilon, N)$-neighborhood of $t$ by

$$\mathcal{V}_{\varepsilon,N}(t) = \left\{k \in \mathbb{Z},\ \left|\frac{k}{N} - t\right| \le \varepsilon\right\}.$$

The idea is to localize the generalized quadratic variation

$$V_{\varepsilon,N}(t) = \sum_{p\in\mathcal{V}_{\varepsilon,N}(t)} (\Delta_p X)^2,$$

where

$$\Delta_p X = \sum_{k=0}^{K} a_k X\left(\frac{k+p}{N}\right).$$

An estimator $\widehat{H}_{\varepsilon,N}(t)$ is then defined by:

$$\widehat{H}_{\varepsilon,N}(t) = \frac{1}{2} + \frac{1}{2}\log_2 \frac{V_{\varepsilon,N/2}(t)}{V_{\varepsilon,N}(t)}.$$

We then have the following Theorem [17].

**Theorem 5.3.1**

1. *Strong consistency.*
   *Take* $\varepsilon = N^{-\alpha}$ *with* $0 < \alpha < 1/2$.

$$\lim_{N\to+\infty} \widehat{H}_{\varepsilon,N}(t) \overset{(a.s.)}{=} H(t).$$

2. *Take* $\alpha = 1/3$.
$$\mathbb{E}(\widehat{H}_{\varepsilon,N}(t) - H(t))^2 = O(\log^2 N\ N^{-2/3}).$$

### 5.3.2 Step-Wise Fractional Function

When pluging a step-wise fractional function $h(t)$ in (5.17), we obtain an a.s. discontinuous process (cf. Sect. 4.3.3). The following construction has therefore been followed. Let us start with the series expansion (4.26) of a multifractional Brownian motion

$$B_h(t) = \frac{1}{C(h(t))^{1/2}} \sum_{\lambda\in\Lambda^+} \chi_\lambda(t, h(t))\eta_\lambda,$$

where

$$\chi_\lambda(x, y) = \int_{\mathbb{R}} \frac{e^{-ix\xi} - 1}{|\xi|^{y+1/2}} \overline{\widehat{\psi_\lambda}(\xi)} \frac{d\xi}{(2\pi)^{1/2}}$$

is defined in (4.25) and $\eta_\lambda$ are independent standard Gaussian random variables. In the case of fractional Brownian motion s, we have $h(t) \equiv H$. Let us now describe the case of a one change-point, the general case being similar. Let $h(t)$ be the function

$$h(t) = H_1 \mathbf{1}_{t\le\theta} + H_2 \mathbf{1}_{t>\theta}$$

for $\theta \in (0, 1)$. Let us recall the definition 4.66 of the step fractional Brownian motion

$$Q_h(t) = \sum_{\lambda\in\Lambda^+} \chi_\lambda(t, h(\lambda))\eta_\lambda.$$

Define the generalized quadratic variations of process $Q_{H(.)}(t)$ on $[s, t]$ at scale $1/N$:

$$V_N(s, t) = \sum_{s\le\frac{p}{N}\le t} (\Delta_p Q_{h(.)})^2,$$

where

$$\Delta_p Q_{h(.)} = \sum_{k=0}^{K} a_k Q_{h(.)}\left(\frac{k+p}{N}\right).$$

We will focus on the estimation of the change-point $\theta$, the estimation of $H_1$ and $H_2$ being similar to what has been done before. The idea is to use the asymptotic of these variations

$$\lim_{N\to+\infty} \frac{\log V_N(s,t)}{2\log N} \overset{(a.s.)}{=} \frac{1}{2} + \inf\{H(u), u \in ]s,t[\}.$$

For a given bandwidth $A$, define

$$D_N(A,t) = \frac{\log V_N(t,t+A)}{2\log N} - \frac{\log V_N(t-A,t)}{2\log N}.$$

We then obtain

- if $t + A < \theta$,

$$\lim_{N\to+\infty} D_N(A,t) \overset{(a.s.)}{=} 0,$$

- if $t - A > \theta$,

$$\lim_{N\to+\infty} D_N(A,t) \overset{(a.s.)}{=} 0,$$

- if $t < \theta < t + A$,

$$\lim_{N\to+\infty} D_N(A,t) \overset{(a.s.)}{=} \inf(H_1, H_2) - H_1,$$

- if $t - A < \theta < t$,

$$\lim_{N\to+\infty} D_N(A,t) \overset{(a.s.)}{=} H_2 - \inf(H_1, H_2).$$

Let $\eta$ be a given threshold. Define then

$$\widehat{\theta}_N = \inf\{t \in [0,1], |D_N(A,t)| \geq \eta\}\ .$$

Let us suppose we know a number $0 < \varepsilon_0 < \theta,\ 1-\theta,\ |H_1 - H_2|$. Then choose $\eta < \varepsilon_0$ and $A < \varepsilon_0$. We obtain [16]

$$\lim_{N\to+\infty} \widehat{\theta}_N \overset{(a.s.)}{=} \theta.$$

## 5.4 Extensions

### *5.4.1 Harmonizable Fractional Lévy Processes*

To sum up the results in this chapter, especially the results derived from Theorems 5.1.3 and 5.2.2, we can roughly say that the estimator (5.5) is optimal in a semi-parametric framework. Nevertheless, Theorems 5.1.3 and 5.2.2 require

Gaussianity. An inspection of the proofs does not make it clear whether Gaussianity is crucial or not. No complete answer to the need of Gaussianity is available at the moment. But partial answers exist. First, these results are clearly not available for processes with infinite variance. Therefore let us focus on second-order processes. We will see that the estimator (5.5) still works (i.e. consistency is true and $1/\sqrt{N}$ rate of convergence for mean square error) on a large class of second-order processes that are locally self-similar with a fractional Brownian motion as a tangent process.

Let us consider the real harmonizable fractional Lévy processes introduced in Sect. 4.4.2,

$$X(x) = \int_{\mathbb{R}} \frac{e^{ix\xi} - 1}{|\xi|^{\frac{1}{2}+H}} dM(\xi).$$

The following theorem ([18]) claims that estimator (5.5) based on generalized quadratic variations is still consistent with $1/\sqrt{N}$ rate of convergence. See also [91]

**Theorem 5.4.1** *Let* $\widehat{H}_N = \frac{1}{2} + \frac{1}{2}\log_2 \frac{V_{N/2}}{V_N}$ *then*

$$\lim_{N\to+\infty} |\widehat{H}_N - H| = O_{\mathbb{P}}(N^{-\frac{1}{2}}).$$

**Remark 5.4.1** *We may wonder whether the estimator (5.5) is consistent for every second-order process. The question of knowing the largest class of processes on which the estimator (5.5) is available is still open.*

## 5.4.2 Intermittency

This book is almost entirely devoted to self-similar, or locally self-similar, processes. (Local) self-similarity is indeed a powerful tool for modeling complex media. Nevertheless, (local) self-similarity does not exhaust the "real world". For a while (e.g. [60]), physicists of turbulence have been working with the kurtosis of local quantity. Such kurtosis has been called intermittency and is defined, in the physicists' language, by the dimensionless ratio

$$Intermittency = \frac{< F^4 >}{< F^2 >^2}, \tag{5.18}$$

the bracket $<, >$ usually is an expectation in time, domain or frequency.

Although the kurtosis of a Gaussian variable has not any interest, we will see that Gaussian processes may exhibit an intermittency index, that can be estimated via a dimensionless ratio like (5.18).

The basic idea is to start from the series expansion (5.6)

$$X(t) = \sum_{j>0} \sum_{k=0}^{2^j} 2^{-jH_1} \varepsilon_{j,k} \psi_{j,k}(t) \ ,$$

and to retain a number of coefficients that exponentially grows to infinity, but such that the number of retained number of coefficients divided by the total number of coefficients exponentially decreases to zero.

Consider $\Lambda_1 = \{j \geq 0, k = 0, \ldots, 2^j - 1\}$ as a dyadic tree: (0, 0) is the common ancestor, the two couples $(2k+1, j+1)$ and $(2k, j+1)$ are the sons of the father $(k, j)$. Let $T$ be a subtree of $\Lambda_1$ and let $T_n$ be the subtree $\{0 \leq j \leq n, k = 0, \ldots, 2^j - 1\}$ of $T$. Assume that the so-called growing number ([98]) exists

$$\delta = \limsup_{n \to +\infty} (\# T_n)^{\frac{1}{n}}.$$

Let us now define the process $Y_T$, where the coefficients of $X$ are retained only along the tree $T$

$$Y_T(x) = \sum_{(j,k) \in T} 2^{-jH_1} \varepsilon_{j,k} \psi_{j,k}(t).$$

Define the $\beta$-variations by

$$V_{N,\beta} = \sum_{p=0}^{N-K} |\varDelta_p Y_T|^{\beta},$$

where

$$\varDelta_p Y_T = \sum_{k=0}^{K} a_k Y_T \left( \frac{k+p}{N} \right).$$

Asymptotically the average number of the coefficients retained at scale $j$ is $\delta^{(2^j)}$. Therefore we may expect the $\beta$-variations at scale the $j$ to be of order $\delta^{(2^j)} \times 2^{-j\beta H}$. The following estimators are then proposed

$$\widehat{H}_N = \frac{1}{2 \log_2 N} \log_2 \frac{V_{N,2}}{V_{N,4}},$$

$$\log(\widehat{\delta}_N) = \frac{1}{N} \log \frac{V_{N,2}^2}{V_{N,4}}.$$

The ratio $\frac{V_{N,2}^2}{V_{N,4}}$ is reminiscent of the intermittency defined in (5.18). So we consider $\widehat{\delta}_N^N$ as an estimator of the intermittency index.

Under suitable assumptions on the functions $\psi_{j,k}$ [20] we have the theorem.

**Theorem 5.4.2**
*Strong consistency.*

$$\lim_{N\to+\infty} \widehat{H}_N \overset{(a.s.)}{=} H,$$

$$\lim_{N\to+\infty} \widehat{\delta}_N \overset{(a.s.)}{=} \delta.$$

## 5.5 Related Topics

In this section we give some applications of the statistical results given in this chapter.

### *5.5.1 A Short Review on Optimal Recovery*

Let $H$ be a Hilbert space, continuously embedded in $C([0, 1])$. The aim is to perform the optimal recovery of $I(f) = \int_0^1 f(t)dt$ from the observations $f\left(\frac{k}{N}\right)$, $k = 0, \ldots, N$ .

Let $B$ be the unit ball of $H$. Let $I_N(f)$ be any estimator (i.e. any measurable function of the observations $f\left(\frac{k}{N}\right)$, $k = 0, \ldots, N$) of $I(f)$. Let us denote the maximal error of $I_N$ by $\sup_{f\in B} |I_N(f) - I(f)|$ . The minimax risk of estimation is given by

$$R_N = \inf_{I_N} \sup_{f\in B} |I_N(f) - I(f)| .$$

Let

$$V_{0,N} = \left\{ f \in H,\ f\left(\frac{k}{N}\right) = 0,\ k = 0, \ldots, N \right\}.$$

We then have the following results ([106, 107]).

1. Characterization of the minimax risk.

$$R_N = \sup_{f\in V_{0,N}\cap B} |I(f)|. \tag{5.19}$$

2. Optimal estimator. Let $A$ be the orthogonal projection of $H$ onto the orthogonal supplementary of $V_{0,N}$. We can check that $A(f)$ is a linear combination of the the observations $f\left(\frac{k}{N}\right)$, $k = 0, \ldots, N$. It follows that the optimal estimator $I_{N,opt}$ that reaches the minimax risk is linear and given by

$$I_{N,opt}(f) = \int_0^1 A(f)(t)dt.$$

Notice that the characterization (5.19) is still available, up to a factor 2, in Banach spaces ([106, 107]).

### 5.5.2 Approximation of Integral of Fractional Processes

Let $X$ be a centered Gaussian process on [0, 1] be such that

$$I(X) = \int_0^1 X(x)dx$$

exists. Now consider the problem of statistical optimal recovery (see [118] for general definitions and results). It means that we are looking for $I_N(X)$ an estimator of $I(X)$ based on the observations $X\left(\frac{k}{N}\right)$, $k = 0, \ldots, N$ that achieves the infimum of $\mathbb{E}(I_N(X) - I(X))^2$. It is a well-known fact (e.g. [44, Chap. 4]) hat the optimal estimator of $I(X)$ is linear. Denote by $H_X$ the Reproducing Kernel Hilbert Space of $X$ (see Sect. 2.1.4) and by $B_X$ its unit ball. We have ([111, 140]):

$$\mathbb{E}(I_N(X) - I(X))^2 = \sup\left\{\left(I_N(h) - \int_0^1 h\right)^2,\ h \in B_X\right\}. \tag{5.20}$$

The optimal estimator that minimizes the right-hand side of (5.20) is therefore the optimal estimator in the Hilbert space $H_X$. From (5.19), we can deduce

$$\inf_{I_N} \mathbb{E}(I_N(X) - I(X))^2 = \sup\left\{\left(\int_0^1 h(x)dx\right)^2,\ h \in B_X,\ h\left(\frac{k}{N}\right) = 0,\ k = 0, \ldots, N\right\}. \tag{5.21}$$

Let us now consider Gaussian processes with $C^K$ sample paths where $K \in \mathbb{N}^*$. For instance take a centered Gaussian process $X$ satisfying the Sacks & Ylvisaker's conditions of order $K$.

$$\begin{cases} \alpha_K^+(t) = \frac{\partial^{2K+1}R}{\partial t^K \partial^{K+1}t}(t, t+), \\ \alpha_K^-(t) = \frac{\partial^{2K+1}R}{\partial t^K \partial^{K+1}t'}(t, t-), \\ \alpha_K(t) = \alpha_K^-(t) - \alpha_K^+(t). \end{cases} \tag{5.22}$$

The (known) fractional parameter is $H = K + 1/2$.

Let $I_N$ be the Euler-Maclaurin formula (e.g. [86]) of order $K$

$$I_N(X) = \frac{1}{N}\sum_{k=0}^{N} a_k X\left(\frac{k}{N}\right),$$

where the $a_k$ for $k = 0$ to $N-1$ are defined by

$$a_k = \begin{cases} \frac{1}{2} - \sum_{j=1}^{K} W_j \quad k = 0, \dots K, \\ 1 + (-1)^{k+1}\sum_{j=k}^{K} \binom{j}{k} W_j \quad 1 \le k \le K, \\ 1 \quad K+1 \le k \le N-K-1, \end{cases}$$

and

$$a_k = a_{N-k} \quad N-K \le k \le N-1,$$

where

$$W_j = \frac{(-1)^j}{(j+1)!}\int_0^1 t(t-1)\dots(t-j)dt \ .$$

The estimator $I_N(X)$ is then asymptotically optimal ([21,122–124]) and satisfies

$$\lim_{N\to+\infty} N^{2K+2}\mathbb{E}(I_N(X) - I(X))^2 = \frac{|B_{2K+2}|}{(2K+2)!}\int_0^1 \alpha_K(t)dt.$$

The rate of convergence is of order $1/N^{2K+2}$ and not $1/N^{2K+1}$.

This result can be extended to processes with an arbitrary fractional index $H$. In this paragraph it will mean a stationary centered Gaussian process $X$ with autocovariance function $r(t) = \mathbb{E}X_t X_0$ such that:

$$r(t) = \sum_{i=0}^{2[H]} \frac{|t|^i}{i!} r^{(i)}(0) - C|t|^{2H} + o(|t|^{2H}).$$

The Euler-MacLaurin estimator $I_N(X)$ of order $[H]$ is then asymptotically optimal ([23, 130]) and satisfies

$$\lim_{N\to+\infty} N^{2H+1}\mathbb{E}(I_N(X) - I(X))^2 = 2C|\zeta(-2H)|,$$

where $\zeta$ is the Riemann zeta function. Notice that, when $H$ is $K + 1/2$, with $K \in \mathbb{N}$ $\zeta(-2K-1) = -\dfrac{B_{2K+2}}{(2K+2)!}$.

We may wonder what happens for multifractional processes. Let $0 < h(t) < 1$ be a $C^3$ function, having a unique minimum in $m \in ]0,1[$, with $h''(m) \neq 0$. Let $\widehat{W}$ be a Fourier transform of a real valued Brownian random measure and define process $X$ by

$$X(x) = \int_{\mathbb{R}} \frac{e^{ix\xi} - 1}{|\xi|^{\frac{1}{2}+h(x)}} \widehat{W}(d\xi).$$

which is up to a multiplicative constant a multifractional Brownian motion. Denote by $I_N$ the trapezoidal predictor. We then have ([71])

$$\lim_{N \to +\infty} \log N N^{2H(m)+1} \mathbb{E}(I_N(X) - I(X))^2 = 4\sqrt{\frac{\pi}{H''(m)}} |\zeta(-2H(m))| \times \int_{\mathbb{R}} \frac{\sin^2(u/2)}{|u|^{1+2H(m)}} du.$$

## 5.6 Exercises

### 5.6.1 Ornstein-Ühlenbeck Process

1. Show that there exists an unique centered Gaussian stationary process of covariance function $r(t) = \mathbb{E}X_t X_0 = \exp(-|t|)$.(See exercise 3.6.4.)
2. Can this process be viewed as a filtered white noise ?

### 5.6.2 Estimation for Smooth Lass Processes

In this exercise we consider again the process defined in exercise 3.6.11 and 4.5.1, for $0 < H < 1$,

$$X_t = \int_{\mathbb{R}} \frac{\varphi(ts)}{|s|^{H+1/2}} W(ds).$$

We assume $\varphi(0) = 0$, and

$$\int_{\mathbb{R}} \frac{|\varphi(s)|^2}{|s|^{2H+1}} ds < \infty.$$

1. Recall why $X$ is a centered Gaussian self-similar process with index $H$.
2. Consider the quadratic variation:

$$V_n = \sum_{k=0}^{n-1} \left( X\left(\frac{k+1}{n}\right) - X\left(\frac{k}{n}\right)\right)^2.$$

Let

$$\widehat{H_n} = \frac{1}{2} + \frac{1}{2}\log_2 \frac{V_{n/2}}{V_n}.$$

Is $\widehat{H_n}$ a consistent estimator of $H$?

### 5.6.3 A Strange Estimator

Let $X$ be a centered Gaussian self-similar process of index $H > 0$. Define

$$\widehat{H_n} = -\frac{\log_2 |X(2^{-n})|}{n}.$$

Show that

$$\lim_{n\to+\infty} \widehat{H_n} \overset{(a.s.)}{=} H.$$

### 5.6.4 Complex Variations

In this exercise $i$ is the complex square root of $-1$.

1. Let $(U, V) \sim \mathcal{N}_2\left((0,0); \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$. Show that there exists $C > 0$, such that, $\forall \rho \in (0,1)$
$$|cov(|U|^i, |V|^i)| \leq C\rho^2.$$

2. Let $B_H$ be a fractional Brownian motion and set
$$\Delta_{p,n} B_H = B_H(p2^{-n}) - 2B_H((p+1)2^{-n}) + B_H((p+2)2^{-n}).$$
Set
$$W_n(B_H) = \frac{2^{iHn}}{2^n - 1} \sum_{p=0}^{2^n-2} |\Delta_{p,n} B_H|^i.$$
Show that
$$\lim_{n\to+\infty} W_n(B_H) = \mathbb{E}|\Delta_{0,0} B_H|^i \quad (a.s.),$$
$$\lim_{n\to+\infty} 2^n \mathrm{var}(W_n(B_H)) = \sum_{k\in\mathbb{Z}} cov(|\Delta_{k,0} B_H|^i, |\Delta_{0,0} B_H|^i).$$

### 5.6.5 Optimal Estimation of Integral of Brownian Motion

Let $B(t)$, $t \in [0, 1]$ be a Brownian motion. Let $K_B$ be its RKHS (cf. Sect. 2.1.4) and let $UB$ be its unit ball. The aim is to estimate

$$I(B) = \int_0^1 B(t)dt$$

from the observations $B(k/n),\ k = 1, \dots, n$.

1. Show that

$$\sup\left\{\left(\int_0^1 h(t)dt\right)^2,\ h(0) = h(1) = 0,\ h \in UB\right\} \geq \frac{1}{12},$$
$$\sup\left\{\left(\int_0^1 h(t)dt\right)^2,\ h(k/n) = 0, k = 0, \dots n\ , h \in UB\right\} \geq \frac{1}{12n^2}.$$

   Hint: Use $h(t) = \sum_{k\geq 0} \dfrac{\sin((2k+1)\pi t)}{(2k+1)^3}$.

2. Let us consider the trapezoidal estimate

$$\widehat{I(B)} = \frac{1}{n}\left(\frac{B(1)}{2} + \sum_{k=1}^{n-1} B(k/n)\right).$$

   Show that the trapezoidal estimate is optimal with respect to the mean square error.

### 5.6.6 Estimation of the Singularity Function

Let $X$ be a process satisfying conditions of Theorem 5.1.3. The aim is to estimate the function $a^2(t)$. Let $e_p$, $p \geq 0$ be an orthonormal basis of $L^2([0, 1])$. Let $\mathcal{F}_s$ be a subspace of $L^2([0, 1])$ equipped with a norm $\|.\|_{\mathcal{F}_s}$. Let $B$ be its unit ball and assume that, $\exists C > 0$, $\forall n \geq 1$

$$\sup_{f\in B} \|f - \sum_0^n I(e_p)e_p\|_{L^2} \leq Cn^{-s}.$$

Define the following estimate of $a^2(t)$

$$\widehat{a^2_{P,N}} = \sum_{p=0}^{P} \widehat{I_{N,P}}(e_p)e_p.$$

Assume that $a^2 \in \mathcal{F}_s$. Show that, with a suitable choice of $P$, there exists $C(a^2) > 0$ such that

$$\mathbb{E}\|a^2 - \widehat{a^2_{P,N}}\|^2_{L^2} \leq C(a^2)N^{-2s/(2s+1)}.$$

# Chapter 6
# Simulations

## 6.1 Introduction

The aim of this chapter is to give simulations of some of fractional fields introduced in the previous chapters. Our main concern here is algorithmic and heuristic : We would like to provide tools to probabilists and statisticians to show samples of fractional fields to practitioners so that they can decide if the proposed fractional models are useful for their own applications. Let us mention here among a large literature two examples of fractional types processes to examples in medicine [84, 132].

Because of the finite nature of the computations performed with computers it is very hard to have exact simulations for fractional fields. Mathematically most of our simulations provide fields $X_n$ that converge to fractional fields when $n \to \infty$ but in most cases when $n$ is finite, even very big, $X_n$ shares very little properties with the limit. Let us give an example Fractional Brownian motion $B_H$ is often simulated only to produce generic sample paths with a given Hölderian roughness $H$. In most simulations of Fractional Brownian motion, the approximation $X_n$ is only sampled on a finite grid and linear interpolations are extrapolated so that the sample paths look like rough functions. Actually in this case for each $n$ the sample of $X_n$ is Lipschitz instead of $H$-Hölderian. Does it mean that simulations are useless in the domain of fractional fields? We don't think so but we don't know mathematical statements that encapsulate what we call the visual meaning of fractional indexes.

Beside these issues we are interested in Sect. 6.3 in the simulation of Gaussian fractional fields. Before we will recall the simulation of Gaussian fractional process in dimension $d = 1$ which is briefly recalled in Sect. 6.2. This section is essentially based on the survey papers [10, 37], and we will only outline simulation techniques that are consequences of the theoretical depiction of our models given in the previous chapters. In particular we will not address the issues of the rate of convergence, since we think that classical norms ($L^2$ for instance) are irrelevant for our purpose. Especially we recall some basic facts for the simulation of fractional Brownian motion but it is clearly not our main point of interest. We will pay more attentions to more exotic models like multifractional Brownian motion or Step fractional Brownian motion

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because the literature is less rich for these models. In the Sect. 6.3 we consider the simulation of Gaussian fractional fields $d > 1$, which is more computationally intensive.

The simulation of Gaussian fractional fields can be addressed roughly in two different ways. If we know analytically the covariance matrix of a finite sample of a fractional Gaussian field then a classical method is the Cholevski decomposition of the covariance $R$ which is a symmetric matrix in a product of a lower and an upper triangular matrix. In this perspective we don't care so much on the fractional properties of the Gaussian random vector. In some cases, especially when $d > 1$ it is the only method we know and we even have to make approximations to run simulations in a reasonable time. But this crude technique has also drawbacks : If you want to zoom a given sample path in a region of interest or if you just want to add points to your finite samples then you have to start the Cholevski decomposition again from the beginning. Actually there exist other techniques of simulations which are recursive and where you can fill-in additional points to your sample if you wish. The most classical one is the Lévy simulation of Brownian motion. It has been generalized to other fractional processes and we will briefly explain the relationship with series expansions we obtained previously in this book for fractional processes.

