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Stefan Schäffler

Generalized Stochastic Processes

Modelling and Applications of Noise Processes

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Stefan Schäffler
Fak. Elektro- und Informationstechnik
Universität der Bundeswehr München
Neubiberg, Germany

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to my grandchild

Anton Rudolf

Introduction

There is nothing so practical
as a good theory.
KURT LEWIN

In 1918, Walter Schottky studied fluctuations of power in vacuum tubes for the first time. Making these fluctuations audible, one obtains a special sound called **noise**. More generally, any physical interference process is called a noise process. Since each technical system is perturbed by a superposition of noise processes, one of the most important and most difficult challenges in engineering consists in the elimination or minimization of parasitic effects caused by noise processes. Obviously, the activation of an airbag in a car must not depend on perturbed measurements or noise processes in electric circuits. While the output of a noise process is not predictable for a concrete situation, it is possible to find probabilistic models for a large class of noise processes which allow the elimination or minimization of the mentioned parasitic effects. A model of shot noise observed by W. Schottky in 1918 is given by a Poissonian white noise for instance. In digital communications, the influence of noise is compensated by channel coding techniques depending on the type of the transmission channel. The better the probabilistic model of a noise process, the better the chance to avoid unpredictable consequences of noise. Therefore, it is indispensable that mathematicians provide effective noise models. On the other hand, it is indispensable that engineers are familiar with the mathematical background of noise modelling in order to handle noise models in an optimal way.

This textbook shall serve a double purpose: first of all, it is a monograph about generalized stochastic processes, a very important but highly neglected part of probability theory which plays an outstanding role in noise modelling. Secondly, this textbook should be a guide to noise modelling for mathematicians and engineers.

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$\| \bullet \|_2$	Euclidean norm
$\mathrm{cl}(M)$	Closure of M
$\mathrm{int}(M)$	Interior of M
$\partial(M)$	Boundary of M
$C^\infty(\mathbb{R}^n, \mathbb{R})$	Set of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$
$\mathrm{supp}(f)$	Support of f
$\mathfrak{D}(\mathbb{R}^n, \mathbb{R})$	Vector space of test functions $\varphi : \mathbb{R}^n \to \mathbb{R}$
$\mathfrak{D}'(\mathbb{R}^n, \mathbb{R})$	Dual space of $\mathfrak{D}(\mathbb{R}^n, \mathbb{R})$
$\mathrm{PV}\left(\int_A f(\mathbf{x})d\mathbf{x}\right)$	Cauchy principal value
$\mathcal{P}(M)$	Power set of M
$(\Omega, \mathcal{S}, \mathbb{P})$	Probability space
$(\Gamma, \mathcal{G})$	Measurable space
$(\Omega, \mathcal{S}, \mu)$	Measure space
$(X_i)_{i\in I}$	Stochastic process
$\mathscr{E}(X)$	Expectation of X
$\mathscr{V}(X)$	Variance of X
$\mathscr{C}(X, Y)$	Covariance of X and Y
AR(p) process	**A**uto**R**egressive process with p parameters
MA(τ) process	**M**oving **A**verage process of depth τ
AWGN	**A**dditive **W**hite **G**aussian **N**oise
LTI system	**L**inear **T**ime **I**nvariant system

Generalized Functions

Stefan Schäffler

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1.1 Test Functions

On the real valued spaces $\mathbb{R}^n$, $n \in \mathbb{N}$, we use the Euclidean norm

$$\| \bullet \|_2 : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto \sqrt{\mathbf{x}^\top \mathbf{x}} := \sqrt{\sum_{i=1}^{n} x_i^2}.$$

A set $A \subseteq \mathbb{R}^n$ is said to be **open**, if there exists a real number $\varepsilon > 0$ for each $\mathbf{x_0} \in A$ such that

$$K_{\mathbf{x_0},\varepsilon} := \{\mathbf{x} \in \mathbb{R}^n;\ \|\mathbf{x} - \mathbf{x_0}\|_2 < \varepsilon\} \subseteq A.$$

If $\mathbf{x_0}$ is a point of $\mathbb{R}^n$, we define an **open neighborhood** of $\mathbf{x_0}$ to be any open subset N of $\mathbb{R}^n$ containing $\mathbf{x_0}$.
A set $A \subseteq \mathbb{R}^n$ is said to be **closed**, if the complement of A given by

$$A^c := \{\mathbf{x} \in \mathbb{R}^n;\ \mathbf{x} \notin A\}$$

is an open set. Let I be any nonempty set and let $A_i \subseteq \mathbb{R}^n$, $i \in I$, be closed sets, then the intersection

$$A := \bigcap_{i \in I} A_i$$

is a closed subset of $\mathbb{R}^n$ as well.
A set $A \subseteq \mathbb{R}^n$ is said to be **bounded**, if there exists an $\varepsilon > 0$ such that

$$A \subseteq K_{\mathbf{0},\varepsilon}.$$

A subset K of $\mathbb{R}^n$, which is closed and bounded, is said to be **compact**.

For $M \subseteq \mathbb{R}^n$, we define the **closure** of M by

$$\mathrm{cl}(M) := \bigcap_{A \in \mathfrak{A}} A,$$

where

$$\mathfrak{A} := \{B \subseteq \mathbb{R}^n;\ B \text{ is a closed set and } M \subseteq B\}.$$

The set $\mathrm{cl}(M)$ represents the smallest closed subset of $\mathbb{R}^n$ with $M \subseteq \mathrm{cl}(M)$. Let I be any nonempty set and let $B_i \subseteq \mathbb{R}^n$, $i \in I$, be open sets, then the union

$$B := \bigcup_{i \in I} B_i$$

is an open subset of $\mathbb{R}^n$ as well.
The **interior** of a set $M \subseteq \mathbb{R}^n$ is defined by

$$\mathrm{int}(M) := \bigcup_{C \in \mathfrak{C}} C,$$

where

$$\mathfrak{C} := \{B \subseteq \mathbb{R}^n;\ B \text{ is an open set and } B \subseteq M\}.$$

The set $\mathrm{int}(M)$ represents the largest open subset of M.
If $M \subseteq \mathbb{R}^n$, then the set

$$\partial(M) := \mathrm{cl}(M) \setminus \mathrm{int}(M) := \{\mathbf{x} \in \mathrm{cl}(M);\ \mathbf{x} \notin \mathrm{int}(M)\}$$

is called the **boundary** of M.
We obtain for each $M \subseteq \mathbb{R}^n$ (see Problem 1.) that
(i) $\mathrm{int}(M) = M$ for any open set M,
(ii) $\mathrm{cl}(M) = M$ for any closed set M,
(iii) $\partial(M) \cap M = \emptyset$ for any open set M,
(iv) $\partial(M) \subseteq M$ for any closed set M.

The boundary ∂M of M is closed, since $\partial(M)^c = \mathrm{int}(M) \cup \mathrm{cl}(M)^c$.
Let

$$f : \mathbb{R}^n \to \mathbb{R}$$

be any function. The **support** of f is given by

$$\mathrm{supp}(f) := \mathrm{cl}(\{\mathbf{x} \in \mathbb{R}^n;\ f(\mathbf{x}) \neq 0\}).$$

Hence, $\mathrm{supp}(f)^c$ is an open set and there exists an $\varepsilon > 0$ for each $\mathbf{x_0} \in \mathrm{supp}(f)^c$ with

$$K_{\mathbf{x_0},\varepsilon} \subseteq \mathrm{supp}(f)^c, \quad \text{and consequently} \quad f(\mathbf{x}) = 0 \text{ for all } \mathbf{x} \in K_{\mathbf{x_0},\varepsilon}.$$

Although there exists an infinite number of points, where the function values of the trigonometric functions `sin` and `cos` are equal to zero, we obtain

$$\mathrm{supp}(\mathtt{sin})^c = \mathrm{supp}(\mathtt{cos})^c = \emptyset.$$

Now, we consider the set of smooth functions (i.e., continuously differentiable of any arbitrary order)

$$f : \mathbb{R}^n \to \mathbb{R},$$

which we denote with $C^\infty(\mathbb{R}^n, \mathbb{R})$. We define an important subset of $C^\infty(\mathbb{R}^n, \mathbb{R})$ using the support of a function.

Definition 1.1 (Test Function)

A function $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ is called a **test function**, if $\mathrm{supp}(\varphi)$ is bounded (and thus compact). The set of all test functions is denoted by $\mathfrak{D}(\mathbb{R}^n, \mathbb{R})$. ◁

It is obvious that

$$\mathrm{supp}\left(\frac{\partial^m \varphi}{\partial x_{i_1} \dots \partial x_{i_m}}\right) \subseteq \mathrm{supp}(\varphi) \quad \text{for all } m \in \mathbb{N},\ i_1, \dots, i_m \in \{1, 2, \dots, n\},$$

where

$$\frac{\partial^m \varphi}{\partial x_{i_1} \dots \partial x_{i_m}} : \mathbb{R}^n \to \mathbb{R}$$

denotes the partial derivative of a test function φ of order m, since $\mathrm{supp}(\varphi)^c$ is an open set, where all partial derivatives of φ vanish.
Looking for a test function h with $h \not\equiv 0$, we consider

$$h : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 0 & \text{for all } x \leq 0 \\ \exp\left(-\frac{1}{x}\right) & \text{for all } x > 0 \end{cases} \quad \text{(Fig. 1.1).}$$

Using

$$k : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto 1 - \|\mathbf{x}\|_2^2, \quad n \in \mathbb{N},$$

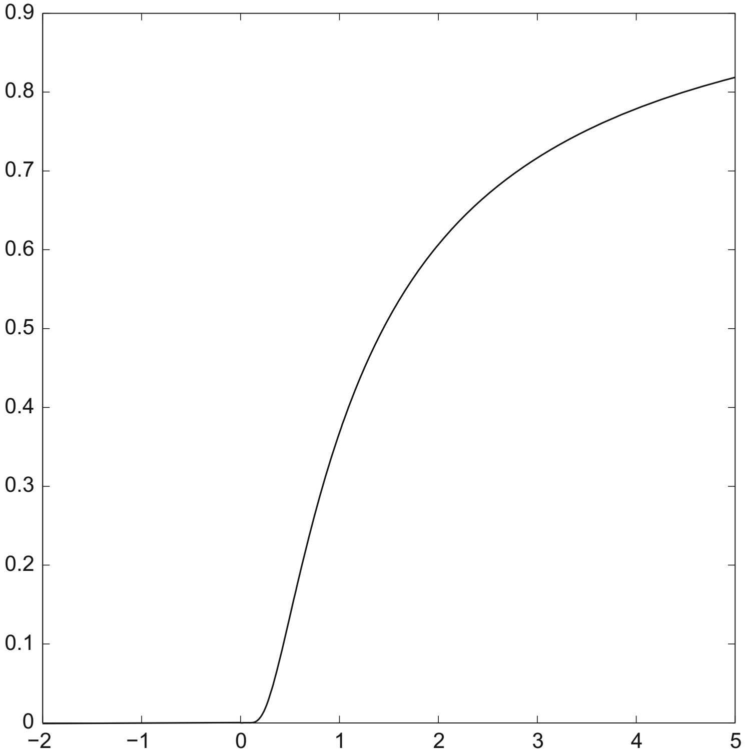

Fig. 1.1 Function h

and

$$\psi := h \circ k : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto \begin{cases} 0 & \text{for all} \quad 1 \leq \|\mathbf{x}\|_2^2 \\ \exp\left(-\frac{1}{1-\|\mathbf{x}\|_2^2}\right) & \text{for all} \quad 1 > \|\mathbf{x}\|_2^2 \end{cases} \quad (\text{Fig. 1.2}),$$

we obtain $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R})$, and ψ is a test function, because

$$\operatorname{supp}(\psi) = \left\{\mathbf{x} \in \mathbb{R}^n;\ \|\mathbf{x}\|_2^2 \leq 1\right\} \quad (= \operatorname{cl}(K_{\mathbf{0},1})).$$

If $\varphi \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R})$, then

$$\lambda\varphi : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto \lambda \cdot \varphi(\mathbf{x})$$

is a test function for each $\lambda \in \mathbb{R}$, because

$$\operatorname{supp}(\lambda\varphi) = \begin{cases} \operatorname{supp}(\varphi) & \text{for } \lambda \neq 0 \\ \emptyset & \text{for } \lambda = 0 \end{cases}.$$

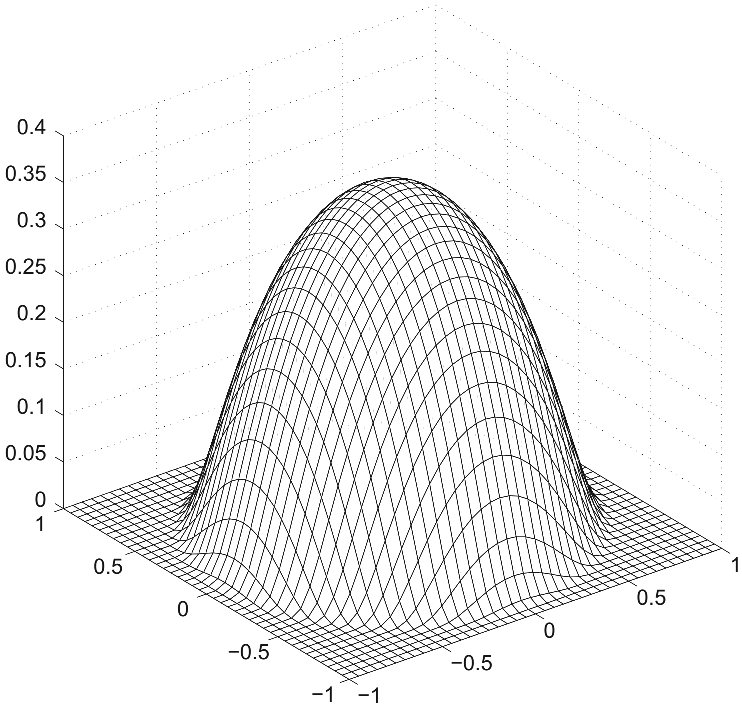

Fig. 1.2 Function $\psi, n = 2$

If $\varphi_1, \varphi_2 \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R})$, then

$$\varphi_1 + \varphi_2 : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x})$$

is a test function as well, because

$$\operatorname{supp}(\varphi_1 + \varphi_2) \subseteq \operatorname{supp}(\varphi_1) \cup \operatorname{supp}(\varphi_2).$$

Consequently, $\mathfrak{D}(\mathbb{R}^n, \mathbb{R})$ is a vector space over $\mathbb{R}$.
Riemann integration of

$$\psi : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto \begin{cases} 0 & \text{for all} \quad 1 \leq \|\mathbf{x}\|_2^2 \\ \exp\left(-\frac{1}{1-\|\mathbf{x}\|_2^2}\right) & \text{for all} \quad 1 > \|\mathbf{x}\|_2^2 \end{cases}$$

leads to

$$0 < I_\psi := \int\limits_{\mathbb{R}^n} \psi(\mathbf{x})\, d\mathbf{x} < \infty.$$

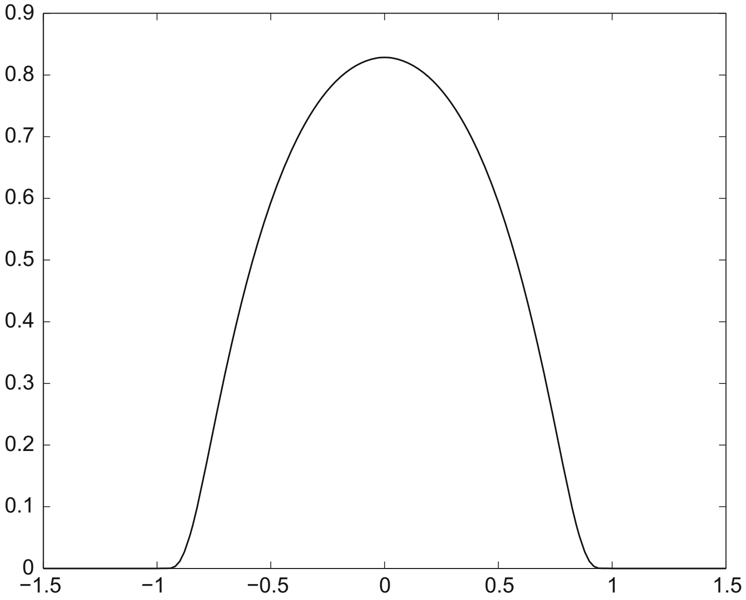

Fig. 1.3 Function $\psi_1, n = 1$

Hence, we obtain for $\psi_1 := \frac{\psi}{I_\psi}$:

$$\int_{\mathbb{R}^n} \psi_1(\mathbf{x})\, d\mathbf{x} = 1 \quad (\text{Fig. 1.3}).$$

Now, we are going to analyze the function

$$\psi_R : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto \frac{\psi_1\left(\frac{\mathbf{x}}{R}\right)}{R^n}$$

for each $R > 0$. A substitution $\mathbf{y} = \frac{\mathbf{x}}{R}$ leads to

$$\int_{\mathbb{R}^n} \psi_R(\mathbf{x})\, d\mathbf{x} = 1.$$

In addition, we get

$$\operatorname{supp}(\psi_R) = \{\mathbf{x} \in \mathbb{R}^n;\ \|\mathbf{x}\|_2 \le R\} = \operatorname{cl}(K_{\mathbf{0},R}).$$

We are able to find new test functions via ψ_R in the following way:
Let

$$f : \mathbb{R}^n \to \mathbb{R}$$

be a continuous function with compact support, then the function

$$\varphi_R : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto \int\limits_{\mathbb{R}^n} f(\boldsymbol{\xi})\psi_R(\boldsymbol{\xi} - \mathbf{x})\, d\boldsymbol{\xi}$$

is a test function for each $R > 0$. To see this, observe that from

$$\int\limits_{\mathbb{R}^n} f(\boldsymbol{\xi})\psi_R(\boldsymbol{\xi} - \mathbf{x})\, d\boldsymbol{\xi} = \int\limits_{\mathrm{cl}(K_{\mathbf{x},R})} f(\boldsymbol{\xi})\psi_R(\boldsymbol{\xi} - \mathbf{x})\, d\boldsymbol{\xi},$$

follows $\varphi_R \in C^\infty(\mathbb{R}^n, \mathbb{R})$ (it is allowed to interchange differentiation and integration in this case). From

$$\varphi_R(\mathbf{x}) = 0 \quad \text{for all} \quad \mathbf{x} \in \{\mathbf{y} \in \mathbb{R}^n;\ \mathrm{cl}(K_{\mathbf{y},R}) \cap \mathrm{supp}(f) = \emptyset\}$$

follows that the support of φ_R is bounded. Choosing the function

$$\mu : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 4 - \big|\, 2|x| - 2 \,\big| & \text{for } -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} \qquad (\text{Fig. 1.4}),$$

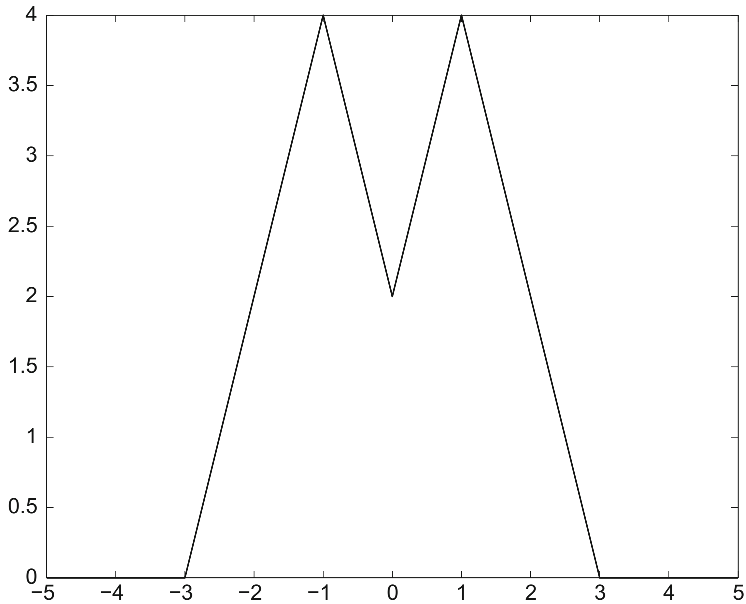

Fig. 1.4 Function μ

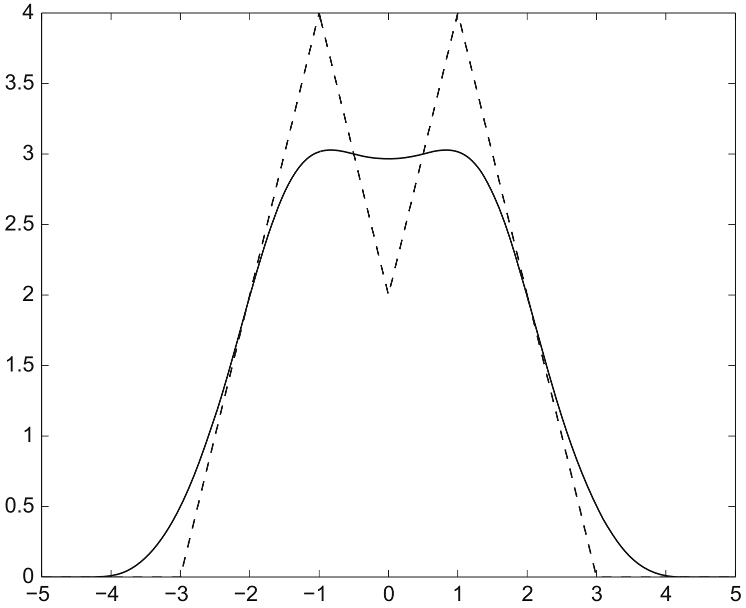

Fig. 1.5 Function $\varphi_{1.5}$

for instance, we obtain the function $\varphi_{1.5}$ for $R = 1.5$ (Fig. 1.5) and the function $\varphi_{0.5}$ for $R = 0.5$ (Fig. 1.6). Our function f can be uniformly approximated as closely as desired by a test function as the following theorem shows.

Theorem 1.2 (Approximation Theorem)
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function with compact support. For each $\varepsilon > 0$, there exists a test function φ such that

$$|f(\mathbf{x}) - \varphi(\mathbf{x})| < \varepsilon \quad \textit{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

With

$$\varphi_R : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto \int_{\mathbb{R}^n} f(\boldsymbol{\xi})\psi_R(\boldsymbol{\xi} - \mathbf{x})\, d\boldsymbol{\xi}$$

we obtain

$$\lim_{R \to 0,\, R>0} \varphi_R = f$$

uniformly in $\mathbb{R}^n$. ◁

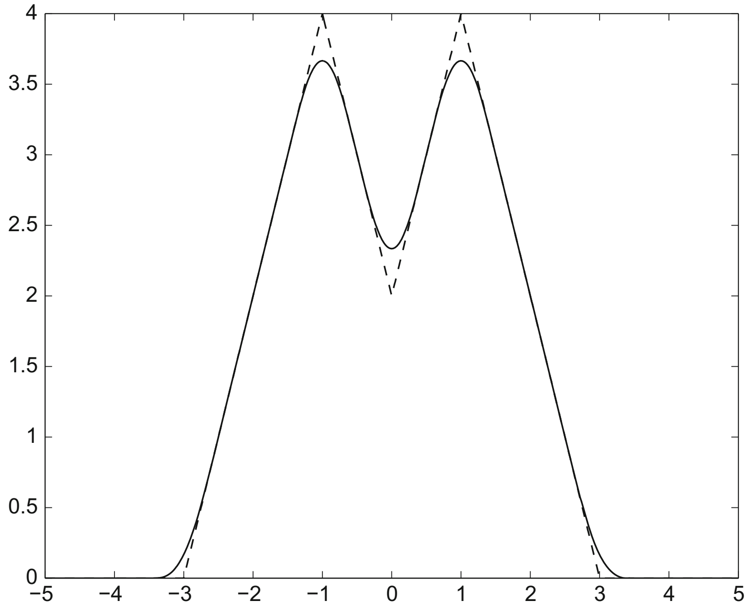

Fig. 1.6 Function $\varphi_{0.5}$

Proof

The function f is uniformly continuous because f has a compact support; hence, there exists a $\delta > 0$ for each $\varepsilon > 0$ with

$$|f(\mathbf{x}) - f(\mathbf{x}')| < \varepsilon \quad \text{for all} \quad \mathbf{x}, \mathbf{x}' \in \mathbb{R}^n \text{ with } \|\mathbf{x} - \mathbf{x}'\|_2 < \delta.$$

Using the second mean value theorem for definite integrals (see [AmEsch08]), we obtain for $\mathbf{x} \in \mathbb{R}^n$:

$$\varphi_R(\mathbf{x}) = \int_{\mathrm{cl}(K_{\mathbf{x},R})} f(\boldsymbol{\xi})\psi_R(\boldsymbol{\xi} - \mathbf{x})\, d\boldsymbol{\xi} = f(\mathbf{x}') \quad \text{for some } \mathbf{x}' \in \mathrm{cl}(K_{\mathbf{x},R}).$$

With $R < \delta$ follows for all $\mathbf{x} \in \mathbb{R}^n$:

$$|f(\mathbf{x}) - \varphi_R(\mathbf{x})| = |f(\mathbf{x}) - f(\mathbf{x}')| < \varepsilon, \quad \text{since } \|\mathbf{x} - \mathbf{x}'\|_2 < \delta.$$

$\square$

In the next section, we describe functions in a more general context.

1.2 Representation of Functions by Functionals

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be **locally integrable**, if $|f|$ is integrable over the closed ball $\mathrm{cl}(K_{\mathbf{x},\varepsilon})$ for each $\mathbf{x} \in \mathbb{R}^n$ and each $\varepsilon > 0$. Now, we are able to assign a functional F to each locally integrable function f in the following way:

$$F_f : \mathfrak{D}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \int_{\mathbb{R}^n} f(\mathbf{x})\varphi(\mathbf{x})\, d\mathbf{x}.$$

Using any nonnegative test function φ with

$$\int_{\mathbb{R}^n} \varphi(\mathbf{x})\, d\mathbf{x} = 1$$

and using any continuous function f, we obtain:

$$F_f(\varphi) = f(\mathbf{x}') \quad \text{for some } \mathbf{x}' \in \mathrm{supp}(\varphi).$$

The mapping F_f is called a **functional**. More general, given any vector space $\mathbb{V}$ over $\mathbb{K}$, then each mapping

$$G : \mathbb{V} \to \mathbb{K}$$

is called a functional. A functional G is said to be **linear**, if

$$\begin{aligned} G(\lambda \cdot \mathbf{x}) &= \lambda \cdot G(\mathbf{x}) \quad \text{for all } \lambda \in \mathbb{K},\ \mathbf{x} \in \mathbb{V}, \\ G(\mathbf{x} + \mathbf{y}) &= G(\mathbf{x}) + G(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{V}, \end{aligned}$$

Thus, F_f is a linear functional for each locally integrable function f.

Let us see how to reconstruct function values of f using F_f. Based on the already introduced test functions ψ_R we consider the test functions

$$\psi_{\mathbf{x},R} : \mathbb{R}^n \to \mathbb{R}, \quad \boldsymbol{\xi} \mapsto \psi_R(\boldsymbol{\xi} - \mathbf{x})$$

and obtain for each continuous function f:

$$F_f(\psi_{\mathbf{x},R}) = f(\mathbf{x}') \quad \text{for some } \mathbf{x}' \in \mathrm{cl}(K_{\mathbf{x},R}).$$

Hence, we get

$$\lim_{R \to 0,\, R>0} F_f(\psi_{\mathbf{x},R}) = f(\mathbf{x}) \quad \text{for all} \quad \mathbf{x} \in \mathbb{R}^n.$$

Choose $n = 1$, f differentiable (and therefore locally integrable), and f' locally integrable. With $\varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R})$, we obtain

$$\int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx = \Big[f(x)\varphi(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx = -\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx$$

using integration by parts, and consequently

$$F_{f'}(\varphi) = -F_f(\varphi') \quad \text{for all } \varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}).$$

The most interesting aspect of this equation lies in the fact that the left hand side is defined only for differentiable functions f, where f prime is locally integrable, while the right hand side is defined for all locally integrable functions f. Using this equation, we are able to give a meaning to the derivative of a function which is not differentiable in terms of classical calculus. This fact represents the principal idea of generalized functions. Coming back to our example

$$\mu : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 4 - \big|2|x| - 2\big| & \text{for } -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases} \qquad (\text{Fig. 1.4}),$$

one gets the function

$$\tilde{\mu} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto -\int_{-\infty}^{\infty} \mu(\xi)\psi_R'(\xi - x)\,d\boldsymbol{\xi}$$

as an approximation of μ prime, which does not exist in terms of classical calculus. The choice $R = 0.5$ leads to the function $\tilde{\mu}$ shown in Fig. 1.7.

The limit of

$$-\int_{-\infty}^{\infty} \mu(\xi)\psi_R'(\xi - x)\,d\boldsymbol{\xi}$$

exists for $R \to 0$. This defines the function shown in Fig. 1.8.

Now, we consider a function $w : \mathbb{R} \to \mathbb{R}$ which is continuous but nowhere differentiable (Fig. 1.9).

Figure 1.10 shows the corresponding function $\tilde{w}$ with $R = 0.01$.

The function w is shaped using paths of one-dimensional Brownian Motions (cf. Chap. 2).

At the end of this section, we investigate some important properties of F_f. To this end, we introduce a concept of convergence for test functions and we define the continuity of a functional.

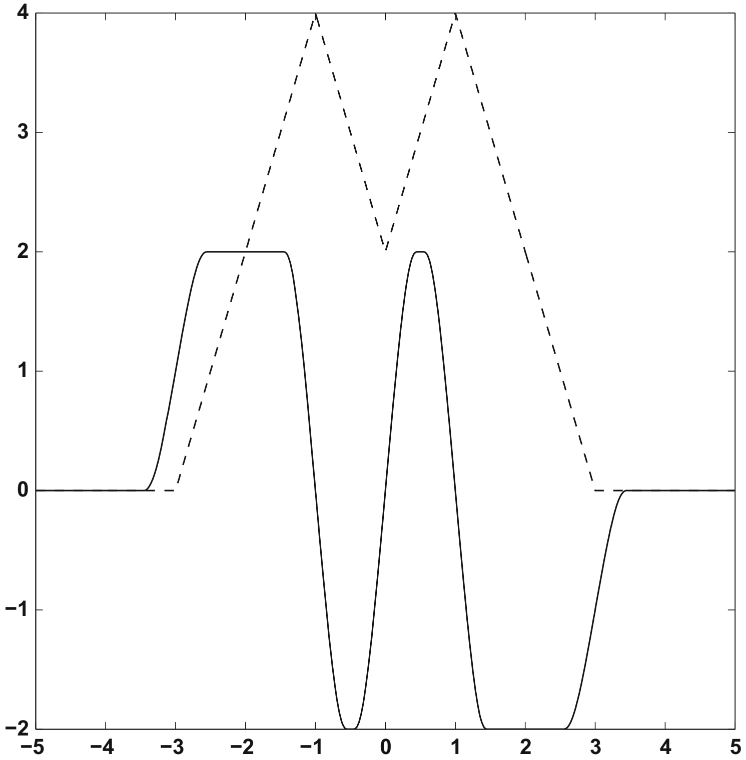

Fig. 1.7 Function $\tilde{\mu}$ for $R = 0.5$

Definition 1.3 (Convergence of Test Functions, Continuous Functionals)
Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a sequence of test functions. This sequence is said to be **convergent** to a test function φ, if there exists a bounded set $M \subset \mathbb{R}^n$ such that

$$\operatorname{supp}(\varphi_i) \subseteq M \quad \text{for all } i \in \mathbb{N},$$

and if the sequence $\{\varphi_i - \varphi\}_{i\in\mathbb{N}}$ and all sequences of partial derivatives of any fixed order of $(\varphi_i - \varphi)$, $i \in \mathbb{N}$, converge to zero uniformly in $\mathbb{R}^n$. A functional

$$F : \mathfrak{D}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$$

is said to be **continuous**, if

$$\lim_{i\to\infty} F(\varphi_i) = F(\varphi)$$

holds for each convergent sequence of test functions $\{\varphi_i\}_{i\in\mathbb{N}}$ with limit φ. ◁

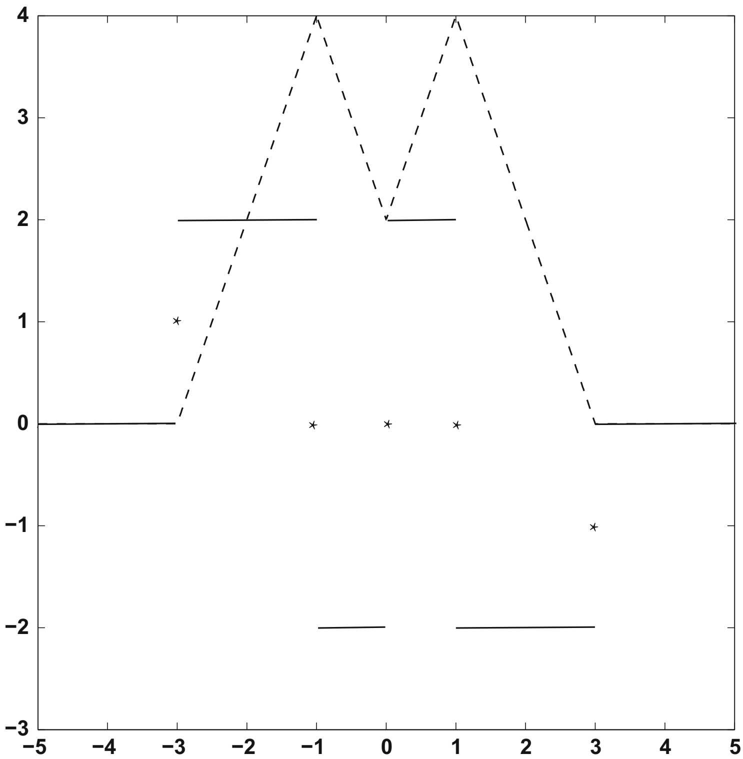

Fig. 1.8 Function $\tilde{\mu}$ for $R \to 0$

Let

$$F : \mathfrak{D}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$$

be linear, then F is continuous, if

$$\lim_{i \to \infty} F(\varphi_i) = 0$$

for all sequences $\{\varphi_i\}_{i \in \mathbb{N}}$ of test functions, which converge to

$$\mathbf{0}_{\mathfrak{D}(\mathbb{R}^n,\mathbb{R})} : \mathbb{R}^n \to \mathbb{R}, \quad \mathbf{x} \mapsto 0.$$

If $\{\varphi_i\}_{i \in \mathbb{N}}$ is a sequence of this type, then there exists some $\rho > 0$ such that

$$\operatorname{supp}(\varphi_i) \subseteq \operatorname{cl}(K_{\mathbf{0},\rho}) \quad \text{for all } i \in \mathbb{N}$$

Fig. 1.9 Function w

and we obtain

$$\lim_{i\to\infty} |F_f(\varphi_i)| = \lim_{i\to\infty} \left| \int_{\mathbb{R}^n} f(\mathbf{x})\varphi_i(\mathbf{x})\, d\mathbf{x} \right| = \lim_{i\to\infty} \left| \int_{\mathrm{cl}(K_{\mathbf{0},\rho})} f(\mathbf{x})\varphi_i(\mathbf{x})\, d\mathbf{x} \right| \leq$$

$$\leq \lim_{i\to\infty} \left(\sup_{\mathbf{x}\in\mathrm{cl}(K_{\mathbf{0},\rho})} \{|\varphi_i(\mathbf{x})|\} \int_{\mathrm{cl}(K_{\mathbf{0},\rho})} |f(\mathbf{x})| d\mathbf{x} \right) = 0.$$

Hence, all linear functionals F_f are continuous.

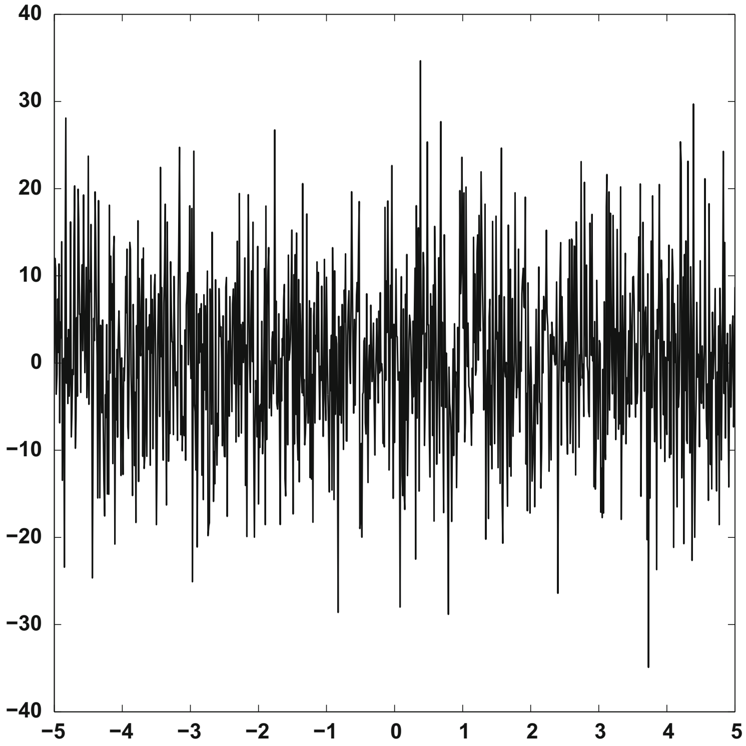

Fig. 1.10 Function $\tilde{w}, R = 0.01$

1.3 Generalized Functions (Distributions)

We are able to assign a linear functional $F_g : \mathfrak{D}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$ to each locally integrable function $g : \mathbb{R}^n \to \mathbb{R}$ as we have seen in the last section. Furthermore, we assigned a linear functional

$$(F_f)' : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto -\int_{-\infty}^{\infty} f(x)\varphi'(x)\, dx$$

to a locally integrable function $f : \mathbb{R} \to \mathbb{R}$, where $(F_f)'$ is interpreted as derivative of f—particulary with regard to functions f which are not differentiable in terms of classical calculus. Let f be differentiable and f' be locally integrable, then we got

$$(F_f)' = F_{f'}.$$

These observations lead to the following definition.

