Z. Wahrscheinlichkeitstheorie verw. Gebiete
41, 153–159 (1977)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

Generalized Ito's Formula and Additive Functionals of Brownian Motion

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Summary. An extension of Ito's formula to convex functions is obtained, and a version of its converse is investigated. By using the generalized Ito's formula obtained here and that obtained by G. Brosamler for higher dimensional Brownian motion, a transparent proof of the correspondence between measures and nonnegative continuous (homogeneous) additive functionals is given.

0. Introduction

Our concern is Ito's formula and the representation of *nonnegative continuous (homogeneous) additive functionals of Brownian motion*, referred to as PCAFB. We shall use $\{X(t), \mathscr{F}_t, P^x\}$ to denote a standard Brownian motion defined on $C[R^n]$. We shall not distinguish between additive functionals which are equivalent. An extension of Ito's formula is obtained for convex functions in Theorem 1, and a converse of Theorem 1 is given in Theorem 2.

The correspondence between measures and PCAFB has been studied before (see Dynkin [2]), but the known approaches are quite complicated. Brosamler [1] extended Ito's formula to the potentials on a Green domain $D \subseteq R^n$, $n \geqq 2$. For a "nice" measure on D the extended Ito's formula leads naturally to a PCAFB. In Proposition 1 and Theorem 3 we show that the one-to-one correspondence of measures and PCAFB can indeed be obtained through the generalized Ito's formula.

1. One Dimensional Case

We shall use $L(t, y)$ to denote the local time of Brownian motion $X(t)$. The generalized Ito's formula is given as follows:

Theorem 1. *Let f be a convex function. Then for all x,*

$$f(X(t)) - f(X(0)) = \int_0^t f'(X(u))\, dX(u) + \tfrac{1}{2} \int_{-\infty}^{\infty} L(t, y)\, d\mu(y), \qquad P^x \text{ a.s.} \tag{1.1}$$

where $d\mu=f''$ *in the weak sense, namely,* $\int_{-\infty}^{\infty} f(x)\,\phi''(x)\,dx=\int_{-\infty}^{\infty} \phi(x)\,d\mu(x)$ *for all* $\phi\in C_c^\infty[R]$, *and* μ *is a locally finite measure.*

Proof. Without loss of generality, we assume the probability measure to be P^0. Let f_n, $n=1,2,\ldots$, be a regularization (see [9] for definition) of f. Since f is convex on R, it is continuous and its derivative f' exists almost everywhere. Indeed, the essential supremum of f' is finite on compact sets (see [3]). Hence f_n converges to f everywhere, and f_n' converges to f' a.e. and boundedly on compact sets. By the bounded convergence theorem, p^0 a.s.

$$\lim_{n\to\infty}\int_0^t (f_n'-f')^2(X(u))\,du=0. \tag{1.2}$$

By Theorem 2 of [4; p. 20],

$$\lim_{n\to\infty}\int_0^t f_n'(X(u))\,dX(u)=\int_0^t f'(X(u))\,dX(u) \qquad \text{(in probability)}. \tag{1.3}$$

Applying Ito's formula to f_n, and letting n go to infinity, we obtain by (1.3),

$$\begin{aligned}&\lim_{n\to\infty}\tfrac{1}{2}\int_0^t f_n''(X(u))\,du\\ &\quad=f(X(t))-f(X(0))-\int_0^t f'(X(u))\,dX(u) \qquad \text{(in probability)}.\end{aligned} \tag{1.4}$$

But $L(t,y)$ is bounded a.s., and $f_n''(y)\,dy$ converges weakly to $d\mu$; hence

$$\lim_{n\to\infty}\int_0^t f_n''(X(u))\,du = \lim_{n\to\infty}\int_{-\infty}^{\infty} L(t,y)\,f_n''(y)\,dy=\int_{-\infty}^{\infty} L(t,y)\,d\mu(y) \quad \text{a.s.} \tag{1.5}$$

Remarks. (A) If $f''=l$, where l is locally integrable, then for all x

$$f(X(t))-f(X(0))=\int_0^t f'(X(u))\,dX(u)+\tfrac{1}{2}\int_0^t l(X(u))\,du \quad \text{a.s. } P^x. \tag{1.6}$$

(B) Let (M_t, G_t, P) be a continuous local martingale, and let f be a convex function. By the argument of Theorem 1, one can obtain:

$$f(M(t))-f(M(0))=\int_0^t f'(M(u))\,dM(u)+A(t) \quad \text{a.s. } P, \tag{1.7}$$

where $A(t)$ is a natural increasing process. Indeed, $A(t)=\frac{1}{2}\int_{-\infty}^{\infty}\bar{L}(t,y)\,d\mu(y)$ whenever a local time $\bar{L}(t,y)$ of $M(t)$ exists.