Examples are given in Sect. 6.4. Then the Sect. 6.5 is devoted to the simulation of non-Gaussian fractional fields. More precisely we will take real harmonizable fractional Lévy fields as an example. At last, in Sect. 6.6 we will wonder on the visual meaning of the fractional index from a quite experimental point of view.

## 6.2 Fractional Gaussian Processes

### *6.2.1 Cholevski Method*

Let $X$ be a centered Gaussian process with covariance function $R(t, t')$ (see Definition 2.1.8). The vector $X\left(\frac{p}{N}\right)$ is a centered Gaussian vector with covariance matrix $\Sigma$:

$$\Sigma_{p,p'} = R\left(\frac{p}{N}, \frac{p'}{N}\right) \quad p, p' = 0, \ldots N.$$

The matrix $\Sigma$ can be decomposed into $\Sigma = L^t L$, where $L$ is a lower triangular matrix. Let $Z_p$, $p = 0, \ldots, N$ be standard centered Gaussian random variables and put $Z =^t (Z_0, \ldots, Z_N)$. We can then easily check that $LZ$ is a centered Gaussian vector with covariance matrix $\Sigma$. This method therefore allows to give an error-free simulation of the $X\left(\frac{p}{N}\right)$, $p = 0, \ldots, N$. This method has a complexity of order $O(N^3)$. When $N$ is rather small, the Cholevski method can be used. Unfortunately, when $N$ becomes large, this method is no more usable. Especially, this is the case when one wants to simulate a Gaussian field.

### 6.2.2 Random Midpoint Displacement

The Brownian motion $B$ has independent, stationary and Gaussian increments. It is therefore straightforward to simulate a Brownian motion over a given discrete uniform grid.

We will indicate here how to simulate a sample path of a Brownian motion at point $\frac{k}{2^j}$ by induction. We set $B(0) = 0$. $B(1)$ is chosen from a centered standard Gaussian variable. Next, $B(1/2)$ is selected from an independent Gaussian variable with mean $(B(0) + B(1))/2$ and variance $1/2$. At the jth stage, the value $B(k/2^j)$ (k odd) is chosen from a Gaussian variable with mean $(B((k-1)/2^j) + B(k/2^j))/2$ and variance $2^{-j}$. This inductive construction of the Brownian motion was originally carried by Lévy [96]. Please note that in this case you can fill-in additional middle points to your sample if you wish. Moreover the simulation of the Brownian motion obtained by taking linear interpolation between the point $k/2^j$ can be viewed as series expansion of the Brownian motion on the so-called Schauder basis. Let us recall a construction described in [79] e.g. We define the Haar function inductively by $H^{(0)}(t) = 1$, $0 \le t \le 1$, and for $j > 1$, $k \in I(j)$ where $I(j)$ is the set of odd numbers between 0 and $2^j$,

$$H_k^{(j)}(t) = \begin{cases} 2^{(j-1)/2}, & \frac{k-1}{2^j} \le t < \frac{k}{2^j}, \\ -2^{(j-1)/2}, & \frac{k}{2^j} \le t < \frac{k+1}{2^j} \\ 0, & \text{otherwise.} \end{cases}$$

Then the Schauder functions $S_k^{(j)}(t) = \int_0^t H_k^{(j)}(u)du$, $0 \le t \le 1$, $k \in I(j)$ and the simulation after $n$th stage is

$$B_t^{(n)}(\omega) = \sum_{j=0}^{n} \sum_{k \in I(j)} \varepsilon_{j,k}(\omega) S_k^{(j)}(t) \tag{6.1}$$

where $\varepsilon_{j,k}$ are i.i.d. centered normalized Gaussian variables.

The random midpoint displacement cannot be trivially generalized to fractional Brownian motion $B_H$: If we take $B_H(k/2^j)$ from a Gaussian variable with mean $(B_H((k-1)/2^j) + B_H(k/2^j))/2$ and variance $2^{-jH}$, the resulting function fails to have stationary increments.

### 6.2.3 *Discretization of Integral Representation in Dimension* 1

Although the crude random midpoint displacement technique fails for Fractional Brownian motions, one can still try to truncate some of the series extension we got in Sect. 3.2.2.4. The idea is to start from a representation by an integral with respect to the Fourier transform of a real Brownian measure

$$B_H(t) = \int_{\mathbb{R}} k(t,\xi)\widehat{W}(d\xi),$$

where $k(t,\xi)$ is for instance the harmonizable kernel

$$k(t,\xi) = \frac{e^{it\xi} - 1}{|\xi|^{H+\frac{1}{2}}}.$$

Then we consider the Riemann-Itô sum associated with the stochastic integral. The simulated process is

$$\widetilde{B_H}(t) = \sum_{|p|\le P} k\left(t, \frac{p}{N}\right)\varepsilon_{p,N}, \tag{6.2}$$

where the $\varepsilon_{p,N} = \widehat{W}\left(\frac{p+1}{N}\right) - \widehat{W}\left(\frac{p}{N}\right)$ are i.i.d. Gaussian centered random variables.

Unfortunately, the kernel $k(t,\xi)$ slowly decreases at infinity and therefore the truncation $|p| \le P$ introduces important errors. Other errors are given by the approximation due to the discretization itself. Therefore these methods are not very effective.

### *6.2.4 Approximate Wavelet Expansion*

Although the previous technique is not very efficient it has been sophisticated to avoid some of the mentioned drawbacks. Please note that in (6.2) the increment $\widehat{W}\left(\frac{p+1}{N}\right) - \widehat{W}\left(\frac{p}{N}\right)$ can be related to an integral of a Haar function with respect to $\widehat{W}$. Hence (6.2) can also be considered as a series expansion truncated for simulation. Several other methods of simulations can be derived from the expansion of the Brownian measure on an orthonormal basis. Expansion in a wavelet basis are of particular interest because the order of magnitude of the wavelet coefficients is known: Expansions in a wavelet basis have been considered for a while (cf. [3, 22]).

It has been shown (cf. Sect. 3.2.2.4) that Fractional Brownian motion can be expanded on a basis

$$B_H(t) = \sum_{\lambda\in\Lambda^+} \varphi_\lambda(t)\eta_\lambda.$$

The computation of the function $\varphi_\lambda$ is quite long and the idea is to replace the functions $\varphi_\lambda$ by an orthonormal basis of $L^2$ $\psi_\lambda$. Unfortunately there is no upper bound for the error. Good candidates for $\psi_\lambda$ are for instance compactly-supported wavelets [47, 48]. The resulting approximate process is then as in (5.6)

$$\widetilde{B_H}(t) = \sum_{j=0}^{J}\sum_{k=0}^{2^j} 2^{-jH}\varepsilon_{j,k}\psi_{j,k}(t) \tag{6.3}$$

where $J$ depends on the required precision. Notice that, if the chosen wavelet is compactly-supported, the sum over index $k$ is indeed finite.

In this paragraph we would like to remark that the randomized Weierstrass function considered in (3.56)

$$\mathcal{W}_r^{al}(t) = \sum_{j=-\infty}^{+\infty} \left(e^{ir^j t} - 1\right) r^{-Hj}(\xi_j + i\eta_j),$$

where the $\xi_j,\ \eta_j$, are i.i.d. standard centered Gaussian random variables and, where $r > 1$, is formally similar to (6.3). For the sake of simplicity let us take the real part of $\mathcal{W}_r^{al}(t)$

$$\begin{aligned}\Re\mathcal{W}_r^{al}(t) &= \sum_{j=-\infty}^{+\infty} (\cos(r^j t) - 1)r^{-Hj}\xi_j - \sin(r^j t)r^{-Hj}\eta_j \\ &= \sum_{j=-\infty}^{+\infty} (\cos(r^j t) - 1)r^{-Hj}\xi_j - \sum_{j=-\infty}^{+\infty} \sin(r^j t)r^{-Hj}\eta_j.\end{aligned}$$

If we denote by

$$\psi_1(x) = (\cos x - 1)\mathbf{1}_{[0,2\pi]}(x).$$

and by

$$\psi_2(x) = \sin(x)\mathbf{1}_{[0,2\pi]}(x),$$

The process $\Re\mathcal{W}_r^{al}(t)$ may be rewritten as

$$W(t) = \sum_{k,j\in\mathbb{Z}} r^{-jH}\psi_1(r^j t - 2k\pi)\xi_j - \sum_{k,j\in\mathbb{Z}} r^{-jH}\psi_2(x)\eta_j.$$

## 6.2.5 *Multifractional Gaussian Processes*

### 6.2.5.1 Multifractional Brownian Motion

Let us now consider the simulation of multifractional Brownian motions. Starting from the integral representation (4.18)

$$B_h(t) = \int_{\mathbb{R}} k(t,\xi)\widehat{W}(d\xi),$$

where

$$k(t,\xi) = \frac{1}{(C(h(t)))^{1/2}} \frac{e^{it\xi} - 1}{|\xi|^{h(t)+\frac{1}{2}}}.$$

Then we have the series expansion (4.26)

$$B_h(t) = \frac{1}{C(h(t))^{1/2}} \sum_{\lambda \in \Lambda^+} \chi_\lambda(t, h(t))\eta_\lambda,$$

where $\chi$ has been defined in the Definition 4.3.3. The truncation of the series in (4.26) leads to a first approximation

$$\widetilde{\widetilde{B_h}}(t) = \sum_{j=0}^{J} \sum_{k \in \mathbb{Z}} \chi_{j,k}(t, h(t))\eta_{j,k},$$

where $J$ depends on the required precision.

Since the computation of $\chi_{j,k}(t, h(t))$ is still time consuming we are actually using the same additional approximation than in the case of the Fractional Brownian motion

$$\widetilde{B_h}(t) = \sum_{j=0}^{J} \sum_{k=0}^{2^j} 2^{-jh(t)} \varepsilon_{j,k} \psi_{j,k}(t)$$

where $\psi_{j,k}$ is an orthonormal basis of $L^2$ that comes from a compactly supported wavelet.

#### 6.2.5.2 Step-Wise Fractional Function

We can extend the previous technique to the case of the step-wise fractional function. Let us recall the Definition (4.66) of the Step fractional Brownian motion

$$Q_h(t) = \sum_{\lambda \in \Lambda^+} \chi_\lambda(t, h(\lambda))\eta_\lambda.$$

where $\chi$ has been defined in Definition 4.3.3. We give the case of one change-point, the general case being similar. Let $h(t)$ be the function:

$$h(t) = H_1 \mathbf{1}_{t \le \theta} + H_2 \mathbf{1}_{t > \theta}.$$

The first approximation is then

$$\widetilde{\widetilde{Q_h(t)}} = \sum_{0 \le j \le J, |k| \le K} \chi_\lambda(t, h(\frac{k}{2^j}))\eta_{j,k}$$

where $\lambda$ has been encoded as $j,\ k$ here. For the sake of simplicity, functions $\chi_\lambda$ can be replaced by $\psi_{j,k}$ an orthonormal basis of $L^2$ that comes from a compactly supported wavelet. We then obtain a second approximation of $Q_h$

$$\widetilde{Q_h}(t) = \sum_{0\le j\le J, |k|\le K} 2^{-jh\left(\frac{k}{2^j}\right)} \psi_{j,k}(t)\varepsilon_{j,k}. \tag{6.4}$$

## 6.3 Fractional Gaussian Fields

### *6.3.1 Discretization of Integral Representation in Dimension d*

The discretization of the integral representation performed in Sect. 6.2.3 in dimension 1 can still be done in higher dimension $d > 1$

$$B_H(t) = \int_{\mathbb{R}^d} k(t,\xi)\widehat{W}(d\xi),$$
$$\xi, t \in \mathbb{R}^d,$$

where $k(t,\xi)$ is for instance the $d$-harmonizable kernel

$$k(t,\xi) = \frac{e^{it.\xi}-1}{||\xi||^{H+\frac{d}{2}}}.$$

The Riemann-Itô sum associated with the stochastic integral is then:

$$\widetilde{B_H}(t) = \sum_{|p|\le P} k\left(t, \frac{p}{N}\right)\varepsilon_{p,N},$$
$$p \in \mathbb{Z}^d, \quad t \in \mathbb{R}^d.$$

### *6.3.2 The Procedure* `Fieldsim` *for Random Fields*

Our approach of Gaussian fractional random fields is based on an exact simulation on a rough grid using a Cholevski method followed by a fast step that yields an approximated simulation on a finer grid. The second step is an improvement and a generalization of the random midpoint displacement used for Brownian motion in Sect. 6.2.2.

**Accurate simulation step.** We first present the accurate simulation part of the procedure. Given a (regular) space discretization $\{M^i, i \in I\}$ of size $n_I$, the problem consists in giving a sample of a centered Gaussian vector of size $n_I$: $(X(M^i))_{i\in I}$ of covariance matrix $\mathbf{R}$ given by $\mathbf{R}_{i,j} = R(M^i, M^j)$, $i, j \in I$. We use the Cholevski method.

**Refined simulation step.** We need to introduce some additional notations. Let $X_{\mathcal{X}_I}(M)$, denote the orthogonal projection of $X(M)$ on the closed linear subspace $\mathcal{X}_I = \overline{sp}\{X(M^i), i \in I\}$, i.e. the linear predictor of $X(M)$ given $X(M^i), i \in I$. The partial innovation $X(M) - X_{\mathcal{X}_I}(M)$ is denoted by $\varepsilon_{\mathcal{X}_I}(M)$. Since $\varepsilon_{\mathcal{X}_I}(M)$ is uncorrelated with any variables of the space $\mathcal{X}_I$, we can obtain "accurate" simulation of $X(M)$ by $X_{\mathcal{X}_I}(M) + \sqrt{Var(\varepsilon_{\mathcal{X}_I}(M))}U$ where $U$ is a centered and reduced Gaussian variable independent of $X(M^i), i \in I$. Notice that the coefficients weight the variables $X(M^i), i \in I$ in $X_{\mathcal{X}_I}(M)$ and the variance of the partial innovation may be determined from the second order structure of the sequence $X(M^i), i \in I, X(M)$ (see [49] for details). The drawback of this approach is when the simulated sequence size increases, we have to keep the record of more and more quantities (filters of several partial innovations and associated variances) and to do more and more computations. Even if that can be done in the $d = 1$ case, it becomes numerically unfeasible when $d \geq 2$. A natural approach to overcome this problem, is to replace in the previous procedure the indexes set $I$ by a set of indexes of neighbors of $M$. We denote by $N_M$ this set. Notice that $X_{\mathcal{X}_{N_M}}(M)$ is the best linear combination of variables of $\mathcal{X}_{N_M}$ approximating $X(M)$ in the sense that the variance of $X(M) - X_{\mathcal{X}_{N_M}}(M)$ is minimum. If we have to use only some variables of the set $\mathcal{X}_{N_M}$ in order to obtain simulation of $X(M)$, the best way is to use $X_{\mathcal{X}_{N_M}}(M) + \sqrt{Var(\varepsilon_{\mathcal{X}_{N_M}}(M))}U$. Let us remark that such a simulated process does not admit anymore $R(\cdot,\cdot)$ as a covariance function, but a covariance function that is a good approximation of $R(\cdot,\cdot)$.

## 6.4 Examples

We present some simulations of Gaussian fractional processes in the following figures. The Gaussian processes have been simulated by the Cholevski method Figs. (6.1, 6.2, 6.3, 6.4, 6.5, 6.6, 6.7, 6.8 and 6.9).

Step fractional Brownian motions below have been simulated with the second approximation of Sect. 6.2.5.2 (Figs. 6.10 and 6.11).

The fields below have been simulated with the `fieldsim` procedure, which is introduced in [32] For fields, two representations have been used: The fields can be represented by a random surface, or by a random colored texture. Especially, if the colors are chosen in a range from blue to white, the simulations may be use to simulate the texture of clouds (Figs. 6.12, 6.13, 6.14 and 6.15).