Definition 1.4 ((Regular) Generalized Function, (Regular) Distribution)
For $\mathfrak{D}(\mathbb{R}^n, \mathbb{R})$, a linear continuous functional

$$F : \mathfrak{D}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}$$

is said to be a **generalized function** or a **distribution**. If there exists a locally integrable function f with

$$F = F_f,$$

then F is called a **regular generalized function** or a **regular distribution**. ◁

The set of all generalized functions is denoted by $\mathfrak{D}'(\mathbb{R}^n, \mathbb{R})$. The derivative F' of F is given by

$$F' : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto -F(\varphi').$$

Defining

$$\lambda F : \mathfrak{D}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto F(\lambda\varphi)\ (= \lambda F(\varphi)), \quad \lambda \in \mathbb{R},\ F \in \mathfrak{D}'(\mathbb{R}^n, \mathbb{R}),$$

and

$$F + G : \mathfrak{D}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto F(\varphi) + G(\varphi), \quad F, G \in \mathfrak{D}'(\mathbb{R}^n, \mathbb{R}),$$

we obtain a vector space $\mathfrak{D}'(\mathbb{R}^n, \mathbb{R})$ over $\mathbb{R}$ (the so-called **dual space** of $\mathfrak{D}(\mathbb{R}^n, \mathbb{R})$). The most important generalized functions, which are not regular, are given by the **Dirac distributions** $\delta_{\mathbf{x_0}}$:

$$\delta_{\mathbf{x_0}} : \mathfrak{D}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \varphi(\mathbf{x_0}), \quad \mathbf{x_0} \in \mathbb{R}^n.$$

It is easy to see, that each $\delta_{\mathbf{x_0}}$ is linear and continuous. Now, we prove by contradiction, that $\delta_{\mathbf{x_0}}$ is not regular for any $\mathbf{x_0} \in \mathbb{R}^n$. Assume there was a locally integrable function f such that

$$\int_{\mathbb{R}^n} f(\mathbf{x})\varphi(\mathbf{x})\, d\mathbf{x} = \varphi(\mathbf{x_0}) \quad \text{for all } \varphi \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}),$$

then there would exist an $\varepsilon > 0$ such that

$$\int_{\mathrm{cl}(K_{\mathbf{x_0},\varepsilon})} |f(\mathbf{x})|\, d\mathbf{x} = d < 1.$$

Using

$$\psi_{\mathbf{x_0},\varepsilon} : \mathbb{R}^n \to \mathbb{R}, \quad \boldsymbol{\xi} \mapsto \psi_\varepsilon(\boldsymbol{\xi} - \mathbf{x_0}),$$

we would obtain

$$\int_{\mathbb{R}^n} |f(\mathbf{x})\psi_{\mathbf{x_0},\varepsilon}(\mathbf{x})|\,d\mathbf{x} \le \sup_{\mathbf{x}\in \mathrm{cl}(K_{\mathbf{x_0},\varepsilon})} \{|\psi_{\mathbf{x_0},\varepsilon}(\mathbf{x})|\} \int_{\mathrm{cl}(K_{\mathbf{x_0},\varepsilon})} |f(\mathbf{x})|\,d\mathbf{x} =$$

$$= \psi_{\mathbf{x_0},\varepsilon}(\mathbf{x_0}) \cdot d < \psi_{\mathbf{x_0},\varepsilon}(\mathbf{x_0}),$$

which is a contradiction to

$$\int_{\mathbb{R}^n} f(\mathbf{x})\psi_{\mathbf{x_0},\varepsilon}(\mathbf{x})\,d\mathbf{x} = \psi_{\mathbf{x_0},\varepsilon}(\mathbf{x_0}).$$

If we approximate a Dirac distribution $\delta_{\mathbf{x_0}}$ by $F_{\psi_{\mathbf{x_0},r}}$, we obtain

$$F_{\psi_{\mathbf{x_0},r}}(\varphi) = \varphi(\mathbf{x}') \quad \text{with } \mathbf{x}' \in \mathrm{cl}(K_{\mathbf{x_0},r}) \quad \text{for all } \varphi \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}).$$

However, the limit of $\psi_{\mathbf{x_0},r}$ for $r \to 0$ does not define a regular function (see Fig. 1.11).

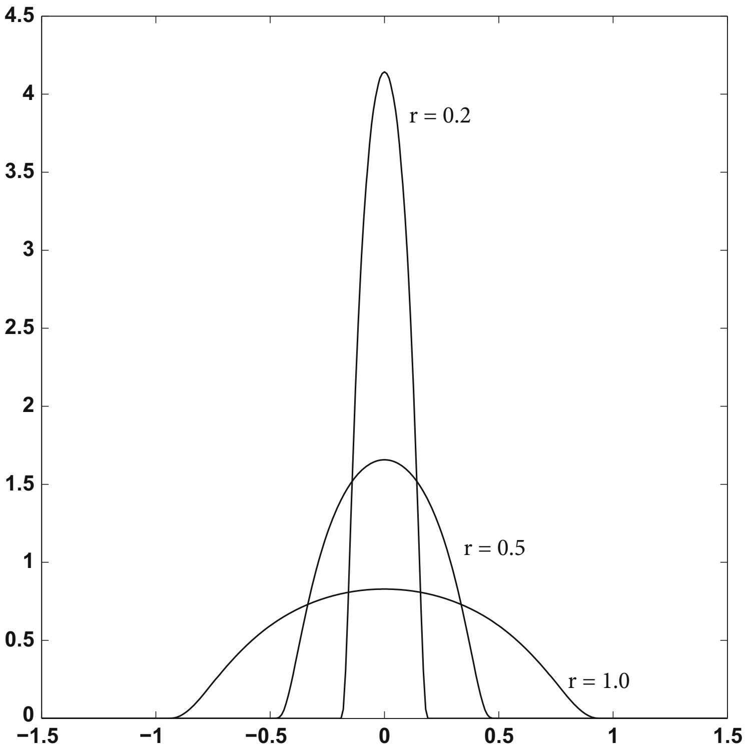

Fig. 1.11 Function $\psi_{0,r}$ for $r = 1.0, 0.5, 0.2$

The derivative of a Dirac distribution is given by

$$\delta_{x_0}' : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto -\varphi'(x_0).$$

Consider the Heaviside function

$$\eta : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 0 & \text{for } x \leq x_0 \\ 1 & \text{otherwise} \end{cases}.$$

We compute

$$F_\eta' : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto -\int_{-\infty}^{\infty} \eta(x)\varphi'(x)\,dx = -\int_{x_0}^{\infty} \varphi'(x)\,dx = \varphi(x_0).$$

Hence, the derivative of the Heaviside function is given by the corresponding Dirac distribution.

Let $y : \mathbb{R} \to \mathbb{R}$ be a solution of a given ordinary differential equation. Using

$$(F_y)' = F_{y'},$$

we may interpret the generalized function F_y as a solution of the proposed ordinary differential equation. In order to find solutions, which are generalized functions but not regular, we introduce an integral calculus for generalized functions. Using $g : \mathbb{R} \to \mathbb{R}$ and

$$G : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \int_{-\infty}^{x} g(\xi)d\xi \quad \text{(existence of the integral is assumed)},$$

we obtain by integration by parts

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{x} g(\xi)d\xi \cdot \varphi(x) \right) dx = \left[G(x) \int_{-\infty}^{x} \varphi(\xi)d\xi \right]_{-\infty}^{\infty} -$$
$$- \int_{-\infty}^{\infty} g(x) \left(\int_{-\infty}^{x} \varphi(\xi)d\xi \right) dx = - \int_{-\infty}^{\infty} g(x) \underbrace{\left(\int_{-\infty}^{x} \varphi(\xi)d\xi \right)}_{=:\hat{\varphi}(x)} dx + c,$$

where the existence of

$$c := \lim_{x \to \infty} G(x) \int_{-\infty}^{x} \varphi(\xi)d\xi$$

is assumed.

Unfortunately, the function $\hat{\varphi}$ defined by

$$\hat{\varphi} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \int_{-\infty}^{x} \varphi(\xi)d\xi, \quad \varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}),$$

is not a test function unless

$$\int_{-\infty}^{\infty} \varphi(\xi)d\xi = 0.$$

Setting

$$\mathfrak{D}_0(\mathbb{R}, \mathbb{R}) := \left\{ \psi \in \mathfrak{D}(\mathbb{R}, \mathbb{R});\ \int_{-\infty}^{\infty} \psi(\xi)d\xi = 0 \right\},$$

we get $\varphi \in \mathfrak{D}_0(\mathbb{R}, \mathbb{R})$ iff φ is the derivative χ' of a test function (see Problem 3.). Defining

$$\mathfrak{D}_1(\mathbb{R}, \mathbb{R}) := \left\{ \psi \in \mathfrak{D}(\mathbb{R}, \mathbb{R});\ \int_{-\infty}^{\infty} \psi(\xi)d\xi = 1 \right\}$$

and using a fixed test function $\psi_1 \in \mathfrak{D}_1(\mathbb{R}, \mathbb{R})$, there exists a unique $\varphi_0 \in \mathfrak{D}_0(\mathbb{R}, \mathbb{R})$ for each $\varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R})$ with

$$\varphi = \varphi_0 + \lambda \psi_1,$$

because

$$\int_{-\infty}^{\infty} \varphi(x)dx = \int_{-\infty}^{\infty} \varphi_0(x)dx + \lambda \int_{-\infty}^{\infty} \psi_1(x)dx = \lambda,$$

and hence

$$\varphi_0 = \varphi - \int_{-\infty}^{\infty} \varphi(x)dx \cdot \psi_1.$$

The mapping

$$P_{\psi_1} : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathfrak{D}_0(\mathbb{R}, \mathbb{R}), \quad \varphi \mapsto \varphi - \int_{-\infty}^{\infty} \varphi(x)dx \cdot \psi_1$$

defines a continuous linear projection for each $\psi_1 \in \mathfrak{D}_1(\mathbb{R}, \mathbb{R})$. Since

$$\int_{-\infty}^{\infty} \varphi'(x)dx = 0 \quad \text{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}),$$

we get

$$P_{\psi_1}(\varphi') = \varphi' \quad \text{for all} \quad \psi_1 \in \mathfrak{D}_1(\mathbb{R}, \mathbb{R}),\ \varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}).$$

Let F be a generalized function and $\psi_1 \in \mathfrak{D}_1(\mathbb{R}, \mathbb{R})$. We define an **indefinite integral G of F** as the generalized function

$$G : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto -F(\hat{\varphi}) \quad \text{with} \quad \hat{\varphi} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \int_{-\infty}^{x} \left(P_{\psi_1}(\varphi)\right)(\xi)d\xi.$$

This is motivated by

$$G'(\varphi) = -G(\varphi') = F(\varphi) \quad \text{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}),$$

since

$$\int_{-\infty}^{x} \left(P_{\psi_1}(\varphi')\right)(\xi)d\xi = \varphi(x) \quad \text{for all} \quad \psi_1 \in \mathfrak{D}_1(\mathbb{R}, \mathbb{R}),\ x \in \mathbb{R}.$$

The choice of a function $\psi_1 \in \mathfrak{D}_1(\mathbb{R}, \mathbb{R})$ corresponds to the choice of an additive constant c for a primitive function in Riemann integration. This becomes apparent, when we look for all generalized functions F with

$$F' = 0.$$

If $\varphi_0 \in \mathfrak{D}_0(\mathbb{R}, \mathbb{R})$, then there exists a test function $\psi \in \mathfrak{D}(\mathbb{R}, \mathbb{R})$ with $\psi' = \varphi_0$ and with

$$F(\varphi_0) = F(\psi') = -F'(\psi) = 0.$$

Using

$$F(\varphi) = F\left(\varphi_0 + \int_{-\infty}^{\infty} \varphi(x)dx \cdot \psi_1\right) = \int_{-\infty}^{\infty} \varphi(x)dx \cdot F(\psi_1),$$

we obtain

$$F : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \int_{-\infty}^{\infty} c\varphi(x)dx \quad \text{with} \quad c = F(\psi_1).$$

The function F is a regular generalized function representing the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto c = F(\psi_1),$$

which is a solution of the ordinary differential equation

$$f' = 0.$$

For locally integrable functions $g \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and $f : \mathbb{R}^n \to \mathbb{R}$, the function

$$gf : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto g(x) \cdot f(x)$$

is locally integrable as well. Hence, we obtain a regular generalized function F_{fg} representing gf by

$$F_{gf} : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \int_{-\infty}^{\infty} g(x)f(x)\varphi(x)dx = \int_{-\infty}^{\infty} f(x)(g(x)\varphi(x))dx.$$

This is the reason, why we define the product $g \cdot F$ of a function $g \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and a generalized function F in the following way:

$$\cdot : C^\infty(\mathbb{R}^n, \mathbb{R}) \times \mathfrak{D}'(\mathbb{R}, \mathbb{R}) \to \mathfrak{D}'(\mathbb{R}, \mathbb{R}), \quad (g, F) \mapsto g \cdot F =: gF$$

with

$$gF : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto F(g\varphi).$$

This approach is correct if gF is linear and continuous. The continuity of gF follows from the Leibniz formula (see [DuisKolk10]) while the linearity is obvious.
Now, we are able to investigate the ordinary differential equation

$$xf' = 0.$$

Using classical calculus, we obtain the complete solution

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto c_1, \quad c_1 \in \mathbb{R}.$$

Consider the generalized function $x\delta_0$, where

$$\delta_0 : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \varphi(0).$$

We get

$$x\delta_0 : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto (x\varphi)(0) = 0.$$

In distributional calculus, the derivative of the Heaviside function η is given by a Dirac distribution. We obtain the generalized function represented by

$$h : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 0 & \text{for } x \leq 0 \\ c_2 & \text{otherwise} \end{cases}, \quad c_2 \in \mathbb{R},$$

as a solution of

$$xf' = 0.$$

Finally, we get the regular generalized functions represented by

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} c_1 & \text{for } x \leq 0 \\ c_1 + c_2 & \text{otherwise} \end{cases}, \quad c_1, c_2 \in \mathbb{R},$$

as solutions of

$$xf' = 0.$$

Now, we investigate the ordinary differential equation

$$-x^3 f' = 2f$$

with generalized solutions

$$f : \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad x \mapsto c \cdot \exp\left(\frac{1}{x^2}\right), \quad c \in \mathbb{R}.$$

Since f is not locally integrable for $c \neq 0$, the regular generalized functions representing f for $c \neq 0$ do not describe solutions of

$$-x^3 f' = 2f.$$

It is possible to show, that this differential equation has a unique distributional solution given by

$$F : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto 0.$$

The transition from classical calculus to distributional calculus in solving the above ordinary differential equation leads to a loss of (not locally integrable) solutions.
On the other hand, under certain circumstances it is possible to assign a generalized function to a not locally integrable function. Assume $\mathbf{x_0} \in \mathbb{R}^n$ and

$$f : \mathbb{R}^n \setminus \{\mathbf{x_0}\} \to \mathbb{R}$$

such that the integral

$$\int\limits_{\{\mathbf{x}\in\mathbb{R}^n;\, \varepsilon\le\|\mathbf{x}-\mathbf{x_0}\|_2\le 1\}} f(\mathbf{x})d\mathbf{x}$$

exists for each $0 < \varepsilon < 1$. Furthermore, assume the existence of the limit

$$\lim_{\varepsilon\to 0} \int\limits_{\{\mathbf{x}\in\mathbb{R}^n;\, \varepsilon\le\|\mathbf{x}-\mathbf{x_0}\|_2\le 1\}} f(\mathbf{x})d\mathbf{x}.$$

Then, the integral

$$\mathrm{PV}\left(\int\limits_{\{\mathbf{x}\in\mathbb{R}^n;\, \|\mathbf{x}-\mathbf{x_0}\|_2\le 1\}} f(\mathbf{x})d\mathbf{x}\right) := \lim_{\varepsilon\to 0} \int\limits_{\{\mathbf{x}\in\mathbb{R}^n;\, \varepsilon\le\|\mathbf{x}-\mathbf{x_0}\|_2\le 1\}} f(\mathbf{x})d\mathbf{x}$$

is called **Cauchy principal value**. If f is integrable over each closed set $\mathrm{cl}(K_{\bar{\mathbf{x}},r})$, which does not contain the point $\mathbf{x_0}$, then we may define a non regular generalized function

$$F : \mathfrak{D}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \quad \mathrm{PV}\left(\int\limits_{\mathrm{cl}(K_{\mathbf{x_0},1})} f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}\right) + \\ + \int\limits_{\{\mathbf{x}\in\mathbb{R}^n;\, \|\mathbf{x}-\mathbf{x_0}\|_2>1\}} f(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}.$$

However, the Cauchy principal value

$$\mathrm{PV}\left(\int_{-1}^{1} \exp\left(\frac{1}{x^2}\right) dx\right)$$

does not exist.

Problems and Solutions

Problems

1. Prove, that

(i) $\mathrm{int}(M) = M$ for any open set M,
(ii) $\mathrm{cl}(M) = M$ for any closed set M,
(iii) $\partial(M) \cap M = \emptyset$ for any open set M,
(iv) $\partial(M) \subseteq M$ for any closed set M.

Specify a set $M \subseteq \mathbb{R}^n$ with

$$\operatorname{int}(\operatorname{cl}(M)) \neq \operatorname{int}(M).$$

2. Consider the regular generalized function

$$F : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \int_{-\infty}^{\infty} \ln(|x|)\varphi(x)dx.$$

Prove, that F' is given by

$$F' : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \mathrm{PV}\left(\int_{-1}^{1} \frac{\varphi(x)}{x}dx\right) + \int_{|x|>1} \frac{\varphi(x)}{x}dx.$$

3. Show that $\varphi \in \mathfrak{D}_0(\mathbb{R}, \mathbb{R})$ iff φ is the derivative of a test function.

4. Consider

$$\delta_0 : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \varphi(0)$$

and denote the derivative of the k-th order of δ_0 by $\delta_0^{(k)}$ (δ_0 for $k = 0$). Show that $x^m \delta^{(k)} \equiv 0$ for $0 \leq k < m$. Discuss the consequences of this identities for the solutions of the ordinary differential equation

$$x^m f' = 0.$$

5. Compute the derivatives of

$$F : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \int_{-\infty}^{\infty} |x|\varphi(x)dx.$$

Solutions

1.

(i) If M is an open set, then $M \in \{B \subseteq \mathbb{R}^n;\, B \text{ is open and } B \subseteq M\}$ and hence, $M \subseteq \operatorname{int}(M)$. The relation $\operatorname{int}(M) \subseteq M$ is obvious.

(ii) If M is a closed set, then $M \subseteq A$ for all

$$A \in \{B \subseteq \mathbb{R}^n;\, B \text{ is closed and } M \subseteq B\}.$$

Hence, $\operatorname{cl}(M) \subseteq M$. The relation $M \subseteq \operatorname{cl}(M)$ is obvious.

(iii) Since $M \subseteq \operatorname{cl}(M)$ and since

$$\partial(M) = \operatorname{cl}(M) \setminus \operatorname{int}(M) = \operatorname{cl}(M) \setminus M, \text{ when } \quad M \text{ is open},$$

we see that

$$\partial(M) \cap M = \emptyset, \text{ when } M \text{ is open.}$$

(iv) Since $\partial(M) \subseteq \mathrm{cl}(M)$ and since $\mathrm{cl}(M) = M$, when M is closed (see (ii)), the assertion is proven.

Choose $n = 1$ and $M = \mathbb{Q}$. We obtain

$$\mathrm{int}(\mathbb{Q}) = \emptyset \quad \text{and} \quad \mathrm{int}(\mathrm{cl}(\mathbb{Q})) = \mathrm{int}(\mathbb{R}) = \mathbb{R}.$$

2. Let $\mathrm{supp}(\varphi) \subseteq [-R, R]$, then we get with $0 < \alpha < R$:

$$F'(\varphi) = -\int_{-\infty}^{\infty} \ln(|x|)\varphi'(x)dx = \lim_{\alpha \to 0} \left(-\left[\varphi(x)\ln(|x|)\right]_{-R}^{-\alpha} - \left[\varphi(x)\ln(|x|)\right]_{\alpha}^{R}\right) +$$

$$+ \lim_{\varepsilon \to 0} \left(\int_{-1}^{-\varepsilon} \frac{\varphi(x)}{x} dx + \int_{\varepsilon}^{1} \frac{\varphi(x)}{x} dx \right) + \int_{|x|>1} \frac{\varphi(x)}{x} dx =$$

$$= \lim_{\alpha \to 0} \left(-\varphi(-\alpha)\ln(\alpha) + \varphi(\alpha)\ln(\alpha)\right) +$$

$$+ \mathrm{PV}\left(\int_{-1}^{1} \frac{\varphi(x)}{x} dx \right) + \int_{|x|>1} \frac{\varphi(x)}{x} dx =$$

$$= \lim_{\alpha \to 0} \left((\varphi(\alpha) - \varphi(-\alpha))\ln(\alpha)\right) + \mathrm{PV}\left(\int_{-1}^{1} \frac{\varphi(x)}{x} dx \right) + \int_{|x|>1} \frac{\varphi(x)}{x} dx.$$

Taylor expansion of

$$h : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \varphi(x) - \varphi(-x)$$

leads to:

$$h(x) = 2\varphi'(\theta)x \quad \text{with} \quad -x \le \theta \le x.$$

Thus, we obtain

$$F'(\varphi) = \lim_{\alpha \to 0} \left(2\alpha\varphi'(\theta)\ln(\alpha)\right) + \mathrm{PV}\left(\int_{-1}^{1} \frac{\varphi(x)}{x} dx \right) + \int_{|x|>1} \frac{\varphi(x)}{x} dx =$$

$$= \mathrm{PV}\left(\int_{-1}^{1} \frac{\varphi(x)}{x} dx \right) + \int_{|x|>1} \frac{\varphi(x)}{x} dx.$$

3. Let $\varphi \in \mathfrak{D}_0(\mathbb{R}, \mathbb{R})$ and

$$\psi : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \int_{-\infty}^{x} \varphi(\xi) d\xi,$$

then one gets

(i) $\psi \in C^\infty(\mathbb{R}, \mathbb{R})$,
(ii) $\psi(x) = 0$ for all $x \leq \min\{\xi; \xi \in \mathrm{supp}(\varphi)\}$,
(iii) $\psi(x) = 0$ for all $x \geq \max\{\xi; \xi \in \mathrm{supp}(\varphi)\}$.

Hence, ψ is a test function and $\varphi = \psi'$.
Let $\varphi = \psi'$, then we obtain

$$\int_{-\infty}^{\infty} \varphi(x)dx = \int_{-\infty}^{\infty} \psi'(x)dx = [\psi(x)]_{-\infty}^{\infty} = 0.$$

4. From

$$\delta_0^{(k)} : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto (-1)^k \varphi^{(k)}(0)$$

follows

$$x^m \delta_0^{(k)} : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto (-1)^k ((x^m \varphi(x))^{(k)}(0).$$

With

$$(x^m \varphi(x))^{(k)} = \sum_{i=0}^{k} \binom{k}{i} (x^m)^{(i)} \varphi^{(k-i)}(x)$$

we obtain

$$(x^m \varphi(x))^{(k)}(0) = 0 \quad \text{for} \quad 0 \leq k < m.$$

Thus, the solutions of the ordinary differential equation

$$x^m f' = 0$$

are given by any linear combination of the $m + 1$ generalized functions

$$F_1,\ F_2,\ F_3 = \delta_0,\ F_4 = \delta_0^{(1)}, \ldots, F_{m+1} = \delta_0^{(m-2)} \quad (\text{for } m > 2),$$

where F_2 denotes the regular generalized function of the Heaviside function and where F_1 denotes the regular generalized function of $y \equiv 1$.

5. Using $F'(\varphi) = -F(\varphi')$, we obtain

$$F^{(p)} : \mathfrak{D}(\mathbb{R},\mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto (-1)^p F\left(\varphi^{(p)}\right), \quad p \in \mathbb{N}.$$

With

$$F : \mathfrak{D}(\mathbb{R},\mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \int_{-\infty}^{\infty} |x|\varphi(x)dx$$

follows

$$F^{(p)} : \mathfrak{D}(\mathbb{R},\mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto (-1)^p \int_{-\infty}^{\infty} |x|\varphi^{(p)}(x)dx =$$

$$= (-1)^p \left(\int_{-\infty}^{0} (-x)\varphi^{(p)}(x)dx + \int_{0}^{\infty} x\varphi^{(p)}(x)dx \right) =$$

$$= (-1)^p \left(\left[(-x)\varphi^{(p-1)}(x)\right]_{-\infty}^{0} + \int_{-\infty}^{0} \varphi^{(p-1)}(x)dx \right) +$$

$$+ (-1)^p \left(\left[x\varphi^{(p-1)}(x)\right]_0^{\infty} - \int_0^{\infty} \varphi^{(p-1)}(x)dx \right) =$$

$$= \begin{cases} (-1)^p 2\varphi^{(p-2)}(0) & \text{for } p \geq 2 \\ \int_0^{\infty} \varphi(x)dx - \int_{-\infty}^{0} \varphi(x)dx & \text{for } p = 1 \end{cases}.$$

Using

$$\kappa : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} -1 & \text{for} \quad x \leq 0 \\ 1 & \text{for} \quad x > 0 \end{cases},$$

the solution for $p = 1$ is given by F_κ.

Stochastic Processes

Stefan Schäffler

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2.1 Background

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space. Each probability space is a model for a random experiment with a nonempty set Ω of the possible **results**. The elements of the σ-field $\mathcal{S}$ over Ω are called **events**, and the function

$$\mathbb{P} : \mathcal{S} \to [0, 1],$$

which assigns a probability $\mathbb{P}(A)$ to each event A, is called a **probability measure**. A nonempty set Γ and a σ-field $\mathcal{G}$ over Γ form a **measurable space** $(\Gamma, \mathcal{G})$. A $\mathcal{S}$-$\mathcal{G}$-measurable function

$$X : \Omega \to \Gamma$$

is called a **random variable**, where X is said to be $\mathcal{S}$-$\mathcal{G}$-measurable, if

$$X^{-1}(A') := \{\omega \in \Omega;\ X(\omega) \in A'\} \in \mathcal{S}$$

for each $A' \in \mathcal{G}$.

Setting

$$\mathbb{P}_X : \mathcal{G} \to [0, 1], \quad A' \mapsto \mathbb{P}\left(X^{-1}(A')\right),$$

we obtain a probability space $(\Gamma, \mathcal{G}, \mathbb{P}_X)$ (see Problem 1.). The probability measure $\mathbb{P}_X$ is called the **pushforward measure** or **image measure** of X.

Let I be a nonempty set and let

$$X_i : \Omega \to \Gamma$$

be a random variable for each $i \in I$, then the family

$$(X_i)_{i \in I} := \{X_i;\ i \in I\}$$

is called a **stochastic process with index set *I***.

Based on the cartesian product Γ^n of Γ, we choose the σ-field $\mathcal{G}^n$ over Γ^n as the smallest σ-field over Γ such that the projections

$$p_k : \Gamma^n \to \Gamma, \quad (\gamma_1, \ldots, \gamma_n) \mapsto \gamma_k, \quad k = 1, \ldots, n,$$

are $\mathcal{G}^n$-$\mathcal{G}$-measurable. The image measure $\mathbb{P}_{X_{i_1,\ldots,i_m}}$ of the random variable

$$X_{i_1,\ldots,i_m} : \Omega \to \Gamma^m, \quad \omega \mapsto (X_{i_1}(\omega), \ldots, X_{i_m}(\omega))$$

is called a **finite-dimensional distribution** of the stochastic process $(X_i)_{i \in I}$ for each $m \in \mathbb{N}$ with $m \leq |I|$ and for pairwise different $i_1, \ldots, i_m \in I$, where $|I|$ denotes the cardinality of I.

Since the σ-field $\mathcal{S}$ over Ω may contain events, which are not relevant for the investigation of a stochastic process $(X_i)_{i \in I}$, one commonly uses the smallest σ-field $\sigma(X_i;\ i \in I)$ over Ω such that each random variable X_i, $i \in I$, is $\sigma(X_i;\ i \in I)$-$\mathcal{G}$-masurable. We obtain

$$\sigma(X_i;\ i \in I) \subseteq \mathcal{S}.$$

The finite-dimensional distributions of a stochastic process $(X_i)_{i \in I}$ determine the probability measure given in $(\Omega, \mathcal{S}, \mathbb{P})$ for all events of the σ-field $\sigma(X_i; i \in I)$. Conversely, fixing the probabilities of all events in $\mathcal{G}^n$, $n \in \mathbb{N}$, in a suitable way, one is able to find a stochastic process, whose finite-dimensional distributions are given by the fixed probabilities. This is the main part of the existence theorem of Kolmogorov, (see [Øks10]), which we recapitulate in the following. Let I be a nonempty set and let Γ be a **polish space** (that is a separable complete metric space). Furthermore, let $\mathcal{G}$ be the σ-field generated by the open subsets of Γ and let $\mathbb{P}_J$ be a probability measure defined on $\mathcal{G}^J := \mathcal{G}^{|J|}$ for each finite nonempty subset J of I. Then there exist a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ and a stochastic process $(X_i)_{i \in I}$, whose finite-dimensional distributions are given by the probability measures $\mathbb{P}_J$, if the following condition holds:

$$\mathbb{P}_K(A) = \mathbb{P}_J\left(p_{JK}^{-1}(A)\right) \quad \text{for all} \quad A \in \mathcal{G}^K,$$

where

$$p_{JK} : \Gamma^J \to \Gamma^K, \quad (\gamma_i)_{i \in J} \mapsto (\gamma_i)_{i \in K}$$

and where K represents a nonempty subset of J. With the choice

$$\Omega = \Gamma^I := \{f; f : I \to \Gamma\}$$

and

$$\mathcal{S} = \mathcal{G}^I := \sigma(p_i;\ i \in I)$$

(the smallest σ-field over Ω such that each projection p_i is $\mathcal{G}^I$-$\mathcal{G}$-measurable), there exists a unique probability measure $\mathbb{P}$ defined on $\mathcal{S}$ such that the finite-dimensional distributions of $(X_i)_{i\in I}$ with $X_i = p_i$, $i \in I$ are given by $\mathbb{P}_J$.

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space, $(\Gamma, \mathcal{G})$ a measurable space, and let $(X_i)_{i\in I}$ be a stochastic process, where

$$X_i : \Omega \to \Gamma, \quad i \in I.$$

The mapping

$$X_\bullet(\omega) : I \to \Gamma, \quad i \mapsto X_i(\omega)$$

is called a **path**, a **realization**, or a **trajectory** of $(X_i)_{i\in I}$. Given a second stochastic process $(Y_i)_{i\in I}$ with

$$Y_i : \Omega \to \Gamma, \quad i \in I,$$

the stochastic processes $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ are called **undistinguishable**, if there exists a set $M \in \mathcal{S}$ such that $\mathbb{P}(M) = 1$ and

$$X_\bullet(\omega) = Y_\bullet(\omega) \quad \text{for all} \quad \omega \in M$$

(all paths are equal ($\mathbb{P}$-)almost surely).

A stochastic process $(X_i)_{i\in I}$ is called a **modification** or a **version** of $(Y_i)_{i\in I}$ and vice versa, if

$$\mathbb{P}(\{\omega \in \Omega;\ X_i(\omega) = Y_i(\omega)\}) = 1 \quad \text{for all} \quad i \in I.$$

A stochastic process $(X_i)_{i\in I}$ is a **modification** of a stochastic process $(Y_i)_{i\in I}$, if they are indistinguishable; the opposite is not true in general (see Problem 2.).

If there exists a commutative composition

$$+ : I \times I \to I,$$

then a stochastic process $(X_i)_{i\in I}$ is called a **stationary stochastic process**, if the finite-dimensional distributions of $(X_i)_{i\in I}$ and $(X_{j+i})_{i\in I}$ are equal for each $j \in I$. This property is called **shift-invariance** of the finite-dimensional distributions.

2

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let $(\mathbb{R}, \mathcal{B})$ be a measurable space with the Borelian σ-field $\mathcal{B}$ over $\mathbb{R}$ (for $\Omega = \mathbb{R}^n$ we are going to use the Borelian σ-field $\mathcal{B}^n$). Using two random variables

$$X, Y : \Omega \to \mathbb{R},$$

the real number

$$\mathscr{E}(X) := \int X d\mathbb{P} \quad \text{(existence assumed)}$$

is said to be the **expectation** of X, and the real number

$$\mathscr{V}(X) := \int (X - \mathscr{E}(X))^2 d\mathbb{P} \quad \text{(existence assumed)}$$

is said to be the **variance** of X. Under the assumption that the variances of X and Y exist, there exists the real number

$$\mathscr{C}(X, Y) := \int (X - \mathscr{E}(X))(Y - \mathscr{E}(Y)) d\mathbb{P},$$

which is called the **covariance** of X and Y. These definitions allow to generalize the notion of stationarity of a stochastic process. A stochastic process $(X_i)_{i \in I}$ with

$$X_i : \Omega \to \mathbb{R}, \quad i \in I,$$

and with a commutative composition

$$+ : I \times I \to I$$

is called a **weakly stationary stochastic process**, if there exist the expectation and the variance for all random variables X_i, $i \in I$, and if the following conditions are fulfilled:

(i) $\mathscr{E}(X_i) = m, \ i \in I,$
(equal expectation for all random variables)

(ii) $\mathscr{C}(X_i, X_{i+h}) = \mathscr{C}(X_j, X_{j+h}), \ i, j, h \in I,$
(the covariance of X_i and X_{i+h} depends only on h)

(iii) $\mathscr{V}(X_i) = \sigma^2, \ i \in I.$
(equal variance for all random variables)

The most important example of a non stationary stochastic process is given by a **Brownian Motion** $(B_t)_{t \in [0,\infty)}$ defined by real random variables

$$B_t : \Omega \to \mathbb{R}, \quad t \in [0, \infty),$$

where the image measures $\mathbb{P}_{B_{t_1,\ldots,t_n}}$, $0 \le t_1 < t_2 < \ldots < t_n$, $n \in \mathbb{N}$, are given by n-dim. Gaussian distributions with

$$\mathscr{E}(B_{t_j}) = 0, \quad j = 1, \ldots, n$$

and

$$\mathscr{C}(B_{t_j}, B_{t_k}) = \min(t_j, t_k), \quad j, k = 1, \ldots, n.$$

It is possible to show that there is a set $N \in \mathcal{S}$ with $\mathbb{P}(N) = 0$ such that all paths $B_\bullet(\omega)$ for $\omega \in \Omega \setminus N$ start at the origin (since $\mathscr{E}(B_0) = \mathscr{V}(B_0) = 0$) and are continuous, but nowhere differentiable. Figure 2.1 shows a typical path of a Brownian Motion for $t \in [0, 10]$.

The function w shown in Fig. 1.9 is composed by two different paths $B_\bullet(\omega_1)$ and $B_\bullet(\omega_2)$ of a Brownian Motion via

$$w : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} B_x(\omega_1) & \text{for } x \ge 0 \\ B_{-x}(\omega_2) & \text{for } x < 0 \end{cases}.$$

Let $\{Z_n\}_{n\in\mathbb{N}_0}$ be a sequence of stochastically independent standard Gaussian distributed random variables defined on a probability space $(\Omega, \mathcal{S}, \mathbb{P})$. Then it is possible to show

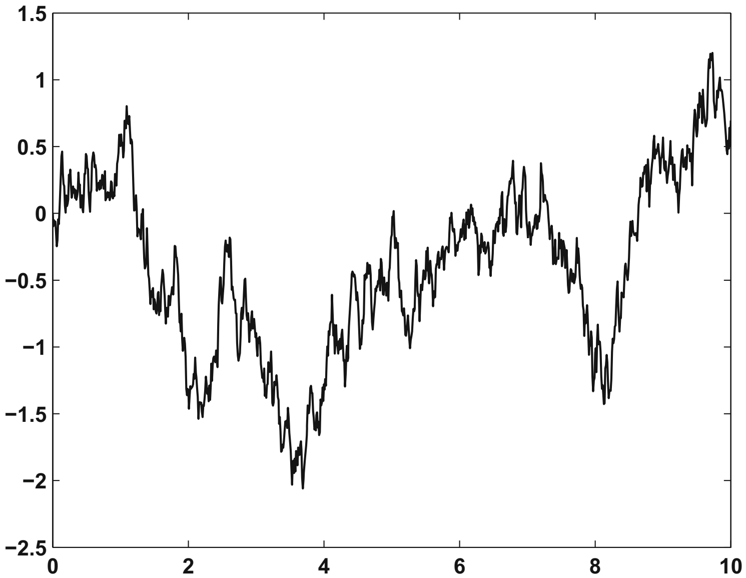


Fig. 2.1 A path of a Brownian Motion for $t \in [0, 10]$

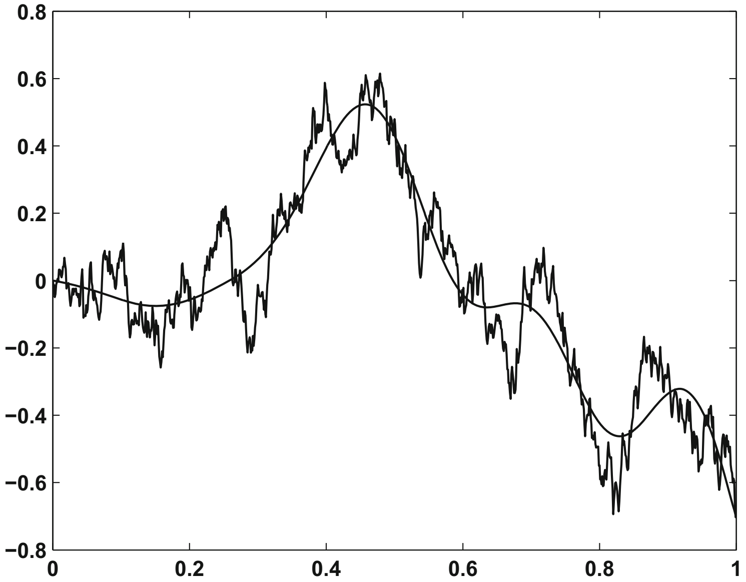


Fig. 2.2 Functions $f_{\hat{\omega},10}$ and $f_{\hat{\omega},1000}$

that the sequence

$$\left\{ Z_0 \cdot t + \sum_{k=1}^{n} Z_k \frac{\sqrt{2}\sin(k\pi t)}{k\pi} \right\}_{n\in\mathbb{N}}$$

converges ($\mathbb{P}$-)almost surely to $(\tilde{B}_t)_{t\in[0,1]}$ uniformly in $[0, 1]$. The random variables $\tilde{B}_t$, $t \in [0, 1]$ represent a Brownian Motion $(B_s)_{s\in[0,\infty)}$ for $s \in [0, 1]$ (see [Hida80], for instance). Figure 2.2 shows

$$f_{\hat{\omega},10} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto Z_0(\hat{\omega}) \cdot x + \sum_{k=1}^{10} Z_k(\hat{\omega}) \frac{\sqrt{2}\sin(k\pi x)}{k\pi}$$

and

$$f_{\hat{\omega},1000} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto Z_0(\hat{\omega}) \cdot x + \sum_{k=1}^{1000} Z_k(\hat{\omega}) \frac{\sqrt{2}\sin(k\pi x)}{k\pi}.$$

A very important example of a (weakly) stationary stochastic process is given by a **discrete white noise process**. Based on a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ and the index set $I = \mathbb{Z}$, a discrete white noise process $(w_i)_{i\in\mathbb{Z}}$ is defined by random variables

$$w_i : \Omega \to \mathbb{R}, \quad i \in \mathbb{Z},$$

with
(i) $\mathscr{E}(w_i) = 0, \quad i \in \mathbb{Z}$,
(ii) $\mathscr{C}(w_i, w_j) = \sigma^2\delta_{ij}$ (Kronecker delta), $\quad i, j \in \mathbb{Z}, \;\; \sigma^2 > 0$.