(C) Levy's result that $|X(t)|-L(t,0)$ is another Brownian motion (see [6]) can be obtained from Theorem 1. Let $f(x)=|x|$. Then $|f'(x)|=1$ when $x\neq 0$, and $f''=2\delta_0$ in the weak sense. Hence for all $x\in R$,

$$|X(t)|-|x|=\int_0^t f'(X(u))\,dX(u)+L(t,0) \quad \text{a.s. } P^x. \tag{1.8}$$

Since the square variation of $\int_0^t f'(X(u))\,dX(u)$ is $\int_0^t [f'(x(u))]^2\,du=t$, it is another Brownian motion.

Similarly, one can obtain Tanaka's formula of expressing $2L(t,0)-X(t)^+$ as a Brownian integral (see [7; p. 68]).

Obviously, Theorem 1 can be extended to linear combinations of convex functions, but can it be extended to a more general class of functions? The answer is negative as shown in the following Theorem.

Theorem 2. *If $(f(X(t)),\mathscr{F}_t,P^0)$ is a continuous local submartingale, then f is a convex function.*

The proof of Theorem 2 follows from the following lemmas.

Lemma 1. *If $(f(X(t)),\mathscr{F}_t,P^0)$ is a continuous local submartingale, then f does not have any proper local maximum.*

Proof. We shall prove the Lemma only when $(f(X(t)),\mathscr{F}_t,P^0)$ is a submartingale; the rest is left for the reader. Since $\limsup_{t\to\infty} X(t)=\infty$ a.s., $\liminf_{t\to\infty} X(t)=-\infty$ a.s., and $f(X(t))$ is sample continuous a.s., f has to be continuous. Then it is bounded on compact sets. By stopping $X(t)$ at $\{-a,a\}$, $a>0$, and using the optional stopping theorem, we obtain

$$\frac{f(a)+f(-a)}{2}\geqq f(0). \tag{1.9}$$

Thus the origin cannot be a proper local maximum. Now suppose that f has a proper local maximum at $c>0$. Let

$$\begin{aligned} &T=\inf\{t\,|\,X(t)=c \text{ or } X(t)=-1\},\\ &S=\inf\{t\,|\,t\geqq T \text{ and } X(t)=c+\varepsilon,\ c-\varepsilon,\ -1\}, \end{aligned} \tag{1.10}$$

where $0<\varepsilon<1$.

By the optional stopping theorem and the boundedness of f on compact sets,

$$E^0[f(X(S))\,|\,\mathscr{F}_T]\geqq f(X(T)). \tag{1.11}$$

By taking the expectation of both sides of (1.11), we obtain,

$$\frac{c}{1+c}f(-1)+\frac{1}{2(1+c)}[f(c-\varepsilon)+f(c+\varepsilon)]\geqq\frac{c}{1+c}f(-1)+\frac{1}{1+c}f(c). \tag{1.12}$$

Hence when ε is small enough we obtain a contradiction to the assumption that c is a local maximum.

Lemma 2. *A continuous function f is convex in R if and only if $f(x)+\alpha x+\beta$ has no proper local maximum for any α and β.*

Lemma 2 can be found in Zygmund [11; p. 22].

Now we are ready to obtain an easy proof of the following well known representation theorem.

Proposition 1. *Let $A(t)$ be a PCAFB. Then there exists a locally finite measure μ such that for all x*

$$A(t) = \int_{-\infty}^{\infty} L(t,y)\, d\mu(y) \quad \text{a.s. } P^x. \tag{1.13}$$

Proof. Since $A(t)$ is a continuous additive functional, by Theorem 2 of Tanaka [10], it follows that for all x,

$$A(t) = f(X(t)) - f(X(0)) + \int_0^t g(X(u))\, dX(u) \quad \text{a.s. } P^x, \tag{1.14}$$

where f is a continuous function and g is locally square integrable. But $A(t)$ is nonnegative, hence it is nondecreasing. Thus, by (1.14), $f(X(t))$ is a local submartingale. Thus f is convex by Theorem 2. Now we can apply Theorem 1 to $f(X(t))$ and obtain,

$$f(X(t)) - f(X(0)) = \int_0^t f'(X(u))\, dX(u) + \tfrac{1}{2} \int_{-\infty}^{\infty} L(t,y)\, d\bar{\mu}(y) \quad \text{a.s. } P^x. \tag{1.15}$$

Combining (1.14) with (1.15), we get

$$A(t) - \int_{-\infty}^{\infty} L(t,y)\, d\mu = \int_0^t (f' - g)(X(u))\, dX(u) \quad \text{a.s. } P^x, \tag{1.16}$$

where $\mu = \frac{1}{2}\bar{\mu}$. The left hand side of (1.16) is a process of bounded variation, but the right hand side is a local martingale; hence both sides vanish.