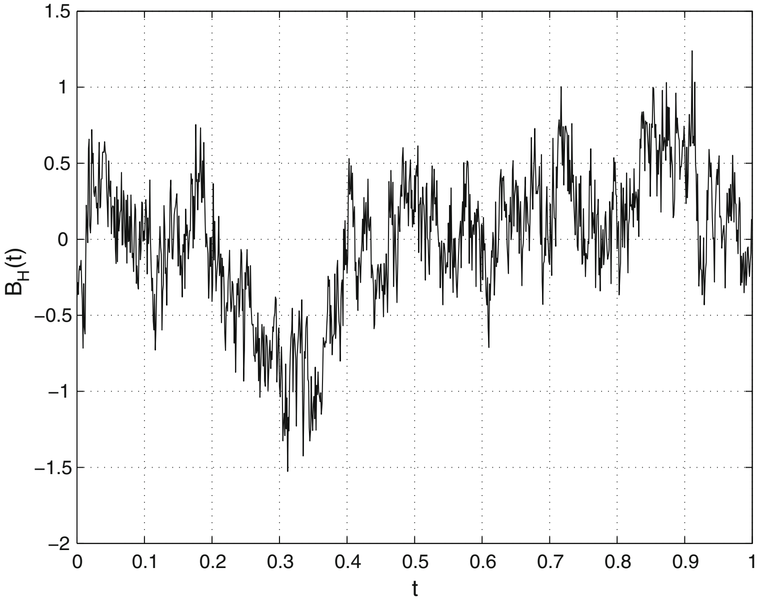


**Fig. 6.1** Fractional Brownian motion with index H = 0.2

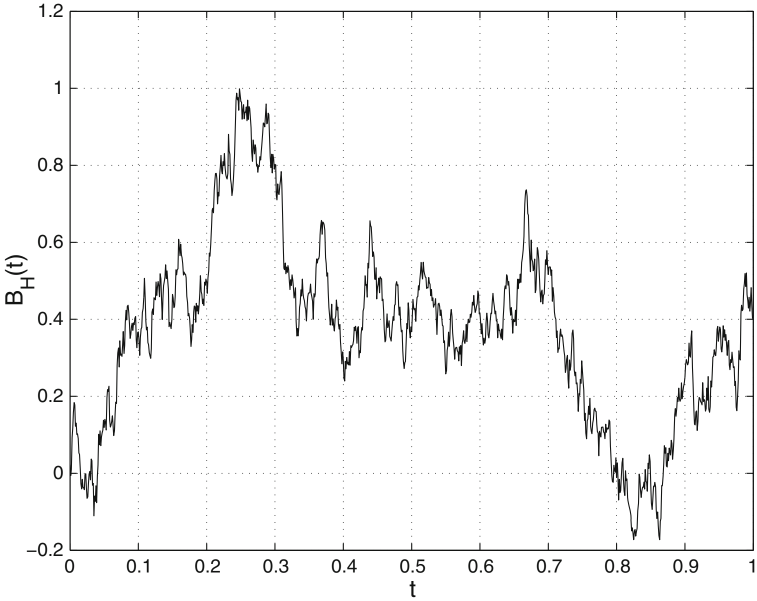


**Fig. 6.2** Brownian motion (H = 0.5)

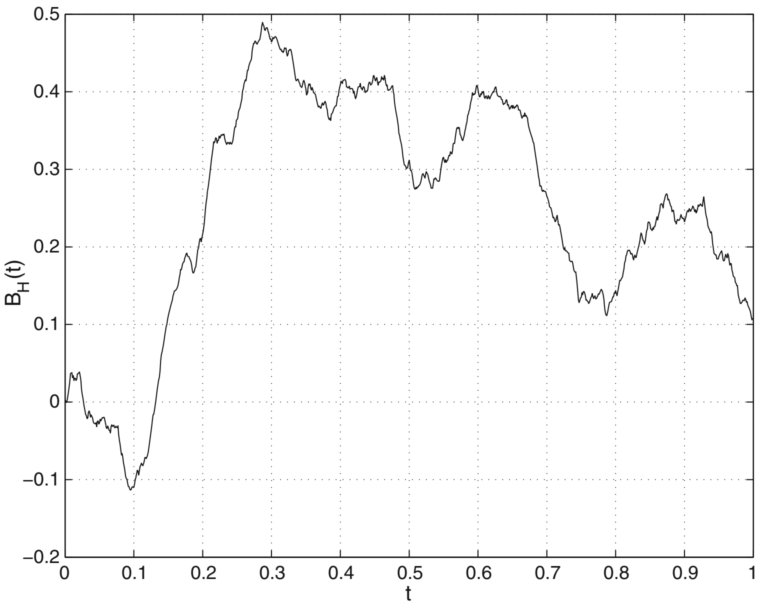


**Fig. 6.3** Fractional Brownian motion with index H = 0.8

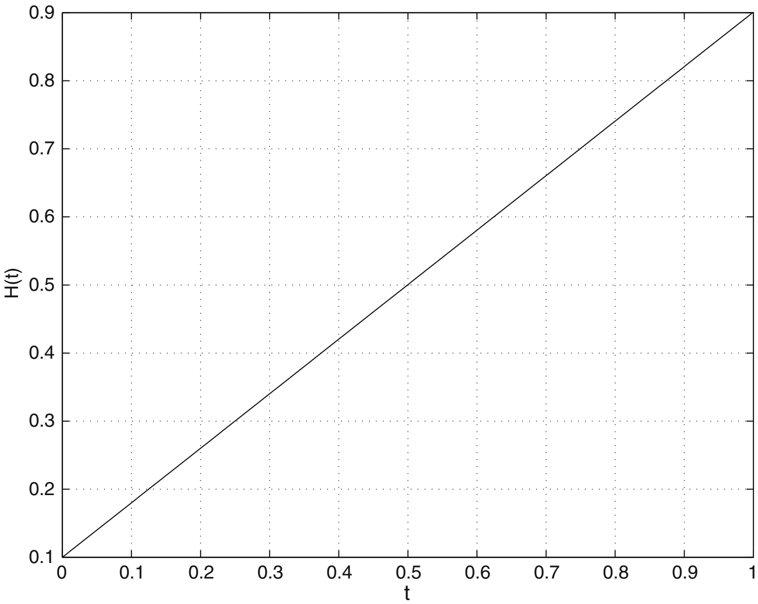


**Fig. 6.4** Linear fractional function h(t)

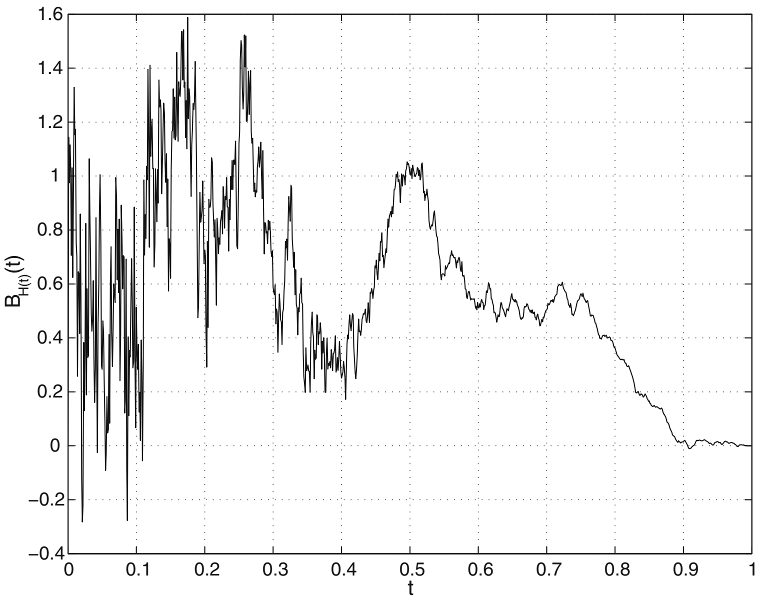


**Fig. 6.5** Multifractional Brownian motion with linear fractional function h(t) of Fig. 6.4

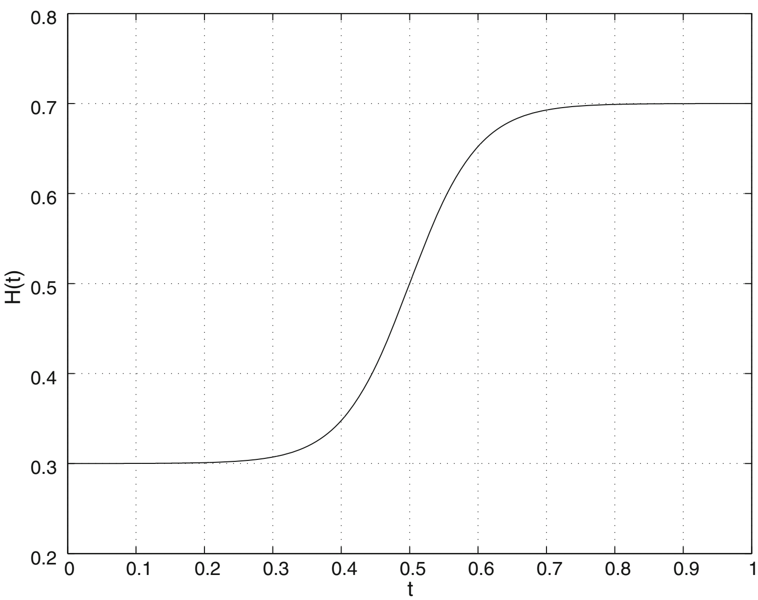


**Fig. 6.6** Logistic fractional function h(t)

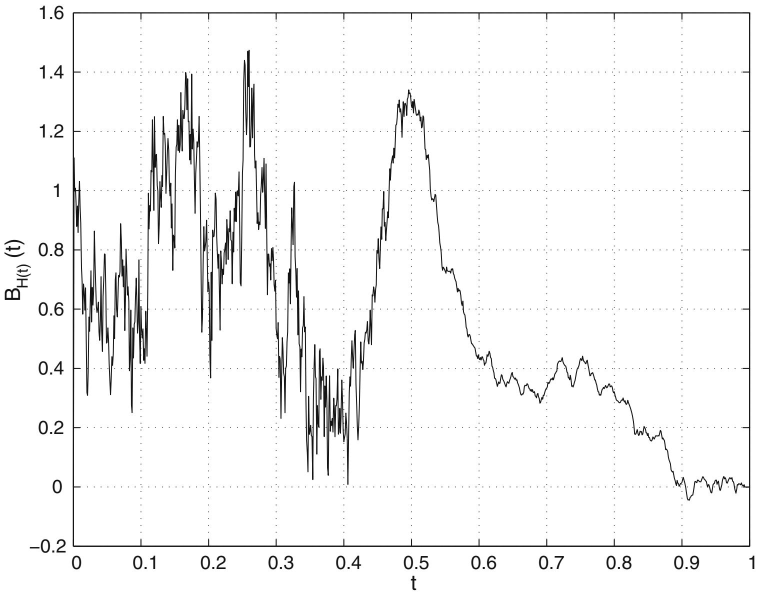


**Fig. 6.7** Multifractional Brownian motion with logistic fractional function h(t) of Fig. 6.6

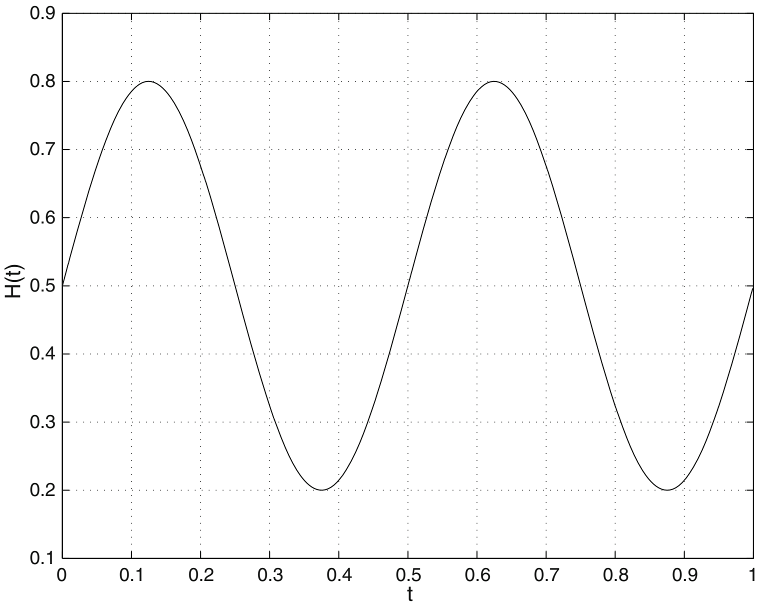


**Fig. 6.8** Sinusoidal fractional function h(t)

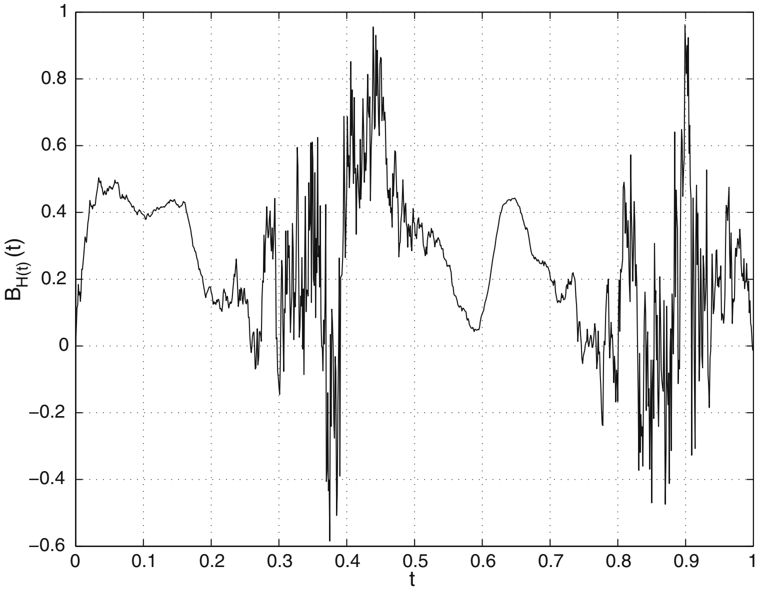


**Fig. 6.9** Multifractional Brownian motion with sinusoidal fractional function h(t) of Fig. 6.8

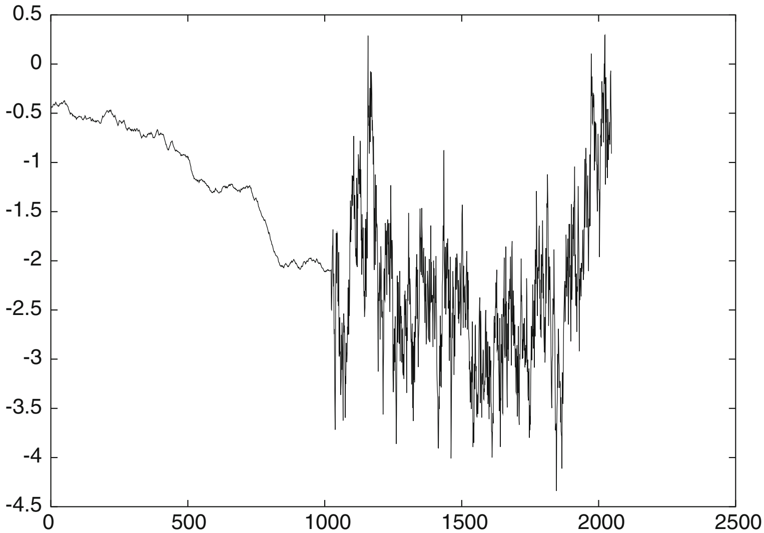


**Fig. 6.10** Step fractional Brownian motion with indexes 0.8 and 0.2

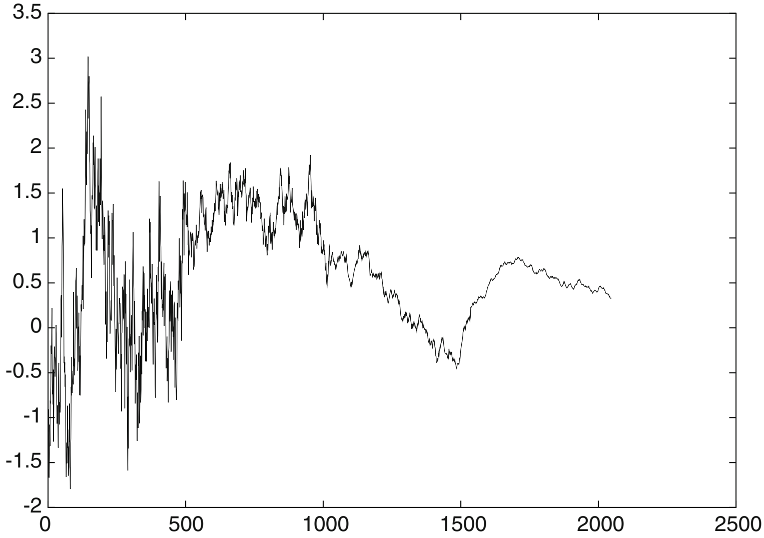


**Fig. 6.11** Step fractional Brownian motion with indexes 0.2, 0.4, 0.6 and 0.8

## 6.5 Simulation of Real Harmonizable Lévy Fields

In this section we consider the simulation of some non-Gaussian fractional fields. We chose to concentrate on the simulation of real harmonizable fractional Lévy fields introduced in Sect. 4.4.2 of Chap. 4.

### Theoretical Results

In this part we recall some theoretical facts of [89] summarized in Sect. 4.4.4. As explained in the Definition 4.4.3, a rhmLf $X_h$ with multifractional function $h$ is defined as the stochastic integral:

$$X_h(x) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{-ix\cdot\xi} - 1}{\|\xi\|^{h(x)+d/2}} M(d\xi),$$

with $M(d\xi)$ a random Lévy measure (cf. Definition 2.1.20). Please remark that we omit here the factor $\frac{1}{(C(h(x)))^{1/2}}$ for simulation purpose.

The field $X_h$ is an infinitely divisible field and infinitely divisible laws can be represented as a generalized shot noise series. An overview of these representations is given in [6]. The control measure, $\nu(dz)$ of the random Lévy measure $M(d\xi)$ is in general a sigma finite measure on the set of complex numbers. If it has finite mass denoted by $\nu(\mathbb{C})$, the generalized shot noise series can be used for simulation

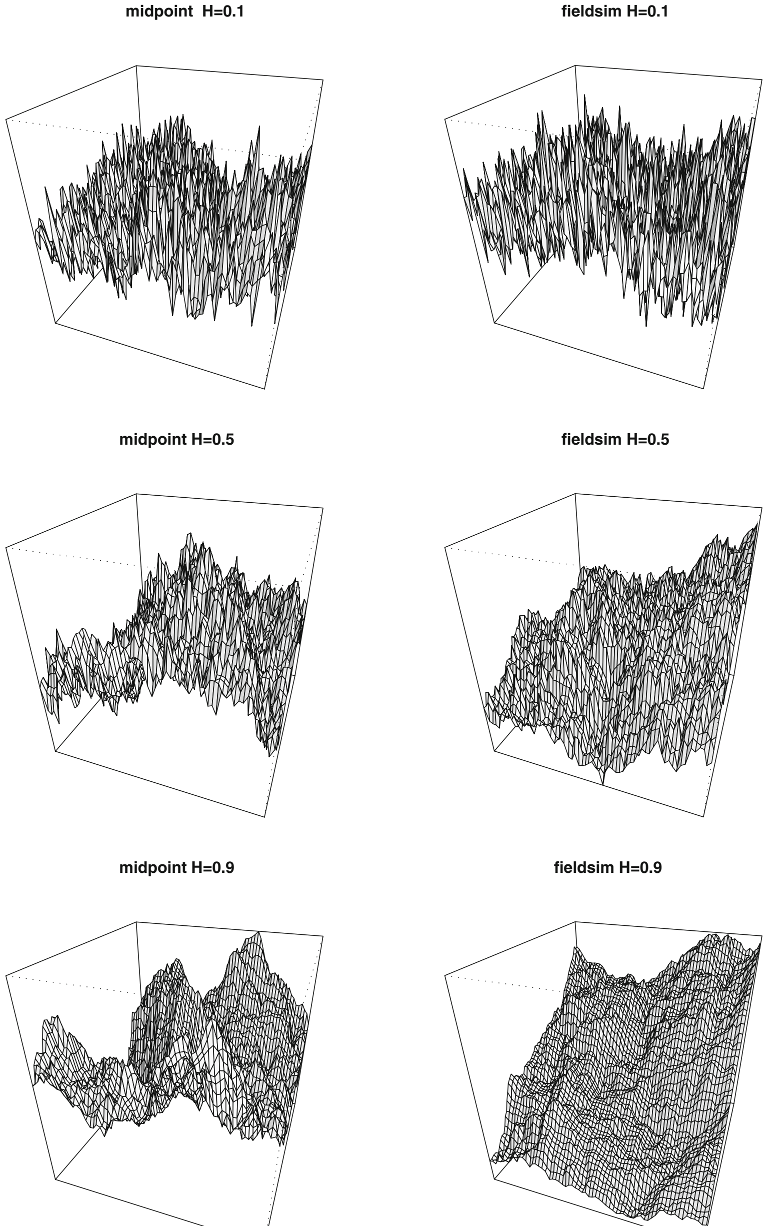


**Fig. 6.12** Fractional Brownian fields with indexes 0.1, 0.5, 0.9

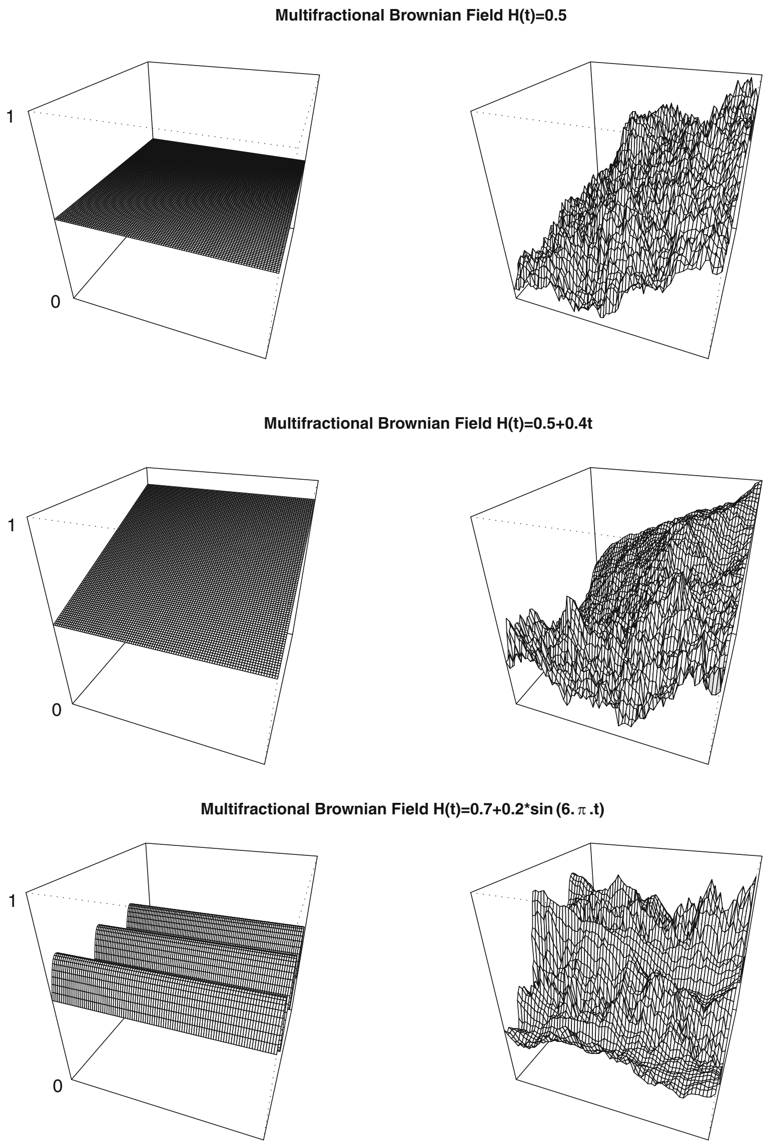


**Fig. 6.13** Multifractional Brownian fields (on the *right*) for the functions $h(t) = 0.5$, $h(t) = 0.5 + 0.4t_1$ and $h(t) = 0.7 + 0.2\sin(2\pi t_1)$ (on the *left*)

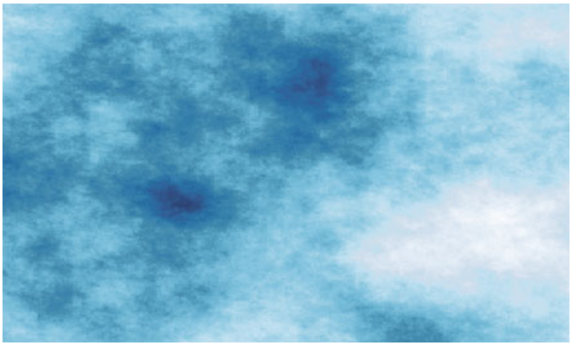

**Fig. 6.14** Cloud simulation: Fractional Brownian fields with index 0.5

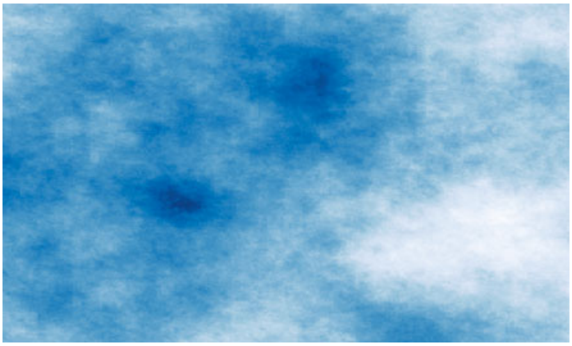

**Fig. 6.15** Cloud simulation: Fractional Brownian fields with index 0.7

purpose. Let us first introduce definitions of random variables used in these series. Let $(Z_n)_{n\geq 1}$, $(U_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ be independent sequences of random variables.

- $\nu$ is supposed to be a finite measure, and $(Z_n)_{n\geq 1}$ is a sequence of i.i.d. random variables such that
$$\mathcal{L}(Z_n) = \frac{\nu(dz)}{\nu(\mathbb{C})}.$$
- $(U_n)_{n\geq 1}$ is a sequence of i.i.d. random variables such that the law of $U_1$ is the uniform probability on the unit sphere in $\mathbb{R}^d$.
- $R_n$ is the $n$th arrival time of a Poisson process with intensity 1.
- $\xi_n = \left(\frac{R_n}{c_d\,\nu(\mathbb{C})}\right)^{1/d} U_n$, where $c_d = \frac{2\pi^{d/2}}{\Gamma(d/2)d}$ is the volume of the unit ball of $\mathbb{R}^d$.

Remark that when $d = 1$, $U_n$ is a symmetric Bernoulli random variable.

Let us now recall the theoretical result that allows us to simulate $X_h$. Next proposition by Lacaux [89] used in the following is given without proof.

**Proposition 6.5.1** *Let $K \subset \mathbb{R}^d$ be a compact set. Then almost surely, the series*

$$Y_h(x) = 2\sum_{n=1}^{+\infty} \Re\left\{ Z_n \frac{e^{-ix\xi_n} - 1}{\|\xi_n\|^{h(x)+\frac{d}{2}}} \right\}, \tag{6.5}$$

*where $\Re$ denotes the real part of a complex number, converges uniformly on $K$ and*

$$\{X_h(x) : x \in K\} \stackrel{d}{=} \{Y_h(x) : x \in K\}.$$

*where $\stackrel{d}{=}$ denotes equality in distribution.*

The rhmLf is then approximated by

$$Y_{h,N}(x) = 2\sum_{n=1}^{N} \Re\left\{ Z_n \frac{e^{-ix\xi_n} - 1}{\|\xi_n\|^{h(x)+\frac{d}{2}}} \right\}. \tag{6.6}$$

One can find in [89] theoretical rates of convergence for $Y_{h,N}$. Theoretical results are given on any compact set $K$ but the algorithms are written on the interval [0, 1] or on the cube $[0, 1]^2$ for sake of simplicity.

### Simulation Procedure

Let us first introduce the formula used for the simulation of one dimensional rhmLf ($d = 1$). The parameters of the simulation are

- $n$: the number of terms in the sum;
- $k$: the number of discretization points;
- $h(t)$ ($t = 1, \ldots, k$): the value of the multifractional function at each discretization point;
- $m = \nu(\mathbb{C})$: the mass value, usually fixed at 1. Actually we choose in the simulation to take $\nu$ the uniform distribution on the unit circle in $\mathbb{C}$.

If we consider the random variables that are standard to simulate:

- $\theta_n$: n-vector generated according to a uniform distribution in [0, 1], $Z_n = e^{i2\pi\theta_n}$;
- $U_n$: n-vector generated with independent symmetric Bernoulli one dimensional marginals;
- $R_n$: n-vector of cumulated sums of samples from exponential distribution with parameter $\lambda = 1$

and if we apply (6.6)

$$Y_{h,n}(t) = \sum_{l=1}^{n} \left( \frac{\cos(2\pi\theta_l)\left[\cos\left(\frac{R_l}{m}t.U_l\right) - 1\right] + \sin(2\pi\theta_l)\left[\sin\left(\frac{R_l}{m}t.U_l\right)\right]}{R_l^{h(t)+1/2}} \right). \tag{6.7}$$

When $d = 2$, the formula (6.7) becomes

$$Y_{h,n}(x, y) = \sum_{l=1}^{n} \left( \frac{\cos(2\pi\theta_l) \left[ \cos\left( \left( \frac{R_l}{\pi m} \right)^{\frac{1}{2}} (x.U_l^1 + y.U_l^2) \right) - 1 \right]}{R_l^{h(x,y)+1}} \right. \\ \left. + \frac{\sin(2\pi\theta_l) \sin\left( \left( \frac{R_l}{\pi m} \right)^{\frac{1}{2}} (x.U_l^1 + y.U_l^2) \right)}{R_l^{h(x,y)+1}} \right), \tag{6.8}$$

where $(U_n^1, U_n^2)$ are uniformly distributed on the unit sphere in $\mathbb{R}^2$. In this case $kx$ is the number of discretization points for the horizontal axis and $ky$ for the vertical axis.

In practice, the difficulty of the simulation lies on both the number of terms in the sum $n$ and in the calculation of scalar products between k-vectors. In our cases, $n$ may vary between $10^3$ and $10^8$, and $k$ between 100 and 1000 which may imply until $10^{16}$ loops for the computations in the extreme case.

## *6.5.1 Using FracSim*

The reader will refer to the Sect. 6.5.2 for details about installation of the Fracsim set.

### 6.5.1.1 Implementation of FracSim

FracSim is a set of R and C functions which allow to perform rhfLfs and rhmLfs simulations. Two classes of functions are implemented:

- R functions: **fracsim.1d()** and **fracsim.2d()** perform initializations thanks to parameters set by the user ($n$, $k$ and $h(t)_{t=1,\ldots,k}$), call C functions for calculation then get back results for graphical representations.
- C functions: **core-1D.c** and **core-2D.c** perform the tasks in the computation which are time consuming because of the number of loops required.

The R environment is the only user interface. The R procedure calls a C subroutine, whose results are returned to R.

In order to make it easier for the reader not used to R language, we detail the call to functions and the command used to produce graphical output in the Sect. 6.5.3. In the following, R commands are preceded by the prompt symbol `R>`.

#### 6.5.1.2 Simulations

One-dimensional case.

*Fractional Lévy*

To simulate rhfLf, one needs to specify one real parameter $0 < H < 1$ or equivalently the multifractional function is constant $h(t) = H, \forall t \in [0, 1]$.

For instance, the command line

```
R> X05 = fracsim.1d(h=0.5,k=1000,n=5000)
```

sets the multifractional function equal to 0.5 on [0,1] and simulates the corresponding process.

The result of the function `fracsim.1d` is an R object of class `list`. It contains the following elements

- the parameters set by the user;
- the vector `t` of discretization points;
- the k-vector `process`, which elements are the rhfLf value at each discretization point.

Thus to produce the graphs on the Fig. 6.16, one have to call the function `plot` as

```
R> plot(x=X05$t,y=X05$proces,type="l")
```

As previously mentioned, we can see on the Fig. 6.16, the greater *h* is, the smoother the sample paths are.