If all random variables w_i, $i \in \mathbb{Z}$, are Gaussian distributed, the stochastic process $(w_i)_{i\in\mathbb{Z}}$ is called a **discrete white Gaussian noise process**; a discrete white Gaussian noise process is a stationary stochastic process. The nomenclature results from the Fourier transform of the **covariance function** $\gamma : \mathbb{Z} \to \mathbb{R}$, $h \mapsto \mathscr{C}(w_0, w_h)$:

$$s : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}, \quad f \mapsto \sum_{h=-\infty}^{\infty} \gamma(h)e^{-i2\pi fh}$$

with inversion formula

$$\gamma(h) = \int_{-1/2}^{1/2} e^{i2\pi fh} s(f)df, \quad h \in \mathbb{Z}.$$

The function s is called the **spectral density function** of $(w_i)_{i\in\mathbb{Z}}$ and the value $s(f)$ quantizes the share of frequency f on the covariance function γ. For a discrete white noise process, we obtain

$$s \equiv \sigma^2.$$

Discrete white noise processes are used to define special classes of stochastic processes in time series analysis (MA-, AR-, and ARMA-processes) (see [Pr81]). For each $k \in \mathbb{N}$, an MA(k)-process $(X_n)_{n\in\mathbb{Z}}$ is defined by k real numbers $b_1, \ldots, b_k$ via

$$X_n : \Omega \to \mathbb{R}, \quad \omega \mapsto \sum_{i=0}^{k} b_i w_{n-i}(\omega), \quad n \in \mathbb{Z},$$

where $b_0 = 1$ and where $(w_i)_{i\in\mathbb{Z}}$ denotes a discrete white noise process.

The covariance function of an MA(k)-process is given by

$$\gamma_{\mathrm{MA}(k)} : \mathbb{Z} \to \mathbb{R}, \quad h \mapsto \begin{cases} \sum\limits_{i=0}^{k-h} b_i b_{i+h} & \text{for } 0 \le h \le k \\ \quad 0 & \text{for } |h| > k \\ \sum\limits_{i=0}^{k+h} b_i b_{i-h} & \text{for } -k \le h < 0 \end{cases}$$

with corresponding spectral density function

$$s_{\mathrm{MA}(k)} : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \mathbb{R}, \quad f \mapsto \sum_{i=0}^{k} b_i^2 + 2\sum_{h=1}^{k} \gamma_{\mathrm{MA}(k)}(h)\cos(2\pi hf).$$

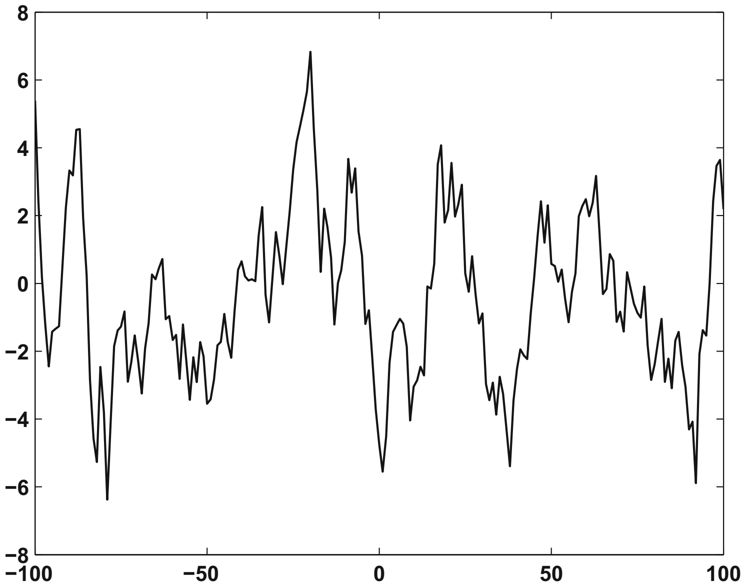


Fig. 2.3 A path of some MA(7)-process

Figure 2.3 shows a path of an MA(7)-process with $b_0, b_1, \ldots, b_7 = 1$ and Fig. 2.4 shows the corresponding spectral density function. Discrete white noise processes are used for the modelling of technical noise processes in a lot of applications.

There are many applications in engineering and natural sciences, where the perturbed quantities are continuous in time. For example, the charge of a capacitor in an RC circuit (see Fig. 2.5) is described by the initial value problem

$$R\dot{Q}(t) + \frac{1}{C}Q(t) = U(t), \quad Q(0) = Q_0, \ t \geq 0$$

with unique solution

$$Q : \mathbb{R}_0^+ \to \mathbb{R}, \quad t \mapsto e^{-\frac{1}{RC}t}\left(\int_0^t \frac{U(\tau)}{R}e^{\frac{1}{RC}\tau}d\tau + Q_0\right).$$

Now, we assume that we can only observe a perturbed solution, different from Q. To model the perturbation, we replace U by the voltage

$$U_\varepsilon : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}, \quad (\omega, t) \mapsto U(t) + \varepsilon_t(\omega),$$

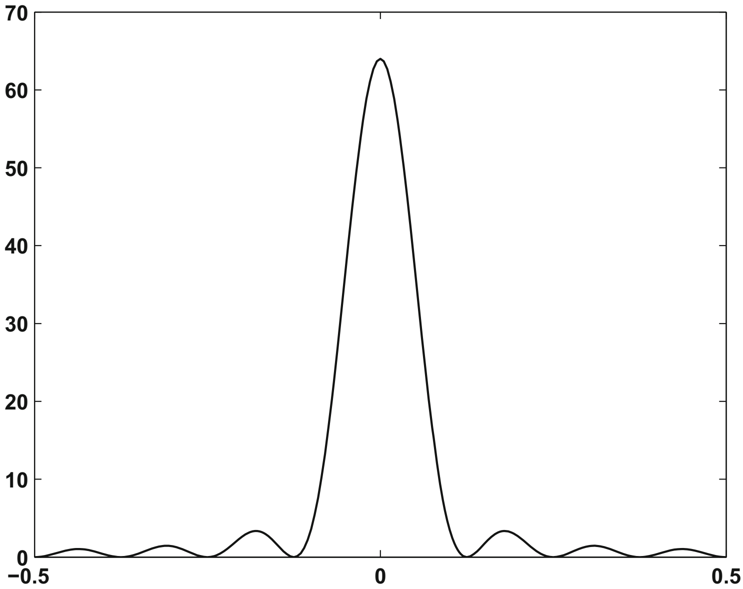


Fig. 2.4 Spectral density function of an MA(7)-process

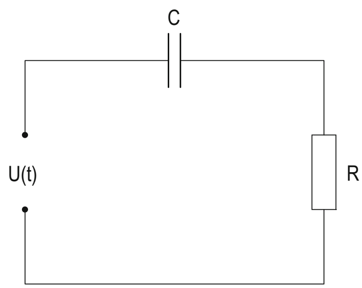


Fig. 2.5 RC circuit

where $(\varepsilon_t)_{t\in\mathbb{R}_0^+}$ is a stochastic process given by

$$\varepsilon_t : \Omega \to \mathbb{R}, \quad t \in \mathbb{R}_0^+.$$

Replacing U by U_ϵ in the solution formula for Q, we obtain a new solution

$$Q_\varepsilon : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}, \quad (\omega, t) \mapsto Q(t) + \frac{e^{-\frac{1}{RC}t}}{R} \int_0^t \varepsilon_\tau e^{\frac{1}{RC}\tau} d\tau,$$

where the integral

$$\int_0^t \varepsilon_\tau e^{\frac{1}{RC}\tau} d\tau$$

has to be defined in an appropriate way. A pathwise Riemann integration is not feasible, since a lot of noise processes are not integrable in this way.

Let $(\varepsilon_t)_{t \in \mathbb{R}_0^+}$ be a weakly stationary stochastic process with $\mathscr{E}(\varepsilon_0) = 0$ and with covariance function (symmetrically expanded on $\mathbb{R}$)

$$\gamma_\varepsilon : \mathbb{R} \to \mathbb{R}, \quad h \mapsto \begin{cases} \mathscr{C}(\varepsilon_0, \varepsilon_h) & \text{for } h \geq 0 \\ \mathscr{C}(\varepsilon_0, \varepsilon_{-h}) & \text{for } h < 0 \end{cases}.$$

Consider a sequence $\{t_0^i, \ldots, t_{k_i}^i\}_{i \in \mathbb{N}}$ of partitionings of the interval $[0, t]$, $t > 0$ with

$$0 = t_0^i < t_1^i < \ldots < t_{k_i}^i = t, \quad i \in \mathbb{N}, \; k_i \in \mathbb{N},$$

and with

$$\lim_{i \to \infty} \max\{t_j^i - t_{j-1}^i; \, j = 1, \ldots, k_i\} = 0.$$

Assuming the existence of a random variable $Y_t : \Omega \to \mathbb{R}$ with finite variance and with

$$\lim_{i \to \infty} \int \left(Y_t - \sum_{j=1}^{k_i} \varepsilon_{t_{j-1}^i} \cdot e^{\frac{1}{RC} t_{j-1}^i} \cdot (t_j^i - t_{j-1}^i) \right)^2 d\mathbb{P} = 0$$

for all sequences $\{t_0^i, \ldots, t_{k_i}^i\}_{i \in \mathbb{N}}$ of the above type, we define

$$\int_0^t \varepsilon_\tau e^{\frac{1}{RC}\tau} d\tau := Y_t.$$

A necessary and sufficient condition for the existence of Y_t is the existence of the Riemann integral:

$$\int_0^t \int_0^t e^{\frac{1}{RC}\tau} e^{\frac{1}{RC}\rho} \gamma_\varepsilon(\tau - \rho) d\tau d\rho \quad (= \mathscr{V}(Y_t)) \quad \text{(see [Pr81])}.$$

Assuming

(i) $\mathscr{E}(\varepsilon_\tau) = 0, \quad \tau \in \mathbb{R}_0^+$,

(ii) $\mathscr{C}(\varepsilon_\tau, \varepsilon_\rho) = \sigma^2 \delta_{\tau\rho}$ (Kronecker delta), $\quad \tau, \rho \in \mathbb{R}_0^+, \quad \sigma^2 > 0$,

we obtain

$$\mathscr{E}(Y_t) = 0 \quad \text{and} \quad \mathscr{V}(Y_t) = \int_0^t \int_0^t e^{\frac{1}{RC}\tau} e^{\frac{1}{RC}\rho} \sigma^2 \delta_{\tau\rho} d\tau d\rho = 0,$$

and consequently $Q_\varepsilon = Q$ (($\mathbb{P}$-)almost surely). The difference between Q and the measurement of the charge thus cannot be explained using a white noise process as additive perturbation of the voltage. Since

$$\int_{-\infty}^{\infty} \gamma_\varepsilon(\tau) e^{-2\pi i f \tau} d\tau = 0 \quad \text{for all} \quad f \in \mathbb{R},$$

there exists no spectral density function for $(\varepsilon_t)_{t\in\mathbb{R}_0^+}$.

Now, we investigate another type of stochastic noise processes called a **bandlimited white noise process** $(u_t)_{t\in\mathbb{R}_0^+}$, which is defined by

$$\mathscr{E}(u_t) = 0 \quad \text{for all} \quad t \in \mathbb{R}_0^+,$$

and by the covariance function

$$\gamma_u : \mathbb{R} \to \mathbb{R}, \quad h \mapsto \begin{cases} 2f_g \alpha \frac{\sin(2\pi f_g h)}{2\pi f_g h} & \text{for } h \geq 0 \\ 2f_g \alpha \frac{\sin(2\pi f_g (-h))}{2\pi f_g (-h)} & \text{for } h < 0 \end{cases}.$$

The corresponding spectral density function is equal to $\alpha > 0$ on the interval $[-f_g, f_g]$ and is zero elsewhere.

The noise voltage of a resistor with resistance R can be modelled by a bandlimited white noise process $(u_{R,t})_{t\in\mathbb{R}_0^+}$ with $f_g = 10^{10}$Hz and $\alpha = 2$RkT, where T is the absolute temperature and where

$$k = 1,38 \cdot 10^{-23} \frac{\text{Ws}}{\text{K}}$$

denotes the Boltzmann constant. The finite-dimensional distributions of $(u_{R,t})_{t\in\mathbb{R}_0^+}$ are assumed to be Gaussian. With $T = 290$K $(= 1685\ ^\circ$C) and $R = 1$kΩ we get $\alpha \approx 8 \cdot 10^{-18} \frac{\text{W}}{\text{Hz}} \Omega$. Figure 2.6 shows the corresponding covariance function γ_R for $-6 \cdot 10^{-10} \leq h \leq 6 \cdot 10^{-10}$ and Fig. 2.7 shows the covariance function γ_R for $-5 \cdot 10^{-9} \leq h \leq 5 \cdot 10^{-9}$. Sampling a path of $(u_{R,t})_{t\in\mathbb{R}_0^+}$ for $0 \leq t \leq 2$ using an equidistant discretization with $\Delta = 0.01$, we obtain realizations of almost uncorrelated random variables (see Fig. 2.8). On a more microscopic scale with $0 \leq t \leq 2 \cdot 10^{-9}$ and $\Delta = 10^{-11}$, the influence of the covariance function of $(u_{R,t})_{t\in\mathbb{R}_0^+}$ becomes significant, as Fig. 2.9 shows. The following important **continuity theorem of Kolmogorov-Chencov** shows, that we are able to choose a modification of $(u_{R,t})_{t\in\mathbb{R}_0^+}$ such that all paths are continuous. This was be done for the paths shown in Figs. 2.8 and 2.9.

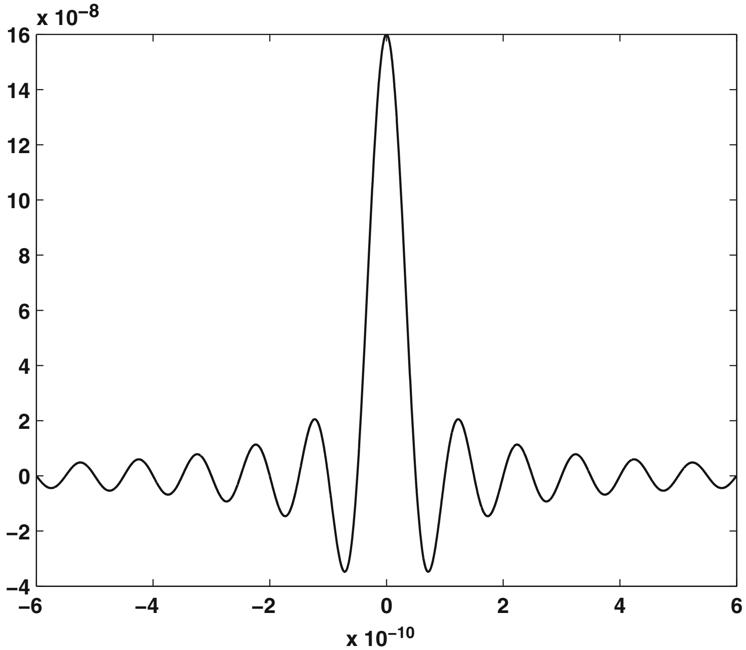


Fig. 2.6 γ_R for $-6 \cdot 10^{-10} \leq h \leq 6 \cdot 10^{-10}$

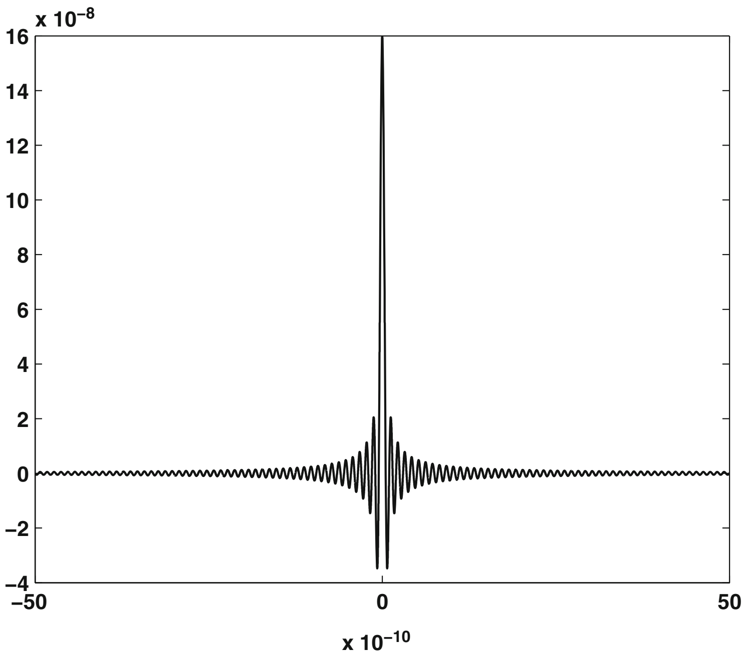


Fig. 2.7 γ_R for $-5 \cdot 10^{-9} \leq h \leq 5 \cdot 10^{-9}$

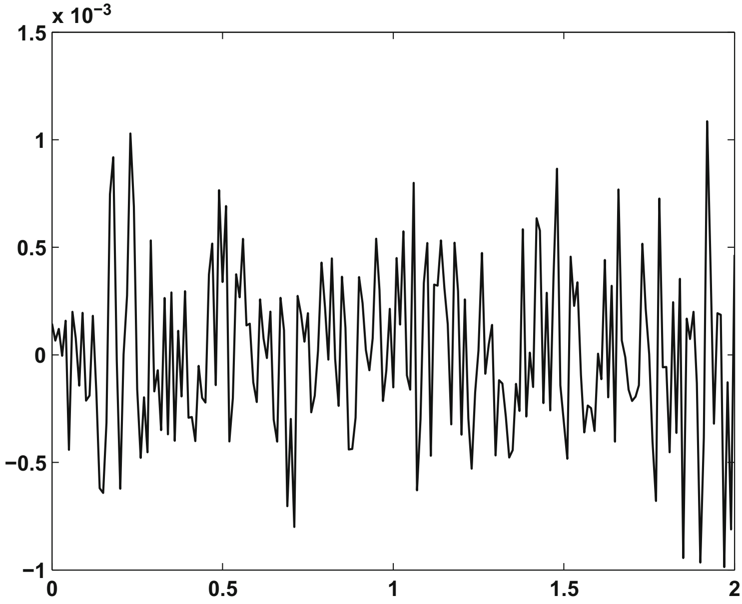


Fig. 2.8 A path of $(u_{R,t})_{t\in\mathbb{R}}$ for $0 \le t \le 2$, $\Delta = 0.01$

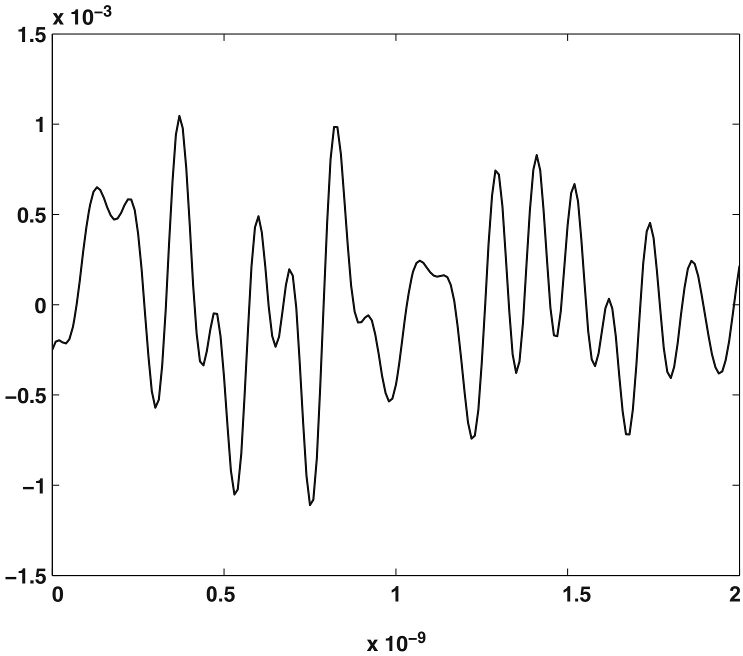


Fig. 2.9 A path of $(u_{R,t})_{t\in\mathbb{R}}$ for $0 \le t \le 2 \cdot 10^{-9}$, $\Delta = 10^{-11}$

Theorem 2.1 (Continuity Theorem of Kolmogorov-Chencov)
Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space, let Γ be a polish space with corresponding metric d, and let $\mathcal{B}_d(\Gamma)$ be the σ-field generated by the open subsets of Γ using the metric d. Furthermore, let $(X_i)_{i\in\mathbb{R}_0^+}$ be a stochastic process given by random variables

$$X_i : \Omega \to \Gamma, \quad i \in \mathbb{R}_0^+.$$

Then we can choose a modification of $(X_i)_{i\in\mathbb{R}_0^+}$ with continuous paths, if there exist real numbers $\zeta, \beta, C > 0$ for each $T \in (0, \infty)$ such that

$$\mathcal{E}\left(d(X_t, X_s)^\zeta\right) \leq C|t-s|^{1+\beta} \quad \textit{for all} \quad s, t \in [0, T].$$

◁

See [StrVar79] for a proof. Using Taylor's theorem

$$1 - \frac{\sin(x)}{x} = \left|\left(1 - \frac{\sin(\xi)}{\xi}\right)''\right| x^2 = \left|\frac{\xi^2 \sin(\xi) + 2\xi\cos(\xi) - 2\sin(\xi)}{\xi^3}\right| x^2 \leq$$
$$\leq \frac{1}{3}x^2 \quad \text{with} \quad \xi \in \begin{cases} [0, x] & \text{for } 0 \leq x \\ [x, 0] & \text{for } x < 0 \end{cases},$$

we obtain:

$$\mathcal{E}((u_t - u_s)^2) = \mathcal{E}(u_t^2) - 2\mathcal{E}(u_t u_s) + \mathcal{E}(u_s^2) = 4f_g\alpha\left(1 - \frac{\sin(2\pi f_g(t-s))}{2\pi f_g(t-s)}\right).$$

Hence, the choice $\zeta = 2$ and $\beta = 1$ shows the applicability of theorem 2.1 to a bandlimited white noise process.

The following application leads us to the theory of generalized stochastic processes, which can also be used to analyze RC circuits.

We assume that a single bit $b \in \{\pm 1\}$ is transmitted by a signal

$$s : [0, 2\pi] \to \mathbb{R}, \quad t \mapsto b \cdot \sin(t).$$

The receiver obtains a noisy signal $\tilde{s}$ in general. In order to detect the transmitted bit, the convolution

$$c_{2\pi} = \frac{1}{\pi}\int_0^{2\pi} \tilde{s}(2\pi - \tau)\sin(2\pi - \tau)d\tau$$

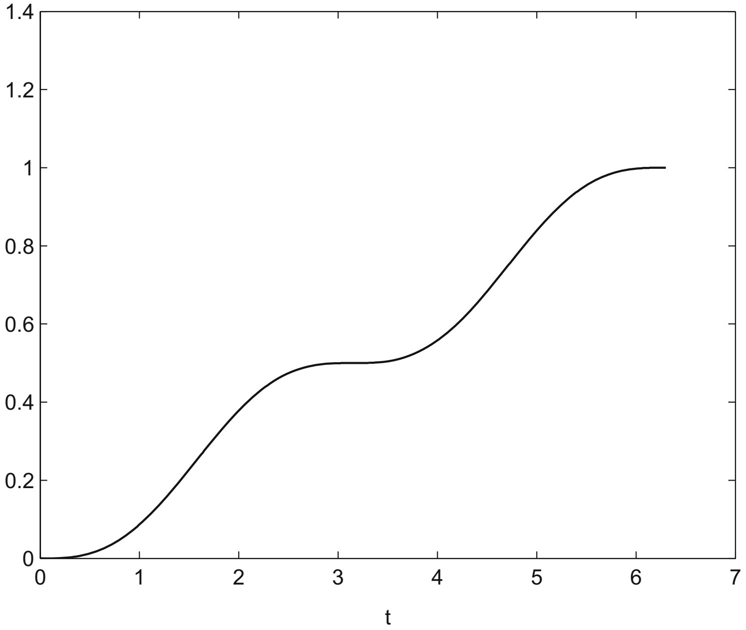


Fig. 2.10 $\frac{1}{\pi}\int\limits_0^t \sin^2(2\pi-\tau)d\tau, \quad 0 \le t \le 2\pi, b=1$

is computed. For $\tilde{s} = s$, we obtain

$$c_{2\pi} = b \quad \text{(see Fig. 2.10)}.$$

If $\tilde{s}$ is obtained from s by adding a stochastic process $(r_t)_{t\in[0,2\pi]}$ modelling a perturbation, then we will get

$$c(\bullet, 2\pi) : \Omega \to \mathbb{R}, \quad \omega \mapsto b + \frac{1}{\pi}\left(\int\limits_0^{2\pi} r_{2\pi-\tau}\sin(2\pi-\tau)d\tau\right)(\omega),$$

where

$$c : \Omega \times [0, 2\pi] \to \mathbb{R}, \qquad (\omega, t) \mapsto \frac{b}{\pi}\int\limits_0^t \sin^2(2\pi-\tau)d\tau +$$
$$+ \frac{1}{\pi}\left(\int\limits_0^t r_{2\pi-\tau}\sin(2\pi-\tau)d\tau\right)(\omega).$$

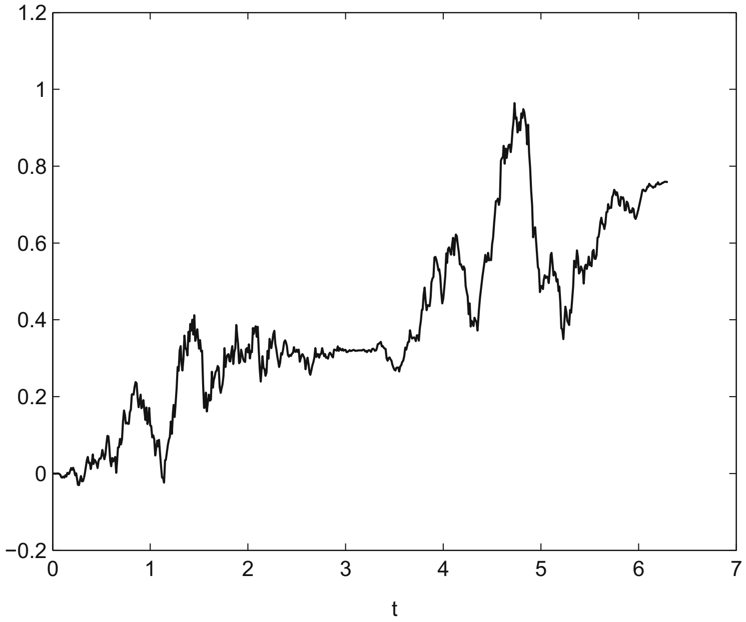


Fig. 2.11 A path of a function $c, b = 1$

Using noise processes $(r_t)_{t\in[0,2\pi]}$ consisting of pairwise uncorrelated random variables with expectation value equal to zero and with identical variance as for the noise process $(\varepsilon_t)_{t\in\mathbb{R}_0^+}$ used in the analysis of the RC circuit above, which arise in space probe communication for instance, one obtains functions c shown in Fig. 2.11. On the other hand, we get

$$\int_0^t r_{2\pi-\tau}\sin(2\pi-\tau)d\tau = 0 \quad (\mathbb{P}\text{-})\text{almost surely} \quad \text{for all} \quad t \in [0, 2\pi]$$

for these processes, again as in the analysis of the RC circuit. In order to resolve this problem, we consider functions

$$f_k : \mathbb{R} \to \mathbb{R}^+, \quad t \mapsto \frac{k}{2}e^{-k|t|}, \quad k > 0.$$

We obtain (see Fig. 2.12)

$$\lim_{k\to\infty} f_k(t) = 0 \quad \text{for all} \quad t \neq 0,$$

$$\lim_{k\to\infty} f_k(0) = \infty,$$

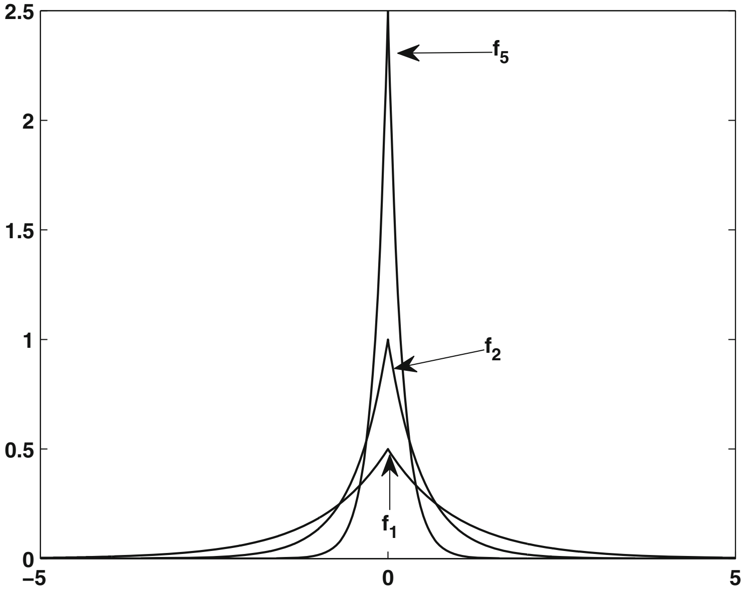


Fig. 2.12 $f_k, k = 1, 2, 5$

and

$$\int_{-\infty}^{\infty} f_k(x)dx = 1 \quad \text{for all} \quad k > 0.$$

Furthermore, the spectral density functions for f_k are given by

$$s_k : \mathbb{R} \to \mathbb{R}, \quad f \mapsto \frac{k^2}{k^2 + (2\pi f)^2}$$

with

$$\lim_{k\to\infty} s_k(f) = 1 \quad \text{for all} \quad f \in \mathbb{R} \quad (\text{see Fig. 2.13}).$$

Since

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} f_k(x)\varphi(x)dx = \varphi(0)$$

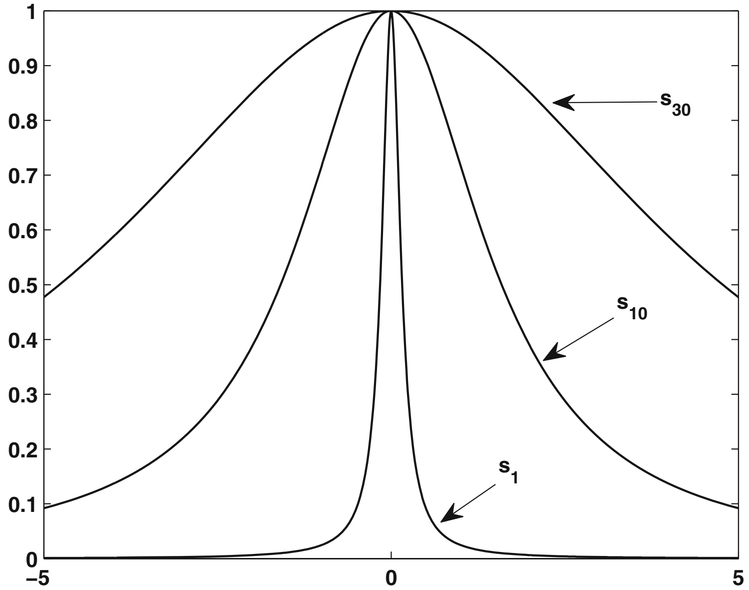


Fig. 2.13 $s_k, k = 1, 10, 30$

for all test functions $\varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R})$, we may interpret the covariance function corresponding to a spectral density function

$$s : \mathbb{R} \to \mathbb{R}, \quad f \mapsto 1 \quad \text{(continuous white noise process)}$$

as a generalized function (namely the Dirac distribution δ_0); this is possible, if we introduce stochastic processes with index set I consisting of the set of test functions $\mathfrak{D}(\mathbb{R}, \mathbb{R})$.

2.2 Stochastic Processes with Index Sets Consisting of Test Functions

Let $\mathfrak{D}(\mathbb{R}, \mathbb{R})$ be the set of all test functions for $n = 1$ (see Definition 1.1). In the following, the set $\mathfrak{D}(\mathbb{R}, \mathbb{R})$ represents the index set I of stochastic processes given by random variables

$$X_\varphi : \Omega \to \mathbb{R}, \quad \varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}),$$

based on a probability space $(\Omega, \mathcal{S}, \mathbb{P})$. Hence, a path of $(X_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ is given by

$$X_\bullet(\omega) : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto X_\varphi(\omega).$$

Analogously to Definition 1.4, we postulate linearity and continuity of

$$(X_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$$

in the following manner.

Definition 2.2 (Generalized Stochastic Process)
A stochastic process $(X_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ defined on a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ and consisting of real valued random variables is called a **generalized stochastic process**, if

$$X_{a\varphi+b\psi} = aX_\varphi + bX_\psi \quad (\mathbb{P}\text{-})\text{almost surely} \quad (\textit{linearity})$$

holds for all $a, b \in \mathbb{R}$ and for all $\varphi, \psi \in \mathfrak{D}(\mathbb{R},\mathbb{R})$, and if $(X_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ is continuous in the following sense: Let $\{\varphi_{ik}\}_{k\in\mathbb{N}}$, $i = 1, \dots, m$, be a convergent sequence of test functions (in the sense of Definition 1.3) for all $m \in \mathbb{N}$ such that

$$\lim_{k\to\infty} \varphi_{ik} = \varphi_i, \quad i = 1, \dots, m,$$

then for every continuous bounded function $g : \mathbb{R}^m \to \mathbb{R}$ holds:

$$\lim_{k\to\infty} \mathscr{E}(g(X_{\varphi_{1k}}, \dots, X_{\varphi_{mk}})) = \mathscr{E}(g(X_{\varphi_1}, \dots, X_{\varphi_m})).$$

(convergence in distribution) ◁

The term *generalized stochastic process* means to use generalized functions to characterize a special type of stochastic processes and does not mean a generalization of stochastic processes.

For each generalized stochastic process there exists a modification such that its paths are generalized functions (see, e.g. [Daw70]).

Let $(\varepsilon_t)_{t\in\mathbb{R}}$ be a stochastic process with locally integrable paths, then we are able to consider a generalized stochastic process $(X_\varphi^\varepsilon)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ defined by

$$X_\varphi^\varepsilon : \Omega \to \mathbb{R}, \quad \omega \mapsto \int_{-\infty}^{\infty} \varepsilon_t(\omega)\varphi(t)dt, \quad \varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R}),$$

where the paths of $(X_\varphi^\varepsilon)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ are generalized functions. Let $(X_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ be a generalized stochastic process with existing expectation $\mathscr{E}(X_\varphi)$ for each $\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})$, then the functional

$$m : \mathfrak{D}(\mathbb{R},\mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \mathscr{E}(X_\varphi),$$

2

is linear (see Problem 3.). In addition, m is a generalized function if m is continuous. Under the assumption of existing variance $\mathscr{V}(X_\varphi)$ for each $\varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R})$, the functional

$$\mathscr{C} : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad (\varphi, \psi) \mapsto \mathscr{E}\left((X_\varphi - m(\varphi))(X_\psi - m(\psi))\right)$$

is called **covariance functional**. Since

$$\mathscr{C}(\lambda\varphi, \mu\psi) = \mathscr{C}(\mu\psi, \lambda\varphi) = \lambda\mu\mathscr{C}(\varphi, \psi) \quad \text{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}),\ \lambda, \mu \in \mathbb{R},$$

and

$$\mathscr{C}(\varphi, \varphi) = \mathscr{E}\left((X_\varphi - m(\varphi))^2\right) \geq 0 \quad \text{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}),$$

the covariance functional $\mathscr{C}$ represents a positive semidefinite symmetrical bilinear form. Investigating $(X_\varphi^\varepsilon)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ given by

$$X_\varphi^\varepsilon : \Omega \to \mathbb{R}, \quad \omega \mapsto \int_{-\infty}^{\infty} \varepsilon_t(\omega)\varphi(t)dt, \quad \varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}),$$

and assuming the existence of $\mathscr{E}(\varepsilon_t)$ and $\mathscr{V}(\varepsilon_t)$ for all $t \in \mathbb{R}$, we obtain

$$\mathscr{E}(X_\varphi^\varepsilon) = \int \int_{-\infty}^{\infty} \varepsilon_t(\omega)\varphi(t)dtd\mathbb{P} = \int_{-\infty}^{\infty} \mathscr{E}(\varepsilon_t)\varphi(t)dt \quad \text{for all} \quad \mathfrak{D}(\mathbb{R}, \mathbb{R}),$$

if the interchange of the order of integration is allowed. Hence, the functional

$$m : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \mathscr{E}(X_\varphi^\varepsilon)$$

represents the function

$$\mathscr{E} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \mathscr{E}(\varepsilon_t)$$

as a generalized function. Computation of the covariance functional leads to

$$\begin{aligned}
&\mathscr{C}(X_\varphi^\varepsilon, X_\psi^\varepsilon) = \mathscr{E}\left((X_\varphi^\varepsilon - \mathscr{E}(X_\varphi^\varepsilon))(X_\psi^\varepsilon - \mathscr{E}(X_\varphi^\varepsilon))\right) = \\
&= \int \left(\int_{-\infty}^{\infty} \varepsilon_t(\omega)\varphi(t)dt - \mathscr{E}(X_\varphi^\varepsilon) \right) \left(\int_{-\infty}^{\infty} \varepsilon_s(\omega)\psi(s)ds - \mathscr{E}(X_\psi^\varepsilon) \right) d\mathbb{P} = \\
&= \int \left(\int_{-\infty}^{\infty} (\varepsilon_t(\omega) - \mathscr{E}(\varepsilon_t))\varphi(t)dt \right) \left(\int_{-\infty}^{\infty} (\varepsilon_s(\omega) - \mathscr{E}(\varepsilon_s))\psi(s)ds \right) d\mathbb{P} =
\end{aligned}$$

$$= \int \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ((\varepsilon_t(\omega) - \mathscr{E}(\varepsilon_t))(\varepsilon_s(\omega) - \mathscr{E}(\varepsilon_s)))\, \varphi(t)\psi(s) dt ds d\mathbb{P} =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int (\varepsilon_t - \mathscr{E}(\varepsilon_t))(\varepsilon_s - \mathscr{E}(\varepsilon_s)) d\mathbb{P} \varphi(t)\psi(s) dt ds =$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathscr{C}(\varepsilon_t, \varepsilon_s)\varphi(t)\psi(s) dt ds.$$

Now, we are able to define the derivative of a generalized stochastic process analogously to the definition of the derivative of a generalized function.