Corollary 1. *Let $A(t)$ be a continuous process. Let $A(\cdot, \omega)$ be of bounded variation almost surely with respect to all P^x, $x \in R$, in any finite time interval. If there exists f and g such that*

$$f(X(t)) - f(X(0)) = \int_0^t g(X(u))\, dX(u) + A(t), \tag{1.17}$$

then

(i) $f = l - h$, *where l and h are convex,*
(ii) $f' = g$ a.e.,
(iii) $A(t) = \frac{1}{2} \int_{-\infty}^{\infty} L(t,y)\, \mu(dy)$, *where $d\mu = f''$ in the weak sense.*

Proof. We can obtain $B(t)$ and $C(t)$, both being PCAFB, such that $A(t) = B(t) - C(t)$. The rest of the proof is similar to that of Proposition 1.

2. Higher Dimensional Case

The following extension of Ito's formula is due to Brosamler [1].

Proposition 2 (Brosamler). *Let p be a Green potential defined on a Green domain $D \subseteq R^n$, $n \geq 2$. Let $D_p = \{y \mid p(y) < \infty\}$. Then, for all $x \in D_p$,*

$$p(X(t))-p(X(0))=\int_0^t \operatorname{grad} p(X(u))\cdot dX(u)-A(t) \quad \text{a.s. } p^x, \tag{2.1}$$

where $A(t)$ is a PCAFB, $t \leqq \inf\{s \mid X(s)\in\partial D\}$.

From now on, we let $D=D_\infty=\bigcup_{k=1}^{\infty} D_k$, where $D_k\subseteq \bar{D}_k\subseteq D_{k+1}$ for all k, and the D_k's are Green domains. Let g_k be the Green function on D_k and let μ be a measure defined on D. Then μ_k is defined to be $\mu | D_k$. Also, we put

$$\tau_k=\inf\{t \mid X(t)\in D_k^c\}, \qquad k=1,2,\ldots,\infty, \tag{2.2}$$

$$p_k=\int_{D_k} g_k(x,y)\,d\mu_k(y), \tag{2.3}$$

$$\mathcal{M}=\{\mu \mid p_k\leqq \operatorname{const}(k,\mu) \text{ for } k=1,2,\ldots\}, \tag{2.4}$$

$$\begin{aligned}\mathcal{A}=\{A(t) \mid\ & A(t) \text{ is a PCAFB, } A(0)=0, \text{ and} \\ & E^x A(\tau_k)\leqq \operatorname{const}(k,A) \text{ for } k=1,2,\ldots\}.\end{aligned} \tag{2.5}$$

Lemma 3. *There exists a one-to-one correspondence between $\mathcal{M}$ and $\mathcal{A}$. Further, the correspondence is given by the generalized Ito's formula, namely, for all $x\in D$,*

$$A(t\wedge\tau_k)=-(p_k(X(t\wedge\tau_k))-p_k(X(0))-\int_0^{t\wedge\tau_k} \operatorname{grad} p_k(X(u))\cdot dX(u)) \quad \text{a.s. } P^x, \tag{2.6}$$

where p_k is defined as in (2.3).

Proof. Given $\mu\in\mathcal{M}$, define p_k as given in (2.3). Applying (2.1) to p_k, we obtain a PCAFB $A^k(t)$. Using Riesz decomposition theorem for p_l and some easy arguments, we obtain for all $x\in D$ and all $l\geqq k$,

$$P^x\{A^l(t\wedge\tau_k)=A^k(t\wedge\tau_k)\}=1. \tag{2.7}$$

Thus, we can patch up $A^k(t)$ and call it $A(t)$. The proof of $A(t)\in\mathcal{A}$ is easy, hence it is omitted.

On the other hand, given $A\in\mathcal{A}$, we define

$$\bar{p}_k(x)=E^x(A(\tau_k)). \tag{2.8}$$

We leave for the reader the proof that $\bar{p}_k$ is lower semicontinuous and superharmonic, and that $\bar{p}_k^*$, the fine boundary function of $\bar{p}_k$ on ∂D_k, vanishes. Hence $\bar{p}_k$, $k=1,2,\ldots$, is a potential. Let $p_{l,k}=\bar{p}_l-\bar{p}_k$ for $l>k$, $S(x,\varepsilon)$ be the surface of the ball centered at x with radius ε, and $T_{S(x,\varepsilon)}=\inf\{t \mid X(t)\in S(x,\varepsilon)\}$. Let $x\in D_k$, and choose ε so that $S(x,\varepsilon)\subseteq D_k$; then

$$\begin{aligned} & E^x[p_{l,k}(X(T_{S(x,\varepsilon)}))] \\ &\quad = E^x[E^{X(T_{S(x,\varepsilon)})}A(\tau_l)-E^{X(T_{S(x,\varepsilon)})}A(\tau_k)] \\ &\quad = E^x[A(T_{S(x,\varepsilon)})+\theta_{T_{S(x,\varepsilon)}}A(\tau_l)]-E^x[A(T_{S(x,\varepsilon)})+\theta_{T_{S(x,\varepsilon)}}A(\tau_k)] \\ &\quad = p_{l,k}(x), \end{aligned} \tag{2.9}$$

where θ is the shift operator.