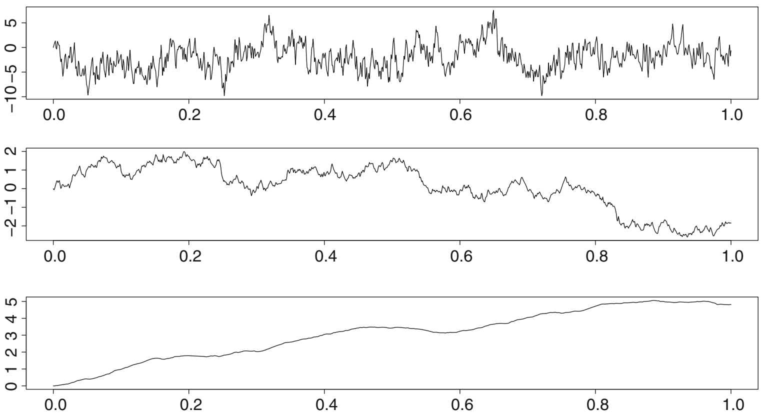


**Fig. 6.16** Examples of fractional Lévy field: From *top* to *bottom* the regularity is set to 0.1, 0.5, 0.9

Multifractional fields

To simulate rhmLfs, one has to give a vector of length `k` for `h`. The two following command lines give examples of

- an increasing regularity function from 0.1 to 0.9 (`Hinc`);
- a sinusoidal regularity function oscillating according to a linear combination of sinus (`Hsin`);

```
R> Hinc = seq(from=0.1,to=0.9,length=1000)
R> Hsin = 0.25+0.25*sin(seq(0,1,length=1000)*(6*pi))
```

We can then call the function `fracsim.1d()` with the vectors previously defined, given as parameter to the argument `h`.

```
R> Xinc = fracsim.1d(h=Hinc,k=1000,n=5000)
R> Xsin = fracsim.1d(h=Hsin,k=1000,n=5000)
```

To highlight the behavior of the rhmLf according to the multifractional function, we represent both regularity functions and the corresponding sample paths on the Fig. 6.17.

As previously mentioned for rhfLfs, the greater the regularity, the smoother the trajectory and this can be observed on one graph: For example when we look at the sinusoidal regularity, the corresponding process is much more perturbed when the regularity is close to zero.

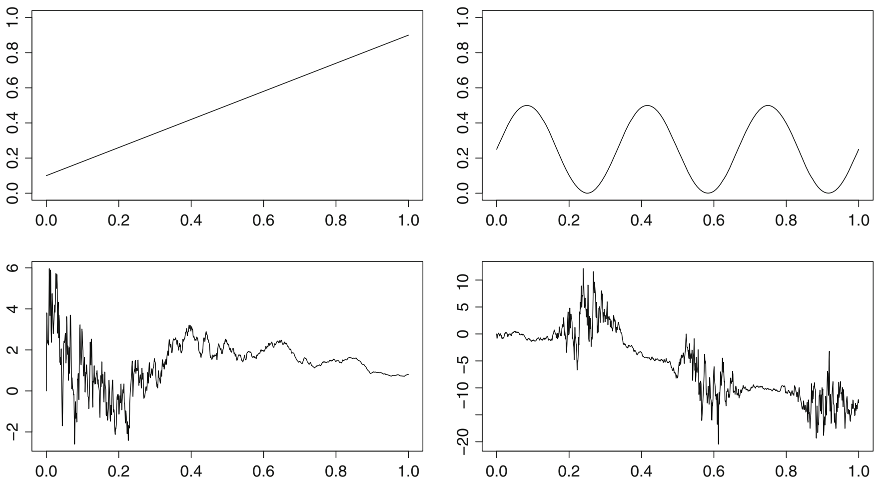


**Fig. 6.17** Two examples of multifractional Lévy fields: Figures on the *top* represent the regularity function, *bottom*, the corresponding processes

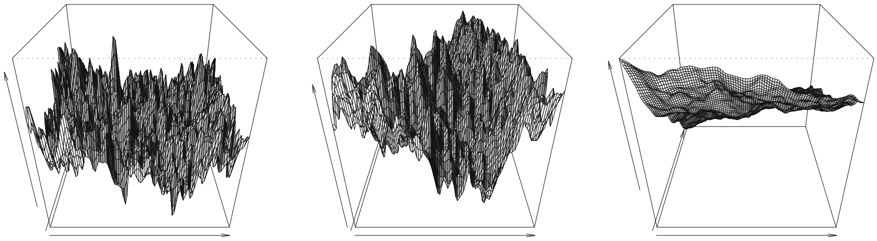

**Fig. 6.18** Examples of fractional Lévy fields: From *left* to *right* h = 0.1, 0.5, 0.9

## Two-dimensional fields

### Fractional fields

RhfLfs can be simulated by calling the function `fracsim.2d()` with one value given as argument `h`. For instance the following command line

```
R> X2d05 = fracsim.2d(h=0.5,kx=100,ky=100,n=100000)
```

sets $h$ constant and equal to 0.5 on a $100 \times 100$ grid with $10^5$ terms in the sum.

The elements of the object `X2d05` provided by the function `fracsim.2d()` are the same as those given by `fracsim.1d()`, except that, in this case, the field values are now located on a grid and are stored in a matrix object.

To represent two dimensional fields, we now use the R function `persp` to draw a surface.

```
R> persp(X2d05,shade=0.5,phi=30)
```

Arguments `shade` and `phi` are optional, they can be set to give a nice representation: `Shade` modifies the shade at a surface facet and `phi` is the colatitude angle defining the viewing direction.

Others functions can be used to represent two dimensional fields, we propose a comparison of various representations at the end of this section.

### Multifractional fields

To simulate two dimensional rhfLfs, one has to give a matrix which dimensions are `kx` $\times$ `ky` as argument for `h`. For example, we can define two matrices corresponding to the cases presented in dimension one with the following command lines.

```
R> Hseq = matrix(rep(seq(0,1,l=100),100),nc=100)
R> Hsin = matrix(rep(0.25+0.25*sin(seq(0,1,l=100)
*(6*pi)),100),nc=100)
```

One can obtain the graph of the regularity function (Fig. 6.19) as well. We call the function `persp` with the same optional arguments used for the representation of fractional two dimensional fields.

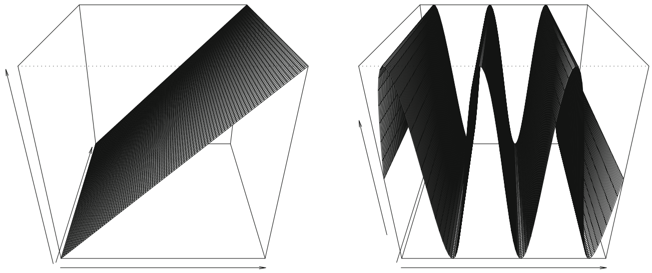

**Fig. 6.19** Two examples of regularity function for multifractional Lévy fields: Increasing on the *left* and sinusoidal on the *right*

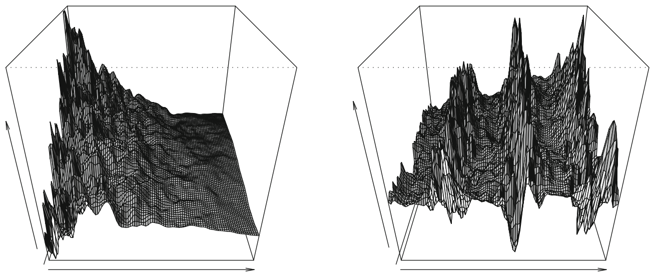

**Fig. 6.20** Two examples of multifractional Lévy fields: Increasing regularity function on the *left* and sinusoidal regularity function on the *right*

Once the matrix containing the regularity function at each point of the grid is built, we can perform the simulation of the field by calling, as in the fractional case, the function `fracsim.2d()`.

```
R> Xseq = fracsim.2d(h=Hseq,kx=100,ky=100,n=100000)
R> Xsin = fracsim.2d(h=Hsin,kx=100,ky=100,n=100000)
```

Figure 6.20 illustrates the behavior of the multifractional process. Considering an increasing regularity (graphic on the left of the Fig. 6.20), the surface looks like a landscape where we go from the mountain with several peaks (low regularity) to the plain (high regularity). With a sinusoidal regularity, the "landscape" looks like an alternation of mountains and valleys.

Technical comments

Graphical representations

The way we represents a two dimensional field has an influence on the visual impression provided. If we only use the `persp` function to draw a surface, it could be

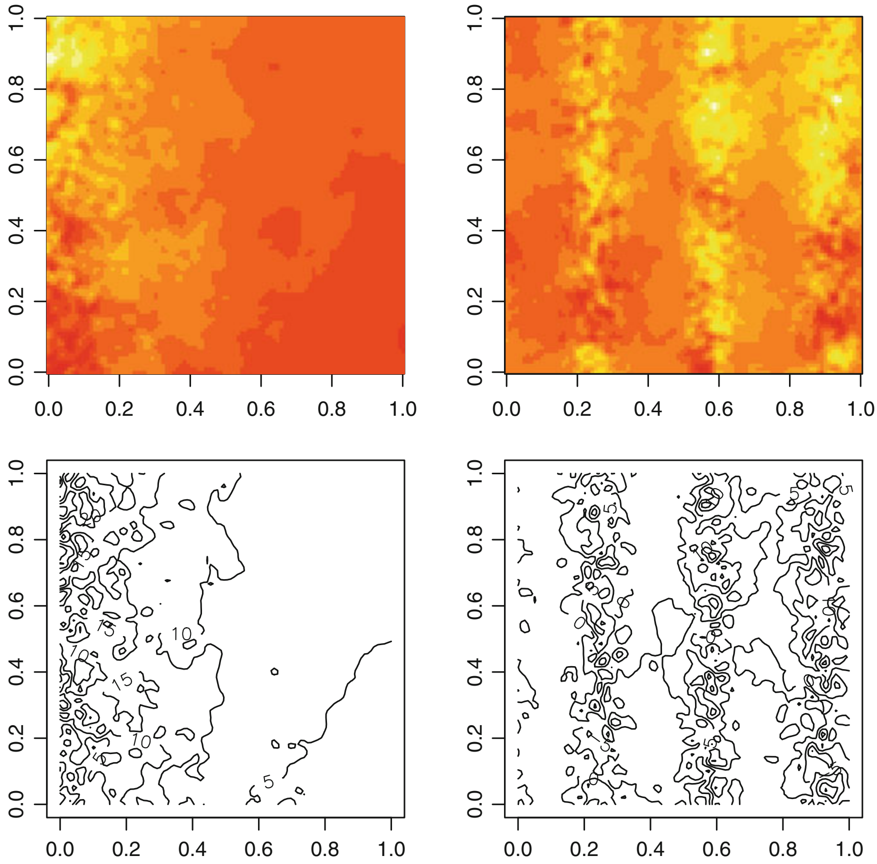


**Fig. 6.21** Image and contour representations of the simulations performed in the previous section (Fig. 6.20)

interesting to modify point of view by changing `phi` (vertical rotation) and `theta` (horizontal rotation) parameters.

However, we can use other kinds of representations such as images or contour line plots as it is done on the Fig. 6.21.

Amongst these various representations, it is not easy to decide which is the best. However, let's note that with a high discretization and a regularity close to zero, the visual impression given by the image is smoother than the one given by the sheet (Fig. 6.22, obtained thanks to improvements using parallel computing). This can be explained by the inability to distinguish various intensity in the same color on the image whereas the peaks appear clearly on the sheet. On the other hand, the image of a two dimensional field highlights more precisely the global trends of the process which is more difficult to observe on the sheet.

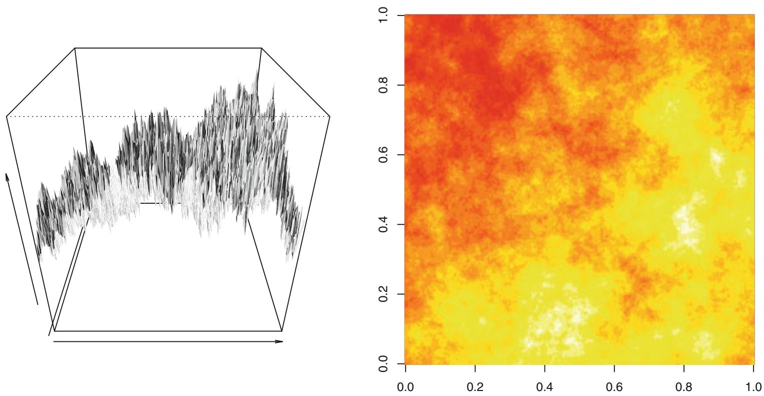


**Fig. 6.22** Surface and image representations of a rhfLf (h=0.4) on a 256 × 256 grid

Durations

For one dimensional simulations, calculations last about few seconds on a 'current use' machine (SGI Origin 2000, RAM: 3840 MO, processors: 195 MHz) for the configuration we performed, and we did not have any trouble with too long calculations.

For two dimensional fields, the calculation durations order of magnitude can be linearly extrapolated from the configuration $\{kx = ky = 100, n = 10000\}$ (total amount of loops is about $10^8$) last about 2 min on the machine previously mentioned.

We can estimate the potential duration for the extreme configuration we would like to perform $\{kx = ky = 500, n = 10^8\}$ at about one year. In [38] it explained how to use parallel computing to get simulations in a reasonable time.

In this section we have proposed an efficient way to produce non Gaussian fields and we hope it can help to model various irregular phenomena. Of special interest is the use of parallel computers that shows that the algorithm is flexible enough. On the other hand there exist other non Gaussian models that could be simulated in a similar way (See [39]). The FracSim set can be downloaded at http://www.jstatsoft.org/v14/i18.

### *6.5.2 Instructions for Using FracSim*

The first step consists in compiling the C functions. On Unix, this can be done through the R system. Just enter the command line above at the Unix prompt.

```
R> CMD SHLIB core-1d.c
R> CMD SHLIB core-2d.c
```

This creates the shared library files `core-1d.o`, `core-1d.so`, `core-2d.o`, `core-2d.so` and `so_locations`.

The shared library (`.so`) have to be loaded into R using the `dyn.load()` function. The lines

```
dyn.load("core-1d.so")
dyn.load("core-2d.so")
```

have been included in the header of the files `FracSim-1D.R` and `FracSim-2D.R` so that the user just has to import them using the `source()` function as follow:

```
source("FracSim-1D.R")
source("FracSim-2D.R")
```

This also creates the functions **fracsim.1d()** and **fracsim.2d()** that perform the simulations.

### 6.5.3 *Description of R Script*

In this section, we give the code used to produce figures of the article.

- Simulations of 1D-rhfLfs with $h = 0.1$, $h = 0.5$ and $h = 0.9$. The results are stored in the three objects `X01`, `X05` and `X09`.

```
R> X01 = fracsim.1d(h=0.1,k=1000,n=5000)
R> X05 = fracsim.1d(h=0.5,k=1000,n=5000)
R> X09 = fracsim.1d(h=0.9,k=1000,n=5000)
```

- Graphical representations of 1D-rhfLfs (Fig. 6.16)

```
R> par(mfrow=c(3,1))
R> plot(x=X01$t,y=X01$process,type="l")
R> plot(X05$t,X05$process,type="l")
R> plot(X09$t,X09$process,type="l")
```

  The first line divides the graphical window horizontally in three parts to put graphics one under each other. Then, we produce plots giving the time discretization vector as abscissa and the process values as ordinate. The option `type="l"` draws a line between each point.
- Simulations of 1D-rhmLfs with increasing and sinusoidal multifractional function

```
R> Hinc = seq(from=0.1,to=0.9,length=1000)
R> Hsin = 0.25+0.25*sin(seq(0,1,length=1000)*(6*pi))
```

  We define an increasing multifractional function with the function `seq` in order to get 1000 (`length`) values regularly spaced between 0.1 (`from`) and 0.9 (`to`). The same function is used in combination with the sinus function to simulate a sinusoidal multifractional function.
- Graphical representations of 1D-rhmLfs (Fig. 6.17)

```
R> par(mfrow=c(2,2))
R> plot(Xinc$t,Hinc,type="l",ylim=c(0,1))
R> plot(Xsin$t,Hsin,type="l",ylim=c(0,1))
R> plot(Xinc$t,Xinc$process,type="l")
R> plot(Xinc$t,Xsin$process,type="l")
```

The graphical window is divided into four parts ($2 \times 2$ grid). The top row contains the plots of the multifractional functions; the vertical axis is set to [0,1] (`ylim`). The bottom row contains the trajectory of the rhmLfs.

- Simulations of 2D-rhfLfs with $h = 0.1$, $h = 0.5$ and $h = 0.9$.

```
R> X2d01 = fracsim.2d(h=0.1,kx=100,ky=100,n=100000)
R> X2d05 = fracsim.2d(h=0.5,kx=100,ky=100,n=100000)
R> X2d09 = fracsim.2d(h=0.9,kx=100,ky=100,n=100000)
```

In these calls, the argument `ky` is optional; if missing, its value is taken to be equal to `kx`.

- Graphical representations of 2D-rhfLfs (Fig. 6.18)

```
R> par(mfrow=c(1,3)
R> persp(X2d01,shade=0.5,phi=30)
R> persp(X2d05,shade=0.5,phi=30)
R> persp(X2d09,shade=0.5,phi=30)
```

The function `persp` produces the plot on a grid determined by the size of the matrix given as the first parameter (`X2d0*`). We used `shade=0.5` to define the shade at a surface facet in order to provide an approximation to daylight illumination as explained in the R help for the `persp` function. The option `phi=30` implies a vertical rotation that sets the point of view above the surface.

- Simulations of 2D-rhmLfs with increasing and sinusoidal multifractional function

```
R> Hseq = matrix(rep(seq(0,1,l=100),100),ncol=100)
R> Hsin = matrix(rep(0.25+0.25*sin(seq(0,1,l=100)*
(6*pi)),100),ncol=100)
```

The two lines above produce two matrices which are the discretization of the multifractional functions in two cases: (1) the regularity increases column by column, (2) the regularity is sinusoidal according to the columns of the grid. By default, the R function `matrix` produces a matrix from a vector filling by column.

```
R> Xseq = fracsim.2d(h=Hseq,kx=100,ky=100,n=100000)
R> Xsin = fracsim.2d(h=Hsin,kx=100,ky=100,n=100000)
```

Once defined the regularity matrix, the call to **fracsim.2d()** is the same as for rhfLfs.

- Graphical representations of multifractional functions for 2D-rhmLfs (Fig. 6.19)

```
R> persp(Hseq,shade=0.5,phi=30)
R> persp(Hsin,shade=0.5,phi=30)
```

- Graphical representations of 2D-rhmLfs (Fig. 6.20)

```
R> persp(Xseq,shade=0.5,phi=30)
R> persp(Xsin,shade=0.5,phi=30)
```

- Image and contour plots (Fig. 6.21)

```
R> par(mfrow=c(2,2)
R> image(Xseq)
R> image(Xsin)
R> contour(Xseq)
R> contour(Xsin)
```

## 6.6 Visual Meaning of the Fractional Index

A practical and important question for applications is to know whether the fractional index has a visual meaning. The aim is to show that we can guess the qualitative behavior of a multifractional function in some cases. The first experiment is as follows. We simulate an approximate step fractional Brownian field with indexes 0.8 and 0.2 generalizing (6.4) when $d = 2$. For $t \in [0, 1]^2$, $k \in \mathbb{N}^2$ we use

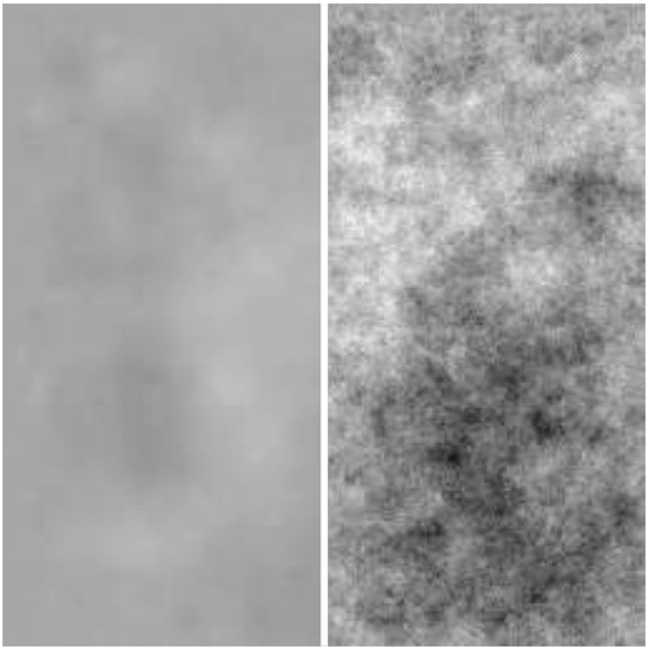

**Fig. 6.23** Step fractional Brownian field with indexes 0.8 and 0.2. Note that, by construction, the image is continuous

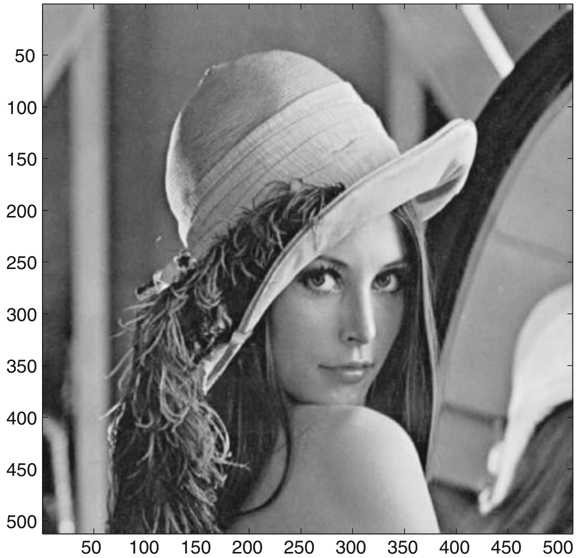


**Fig. 6.24** Original Lenna's picture

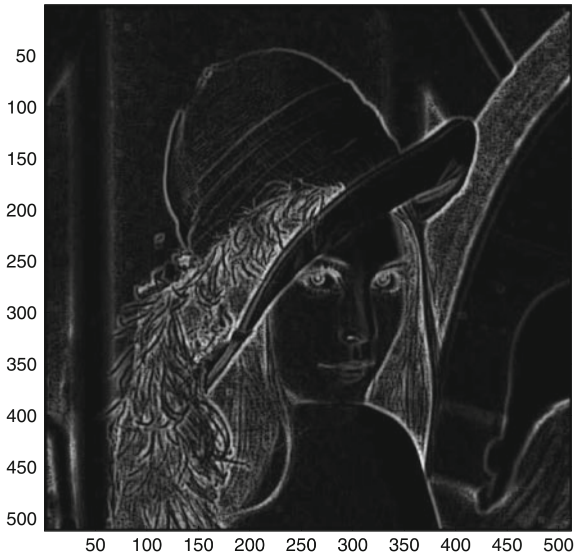


**Fig. 6.25** Fractional Lenna's picture (cf. Fig. 6.24)

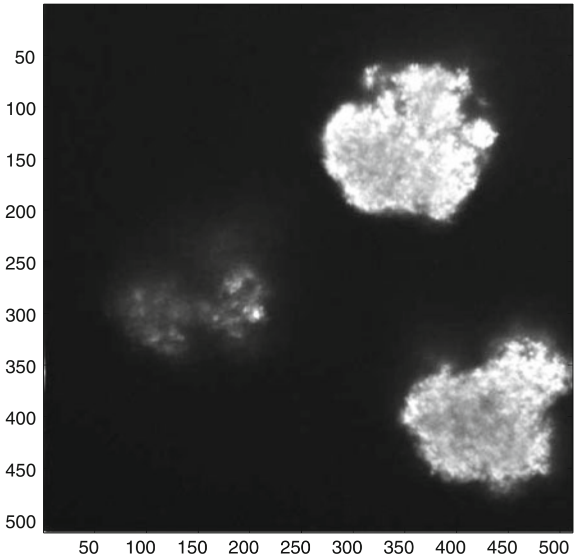


**Fig. 6.26** Original picture of cells

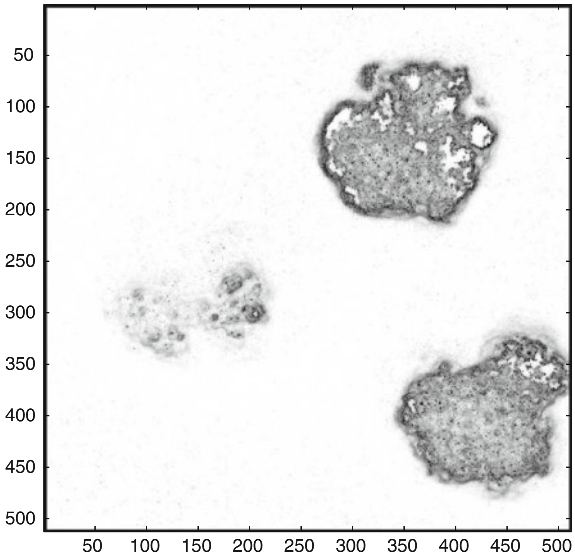


**Fig. 6.27** Fractional picture of cells (cf. Fig. 6.26)

$$\widetilde{Q_h}(t) = \sum_{0 \le j \le J, k_1, k_2 \le K} 2^{-jh\left(\frac{k}{2^j}\right)} \psi_{j,k}(t) \varepsilon_{j,k} \,. \tag{6.9}$$

where $h\left(\frac{k}{2^j}\right) = 0.8$ if $k_1 \le 2^{j-1}$ and $h\left(\frac{k}{2^j}\right) = 0.2$ if $k_1 > 2^{j-1}$. By construction, this field is centered and continuous. Therefore, the two parts of the Fig. 6.23 are due to the difference of fractional indexes between the right and the left side.