Definition 2.3 (Derivative of a Generalized Stochastic Process)

Let $(X_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ be a generalized stochastic process defined on a probability space $(\Omega, \mathcal{S}, \mathbb{P})$, then the generalized stochastic process $(-X_{\varphi'})_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ is called **derivative** of $(X_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ (denoted by $(X'_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$). ◁

Let $(X_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ be a generalized stochastic process such that the expectation $\mathscr{E}(X_\varphi)$ exists for each $\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})$, then using

$$m : \mathfrak{D}(\mathbb{R},\mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \mathscr{E}(X_\varphi)$$

we compute

$$\mathscr{E}(X'_\varphi) = \mathscr{E}(-X_{\varphi'}) = -m(\varphi') = m'(\varphi) \quad \text{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R}).$$

If the variance $\mathscr{V}(X_\varphi)$ exists for all $\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})$, then we are able to compute the covariance functional $\mathscr{C}'$ of $(X'_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$:

$$\begin{aligned}
\mathscr{C}'(\varphi, \psi) &= \mathscr{E}\left((X'_\varphi - m'(\varphi))(X'_\psi - m'(\psi))\right) = \\
&= \mathscr{E}\left((-X_{\varphi'} + m(\varphi'))(-X_{\psi'} + m(\psi'))\right) = \\
&= \mathscr{E}\left((X_{\varphi'} - m(\varphi'))(X_{\psi'} - m(\psi'))\right) = \\
&= \mathscr{C}(\varphi', \psi') \quad \text{for all} \quad \varphi, \psi \in \mathfrak{D}(\mathbb{R},\mathbb{R}).
\end{aligned}$$

The following theorem deals with the derivative of a Brownian Motion. For this purpose, we summarize the properties of a Brownian Motion. Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let $(B_t)_{t \in [0,\infty)}$ be a Brownian Motion consisting of random variables

$$B_t : \Omega \to \mathbb{R}, \quad t \in [0, \infty).$$

Then, the image measure $\mathbb{P}_{B_{t_1,\ldots,t_n}}$ with $0 \leq t_1 < t_2 < \ldots < t_n$, $n \in \mathbb{N}$, is given by an n-dim. Gaussian distribution with

$$\mathscr{E}(B_{t_j}) = 0, \quad j = 1, \ldots, n,$$

and with

$$\mathscr{C}(B_{t_j}, B_{t_k}) = \min(t_j, t_k), \quad j, k = 1, \ldots, n.$$

Hence, the increments

$$B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_n} - B_{t_{n-1}}$$

are stochastically independent (see Problem 7.). Furthermore, ($\mathbb{P}$-)almost all paths

$$B_\bullet(\omega) : [0, \infty) \to \mathbb{R}, \quad \omega \in \Omega,$$

of a Brownian Motion are continuous, but nowhere differentiable (using the classical calculus). Since the random variables

$$\frac{B_{t_2} - B_{t_1}}{t_2 - t_1}, \frac{B_{t_3} - B_{t_2}}{t_3 - t_2}, \ldots, \frac{B_{t_n} - B_{t_{n-1}}}{t_n - t_{n-1}}$$

are stochastically independent and since

$$\mathscr{E}\left(\frac{B_{t_k} - B_{t_{k-1}}}{t_k - t_{k-1}}\right) = 0, \quad \mathscr{V}\left(\frac{B_{t_k} - B_{t_{k-1}}}{t_k - t_{k-1}}\right) = \frac{1}{t_k - t_{k-1}}, \quad k = 2, \ldots, n,$$

we would expect the derivative of a Brownian Motion to be a stationary stochastic process consisting of stochastically independent random variables with expectation equal to zero and with infinite variance. In order to compute the derivative of a Brownian Motion, we have to represent a Brownion Motion as a generalized stochastic process. To this end, we consider a generalized stochastic process $(B_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ with

$$B_\varphi : \Omega \to \mathbb{R}, \quad \omega \mapsto \int_0^\infty B_t(\omega)\varphi(t)dt \quad \text{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}).$$

The approximation of the above integrals by Riemann sums shows that all finite-dimensional distributions of $(B_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ are Gaussian with vanishing expectation. The covariance functional is given by

$$\mathscr{C}(\varphi, \psi) = \int_0^\infty \int_0^\infty \min(t, s)\varphi(t)\psi(s)dtds \quad \text{for all} \quad \varphi, \psi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}).$$

Theorem 2.4 (Derivative of a Brownian Motion)
Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let $(B_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ be a generalized stochastic process with Gaussian finite-dimensional distributions, with

$$\mathscr{E}\left(B_\varphi\right) = 0 \quad \textit{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R}),$$

and with

$$\mathscr{C}(\varphi, \psi) = \int_0^\infty \int_0^\infty \min(t,s)\varphi(t)\psi(s)dtds \quad \textit{for all} \quad \varphi, \psi \in \mathfrak{D}(\mathbb{R},\mathbb{R}),$$

then $(B'_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ is a generalized stochastic process with Gaussian finite-dimensional distributions, with

$$\mathscr{E}\left(B'_\varphi\right) = 0 \quad \textit{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R}),$$

and with covariance functional

$$\mathscr{C}'(\varphi, \psi) = \int_0^\infty \varphi(t)\psi(t)dt \quad \textit{for all} \quad \varphi, \psi \in \mathfrak{D}(\mathbb{R},\mathbb{R}).$$

◁

Before proving this theorem, we analyze the covariance functional

$$\mathscr{C}'(\varphi, \psi) = \int_0^\infty \varphi(t)\psi(t)dt \quad \text{for all} \quad \varphi, \psi \in \mathfrak{D}(\mathbb{R},\mathbb{R}).$$

Assume the existence of a function $c : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ such that

$$\mathscr{C}'(\varphi, \psi) = \int_0^\infty \int_0^\infty c(s,t)\varphi(t)\psi(s)dtds \quad \text{for all} \quad \varphi, \psi \in \mathfrak{D}(\mathbb{R},\mathbb{R}).$$

Hence, $\mathscr{C}'$ would be the representation of the covariance function c of $(B'_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ as generalized function.

From

$$\begin{aligned}\mathscr{C}'(\varphi, \psi) &= \int_0^\infty \varphi(t)\psi(t)dt = \int_0^\infty \int_0^\infty c(s,t)\varphi(t)\psi(s)dtds = \\ &= \int_0^\infty \varphi(t) \int_0^\infty c(s,t)\psi(s)dsdt \quad \text{for all} \quad \varphi, \psi \in \mathfrak{D}(\mathbb{R},\mathbb{R})\end{aligned}$$

we obtain the following property of c:

$$\int_0^\infty c(s,t)\psi(s)ds = \psi(t) \quad \text{for all} \quad \psi \in \mathfrak{D}(\mathbb{R},\mathbb{R}),\ t \in \mathbb{R}_0^+.$$

This equation describes the Dirac distribution

$$\delta_t : \mathfrak{D}(\mathbb{R},\mathbb{R}) \to \mathbb{R}, \quad \psi \mapsto \psi(t), \quad t \in \mathbb{R}_0^+.$$

Therefore, the function c can be interpreted as pointwise limit of

$$c_r : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}, \quad (s,t) \mapsto \frac{1}{\sqrt{2\pi r^2}} e^{-\frac{(s-t)^2}{2r^2}}$$

for $r \to 0$ (see Figs. 2.14 and 2.15), as we expected. For the spectral density functions s_r of

$$g_r : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{1}{\sqrt{2\pi r^2}} e^{-\frac{x^2}{2r^2}}$$

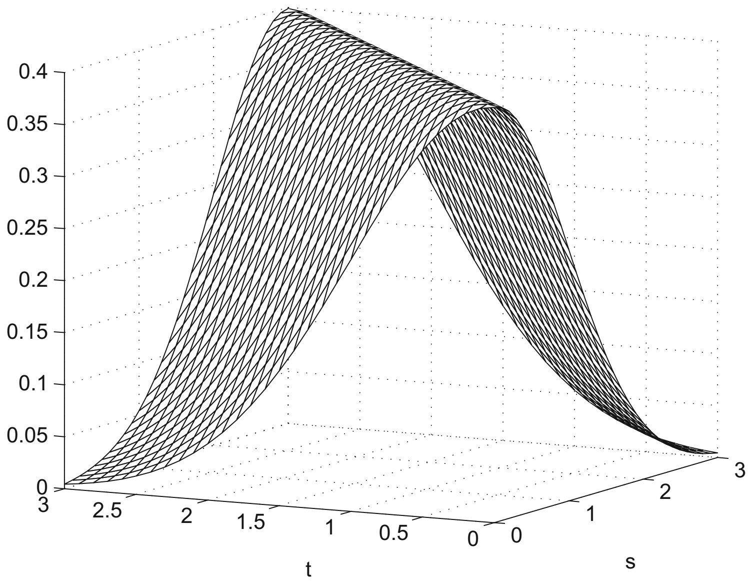


Fig. 2.14 Function c_1

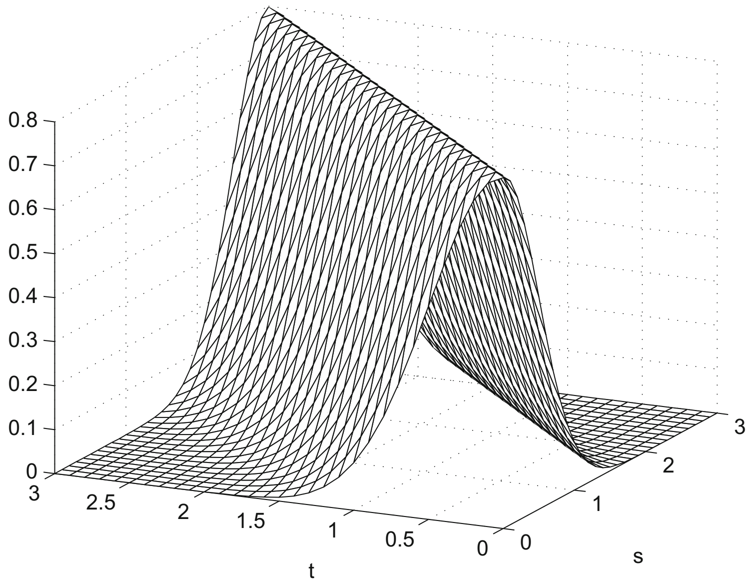


Fig. 2.15 Function $c_{0.5}$

we obtain

$$\lim_{r \longrightarrow 0} s_r(f) = \lim_{r \longrightarrow 0} e^{-2\pi^2 f^2 r^2} = 1 \quad \text{for all} \quad f \in \mathbb{R}.$$

Proof (of Theorem 2.4)
Since $B'_\varphi = -B_{\varphi'}$, the finite-dimensional distributions of $(B'_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ are Gaussian with

$$\mathscr{E}\left(B'_\varphi\right) = 0 \quad \text{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R}).$$

The covariance functional of $(B_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ is given by

$$\begin{aligned}
\mathscr{C}(\varphi,\psi) &= \iint\limits_{0 \le s \le t} \min(t,s)\varphi(t)\psi(s)dtds + \iint\limits_{0 \le t \le s} \min(t,s)\varphi(t)\psi(s)dtds = \\
&= \int_0^\infty \int_0^t s\varphi(t)\psi(s)dsdt + \int_0^\infty \int_0^s t\varphi(t)\psi(s)dtds = \\
&= \int_0^\infty \varphi(t) \int_0^t s\psi(s)dsdt + \int_0^\infty \psi(s) \int_0^s t\varphi(t)dtds.
\end{aligned}$$

Integrating by parts with $v' = \varphi$ and $u = \int_0^t s\psi(s)ds$ and with $v' = \psi$ and $u = \int_0^s t\varphi(t)dt$ respectively, we obtain

$$
\begin{aligned}
\mathscr{C}(\varphi, \psi) &= \int_0^\infty \varphi(t) \int_0^t s\psi(s)dsdt + \int_0^\infty \psi(s) \int_0^s t\varphi(t)dtds = \\
&= \int_0^\infty (\Phi(\infty) - \Phi(t))t\psi(t)dt + \int_0^\infty (\Psi(\infty) - \Psi(t))t\varphi(t)dt = \\
&= \int_0^\infty t\left((\Phi(\infty) - \Phi(t))\psi(t) + (\Psi(\infty) - \Psi(t))\varphi(t)\right) dt,
\end{aligned}
$$

where

$$
\Phi : \mathbb{R}_0^+ \to \mathbb{R}, \quad t \mapsto \int_0^t \varphi(s)ds \quad \text{and} \quad \Phi(\infty) := \int_0^\infty \varphi(s)ds
$$

and where

$$
\Psi : \mathbb{R}_0^+ \to \mathbb{R}, \quad t \mapsto \int_0^t \psi(s)ds \quad \text{and} \quad \Psi(\infty) := \int_0^\infty \psi(s)ds.
$$

Integration by parts with $v' = (\Phi(\infty) - \Phi(t))\psi(t) + (\Psi(\infty) - \Psi(t))\varphi(t)$ (and hence, $v = -(\Phi(\infty) - \Phi(t))(\Psi(\infty) - \Psi(t))$ and $u = t$ leads to

$$
\begin{aligned}
\mathscr{C}(\varphi, \psi) &= \lim_{t\to\infty} tv(t) + \int_0^\infty (\Phi(\infty) - \Phi(t))(\Psi(\infty) - \Psi(t))dt = \\
&= -\lim_{t\to\infty} t^2 v'(t) + \int_0^\infty (\Phi(\infty) - \Phi(t))(\Psi(\infty) - \Psi(t))dt = \\
&= \int_0^\infty (\Phi(\infty) - \Phi(t))(\Psi(\infty) - \Psi(t))dt.
\end{aligned}
$$

Therefore, we obtain

$$
\mathscr{C}'(\varphi, \psi) = \mathscr{C}\left(\varphi', \psi'\right) = \int_0^\infty \varphi(t)\psi(t)dt \quad \text{for all} \quad \varphi, \psi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}).
$$

□

A generalized stochastic process with Gaussian finite-dimensional distributions is called a **Gaussian generalized stochastic process**. The derivative of a Gaussian generalized stochastic process is again a Gaussian generalized stochastic process.

Definition 2.5 (Gaussian White Noise Process)
A Gaussian generalized stochastic process $(R_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ with

$$\mathscr{E}\left(R_\varphi\right) = 0 \quad \text{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R}),$$

and with covariance functional

$$\mathscr{C}\,(\varphi,\psi) = \int_0^\infty \varphi(t)\psi(t)dt \quad \text{for all} \quad \varphi,\psi \in \mathfrak{D}(\mathbb{R},\mathbb{R})$$

is called a **Gaussian white noise process on** $\mathbb{R}_0^+$. If the covariance functional is given by

$$\mathscr{C}\,(\varphi,\psi) = \int_{-\infty}^\infty \varphi(t)\psi(t)dt \quad \text{for all} \quad \varphi,\psi \in \mathfrak{D}(\mathbb{R},\mathbb{R}),$$

then $(R_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ is called a **Gaussian white noise process on** $\mathbb{R}$. ◁

Consider a generalized stochastic process $(X_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ with covariance functional $\mathscr{C}\,(\varphi,\psi)$ such that

$$\mathscr{C}\,(\varphi,\psi) = 0 \quad \text{for all} \quad \varphi,\psi \in \mathfrak{D}(\mathbb{R},\mathbb{R}) \text{ with } \operatorname{supp}(\varphi)\cap\operatorname{supp}(\psi) = \emptyset,$$

then $(X_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ is called a **generalized stochastic process with independent values**. A Gaussian white noise process is of this type.
A very important class of stochastic processes is given by **Poisson processes**. Based on a probability space $(\Omega,\mathcal{S},\mathbb{P})$, a random variable

$$X : \Omega \to \mathbb{N}_0$$

(thus, an $\mathcal{S}$-$\mathcal{P}(\mathbb{N}_0)$-measurable mapping) is said to be **Poisson distributed**, if for each $\lambda > 0$ the image measure $\mathbb{P}_X$ of X is given by

$$\mathbb{P}_X(\{k\}) = \frac{\lambda^k}{k!}e^{-\lambda}, \qquad k \in \mathbb{N}_0.$$

We obtain (see Problem 4.):

$$\mathscr{E}(X) = \lambda \quad \text{und} \quad \mathscr{V}(X) = \lambda.$$

A Poisson distributed random variable counts the number of events, which occur mutually independent at a fixed time interval using a constant mean rate λ. A stochastic process $(P_t)_{t\in[0,\infty)}$ consisting of random variables

$$P_t : \Omega \to \mathbb{N}_0, \quad t \in [0,\infty),$$

is called **Poisson process**, if the random variable P_t is Poisson distributed with

$$\mathscr{E}(P_t) = \mathscr{V}(P_t) = \lambda t, \quad t \in [0,\infty), \quad \lambda > 0,$$

and if

$$\mathscr{C}(P_s, P_t) = \lambda \cdot \min\{s,t\} \quad s,t \in [0,\infty).$$

Let $(P_t)_{t\in[0,\infty)}$ be a Poisson process with parameter $\lambda > 0$. Investigating the generalized stochastic process $(P_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ given by

$$P_\varphi : \Omega \to \mathbb{R}, \quad \omega \mapsto \int_0^\infty P_t(\omega)\varphi(t)dt, \quad \text{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R}),$$

we obtain for the derivative $(P'_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$:

$$\mathscr{E}(P'_\varphi) = -\int_0^\infty \lambda t\varphi'(t)dt = \lambda \int_0^\infty \varphi(t)dt$$

and

$$\mathscr{C}'(\varphi,\psi) = \mathscr{C}\left(\varphi',\psi'\right) = \lambda \int_0^\infty \varphi(t)\psi(t)dt \quad \text{for all} \quad \varphi,\psi \in \mathfrak{D}(\mathbb{R},\mathbb{R}).$$

The generalized stochastic process $(P'_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ is called **Poissonian white noise process**.
As an example of a Poisson process consider the random emission of electrons in electronic devices like diodes or transistors, which is modelled by a Poisson process $(P_t)_{t\in[0,\infty)}$ with parameter λ, then each electron causes a current pulse described by a function ρ. This current pulse is given by

$$\rho_\alpha : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} \frac{\alpha e}{z^2}t & \text{for } 0 \le t \le z \\ 0 & \text{elsewhere} \end{cases}$$

in a vacuum tube for instance, where e denotes the charge of the electron, α denotes a tube specific constant, and where z denotes the transition time of the electron from the cathode to the anode (see [BeiMont03]). Hence, the power induced by emitted electrons defines a stochastic process $(I_t)_{t\in[0,\infty)}$ given by

$$I_t : \Omega \to \mathbb{R}, \quad \omega \mapsto \begin{cases} \sum_{k=1}^{P_t(\omega)} \rho(t - T_k(\omega)) & \text{for } P_t(\omega) > 0 \\ 0 & \text{elsewhere} \end{cases}, \quad t \in [0, \infty),$$

where

$$T_1, T_2, \ldots : \Omega \to [0, \infty)$$

denote the moments, at which the electrons have been emitted. The number of summands in the above sum is given by the number of electrons, which have been emitted within the time interval $[0, t]$. From properties of Poisson processes (see, e.g. [Bei97]) we know that the random variables $T_1, \ldots, T_n$ are stochastically independent and uniformly distributed on $[0, t]$, if exactly n electrons have been emitted within the time interval $[0, t]$. Under this assumption, we are able to compute the distribution function of I_t, if the function ρ is known:

$$\mathbb{P}(\{\omega \in \Omega;\, I_t(\omega) \le c\}) = \sum_{n=0}^{\infty} \mathbb{P}(\{\omega \in \Omega;\, I_t(\omega) \le c\} | P_t = n) \frac{(\lambda t)^n}{n!} e^{-\lambda t} =$$

$$= e^{-\lambda t} + \sum_{n=1}^{\infty} \mathbb{P}\left(\left\{\omega \in \Omega;\, \sum_{k=1}^{n} \rho(t - T_k(\omega)) \le c\right\}\right) \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \quad c \ge 0.$$

The expectation and the covariance function of I_t are given by (**Campbell formulas**):

$$\mathscr{E}(I_t) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} \mathscr{E}(\rho(t - T_k)) \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \sum_{n=1}^{\infty} n \int_0^t \frac{\rho(t-\tau)}{t} d\tau \frac{(\lambda t)^n}{n!} e^{-\lambda t} =$$

$$= \int_0^t \rho(\tau) d\tau \frac{1}{t} \sum_{n=1}^{\infty} n \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \lambda \int_0^t \rho(\tau) d\tau.$$

and

$$\mathscr{C}(I_t, I_s) = \lambda \int_0^{\min(t,s)} \rho(\tau)\rho(|t - s| + \tau) d\tau.$$

Using $\rho = \rho_\alpha$, we obtain for $t = s + h$, $h \geq 0$:

$$\mathscr{E}(I_t) = \begin{cases} \lambda \frac{\alpha e}{2z^2} t^2 & \text{for } t \leq z \\ \lambda \frac{\alpha e}{2} & \text{elsewhere} \end{cases}, \quad t \in [0, \infty),$$

and

$$\mathscr{C}(I_t, I_s) = \begin{cases} \lambda \frac{\alpha^2 e^2}{z^4} \int\limits_0^{z-h} \tau(h+\tau)d\tau & \text{for } h \leq z \leq t \\ \lambda \frac{\alpha^2 e^2}{z^4} \int\limits_0^{s} \tau(h+\tau)d\tau & \text{for } \quad z > t \\ 0 & \text{for } \quad z < h \end{cases}$$

$$= \begin{cases} \lambda \frac{\alpha^2 e^2}{3z} \left(1 - \frac{3h}{2z} + \frac{h^3}{2z^3}\right) & \text{for } h \leq z \leq t \\ \lambda \frac{\alpha^2 e^2}{6z^4} \left(3hs^2 + 2s^3\right) & \text{for } \quad z > t \\ 0 & \text{for } \quad z < h \end{cases}.$$

Since the transition time z in ρ_α is very small, the stochastic process I_t is usually approximated by a Poissonian white noise process, which is called a **shot noise**. This can be done for each type of commonly used functions ρ.

2.3 Gaussian Generalized Stochastic Processes

In this section, we investigate a very important class of stochastic processes, namely Gaussian generalized stochastic processes.

Theorem 2.6 (Gaussian Generalized Stochastic Process)
Let

$$m : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$$

be a continuous linear functional and let

$$\mathscr{C} : \mathfrak{D}(\mathbb{R}, \mathbb{R}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$$

(continued)

Theorem 2.6 (continued)
be a continuous positive semidefinite symmetrical bilinear form, then there exist a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ *and a Gaussian generalized stochastic process* $(G_\varphi)_{\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})}$ *with expectation*

$$\mathscr{E}\left(G_\varphi\right) = m(\varphi) \quad \textit{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}),$$

and with covariance functional $\mathscr{C}$*. This generalized stochastic process is unique up to modifications.* ◁

Proof
We only sketch the main idea of the proof. For details see [GelWil64]. First, let $\mathscr{C}$ be a continuous positive definite symmetrical bilinear form, then the matrix

$$\mathbf{C} = \begin{pmatrix} \mathscr{C}(\varphi_1, \varphi_1) & \mathscr{C}(\varphi_1, \varphi_2) & \cdots & \mathscr{C}(\varphi_1, \varphi_{n-1}) & \mathscr{C}(\varphi_1, \varphi_n) \\ \mathscr{C}(\varphi_2, \varphi_1) & \mathscr{C}(\varphi_2, \varphi_2) & \cdots & \mathscr{C}(\varphi_2, \varphi_{n-1}) & \mathscr{C}(\varphi_2, \varphi_n) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathscr{C}(\varphi_{n-1}, \varphi_1) & \mathscr{C}(\varphi_{n-1}, \varphi_2) & \cdots & \mathscr{C}(\varphi_{n-1}, \varphi_{n-1}) & \mathscr{C}(\varphi_{n-1}, \varphi_n) \\ \mathscr{C}(\varphi_n, \varphi_1) & \mathscr{C}(\varphi_n, \varphi_2) & \cdots & \mathscr{C}(\varphi_n, \varphi_{n-1}) & \mathscr{C}(\varphi_n, \varphi_n) \end{pmatrix}$$

is positive definite, if the test functions $\varphi_1, \ldots, \varphi_n$, $n \in \mathbb{N}$, are linearly independent. Hence, there exist a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ and a random variable

$$Y : \Omega \to \mathbb{R}^n,$$

which is unique ($\mathbb{P}$-)almost surely and which is Gaussian distributed with expectation

$$\mathscr{E}(Y) = \begin{pmatrix} m(\varphi_1) \\ \vdots \\ m(\varphi_n) \end{pmatrix}$$

and with covariance matrix $\mathbf{C}$. We choose

$$\begin{pmatrix} G_{\varphi_1} \\ \vdots \\ G_{\varphi_n} \end{pmatrix} = Y.$$

If the test functions $\varphi_1, \ldots, \varphi_n$ are not linearly independent, then there exist a natural number $q < n$ and q linearly independent test functions $\psi_1, \ldots, \psi_q$ with

$$\varphi_i = \sum_{k=1}^{q} b_{i,k} \psi_k, \quad b_{i,k} \in \mathbb{R}, \quad i = 1, \ldots, n, \quad k = 1, \ldots, q.$$

Using Definition 2.2, we are able to choose

$$G_{\varphi_i} = \sum_{k=1}^{q} b_{i,k} G_{\psi_k} \quad \text{for all } \ i = 1, \dots, n.$$

Therefore, the distribution of $(G_{\varphi_1}, \dots, G_{\varphi_n})$ is uniquely determined by the distribution of

$$(G_{\psi_1}, \dots, G_{\psi_k}).$$

Furthermore, the random vector $(G_{\psi_1}, \dots, G_{\psi_k})$ is q-dimensional Gaussian distributed with expectation

$$\begin{pmatrix} m(\psi_1) \\ \vdots \\ m(\psi_q) \end{pmatrix}$$

and with positive definite covariance matrix

$$\mathbf{K} = \begin{pmatrix} \mathscr{C}(\psi_1, \psi_1) & \mathscr{C}(\psi_1, \psi_2) & \cdots & \mathscr{C}(\psi_1, \psi_{q-1}) & \mathscr{C}(\psi_1, \psi_q) \\ \mathscr{C}(\psi_2, \psi_1) & \mathscr{C}(\psi_2, \psi_2) & \cdots & \mathscr{C}(\psi_2, \psi_{q-1}) & \mathscr{C}(\psi_2, \psi_q) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathscr{C}(\psi_{q-1}, \psi_1) & \mathscr{C}(\psi_{q-1}, \psi_2) & \cdots & \mathscr{C}(\psi_{q-1}, \psi_{q-1}) & \mathscr{C}(\psi_{q-1}, \psi_q) \\ \mathscr{C}(\psi_q, \psi_1) & \mathscr{C}(\psi_q, \psi_2) & \cdots & \mathscr{C}(\psi_q, \psi_{q-1}) & \mathscr{C}(\psi_q, \psi_q) \end{pmatrix}.$$

Now let $\mathscr{C}$ be a continuous positive *semi*definite symmetrical bilinear form, then we are able to represent test functions $\varphi_1, \dots, \varphi_n$ by

$$\varphi_i = \sum_{k=1}^{p} b_{i,k} \psi_k + \sum_{k=p+1}^{K} b_{i,k} \xi_k, \quad b_{i,k} \in \mathbb{R}, \quad i = 1, \dots, n, \quad k = 1, \dots, K,$$

where

(i) the test functions $\psi_1, \dots, \psi_p, \xi_{p+1}, \dots, \xi_K$ form a basis of $\operatorname{span}\{\varphi_1, \dots, \varphi_n\}$,

(ii) the covariance functional $\mathscr{C}$ restricted to $\operatorname{span}\{\psi_1, \dots, \psi_p\}$ is represented by a positive definite matrix,

(iii) the covariance functional $\mathscr{C}$ restricted to $\operatorname{span}\{\xi_{p+1}, \dots, \xi_K\}$ is represented by the zero matrix.

Using Definition 2.2 again, we are able to choose

$$G_{\varphi_i} = \sum_{k=1}^{p} b_{i,k} G_{\psi_k} + \sum_{k=p+1}^{K} b_{i,k} G_{\xi_k}, \quad b_{i,k} \in \mathbb{R}, \quad i = 1, \dots, n, \quad k = 1, \dots, K,$$

and we obtain

$$G_{\varphi_i} = \sum_{k=1}^{p} b_{i,k} G_{\psi_k} + \sum_{k=p+1}^{K} b_{i,k} m(\xi_k), \quad b_{i,k} \in \mathbb{R}, \quad i = 1, \ldots, n, \quad k = 1, \ldots, K,$$

($\mathbb{P}$-)almost surely. □

Consider a stochastic process $(\mathbf{B}_t)_{t\in[0,\infty)}$ given by random variables

$$\mathbf{B}_t : \Omega \to \mathbb{R}^n, \quad t \in [0, \infty), \quad n \in \mathbb{N},$$

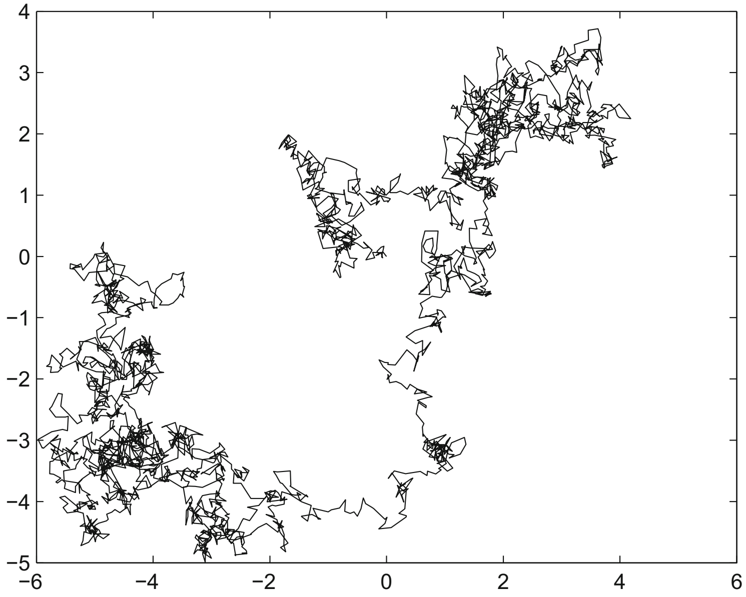
where the image measure $\mathbb{P}_{\mathbf{B}_t}$ is given by an n-dimensional Gaussian distribution with

$$\mathscr{E}(\mathbf{B}_t) = \mathbf{0}, \quad t \in [0, \infty),$$

and with covariances

$$\mathscr{C}((\mathbf{B}_t)_i, (\mathbf{B}_s)_j) = \begin{cases} \min(s,t) & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}, \quad i, j \in \{1, \ldots, n\},\ s, t \in [0, \infty),$$

then $(\mathbf{B}_t)_{t\in[0,\infty)}$ is called an n-dimensional Brownian Motion. ◘ Figure 2.16 shows a path of a two-dimensional Brownian Motion. A generalized stochastic process



◘ **Fig. 2.16** A path of a two-dimensional Brownian Motion

$(\mathbf{B}_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ representing an n-dimensional Brownian Motion is defined component for component:

$$\mathbf{B}_\varphi : \Omega \to \mathbb{R}^n, \quad \omega \mapsto \begin{pmatrix} \int_0^\infty (\mathbf{B}_t)_1(\omega)\varphi(t)dt \\ \vdots \\ \int_0^\infty (\mathbf{B}_t)_n(\omega)\varphi(t)dt \end{pmatrix}.$$

Using

$$\mathbf{R}_\varphi : \Omega \to \mathbb{R}^n, \quad \omega \mapsto \begin{pmatrix} -\int_0^\infty (\mathbf{B}_t)_1(\omega)\varphi'(t)dt \\ \vdots \\ -\int_0^\infty (\mathbf{B}_t)_n(\omega)\varphi'(t)dt \end{pmatrix}$$

we obtain an n-dimensional Gaussian white noise process $(\mathbf{R}_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$.

2.4 Wiener Integration

There are many applications in engineering and natural sciences, where noise processes are not directly observable. Considering the charge of a capacitor in an RC circuit for instance, noise effects are observable after the integration of an initial value problem. We observed the same fact investigating the transmission of signals (see ▶ Sect. 2.1). Therefore, we have to develop an integration calculus for functions multiplicatively disturbed by a Gaussian white noise process (so-called **Wiener integration**). Since Gaussian white noise processes are defined as generalized stochastic processes, we are not able to use a classical approach (e.g. Riemann sums) for the definition of a Wiener integral. On the other hand, we know that a Gaussian white noise process can be interpreted as derivative of a Brownian Motion.

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space, $(B_t)_{t\in[0,\infty)}$ be a one-dimensional Brownian Motion, and let

$$f : [0, T] \to \mathbb{R}, \quad T > 0,$$

be a continuous function, which is multiplicatively disturbed by a Gaussian white noise process, then we are able to approximate the disturbed function r. For that, we use a sequence of partitions $\{t_0^i, \ldots, t_{k_i}^i\}_{i\in\mathbb{N}}$ with

$$0 = t_0^i < t_1^i < \ldots < t_{k_i}^i = T, \quad i \in \mathbb{N},\ k_i \in \mathbb{N},$$

and with

$$\lim_{i\to\infty} \max\{t_j^i - t_{j-1}^i; j = 1, \ldots, k_i\} = 0.$$

For each $i \in \mathbb{N}$, we consider random variables

$$r_i : [0,T] \times \Omega \to \mathbb{R}, \quad (t,\omega) \mapsto \sum_{j=0}^{k_i-1} f(t_j^i) \frac{B_{t_{j+1}^i}(\omega) - B_{t_j^i}(\omega)}{t_{j+1}^i - t_j^i} I_{[t_j^i, t_{j+1}^i)}(t),$$

where

$$I_{[t_j^i, t_{j+1}^i)} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} 1 & \text{for } t \in [t_j^i, t_{j+1}^i) \\ 0 & \text{for } t \notin [t_j^i, t_{j+1}^i) \end{cases}, \quad i \in \mathbb{N},\ j \in \{0, \ldots, k_i - 1\},$$

and we use step functions

$$f_i : [0,T] \to \mathbb{R}, \quad t \mapsto \sum_{j=0}^{k_i-1} f(t_j^i) I_{[t_j^i, t_{j+1}^i)}(t), \quad i \in \mathbb{N}.$$

We have approximated the function f by step functions and we have replaced the Gaussian white noise process by difference quotients. Now, it is natural to define the Wiener integral for r_i in the following way:

$$\begin{aligned} \left(\int_0^T f_i(t)dB_t\right)(\omega) &:= \int_0^T r_i(t,\omega)dt = \\ &= \sum_{j=0}^{k_i-1} f(t_j^i) \frac{B_{t_{j+1}^i}(\omega) - B_{t_j^i}(\omega)}{t_{j+1}^i - t_j^i} (t_{j+1}^i - t_j^i) = \\ &= \sum_{j=0}^{k_i-1} f(t_j^i) \left(B_{t_{j+1}^i}(\omega) - B_{t_j^i}(\omega)\right), \quad \omega \in \Omega. \end{aligned}$$

Therefore, the random variable

$$\int_0^T f_i(t)dB_t : \Omega \to \mathbb{R}$$

is Gaussian distributed with expectation

$$\mathscr{E}\left(\int_0^T f_i(t)dB_t\right) = 0$$

and with variance

$$\mathscr{V}\left(\int_0^T f_i(t)dB_t\right) = \mathscr{E}\left(\left(\int_0^T f_i(t)dB_t\right)^2\right) = \sum_{j=0}^{k_i-1} f^2(t_j^i)\left(t_{j+1}^i - t_j^i\right).$$

Obviously, for $T = 0$ we define

$$\int_0^0 f_i(t)dB_t = 0 \quad (\mathbb{P}\text{-})\text{almost surely.}$$

Since almost all paths of a Brownian Motion are of unbounded variation on each nontrivial finite interval, we are not able to compute the limit

$$\lim_{i\to\infty}\left(\int_0^T f_i(t)dB_t\right)(\omega)$$

pointwise in order to define a random variable

$$\int_0^T f(t)dB_t : \Omega \to \mathbb{R}.$$

On the other hand, we are able to define a Gaussian distributed random variable

$$X_T : \Omega \to \mathbb{R}$$

with

$$\mathscr{E}(X_T) = 0 \quad \text{and} \quad \mathscr{V}(X_T) = \int_0^T f^2(t)dt,$$

which is motivated by the fact that

$$\lim_{i\to\infty} \mathscr{E}\left(\left(\int_0^T f_i(t)dB_t\right)^2\right) = \lim_{i\to\infty}\sum_{j=0}^{k_i-1} f^2(t_j^i)\left(t_{j+1}^i - t_j^i\right) = \int_0^T f^2(t)dt.$$

Hence, the sequence

$$\left\{\int_0^T f_i(t)dB_t\right\}_{i\in\mathbb{N}}$$

converges **in the mean square** to X_T and we define

$$\int_0^T f(t)dB_t := X_T.$$

We obtain a stochastic process $(X_t)_{t\in[0,\infty)}$ with

$$\mathscr{E}(X_t) = 0 \quad \text{and with} \quad \mathscr{C}(X_s, X_t) = \int_0^{\min(s,t)} f^2(\tau)d\tau.$$

The covariance function $\mathscr{C}$ can be determined by the fact that the increments of a Brownian Motion are stochastically independent. We obtain the convergence in the mean square of the sequence

$$\left\{\int_0^T f_i(t)dB_t\right\}_{i\in\mathbb{N}}$$

to X_T as follows:

Let $\mathscr{L}_2((\Omega,\mathcal{S},\mathbb{P}),\mathbb{R})$ be the set of all random variables

$$X : \Omega \to \mathbb{R} \quad \text{with} \quad \int X^2 d\mathbb{P} < \infty,$$

then we get a seminorm on $\mathscr{L}_2((\Omega,\mathcal{S},\mathbb{P}),\mathbb{R})$ by

$$\|\bullet\|_{\mathscr{L}_2} : \mathscr{L}_2((\Omega,\mathcal{S},\mathbb{P}),\mathbb{R}) \to \mathbb{R}, \quad X \mapsto \sqrt{\int X^2 d\mathbb{P}}$$

(see Definition 3.2) and we obtain a complete seminormed vector space

$$\mathscr{L}_2((\Omega,\mathcal{S},\mathbb{P}),\mathbb{R})$$

(see [Deck06]). Since

$$\left\{\int_0^T f_i(t)dB_t\right\}_{i\in\mathbb{N}}$$

defines a Cauchy sequence based on $\| \bullet \|_{\mathcal{L}_2}$, the sequence

$$\left\{ \int_0^T f_i(t) dB_t \right\}_{i \in \mathbb{N}}$$

converges in the mean square. Let Y_T and X_T be two limits of this sequence, then we obtain

$$\|Y_T - X_T\|_{\mathcal{L}_2}^2 = \int (Y_T - X_T)^2 d\mathbb{P} = 0,$$

and hence

$$Y_T = X_T \quad (\mathbb{P}\text{-})\text{almost surely.}$$

A proof of the Gaussian distribution of X_T is given in [Deck06]. Since the limit of

$$\left\{ \int_0^T f_i(t) dB_t \right\}_{i \in \mathbb{N}}$$

is fixed up to a null set, we are able to choose $(X_t)_{t \in [0,\infty)}$ with

$$\lim_{i \to \infty} \mathcal{E}\left(\left(X_t - \int_0^t f_i(\tau) dB_\tau \right)^2 \right) = 0, \quad t \in [0, \infty),$$

such that all paths are continuous (using again the continuity theorem of Kolmogorov-Chencov). This is done in the following. The Gaussian distribution of the random variables X_t, $t \in [0, \infty)$, and the fact that

$$\mathcal{E}(X_t) = 0 \quad \text{and} \quad \mathcal{C}(X_s, X_t) = \int_0^{\min(s,t)} f^2(\tau) d\tau$$

guarantee that the Wiener integral is well defined.