Then $p_{l,k}$ is harmonic on D_k and hence $\Delta \bar{p}_l = \Delta \bar{p}_k$ for all $l > k$ on D_k. Hence we can define a measure μ on D by letting $\mu_k = \Delta \bar{p}_k$. Clearly, $\mu \in \mathcal{M}$. Now given $A \in \mathcal{A}$, by the above arguments, we obtain $\mu \in \mathcal{M}$ such that

$$E^x(A(\tau_k)) = \int g_k(x, y)\, d\mu_k(y) \equiv p_k(x) = \bar{p}_k(x). \tag{2.10}$$

By (2.1) and (2.7) we can find $\bar{A}(t)$ such that for all x, p^x a.s.

$$\bar{A}(t \wedge \tau_k) = -(p_k(X(t \wedge \tau_k)) - p_k(X(0)) - \int_0^{t \wedge \tau_k} \operatorname{grad} p_k(X(u)) \cdot dX(u)). \tag{2.11}$$

Clearly, $E^x[\bar{A}(\tau_k)] = E^x[A(\tau_k)] = p_k(x)$. But $(A(t \wedge \tau_k) - \bar{A}(t \wedge \tau_k), \mathscr{F}_{t \wedge \tau_k})$ is a p^x-martingale for all x, and $A(t \wedge \tau_k) - \bar{A}(t \wedge \tau_k)$ is of bounded variation. Hence $A(t)$ and $\bar{A}(t)$ must be equivalent when $t < \tau_\infty$.

Remarks. (i) By easy computations one can show that all W-measures (for definition, see [2]) are in $\mathcal{M}$ and that all W-functionals are in $\mathcal{A}$.

(ii) By using Lemma 3 and Remark (i), one can show the one-to-one correspondence between W-measure and W-functionals.

(iii) By using (2.6) one can obtain results similar to those given in [2; p. 261]. The proofs are much simpler.

Theorem 3. *Let $A(t)$ be an S-functional on D. Then there exists potentials $\{p_k\}_{k=1}^{\infty}$ such that for almost all $x \in D$, and for all k*

$$A(t \wedge \tau_k) = -(p_k(X(t \wedge \tau_k)) - p_k(X(0)) - \int_0^{t \wedge \tau_k} \operatorname{grad} p_k(X(u)) \cdot dX(u)) \quad \text{a.s. } P^x. \tag{2.12}$$

Proof. By Theorem 8.5 and Section 8.14 of [2], we know there exists an S-measure μ which corresponds to $A(t)$. That is, there exist closed sets Γ_k, which increase to D, such that:

(i) $\mu | \Gamma_k$ is a W-measure for all k and hence $\mu|_{\Gamma_k} \in \mathcal{M}$,

(ii) $T_k \uparrow \tau_\infty$ a.s. for almost all P^x as $k \uparrow \infty$, where

$$T_k = \inf\{t \mid X(t) \in \Gamma_k^c\}.$$

Let $E_k = D_k \cap \Gamma_k$ and $\nu_k = \mu|_{E_k}$ and let

$$q_k = \int_{D_k} g_k(x, y)\, d\nu_k(y). \tag{2.13}$$

By Lemma 3, there exists $\bar{A} \in \mathcal{A}$ such that for $t < \tau_k$ and for all $x \in D$,

$$\bar{A}(t) = -(q_k(X(t)) - q_k(X(0)) - \int_0^t \operatorname{grad} q_k(X(u)) \cdot dX(u)) \quad \text{a.s. } P^x. \tag{2.14}$$

By Theorem 8.15 and Section 8.14 of [2] again, we obtain $\bar{A}(t) = A(t)$ for $t < \tau_\infty$, P^x a.s. for almost all x, indeed for all $x \in \bigcup_{k=1}^{\infty} \Gamma_k$.

Acknowledgement. This paper is based on part of the author's thesis completed at the University of Minnesota under the supervision of Professor Steven Orey. The author would like to thank Professor Steven Orey for his guidance and encouragement during the preparation of this paper.

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Received April 7, 1977