Another kind of visual experiments can be done with real world data. We take a picture. For every pixel, we estimate the fractional index in a neighborhood of the pixel, let us say on a $8 \times 8$ square. Then, the value of the pixel is replaced by the value of the fractional index. Afterwards, we need to normalize the new image, for instance by multiplying the fractional index by 256. We present two examples below: Figs. 6.24 and 6.26 are the original images, Figs. 6.25 and 6.27 are the pictures of the fractional indexes. We could expect that the transformation will destroy the structure of the pictures. The following examples show that surprisingly the edges of the big objects in the pictures are preserved.

# Appendix A

## A.1 Localization Property for the Vaguelettes

In this appendix we give the proof of Proposition 4.3.6, which is now recalled

**Proposition A.1.1** *Let $x \in \mathbb{R}$, $y \in (0,1)$ and $\lambda \in \Lambda^+$*

$$\chi_\lambda(x,y) = \int_{\mathbb{R}} \frac{e^{-ix\xi}-1}{|\xi|^{y+1/2}} \overline{\widehat{\psi_\lambda}(\xi)} \frac{d\xi}{(2\pi)^{1/2}} \tag{A.1}$$

*and for $\lambda = (0,k,0)$*

$$\tilde{\chi}_\lambda(x,y) = \int_{\mathbb{R}} \frac{e^{-ix\xi}-1+ix\xi-1/2x^2\xi^2}{|\xi|^{y+1/2}} \overline{\widehat{\psi_\lambda}(\xi)} \frac{d\xi}{(2\pi)^{1/2}}. \tag{A.2}$$

*Then for $j > 0$ $\chi_\lambda$ is an analytic function and satisfies for every $K \in \mathbb{N}^*$*

$$|\chi_\lambda(x,y)| \le C(K)2^{-yj}\left(\frac{1}{1+|2^jx-k|^K} + \frac{1}{1+|k|^K}\right) \tag{A.3}$$

*and for $x,\ x' \in \mathbb{R}$, and $y,\ y' \in (0,1)$*

$$|\chi_\lambda(x,y) - \chi_\lambda(x',y')| \le C(K)2^{-\min(y,y')j}\left[\frac{2^j|x-x'|+j|y-y'|}{1+|2^jx-k|^K} + \frac{j|y-y'|}{1+|k|^K}\right]. \tag{A.4}$$

*Moreover $\tilde{\chi}_\lambda(x,y)$ is a $C^2$ function and satisfies for $K = 0,\ 1,\ 2$*

$$|\tilde{\chi}_\lambda(x,y)| \le C(K)\left(\frac{1}{1+|x-k|^K} + \frac{1}{1+|k|^K}\right) \tag{A.5}$$

S. Cohen and J. Istas, *Fractional Fields and Applications*,
Mathématiques et Applications 73, DOI: 10.1007/978-3-642-36739-7,

*and*

$$
\begin{aligned}
&|\tilde{\chi}_\lambda(x,y)-\tilde{\chi}_\lambda(x',y')| \\
&\quad\le C(K)\left[\frac{|x-x'|}{1+|x-k|^K}+|y-y'|\left(\frac{1}{1+|x-k|^K}+\frac{1}{1+|k|^K}\right)\right].
\end{aligned}
\tag{A.6}
$$

*Proof of Proposition 4.3.6* Let us first show that $\chi_\lambda$ is analytic since the support of $\widehat{\psi_\lambda}$ is compact and the integrand in (A.1) is an analytic function of $(x, y)$.

Let us prove (A.3). Since $\lambda \in \Lambda^+$

$$
\widehat{\psi^{(1)}_{j,k}}(\xi)=2^{-j/2}e^{-i2^{-j}k\xi}\widehat{\psi^{(1)}}(2^{-j}\xi).
\tag{A.7}
$$

Because of the support of $\widehat{\psi^{(1)}}$

$$
supp(\widehat{\psi}^{(1)}_{j,k})\subset\left\{\frac{2^{j+1}\pi}{3}\le|\xi|\le\frac{2^{j+3}\pi}{3}\right\}
\tag{A.8}
$$

For the sake of simplicity let us drop the exponent in $\psi^{(1)}$ when we study $\chi_\lambda$. Hence the integral in (A.1) can be split in two parts $I = I_1 - I_0$, where

$$
I_1=\int_{\mathbb{R}}\frac{e^{-ix\xi}}{|\xi|^{y+1/2}}\overline{\widehat{\psi_{j,k}}(\xi)}\frac{d\xi}{(2\pi)^{1/2}}
\tag{A.9}
$$

and

$$
I_0=\int_{\mathbb{R}}\frac{1}{|\xi|^{y+1/2}}\overline{\widehat{\psi_{j,k}}(\xi)}\frac{d\xi}{(2\pi)^{1/2}}
\tag{A.10}
$$

that are defined.

Let us first consider $I_1$.

$$
\begin{aligned}
I_1&=2^{-j/2}\int_{\mathbb{R}}e^{-i(x-2^{-j}k)\xi}\frac{\overline{\widehat{\psi}(2^{-j}\xi)}d\xi}{(2\pi)^{1/2}|\xi|^{y+1/2}} \\
&=2^{-yj}\int_{\mathbb{R}}e^{-i(2^jx-k)\eta}\frac{\overline{\widehat{\psi}(\eta)}d\eta}{(2\pi)^{1/2}|\eta|^{y+1/2}} \\
&\le 2^{-yj}\int_{\mathbb{R}}\frac{|\widehat{\psi}(\eta)|}{(2\pi)^{1/2}|\eta|^{y+1/2}}d\eta
\end{aligned}
\tag{A.11}
$$

where the change of variable $\eta = 2^{-j}\xi$ has been used. The integral in the last line is defined thanks to the support of $\psi$. If we consider $x = 0$ in the inequality (A.9) we can bound the second part of $\chi_\lambda(x, y)$ by the same bound, then (A.3) is proved when $K = 0$.

To have (A.3) for $K \geq 1$ an integration by parts is used. One gives some details for $K = 1$. Let us go back to (A.11), when $x \neq 2^{-j}k$

$$I = 2^{-j/2} \int_{\mathbb{R}} e^{-i(x-2^{-j}k)\xi} \frac{\overline{\widehat{\psi}(2^{-j}\xi)}d\xi}{(2\pi)^{1/2}|\xi|^{y+1/2}} \tag{A.12}$$
$$= 2^{-j/2} \int_{\mathbb{R}} \frac{-e^{-i(x-2^{-j}k)\xi}}{i(2^{-j}k - x)} \frac{\partial}{\partial \xi} \left( \frac{\overline{\widehat{\psi}(2^{-j}\xi)}}{(2\pi)^{1/2}|\xi|^{y+1/2}} \right) d\xi$$

because $\psi$ has a compact support. Then

$$\frac{\partial}{\partial \xi} \left( \frac{\overline{(\partial/\partial\xi)\,\widehat{\psi}(2^{-j}\xi)}}{(2\pi)^{1/2}|\xi|^{y+1/2}} \right) = \frac{2^{-j}\overline{(\partial/\partial\xi)\,\widehat{\psi}(2^{-j}\xi)}}{(2\pi)^{1/2}|\xi|^{y+1/2}} - \frac{(y+1/2)\overline{\widehat{\psi}(2^{-j}\xi)}}{(2\pi)^{1/2}|\xi|^{y+3/2}} \tag{A.13}$$

Then

$$I = 2^{-j/2} \int_{\mathbb{R}} \frac{-e^{-i(x-2^{-j}k)\xi}}{i(2^{-j}k - x)} \frac{2^{-j}\overline{(\partial/\partial\xi)\,\widehat{\psi}(2^{-j}\xi)}}{(2\pi)^{1/2}|\xi|^{y+1/2}} d\xi \tag{A.14}$$
$$- 2^{-j/2} \int_{\mathbb{R}} \frac{-e^{-i(x-2^{-j}k)\xi}}{i(2^{-j}k - x)} \frac{(y+1/2)\overline{\widehat{\psi}(2^{-j}\xi)}}{(2\pi)^{1/2}|\xi|^{y+3/2}} d\xi \tag{A.15}$$

The right hand term of (A.14) can be bounded by $2^{-(y+1)j} \int_{\mathbb{R}} \frac{\overline{|(\partial/\partial\xi)\,\widehat{\psi}(\eta)|}}{(2\pi)^{1/2}|\eta|^{y+1/2}}$ with the same change of variable we did in (A.11). This bound is also valid for (A.15) since $y + 1/2$ in (A.14) becomes $y + 3/2$ in (A.15). Hence

$$|I_1| \leq \frac{C(1)2^{-jy}}{|2^j x - k|} \tag{A.16}$$

Combining this inequality with the one when $x = 0$ yields (A.3) when $K = 1$. The general case for $K > 1$ relies on $K$ integrations by parts.

Let us now prove (A.4). When $y = y'$ one gets

$$|\chi_\lambda(x, y') - \chi_\lambda(x', y')| = \left| \int_{\mathbb{R}} \frac{(e^{-ix\xi} - e^{-ix'\xi})\overline{\widehat{\psi}_\lambda(\xi)}}{(2\pi)^{1/2}|\xi|^{y'+1/2}} d\xi \right| \tag{A.17}$$
$$\leq |x - x'| \int_{\mathbb{R}} \frac{2^{-j/2}|\xi\widehat{\psi}(2^{-j}\xi)|}{(2\pi)^{1/2}|\xi|^{y'+1/2}} d\xi \tag{A.18}$$
$$\leq C|x - x'|2^j 2^{-y'j}. \tag{A.19}$$

And when $x = x'$

$$|\chi_\lambda(x,y)-\chi_\lambda(x,y')| = \left|\int_{\mathbb{R}}\left(\frac{1}{|\xi|^{y+1/2}}-\frac{1}{|\xi|^{y'+1/2}}\right)\frac{(e^{-ix\xi}-1)\overline{\widehat{\psi}_\lambda(\xi)}}{(2\pi)^{1/2}}d\xi\right|$$

Then for some $\eta(y,y',\xi)\geq\min(y,y')$

$$|\chi_\lambda(x,y)-\chi_\lambda(x,y')| = |y-y'|\left|\int_{\mathbb{R}}\frac{(e^{-ix\xi}-1)|\ln(|\xi|)|\overline{\widehat{\psi}_\lambda(\xi)}}{(2\pi)^{1/2}|\xi|^{\eta+1/2}}d\xi\right| \tag{A.20}$$

If we apply the technique used for (A.11) to (A.20), one gets

$$\begin{aligned}&|\chi_\lambda(x,y)-\chi_\lambda(x,y')|\\ &\quad\leq C(K)|y-y'|j2^{-j\min(y,y')}\left(\frac{1}{1+|2^jx-k|^K}+\frac{1}{1+|k|^K}\right)\end{aligned} \tag{A.21}$$

Combining (A.17) and (A.21) yields (A.4).

Let us now consider (A.5), since the support of $\widehat{\psi}_\lambda$ for $\lambda=(0,k,0)$ contains 0 we have to take care of the pole $|\xi|^{-(y'+1/2)}$ in the integration by parts that yields the factor $\frac{1}{1+|2^jx-k|^K}$. Actually the fact that

$$\frac{\partial^2}{\partial\xi^2}\left(\frac{e^{-ix\xi}-1+ix\xi-1/2x^2\xi^2}{|\xi|^{y+1/2}}\right)\leq C|\xi|^{1/2-y} \tag{A.22}$$

yields (A.5), (A.6) is obtained similarly.

## A.2 Proof of Theorem 5.1.3

The study of the generalized quadratic variations the $V_N$ requires estimates of their expectations and variances. Note that

$$\begin{aligned}\mathbb{E}V_N &= \sum_{p=0}^{N-K}\mathbb{E}\left(\sum_{k=0}^{K}a_kX\left(\frac{k+p}{N}\right)\right)^2\\ &= \sum_{p=0}^{N-K}\int_{\mathbb{R}}\left|\sum_{k=0}^{K}a_kg\left(\frac{k+p}{N},\xi\right)e\left(\xi\frac{k+p}{N}\right)\right|^2d\xi,\end{aligned} \tag{A.23}$$

where function $e$ stands for $e(\xi)=e^{i\xi}-1$.

Using the expansion for $g(t,\xi)$, we can express $\mathbb{E}V_N$ as a sum of terms, each of them being of the following form ($p=0,\ldots,N-K$)

$$I(S,S')_{p,p'} = \int_{\mathbb{R}} \sum_{k,k'=0}^{K} a_k a_{k'} S\left(\frac{k+p}{N}, Nu\right) S'\left(\frac{k'+p'}{N}, Nu\right)$$
$$e((k+p)u)\overline{e}((k'+p')u))N du. \tag{A.24}$$

A change of variables $u = N\xi$ has been performed and each $S, S'$ stands for one of the functions at the right hand-side of (5.3). So that either $S(t,\xi) = \dfrac{a(t)}{|\xi|^{H+1/2}}$ or $S(t,\xi)$ is bounded by such a term, and the same holds for $S'$.

The following condition is implied by **Hyp** 5.1.1 if $S$ stands for one of the functions at the right hand-side of (5.3).

$S(t,\xi) \in C^{2,2}([0,1]\times\mathbb{R}^*)$ and

$$\left|\frac{\partial^{i+j}}{\partial t^i \partial \xi^j} S(t,\xi)\right| \le \frac{C}{|\xi|^{\frac{1}{2}+\delta+j}}\ , \tag{A.25}$$

for $i = 0$ to 2 and $j = 0$ to 2 with $0 < \delta < 1$, and the same holds for $S'$.

Since $X$ is a Gaussian process, the variance of $V_N$ is given by

$$\mathrm{var}(V_N) = 2\sum_{p,p'=0}^{N-K}\left(\int_{\mathbb{R}} \sum_{k,k'=0}^{K} a_k a_{k'} g\left(\frac{k+p}{N},\xi\right) g\left(\frac{k'+p'}{N},\xi\right)\right.$$
$$\left. e\left(\xi\frac{k+p}{N}\right)\overline{e}\left(\xi\frac{k'+p'}{N}\right) d\xi\right)^2$$

which is a sum of terms of the form $I(S,S')_{p,p'}$. The estimation of the expectation and variance of $V_N$ first requires the estimation of the $I(S,S')_{p,p'}$. We therefore state the following result on the $I(S,S')_{p,p'}$ now.

**Lemma A.2.1** *The following bound*

$$|I(S,S')_{p,p'}| \le \frac{C}{N^{\delta+\delta'}(1+(p-p')^2)}$$

*holds for $N$ large enough, if* (A.25) *holds.*

*Proof of Lemma A.2.1* We use a Taylor expansion of $S$ of order 2 for $0 \le k \le K$

$$S\left(\frac{k+p}{N}, Nu\right) = \sum_{j=0}^{2} \frac{\partial^j}{\partial t^j} S\left(\frac{p}{N}, Nu\right)\frac{k^j}{N^j j!} + \frac{k^2}{2N^2}\frac{\partial^2}{\partial t^2} S\left(\frac{\kappa+p}{N}, Nu\right),$$

where $0 \le \kappa \le K$. The same holds for $S'$. We then obtain the following expansion for $I(S,S')_{p,p'}$

$$
\begin{aligned}
I(S,S')_{p,p'} = &\sum_{j=0}^{2} N^{-j} \sum_{j_1+j_2=j} \frac{1}{j_1!j_2!} \\
&\times \int_{\mathbb{R}} \left[ \sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2} e((k+p)u)\overline{e}((k'+p')u) \right] \\
&\times \frac{\partial^{j_1}}{\partial t^{j_1}} S\left(\frac{p}{N}, Nu\right) \frac{\partial^{j_2}}{\partial t^{j_2}} S'\left(\frac{p'}{N}, Nu\right) N du \qquad \text{(A.26)} \\
&+ \sum_{j=2}^{4} N^{-j} \sum_{j_1+j_2=j} \frac{1}{j_1!j_2!} \\
&\times \int_{\mathbb{R}} \sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2} e((k+p)u)\overline{e}((k'+p')u) \qquad \text{(A.27)} \\
&\times \frac{\partial^{j_1}}{\partial t^{j_1}} S\left(\frac{\epsilon(j_1)\kappa + p}{N}, Nu\right) \frac{\partial^{j_2}}{\partial t^{j_2}} S'\left(\frac{\epsilon(j_2)\kappa' + p'}{N}, Nu\right) N du,
\end{aligned}
$$

where $\epsilon(j) = 1$ if $j = 2$, 0 otherwise.

Let us first consider the case $p = p'$. We have to bound $|I(S, S')_{p,p}|$ by $C\ N^{-\delta-\delta'}$. We clearly have

$$\sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2} e((k+p)u)\overline{e}((k'+p)u) = \sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2} e^{iu(k-k')}.$$

Each integral of (A.26) is bounded by a term involving

$$\int_{\mathbb{R}} \left| \sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2} e^{iu(k-k')} \right| \frac{du}{|u|^{\delta+\delta'+1}}. \qquad \text{(A.28)}$$

The function $\sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2} e^{iu(k-k')}$ and its derivatives up to order 2 vanish at $u = 0$ hence $\sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2} e^{iu(k-k')} = o(u^2)$ when $|u| \to 0$. Then

$$\int_{0^+}^{1} \left| \sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2} e^{iu(k-k')} \right| \frac{du}{|u|^{\delta+\delta'+1}} < +\infty,$$

since $\delta + \delta' < 2$. Moreover, since $\delta + \delta' > 0$ the integral (A.28) is convergent at infinity and therefore each term of line (A.26) is an $O(N^{-j-\delta-\delta'})$. It remains to bound (A.27). Each integral of (A.27) is bounded by

$$CN^{-\delta-\delta'}\left|\sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2}\right| \int_{\mathbb{R}} |e((k+p)u)e((k'+p)u)| \frac{du}{|u|^{\delta+\delta'+1}}.$$

As $|u| \to \infty$, $e((k+p)u) = O(1)$ hence

$$\int_1^{+\infty} |e((k+p)u)e((k'+p')u)| \frac{du}{|u|^{\delta+\delta'+1}} = O(1),$$

and as $|u| \to 0$, $e((k+p)u) = O((k+p)u$ hence

$$\int_{0^+}^{1} |e((k+p)u)e((k'+p')u)| \frac{du}{|u|^{\delta+\delta'+1}} = O(pp'),$$

Since $p < N$, (A.27) is bounded by $O(N^{-\delta-\delta'})$. Lemma A.2.1 is proved for $p = p'$.

It remains to prove Lemma A.2.1 when $p \neq p'$. Expression (A.26) leads to the integral factor

$$\int_{\mathbb{R}} e^{iu(p-p')} \left[\sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2} e^{iu(k-k')}\right] \frac{\partial^{j_1}}{\partial t^{j_1}} S\left(\frac{p}{N}, Nu\right) \frac{\partial^{j_2}}{\partial t^{j_2}}$$
$$S'\left(\frac{p'}{N}, Nu\right) N du.$$

We integrate by parts twice and this gives

$$\int_{\mathbb{R}} \frac{e^{iu(p-p')}}{(p-p')^2} \frac{\partial^2}{\partial u^2} \left[\left(\sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2} e^{iu(k-k')}\right) \frac{\partial^{j_1}}{\partial t^{j_1}} S\left(\frac{p}{N}, Nu\right) \frac{\partial^{j_2}}{\partial t^{j_2}}\right.$$
$$\left. S'\left(\frac{p'}{N}, Nu\right)\right] N du.$$

To prove that the previous integral converges, and that all terms coming from integrated terms in the integration by parts vanish, we only have to prove the absolute convergence of the terms given by the second derivative with respect to $u$ of

$$\psi(u, j_1, j_2) \frac{\partial^{j_1}}{\partial t^{j_1}} S\left(\frac{p}{N}, Nu\right) \frac{\partial^{j_2}}{\partial t^{j_2}} S'\left(\frac{p'}{N}, Nu\right), \tag{A.29}$$

where $\psi(u, j_1, j_2) = \sum_{k,k'=0}^{K} a_k a_{k'} k^{j_1} k'^{j_2} e^{iu(k-k')}$. Clearly, as $|u|$ goes to $\infty$, $\left|\frac{\partial^i}{\partial u^i} \psi(u, j_1, j_2)\right| = O(1)$ for $i = 0, 1, 2$. This implies the convergence of (A.29) as

$|u| \to \infty$. To have convergence when $|u| \to 0$ let us remark that $\left|\frac{\partial^i}{\partial u^i}\psi(u, j_1, j_2)\right|$ $= O(|u|^{4-i})$, when $|u| \to 0$. Then

$$\left|\frac{\partial^{i_1}\psi(u, j_1, j_2)}{\partial u^{i_1}} \frac{\partial^{i_2+j_1}}{\partial t^{j_1}\partial u^{i_2}} S\left(\frac{p}{N}, Nu\right) \frac{\partial^{i_3+j_2}}{\partial t^{j_2}\partial u^{i_3}} S'\left(\frac{p'}{N}, Nu\right)\right| \\ \leq \frac{C|u|^{4-(i_1+i_2+i_3)-(\delta+\delta'+1)}}{N^{\delta+\delta'+1+i_2+i_3}},$$

for $i_1 + i_2 + i_3 = 2$. Hence each term of the first line (cf. (A.26)) of the expansion of $I(S, S')_{p,p'}$ is of order $\frac{1}{N^{\delta+\delta'+j}(p-p')^2}$. We use a similar upper bound $O(N^{-\delta-\delta'-2})$ for the second line (cf. (A.27)) of the expansion of $I(S, S')_{p,p'}$. Since $p, p' < N$, we have proved Lemma A.2.1 for $p \neq p'$.

A second technical lemma relates the asymptotic behavior of $I(S, S)_{p,p'}$ when $S(t, \xi) = \frac{a(t)}{|\xi|^{H+\frac{1}{2}}}$ to the function

$$F_\gamma(x) = \int_{\mathbb{R}} \sum_{k,k'=0}^{K} a_k a_{k'} \frac{e^{i(x+k-k')u}}{|u|^{\gamma+1}} du.$$

**Lemma A.2.2** *If $a$ is $C^2$ and $S(t, \xi) = \frac{a(t)}{|\xi|^{H+1/2}}$*

$$I(S, S)_{p,p'} = N^{-\delta-\delta'} a(p\Delta)a(p'\Delta)F_{\delta+\delta'}(p-p') \\ + O\left(\frac{1}{N^{\delta+\delta'+1}(1+(p-p')^2)}\right).$$

*Proof of Lemma A.2.2*: To begin with, we use the Taylor expansion of $a$ at order 2 to get the expansions (A.26) and (A.27)

$$I_{p,p'} = \int_{\mathbb{R}} \left[\sum_{k,k'=0}^{K} a_k a_{k'} e((k+p)u)\overline{e}((k'+p')u)\right] \frac{a(p\Delta)a(p'\Delta)}{N^{\delta+\delta'}|u|^{\delta+\delta'+1}} du \\ + O\left(\frac{1}{N^{\delta+\delta'+1}(1+(p-p')^2)}\right).$$

Using

$$\sum_{k,k'=0}^{K} a_k a_{k'} e((k+p)u)\overline{e}((k'+p')u) = \left(\sum_{k,k'=0}^{K} a_k a_{k'} e^{i(k-k')u}\right) e^{i(p-p')u},$$

and Lemma A.2.2 follows.