Now, let us come back to the analysis of an RC circuit with $C = 1\mu F$, $R = 1k\Omega$, $Q_0 = 0$, and $U(t) \equiv 100V$ direct voltage (see Fig. 2.17), then we obtain a time-dependent charge of the capacitor shown in Fig. 2.18 given by

$$Q : \mathbb{R}_0^+ \to \mathbb{R}, \quad t \mapsto e^{-\frac{1}{RC}t} \left(\int_0^t \frac{U(\tau)}{R} e^{\frac{1}{RC}\tau} d\tau + Q_0 \right).$$

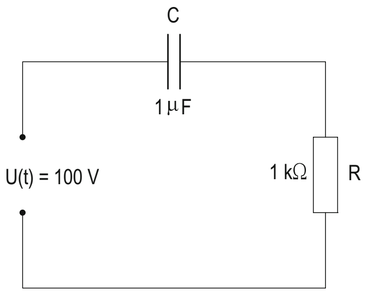


Fig. 2.17 RC circuit

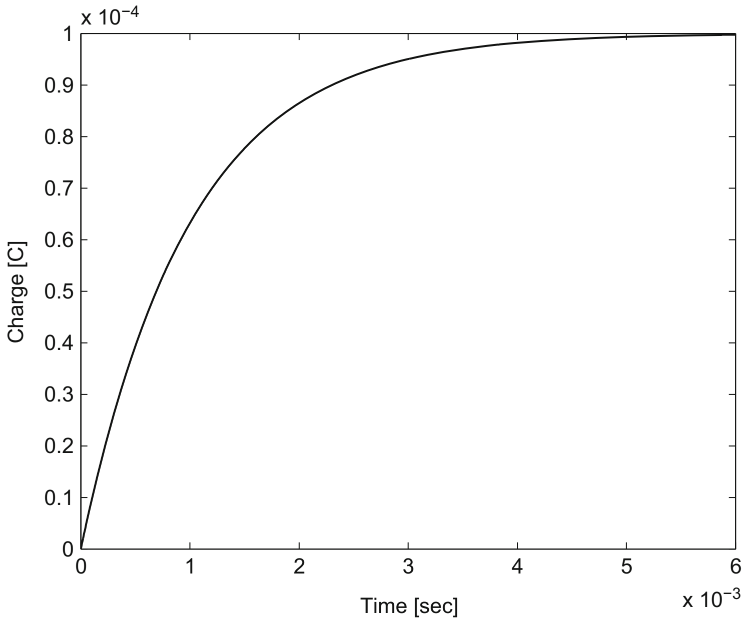


Fig. 2.18 Charge Q

Under the assumption that the voltage U is additively disturbed by a Gaussian white noise process, we get the disturbed charge

$$\hat{Q} : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}, \quad (\omega, t) \mapsto Q(t) + \frac{e^{-\frac{1}{RC}t}}{R} \left(\int_0^t e^{\frac{1}{RC}\tau} dB_\tau \right)(\omega).$$

Figure 2.19 shows a path of the disturbance

$$S : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}, \quad (\omega, t) \mapsto \frac{e^{-\frac{1}{RC}t}}{R} \left(\int_0^t e^{\frac{1}{RC}\tau} dB_\tau \right)(\omega)$$

2

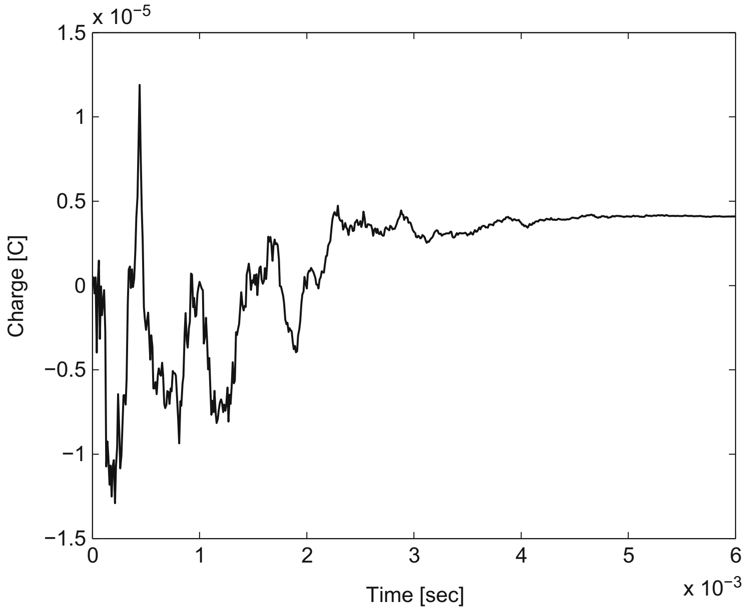


Fig. 2.19 A path of S

and Fig. 2.20 shows the corresponding path of $\hat{Q}$. The numerical simulation of complex electric circuits including random noise effects, which leads to the numerical solution of systems of stochastic algebro-differential equations, is very important for the analysis of the reliability of these circuits.

In the following, we consider the transmission of a single bit $b \in \{\pm 1\}$ by a signal

$$s : [0, 2\pi] \to \mathbb{R}, \quad t \mapsto b \cdot \sin(t)$$

again. The receiver obtains a noisy signal $\tilde{s}$. In order to detect the transmitted bit, the convolution

$$c_{2\pi} = \frac{1}{\pi} \int_0^{2\pi} \tilde{s}(2\pi - \tau) \sin(2\pi - \tau) d\tau$$

is computed, where

$$c_{2\pi} = b \quad \text{for} \quad s = \tilde{s} \quad (\text{Fig. 2.21}).$$

In many applications, the convolved noisy signal is typically given by functions shown in Fig. 2.22, which are obtained from s by adding a Gaussian white noise process. Hence, the convolved noisy signal is given by a path of the stochastic process

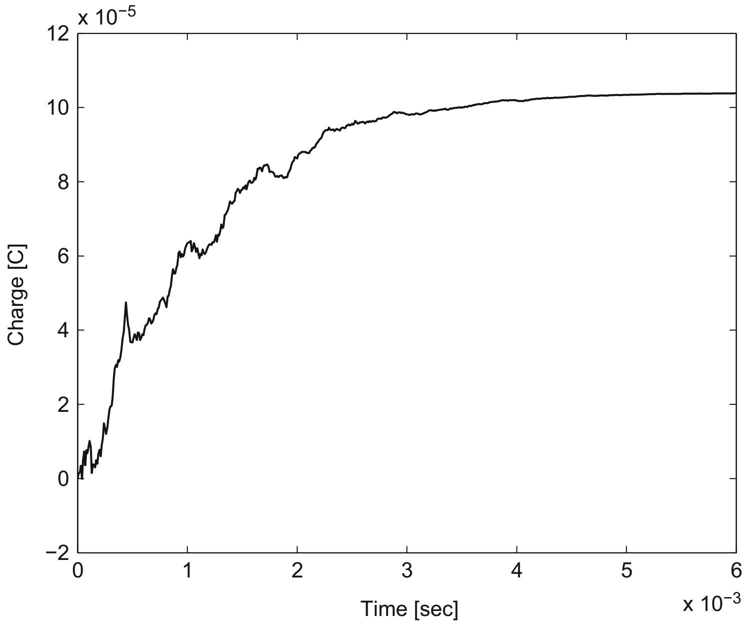


Fig. 2.20 Corresponding path of $\hat{Q}$

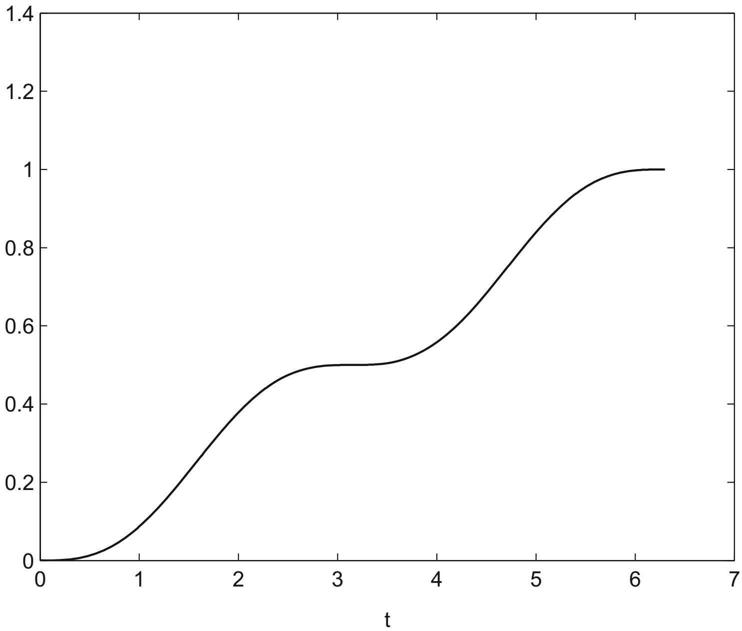


Fig. 2.21 $\frac{1}{\pi}\int_0^t \sin^2(2\pi - \tau)d\tau, \quad 0 \leq t \leq 2\pi, b = 1$

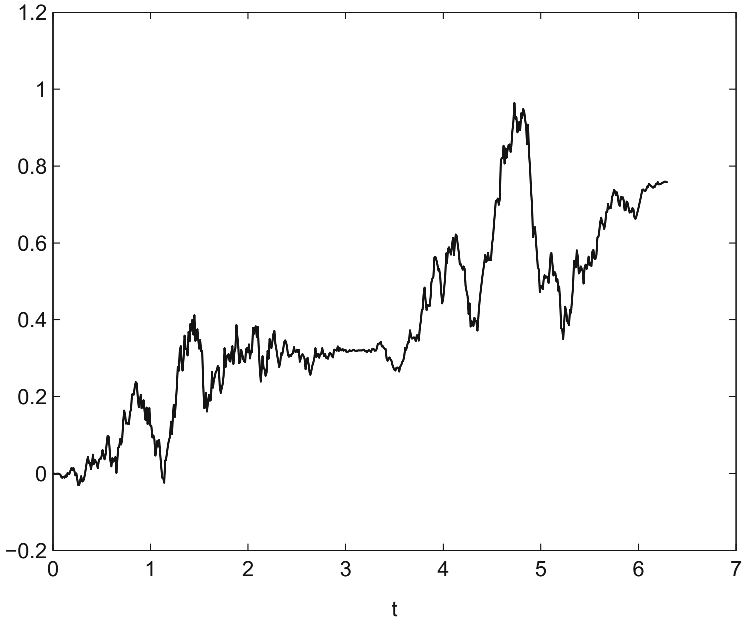


Fig. 2.22 Convolved noisy signal, $b = 1$

defined by

$$c : \Omega \times [0, 2\pi] \to \mathbb{R}, \quad (\omega, t) \mapsto \frac{b}{\pi} \int_0^t \sin^2(2\pi - \tau) d\tau + \frac{1}{\pi} \left(\int_0^t \sin(2\pi - \tau) dB_\tau \right) (\omega).$$

Figure 2.23 shows the corresponding path of

$$R : \Omega \times [0, 2\pi] \to \mathbb{R}, \quad (\omega, t) \mapsto \frac{1}{\pi} \left(\int_0^t \sin(2\pi - \tau) dB_\tau \right) (\omega).$$

A decision relating to the transmitted value of b ($b = +1$ or $b = -1$) depends on a realization of the random variable

$$c(\bullet, 2\pi) : \Omega \to \mathbb{R}, \quad \omega \mapsto b + \frac{1}{\pi} \left(\int_0^{2\pi} \sin(2\pi - \tau) dB_\tau \right) (\omega),$$

and therefore on a measurement of the convolved noisy signal at $t = 2\pi$ (if > 0: $b = 1$, if < 0: $b = -1$).

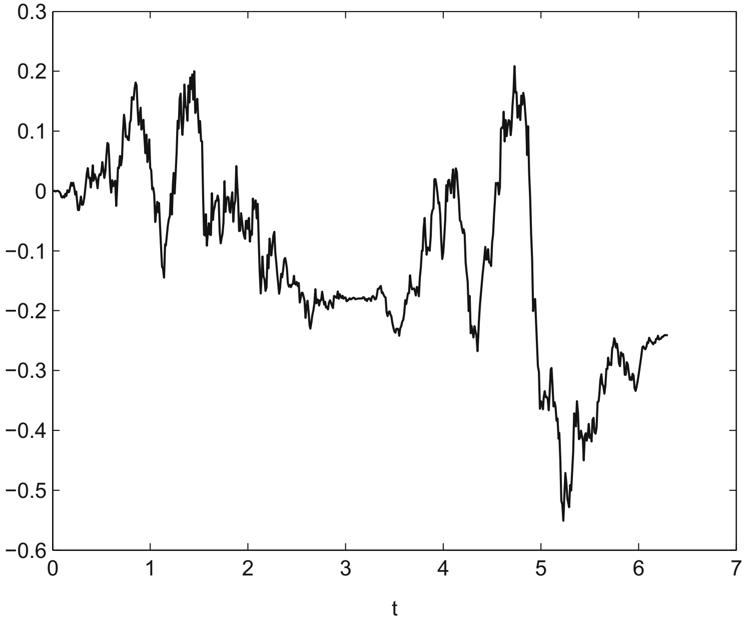


Fig. 2.23 Path of R

We get:

$$\mathscr{E}(c(\bullet, 2\pi)) = b, \quad \mathscr{V}(c(\bullet, 2\pi)) = \frac{1}{\pi}.$$

The value

$$\frac{E_S}{E_R} := \frac{1}{\mathscr{V}(c(\bullet, 2\pi))}$$

is called **signal-to-noise-ratio** and represents the quotient of the signal energy by the noise energy. The greater this quotient the greater the probability for a correct decision. The proposed convolution of the noisy signal maximizes the signal-to-noise-ratio. A communication channel of this type is called an **AWGN**-channel (**A**dditive **W**hite **G**aussian **N**oise). The error probability $\mathbb{P}_e$ for a wrong decision is given by

$$\mathbb{P}_e = \frac{1}{\sqrt{2\pi \mathscr{V}(c(\bullet, 2\pi))}} \int_0^\infty \exp\left(-\frac{(x+1)^2}{2 \cdot \mathscr{V}(c(\bullet, 2\pi))}\right) dx =$$

$$= \frac{1}{\sqrt{2\pi \mathscr{V}(c(\bullet, 2\pi))}} \int_{-\infty}^0 \exp\left(-\frac{(x-1)^2}{2 \cdot \mathscr{V}(c(\bullet, 2\pi))}\right) dx.$$

For

$$\mathscr{V}(c(\bullet, 2\pi)) = \frac{1}{\pi}$$

we obtain

$$\mathbb{P}_e \approx 3,84\%.$$

Since the Poissonian white noise process is very useful in practice as well, we introduce the Wiener integral based on a Poisson process. The main differences to the Wiener integration based on a Brownian Motion are given by the fact that the expectation of a Poissonian white noise process is not equal to zero and by the fact that the paths of a Poissonian process are step functions which are continuous from the right.

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space, $(P_t)_{t\in[0,\infty)}$ be a Poissonian process with parameter $\lambda > 0$, and let

$$f : [0, T] \to \mathbb{R}, \quad T > 0,$$

be a continuous function, which is multiplicatively disturbed by a Poissonian white noise process, then we are able to approximate the disturbed function w. For that, we use a sequence of partitions $\{t_0^i, \ldots, t_{k_i}^i\}_{i\in\mathbb{N}}$ with

$$0 = t_0^i < t_1^i < \ldots < t_{k_i}^i = T, \quad i \in \mathbb{N},\ k_i \in \mathbb{N},$$

and with

$$\lim_{i\to\infty} \max\{t_j^i - t_{j-1}^i; j = 1, \ldots, k_i\} = 0.$$

For each $i \in \mathbb{N}$, we consider random variables

$$w_i : [0, T] \times \Omega \to \mathbb{R}, \quad (t, \omega) \mapsto \sum_{j=0}^{k_i-1} f(t_j^i) \frac{P_{t_{j+1}^i}(\omega) - P_{t_j^i}(\omega)}{t_{j+1}^i - t_j^i} I_{[t_j^i, t_{j+1}^i)}(t),$$

and we use step functions

$$f_i : [0, T] \to \mathbb{R}, \quad t \mapsto \sum_{j=0}^{k_i-1} f(t_j^i) I_{[t_j^i, t_{j+1}^i)}(t), \quad i \in \mathbb{N}.$$

We have approximated the function f by step functions and we have replaced the Poissonian white noise process by difference quotients. Now, it is natural to define the

Wiener integral for w_i in the following way:

$$\left(\int_0^T f_i(t)dP_t\right)(\omega) := \int_0^T w_i(t,\omega)dt =$$

$$= \sum_{j=0}^{k_i-1} f(t_j^i)\frac{P_{t_{j+1}^i}(\omega) - P_{t_j^i}(\omega)}{t_{j+1}^i - t_j^i}(t_{j+1}^i - t_j^i) =$$

$$= \sum_{j=0}^{k_i-1} f(t_j^i)\left(P_{t_{j+1}^i}(\omega) - P_{t_j^i}(\omega)\right), \quad \omega \in \Omega.$$

Obviously, for $T = 0$, we define

$$\int_0^0 f_i(t)dP_t = 0 \quad (\mathbb{P}\text{-})\text{almost surely}.$$

We obtain

$$\mathscr{E}\left(\int_0^T f_i(t)dP_t\right) = \lambda\sum_{j=0}^{k_i-1} f\left(t_j^i\right)\left(t_{j+1}^i - t_j^i\right)$$

and

$$\mathscr{V}\left(\int_0^T f_i(t)dP_t\right) = \mathscr{E}\left(\left(\int_0^T f_i(t)dP_t - \lambda\sum_{j=0}^{k_i-1} f\left(t_j^i\right)\left(t_{j+1}^i - t_j^i\right)\right)^2\right) =$$

$$= \mathscr{E}\left(\left(\sum_{j=0}^{k_i-1} f\left(t_j^i\right)\left(P_{t_{j+1}^i} - P_{t_j^i} - \lambda\left(t_{j+1}^i - t_j^i\right)\right)\right)^2\right) =$$

$$= \lambda\sum_{j=0}^{k_i-1} f^2\left(t_j^i\right)\left(t_{j+1}^i - t_j^i\right).$$

We get a stochastic process

$$\left(\int_0^t f(\tau)dP_\tau\right)_{t\in[0,\infty)}$$

given by pathwise computation of the limit

$$\left(\int_0^t f(\tau)dP_\tau\right) : \Omega \to \mathbb{R}, \quad \omega \mapsto \lim_{i\to\infty} \sum_{j=0}^{k_i-1} f(t_j^i)\left(P_{t_{j+1}^i}(\omega) - P_{t_j^i}(\omega)\right),$$

which defines a **Riemann-Stieltjes integration** for each path.

It follows:

$$\mathscr{E}\left(\int_0^t f(\tau)dP_\tau\right) = \lambda \int_0^t f(\tau)d\tau$$

and

$$\mathscr{C}\left(\int_0^t f(\tau)dP_\tau, \int_0^s f(\tau)dP_\tau\right) = \lambda \int_0^{\min(s,t)} f^2(\tau)d\tau, \quad s, t \in [0, \infty).$$

This type of integration is necessary for the numerical simulation of electric circuits with respect to shot noise.

2.5 AR(p)-Processes

The modelling of technical problems using physical conservation laws leads to differential equations in general. A large class of models is given by the following type of initial value problems:

Assume $p \in \mathbb{N}$, $a_0, \ldots, a_p \in \mathbb{R}$, and let $f : [0, \infty) \to \mathbb{R}$ be a continuous function; we look for a p times continuously differentiable function

$$y : [0, \infty) \to \mathbb{R}$$

solving the initial value problem

$$a_p y^{(p)}(x) + a_{p-1} y^{(p-1)}(x) + \ldots + a_1 y'(x) + a_0 y(x) = f(x) \quad \text{for all} \quad x \in [0, \infty),$$

where $y(0) = y_0, \ldots, y^{(p-1)}(0) = y_0^{(p-1)}$ and where $y^{(k)}$ denotes the derivative of y of order k. The derivatives at $x = 0$ are defined by the limits from the right. We know from the calculus of ordinary differential equations (see, e.g. [Wal00]) that one obtains p linearly independent functions

$$y_1, \ldots, y_p : [0, \infty) \to \mathbb{R}$$

using the ansatz $y(x) = e^{\lambda x}$ such that the solution y_h of the homogeneous initial value problem

$$a_p y^{(p)}(x) + a_{p-1} y^{(p-1)}(x) + \ldots + a_1 y'(x) + a_0 y(x) = 0 \quad \text{for all} \quad x \in [0, \infty)$$

with $y(0) = y_0, \ldots, y^{(p-1)}(0) = y_0^{(p-1)}$ is given by a special linear combination of the functions $y_1, \ldots, y_p$:

$$y_h = c_1 y_1 + \ldots + c_p y_p.$$

The solution of the inhomogeneous initial value problem is given by

$$y : [0, \infty) \to \mathbb{R}, \quad x \mapsto \sum_{i=1}^{p} c_i y_i(x) + \sum_{i=1}^{p} (-1)^{i+p} y_i(x) \int_0^x \frac{\det(\mathbf{Y}_i(\xi))}{\det(\mathbf{Y}(\xi))} f(\xi) d\xi,$$

where

$$\mathbf{Y}(x) := \begin{pmatrix} y_1(x) & y_2(x) & \cdots & y_p(x) \\ y_1'(x) & y_2'(x) & \cdots & y_p'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(p-1)}(x) & y_2^{(p-1)}(x) & \cdots & y_p^{(p-1)}(x) \end{pmatrix}$$

and where $\mathbf{Y}_i(x)$ denotes a matrix given by cancelling the ith column and the last row of $\mathbf{Y}(x)$.

Now, we consider the initial value problem

$$a_p y^{(p)}(t) + a_{n-1} y^{(p-1)}(t) + \ldots + a_1 y'(t) + a_0 y(t) = f(t) \quad \text{for all} \quad t \in [0, \infty)$$

with initial values $y(t_0) = y_0, \ldots, y^{(n-1)}(t_0) = y_0^{(p-1)}$, where the function f is defined as a sum of a continuous function $g : [0, \infty) \to \mathbb{R}$ and a Gaussian white noise process defined on $\mathbb{R}_0^+$. Then we can find a stochastic process $(Y_t)_{t \in [0,\infty)}$ given by

$$\begin{aligned} Y : \Omega \times [0, \infty) \to \mathbb{R}, \quad t \mapsto & \sum_{i=1}^{p} c_i y_i(t) + \\ & + \sum_{i=1}^{p} (-1)^{i+p} y_i(t) \int_0^t \frac{\det(\mathbf{Y}_i(\xi))}{\det(\mathbf{Y}(\xi))} g(\xi) d\xi + \\ & + \sum_{i=1}^{p} (-1)^{i+p} y_i(t) \left(\int_0^t \frac{\det(\mathbf{Y}_i(\xi))}{\det(\mathbf{Y}(\xi))} dB_\xi \right)(\omega), \end{aligned}$$

as a solution of the above initial value problem using Wiener integration. We set

$$\det(\mathbf{Y}_i(t)) \equiv 1$$

for $p = 1$. The stochastic process $(Y_t)_{t\in[0,\infty)}$ is called an **AR(p)-process** (**Auto****R**egressive process) with parameters $a_0, \ldots, a_p$, with initial values $y_0, \ldots, y_0^{(p-1)}$, and with the inhomogeneity $g : [0,\infty) \to \mathbb{R}$. A Brownian Motion is defined as an AR(1)-process with $a_0 = 0$, $a_1 = 1$, $y_0 = 0$ and with $g \equiv 0$. The charge of a capacitor in an RC circuit

$$Y : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}, \quad (\omega, t) \mapsto Q(t) + \frac{e^{-\frac{1}{RC}t}}{R}\left(\int_0^t e^{\frac{1}{RC}\tau} dB_\tau\right)(\omega)$$

defines an AR(1)-process with $a_0 = \frac{1}{C}$, $a_1 = R$, $y_0 = Q_0$ and with

$$g : [0,\infty) \to \mathbb{R}, \quad t \mapsto U(t).$$

The covariance function of $(Y_t)_{t\in[0,\infty)}$ is given by:

$$\mathscr{C}(Y_t, Y_{t+h}) = \frac{C}{2R}\left(e^{-\frac{1}{RC}|h|} - e^{-\frac{1}{RC}(2t+h)}\right)$$

(see Fig. 2.24 for C = 1μF and R = 1kΩ). Since

$$\lim_{t\to\infty} \mathscr{C}(Y_t, Y_{t+h}) = \frac{C}{2R}e^{-\frac{1}{RC}|h|}, \quad h \in \mathbb{R},$$

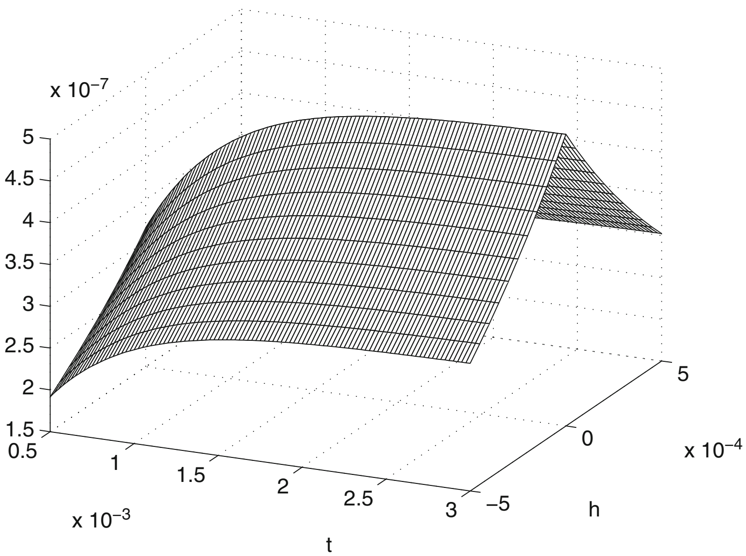


Fig. 2.24 $\mathscr{C}(Y_t, Y_{t+h})$, $C = 1\mu$F, and $R = 1$kΩ

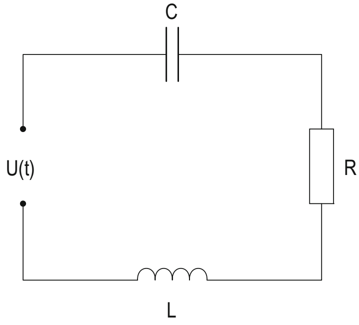


Fig. 2.25 LRC circuit

the stochastic process $(Y_t)_{t\in[0,\infty)}$ is called an **asymptotically stationary process** (see Appendix B for the corresponding spectral density function). A given AR(p)-process defines an asymptotically stationary process, iff each root of the polynomial

$$\sum_{i=0}^{p} \lambda^i a_i$$

has a negative real part.

Considering the capacitor voltage U_c in a LRC circuit (see Fig. 2.25), we have to solve the initial value problem

$$LC\ddot{U}_C(t) + RC\dot{U}_C(t) + U_C(t) = U(t), \quad t \geq 0,\ U_C(0) = U_0,\ \dot{U}_C(0) = \dot{U}_0.$$

Assuming again that the voltage U is additively disturbed by a Gaussian white noise process on $\mathbb{R}_0^+$, we obtain an AR(2)-process $(Y_t)_{t\in[0,\infty)}$ with $a_0 = 1$, $a_1 = RC$, $a_2 = LC$, $y_0 = U_0$, $y_0^{(1)} = \dot{U}_0$ and with

$$g : [0,\infty) \to \mathbb{R}, \quad t \mapsto U(t).$$

This AR(2)-process defines an asymptotically stationary process, iff $C, L, R > 0$. For

$$C, L, R > 0 \quad \text{and} \quad R^2C = 4L,$$

we obtain the matrix

$$\mathbf{Y}(t) = \begin{pmatrix} e^{-\frac{R}{2L}t} & te^{-\frac{R}{2L}t} \\ -\frac{R}{2L}e^{-\frac{R}{2L}t} & e^{-\frac{R}{2L}t} - t\frac{R}{2L}e^{-\frac{R}{2L}t} \end{pmatrix}$$

and after some calculations

$$\lim_{t\to\infty} \mathscr{C}(Y_t, Y_{t+h}) = \left(2\frac{L^3}{R^3} + \frac{L^2}{R^2}|h|\right) e^{-\frac{R}{2L}|h|}, \quad h \in \mathbb{R} \quad \text{(see Fig. 2.26).}$$

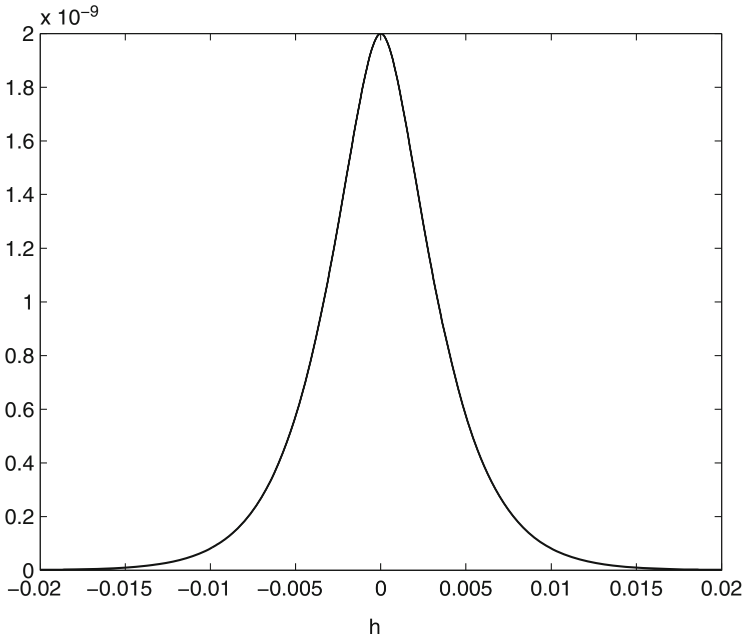


Fig. 2.26 Asymptotic covariance function with $R = 10\Omega$ and $L = 0.01$H

The just developed asymptotic covariance function is the solution of the initial value problem

$$LC\gamma''(h) + RC\gamma'(h) + \gamma(h) = 0, \quad \gamma(0) = 2\frac{L^3}{R^3}, \ \gamma'(0) = 0,$$

for $h \geq 0$.

In 1908, AR(p)-processes were investigated for the first time by Paul Langevin. He considered the equation

$$m\ddot{x}(t) + \zeta\dot{x}(t) = f(t)$$

as a model of an one-dimensional particle with mass m in a resting medium (gas or liquid). The function f is chosen as a Gaussian white noise process (hence, $g \equiv 0$), which is a model for the collision between the particle and the molecules of the medium.

At the end of this section, we investigate a generalization of AR(p)-processes. With $n \in \mathbb{N}$ and $T \in (0, \infty)$, assume that

$$\mathbf{A} : [0, T) \to \mathbb{R}^{n,n} \quad \text{and} \quad \mathbf{f} : [0, T) \to \mathbb{R}^n$$

are continuous functions. Then, we are looking for a solution

$$\mathbf{y} : [0, T) \to \mathbb{R}^n$$

of

$$\dot{\mathbf{y}}(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{f}(t) \quad \text{for all} \quad t \in [0, T),$$

with $\mathbf{y}(0) = \mathbf{y}_0$, where $\dot{\mathbf{y}}$ denotes differentiation component for component. We know from the calculus of systems of ordinary differential equations (see again [Wal00]) that one obtains p linearly independent functions

$$\mathbf{y}_i : [0, T) \to \mathbb{R}^n, \quad i = 1, \ldots, n,$$

such that the solution $\mathbf{y}_\mathbf{h}$ of the homogeneous initial value problem

$$\dot{\mathbf{y}}(t) = \mathbf{A}(t)\mathbf{y}(t) \quad \text{for all} \quad t \in [0, T)$$

with $\mathbf{y}(0) = \mathbf{y}_0$ is given by a special linear combination of functions $\mathbf{y}_1, \ldots, \mathbf{y}_n$:

$$\mathbf{y}_\mathbf{h} = c_1\mathbf{y}_1 + \ldots + c_n\mathbf{y}_n$$

and vice versa. Using

$$\mathbf{W}(t) := (\mathbf{y}_1(t), \ldots, \mathbf{y}_n(t)) \in \mathbb{R}^{n,n}, \quad t \in [0, T),$$

and $\mathbf{c} = (c_1, \ldots, c_n)^\top$, the solution of the inhomogeneous initial value problem is given by

$$\mathbf{y} : [0, T) \to \mathbb{R}^n, \quad t \mapsto \mathbf{W}(t) \left(\int_0^t \mathbf{W}(\tau)^{-1}\mathbf{f}(\tau)d\tau + \mathbf{c} \right).$$

If the function $\mathbf{f}$ is defined as a sum of a continuous function $\mathbf{g} : [0, T) \to \mathbb{R}^n$ and an n-dim. Gaussian white noise process defined on $\mathbb{R}_0^+$, then we can find a stochastic process $(\mathbf{Y}_t)_{t\in[0,T)}$ given by

$$\mathbf{Y} : \Omega \times [0, T) \to \mathbb{R}^n, \quad t \mapsto \mathbf{W}(t) \left(\int_0^t \mathbf{W}(\tau)^{-1}\mathbf{g}(\tau)d\tau + \mathbf{c} \right) +$$
$$+ \mathbf{W}(t) \begin{pmatrix} \int_0^t \left(\mathbf{W}(\tau)^{-1}\right)_{1,1} d(\mathbf{B}_1)_\tau + \ldots + \int_0^t \left(\mathbf{W}(\tau)^{-1}\right)_{1,n} d(\mathbf{B}_n)_\tau \\ \vdots \\ \int_0^t \left(\mathbf{W}(\tau)^{-1}\right)_{n,1} d(\mathbf{B}_1)_\tau + \ldots + \int_0^t \left(\mathbf{W}(\tau)^{-1}\right)_{n,n} d(\mathbf{B}_n)_\tau \end{pmatrix} (\omega),$$

as a solution of the above initial value problem, where $(\mathbf{B}_t)_{t\in[0,\infty)}$ denotes an n-dim. Brownion Motion.

With $n = 1$, $T = 1$, and

$$A : [0, 1) \to \mathbb{R}, \quad t \mapsto -\frac{1}{1-t}, \quad g \equiv 0$$

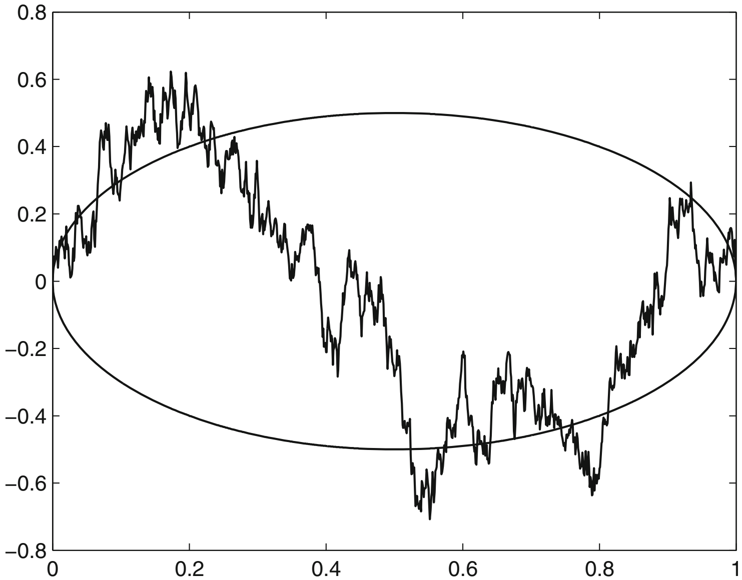
we obtain the so-called **Brownian Bridge**

$$Y : \Omega \times [0, 1) \to \mathbb{R}, \quad (\omega, t) \mapsto (1-t)\left(\int_0^t \frac{1}{1-u} dB_u\right)(\omega).$$

The covariance function is given by

$$\mathscr{C}(Y_s, Y_t) = \int_0^{\min(s,t)} \frac{(1-s)(1-t)}{(1-u)^2} du = (1-s)(1-t)\left(\frac{1}{1-\min(s,t)} - 1\right) =$$
$$= \min(s,t) - st.$$

A path of a Brownian Bridge and the function $\pm\sqrt{\mathscr{V}(Y_t)}$, $t \in [0, 1)$ is shown in ◘ Fig. 2.27. AR(p)-processes are defined by initial value problems in combination with Gaussian white noise processes. In Chap. 1, we have solved ordinary differential



◘ **Fig. 2.27** A path of a Brownian Bridge equations using the calculus of generalized functions. This approach is not applicable for AR(p)-processes, because it is not possible to take into account the initial values using the calculus of generalized functions.

2.6 MA(τ)-Processes

Analyzing the transmission of a single bit using an AWGN-channel, we investigated the random variable

$$Y : \Omega \to \mathbb{R}, \quad \omega \mapsto \left(\int_0^{2\pi} \sin(2\pi - u) dB_u \right)(\omega).$$

Choosing $\tau > 0$ and a continuous function

$$f : [0, \tau] \to \mathbb{R},$$

a stochastic process $(X_\theta)_{\theta \in [0,\infty)}$ with

$$X_\theta : \Omega \to \mathbb{R}, \quad \omega \mapsto \left(\int_0^{\theta} f(\theta - u) dB_u \right)(\omega), \quad \theta \in [0, \tau),$$

and with

$$X_\theta : \Omega \to \mathbb{R}, \quad \omega \mapsto \left(\int_{\theta-\tau}^{\theta} f(\theta - u) dB_u \right)(\omega), \quad \theta \in [\tau, \infty)$$

is called an **MA(τ)-process** (**M**oving **A**verage process with depth τ), where $(B_t)_{t \in [0,\infty)}$ denotes a one-dimensional Brownian Motion. The covariance $\mathscr{C}(X_t, X_s)$ is given by

$$\mathscr{C}(X_t, X_s) = \begin{cases} 0 & \text{for} \quad 0 \le s < t - \tau \\ \int_0^s f(t-u) f(s-u) du & \text{for} \quad 0 \le s \le t \le \tau \\ \int_{t-\tau}^s f(t-u) f(s-u) du & \text{for} \quad 0 < t - \tau \le s \end{cases}.$$

Using $h := t - s \geq 0$ and using the substitution $u = s - x$, we get:

$$\mathscr{C}(X_{s+h}, X_s) = \begin{cases} 0 & \text{for} \quad h > \tau \\ \int_0^s f(h+x)f(x)dx & \text{for} \quad h \leq \tau - s \\ \int_0^{\tau-h} f(h+x)f(x)dx & \text{for} \quad h \leq \tau < s+h \end{cases}.$$

Hence, each MA(τ)-process $(X_\theta)_{\theta\in[0,\infty)}$ is a stationary process for $\theta \geq \tau$. Figure 2.28 shows the covariance function using $\tau = 2\pi, f = \sin$, and $\theta \geq 2\pi$. Of course it is possible to consider MA(τ)-processes using a Poissonian white noise process:

$$X_\theta : \Omega \to \mathbb{R}, \quad \omega \mapsto \int_0^\theta f(\theta - u)dP_u, \quad \theta \in [0, \tau),$$

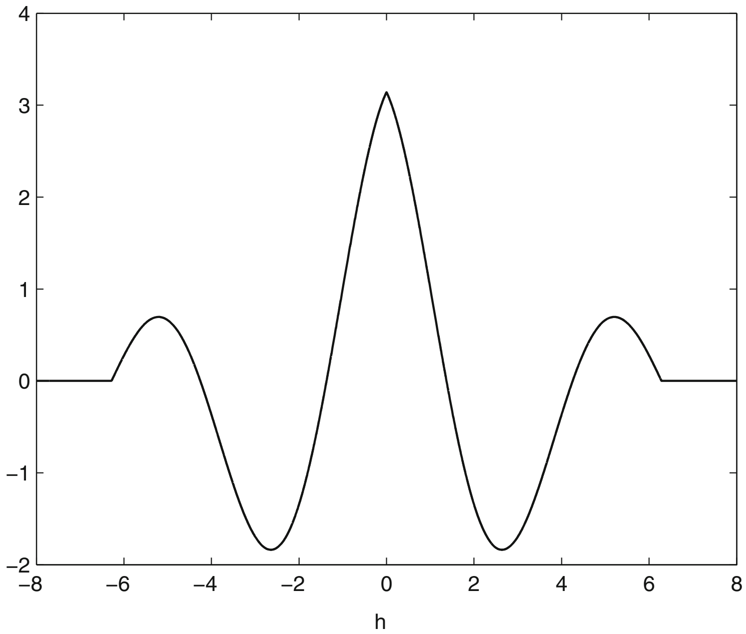


Fig. 2.28 Covariance function of an MA(2π)-process with $f = \sin$ and $\theta \geq 2\pi$

and

$$X_\theta : \Omega \to \mathbb{R}, \quad \omega \mapsto \int\limits_{\theta-\tau}^{\theta} f(\theta - u) dP_u, \quad \theta \in [\tau, \infty).$$

We obtain:

$$\mathscr{E}(X_\theta) = \begin{cases} \lambda \int\limits_0^\theta f(\theta - x)dx & \text{for} \quad \theta \in [0, \tau) \\ \lambda \int\limits_{\theta-\tau}^{\theta} f(\theta - x)dx & \text{for} \quad \theta \geq \tau \end{cases}$$

and

$$\mathscr{C}(X_{s+h}, X_s) = \begin{cases} 0 & \text{for} \quad h > \tau \\ \lambda \int\limits_0^s f(h+x)f(x)dx & \text{for} \quad h \leq \tau - s \\ \lambda \int\limits_0^{\tau-h} f(h+x)f(x)du & \text{for} \quad h \leq \tau \leq s+h \end{cases}.$$

Choosing

$$f = \rho_\alpha : \mathbb{R} \to \mathbb{R}, \quad u \mapsto \begin{cases} \frac{\alpha e}{z^2} u & \text{for} \quad 0 \leq u \leq z \\ 0 & \text{elsewhere} \end{cases},$$

for instance, it follows with $\tau = z$:

$$\mathscr{E}(X_\theta) = \begin{cases} \lambda \frac{\alpha e}{2z^2} \theta^2 & \text{for} \quad \theta \in [0, \tau) \\ \lambda \frac{\alpha e}{2} & \text{for} \quad \theta \geq \tau \end{cases}$$

and

$$\mathscr{C}(X_{s+h}, X_s) = \begin{cases} \lambda \frac{\alpha^2 e^2}{3z} \left(1 - \frac{3h}{2z} + \frac{h^3}{2z^3}\right) & \text{for} \quad h \leq z \leq s+h \\ \lambda \frac{\alpha^2 e^2}{z^4} \left(\frac{s^3}{3} + \frac{hs^2}{2}\right) & \text{for} \quad h \leq z - s \\ 0 & \text{for} \quad h > z \end{cases}.$$

Hence, the power induced by emitted electrons (see ▶ Sect. 2.2) given by

$$I_t : \Omega \to \mathbb{R}, \quad \omega \mapsto \begin{cases} \sum_{k=1}^{P_t(\omega)} \rho(t - T_k(\omega)) & \text{for } P_t(\omega) > 0 \\ 0 & \text{elsewhere} \end{cases}, \quad t \in [0, \infty)$$

forms a MA(z)-process based on a Poissonian white noise process.
From a system theoretical point of view, a MA(τ)-process $(X_\theta)_{\theta \in [0,\infty)}$ with

$$X_\theta : \Omega \to \mathbb{R}, \quad \omega \mapsto \left(\int_0^\theta f(\theta - u) dB_u \right)(\omega), \quad \theta \in [0, \tau),$$

and with

$$X_\theta : \Omega \to \mathbb{R}, \quad \omega \mapsto \left(\int_{\theta-\tau}^\theta f(\theta - u) dB_u \right)(\omega), \quad \theta \in [\tau, \infty),$$

is given, iff a Gaussian white noise process is transformed by a **LTI**-system (**L**inear **T**ime **I**nvariant). The function

$$f : [0, \tau] \to \mathbb{R}$$

is called **impulse response**.

Problems and Solutions

Problems

1. Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space, $(\Gamma, \mathcal{G})$ be a measurable space, and let

$$X : \Omega \to \Gamma$$

be an $\mathcal{S}$-$\mathcal{G}$-measurable function.
Prove that

$$\mathbb{P}_X : \mathcal{G} \to [0, 1], \quad A' \mapsto \mathbb{P}\left(X^{-1}(A')\right)$$

defines a probability measure on $\mathcal{G}$.

2. Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space, $(\Gamma, \mathcal{G})$ be a measurable space, and let I be a nonempty set. Assume that two stochastic processes $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ with

$$X_i, Y_i : \Omega \to \Gamma, \quad i \in I,$$

are given.

Prove that $(X_i)_{i\in I}$ is a modification of $(Y_i)_{i\in I}$, if both are indistinguishable; prove that the opposite is not true in general.

3. Let $(X_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R},\mathbb{R})}$ be a generalized stochastic process such that the expectation value $\mathscr{E}(X_\varphi)$ exists for each $\varphi \in \mathfrak{D}(\mathbb{R},\mathbb{R})$. Prove that the functional

$$m : \mathfrak{D}(\mathbb{R},\mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \mathscr{E}(X_\varphi)$$

is linear.

Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a sequence of test functions, which converges to the test function φ. Prove that

$$\lim_{k\to\infty} \mathscr{E}(\sin(X_{\varphi_k})^n) = \mathscr{E}(\sin(X_\varphi)^n)$$

for each $n \in \mathbb{N}$.

4. Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let

$$X : \Omega \to \mathbb{N}_0$$

be a Poisson distributed random variable with image measure

$$\mathbb{P}_X(\{k\}) = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k \in \mathbb{N}_0, \quad \lambda > 0.$$

Prove that

$$\mathscr{E}(X) = \lambda \quad \text{and} \quad \mathscr{V}(X) = \lambda.$$

5. Compute the covariance

$$\mathscr{C}(X_{h+2\pi}, X_{2\pi}) = \begin{cases} 0 & \text{for} \quad 2\pi < h \\ \int\limits_0^{2\pi-h} \sin(h+x)\sin(x)dx & \text{for} \quad 0 \le h \le 2\pi \end{cases}$$

explicitly.

6. Prove the existence of a Brownian Motion using the existence theorem of Kolmogorov.

7. Prove that the increments of a one-dimensional Brownian Motion are stochastically independent.

8. Assume that a single bit $b \in \{\pm 1\}$ is transmitted by a signal

$$s : [0, 2k\pi] \to \mathbb{R}, \quad t \mapsto b \cdot \sin(t), \quad k \in \mathbb{N}.$$

The receiver obtains a noisy signal $\tilde{s}$ in general. In order to detect the transmitted bit, the convolution

$$c_{2k\pi} = \frac{1}{k\pi} \int_0^{2k\pi} \tilde{s}(2k\pi - \tau) \sin(2k\pi - \tau) d\tau$$

is computed. For $\tilde{s} = s$, we obtain

$$c_{2k\pi} = b.$$

Using an AWGN-channel, we will get

$$c : \Omega \times [0, 2k\pi] \to \mathbb{R}, \quad (\omega, t) \mapsto \quad \frac{b}{k\pi} \int_0^t \sin^2(2k\pi - \tau) d\tau + \\ + \frac{1}{k\pi} \left(\int_0^t \sin(2k\pi - \tau) dB_\tau \right) (\omega).$$

If $c(\hat{\omega}, 2k\pi) \geq 0$, we choose $b = 1$. If $c(\hat{\omega}, 2k\pi) < 0$, we choose $b = -1$, where $\hat{\omega}$ denotes the realized path of the noise.

Compute the probability for a wrong decision.

Solutions

1. As $\mathbb{P}(A) \geq 0$ for all $A \in \mathcal{S}$, we get

$$\mathbb{P}_X (A') = \mathbb{P} \left(X^{-1}(A') \right) \geq 0 \quad \text{for all} \quad A \in \mathcal{G}.$$

Furthermore, one obtains:

$$\mathbb{P}_X (\Gamma) = \mathbb{P} \left(X^{-1}(\Gamma) \right) = \mathbb{P} (\Omega) = 1$$

and

$$\mathbb{P}_X (\emptyset) = \mathbb{P} \left(X^{-1}(\emptyset) \right) = \mathbb{P} (\emptyset) = 0.$$

Let $\{A'_i\}_{i\in\mathbb{N}}$ be a sequence of pairwise disjoint sets with $A'_i \in \mathcal{G}, \quad i \in \mathbb{N}$, then $\left\{X^{-1}(A'_i)\right\}_{i\in\mathbb{N}}$ is a set of pairwise disjoint sets with $X^{-1}(A'_i) \in \mathcal{S}$, $i \in \mathbb{N}$, and we get:

$$\mathbb{P}_X\left(\bigcup_{i=1}^{\infty} A'_i\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} X^{-1}(A'_i)\right) = \sum_{i=1}^{\infty}\mathbb{P}\left(X^{-1}(A'_i)\right) = \sum_{i=1}^{\infty}\mathbb{P}_X\left(A'_i\right).$$

2. If $(X_i)_{i\in I}$ and $(Y_i)_{i\in I}$ are indistinguishable, then there exists a set $M \in \mathcal{S}$ with $\mathbb{P}(M) = 1$ and with

$$X_\bullet(\omega) = Y_\bullet(\omega) \quad \text{for all} \quad \omega \in M.$$

Therefore,

$$M \subseteq \{\omega \in \Omega;\, X_i(\omega) = Y_i(\omega)\} \quad \text{for all} \quad i \in I$$

and

$$\mathbb{P}\left(\{\omega \in \Omega;\, X_i(\omega) = Y_i(\omega)\}\right) = 1 \quad \text{for all} \quad i \in I.$$

Choose $\Omega = \mathbb{R}$, $\mathcal{S} = \mathcal{B}$, and $\mathbb{P}$ given by an $\mathcal{N}(0,1)$ Gaussian distribution.
Using $I = \mathbb{R}$, using the random variables

$$X_t : \mathbb{R} \to \mathbb{R}, \quad \omega \mapsto 0 \quad \text{for all} \quad t \in \mathbb{R},$$

and using

$$Y_t : \mathbb{R} \to \mathbb{R}, \quad \omega \mapsto \begin{cases} 0 & \text{for } t \neq \omega \\ 1 & \text{for } t = \omega \end{cases}, \quad t \in \mathbb{R},$$

we obtain

$$\mathbb{P}\left(\{\omega \in \Omega;\, X_t(\omega) = Y_t(\omega)\}\right) = \mathbb{P}(\Omega \setminus \{t\}) = 1 \quad \text{for all} \quad t \in \mathbb{R},$$

but

$$\{\omega \in \Omega;\, X_\bullet(\omega) = Y_\bullet(\omega)\} = \emptyset.$$

3. Since

$$X_{\lambda\varphi+\mu\psi} = \lambda X_\varphi + \mu X_\psi \quad (\mathbb{P}\text{-})\text{almost surely},$$

we get

$$\mathscr{E}(X_{\lambda\varphi+\mu\psi}) = \lambda\mathscr{E}(X_\varphi) + \mu\mathscr{E}(X_\psi).$$

2

Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a sequence of test functions, which converges to a test function φ, then using

$$g:\mathbb{R}\to\mathbb{R},\quad x\mapsto \sin(x)^n$$

and using Definition 2.2 we obtain:

$$\lim_{k\to\infty}\mathscr{E}(\sin(X_{\varphi_k})^n)=\lim_{k\to\infty}\mathscr{E}(g(X_{\varphi_k}))=\mathscr{E}(g(X_\varphi))=\mathscr{E}(\sin(X_\varphi)^n).$$

4.

$$\mathscr{E}(X)=\sum_{k=0}^{\infty}k\frac{\lambda^k}{k!}e^{-\lambda}=\lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}=\lambda.$$

$$\begin{aligned}\mathscr{E}\left(X^2\right)&=\sum_{k=0}^{\infty}k^2\frac{\lambda^k}{k!}e^{-\lambda}=e^{-\lambda}\sum_{k=1}^{\infty}k\frac{\lambda^k}{(k-1)!}=\\&=e^{-\lambda}\left(\sum_{k=1}^{\infty}(k-1)\frac{\lambda^k}{(k-1)!}+\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}\right)=\\&=e^{-\lambda}\left(\sum_{k=2}^{\infty}\frac{\lambda^k}{(k-2)!}+\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}\right)=\\&=\lambda^2e^{-\lambda}\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!}+\lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}=\\&=\lambda^2+\lambda.\end{aligned}$$

With

$$\mathscr{V}(X)=\mathscr{E}\left(X^2\right)-(\mathscr{E}(X))^2,$$

it follows that

$$\mathscr{V}(X)=\lambda.$$

5. Since

$$\sin(x)\sin(y)=\frac{1}{2}(\cos(x-y)-\cos(x+y))\quad\text{for all}\quad x,y\in\mathbb{R},$$

we obtain

$$\mathscr{C}(X_{h+2\pi}, X_{2\pi}) = \begin{cases} 0 & \text{for} \quad 2\pi < h \\ \int\limits_0^{2\pi-h} \sin(h+x)\sin(x)dx & \text{for} \quad 0 \le h \le 2\pi \end{cases}$$

$$= \begin{cases} 0 & \text{for} \quad 2\pi < h \\ \int\limits_0^{2\pi-h} \frac{1}{2}\left(\cos(h) - \cos(h+2x)\right)dx & \text{for} \quad 0 \le h \le 2\pi \end{cases}$$

$$= \begin{cases} 0 & \text{for} \quad 2\pi < h \\ \frac{1}{2}\left(\cos(h)(2\pi - h) + \sin(h)\right) & \text{for} \quad 0 \le h \le 2\pi \end{cases}.$$

6. Since $\mathbb{R}$ is a complete metric space and since $\mathrm{cl}(\mathbb{Q}) = \mathbb{R}$, $\mathbb{R}$ is a polish space. Let

$$J = \{t_1, \ldots, t_{|J|}\} \subset [0, \infty)$$

be a nonempty finite set and let

$$K = \{\tau_1, \ldots, \tau_{|K|}\} \subset J$$

be a nonempty subset of J such that $\nu := (|J| - |K|) > 0$ and

$$A \in \mathcal{B}^{|K|}.$$

Using

$$\mathbf{C} = \begin{pmatrix} t_1 & (t_2 - t_1) & \cdots & (t_{|J|-1} - t_1) & (t_{|J|} - t_1) \\ (t_2 - t_1) & t_2 & \cdots & (t_{|J|-1} - t_2) & (t_{|J|} - t_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (t_{|J|-1} - t_1) & (t_{|J|-1} - t_2) & \cdots & t_{|J|-1} & (t_{|J|} - t_{|J|-1}) \\ (t_{|J|} - t_1) & (t_{|J|} - t_2) & \cdots & (t_{|J|} - t_{|J|-1}) & t_{|J|} \end{pmatrix},$$

one gets

$$\mathbb{P}_J(p_{JK}^{-1}(A)) = \int\limits_M \underbrace{\frac{1}{\sqrt{(2\pi)^{|J|})\det(\mathbf{C})}} \exp\left(-\frac{\mathbf{x}^T\mathbf{C}^{-1}\mathbf{x}}{2}\right)}_{=:n(\mathbf{x})} d\lambda^{|J|}(\mathbf{x}),$$

where

$$M = \left\{ x \in \mathbb{R}^{|J|}; (x_{\tau_1}, \ldots, x_{\tau_{|K|}}) \in A \right\}.$$

Hence, we obtain:

$$\mathbb{P}_J \left(p_{JK}^{-1}(A) \right) = \int_A \left(\int_{\mathbb{R}^\nu} n(\mathbf{x}) d\lambda^\nu (\mathbf{x} \setminus (x_{\tau_1}, \ldots, x_{\tau_{|K|}})) \right) d\lambda^{|K|}(x_{\tau_1}, \ldots, x_{\tau_{|K|}}) =$$

$$= \int_A \frac{1}{\sqrt{(2\pi)^{|K|})\det(\mathbf{C}_K)}} \exp\left(-\frac{\mathbf{x}^T \mathbf{C}_K^{-1} \mathbf{x}}{2} \right) d\lambda^{|K|}(\mathbf{x}) =$$

$$= \mathbb{P}_K(A),$$

where

$$\mathbf{C}_K = \begin{pmatrix} \tau_1 & (\tau_2 - \tau_1) & \cdots & (\tau_{|K|-1} - \tau_1) & (\tau_{|K|} - \tau_1) \\ (\tau_2 - \tau_1) & \tau_2 & \cdots & (\tau_{|K|-1} - \tau_2) & (\tau_{|K|} - t_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (\tau_{|K|-1} - \tau_1) & (\tau_{|K|-1} - \tau_2) & \cdots & \tau_{|K|-1} & (\tau_{|K|} - \tau_{|K|-1}) \\ (\tau_{|K|} - \tau_1) & (\tau_{|K|} - \tau_2) & \cdots & (\tau_{|K|} - \tau_{|K|-1}) & \tau_{|K|} \end{pmatrix}.$$

7. For $0 \le t_1 < t_2 < \ldots < t_n$, $n \ge 2$, the increments

$$B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_n} - B_{t_{n-1}}$$

of a one-dimensional Brownian Motion $(B_t)_{t \in [0,\infty)}$ are stochastically independent, iff they are pairwise uncorrelated. Assuming $t_i > t_j$, one obtains:

$$\mathscr{C}(B_{t_{i+1}} - B_{t_i}, B_{t_{j+1}} - B_{t_j}) =$$

$$= \mathscr{C}(B_{t_{i+1}}, B_{t_{j+1}}) - \mathscr{C}(B_{t_i}, B_{t_{j+1}}) - \mathscr{C}(B_{t_{i+1}}, B_{t_j}) + \mathscr{C}(B_{t_i}, B_{t_j}) =$$

$$= t_{j+1} - t_{j+1} - t_j + t_j = 0.$$

8. Since the random variable $c(\bullet, 2k\pi)$ is Gaussian distributed with

$$\mathscr{E}(c(\bullet, 2k\pi)) = b \quad \text{and} \quad \mathscr{V}(c(\bullet, 2k\pi)) = \frac{1}{k\pi},$$

it follows that

$$\mathbb{P}_e = \frac{1}{\sqrt{2\pi\mathcal{V}(c(\bullet, 2k\pi))}} \int\limits_0^{\infty} \exp\left(-\frac{(x+1)^2}{2\cdot\mathcal{V}(c(\bullet, 2k\pi))}\right) dx =$$

$$= \frac{1}{\sqrt{2\pi\mathcal{V}(c(\bullet, 2\pi))}} \int\limits_{-\infty}^{0} \exp\left(-\frac{(x-1)^2}{2\cdot\mathcal{V}(c(\bullet, 2\pi))}\right) dx =$$

$$= \sqrt{\frac{k}{2}} \int\limits_0^{\infty} \exp\left(-\frac{k\pi(x+1)^2}{2}\right) dx.$$

Stochastic Differential Equations

Stefan Schäffler

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3.1 Itô Integration

Considering the capacitor voltage U_c in an LRC circuit (see Fig. 3.1), we have to solve the initial value problem

$$LC\ddot{U}_C(t) + RC\dot{U}_C(t) + U_C(t) = U(t), \quad t \geq 0,\ U_C(0) = U_0,\ \dot{U}_C(0) = \dot{U}_0.$$

Assuming again that the voltage U is additively disturbed by a Gaussian white noise process on $\mathbb{R}_0^+$, we obtain an AR(2)-process $(Y_t)_{t\in[0,\infty)}$ with
$a_0 = 1$, $a_1 = RC$, $a_2 = LC$, $y_0 = U_0$, $y_0^{(1)} = \dot{U}_0$ and with

$$g : [0, \infty) \to \mathbb{R}, \quad t \mapsto U(t).$$

If we take into account the noise voltage of the resistor R, we have to replace R by $R + u_{R,t}$, where $(u_{R,t})_{t\in\mathbb{R}_0^+}$ denotes a bandlimited white noise process. Since all paths of $(u_{R,t})_{t\in\mathbb{R}_0^+}$ are continuous, the solution of the homogeneous equation

$$LC\ddot{U}_C(t) + (R + u_{R,t})C\dot{U}_C(t) + U_C(t) = 0, \quad t \geq 0,\ U_C(0) = U_0,\ \dot{U}_C(0) = \dot{U}_0$$

can be computed pathwise using the theory of ordinary differential equations for systems like

$$\dot{\mathbf{y}}(t) = \mathbf{A}(t)\mathbf{y}(t) \quad \text{for all} \quad t \in [0, \infty)$$

with $\mathbf{y}(0) = \mathbf{y}_0$.

On the other hand, it is not possible to compute the inhomogeneous solution by Wiener integration, because the corresponding integral kernel depends on $\omega \in \Omega$ also.

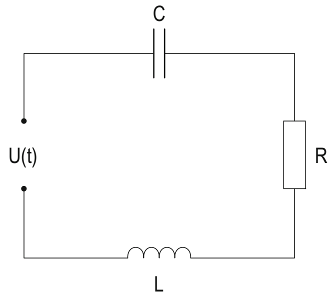

Fig. 3.1 LRC circuit

We know that a bandlimited white noise process $(u_t)_{t\in\mathbb{R}_0^+}$, is defined by

$$\mathscr{E}(u_t) = 0 \quad \text{for all} \quad t \in \mathbb{R}_0^+,$$

and by the covariance function

$$\gamma_u : \mathbb{R} \to \mathbb{R}, \quad h \mapsto \begin{cases} 2f_g\alpha\frac{\sin(2\pi f_g h)}{2\pi f_g h} & \text{for } h \geq 0 \\ 2f_g\alpha\frac{\sin(2\pi f_g(-h))}{2\pi f_g(-h)} & \text{for } h < 0 \end{cases}.$$

The corresponding spectral density function is equal to $\alpha > 0$ on the interval $[-f_g, f_g]$ and is zero elsewhere.

Bandlimited white noise processes are assumed to be Gaussian processes in practice. Furthermore, the limit $f_g \to \infty$ is commonly used. Therefore, we have to develop a new integration calculus with a Gaussian white noise process as a multiplicative part of the integration kernel. This new integration calculus is called **Itô integration** and will be introduced in the following (see [Beh13]).

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let $(\Gamma, \mathcal{G})$ be a measurable space. Let I be a totally ordered nonempty set and consider $(X_i)_{i\in I}$, where the mappings

$$X_i : \Omega \to \Gamma, \quad i \in I,$$

define a stochastic process.

Now, we introduce a family $(\mathcal{F}_i)_{i\in I}$ of σ-fields over Ω given by

$$\mathcal{F}_j = \sigma(X_i;\, i \leq j), \quad i, j \in I.$$

Hence, the σ-field $\mathcal{F}_j$ is the smallest σ-field over Ω such that the random variables X_i, $i \leq j$, are $\mathcal{F}_j$-$\mathcal{G}$-measurable and $\mathcal{F}_j$ can be interpreted as an accumulated information for $(X_i)_{i\in I}$ up to $i = j$. The family $(\mathcal{F}_i)_{i\in I}$ is called the **natural filtration** of $(X_i)_{i\in I}$. We immediately get

$$\mathcal{F}_i \subseteq \mathcal{F}_j \subseteq \mathcal{S} \quad \text{for all} \quad i \leq j, \quad i, j \in I.$$

Any family of σ-fields over Ω with this property is called a **filtration**. Let $(Y_i)_{i\in I}$ be a second stochastic process with

$$Y_i : \Omega \to \Gamma, \quad i \in I,$$

then $(Y_i)_{i\in I}$ is said to be **adapted** to a filtration $(\mathcal{F}_i)_{i\in I}$, if the random variable Y_i is $\mathcal{F}_i$-$\mathcal{G}$-measurable for each $i \in I$. Therefore, each stochastic process is adapted to its natural filtration.

Let $(B_t)_{t\in[0,\infty)}$ be a one-dimensional Brownian Motion and let

$$\left(\mathcal{F}_t^B\right)_{t\in[0,\infty)}$$

be its natural filtration. Furthermore, let $(Z_t)_{t\in[0,\infty)}$ be a stochastic process with the following property:

For each $T > 0$ there exists a partition

$$0 = t_0 < t_1 < \ldots < t_{k_T} = T, \quad k_T \in \mathbb{N},$$

with

$$Z_t : \Omega \to \mathbb{R}, \quad \omega \mapsto \sum_{j=0}^{k_T-1} Z_j'(\omega) \cdot I_{[t_j,t_{j+1})}(t), \quad t \in [0,\infty),$$

where

$$I_{[t_j,t_{j+1})} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} 1 & \text{for } t \in [t_j, t_{j+1}) \\ 0 & \text{for } t \notin [t_j, t_{j+1}) \end{cases}, \quad j \in \{0,\ldots,k_T-1\}.$$

In addition, we assume that the random variable

$$Z_j' : \Omega \to \mathbb{R}$$

is $\mathcal{F}_{t_j}^B$-$\mathcal{B}$-measurable and bounded for each $j \in \{0,\ldots,k_T-1\}$ (Z_0' is constant ($\mathbb{P}$-)almost surely). Consequently, the stochastic process $(Z_t)_{t\in[0,\infty)}$ is adapted to $\left(\mathcal{F}_t^B\right)_{t\in[0,\infty)}$ and its paths are step functions (see ◘ Figs. 3.2 and 3.3, for instance). Now, we introduce the Itô integration for integral kernels consisting of the product of $(Z_t)_{t\in[0,\infty)}$ and a Gaussian white noise process. To this end, we approximate a Gaussian white noise process by difference quotients again and obtain

$$\int_0^T Z_t dB_t := \sum_{j=0}^{k_T-1} Z_j' \frac{B_{t_{j+1}} - B_{t_j}}{t_{j+1} - t_j} \cdot (t_{j+1} - t_j) = \sum_{j=0}^{k_T-1} Z_j' \cdot (B_{t_{j+1}} - B_{t_j}).$$

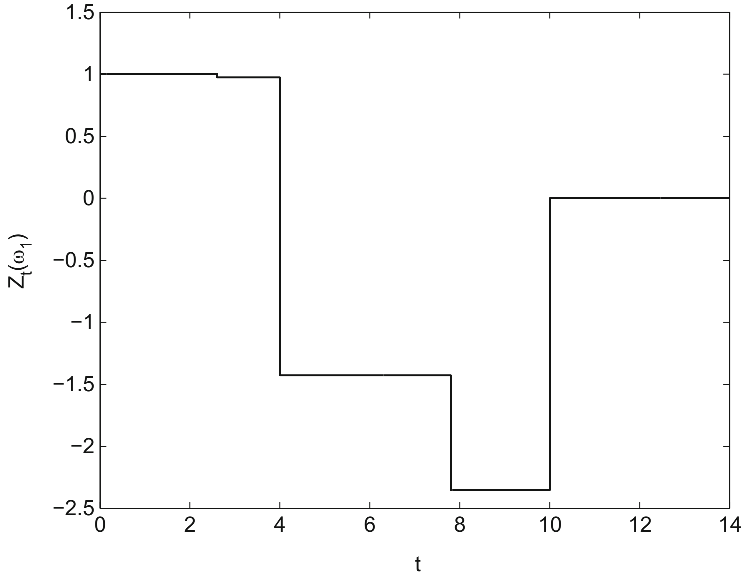

Fig. 3.2 $Z_t(\omega_1), 0 \leq t \leq 14$

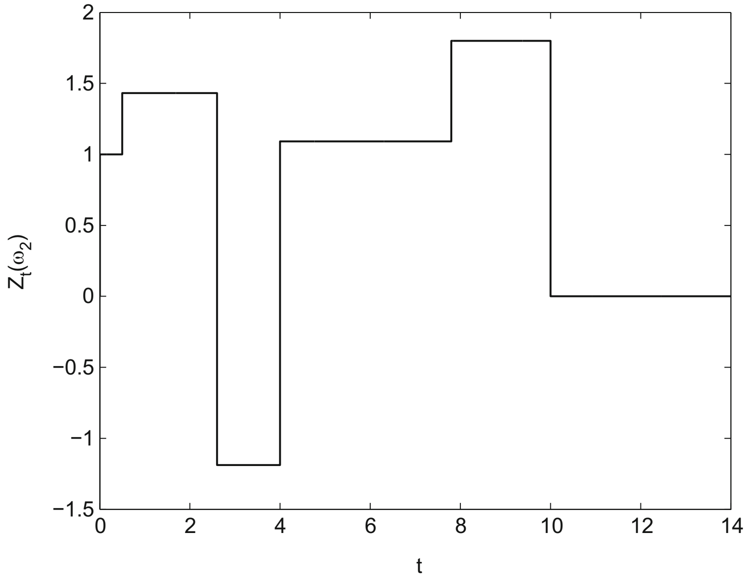

Fig. 3.3 $Z_t(\omega_2), 0 \leq t \leq 14$

Since Z'_j is $\mathcal{F}^B_{t_j}$-$\mathcal{B}$-measurable and since the increments of a Brownion Motion are stochastically independent, we obtain

$$\mathscr{E}\left(\sum_{j=0}^{k_T-1} Z'_j \cdot (B_{t_{j+1}} - B_{t_j})\right) = \sum_{j=0}^{k_T-1} \mathscr{E}(Z'_j) \cdot \mathscr{E}((B_{t_{j+1}} - B_{t_j})) = 0,$$

Hence, the random variables Z'_j and $(B_{t_{j+1}} - B_{t_j})$ are stochastically independent. The following theorem is of great importance.

Theorem 3.1 (Itô Isometry)
Let $(B_t)_{t\in[0,\infty)}$ be a one-dimensional Brownian Motion and let $\left(\mathcal{F}^B_t\right)_{t\in[0,\infty)}$ be its natural filtration. Furthermore, let $(Z_t)_{t\in[0,\infty)}$ be a stochastic process with the following property:
For each $T > 0$ there exists a partition

$$0 = t_0 < t_1 < \ldots < t_{k_T} = T, \quad k_T \in \mathbb{N},$$

with

$$Z_t : \Omega \to \mathbb{R}, \quad \omega \mapsto \sum_{j=0}^{k_T-1} Z'_j(\omega) \cdot I_{[t_j,t_{j+1})}(t), \quad t \in [0,\infty),$$

where

$$I_{[t_j,t_{j+1})} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} 1 \ \text{for}\ t \in [t_j, t_{j+1}) \\ 0 \ \text{for}\ t \notin [t_j, t_{j+1}) \end{cases}, \quad j \in \{0, \ldots, k_T - 1\}.$$

In addition, we assume that the random variable

$$Z'_j : \Omega \to \mathbb{R}$$

is $\mathcal{F}^B_{t_j}$-$\mathcal{B}$-measurable and bounded for each $j \in \{0, \ldots, k_T - 1\}$.
Using

$$\int_0^T Z_t dB_t := \sum_{j=0}^{k_T-1} Z'_j \cdot (B_{t_{j+1}} - B_{t_j})$$

we obtain

(continued)

3

Theorem 3.1 (continued)

$$\mathscr{E}\left(\left(\int_0^T Z_t dB_t\right)^2\right) = \mathscr{E}\left(\int_0^T Z_t^2 dt\right) \quad \textit{(Itô isometry)},$$

where

$$\int_0^T Z_t^2 dt : \Omega \to \mathbb{R}, \quad \omega \mapsto \int_0^T (Z_t(\omega))^2 dt.$$

◁

Proof

$$\mathscr{E}\left(\left(\int_0^T Z_t dB_t\right)^2\right) = \mathscr{E}\left(\left(\sum_{j=0}^{k_T-1} Z'_j \cdot (B_{t_{j+1}} - B_{t_j})\right)^2\right) =$$

$$= \sum_{\substack{i,j=0\\ i\neq j}}^{k_T-1} \mathscr{E}\left(Z'_i \cdot (B_{t_{i+1}} - B_{t_i}) \cdot Z'_j \cdot (B_{t_{j+1}} - B_{t_j})\right) +$$

$$+ \mathscr{E}\left(\sum_{j=0}^{k_T-1} \left(Z'_j \cdot (B_{t_{j+1}} - B_{t_j})\right)^2\right) =$$

$$= \sum_{j=0}^{k_T-1} \mathscr{E}\left(\left(Z'_j \cdot (B_{t_{j+1}} - B_{t_j})\right)^2\right) = \sum_{j=0}^{k_T-1} \mathscr{E}\left((Z'_j)^2\right) \cdot (t_{j+1} - t_j) =$$

$$= \mathscr{E}\left(\sum_{j=0}^{k_T-1} (Z'_j)^2 \cdot (t_{j+1} - t_j)\right) = \mathscr{E}\left(\int_0^T Z_t^2 dt\right).$$

□

Naturally, we define for $T = 0$:

$$\int_0^0 Z_t dB_t = 0 \quad (\mathbb{P}\text{-})\text{almost surely}.$$

Now, we show that Itô integration is well defined. Choose $T > 0$ and choose two partitions

$$0 = t_0 < t_1 < \ldots < t_{k_T} = T, \quad k_T \in \mathbb{N},$$

and

$$0 = \tau_0 < \tau_1 < \ldots < \tau_{p_T} = T, \quad p_T \in \mathbb{N},$$

of the interval $[0, T]$. Assume that the random variable

$$Z'_j : \Omega \to \mathbb{R}$$

is $\mathcal{F}^B_{t_j}$-$\mathcal{B}$-measurable for each $j \in \{0, \ldots, k_T - 1\}$ and that the random variable

$$Z''_i : \Omega \to \mathbb{R}$$

is $\mathcal{F}^B_{\tau_i}$-$\mathcal{B}$-measurable for each $i \in \{0, \ldots, p_T - 1\}$. We have to show that

$$\int_0^T \left(\sum_{j=0}^{k_T-1} Z'_j \cdot I_{[t_j,t_{j+1})}(t) \right) dB_t = \int_0^T \left(\sum_{j=0}^{p_T-1} Z''_j \cdot I_{[\tau_j,\tau_{j+1})}(t) \right) dB_t$$

under the assumption that

$$\sum_{j=0}^{k_T-1} Z'_j \cdot I_{[t_j,t_{j+1})}(t) = \sum_{j=0}^{p_T-1} Z''_j \cdot I_{[\tau_j,\tau_{j+1})}(t) \quad (= Z_t)$$

for each $t \in [0, T]$. Consider the partition

$$0 = \eta_0 < \eta_1 < \ldots < \eta_{l_T} = T, \quad l_T \in \mathbb{N},$$

where

$$t_j, \tau_i \in \{\eta_0, \ldots, \eta_{l_T}\}, \quad j \in \{0, \ldots, k_T\}, \; i \in \{0, \ldots, p_T\}.$$

With

$$\bar{t}_j := \max\{t \in \{t_0, \ldots, t_{k_T}\};\; t \le \eta_j\}, \quad j \in \{1, \ldots, l_T\},$$

and with

$$\bar{\tau}_j := \max\{\tau \in \{\tau_0, \ldots, \tau_{p_T}\};\; \tau \le \eta_j\}, \quad j \in \{1, \ldots, l_T\},$$

we obtain

$$\int_0^T \left(\sum_{j=0}^{k_T-1} Z_j' \cdot I_{[t_j,t_{j+1})}(t) \right) dB_t = \sum_{j=0}^{k_T-1} Z_j' \left(B_{t_{j+1}} - B_{t_j} \right) =$$

$$= \sum_{j=0}^{l_T-1} Z_{\tilde{\tau}_j}' \left(B_{\eta_{j+1}} - B_{\eta_j} \right) = \sum_{j=0}^{l_T-1} Z_{\tilde{\tau}_j}'' \left(B_{\eta_{j+1}} - B_{\eta_j} \right) =$$

$$= \sum_{j=0}^{p_T-1} Z_j'' \left(B_{\tau_{j+1}} - B_{\tau_j} \right) = \int_0^T \left(\sum_{j=0}^{p_T-1} Z_j'' \cdot I_{[\tau_j,\tau_{j+1})}(t) \right) dB_t,$$

since

$$\sum_{j=0}^{k_T-1} Z_j' \cdot I_{[t_j,t_{j+1})}(t) = \sum_{j=0}^{p_T-1} Z_j'' \cdot I_{[\tau_j,\tau_{j+1})}(t), \quad t \in [0,T].$$

In the following step, we are going to enlarge the set of integral kernels for Itô integration. To this end, we need two important functional analytic concepts.