Now we describe the asymptotic behavior of the expectation and variance of the $V_N$ as $N$ goes to infinity.

**Proposition A.2.1** *As $N \to \infty$ the following convergences hold*

$$N^{2H-1}\mathbb{E}V_N \to F_{2H}(0)\int_0^1 a^2(t)dt, \tag{A.30}$$

$$N^{4H-1}var(V_N) \to 2\sum_{q=-\infty}^{\infty} F_{2H}^2(q)\int_0^1 a^4(t)dt. \tag{A.31}$$

*Proof of Proposition A.2.1*: Let us recall that (A.23) expresses $\mathbb{E}V_N$ as a sum of terms that can be written as $I(S, S')_{p,p'}$. Using expansion (5.3), each integral

$$\int_{\mathbb{R}} \sum_{k,k'=0}^{K} a_k a_{k'} g\left(\frac{k+p}{N}, \xi\right) g\left(\frac{k'+p}{N}, \xi\right) e\left(\xi\frac{k+p}{N}\right) \overline{e}\left(\xi\frac{k'+p}{N}\xi\right) d\xi$$

can be written

$$\begin{aligned} I = I_{p,p}&\left(\frac{a(t)}{|\xi|^{H+1/2}}, \frac{a(t)}{|\xi|^{H+1/2}}\right) \\ &+ 2I_{p,p}\left(\frac{a(t)}{|\xi|^{H+1/2}}, \varepsilon(t,\xi)\right) + I_{p,p}(\varepsilon(t,\xi), \varepsilon(t,\xi)), \end{aligned} \tag{A.32}$$

where a change of variables $u = \dfrac{\xi}{N}$ has been performed.

Clearly, $I_{p,p}\left(\dfrac{a(t)}{|\xi|^{H+1/2}}, \dfrac{a(t)}{|\xi|^{H+1/2}}\right)$ is the preponderant term as $N \to \infty$. Applying Lemma A.2.2 to this term and Lemma A.2.1 to the others, we get

$$\mathbb{E}V_N = \sum_{p=0}^{N-K}\left[N^{-2H}a^2\left(\frac{p}{N}\right)F_{2H}(0) + O\left(N^{-2H-1}\right)\right]. \tag{A.33}$$

By standard results on Riemann's sums

$$\frac{1}{N}\sum_{p=0}^{N-K} a^2\left(\frac{p}{N}\right) = \int_0^1 a^2(t)dt + O\left(\frac{1}{N}\right).$$

And the first part of Proposition A.2.1 follows.

Let us now prove the asymptotic behavior of the variance of the $V_N$, which is quite similar to the previous proof. Expansion (5.3), Lemmas A.2.1 and A.2.2 imply

$$\mathrm{var}(V_N) = 2\sum_{p,p'=0}^{N-K}\left\{N^{-2H} a\left(\frac{p}{N}\right) a\left(\frac{p'}{N}\right) F_{2H}(p-p') \right.$$
$$\left. + O\left(\frac{1}{N^{2H+1}}\right)(1+(p-p')^2)^{-1}\right\}^2.$$

We focus on the main term $2N^{-4H}\sum_{p,p'=0}^{N-K} a^2\left(\frac{p}{N}\right) a^2\left(\frac{p'}{N}\right) F_{2H}^2(p-p')$. Set $q = p - p'$ and $q' = p + p'$. The sum is then split for $|q| < Q$ and $|q| \geq Q$, where $Q$ is prescribed later. The second part is then

$$N^{-4H}\sum_{q'=0}^{2(N-K)}\sum_{|q|\geq Q} a^2\left(\frac{q+q'}{2N}\right) a^2\left(\frac{q-q'}{2N}\right) F_{2H}^2(q).$$

Since $|F_{2H}(q)| \leq \dfrac{C}{1+q^2}$, this sum can be bounded by

$$O\left(N^{-4H}\sum_{q'=0}^{2(N-K)}\sum_{|q|\geq Q}\frac{1}{(1+q^2)^2}\right).$$

Fix $Q$ large enough, this sum is then less than $O\left(\dfrac{1}{N^{4H-1}Q^3}\right)$. Consider now the second part $T = N^{-4H}\sum_{q'=0}^{2(N-K)}\sum_{|q|\leq Q} a^2\left(\frac{q+q'}{2N}\right) a^2\left(\frac{q-q'}{2N}\right) F_{2H}^2(q)$. Permuting the sums and using standard results on Riemann's sums,

$$T = \frac{1}{N^{4H-1}}\sum_{|q|\leq Q} F_{2H}^2(q)\left(\int_0^1 a^2\left(s+\frac{q}{2N}\right) a^2\left(s-\frac{q}{2N}\right) ds + O\left(\frac{1}{N}\right)\right).$$

By a Taylor's expansion of $a$, we get

$$T = \frac{1}{N^{4H-1}}\sum_{|q|\leq Q} F_{2H}^2(q)\left(\int_0^1 a^4(s) ds + O\left(\frac{q}{N}\right)\right).$$

Since $|F_{2H}(q)| \leq \dfrac{C}{1+q^2}$, $T$ is asymptotically equivalent to

$$\frac{1}{N^{4H-1}} \int_0^1 a^4(s)ds \sum_{q=-\infty}^{q=+\infty} F_{2H}^2(q).$$

Hence

$$\text{var}(V_N) = \frac{2}{N^{4H-1}} \int_0^1 a^4(s)ds \sum_{q=-\infty}^{q=+\infty} F_{2H}^2(q) + O\left(\frac{1}{N^{3H+\eta-1}}\right),$$

and Proposition A.2.1 is proved.

We can now prove the almost sure convergence and central limit theorem for generalized quadratic variations.

**Proposition A.2.2** *The following limit holds*

$$\lim_{N\to\infty} \frac{V_N}{\mathbb{E}V_N} = 1 \quad a.s., \tag{A.34}$$

*and*

$$\frac{V_N - \mathbb{E}V_N}{\sqrt{var(V_N)}} \textit{converges to a centered Gaussian variable} \tag{A.35}$$

*as* $N \to \infty$.

*Proof of PropositionA.2.2*: The generalized quadratic variation $V_N$ can be written

$$\begin{aligned} V_N &= \sum_{p=0}^{N-K} \left( \sum_{k=0}^{K} a_k X\left(\frac{k+p}{N}\right)\right)^2 \\ &= Tr({}^tYY). \end{aligned}$$

where $Y$ is the $\mathbb{R}^{N-K+1}$ valued random column vector defined by

$$Y_{p,N} = \sum_{k=0}^{K} a_k X\left(\frac{k+p}{N}\right).$$

Since $M = \mathbb{E}(Y^tY)$ is a $(N-K+1)\times(N-K+1)$ symmetric non-negative matrix, we can find a diagonal matrix $Diag(\lambda_{p,N})$ with non negative eigenvalues $\lambda_{p,N}$ of $M$ on the diagonal, and an orthogonal $(N-K+1)\times(N-K+1)$ matrix $O$ such that

$$Diag(\lambda_{p,N}) =^t OMO.$$

For each $N$ let us suppose that the rank of $M$ is $r_N$ and assume that $\lambda_{p,N} = 0$ for $p > r_N$. Denote by $Diag\left(\frac{1}{\sqrt{\lambda_{p,N}}}\right)$ the diagonal matrix with diagonal term $\frac{1}{\sqrt{\lambda_{p,N}}}$ if $p \leq r_N$, and 0 elsewhere. Let $\xi$ be the random vector defined by

$$\xi_p = \left(Diag\left(\frac{1}{\sqrt{\lambda_{p,N}}}\right) OY\right)_p \quad \text{if} \quad p \leq r_N$$

and by $\xi_{r_N+1}, \ldots, \xi_{N-K}$ arbitrary independent centered standard Gaussian random variables independent of $\xi_p$ for $p \leq r_N$. The $\xi_p$ $(p = 0, \ldots, N-K)$ are identically independent centered Gaussian variables with variance 1. Then

$$\begin{aligned} V_N &= Tr({}^tYY) \\ &= \sum_{p=0}^{N-K} \lambda_{p,N} \xi_p^2 \,. \end{aligned} \tag{A.36}$$

Hence

$$\mathrm{var}(V_N) = \mathrm{var}(\xi_0^2) \sum_{p=0}^{N-K} \lambda_{p,N}^2 .$$

Then

$$\begin{aligned} \mathbb{E}(V_N - \mathbb{E}V_N)^4 &= \mathbb{E}(\xi_0^2 - 1)^4 \sum_{p=0}^{N-K} \lambda_{p,N}^4 + \sum_{p,p'=0}^{N-K} \mathbb{E}(\xi_0^2 - 1)^2 \lambda_{p,N}^2 \lambda_{p',N}^2 \\ &\leq C \left(\sum_{p=0}^{N-K} \lambda_{p,N}^2\right)^2 \\ &= C \mathrm{var}^2 V_N . \end{aligned}$$

Using Proposition A.2.1 and Borel-Cantelli's Lemma, the almost sure convergence is proved.

Let us now prove the second part of Proposition A.2.2, which describes the rate of convergence in (A.34). The following central limit theorem is used

**Lemma A.2.3** *Consider the sequence of variable $S_N$ defined by:*

$$S_N = \sum_{p=0}^{N-K} \lambda_{p,N} (\xi_p^2 - 1),$$

*where the $\xi_p$ are i.i.d. centered normalized Gaussian variables and the $\lambda_{p,N}$ are positive. If $\max_{p=0,\ldots,N-K} \lambda_{p,N} = o(\sqrt{var(S_N)})$, then $S_N/\sqrt{var(S_N)}$ converges in distribution to a centered normalized Gaussian variable.*

Lemma A.2.3 is easily proved using a Taylor expansion of the characteristic function of $S_N$. One can also apply the central limit theorem with Lindeberg condition. (See e.g. [57] example (f) in Chap. VIII.4.) Consequently we only have to prove that $\max\limits_{p=0,\ldots,N-K} \lambda_{p,N} = o(\sqrt{\operatorname{var}(V_N)})$. In order to bound the largest eigenvalue of the correlation matrix, a classical linear algebra lemma is used, that claims that the largest eigenvalue of a matrix $C$ is bounded by $\max\limits_i \sum\limits_j |C_{i,j}|$. Applying this Lemma to matrix $M = (m_{p,p'})$ leads to consider

$$\begin{aligned} m_{p,p'} &= \mathbb{E}(Y_{p,N} Y_{p',N}) \\ &= \mathbb{E} \sum_{k,k'=0}^{N-K} a_k a_{k'} X\left(\frac{k+p}{N}\right) X\left(\frac{k'+p'}{N}\right). \end{aligned}$$

By Lemma A.2.1

$$\begin{aligned} \sum_{p'=0}^{N-K} |\mathbb{E} \sum_{k,k'=0}^{K} a_k a_{k'} X\left(\frac{k+p}{N}\right) X\left(\frac{k'+p'}{N}\right)| &\le \sum_{p'=0}^{N-K} \frac{C}{N^{2H}(1+(p-p')^2)} \\ &\le \frac{C}{N^{2H}}. \end{aligned}$$

Hence

$$\max_{p=0,\ldots,N-K} \sum_{p'=0}^{N-K} |\mathbb{E} \sum_{k,k'=0}^{K} a_k a_{k'} X\left(\frac{k+p}{N}\right) X\left(\frac{k'+p'}{N}\right)| = o(\sqrt{\operatorname{var}(V_N)}) .$$

and Lemma A.2.3 is applied to get the convergence in distribution of $\dfrac{V_N - \mathbb{E}V_N}{\sqrt{\operatorname{var}(V_N)}}$ in Proposition A.2.2. With the same arguments, the asymptotic normality of a linear combination of $V_N$ and $V_{N/2}$ with positive weights is obtained. The same can be of course done with negative weights.

Denote by $P_n$ (resp. $\Psi_n$) the distribution (resp. the Laplace transform) of the couple $(V_N, V_{N/2})$. We know that $\Psi_n(x, y)$ converges to some $\Psi(x, y)$ when $xy \ge 0$, where $\Psi$ is the Laplace transform of a Gaussian distribution $\Phi$. Assume that $P_n$ converges to a distribution $P$. The Laplace transform of $P$ is equal to $\Psi(x, y)$ when $xy \ge 0$. The Laplace transform is defined on a convex set, so this set is the whole plan. It is analytic in its support, so it is equal to $\Psi$ on the whole plan. The unique possible limiting distribution is $\Psi$. But the sequence is tight, because its marginals are tight, so it converges to the Gaussian distribution $\Psi$.

The previous results are then applied to the estimator of $H$. Its convergence (a.s.) to $H$ is a consequence of the logarithmic behavior of $\mathbb{E}V_N$ and of (A.34). To estimate the rate of convergence of $\widehat{H}_N$ to $H$, we obtain a central limit theorem for $\sqrt{N}(\widehat{H}_N - \mathbb{E}\widehat{H}_N)$, which is a function of the couple $(V_N, V_{N/2})$. We deduce this asymptotic normality from the remark above and Theorem 3.311 of [45]. The proof is completed by showing that the bias term $|\mathbb{E}\widehat{H}_N - H|$ is of order $O(N^{-\eta})$ which is preponderant since $\eta > 1/2$.

# Appendix B

## B.1 Solution to Exercise 2.3.1

Let us suppose $\mathbb{E}X^2 > 0$ and let us denote by $Z = Y + cX$ where $c = -\frac{\mathbb{E}XY}{E\mathbb{X}^2}$. This choice of $c$ implies that $X$ and $Z$ are independent. Then

$$\begin{aligned}\mathbb{P}(X \geq a,\ Y \geq b) &\leq \mathbb{P}(X \geq a,\ Z \geq b + ca)\\ &\leq \mathbb{P}(X \geq a)\mathbb{P}(Z \geq b)\\ &\leq \mathbb{P}(X \geq a)\mathbb{P}(Y \geq b),\end{aligned}$$

where the last inequality comes from $\mathbb{E}Z^2 \leq \mathbb{E}Y^2$.

## B.2 Solution to Exercise 2.3.2

1. By definition

$$\begin{aligned}\mathbb{P}(|X| > r) &= 2\int_r^{+\infty} \exp(-\frac{t^2}{2})dt/\sqrt{2\pi}\\ &= 2\int_r^{+\infty} t^{-1}\exp(-\frac{t^2}{2})tdt/\sqrt{2\pi}.\end{aligned}$$

Then one can use integration by parts to get

$$\begin{aligned}\mathbb{P}(|X| > r) &= \frac{2\exp(-\frac{r^2}{2})}{\sqrt{2\pi}r} - 2\int_r^{+\infty} t^{-2}\exp(-\frac{t^2}{2})dt/\sqrt{2\pi}\\ &= \frac{2\exp(-\frac{r^2}{2})}{\sqrt{2\pi}r} - \frac{2\exp(-\frac{r^2}{2})}{\sqrt{2\pi}r^3} + 2\int_r^{+\infty} 3t^{-4}\exp(-\frac{t^2}{2})dt/\sqrt{2\pi}.\end{aligned}$$

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Mathématiques et Applications 73, DOI: 10.1007/978-3-642-36739-7,

Since the integrals in the two previous lines are obviously non negative we have proved the inequalities (2.78).

2. Since $X_j/\sqrt{\mathbb{E}X_j^2}$ have a unit variance, one can assume that $\mathbb{E}X_j^2 = 1$ for all $j \in \mathbb{N}$. Because of the previous question we know that for $n \geq 2$

$$\mathbb{P}(|X_j| > \sqrt{2\lambda \log(n)}) \leq \frac{\sqrt{2}\exp\left(-(\sqrt{2\lambda \log(n)})^2/2\right)}{\sqrt{2\pi\lambda \log(n)}}.$$

Then for $\lambda > 1,\ n \geq 2$

$$\mathbb{P}(|X_j| > \sqrt{2\lambda \log(n)}) \leq \frac{n^{-\lambda}}{\sqrt{\pi \log(2)}}$$

and inequality (2.79) is proved.

3. For $\lambda > 1$ $\sum_{n\in\mathbb{N}} \mathbb{P}(\sup_{1\leq j\leq n} |X_j| > \sqrt{2\lambda \log(n)}) < +\infty$ and, because of Borel-Cantelli Lemma, there exists almost surely a random integer $n_0(\omega)$ such that

$$\sup_{1\leq j\leq n} |X_j| \leq \sqrt{2\lambda \log(n)}$$

for $n > n_0(\omega)$. Let

$$M_n = \sup_{1\leq j\leq n} |X_j|/\sqrt{\log(n)}$$

for $n \geq 2$, and

$$C(\omega) = \max(\sqrt{2\lambda}, \sup_{2\leq j\leq n_0(\omega)} M_n),$$

then inequality (2.80) holds almost surely.

4. One can apply the previous result to

$$\sup_{1\leq j'\leq j,\ -|k|\leq k'\leq |k|} |\eta_{j',k'}|$$

and there exist positive random variable $C, C'$ such that

$$\begin{aligned}|\eta_{j,k}| &\leq \sup_{1\leq j'\leq j,\ 1\leq k'\leq k} |\eta_{j',k'}| \\ &\leq C(\omega)\log((j+1)(2|k|+1))^{1/2} \\ &\leq C'(\omega)(\log(j+1)^{1/2} + \log(|k|+1)^{1/2})\end{aligned}$$

almost surely.

## B.3 Solution to Exercise 2.3.3

1. For every $i \neq j \in I$ let us consider $\sigma(H_i)$-random variables $Y_i$ then $(Y_i, Y_j)$ is a Gaussian random vector and the characteristic function of $(Y_i, Y_j)$ is the product of the characteristic functions of $Y_i$ and $Y_j$ if and only if $Y_i$ and $Y_j$ are orthogonal and their covariance matrix is diagonal.
2. If $X_i$ is in $H_i$ they are $\sigma(H_i)$-measurable. Hence by conditional independence

$$\mathbb{E}(X_i\overline{X_j}|\sigma(K)) = \mathbb{E}(X_i|\sigma(K))\overline{\mathbb{E}(X_j|\sigma(K))}.$$

   But

$$\mathbb{E}(\mathbb{E}(X_i\overline{X_j}|\sigma(K))) = \mathbb{E}(X_i\overline{X_j})$$

   and

$$\mathbb{E}(X_i|\sigma(K)) = P_K(X_i).$$

   Hence the necessity of the condition is proved.
3. If $H_i$ for $i \in I$ are orthogonally secant with respect to $K$ then the orthogonal complement $H_i'$ of $K$ in $H_i$ are orthogonal and by the first part of the exercise the corresponding sigma field $\sigma(H_i')$ and $\sigma(K)$ are independent. Since $\sigma(H_i)$ is the smallest field containing $\sigma(H_i')$ and $\sigma(K)$; $\sigma(H_i)$s are conditionally independent with respect to $\sigma(K)$.

## B.4 Solution to Exercise 2.3.4

1 and 2. A covariance function is non-negative definite. It follows immediately from the definition of a non-negative definite function that $\lambda R$ and $R + Q$ are non-negative.

3. Let us now study $RQ$. For $t_1, \ldots, t_n \in T$, define the matrix $\Sigma_{2n}$ by:

$$\begin{aligned}
&\Sigma_{2n}(i, j) = R(t_i, t_j) \quad 1 \le i, j \le n,\\
&\Sigma_{2n}(i, j) = 0 \quad 1 \le i \le n, n < j \le 2n,\\
&\Sigma_{2n}(i, j) = 0 \quad n < i \le 2n, 1 \le j \le n,\\
&\Sigma_{2n}(i, j) = Q(t_i, t_j) \quad n < i \le 2n, n < j \le 2n.
\end{aligned}$$

The matrix $\Sigma_{2n}$ is clearly a symmetric non-negative definite matrix. It is the covariance matrix of the centered Gaussian vector

$$(X_1^R, \ldots, X_n^R, X_{n+1}^Q, \ldots, X_{2n}^Q),$$

where $(X_1^R, \ldots, X_n^R)$ is a centered Gaussian random vector with covariance matrix $R(t_i, t_j)$ and $(X_{n+1}^Q, \ldots, X_{2n}^Q)$ is a centered Gaussian random vector with covariance matrix $Q(t_i, t_j)$ independent of $X^R$. For $i = 1, \ldots, n$, define $Y_i = X_i^R X_{i+n}^Q$. Then $\mathbb{E}(Y_i Y_j) = R(t_i, t_j)Q(t_i, t_j)$ and

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j R(t_i, t_j) Q(t_i, t_j) = \mathbb{E}\left(\sum_{i=1}^{n} \lambda_i Y_i\right)^2 \geq 0 .$$

Function $RQ$ is a non-negative definite function, it is therefore a covariance function.

## B.5 Solution to Exercise 2.3.5

1. a. Function $r(t)$ is real, non-negative, $r(0) = 1$, $r(-t) = r(t)$ and convex. By Pólya's criterion, $r$ is a characteristic function. By Bochner's theorem, it is a non-negative definite function. It is therefore a covariance function.
   b. We know from Corollary 2.1.1 that $e^{-|t|^\alpha}$ is a characteristic function since $0 < \alpha \leq 2$. It is therefore a covariance function. Let us check that $t \to \cos(t)$ is a non-negative definite function

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \cos(t_i - t_j) = |\sum_{i=1}^{n} \lambda_i e^{it_i}|^2 .$$

   By exercise 2.3.4, the product $e^{-|t|^\alpha} \cos(t)$ is a covariance function.
   c. Function $t \to 1/(1 + t^2)$ is the Fourier transform of the positive function $\pi e^{-|\lambda|}$, it is a covariance function. By exercise 2.3.4, the product $1/(1+t^2)^n$ is a covariance function.
   d. The modified Bessel function of the second kind can be represented as [1]

$$K_\nu(t) = \frac{\Gamma(\nu + 1/2)2^\nu}{\sqrt{\pi}} \int_0^{+\infty} \frac{\cos(ut)}{(1 + u^2)^{\nu+1/2}} du.$$

   Then

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j K_\nu(t_i - t_j)$$
$$= \frac{\Gamma(\nu + 1/2)2^\nu}{\sqrt{\pi}} \int_0^{+\infty} \frac{\sum_{i=1}^{n} |\lambda_i e^{iut_i}|^2}{(1 + u^2)^{\nu+1/2}} du \geq 0.$$

2. Since $f(r(t)) = \sum_{n\geq 1} f_n r^n(t)$, it follows directly from exercise 2.3.4.
3. Let $t_1 = 0.11$ and $t_2 = 0.1$. Let us consider for instance a function $H$ such that $H(t_1) = 0,2$, $H(t_2) = 0,9$, $\lambda_1 = -0.7$, $\lambda_2 = 1$. One then checks that $\sum_{i,j=1}^{2} \lambda_i \lambda_j R(t_i, t_j)$ is negative. So the answer is no in general. As far as we know it is an open question to find necessary and sufficient conditions on the function $H$ to have a covariance function.

## B.6 Solution to Exercise 2.3.6

1. Since

$$
\begin{aligned}
d(t,s) &= \mu(H_t \Delta H_s),\\
d(t,s) &= \int_{\mathbb{S}^2} |1_{H_t} - 1_{H_s}| d\mu\\
&= \int_{\mathbb{S}^2} |1_{H_t} - 1_{H_s}|^2 d\mu.
\end{aligned}
$$

With $\sum_1^n \lambda_i = 0$

$$
\sum_{i,j=1}^{n} \lambda_i \lambda_j d(t_i, t_j) = -2 \int_{\mathbb{S}^2} |\sum_{i=1}^{n} \lambda_i 1_{H_{t_i}}|^2 d\mu \leq 0.
$$

Function $d$ is of negative type, it follows from Schoenberg's theorem that $R_{1/2}$ is a covariance function.

**NB:** this is the way P. Lévy introduces the Spherical Brownian [96].

2. Formula (2.7) implies that $d^{2H}$ is of negative type for $H \leq 1/2$. Schoenberg's theorem implies that $R_H$ is a covariance function for $H \leq 1/2$.
3. Let $t_1, t_2, t_3, t_4$ be four points on a great circle such that

$$
\begin{aligned}
d(t_1, t_2) &= d(t_1, t_3)\\
&= d(t_2, t_4)\\
&= d(t_3, t_4)\\
&= 1,
\end{aligned}
$$

and

$$
\begin{aligned}
d(t_1, t_4) &= d(t_2, t_3)\\
&= 2\,.
\end{aligned}
$$

Let $\lambda_1 = \lambda_4 = 1$ and $\lambda_2 = \lambda_3 = -1$. Then $\sum_{i,j=1}^{4} \lambda_i \lambda_j d(t_i, t_j) = -8 + 42^{2H}$ is positive for $H > 1/2$.
**NB:** the same is true for the hyperbolic space [72].
4. It follows directly from Schoenberg's theorem.
5. It is the covariance of a fractional Brownian motion indexed by $\mathbb{R}^3$ and restricted to the sphere.