Definition 3.2 (Pseudometric, Seminorm)

Let X be a nonempty set. A tuple (X, d) is called a **pseudometric space**, if there exists a mapping $d : X \times X \to \mathbb{R}$ such that

(i) $d(x,x) = 0 \quad$ for all $x \in X$,

(ii) $d(x,y) = d(y,x) \quad$ for all $x, y \in X$,

(iii) $d(x,y) \leq d(x,z) + d(z,y) \quad$ for all $x, y, z \in X$.

The mapping d is called a **pseudometric** on X. If

$$d(x,y) = 0 \quad \Longrightarrow \quad x = y \quad \text{for all} \quad x, y \in X$$

holds in addition, the tuple (X, d) is called a metric space. The (pseudo)metric space (X, d) is called complete, if each Cauchy sequence $\{x_i\}_{i \in \mathbb{N}}$, $x_i \in X$, converges to an element of X, i.e. there exists an $x \in X$ with

$$\lim_{i \to \infty} d(x_i, x) = 0.$$

Let X be a vector space over $\mathbb{R}$. A tuple $(X, \| \bullet \|)$ is called a **seminormed space**, if there exists a mapping $\| \bullet \| : X \to \mathbb{R}$ such that

(i) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\| \quad$ for all $x \in X$, $\lambda \in \mathbb{R}$,

(ii) $\|x + y\| \leq \|x\| + \|y\| \quad$ for all $x, y, \in X$.

(continued)

Definition 3.2 (continued)

The mapping $\| \bullet \|$ is called a **seminorm** on X. If

$$\|x\| = 0 \quad \Longrightarrow \quad x = 0 \quad \text{for all} \quad x \in X$$

holds in addition, the tuple $(X, \| \bullet \|)$ is called a normed space. The (semi)normed space $(X, \| \bullet \|)$ is called complete, if each Cauchy sequence $\{x_i\}_{i\in\mathbb{N}}$, $x_i \in X$, converges to an element of X, i.e. there exists an $x \in X$ with

$$\lim_{i\to\infty} \|x_i - x\| = 0.$$

$\triangleleft$

Now, we are able to enlarge the set of integral kernels for Itô integration using the following extension theorem.

Lemma 3.3 (Extension Theorem) *Let (X, d) be a pseudometric space and let (X', d') be a complete pseudometric space. Furthermore, let X_0 be a dense subset of X and let*

$$f : X_0 \to X'$$

be a Lipschitz continuous mapping, i.e. there exists a constant $L \geq 0$ with

$$d'(f(x), f(y)) \leq L \cdot d(x, y) \quad \textit{for all} \quad x, y \in X_0.$$

With these assumptions there exists a Lipschitz continuous extension F of f (with the same constant L) defined on X:

$$F : X \to X', \quad x \mapsto f(x) \quad \textit{for all} \quad x \in X_0.$$

If

$$d'(f(x), f(y)) = d(x, y) \quad \textit{for all} \quad x, y \in X_0 \quad (f \textit{ is an isometry}),$$

then F is an isometry too. $\triangleleft$

Proof

For each $x \in X$, there exists a convergent sequence $\{x_i\}_{i\in\mathbb{N}}$, $x_i \in X_0$, with limit x, because X_0 is a dense subset of X. Therefore, $\{x_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence and from

$$d'(f(x_n), f(x_m)) \leq L \cdot d(x_n, x_m)$$

follows that $\{f(x_i)\}_{i\in\mathbb{N}}$ is a Cauchy sequence, too. Since X' is complete, there exists an $x' \in X'$ with

$$x' = \lim_{i\to\infty} f(x_i)$$

and we are able to choose

$$F(x) = \xi' \quad \text{with} \quad \xi' \in \{y' \in X';\, d'(x', y') = 0\}.$$

Now, we have to show that the mapping $F : X \to X'$ is well defined. Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ be two sequences with elements in X_0, which converge to $x \in X$, then we obtain

$$d'(f(x_i), f(y_i)) \leq L \cdot d(x_i, y_i) \stackrel{i\to\infty}{\longrightarrow} 0.$$

Since the constant sequence $\{x\}_{i\in\mathbb{N}}$ converges to x, we are able to choose

$$F(x) = f(x) \quad \text{for all} \quad x \in X_0.$$

The fact that F is Lipschitz continuous with the same constant L as f follows from the continuity of the pseudometrics d and d': Let $\{x_i\}_{i\in\mathbb{N}}$ and $\{z_i\}_{i\in\mathbb{N}}$ be two sequences with elements in X_0, where $\{x_i\}_{i\in\mathbb{N}}$ converges to $x \in X$ and where $\{z_i\}_{i\in\mathbb{N}}$ converges to $z \in X$, then it holds

$$\begin{aligned} d'(F(x), F(z)) &= d'(\lim_{i\to\infty} f(x_i), \lim_{i\to\infty} f(z_i)) = \lim_{i\to\infty} d'(f(x_i), f(z_i)) \leq \\ &\leq \lim_{i\to\infty} L \cdot d(x_i, z_i) = L \cdot d(\lim_{i\to\infty} x_i, \lim_{i\to\infty} z_i) = L \cdot d(x, z). \end{aligned}$$

Analogously it can be shown that F is an isometry if f is an isometry. $\square$

In order to enlarge the set of integral kernels for Itô integration, we have to choose the spaces X_0, X', the pseudometrics d and d', and the mapping f in an appropriate way. X_0 is chosen as the set of all elementary stochastic processes $\mathfrak{E}([0, T])$ on $[0, T]$ defined as follows:
Let $(B_t)_{t\in[0,\infty)}$ be a one-dimensional Brownian Motion and let $\left(\mathcal{F}_t^B\right)_{t\in[0,\infty)}$ be its natural filtration. Furthermore, let $(Z_t)_{t\in[0,\infty)}$ be a stochastic process with the property: There exists a partition

$$0 = t_0 < t_1 < \ldots < t_{k_T} = T, \quad k_T \in \mathbb{N},$$

with

$$Z_t : \Omega \to \mathbb{R}, \quad \omega \mapsto \sum_{j=0}^{k_T-1} Z_j'(\omega) \cdot I_{[t_j, t_{j+1})}(t), \quad t \in [0, \infty),$$

where

$$I_{[t_j,t_{j+1})} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} 1 & \text{for } t \in [t_j, t_{j+1}) \\ 0 & \text{for} t \notin [t_j, t_{j+1}) \end{cases}, \quad j \in \{0, \ldots, k_T - 1\}.$$

In addition, we assume that the random variable

$$Z_j' : \Omega \to \mathbb{R}$$

is $\mathcal{F}_{t_j}^B$-$\mathcal{B}$-measurable and bounded for each $j \in \{0, \ldots, k_T - 1\}$.
Then, the stochastic process $(Z_t)_{t\in[0,\infty)}$ is called **elementary stochastic process** on $[0, T]$. The mapping

$$Z : [0, T] \times \Omega \to \mathbb{R}, \quad (t, \omega) \mapsto Z_t(\omega),$$

is $(\mathcal{B}_{[0,T]} \times \mathcal{S})$-$\mathcal{B}$-measurable, where $\mathcal{B}_{[0,T]} \times \mathcal{S}$ denotes the smallest σ-field over $[0, T] \times \Omega$ such that

$$p_1 : [0, T] \times \Omega \to [0, T], \quad (t, \omega) \mapsto t,$$

is $(\mathcal{B}_{[0,T]} \times \mathcal{S})$-$\mathcal{B}$-measurable and

$$p_2 : [0, T] \times \Omega \to \Omega, \quad (t, \omega) \mapsto \omega,$$

is $(\mathcal{B}_{[0,T]} \times \mathcal{S})$-$\mathcal{S}$-measurable. There exists the product measure

$$(\lambda \times \mathbb{P}) : \mathcal{B}_{[0,T]} \times \mathcal{S} \to \mathbb{R}$$

uniquely defined by

$$(\lambda \times \mathbb{P})(M_1 \times M_2) = \lambda(M_1) \cdot \mathbb{P}(M_2) \quad \text{for all} \quad M_1 \in \mathcal{B}_{[0,T]}, M_2 \in \mathcal{S}.$$

A seminorm $d = \| \bullet \|_{[0,T]\times\Omega}$ is given by

$$\| \bullet \|_{[0,T]\times\Omega} : \mathfrak{E}([0, T]) \to \mathbb{R}, \quad (Z_t)_{t\in[0,\infty)} \mapsto \sqrt{\int\limits_{[0,T]\times\Omega} Z_t^2 d(\lambda(t) \times \mathbb{P})}.$$

We obtain

$$\|(Z_t)_{t\in[0,\infty)}\|_{[0,T]\times\Omega}^2 = \mathscr{E}\left(\int_0^T Z_t^2 dt\right) \quad \text{for all} \quad (Z_t)_{t\in[0,\infty)} \in \mathfrak{E}([0, T]).$$

X is chosen as the set $\mathscr{L}_2((\Omega, \mathcal{S}, \mathbb{P}), \mathbb{R})$ of all random variables

$$Y : \Omega \to \mathbb{R} \quad \text{with} \quad \int X^2 d\mathbb{P} < \infty.$$

A seminorm $d' = \| \bullet \|_{\mathscr{L}_2}$ is given by

$$\| \bullet \|_{\mathscr{L}_2} : \mathscr{L}_2((\Omega, \mathcal{S}, \mathbb{P}), \mathbb{R}) \to \mathbb{R}, \quad Y \mapsto \sqrt{\int Y^2 d\mathbb{P}}.$$

We obtain

$$\left\| \int_0^T Z_t dB_t \right\|_{\mathscr{L}_2}^2 = \mathscr{E}\left(\left(\int_0^T Z_t dB_t \right)^2 \right).$$

The mapping f is chosen by

$$f : \mathfrak{E}([0,T]) \to \mathscr{L}_2((\Omega, \mathcal{S}, \mathbb{P}), \mathbb{R}), \quad (Z_t)_{t \in [0,\infty)} \mapsto \int_0^T Z_t dB_t.$$

By Theorem 3.1 (Itô isometry) f is an isometry. The application of Lemma 3.3 yields

$$F : \mathrm{cl}(\mathfrak{E}([0,T])) \to \mathscr{L}_2((\Omega, \mathcal{S}, \mathbb{P}), \mathbb{R}), \; (X_t)_{t \in [0,\infty)} \mapsto F\left((X_t)_{t \in [0,\infty)}\right) =: \int_0^T X_t dB_t.$$

We have extended the set of integral kernels for Itô integration to the closure $\mathrm{cl}(\mathfrak{E}([0,T]))$ of $\mathfrak{E}([0,T])$, where the Itô isometry is preserved. Now, we have to investigate, which stochastic processes are elements of $\mathrm{cl}(\mathfrak{E}([0,T]))$.
Let $(X_t)_{t \in [0,\infty)}$ be a stochastic process, $(B_t)_{t \in [0,\infty)}$ a one-dimensional Brownian Motion and let

$$\left(\mathcal{F}_t^B\right)_{t \in [0,\infty)}$$

its natural filtration. Assume that

(i)

$$X : [0,T] \times \Omega \to \mathbb{R}, \quad (t, \omega) \mapsto X_t(\omega)$$

is $(\mathcal{B}_{[0,T]} \times \mathcal{S})$-$\mathcal{B}$-measurable and

$$\int\limits_{[0,T]\times\Omega} X_t^2 d(\lambda(t) \times \mathbb{P}) < \infty,$$

(ii) X_t is $\mathcal{F}_t$-$\mathcal{B}$-measurable for each $t \in [0, T]$ and

$$\int X_t^2 d\mathbb{P} < \infty,$$

then $(X_t)_{t\in[0,\infty)} \in \mathrm{cl}(\mathfrak{E}([0, T]))$. We sketch the proof in three steps (for details, see [Øks10]). The corresponding limits are computed using Theorem A.9.

First step: Assume that the paths of $(X_t)_{t\in[0,\infty)}$ are continuous on the interval $[0, T]$ and that the mapping $X : [0, T] \times \Omega \to \mathbb{R}$ is bounded. Using the partition

$$t_i = i\frac{T}{n}, \quad i = 0, \ldots, n, \quad n \in \mathbb{N},$$

we consider

$$Y_t^{(n)} : \Omega \to \mathbb{R}, \quad \omega \mapsto \sum_{i=0}^{n-1} X_{t_i}(\omega) \cdot I_{[t_i,t_{i+1})}(t), \quad t \in [0, \infty).$$

Since the paths of $(X_t)_{t\in[0,\infty)}$ are uniformly continuous on the interval $[0, T]$ and since X is bounded, one gets

$$\lim_{n\to\infty} \left\| \left(Y_t^{(n)}\right)_{t\in[0,\infty)} - (X_t)_{t\in[0,\infty)} \right\|_{[0,T]\times\Omega} = 0.$$

Second step: Assume that the paths of $(X_t)_{t\in[0,\infty)}$ are continuous on the interval $[0, T]$. For $n \in \mathbb{N}$, we define:

$$Y_t^{[n]} : \Omega \to \mathbb{R}, \quad \omega \mapsto \begin{cases} X_t(\omega) & \text{for} \quad |X_t(\omega)| \le n \\ n & \text{for} \quad X_t(\omega) > n \\ -n & \text{for} \quad X_t(\omega) < -n \end{cases}, \quad \text{for all} \quad t \in [0, \infty).$$

From step one we get

$$\left(Y_t^{[n]}\right)_{t\in[0,\infty)} \in \mathrm{cl}(\mathfrak{E}([0, T]))$$

and

$$\lim_{n\to\infty} \left\| \left(Y_t^{[n]}\right)_{t\in[0,\infty)} - (X_t)_{t\in[0,\infty)} \right\|_{[0,T]\times\Omega} = 0.$$

Third step: The proof is completed by the fact, that the set of all real valued continuous functions defined on the interval $[0,T]$ is a dense subset of the set $\mathscr{L}_2([0,T],\mathbb{R})$ of all functions

$$f : [0,T] \to \mathbb{R} \quad \text{with} \quad \int\limits_{[0,T]} f^2 d\lambda < \infty.$$

Finally, we obtain a stochastic process $\left(I_t^Z\right)_{t\in[0,\infty)}$ with

$$I_t^Z : \Omega \to \mathbb{R}, \quad \omega \mapsto \begin{cases} 0 & \text{for} \quad t = 0 \\ \int\limits_0^t Z_\tau dB_\tau & \text{for} \quad t > 0 \end{cases}$$

by varying T. It is always possible to choose a modification of $\left(I_t^Z\right)_{t\in[0,\infty)}$ with continuous paths (using Doob's martingale inequality, see [Øks10]), which is done in the following.
If we choose

$$(X_t)_{t\in[0,\infty)} = (B_t)_{t\in[0,\infty)},$$

for instance, we obtain for $t \in [0,\infty)$ (see Problem 1.):

$$\int\limits_0^t B_\tau dB_\tau = \frac{1}{2}\left(B_t^2 - t\right) \quad (\mathbb{P}\text{-})\text{almost surely} \quad (\text{see Fig. 3.4}),$$

which we can reformulate as

$$B_t^2 = \int\limits_{[0,t]} 1 d\lambda + 2\int\limits_0^t B_\tau dB_\tau \quad (\mathbb{P}\text{-})\text{almost surely.}$$

A stochastic process, which is defined as a sum of a $\mathcal{F}_0$-$\mathcal{B}$-measurable random variable X_0 (equal to zero in the above example), of a pathwise Lebesgue integral and of an Itô integral is called an **Itô differential**. Practical examples of Itô differentials will be discussed later. Now, we have to introduce Itô integration based on a Poisson process $(P_t)_{t\in[0,\infty)}$ instead of a Brownian Motion in order to incorporate shot noise in circuit

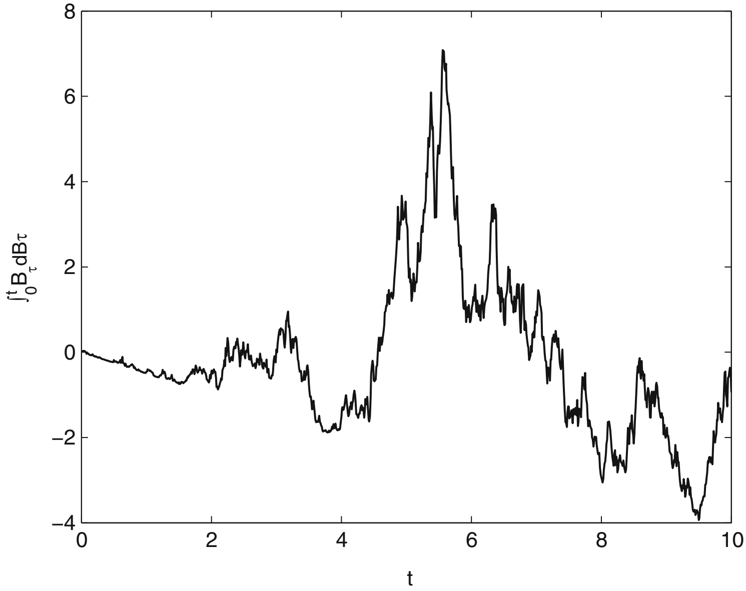

Fig. 3.4 A path of $\int\limits_0^t B_\tau dB_\tau, 0 \leq t \leq 10$

simulation. Since

$$\mathscr{E}(P_t) = \nu \cdot t \neq 0 \quad \text{for} \quad \nu \neq 0,\ t \in (0, \infty),$$

we initially have to use a **compensated Poisson process** $\left(\hat{P}_t\right)_{t\in[0,\infty)}$ given by

$$\hat{P}_t(\omega) := P_t(\omega) - \nu \cdot t \quad \text{for all} \quad \omega \in \Omega, \quad t \in [0, \infty).$$

A path of $\left(\hat{P}_t\right)_{t\in[0,\infty)}$ for $0 \leq t \leq 10$ is shown in Fig. 3.5.

Using the canonical filtration

$$\left(\mathcal{F}_t^{\hat{P}}\right)_{t\in[0,\infty)}$$

of $\left(\hat{P}_t\right)_{t\in[0,\infty)}$ we are able to define Itô integration based on a compensated Poisson process $(P_t)_{t\in[0,\infty)}$ instead of a Brownian Motion analogously to the approach above for a Brownian Motion (see Problem 4.). Since

$$\hat{P}_t = P_t(\omega) - \nu \cdot t,$$

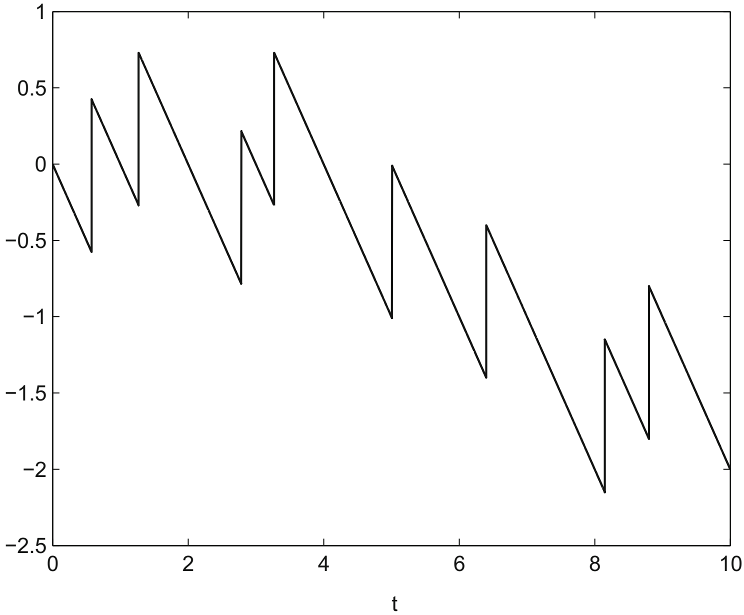

Fig. 3.5 A path of $\left(\hat{P}_t\right)_{t\in[0,\infty)}$, $0 \leq t \leq 10$

it is natural to define for each $t \in [0, \infty)$:

$$\left(\int_0^t X_\tau dP_\tau\right)(\omega) := \nu \cdot \int_{[0,t]} X_\tau(\omega) d\lambda(\tau) + \left(\int_0^t X_\tau d\hat{P}_\tau\right)(\omega) \quad \text{for all} \quad \omega \in \Omega,$$

which is an Itô differential again.

At this point, it is obvious to ask which stochastic processes can be used (in addition to a Brownian Motion and a compensated Poisson process) to define Itô integration in view of noise modelling. In stochastic analysis it is common to focus on the following class of stochastic processes.

Definition 3.4 (Lévy Martingale)

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let $(L_t)_{t\in[0,\infty)}$ be a real valued stochastic process with the following properties

(i) $L_0 \equiv 0$,

(ii) the increments

$$L_{t_2} - L_{t_1}, L_{t_3} - L_{t_2}, \ldots, L_{t_n} - L_{t_{n-1}}$$

(continued)

Definition 3.4 (continued)

are stochastically independent for $0 \leq t_1 < t_2 < \ldots < t_n$, $n \in \mathbb{N}$,

(iii) $(L_{t+h} - L_t)$ and L_h are identically distributed for each $t \in [0, \infty)$ and each $h > 0$,

(iv) $\mathscr{E}(L_1) = 0$,

(v) for each $\varepsilon > 0$ and each $s > 0$ holds

$$\lim_{t \to s} \mathbb{P}\left(\omega \in \Omega;\ |L_t(\omega) - L_s(\omega)| > \varepsilon\right) = 0,$$

then $(L_t)_{t\in[0,\infty)}$ is called a **Lévy martingale**.

◁

From (ii) and (iii) follows that the difference quotients

$$\frac{L_{t_2} - L_{t_1}}{t_2 - t_1}, \frac{L_{t_3} - L_{t_2}}{t_3 - t_2}, \ldots, \frac{L_{t_n} - L_{t_{n-1}}}{t_n - t_{n-1}}$$

are stochastically independent and that they are identically distributed, if

$$t_i - t_{i-1} = h, \quad i \in \{2, 3, \ldots, n\}, \quad h > 0.$$

From $\mathscr{E}(L_1) = 0$, we obtain after some calculations

$$\mathscr{E}(L_t) = 0 \quad \text{for all} \quad t \in [0, \infty).$$

The last property defines a stochastical continuity condition for $(L_t)_{t\in[0,\infty)}$. It is possible to show (see [Protter10], Chapter 1, Theorem 30) that for each Lévy martingale $(L_t)_{t\in[0,\infty)}$ there exists a unique modification such that this modification is again a Lévy martingale with paths, which are continuous from the right and with existing limit from the left for all $t > 0$. Only such Lévy martingales are considered in the following. Furthermore, it is possible to show that a stochastic process $(L_t)_{t\in[0,\infty)}$ given by

$$L_t = \sigma \cdot B_t \quad \text{for all} \quad t \in [0, \infty),$$

where $(B_t)_{t\in[0,\infty)}$ is a one-dimensional Brownian Motion and $\sigma > 0$, is the unique Lévy martingale with continuous paths. Given a Lévy martingale $(L_t)_{t\in[0,\infty)}$ and a nonempty Borel subset set $M \in \mathcal{B}$ of $\mathbb{R}$ with $0 \notin M$, we introduce the stochastic process $\left(P_t^M\right)_{t\in[0,\infty)}$, where $P_t^M(\omega)$ is the number of jumps along the path $L_\bullet(\omega)$ of $(L_t)_{t\in[0,\infty)}$ between $0 \leq s \leq t$ such that $h \in M$ holds with

$$h := L_\tau(\omega) - \lim_{\substack{t \to \tau \\ t < \tau}} L_t(\omega).$$

From the properties of Lévy martingales follows that $\left(P_t^M\right)_{t\in[0,\infty)}$ is a Poisson process with parameter

$$\nu = \mathscr{E}\left(P_1^M\right),$$

and $(\hat{P}_t^M)_{t\in[0,\infty)}$ denotes the corresponding compensated Poisson process. If the jumps along the paths of $(L_t)_{t\in[0,\infty)}$ are bounded in absolute value (by $C > 0$, for instance), we are able to consider the random variable

$$Z_t^n = \sum_{k=1}^{n-1} \frac{kC}{n} \left(\hat{P}_t^{\left(\frac{kC}{n}, \frac{(k+1)C}{n}\right]} - \hat{P}_t^{\left[-\frac{(k+1)C}{n}, -\frac{kC}{n}\right)} \right), \qquad n \in \mathbb{N},\ n \geq 2,\ t \in [0,\infty).$$

The sequence $(Z_t^n)_{n\in\mathbb{N}}$ of random variables converges in the mean square to a random variable Z_t (see [Protter10], Chapter 1, Theorem 41) for each $t \in [0,\infty)$, and hence, we obtain a stochastic process $(Z_t)_{t\in[0,\infty)}$ and the **Lévy-Itô decomposition**:
There exists a $\sigma \geq 0$ with

$$L_t = \sigma \cdot B_t + Z_t \quad \text{for all} \quad t \in [0,\infty).$$

Furthermore, for all $s, t \in [0,\infty)$ holds:

$$\mathscr{V}(L_1) < \infty \quad \text{and} \quad \mathscr{C}(L_s, L_t) = \min(s,t)\mathscr{V}(L_1).$$

A path of a Lévy martingale is shown in Fig. 3.6, while Fig. 3.7 shows the corresponding path of $(\sigma B_t)_{t\in[0,\infty)}$ and Fig. 3.8 shows the corresponding path of $(Z_t)_{t\in[0,\infty)}$.

Lévy martingales with Lévy-Itô decomposition define a class of stochastic processes, which can be used for the definition of Itô integration (see [Protter10]).

Now, we introduce a stochastic analogon to the following result from classical analysis:

Let $x_0 \in \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function, then with

$$F : [0,\infty) \to \mathbb{R}, \quad t \mapsto x_0 + \int_0^t f(x)dx$$

and with

$$G : [0,\infty) \to \mathbb{R}, \quad t \mapsto g(t, F(t))$$

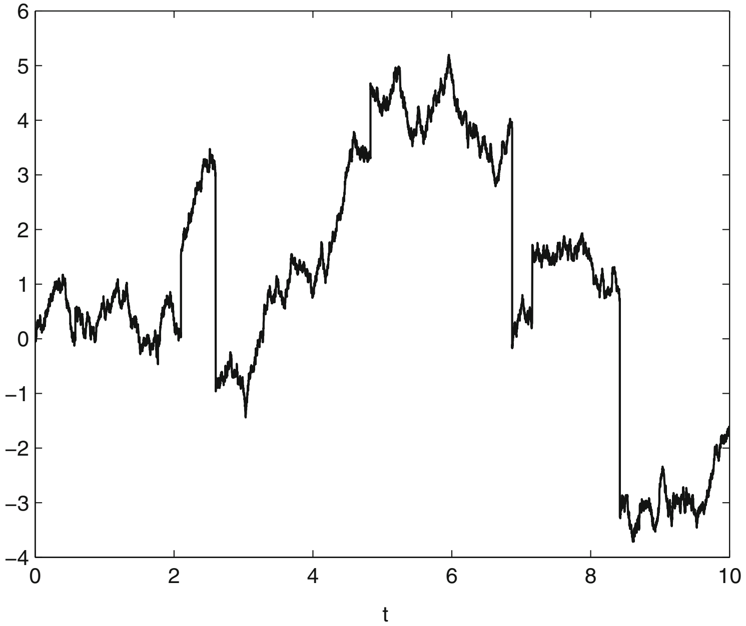

Fig. 3.6 A path of a Lévy martingale, $0 \leq t \leq 10$

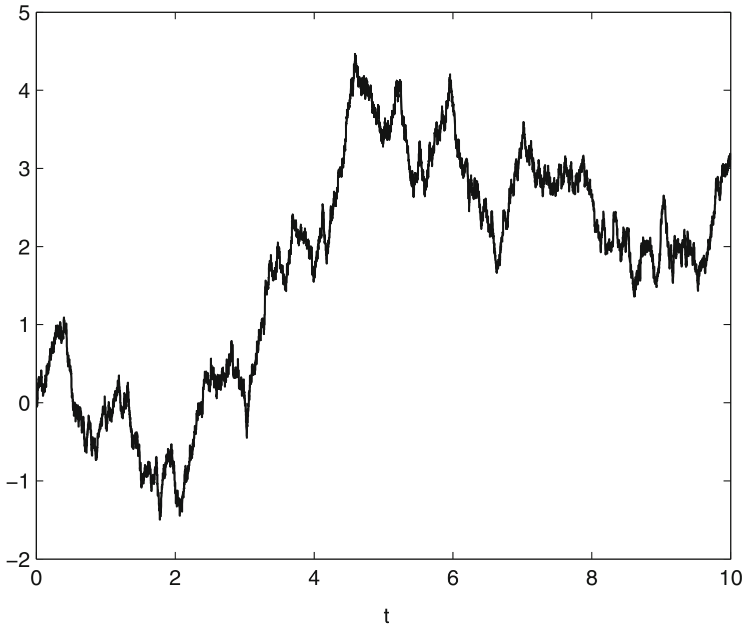

Fig. 3.7 A path of $(\sigma B_t)_{t\in[0,\infty)}$, $0 \leq t \leq 10$

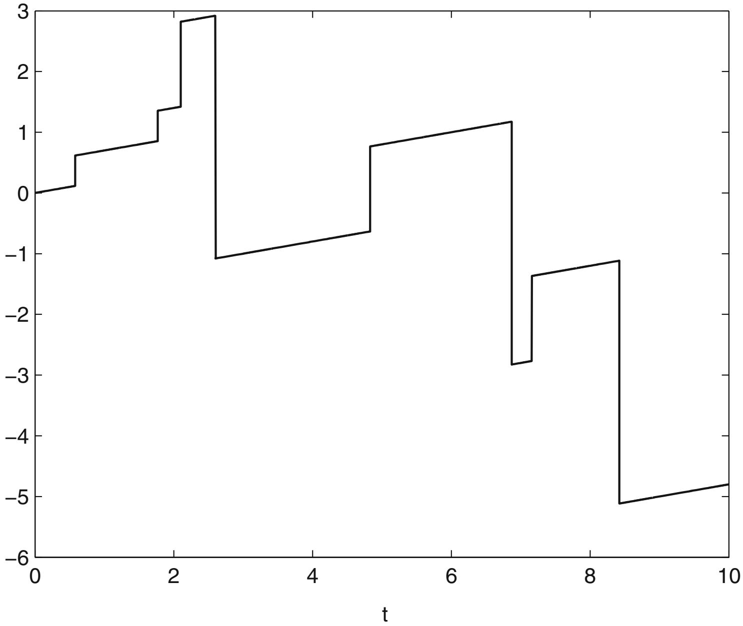

Fig. 3.8 A path of $(Z_t)_{t\in[0,\infty)}$, $0 \leq t \leq 10$

we get the function values of G by

$$G(t) = \tilde{x}_0 + \int_0^t \tilde{f}(x)dx \quad \text{for all} \quad t \in [0,\infty),$$

where

$$\tilde{x}_0 = g(0, x_0),$$
$$\tilde{f} : [0,\infty) \to \mathbb{R}, \quad x \mapsto \frac{\partial g}{\partial x_1}(x, F(x)) + f(x)\frac{\partial g}{\partial x_2}(x, F(x)).$$

Theorem 3.5 (Itô' s Lemma)
Let $(B_t)_{t\in[0,\infty)}$ be a one-dimensional Brownian Motion and let $\left(\mathcal{F}_t^B\right)_{t\in[0,\infty)}$ be its natural filtration. Furthermore, let

$$(U_t)_{t\in[0,\infty)}, (V_t)_{t\in[0,\infty)} \in \mathrm{cl}(\mathfrak{E}([0,T]))$$

(continued)

Theorem 3.5 (continued)
be two stochastic processes with continuous paths, let X_0 be a $\mathcal{F}_0$-$\mathcal{B}$-measurable random variable, and let $(X_s)_{s\in[0,T]}$ be an Itô differential given by

$$X_\tau(\omega) = X_0(\omega) + \int_{[0,\tau]} U_s(\omega)d\lambda(s) + \left(\int_0^\tau V_s dB_s\right)(\omega), \quad \omega \in \Omega,\ \tau \in [0,T].$$

Using a twice continuously differentiable function

$$g : [0,T] \times \mathbb{R} \to \mathbb{R}, \quad (x_1, x_2) \mapsto g(x_1, x_2)$$

we obtain with

$$\tilde{X}_\tau : \Omega \to \mathbb{R}, \quad \omega \mapsto g(\tau, X_\tau(\omega)), \quad \tau \in [0,T] :$$

$$\tilde{X}_\tau(\omega) = \tilde{X}_0(\omega) + \int_{[0,\tau]} \tilde{U}_s(\omega)d\lambda(s) + \left(\int_0^\tau \tilde{V}_s dB_s\right)(\omega), \quad \omega \in \Omega,\ \tau \in [0,T],$$

where

$$\begin{aligned}
\tilde{X}_0 &= g(0, X_0),\\
\tilde{U}_s &= \frac{\partial g}{\partial x_1}(s, X_s) + U_s \frac{\partial g}{\partial x_2}(s, X_s) + \frac{1}{2} V_s^2 \frac{\partial^2 g}{\partial x_2^2}(s, X_s).\\
\tilde{V}_s &= V_s \frac{\partial g}{\partial x_2}(s, X_s).
\end{aligned}$$

◁

Proof
Let $\tau \in (0,T]$ and let $\{t_0^i, \ldots, t_{k_i}^i\}_{i\in\mathbb{N}}$ be a sequence of partitions with

$$0 = t_0^i < t_1^i < \ldots < t_{k_i}^i = \tau, \quad i \in \mathbb{N},\ k_i \in \mathbb{N},$$

and with

$$\lim_{i\to\infty} \max\{t_j^i - t_{j-1}^i;\ j = 1, \ldots, k_i\} = 0.$$

Taylor expansion of g gives

$$
\begin{aligned}
g\left(t_{j+1}^i, X_{t_{j+1}^i}\right) - g\left(t_j^i, X_{t_j^i}\right) = {} & \frac{\partial g}{\partial x_1}\left(t_j^i, X_{t_j^i}\right)\left(t_{j+1}^i - t_j^i\right) + \\
& + \frac{\partial g}{\partial x_2}\left(t_j^i, X_{t_j^i}\right)\left(X_{t_{j+1}^i} - X_{t_j^i}\right) + \\
& + \frac{1}{2}\frac{\partial^2 g}{\partial x_1^2}\left(t_j^i, X_{t_j^i}\right)\left(t_{j+1}^i - t_j^i\right)^2 + \\
& + \frac{\partial^2 g}{\partial x_1 \partial x_2}\left(t_j^i, X_{t_j^i}\right)\left(t_{j+1}^i - t_j^i\right)\left(X_{t_{j+1}^i} - X_{t_j^i}\right) \\
& + \frac{1}{2}\frac{\partial^2 g}{\partial x_2^2}\left(t_j^i, X_{t_j^i}\right)\left(X_{t_{j+1}^i} - X_{t_j^i}\right)^2 + o\left(\left\| \begin{matrix} t_{j+1}^i - t_j^i \\ X_{t_{j+1}^i} - X_{t_j^i} \end{matrix} \right\|_2^2\right).
\end{aligned}
$$

Summation from $j = 0$ to $j = k_i$ and computation of the limit for $i \to \infty$ produces for the left hand side

$$
\lim_{i\to\infty} \sum_{j=0}^{k_i - 1} \left(g\left(t_{j+1}^i, X_{t_{j+1}^i}\right) - g\left(t_j^i, X_{t_j^i}\right)\right) = g\left(\tau, X_\tau\right) - g\left(0, X_0\right).
$$

Furthermore, we get

$$
\lim_{i\to\infty} \sum_{j=0}^{k_i - 1} \frac{\partial g}{\partial x_1}\left(t_j^i, X_{t_j^i}\right)\left(t_{j+1}^i - t_j^i\right) = \int_{[0,\tau]} \frac{\partial g}{\partial x_1}(s, X_s) d\lambda(s).
$$

For the next summand, we obtain

$$
\begin{aligned}
& \sum_{j=0}^{k_i - 1} \frac{\partial g}{\partial x_2}\left(t_j^i, X_{t_j^i}\right)\left(X_{t_{j+1}^i} - X_{t_j^i}\right) = \\
= & \sum_{j=0}^{k_i - 1} \left(\frac{\partial g}{\partial x_2}\left(t_j^i, X_{t_j^i}\right) \int_{[t_j, t_{j+1}]} U_s(\omega) d\lambda(s) + \frac{\partial g}{\partial x_2}\left(t_j^i, X_{t_j^i}\right)\left(\int_{t_j}^{t_{j+1}} V_s dB_s\right)(\omega)\right) = \\
= & \sum_{j=0}^{k_i - 1} \left(\int_{[t_j, t_{j+1}]} \frac{\partial g}{\partial x_2}\left(t_j^i, X_{t_j^i}\right) U_s(\omega) d\lambda(s) + \int_{t_j}^{t_{j+1}} \left(\frac{\partial g}{\partial x_2}\left(t_j^i, X_{t_j^i}\right) V_s dB_s\right)(\omega)\right) \\
\underset{i\to\infty}{\longrightarrow} & \int_{[0,\tau]} U_s \frac{\partial g}{\partial x_2}\left(s, X_s\right) d\lambda(s) + \int_0^\tau V_s \left(\frac{\partial g}{\partial x_2}\left(s, X_s\right)\right) dB_s
\end{aligned}
$$

(convergence in the mean square).