## B.7 Solution to Exercise 2.3.7

Since $X$ is Gaussian, $B_X$ is itself Gaussian. Take $t \in [0, 1]$. Let $Z$ be equal to $X(t) - B_X(t)$. By standard results on Gaussian regression, there exists two constants $\alpha$ and $\beta$ such that $Z = \alpha + \beta X(1)$ and $Z$ and $(1, X(1))$ are independent. First we obtain $\alpha = \mathbb{E}\mathbb{Z} = \mathbb{E}X(t) = 0$. Second, since $\mathbb{E}((X(t) - Z)X(1)) = 0$, we obtain $\beta = R(1, t)$.

For instance, when $X$ is fractional Brownian motionof index $H$, (See the definition in Corollary 3.2.1.) fractional Brownian Bridge is given by:

$$B_X(t) = X(t) - \frac{1 + t^{2H} - |1 - t|^{2H}}{2} X(1).$$

## B.8 Solution to Exercise 2.3.8

If you take two finite dimensional distributions of $X$ and $Y$ the set where the random vectors are different is negligible, hence their distributions are equal.

## B.9 Solution to Exercise 2.3.9

Write the variance of the increments of the sum. Since the processes $X$ and $Y$ are independent

$$\mathbb{E}((X_t + Y_t) - (X_s + Y_s))^2 = \mathbb{E}((X_t - X_s)^2 + \mathbb{E}((Y_t - Y_s)^2.$$

## B.10 Solution to Exercise 2.3.10

Let $\varphi_n, n \geq 0$ and $\lambda_n^2, n \geq 0$ be the eigenfunctions and eigenvalues of the covariance operator of $X$. $K_X$ can be defined as follows

$$K_X = \left\{ f = \sum_{n \geq 1} f_n \varphi_n, \quad \sum_{n \geq 1} \frac{f_n^2}{\lambda_n^2} < \infty \right\}.$$

This is the same condition as Equation (2.24).

## B.11 Solution to Exercise 3.6.1

1. If $\varepsilon > 0$,

$$\begin{aligned} S(\varepsilon t) &= \sup_{0 < s \leq \varepsilon t,\ s \in \mathbb{Q}} X(s) \\ &= \sup_{0 < \varepsilon u \leq \varepsilon t,\ \varepsilon u \in \mathbb{Q}} X(\varepsilon u) \\ &\overset{(d)}{=} \varepsilon^H \sup_{0 < u \leq t,\ u \in \mathbb{Q}} X(u) \\ &= \varepsilon^H S(t). \end{aligned}$$

2. If $\varepsilon > 0$,

$$I(\varepsilon t) = \inf\{s \in \mathbb{Q} \text{ s.t. } X(s) \geq \varepsilon t\}.$$

Let $\mu = \varepsilon^{1/H}$ then

$$\begin{aligned} I(\varepsilon t) &= \inf\{s \in \mathbb{Q} \text{ s.t. } \left(\tfrac{1}{\mu}\right)^H X(s) \geq t\} \\ &\overset{(d)}{=} \inf\{s \in \mathbb{Q} \text{ s.t.} X\left(\left(\tfrac{1}{\mu}\right) s\right) \geq t\} \\ &= \tfrac{1}{\mu} \inf\{s \in \mathbb{Q} \text{ s.t. } X(s) \geq t\}. \end{aligned}$$

One can check that $I$ is non-decreasing since if $t > t'$

$$\{s \in \mathbb{Q} \text{ s.t. } X(s) \geq \varepsilon t\} \subset \{s \in \mathbb{Q} \text{ s.t. } X(s) \geq \varepsilon t'\},$$

and right-continuity of $I$ follows from

$$\{s \in \mathbb{Q} \text{ s.t. } X(s) \geq \varepsilon t\} = \bigcap_{n > 0} \{s \in \mathbb{Q} \text{ s.t. } X(s) \geq \varepsilon t + 1/n\}.$$

3. if $X$ and $Y$ are independent then

$$(X, Y(\varepsilon.)) \overset{(d)}{=} (X, \varepsilon^H Y).$$

Let us take a bounded measurable function $f$

$$\begin{aligned}\mathbb{E}f(X(Y(\varepsilon t))) &= \mathbb{E}(\mathbb{E}(f(X(Y(\varepsilon t)))|Y)) \\ &= \mathbb{E}(\mathbb{E}(f(X(\varepsilon^{H'}Y(t)))|Y))\end{aligned}$$

because of the identity in distribution just recalled. Then

$$\begin{aligned}\mathbb{E}f(X(Y(\varepsilon t))) &= \mathbb{E}(\mathbb{E}f(\varepsilon^{HH'}X(Y(t)))|Y)) \\ &= \mathbb{E}f(\varepsilon^{HH'}X(Y(t))).\end{aligned}$$

## B.12 Solution to Exercise 3.6.2

Function $x \to \log|t-x| - \log|x|$ belongs to $L^2$: process $X$ is well defined. $X$ is a centered Gaussian process.

$$\mathbb{E}(X(t) - X(s))^2 = \int_{\mathbb{R}} (\log|t-x| - \log|s-x|)^2 dx.$$

The change of variables $-(t-s)v = s - x$ leads to:

$$\mathbb{E}(X(t) - X(s))^2 = |t-s| \int_{\mathbb{R}} (\log|1-v| - \log|v|)^2 dv.$$

$X$ is a Brownian motion.

## B.13 Solution to Exercise 3.6.3

1. Because of the exercise (2.83) the Markov property is equivalent to

$$\mathbb{E}(P_{\sigma(X_t)}(U)P_{\sigma(X_t)}(V)) = \mathbb{E}(UV) \tag{B.1}$$

for every $U \in \sigma(X_s, s \le t)$ and every $V \in \sigma(X_s, s \ge t)$. In (B.1) $P_{\sigma(X_t)}$ is the orthogonal projection on the vector space spanned by $X_t$. Then (B.1) can be rewritten

$$\mathbb{E}(P_{\sigma(X_t)}(X_s)P_{\sigma(X_t)}(X_u)) = \mathbb{E}(X_s X_u) \tag{B.2}$$

for every $s \le t \le u$. The characterization of the Markov property follows from:

$$P_{\sigma(X_t)}(X_v) = \frac{K(v,t)}{K(t,t)} X_t$$

if $K(t,t) > 0$ and is vanishing otherwise.

2. Unless $H = 1/2$ the Eq. (3.85) is not fulfilled for

$$K(s,t) = \frac{C_H}{2}\left\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right\}.$$

One can take for instance $a > 0$ and $s = a$, $t = 2a$, $u = 3a$ to check this last claim.

3. Since $K$ is positive definite then

$$0 \le K(t,t) = \phi(t)\overline{\psi(t)} = |\psi(t)|^2 \left(\frac{\phi}{\psi}\right)(t)$$

implies that $\frac{\phi}{\psi}$ is a non-negative real valued function. Then Cauchy Schwarz inequality

$$|K(s,t)|^2 \le K(t,t)K(s,s)$$

yields that this function is non-decreasing.

4. A straightforward computation using the covariance of the Brownian motion.
5. If the condition (3.86) is fulfilled then $K$ is a continuous function from $I \times I$ and $X$ is $L^2$ continuous, it is straightforward to derive (B.2), for a covariance satisfying (3.86). Reciprocally $0 < \mathbb{E}(X_t^2) = K(t,t)$ implies that $K$ is non vanishing on the diagonal of $I \times I$. Then by continuity of $K$ it is non-vanishing on a neighborhood $N$ of the diagonal. Since for every $s < t$ one can find an increasing sequence $s = t_0 < t_1 < \ldots < t_n = t$ such that for every $i$ $(t_i, t_{i+1}) \in N$, then

$$K(s,t) = \frac{\prod_{i=0}^{n-1} K(t_i, t_{i+1})}{\prod_{i=1}^{n} K(t_i, t_i)}$$

and this implies that for every $s$ and $t$ $K(s,t) \neq 0$. Hence one can define for an arbitrary $t_0$

$$\phi(s) = \begin{cases} K(s,t_0) & \text{if } s \le t_0 \\ \frac{K(s,s)K(t_0,t_0)}{K(t_0,s)} & \text{if } s \ge t_0 \end{cases}$$

and

$$\psi(u) = \begin{cases} \frac{K(u,u)}{K(t_0,u)} & \text{if } s \le t_0 \\ \frac{K(u,t_0)}{K(t_0,t_0)} & \text{if } s \ge t_0. \end{cases}$$

They are non-vanishing continuous functions such that

$$K(s,u) = \phi(s)\overline{\psi}(u)$$

if $s \leq u$.

## B.14 Solution to Exercise 3.6.4

Since $X$ is stationary the covariance function $K(t,s) = r(t-s)$. We assume $r(0) = \mathbb{E}(X(0)^2) > 0$ or $X$ is zero for every $t$. According to question 1. of exercise 3.6.3 for all $s \leq t \leq u$

$$r(u-s) = \frac{r(t-s)r(u-t)}{r(0)}.$$

Let us define $\forall x \geq 0,\ \tilde{r}(x) = \frac{r(x)}{r(0)}$. Then $\tilde{r}$ is continuous function such that $\tilde{r}(x+y) = \tilde{r}(x)\tilde{r}(y)$. Hence there exist $\lambda \in \mathbb{R}$ such that $\tilde{r}(x) = e^{\lambda x} \quad \forall x \geq 0$. Moreover for $h > 0$ $r(-h) = \mathbb{E}X(t-h)X(t) = \mathbb{E}X(t)X(t+h) = r(h)$, then $r(t) = Ce^{\lambda|t|}$ for some non-negative constant $C$. Those processes are called Ornstein Ühlenbeck process when $\lambda < 0$.

## B.15 Solution to Exercise 3.6.5

1. There exists $C > 0$ such that $\frac{1-e^{-z}}{z^{K+1}} < \frac{C}{z^{K+1}}$ when $z \to +\infty$. Then $\frac{1-e^{-z}}{z^{K+1}} \sim \frac{1}{z^K}$ when $z \to 0$. Hence the integral defining $C(K)$ is convergent.
2. Apply the change of variable $z = xy$ to (3.87).
3. The representation formula (3.88) is obtained when one applies (3.87) to $(|t|^{2H} + |s|^{2H})^K$ and to $|t-s|^{2HK}$.
4. $R'(t,s) = |t|^{2H} + |s|^{2H} - |t-s|^{2H}$ is up to a multiplicative constant the covariance of fractional Brownian motion and hence of non-negative type. Then $\phi(t,s) = e^{R'(t,s)} - 1$ is of non-negative type because $e^z - 1$ has positive coefficients in its power series expansion in $z = 0$ and because of 2. in exercise 2.3.5.
5. Since $R$ is of non-negative type the theory for Gaussian processes yields the result.
6. We can easily check that $X(0) = 0$ (a.s.) and $R(\lambda t, \lambda s) = \lambda^{2HK} R(t,s)$ for $\lambda > 0$.
7. Fractional Brownian motion is the unique Gaussian self-similar process with stationary increments. $X$ has stationary increments if and only if $K = 1$.

8. $\mathbb{E}(X(t)-X(s))^2$
$$= 2^{1-K}|t-s|^{2HK} + (|t|^{2HK} + |s|^{2HK} - 2^{1-K}(|t|^{2H} + |s|^{2H})^K).$$

   Since $\left(|t|^{2HK} + |s|^{2HK} - 2^{1-K}(|t|^{2H} + |s|^{2H})^K\right)$ is non-negative by concavity:
$$\mathbb{E}(X(t)-X(s))^2 \le 2^{1-K}|t-s|^{2HK},$$

9. Kolmogorov-Chentsov's Theorem 2.1.7 proves that $X$ is at least $HK$ locally Hölder continuous.
10. It remains to prove that index $HK$ is the "best" Hölder exponent. Fix a point $t$. Let $\sigma_n^2 = \mathbb{E}(X(t+1/n)-X(t))^2$. There exists a positive constant $C$ such that $\sigma_n > Cn^{-HK}$ as $n \to +\infty$. Then one can show that for $\gamma > HK$
$$\lim_{n\to\infty} \frac{|1/n|^\gamma}{X(t+1/n)-X(t)} \overset{(d)}{=} 0$$

    as in (3.41). The end of the proof is similar to the the end of the proof of Theorem 3.2.3.

## B.16 Solution to Exercise 3.6.6

Since the Poisson measure are independently scattered, the process has independent increments. The mean measure of the compensated Poisson measure used to construct the random Lévy or stable measure is the Lebesgue measure, the characteristic function (2.33) implies the stationarity of the increments. Moreover, one can deduce from the Lévy Kintchine formula that the (non-random) Lévy measure of the Lévy process is $\nu$.

## B.17 Solution to Exercise 3.6.7

1. Let us compute the characteristic function of $\int f(\xi)M(d\xi)$, which is denoted by
$$\Phi(u,v) = \mathbb{E}\exp\left(i(u\Re\int f(\xi)M(d\xi) + v\Im\int f(\xi)M(d\xi))\right).$$

   Because of Definition 2.1.20 and of (2.47)

$$\begin{aligned}\Phi(u,v) &= \mathbb{E}\exp\left(i(u\int \Re(f(\xi)z+f(-\xi)\bar{z})\tilde{N}(d\xi,dz)\right.\\ &\qquad \left. +v\int \Im(f(\xi)z+f(-\xi)\bar{z})\tilde{N}(d\xi,dz))\right)\\ &= \exp\left(\int_{\mathbb{R}^d\times\mathbb{C}}[\exp(g_{u,v}(\xi,z))-1-g_{u,v}(\xi,z)]d\xi d\nu(z)\right),\end{aligned}$$

where $g_{u,v}(\xi,z)) = i(u\Re(f(\xi)z+f(-\xi)\bar{z})+v\Im(f(\xi)z+f(-\xi)\bar{z})$. Then,

$$\begin{aligned}&\log(\Phi(u,v)) =\\ &\qquad \int_{\mathbb{R}^d\times[0,2\pi]\times\mathbb{R}^+_*}[\exp(g_{u,v}(\xi,\rho e^{i\theta}))-1-g_{u,v}(\xi,\rho e^{i\theta})]d\xi d\theta d\nu_\rho(d\rho).\end{aligned}$$

If we replace $f$ by $-f$ in $g_{u,v}$, one can rewrite the product $-f(\xi)\rho e^{i\theta} = f(\xi)\rho e^{i(\theta+\pi)}$. Hence, the characteristic function of $\int f(\xi)M(d\xi)$ is the same than the characteristic function of $-\int f(\xi)M(d\xi)$, because of the invariance of the measure $d\theta$ with respect to the translation of magnitude $\pi$.

2. Please note that both integrals $-\int f(\xi)M(d\xi)$ and $\int f(\xi)\exp(ia(\xi))M(d\xi)$ are real valued. Then the characteristic function of $\int f(\xi)\exp(ia(\xi))M(d\xi)$

$$\Phi(u) = \mathbb{E}\exp\left(i(u\int f(\xi)\exp(ia(\xi))M(d\xi))\right)$$

is given by

$$\begin{aligned}\int_{\mathbb{R}^d\times[0,2\pi]\times\mathbb{R}^+_*}&[\exp(iu2\Re(f(\xi)e^{ia(\xi)}\rho e^{i\theta}))-1\\ &-iu2\Re(f(\xi)e^{ia(\xi)}\rho e^{i\theta})]d\xi d\theta d\nu_\rho(d\rho).\end{aligned}$$

It is equal to $\mathbb{E}\exp\big(i(u\int f(\xi)M(d\xi))\big)$ because of the invariance of the measure $d\theta$ with respects to the translation of magnitude $a(\xi)$.

3. Because of (2.48)

$$\mathbb{E}|\int_{\mathbb{R}^d} f(\xi)M(d\xi)|^2 = \int_{\mathbb{R}^d}|f(\xi)|^2 d\xi\int_{[0,2\pi]\times\mathbb{R}^+_*}\rho^2 d\theta d\nu_\rho(d\rho).$$

Hence we get (2.55).

## B.18 Solution to Exercise 3.6.8

1. $x \mapsto (x^2+1)^{-s/2}$ is a non-negative function integrable with respect to the Lebesgue measure on $\mathbb{R}$. Hence its Fourier transform is a continuous function $f_s(\omega)$ such that $f_s(\omega) \to 0$ when $|\omega| \to \infty$. Actually $f_s$ is a modified Bessel function, but we will not use this fact.
2. First

$$\left(|x-y|^2 + |X(x)-X(y)|^2\right)^{-s/2} = |x-y|^{-s} \int_{\mathbb{R}} e^{i\omega \frac{X(x)-X(y)}{|x-y|}} f_s(\omega) d\omega.$$

Putting $\lambda = |x-y|^{H-1}\omega$, we have

$$\mathbb{E}I_s(m) = \iint_{[0,1]^2} |x-y|^{1-H-s} \int_{\mathbb{R}} \Re\mathbb{E}\left(e^{i\lambda \frac{X(x)-X(y)}{|x-y|^H}}\right) f_s(|x-y|^{1-H}\lambda) d\lambda dx dy$$

and

$$\mathbb{E}I_s(m) = \iint_{[0,1]^2} |x-y|^{1-H-s} \int_{\mathbb{R}} \Re\mathbb{E}\left(e^{i\lambda X(1)}\right) f_s(|x-y|^{1-H}\lambda) d\lambda dx dy.$$

It follows that:

$$\mathbb{E}I_s(m) \le \sup_{\omega\in\mathbb{R}} |f_s(\omega)| \int_{\mathbb{R}} |\mathbb{E}(e^{i\lambda X(1)})| d\lambda \iint_{[0,1]^2} |x-y|^{1-H-s} dx dy.$$

3. Let us take $s$ such that $2-H > s > 1$ then $1-H-s > -1$ and $\iint_{[0,1]^2} |x-y|^{1-H-s} dx dy < \infty$. Therefore $\mathbb{E}I_s(m) < \infty$. Because of Frostman's Lemma (cf. Lemma 2.2.2) $\dim_H\{(x, X(x)),\ x \in [0,1]\} \ge s$. Hence $\dim_H\{(x, X(x)),\ x \in [0,1]\} \ge 2-H$.
4. Since $X$ is $H$-Hölder continuous one can apply Lemma 2.2.1 to get $\dim_H\{(x, X(x)),\ x \in [0,1]\} \le 2-H$.

## B.19 Solution to Exercise 3.6.9

- Upper bound.
  We split the interval [0, 1] into sub-intervals

$$[k3^{-n}, (k+1)3^{-n}),\ k = 0, \ldots, 3^n - 1.$$

We need $2^n$ sub-intervals to cover $F$. On such a sub-interval, the oscillation of $X$ is overestimated by $C3^{-nH'}$ for every $H' < H$ (see Theorem 3.2.3). We need $C2^n 3^{n(1-H')}$ squares of size $3^{-n}$ to cover $\{(x, X(x)),\ x \in F\}$. The Hausdorff dimension of $\{(x, X(x)),\ x \in F\}$ is overestimated by $1 + \log_3 2 - H$.

- Lower bound. Equation (3.52) leads to:

$$\mathbb{E}((X(t) - X(s))^2 + (t-s)^2)^{-s/2} \le C|t-s|^{1-s-H}.$$

Let $\mu$ be a probability measure supported by $F$ and

$$I_s(\mu) = \iint_{F\times F} ((X(t) - X(s))^2 + (t-s)^2)^{-s/2} d\mu(t) d\mu(s).$$

Then

$$\begin{aligned}\mathbb{E}I_s(\mu) &= \iint_{F\times F} \mathbb{E}((X(t) - X(s))^2 + (t-s)^2)^{-s/2} d\mu(t) d\mu(s)\\ &\le C \iint_{F\times F} |t-s|^{1-s-H} d\mu(t) d\mu(s)\ .\end{aligned}$$

We now apply Frostman's Lemma (cf. Lemma 2.2.2). Since $\dim_H(F) = \log_3 2$, for s such that $H + s - 1 > \log_3 2\ \mathcal{H}(F) = 0$. Then for every $s' < s$ there exists a probability measure $\mu$ such that

$$\iint_{F\times F} |t-s|^{1-s'-H} d\mu(t) d\mu(s) < \infty.$$

For this measure $\mu$ $\mathbb{E}I_s'(\mu) < \infty$ and almost surely $I_s'(\mu) < \infty$. This implies $\dim_H\{(x, X(x)),\ x \in F\} \ge 1 + \log_3 2 - H$.

It follows that:

$$\dim_H\{(x, X(x)),\ x \in F\} \overset{(a.s.)}{=} 1 + \log_3 2 - H.$$

## B.20 Solution to Exercise 3.6.10

1. First

$$\begin{aligned}&\int_0^1 x^{2H} \cos(2\pi x) dx\\ &= \int_0^{1/4} \left( x^{2H} - \left(\frac{1}{2} - x\right)^{2H} + (1-x)^{2H} - \left(\frac{1}{2} + x\right)^{2H} \right) \cos(2\pi x) dx\end{aligned}$$

Since $x \mapsto x^{2H}$ is concave for $0 < H \le \frac{1}{2}$ we get

$$\left(x^{2H} - \left(\frac{1}{2} - x\right)^{2H} + (1-x)^{2H} - \left(\frac{1}{2} + x\right)^{2H}\right) \le 0$$

for $0 \le x \le 1/4$ and $c_2 \le 0$. One can use similar arguments to show that $c_{2n} \le 0$ for any integer $n$. The integral in the definition of $c_{2n+1}$ shall be split into two parts and we use the fact that $x \mapsto x^{2H}$ is increasing to show that $c_{2n+1} \le 0$. In [40] fractional fields parameterized by Euclidean spheres $S^d$ are studied, and series expansion of $\theta \mapsto |\theta|^{2H}$ when $\theta \in S^d$ are used for this study. In this exercise $X(t)$ can be considered as a fractional field parameterized by the circle $S^1$ because of the $2\pi$ periodicity of $X$. So the results on $c_n$ are a special case of the Lévy Kintchine formula for $\theta \mapsto |\theta|^{2H}$ when $\theta \in S^1$ as seen in [40].

2. Since $x \mapsto |x|^{2H}$ is a continuous function $C^1$ except for $x = 0$. We can apply Dirichlet theorem for Fourier series that yields the convergence of the Fourier series to $|x|^{2H}$. When $x = 0$ we use the Dirichlet criterion, which is in this particular case $\int_0^1 \frac{x^{2H}}{x} dx < \infty$, to have the convergence of $\sum_{n=1}^{+\infty} c_n$ and we denotes its sum $-c_0$ which is non-positive because of 1.
3. Let us consider the series

$$S(t) = \sum_{n=1}^{+\infty} [\xi_n(\cos(\pi nt) - 1) - \eta_n \sin(\pi nt)]\sqrt{\frac{-c_n}{2}}.$$

$\forall t \in [-1, 1]$,

$$\begin{aligned}\mathbb{E}(S(t)^2) &= \sum_{n=1}^{+\infty} (\cos(\pi nt) - 1)c_n \\ &= |t|^{2H} - c_0 + c_0.\end{aligned}$$

Hence the series defining $S(t)$ converge in $L^2$ and obviously $S(t) = \Re(X(t))$. With the same type of argument one can show the convergence in $L^2$ of $\Im(X(t))$. Moreover $\forall t,\ s \in \mathbb{R}$,

$$X(t) - X(s) = \sum_{n=1}^{+\infty} \left([\xi_n + i\eta_n]e^{in\pi s}\right)(e^{in\pi(t-s)} - 1).$$

But $((\xi_n + i\eta_n)e^{in\pi s})_n \overset{(d)}{=} (\xi_n + i\eta_n)_n$, and $X$ has stationary increments. In particular $\mathbb{E}((\Re X(t) - \Re X(s))^2) = |t-s|^{2H}$ for all $t,\ s$ such that $-1 < t-s < 1$.

## B.21 Solution to Exercise 3.6.11

1. If $X(0)$ is defined
$$\int_{\mathbb{R}} \frac{\varphi(0)^2}{|s|^{2H+1}} ds < \infty$$
which implies $\varphi(0) = 0$. Similarly $X(t)$ is defined
$$\int_{\mathbb{R}} \frac{\varphi(ts)^2}{|s|^{2H+1}} ds < \infty$$
which is equivalent to
$$\int_{\mathbb{R}} \frac{|\varphi(s')|^2}{|s'|^{2H+1}} ds' < \infty$$
by the change of variables $s' = t$.
2. The process $X$ is a Gaussian centered process, hence, it is enough to compute the covariance function of this process to get self-similarity,
$$\mathbb{E}(X(\varepsilon t)X(\varepsilon t')) = \int_{\mathbb{R}} \frac{\varphi(\varepsilon ts)\varphi(\varepsilon t's)}{|s|^{2H+1}} ds \tag{B.3}$$
because of the isometry property of the random Brownian measure. Then by the change of variable $s' = \varepsilon s$ it is clear that
$$\mathbb{E}(X(\varepsilon t)X(\varepsilon t')) = \varepsilon^{2H}\mathbb{E}(X(t)X(t')), \tag{B.4}$$
hence, $X$ is $H$-self-similar.
3. We will apply Theorem 2.1.8. Let us consider $[A, B]$ with $A,\ B > 0$. Then for $t,\ t' \in [A, B]$ there exists a constant $C > 0$ such that
$$|\varphi(ts) - \varphi(t's)| < C|t - t'|$$
for all $s \in \mathbb{R}$. Hence
$$\mathbb{E}(X(t') - X(t))^2 < C^2(t - t')^2$$
and there exists $p \in \mathbb{N}$ such that
$$\mathbb{E}(X(t') - X(t))^{2p} < C_p(t - t')^{2p}$$
for a constant $C_p$ that depends on $p$. The first inequality in the assumption of Theorem 2.1.8 is satisfied. Since $\varphi$ is $C^2$. (We can take a weaker assumption.) we have $|\varphi((t + h)s) + \varphi((t - h)s) - 2\varphi(ts)| < Ch^2$ for $s \in \mathbb{R}$ and $t \in [A, B]$.

Hence
$$\mathbb{E}(X(t+h)+X(t-h)-2X(t))^2 < C^2h^4$$

and the second inequality in the assumption of Theorem 2.1.8 is satisfied.

4. Let us check that the process is defined for the function $\varphi(\lambda)=\min(|\lambda|,1)$. First $\varphi(0)=0$ then
$$\frac{\varphi^2(\lambda)}{|\lambda|^{2H+1}} \sim |\lambda|^{1-2H}$$
as $|\lambda| \to 0$, and
$$\frac{\varphi^2(\lambda)}{|\lambda|^{2H+1}} \sim |\lambda|^{-1-2H}$$
as $|\lambda| \to +\infty$, which yields
$$\int_{\mathbb{R}} \frac{|\varphi(s)|^2}{|s|^{2H+1}} ds < +\infty.$$
Since $\varphi$ is an even function, $X(t) \overset{(a.s.)}{=} X(-t)$. If we suppose that $X$ has stationary increments then
$$X(t)-X(-t) \overset{(d)}{=} X(2t),$$
since $X(0) \overset{(a.s.)}{=} 0$. But
$$\mathbb{E}X^2(2t)) = \int_{\mathbb{R}} \frac{|\varphi(2ts)|^2}{|s|^{2H+1}} ds > 0$$
which yields a contradiction.

## B.22 Solution to Exercise 4.5.1

1. Because of exercise 3.6.11 we know that $X$ is $H$-self-similar and $X(0)=0$. Hence
$$\left(\frac{X(\varepsilon u)}{\varepsilon^H}\right)_{u\in\mathbb{R}} \overset{(d)}{=} (X(u))_{u\in\mathbb{R}}$$
and the process is $H$-locally asymptotically self-similar at $t=0$ and it is its own tangent field.
2. Let us consider the process
$$Y(t) = \int_{\mathbb{R}} \frac{s\varphi'(ts)}{|s|^{H+1/2}} W(ds)$$

which is obtained by a formal derivation of the integral defining $X$. One can check that $0 < \int_{\mathbb{R}} \frac{(\varphi')^2(s)}{|s|^{2H}} ds < +\infty$. Hence $Y$ is well defined and non degenerated. For $t \neq 0$ and $\forall u \in \mathbb{R}$

$$\lim_{\varepsilon \to 0} \left( \frac{X(t + \varepsilon u) - X(t)}{\varepsilon} - Y(t)u \right) = 0$$

in $L^2$. Because of the almost sure convergence proved in exercise 3.6.11, the convergence is also almost sure. Hence for $t \neq 0$ $X$ is 1-locally asymptotically self-similar and the tangent process at point $t$ is $(Y(t)u)_{u \in \mathbb{R}}$.

## B.23 Solution to Exercise 4.5.2

1. Let us recall that when you fix the second variable of a Hurst process, you get a standard fractional Brownian field in the first variable. Then the definition of $\tilde{B}$ yields the results.
2. For $x \neq y$

$$\mathbb{E}(Y_{\varepsilon,1}(x,u)Y_{\varepsilon,1}(y,v)) \\ = \frac{\varepsilon^{-h(x)-h(y)}}{C(h(x))^{1/2}C(h(y))^{1/2}} \int_{\mathbb{R}^d} \frac{e^{(y-x).\xi}(e^{-i\varepsilon u.\xi} - 1)(e^{-i\varepsilon v.\xi} - 1)}{\|\xi\|^{d+h(x)+h(y)}} \frac{d\xi}{(2\pi)^{d/2}}.$$

Hence,

$$\mathbb{E}(Y_{\varepsilon,1}(x,u)Y_{\varepsilon,1}(y,v)) \\ = \frac{1}{C(h(x))^{1/2}C(h(y))^{1/2}} \int_{\mathbb{R}^d} \frac{e^{a_\varepsilon}(e^{-iu.\xi} - 1)(e^{-iv.\xi} - 1)}{\|\xi\|^{d+h(x)+h(y)}} \frac{d\xi}{(2\pi)^{d/2}}$$

where $a_\varepsilon = \frac{y-x}{\varepsilon}$. Since $x \neq y$ $|a_\varepsilon| \to \infty$ when $\varepsilon \to 0^+$. Then using Riemann Lebesgue Lemma one easily concludes that

$$\lim_{\varepsilon \to 0^+} \mathbb{E}(Y_{\varepsilon,1}(x,u)Y_{\varepsilon,1}(x,v)) = 0.$$

3. Using the isometry property of random Brownian measures we get

$$\mathbb{E}(Y_{\varepsilon,2}(x,u))^2 = I_{22}$$

defined before (4.34) in the proof of Proposition 4.3.4, where we take $u_1 = 0$. Because of (4.41), we get the result. Please remark that (4.41) is a stronger result than the one we need. Hence the proof of this part can be made without using (4.41) and will be shorter than the proof of (4.41).

4. We first remark that

$$\frac{B_h(x+\varepsilon u)-B_h(x)}{\varepsilon^{h(x)}} = Y_{\varepsilon,1}+Y_{\varepsilon,2},$$

Then 1 and 2 yield

$$\lim_{\varepsilon\to 0^+}\mathbb{E}(Y_{\varepsilon,1}(x,u)Y_{\varepsilon,1}(y,v)) = \mathbb{E}(\tilde{B}(x,u)\tilde{B}(y,v))$$

for all $x, y \in \mathbb{R}^d$. Hence

$$\lim_{\varepsilon\to 0^+}\big(Y_{\varepsilon,1}(x,u)\big)_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d} \overset{(d)}{=} (\tilde{B}(x,u))_{(x,u)\in\mathbb{R}^d\times\mathbb{R}^d}.$$

Because of 3. $Y_{\varepsilon,2}$ converges in probability to 0 and the sum of $Y_{\varepsilon,1}+Y_{\varepsilon,2}$ converges to the same limit as $Y_{\varepsilon,1}$ in distribution. Moreover

## B.24 Solution to Exercise 4.5.3

1. Because of (4.19)

$$\lim_{t\to 1^-} C(h(t)) = C\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{\sqrt{2}\sin(\frac{\pi}{2})} = \sqrt{\frac{\pi}{2}}.$$

Hence we get the first part of 1. Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we get the second part of 1.

2. Let $(X(x,y))_{(x,y)\in\mathbb{R}\times(0,1)}$ be the Hurst process defined in Definition 4.3.4, then

$$\mathbb{E}\left(\frac{X(1,\frac{1}{2})}{(C(\frac{1}{2}))^{\frac{1}{2}}} - \frac{X(1,\frac{1}{4})}{(C(\frac{1}{4}))^{\frac{1}{2}}}\right)^2$$

$$= \int_{\mathbb{R}} 2(1-\cos(\xi))\left(\frac{1}{(\frac{\pi}{2})^{\frac{1}{4}}|\xi|} - \frac{1}{|\xi|^{\frac{3}{4}}}\right)^2 \frac{d\xi}{\sqrt{2\pi}} > 0.$$

Since $B_h(t) = \frac{1}{(C(h(t)))^{1/2}}X(t,h(t))$, if we denote by

$$B_h(1^-) = \lim_{t\to 1^-} B_h(t)$$

respectively by $B_h(1^+) = \lim_{t\to 1^+} B_h(t)$, we get that

$$\mathbb{E}(B_h(1^-) - B_h(1^+))^2 > 0.$$

Since $B_h(1^-) - B_h(1^+)$ is a Gaussian random variable it is almost surely positive.

## B.25 Solution to Exercise 4.5.4

1. Since $h$ is continuous on $\mathcal{K}$ the minimum $a = \inf\{h(t),\ \ t \in \mathcal{K}\}$ is positive. Because of (4.48) and of (2.81)

$$\begin{aligned}
&\Big| \sum_{j \geq J, k \in \mathbb{Z}} \chi_{j,k,1}(t, h(t))\eta_{j,k,1} \Big| \\
&\qquad \leq C \sum_{j \geq J} 2^{-aj} \sum_{k \in \mathbb{Z}} \left( \frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1 + |2^j t - k|^2} \right. \\
&\qquad\qquad \left. + \frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1 + |k|^2} \right).
\end{aligned}$$

and we get

$$\Big| \sum_{j \geq J, k \in \mathbb{Z}} \chi_{j,k,1}(t, h(t))\eta_{j,k,1} \Big| \leq C 2^{-a' J}$$

where $a' < a$. It shows the almost sure uniform convergence on $\mathcal{K}$.

2. As in the previous question

$$\begin{aligned}
&\Big| \sum_{|k| \geq K} \chi_{j,k,1}(t, h(t))\eta_{j,k,1} \Big| \leq C 2^{-aj} \\
&\sum_{|k| \geq K} \left( \frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1 + |2^j t - k|^2} + \frac{\log(1+j)^{1/2} + \log(1+|k|)^{1/2}}{1 + |k|^2} \right).
\end{aligned}$$

Let us show the almost sure uniform convergence on $\mathcal{K}$ of $\sum_{|k| \geq K} \frac{1}{1+|2^j t - k|^2}$. Let us denote by $k'(K) = \inf\{|2^j t - k|,\ \ \forall t \in \mathcal{K},\ \ \forall k \in \mathbb{Z},\ \ |k| \geq K\}$, and let us remark that $\lim_{K \to +\infty} k'(K) = \infty$ since $\mathcal{K}$ is compact. Hence

$$\sum_{|k| \geq K} \frac{1}{1 + |2^j t - k|^2} \leq C \sum_{k \geq k'(K)} \frac{1}{k^2} \to 0$$

when $K \to +\infty$. The uniform convergence on $\mathcal{K}$ of $\sum_{|k| \geq K} \chi_{j,k,1}(t, h(t))\eta_{j,k,1}$ is a consequence of the previous fact.

3. Because of (4.50)

$$\Big|\sum_{|k|\geq K}\tilde{\chi}_{0,k,0}(t,h(t))\eta_{j0k,0}\Big| \leq C\sum_{|k|\geq K}\left(\frac{1}{1+|t-k|^2}+\frac{1}{1+|k|^2}\right)|\eta_{0,k,0}|$$

which almost surely uniformly converge on $\mathcal{K}$ by similar arguments as the one we used in the previous question.

4. Let us consider $M_K(t)=\sum_{0<k\leq K}(\tilde{\chi}_{0,k,0}(t,h(t))-\chi_{0,k,0}(t,h(t)))\eta_{0,k,0}$. As in the proof of Theorem 3.2.1

$$M_K(t)=\mathbb{E}\left(\Big(\int_{|\xi|\leq\frac{4\pi}{3}}\frac{it\xi-1/2t^2\xi^2}{|\xi|^{1/2+h(t)}}\widehat{W^+}(d\xi)|\mathcal{F}_K\right)$$

where $\mathcal{F}_K=\sigma(\eta_{0,k,0},\ 0\leq k\leq K)$. Moreover the process

$$X^{\flat\flat}(t,h(t))=\int_{|\xi|\leq\frac{4\pi}{3}}\frac{it\xi-1/2t^2\xi^2}{|\xi|^{1/2+h(t)}}\widehat{W^+}(d\xi)$$

is almost surely continuous. Hence because of Proposition V-2-6 p 104 in [110], $M_K$ converges almost surely uniformly on every compact. The same property is obviously true for

$$\sum_{0<|k|\leq K}(\tilde{\chi}_{0,k,0}(t,h(t))-\chi_{0,k,0}(t,h(t)))\eta_{0,k,0}.$$

Hence we get the result.

5. Since the normalization function is analytic and non-vanishing, and $h$ is continuous $\frac{1}{(C(h(t)))^{1/2}}$ is bounded on $\mathcal{K}$ and the result is a straightforward consequence of 1. 2. and 4.

## B.26 Solution to Exercise 5.6.1

1. The Fourier transform of $\exp(-|t|)$ is $f(\xi)=\dfrac{1}{\pi(1+\xi^2)}$. This function is positive. By Bochner's theorem, there exists an unique centered Gaussian stationary process $X$ such that $\mathbb{E}X(t)X(0)=\exp(-|t|)$.
2. Let $Y(t)=X(t)-X(0)$.

$$Y(t)=\int_{\mathbb{R}}\sqrt{f(\xi)}(e^{it\xi}-1)\widehat{W}(d\xi).$$

One writes $f$ as

$$f(\xi) = \frac{1}{\pi|\xi|^2} + \varepsilon(\xi),$$
$$\varepsilon(\xi) = \frac{1}{\pi|\xi|^2}\left(\frac{1}{1+\frac{1}{|\xi|^2}} - 1\right).$$

One easily checks that the function $\varepsilon$ satisfies **Hyp** 5.1.1 with $H = 1/2$ and $0 < \eta < 4$.

## B.27 Solution of Exercise 5.6.2

1. Self-similarity was proved in exercise 5.6.11.
2. Process $X$ is $C^1$ since $\phi$ is $C^2$. A Taylor expansion leads to

$$V_n = \sum_{k=0}^{n-1} X'^2(\theta_k),$$

with $\theta_k \in (k/n, (k+1)/n)$. For a given sample path, $\frac{1}{n}V_n$ is a Riemann sum that converges to $\int_0^1 X'^2(t)dt$. $\widehat{H_n}$ converges to 1 and is of course not consistent.

## B.28 Solution of Exercise 5.6.3

Let $h > 0$. Since $X$ is a Gaussian self-similar process, $X(h)/h^H$ is centered Gaussian variable $Y$, the variance of which does not depend on $h$. We then have

$$\mathbb{E}\log_2|2^{nH}X(2^{-n})| = \mathbb{E}\log_2|Y|,$$
$$\operatorname{var}\log_2|2^{nH}X(2^{-n})| = \operatorname{var}\log_2|Y|.$$

Borel-Cantelli lemma then implies that $\widehat{H_n}$ converges, a.s., as $n \to +\infty$, to $H$.

## B.29 Solution of Exercise 5.6.4

1. Let $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. Then

$$det(\Sigma) = 1 - \rho^2,$$
$$\Sigma^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$$

Then

$$\mathbb{E}(|U|^i|V|^{-i})$$
$$= \frac{1}{2\pi\sqrt{1-\rho^2}}\int_{\mathbb{R}^2}|x|^i|y|^{-i}\exp\left\{-\frac{1}{2(1-\rho^2)}(x^2+y^2-2\rho xy)\right\}dxdy,$$

and

$$cov(|U|^i,|V|^i) = \frac{1}{2\pi}\int_{\mathbb{R}^2}|x|^i|y|^{-i}\exp(-(x^2+y^2)/2)A_\rho(x,y)dxdy,$$

where

$$A_\rho(x,y) = \frac{1}{\sqrt{1-\rho^2}}\exp\left(-\frac{\rho^2}{2(1-\rho^2)}(x^2+y^2)\right)\exp\left(\frac{\rho xy}{1-\rho^2}\right) - 1.$$

Since

$$\int_{\mathbb{R}}|x|^i x e^{-x^2/2}dx = 0,$$

one gets

$$cov(|U|^i,|V|^i) = \frac{1}{2\pi}\int_{\mathbb{R}^2}|x|^i|y|^{-i}\exp(-(x^2+y^2)/2)B_\rho(x,y)dxdy,$$

where

$$B_\rho(x,y) = A_\rho(x,y) - \rho xy.$$

One then checks that

$$\lim_{\rho\to 0}\frac{B_\rho(x,y)}{\rho^2} = \frac{1}{2} + x^2y^2/2 - (x^2+y^2)/2,$$

$B_0(x,y) = 0$, $\left(\frac{\partial B_\rho(x,y)}{\partial\rho}\right)_{\rho=0} = 0$, and

$$\frac{\partial^2 B_\rho(x,y)}{\partial\rho^2} = P_\rho(x,y)\exp\left(-\frac{\rho^2}{2(1-\rho^2)}(x^2+y^2)\right)\exp\left(\frac{\rho xy}{1-\rho^2}\right),$$

where $P_\rho(x,y)$ is fourth degree polynomial that depends continuously on $\rho$. Let $|\rho| \leq 1/2$. Using a Taylor expansion of order two, there exists a fourth degree polynomial $P(x, y)$ and $0 < C < 1/2$ such that

$$\left| \frac{B_\rho(x, y)}{\rho^2} \right| \leq |P(x, y)| e^{C|xy|}.$$

One then concludes with the dominated convergence theorem.

2. Since $B_H$ is a fractional Brownian motion and we know the covariance function (3.9) of fractional Brownian motion, one can compute

$$cov(\Delta_{k,0} B_H, \Delta_{0,0} B_H) = \frac{V}{2} \left\{ -|k+2|^{2H} + 4|k+1|^{2H} - 6|k|^{2H} \right.$$
$$\left. +4|k-1|^{2H} - |k-2|^{2H} \right\}$$

and a Taylor expansion shows that

$$cov(\Delta_{k,0} B_H, \Delta_{0,0} B_H) = \frac{V}{2} |k|^{2H-4} + o(|k|^{2H-4}) \tag{B.5}$$

as $|j| \to \infty$.
Since process $B_H$ is $H$-self-similar with stationary increments

$$\mathbb{E} W_n(B_H) = \mathbb{E} |\Delta_{0,0} B_H|^i.$$

Then

$$\begin{aligned}\mathbb{E}|W_n(B_H)|^2 &= \frac{1}{(2^n - 1)^2} \sum_{p, p'=0}^{2^n - 2} \mathbb{E}|\Delta_{p,0} B_H|^i |\Delta_{p',0} B_H|^{-i} \\ &= \frac{1}{2^n - 1} \sum_{|k| \leq 2^n - 2} \left(1 - \frac{|k|}{2^n - 1}\right) \mathbb{E}|\Delta_{k,0} B_H|^i |\Delta_{0,0} B_H|^{-i}.\end{aligned}$$

It follows

$$\mathbb{E}|W_n(B_H) - \mathbb{E}|\Delta_{0,0} B_H|^i|^2 \leq$$
$$\frac{1}{2^n - 1} \sum_{|k| \leq 2^n - 2} \left(1 - \frac{|k|}{2^n - 1}\right) cov(|\Delta_{k,0} B_H|^i, |\Delta_{0,0} B_H|^i).$$

Using the first question with $\rho = \frac{cov(\Delta_{k,0} B_H, \Delta_{0,0} B_H)}{\mathrm{var}(\Delta_{0,0} B_H)}$ we get

$$|cov(|\Delta_{k,0} B_H|^i, |\Delta_{0,0} B_H|^i)| \leq C k^{4H-8},$$
$$\sum_{k \in \mathbb{Z}} |cov(|\Delta_{k,0} B_H|^i, |\Delta_{0,0} B_H|^i)| < \infty.$$

The dominated convergence theorem then yields

$$\lim_{n\to+\infty} 2^n \mathbb{E}|W_n(B_H) - \mathbb{E}|\Delta_{0,0}B_H|^i|^2 = \sum_{k\in\mathbb{Z}} cov(|\Delta_{k,0}B_H|^i, |\Delta_{0,0}B_H|^i).$$

Let us now apply Bienaymé-Tchebicheff's inequality. For all $\delta > 0$

$$\mathbb{P}(|W_n(B_H) - \mathbb{E}|\Delta_{0,0}B_H|^i| \geq \delta) \leq \frac{\mathbb{E}|W_n(B_H) - \mathbb{E}|\Delta_{0,0}B_H|^i|^2}{\delta^2}.$$

Borel-Cantelli's lemma then yields

$$\lim_{n\to+\infty} W_n(B_H) = \mathbb{E}|\Delta_{0,0}B_H|^i \quad (a.s.).$$

## B.30 Solution of Exercise 5.6.5

1. We first check that $h(0) = h(1) = 0$ and $h$ is in $C^1(0, 1)$. Then

$$\begin{aligned}\int_0^1 h(t)dt &= \frac{2}{\pi}\sum_{k\geq 0}\frac{1}{(2k+1)^4}\\ &= \frac{\pi^3}{48},\\ \int_0^1 h'^2(t)dt &= \sum_{k\geq 0}\frac{\pi^2}{(2k+1)^4}\int_0^1 \cos^2((2k+1)\pi t)dt\\ &= \frac{\pi^6}{192}.\end{aligned}$$

Therefore

$$\frac{\left(\int_0^1 h(t)dt\right)^2}{\int_0^1 h'^2(t)dt} = 1/12.$$

2. Let us compute the mean square error of $\widehat{I(B)}$.

$$I(B) - \widehat{I(B)} = -\frac{B(1)}{2n} + \sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}(B(t) - B(k/n))dt,$$

Let us use the independence of the increments of $B$.

$$\mathbb{E}(I(B)-\widehat{I(B)})^2 = \frac{1}{4n^2} - \frac{1}{n}\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}(t-k/n)dt$$
$$+\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}\int_{k/n}^{(k+1)/n}\mathbb{E}(B(t)-B(k/n))(B(t')-B(k/n))dtdt'.$$

Then $\mathbb{E}(I(B)-\widehat{I(B)})^2 = -\frac{1}{4n^2}+\frac{1}{n^2}\int_0^1\int_0^1 \inf(x,y)dxdy$ and we get $\mathbb{E}(I(B)-\widehat{I(B)})^2 = \frac{1}{12n^2}$. It follows that the mean square error of $\widehat{I(B)}$ is equal to the bound of the first question. According to Sect. 5.5.1 , the trapezoidal estimate is optimal with respect to the mean square error.

## B.31 Solution of Exercise 5.6.6

Since the $e_p$, $p \geq 0$ is an orthonormal basis of $L^2([0, 1])$

$$\mathbb{E}\|a^2-\widehat{a^2_{P,N}}\|^2_{L^2} = \sum_{p\geq P+1} I^2(e_p) + \sum_{p=0}^{P}\mathbb{E}(I(e_p)-\widehat{I_{N,P}})^2.$$

Since $a^2 \in \mathcal{F}_s$, there exists $C_1 > 0$ such that

$$\sum_{p\geq P+1} I^2(e_p) \leq C_1 P^{-2s}.$$

According to Theorem 4.1.6, and using

$$\left(\int_0^1 a^2 e_p\right)^2 \leq \|a^2\|^2_{L^2},$$

there exists $C_2 > 0$ such that

$$\mathbb{E}(I(e_p)-\widehat{I_{N,P}})^2 \leq C_2/N.$$

To sum up

$$\mathbb{E}\|a^2-\widehat{a^2_{P,N}}\|^2_{L^2} \leq C_1 P^{-2s} + C_2 P/N.$$

One concludes with the choice $P = N^{1/(2s+1)}$.

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