With

$$S(\omega) := \max_{0 \le s \le \tau} \left\{ \left| \frac{\partial^2 g}{\partial x_1^2} (s, X_s(\omega)) \right| \right\}$$

follows

$$\sum_{j=0}^{k_i - 1} \left| \frac{1}{2} \frac{\partial^2 g}{\partial x_1^2} \left(t_j^i, X_{t_j^i}(\omega) \right) \right| \left(t_{j+1}^i - t_j^i \right)^2 \quad \le \quad S(\omega) \max_{j=0,\dots,k_i} \{ t_{j+1}^i - t_j^i \} \sum_{j=0}^{k_i-1} (t_{j+1}^i - t_j^i) =$$

$$= S(\omega) \max_{j=0,\dots,k_i-1} \{ t_{j+1}^i - t_j^i \} \tau \underset{i \to \infty}{\longrightarrow} 0 \quad \text{for all} \quad \omega \in \Omega,$$

and analogously

$$\lim_{i \to \infty} \sum_{j=0}^{k_i - 1} \left| \frac{\partial^2 g}{\partial x_1 \partial x_2} \left(t_j^i, X_{t_j^i} \right) \right| \left(t_{j+1}^i - t_j^i \right) \left(X_{t_{j+1}^i} - X_{t_j^i} \right) = 0 \quad \text{for all} \quad \omega \in \Omega.$$

The approximation

$$X_{t_{j+1}^i} - X_{t_j^i} \approx U_{t_j^i} \left(t_{j+1}^i - t_j^i \right) + V_{t_j^i} \left(B_{t_{j+1}^i} - B_{t_j^i} \right)$$

and

$$g_j := \frac{1}{2} \frac{\partial^2 g}{\partial x_2^2} \left(t_j^i, X_{t_j^i} \right)$$

lead to

$$\sum_{j=0}^{k_i - 1} \frac{1}{2} \frac{\partial^2 g}{\partial x_2^2} \left(t_j^i, X_{t_j^i} \right) \left(X_{t_{j+1}^i} - X_{t_j^i} \right)^2 \approx$$

$$\approx \sum_{j=0}^{k_i-1} g_j U_{t_j^i}^2 \left(t_{j+1}^i - t_j^i \right)^2 + 2 \sum_{j=0}^{k_i-1} g_j U_{t_j^i} V_{t_j^i} \left(t_{j+1}^i - t_j^i \right) \left(B_{t_{j+1}^i} - B_{t_j^i} \right) +$$

$$+ \sum_{j=0}^{k_i-1} g_j V_{t_j^i}^2 \left(B_{t_{j+1}^i} - B_{t_j^i} \right)^2.$$

We know that the first two summands converge to zero in the mean square. Since

$$\mathscr{E}\left(\left(\sum_{j=0}^{k_i-1} g_j V_{t_j^i}^2 \left(B_{t_{j+1}^i} - B_{t_j^i}\right)^2 - \sum_{j=0}^{k_i-1} g_j V_{t_j^i}^2 \left(t_{j+1}^i - t_j^i\right)\right)^2\right) =$$

$$= \mathscr{E}\left(\left(\sum_{j=0}^{k_i-1} g_j V_{t_j^i}^2 \left(\left(B_{t_{j+1}^i} - B_{t_j^i}\right)^2 - \left(t_{j+1}^i - t_j^i\right)\right)\right)^2\right) =$$

$$= \sum_{j=0}^{k_i-1} \mathscr{E}\left(g_j^2 V_{t_j^i}^4\right) \mathscr{E}\left(\left(\left(B_{t_{j+1}^i} - B_{t_j^i}\right)^2 - \left(t_{j+1}^i - t_j^i\right)\right)^2\right) =$$

$$= \sum_{j=0}^{k_i-1} \mathscr{E}\left(g_j^2 V_{t_j^i}^4\right) \mathscr{E}\left(\left(B_{t_{j+1}^i} - B_{t_j^i}\right)^4\right) -$$

$$-2 \sum_{j=0}^{k_i-1} \mathscr{E}\left(g_j^2 V_{t_j^i}^4\right) \mathscr{E}\left(\left(B_{t_{j+1}^i} - B_{t_j^i}\right)^2 \left(t_{j+1}^i - t_j^i\right)\right) +$$

$$+ \sum_{j=0}^{k_i-1} \mathscr{E}\left(g_j^2 V_{t_j^i}^4\right) \left(t_{j+1}^i - t_j^i\right)^2$$

and since

$$\mathscr{E}\left((B_t - B_s)^4\right) = 3(t-s)^2, \quad t \geq s \quad \text{(fourth moment of the Gaussian distribution)},$$

follows

$$\lim_{i\to\infty} \mathscr{E}\left(\left(\sum_{j=0}^{k_i-1} g_j V_{t_j^i}^2 \left(B_{t_{j+1}^i} - B_{t_j^i}\right)^2 - \sum_{j=0}^{k_i-1} g_j V_{t_j^i}^2 \left(t_{j+1}^i - t_j^i\right)\right)^2\right) =$$

$$= \lim_{i\to\infty} 2 \sum_{j=0}^{k_i-1} \mathscr{E}\left(g_j^2 V_{t_j^i}^4\right) \left(t_{j+1}^i - t_j^i\right)^2 = 0,$$

and hence

$$\sum_{j=0}^{k_i-1} g_j V_{t_j^i}^2 \left(B_{t_{j+1}^i} - B_{t_j^i}\right)^2$$

converges in the mean square to

$$\int_{[0,\tau]} \frac{1}{2} V_s^2 \frac{\partial^2 g}{\partial x_2^2}(s, X_s) d\lambda(s).$$

The limits in this proof are computed using Theorem A.9. □

Investigating the example

$$\begin{aligned} X_0 : \Omega &\to \mathbb{R}, \quad & \omega &\mapsto 0, \\ U : [0,\infty) \times \Omega &\to \mathbb{R}, \quad & (t,\omega) &\mapsto 0, \\ V : [0,\infty) \times \Omega &\to \mathbb{R}, \quad & (t,\omega) &\mapsto 1, \\ g : [0,T] \times \mathbb{R} &\to \mathbb{R}, \quad & (x_1,x_2) &\mapsto x_2^3, \end{aligned}$$

we obtain

$$X_\tau = B_\tau \quad (\mathbb{P}\text{-})\text{almost surely}, \quad \tau \in [0,T].$$

and

$$B_\tau^3 = 3 \int\limits_{[0,\tau]} B_s d\lambda(s) + 3 \int\limits_0^\tau B_s^2 dB_s, \quad (\mathbb{P}\text{-})\text{almost surely}, \quad \tau \in [0,T].$$

In other words

$$\int\limits_0^\tau B_s^2 dB_s = \frac{1}{3} B_\tau^3 - \int\limits_{[0,\tau]} B_s d\lambda(s), \quad (\mathbb{P}\text{-})\text{almost surely}, \quad \tau \in [0,T].$$

Itô's lemma can be formulated for compensated Poisson processes and - more generally - for Lévy martingales also (see [HackThal94]).

3.2 Existence and Uniqueness

In the following, we consider an n-dim. Brownian Motion $(\mathbf{B}_t)_{t\in[0,\infty)}$ consisting of random variables

$$\mathbf{B}_t : \Omega \to \mathbb{R}^n, \quad t \in [0,\infty), \quad n \in \mathbb{N}.$$

The image measure $\mathbb{P}_{\mathbf{B}_t}$ is given by an n-dim. Gaussian distribution with

$$\mathscr{E}(\mathbf{B}_t) = \mathbf{0}, \quad t \in [0,\infty),$$

and with

$$\mathscr{C}((\mathbf{B}_t)_i, (\mathbf{B}_s)_j) = \begin{cases} \min(s,t) & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \quad i,j \in \{1,\ldots,n\}, \; s,t \in [0,\infty).$$

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$(\mathcal{F}_t^{\mathbf{B}})_{t\in[0,\infty)}$ denotes the natural filtration

$$\mathcal{F}_t^{\mathbf{B}} = \sigma(\mathbf{B}_s;\ s \le t), \quad s, t \in [0, \infty),$$

of $(\mathbf{B}_t)_{t\in[0,\infty)}$. Using suitable functions

$$\mathbf{h} : \mathbb{R}^k \times [0, b] \to \mathbb{R}^k \quad \text{and} \quad \mathbf{G} : \mathbb{R}^k \times [0, b] \to \mathbb{R}^{k,n}, \quad k \in \mathbb{N},$$

we are going to find a stochastic process $(\mathbf{X}_t)_{t\in[0,b]}$ with

$$\mathbf{X}_t : \Omega \to \mathbb{R}^k, \quad t \in [0, b],$$

which is adapted to $(\mathcal{F}_t^{\mathbf{B}})_{t\in[0,b]}$ and which is defined by

$$\mathbf{X}_t(\omega) = \mathbf{X}^0(\omega) + \begin{pmatrix} \int\limits_{[0,t]} \mathbf{h}_1(\mathbf{X}_\tau(\omega), \tau)d\lambda(\tau) \\ \vdots \\ \int\limits_{[0,t]} \mathbf{h}_k(\mathbf{X}_\tau(\omega), \tau)d\lambda(\tau) \end{pmatrix} +$$

$$+ \begin{pmatrix} \left(\int\limits_0^t \mathbf{G}_{1,1}(\mathbf{X}_\tau, \tau)d(B_1)_\tau\right)(\omega) + \ldots + \left(\int\limits_0^t \mathbf{G}_{1,n}(\mathbf{X}_\tau, \tau)d(B_n)_\tau\right)(\omega) \\ \vdots \\ \left(\int\limits_0^t \mathbf{G}_{k,1}(\mathbf{X}_\tau, \tau)d(B_1)_\tau\right)(\omega) + \ldots + \left(\int\limits_0^t \mathbf{G}_{k,n}(\mathbf{X}_\tau, \tau)d(B_n)_\tau\right)(\omega) \end{pmatrix}.$$

This equation is called a **stochastic differential equation** and is usually denoted by

$$d\mathbf{X}_t = \mathbf{h}(\mathbf{X}_t, t)dt + \mathbf{G}(\mathbf{X}_t, t)d\mathbf{B}_t, \quad \mathbf{X}_0 = \mathbf{X}^0, \quad t \in [0, b].$$

Consider again the initial value problem

$$LC\ddot{U}_C(t) + RC\dot{U}_C(t) + U_C(t) = U(t), \quad t \ge 0,\ U_C(0) = U_0,\ \dot{U}_C(0) = \dot{U}_0,$$

or equivalently

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} x_2(t) \\ -\frac{1}{LC}x_1(t) - \frac{R}{L}x_2(t) + \frac{1}{LC}u(t) \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} U_0 \\ \dot{U}_0 \end{pmatrix}.$$

If we take into account the noise voltage of the resistor R approximated by a Gaussian white noise process with a spectral density function, which is a constant σ_R, and if we assume again that the voltage U is additively disturbed by a Gaussian white noise with a spectral density function, which is a constant σ_U, then we obtain a stochastic differential equation

$$\mathbf{X}_t(\omega) = \begin{pmatrix} U_0 \\ \dot{U}_0 \end{pmatrix} + \begin{pmatrix} \int\limits_{[0,t]} (X_2)_\tau(\omega) d\lambda(\tau) \\ \int\limits_{[0,t]} \left(-\frac{1}{LC}(X_1)_\tau(\omega) - \frac{R}{L}(X_2)_\tau(\omega) + \frac{1}{LC}u(\tau)\right) d\lambda(\tau) \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 \\ \sigma_U \left(\int\limits_0^t \frac{1}{LC} d(B_1)_\tau\right)(\omega) - \sigma_R \left(\int\limits_0^t \frac{(X_2)_\tau}{L} d(B_2)_\tau\right)(\omega) \end{pmatrix}, \quad t \in [0, b],$$

where we assume that the two noise processes are stochastically independent. We obtain in this case

$$\mathbf{h} : \mathbb{R}^2 \times [0, b] \to \mathbb{R}^2, \quad (\mathbf{x}, t) \mapsto \begin{pmatrix} x_2 \\ -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{LC}u(t) \end{pmatrix}$$

and

$$\mathbf{G} : \mathbb{R}^2 \times [0, b] \to \mathbb{R}^{2,2}, \quad (\mathbf{x}, t) \mapsto \begin{pmatrix} 0 & 0 \\ \frac{\sigma_U}{LC} & -\frac{\sigma_R}{L}x_2 \end{pmatrix}.$$

A proof of the existence and uniqueness of a solution of a stochastic differential equation

$$d\mathbf{X}_t = \mathbf{h}(\mathbf{X}_t, t)dt + \mathbf{G}(\mathbf{X}_t, t)d\mathbf{B}_t, \quad \mathbf{X}_0 = \mathbf{X}^0, \quad t \in [0, b],$$

is usually based on a generalization of the Picard-Lindelöf theorem (see, e.g. [Beh13] and [Øks10]). Now, we introduce an approach which generalizes an elegant proof for the existence and uniqueness of solutions of initial value problems formulated in [Wal00]. Let

$$\mathbf{h} : \mathbb{R}^k \times [0, b] \to \mathbb{R}^k \quad \text{and} \quad \mathbf{G} : \mathbb{R}^k \times [0, b] \to \mathbb{R}^{k,n}, \quad k \in \mathbb{N},$$

be Lipschitz continuous functions with common Lipschitz constant $L > 0$. In other words, we assume

$$\|\mathbf{h}(\mathbf{x}, \tau) - \mathbf{h}(\mathbf{y}, \tau)\|_F + \|\mathbf{G}(\mathbf{x}, \tau) - \mathbf{G}(\mathbf{y}, \tau)\|_F \le L\|\mathbf{x} - \mathbf{y}\|_F$$

for all $\tau \in [0, b]$, where the **Frobenius norm** $\| \bullet \|_F$ is given by:

$$\| \bullet \|_F : \mathbb{R}^{n,m} \to \mathbb{R}, \quad \mathbf{A} = (a_{ij}) \mapsto \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2}$$

for all $n, m \in \mathbb{N}$ ($\mathbb{R}^n$ is identified by $\mathbb{R}^{n,1}$). Now, we consider the set $\mathfrak{Y}$ of all stochastic processes $(\mathbf{Y}_t)_{t\in[0,b]}$ given by

$$\mathbf{Y}_t : \Omega \to \mathbb{R}^k, \quad t \in [0, b],$$

with the following properties

(i)
$$\mathbf{Y} : [0, b] \times \Omega \to \mathbb{R}^k, \quad (t, \omega) \mapsto \mathbf{Y}_t(\omega)$$
is $(\mathcal{B}_{[0,b]} \times \mathcal{S})$-$\mathcal{B}^k$-measurable,

(ii) $\mathbf{Y}_t$ is $\mathcal{F}_t$-$\mathcal{B}^k$-measurable for each $t \in [0, b]$,

(iii) all paths of $(\mathbf{Y}_t)_{t\in[0,b]}$ are continuous,

(iv) $\sup\limits_{0\le t\le b} \left\{ \mathscr{E}\left(\|\mathbf{Y}_t\|_F^2\right)\right\} < \infty.$

For each $K \in [0, \infty)$, we obtain a seminormed space $\left(\mathfrak{Y}, \|\bullet\|_{(K)}\right)$ by

$$\|\bullet\|_{(K)} : \mathfrak{Y} \to \mathbb{R}, \quad (\mathbf{Y}_t)_{t\in[0,b]} \mapsto \sqrt{\sup_{0\le t\le b}\left\{ \mathscr{E}\left(\|\mathbf{Y}_t\|_F^2\right) \cdot e^{-Kt}\right\}}.$$

Important is the fact that the choice $K = 4(b + 1)L^2$ leads to a contraction mapping

$$T : \mathfrak{Y} \to \mathfrak{Y}, \quad (\mathbf{Y}_t)_{t\in[0,b]} \mapsto (\mathbf{Z}_t)_{t\in[0,b]}$$

with

$$\mathbf{Z}_t = \mathbf{X}^0 + \underbrace{\begin{pmatrix} \int\limits_{[0,t]} \mathbf{h}_1(\mathbf{Y}_\tau, \tau)d\lambda(\tau) \\ \vdots \\ \int\limits_{[0,t]} \mathbf{h}_k(\mathbf{Y}_\tau, \tau)d\lambda(\tau) \end{pmatrix}}_{=:\int\limits_{[0,t]} \mathbf{h}(\mathbf{Y}_\tau,\tau)d\lambda(\tau)} +$$

$$+ \underbrace{\begin{pmatrix} \int\limits_0^t \mathbf{G}_{1,1}(\mathbf{Y}_\tau, \tau)d(\mathbf{B}_1)_\tau + \ldots + \int\limits_0^t \mathbf{G}_{1,n}(\mathbf{Y}_\tau, \tau)d(\mathbf{B}_n)_\tau \\ \vdots \\ \int\limits_0^t \mathbf{G}_{k,1}(\mathbf{Y}_\tau, \tau)d(\mathbf{B}_1)_\tau + \ldots + \int\limits_0^t \mathbf{G}_{k,n}(\mathbf{Y}_\tau, \tau)d(\mathbf{B}_n)_\tau \end{pmatrix}}_{=:\int\limits_0^t \mathbf{G}(\mathbf{Y}_\tau,\tau)d\mathbf{B}(\tau)}.$$

Hence, the mapping T has a unique fixed point $(\mathbf{X}_t)_{t\in[0,b]}$ in $\mathrm{cl}(\mathfrak{Y})$ which is the solution of the above stochastic differential equation (uniqueness with regard to the seminorm $\| \bullet \|_{(K)}$). The proof, which we give in the following, uses two inequalities in seminormed spaces (see Problem 3.).
Let $(V, \| \bullet \|)$ be a seminormed space, then we obtain:

$$\|x+y+z\|^2 \le 3\left(\|x\|^2 + \|y\|^2 + \|z\|^2\right) \quad \text{for all} \quad x, y, z \in V,$$

and

$$\|x+y\|^2 \le 2\left(\|x\|^2 + \|y\|^2\right) \quad \text{for all} \quad x, y \in V.$$

Furthermore, for a continuous function

$$f : [a,b] \to \mathbb{R}$$

we use the inequality

$$\left(\int_{[a,b]} f d\lambda\right)^2 \le (b-a) \int_{[a,b]} f^2 d\lambda,$$

which follows from the Cauchy-Schwarz inequality. In a first step, we prove that the image $T((\mathbf{Y}_t)_{t\in[0,b]})$ of $(\mathbf{Y}_t)_{t\in[0,b]}$ is an element of $\mathfrak{Y}$. Hence, we have to show that

$$\sup_{0\le t\le b}\left\{\mathscr{E}\left(\|(T((\mathbf{Y}_s)_{s\in[0,b]}))_t\|_F^2\right)\right\} < \infty.$$

$$\sup_{0\le t\le b}\left\{\mathscr{E}\left(\|\mathbf{Z}_t\|_F^2\right)\right\} =$$

$$= \sup_{0\le t\le b}\left\{\mathscr{E}\left(\left\|\mathbf{X}^0 + \int_{[0,t]} \mathbf{h}(\mathbf{Y}_\tau,\tau)d\lambda(\tau) + \int_0^t \mathbf{G}(\mathbf{Y}_\tau,\tau)d\mathbf{B}(\tau)\right\|_F^2\right)\right\} \le$$

$$\le 3 \sup_{0\le t\le b}\left\{\mathscr{E}\left(\|\mathbf{X}^0\|_F^2 + \left\|\int_{[0,t]} \mathbf{h}(\mathbf{Y}_\tau,\tau)d\lambda(\tau)\right\|_F^2 + \left\|\int_0^t \mathbf{G}(\mathbf{Y}_\tau,\tau)d\mathbf{B}(\tau)\right\|_F^2\right)\right\} \le$$

3

$$\leq 3 \sup_{0\leq t\leq b}\left\{\mathscr{E}\left(\left\|\mathbf{X}^0\right\|_F^2 + t\int\limits_{[0,t]}\|\mathbf{h}(\mathbf{Y}_\tau,\tau)\|_F^2 d\lambda(\tau) + \underbrace{\int\limits_{[0,t]}\|\mathbf{G}(\mathbf{Y}_\tau,\tau)\|_F^2 d\lambda(\tau)}_{\text{using the Itô isometry}}\right)\right\} \leq$$

$$\leq 3 \sup_{0\leq t\leq b}\left\{\mathscr{E}\left(\left\|\mathbf{X}^0\right\|_F^2 + t\int\limits_{[0,t]}\|\mathbf{h}(\mathbf{Y}_\tau,\tau)-\mathbf{h}(\mathbf{0},\tau)+\mathbf{h}(\mathbf{0},\tau)\|_F^2 d\lambda(\tau) + \right.\right.$$

$$\left.\left. + \int\limits_{[0,t]}\|\mathbf{G}(\mathbf{Y}_\tau,\tau)-\mathbf{G}(\mathbf{0},\tau)+\mathbf{G}(\mathbf{0},\tau)\|_F^2 d\lambda(\tau)\right)\right\} \leq$$

$$\leq 3 \sup_{0\leq t\leq b}\left\{\mathscr{E}\left(\left\|\mathbf{X}^0\right\|_F^2 + 2b\int\limits_{[0,t]}\|\mathbf{h}(\mathbf{Y}_\tau,\tau)-\mathbf{h}(\mathbf{0},\tau)\|_F^2 d\lambda(\tau) + \underbrace{2b\int\limits_{[0,b]}\|\mathbf{h}(\mathbf{0},\tau)\|_F^2 d\lambda(\tau)}_{=:C_1} + \right.\right.$$

$$\left.\left. + 2\int\limits_{[0,t]}\|\mathbf{G}(\mathbf{Y}_\tau,\tau)-\mathbf{G}(\mathbf{0},\tau)\|_F^2 d\lambda(\tau) + \underbrace{2\int\limits_{[0,b]}\|\mathbf{G}(\mathbf{0},\tau)\|_F^2 d\lambda(\tau)}_{=:C_2}\right)\right\} \leq$$

$$\leq 3C_1 + 3C_2 + 3\mathscr{E}\left(\left\|\mathbf{X}^0\right\|_F^2\right) + 6L^2(b+1)\sup_{0\leq t\leq b}\left\{\mathscr{E}\left(\|\mathbf{Y}_t\|_F^2\right)\right\} < \infty.$$

In the final step, we show that the mapping T is a contraction. Choose $t \in [0, b]$.

$$\mathscr{E}\left(\|(T((\mathbf{Y}_s)_{s\in[0,b]}))_t - (T((\mathbf{Z}_s)_{s\in[0,b]}))_t\|_F^2\right) =$$

$$= \mathscr{E}\left(\left\|\int\limits_{[0,t]}(\mathbf{h}(\mathbf{Y}_\tau,\tau)-\mathbf{h}(\mathbf{Z}_\tau,\tau))\,d\lambda(\tau) + \int_0^t(\mathbf{G}(\mathbf{Y}_\tau,\tau)-\mathbf{G}(\mathbf{Z}_\tau,\tau))\,d\mathbf{B}(\tau)\right\|_F^2\right) \leq$$

$$\leq 2\mathscr{E}\left(\left\|\int\limits_{[0,t]}(\mathbf{h}(\mathbf{Y}_\tau,\tau)-\mathbf{h}(\mathbf{Z}_\tau,\tau))\,d\lambda(\tau)\right\|_F^2 + \left\|\int_0^t(\mathbf{G}(\mathbf{Y}_\tau,\tau)-\mathbf{G}(\mathbf{Z}_\tau,\tau))\,d\mathbf{B}(\tau)\right\|_F^2\right) \leq$$

$$\leq 2\mathscr{E}\left(t\int\limits_{[0,t]}\|\mathbf{h}(\mathbf{Y}_\tau,\tau)-\mathbf{h}(\mathbf{Z}_\tau,\tau)\|_F^2 d\lambda(\tau) + \int_0^t\|\mathbf{G}(\mathbf{Y}_\tau,\tau)-\mathbf{G}(\mathbf{Z}_\tau,\tau)\|_F^2 d\lambda(\tau)\right) \leq$$

$$\le 2\mathscr{E}\left(tL^2 \int\limits_{[0,t]} \|\mathbf{Y}_\tau - \mathbf{Z}_\tau\|_F^2 d\lambda(\tau) + L^2 \int\limits_0^t \|\mathbf{Y}_\tau - \mathbf{Z}_\tau\|_F^2 d\lambda(\tau) \right) =$$

$$= 2(t+1)L^2\mathscr{E}\left(\int\limits_{[0,t]} \|\mathbf{Y}_\tau - \mathbf{Z}_\tau\|_F^2 d\lambda(\tau) \right) \le$$

$$\le 2(b+1)L^2 \int\limits_{[0,t]} \mathscr{E}\left(\|\mathbf{Y}_\tau - \mathbf{Z}_\tau\|_F^2 \right) d\lambda(\tau) =$$

$$= 2(b+1)L^2 \int\limits_{[0,t]} \mathscr{E}\left(\|\mathbf{Y}_\tau - \mathbf{Z}_\tau\|_F^2 \right) e^{-4(b+1)L^2\tau} e^{4(b+1)L^2\tau} d\lambda(\tau) \le$$

$$\le 2(b+1)L^2 \int\limits_{[0,t]} \sup_{0\le\tau\le b} \left\{ \mathscr{E}\left(\|\mathbf{Y}_\tau - \mathbf{Z}_\tau\|_F^2 \right) e^{-4(b+1)L^2\tau} \right) e^{4(b+1)L^2\tau} d\lambda(\tau) =$$

$$= 2(b+1)L^2 \|(\mathbf{Y}_t)_{t\in[0,b]} - (\mathbf{Z}_t)_{t\in[0,b]}\|^2_{(2(b+1)L^2)} \int\limits_{[0,t]} e^{4(b+1)L^2\tau} d\lambda(\tau) \le$$

$$\le \frac{1}{2} \|(\mathbf{Y}_t)_{t\in[0,b]} - (\mathbf{Z}_t)_{t\in[0,b]}\|^2_{(2(b+1)L^2)} e^{4(b+1)L^2 t}.$$

Finally, we obtain by multiplication with $e^{-4(b+1)L^2 t}$ and using $K = 4(b+1)L^2$:

$$\|T((\mathbf{Y}_s)_{s\in[0,b]}) - T((\mathbf{Z}_s)_{s\in[0,b]})\|_{(K)} \le \frac{1}{\sqrt{2}} \|(\mathbf{Y}_s)_{s\in[0,b]} - (\mathbf{Z}_s)_{s\in[0,b]}\|_{(K)}.$$

Simple growth and decay models are usually given by

$$\dot{N}(t) = \lambda N(t), \quad N(0) = N_0, \quad t \in [0, \infty),$$

where λ denotes the rate of growth ($\lambda > 0$) or the rate of decay ($\lambda < 0$). If we assume that λ is not a constant but additively disturbed by a Gaussian white noise process with spectral density function $\nu > 0$, then we obtain the so-called **geometric Brownian Motion** $(X_t)_{t\in[0,\infty)}$ defined by

$$dX_t = \lambda X_t dt + \nu X_t dB_t, \quad X_0 = N_0, \quad t \in [0, \infty).$$

The solution of this stochastic differential equation is given by (see [Øks10])

$$X_t : \Omega \to \mathbb{R}, \quad \omega \mapsto N_0 \exp\left(\left(\lambda - \frac{\nu^2}{2} \right) t + \nu B_t(\omega) \right), \quad t \in [0, \infty).$$

3

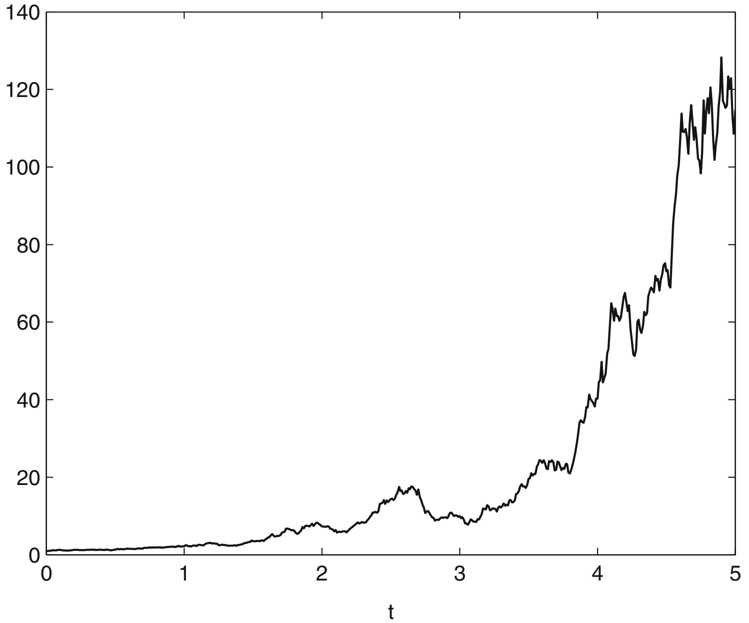

Fig. 3.9 A path of a geometric Brownian Motion, $\lambda = 1$, $\nu = \frac{1}{2}$, $N_0 = 1$

Figure 3.9 shows a path of a geometric Brownian Motion with $\lambda > 0$, while Fig. 3.10 shows a path of a geometric Brownian Motion with $\lambda < 0$.

Geometric Brownian Motions play an important role for the modelling of trends of prices for financial products (see again [Øks10]).

For a lot of applications it is not possible to solve stochastic differential equations analytically. Therefore, it is necessary to compute approximations of a solution numerically. In this book, it is not possible to deal with the numerical analysis of stochastical differential equations in detail (see the definitive book [KlPl11] for this area with more than 600 pages). Nevertheless, we are going to introduce some main ideas.

Consider a twice continuously differentiable curve

$$\mathbf{x} : [0, \infty) \to \mathbb{R}^2, \quad t \mapsto \mathbf{x}(t).$$

The curvature of this curve is defined by

$$\kappa : [0, \infty) \to \mathbb{R}, \quad t \mapsto \frac{\dot{x}_1(t)\ddot{x}_2(t) - \ddot{x}_1(t)\dot{x}_2(t)}{(\dot{x}_1(t)^2 + \dot{x}_2(t)^2)^{\frac{3}{2}}}$$

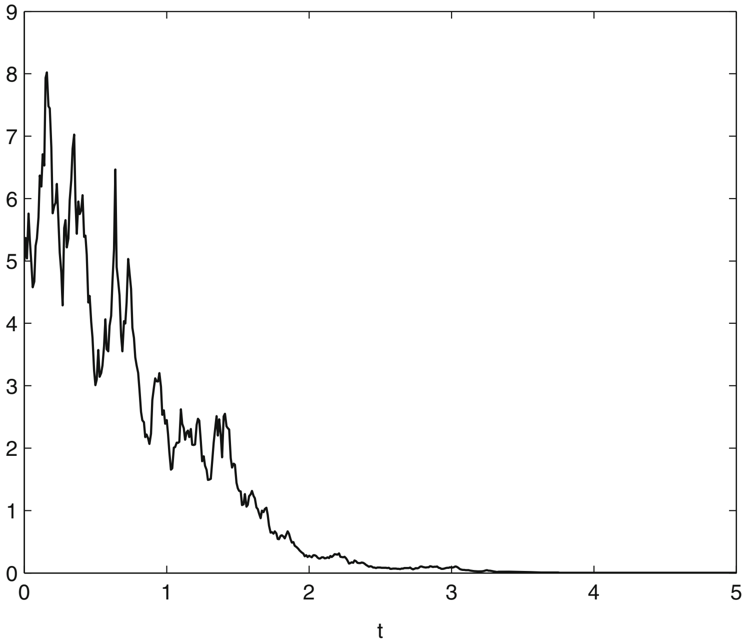

Fig. 3.10 A path of a geometric Brownian Motion, $\lambda = -1, \nu = 1, N_0 = 5$

and defines a measure for the deviation of the curve from a straight line at $t \geq 0$ (this can be done for curves in $\mathbb{R}^n$, $n \in \mathbb{N}$, also). Assume that a curve is defined by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

and assume that this initial value problem has a unique solution but is not solvable analytically. The choice of a suitable method for the numerical approximation of the solution depends mainly on the curvature of the solution as the following examples show.

A self-evident approach for the numerical approximation of the solution of an initial value problem

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

is given by the Euler method. Assume that one has computed an approximation $\mathbf{x}_{\text{app}}(\bar{t})$ of $\mathbf{x}(\bar{t})$. The Euler method with step size $h > 0$ defines

$$\mathbf{x}_{\text{app}}(\bar{t} + h) = \mathbf{x}_{\text{app}}(\bar{t}) + h\mathbf{f}(\mathbf{x}_{\text{app}}(\bar{t}), \bar{t})$$

as an approximation of $\mathbf{x}(\bar{t}+h)$. This method arises from the replacement of

$$\int_{\bar{t}}^{\bar{t}+h} \mathbf{f}(\mathbf{x}(t), t)dt \quad \text{with} \quad h\mathbf{f}(\mathbf{x}_{\text{app}}(\bar{t}), \bar{t})$$

in the integral form

$$\mathbf{x}(\bar{t}+h) = \mathbf{x}_{\text{app}}(\bar{t}) + \int_{\bar{t}}^{\bar{t}+h} \mathbf{f}(\mathbf{x}(t), t)dt$$

of the initial value problem

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t), \quad \mathbf{x}(\bar{t}) = \mathbf{x}_{\text{app}}(\bar{t}).$$

Now, we consider a first example

$$\dot{\mathbf{x}}(t) = -\begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{pmatrix} \mathbf{x}(t) \ (= -\mathbf{N}\mathbf{x}(t)), \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

with the unique solution

$$\mathbf{x} : [0, \infty) \to \mathbb{R}^2, \quad t \mapsto \begin{pmatrix} e^{-t} + e^{-2t} \\ e^{-t} - e^{-2t} \end{pmatrix}$$ (Fig. 3.11).

Figure 3.12 shows the curvature κ of this curve. We obtain

$$0 < \kappa(t) < 1.5, \quad t \in [0, \infty).$$

Using the Euler method with constant step size $h > 0$, one obtains with $\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

$$\begin{aligned} \mathbf{x}_{\text{app}}(0 + ih) &= \mathbf{x}_{\text{app}}(0 + (i-1)h) - h\mathbf{N}\mathbf{x}_{\text{app}}(0 + (i-1)h) = \\ &= (\mathbf{I}_2 - h\mathbf{N})\mathbf{x}_{\text{app}}(0 + (i-1)h) = \\ &= (\mathbf{I}_2 - h\mathbf{N})^i \begin{pmatrix} 2 \\ 0 \end{pmatrix}. \end{aligned}$$

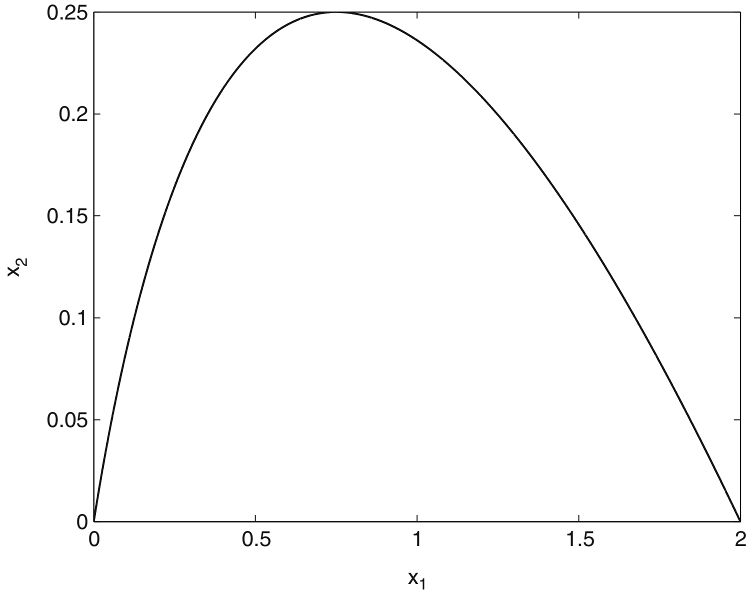

Fig. 3.11 The curve $\mathbf{x}$

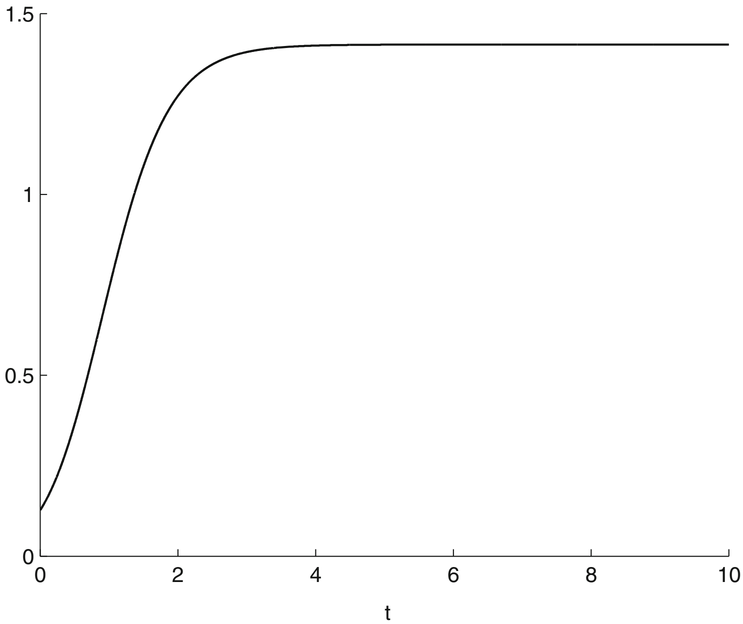

Fig. 3.12 Curvature of $\mathbf{x}$

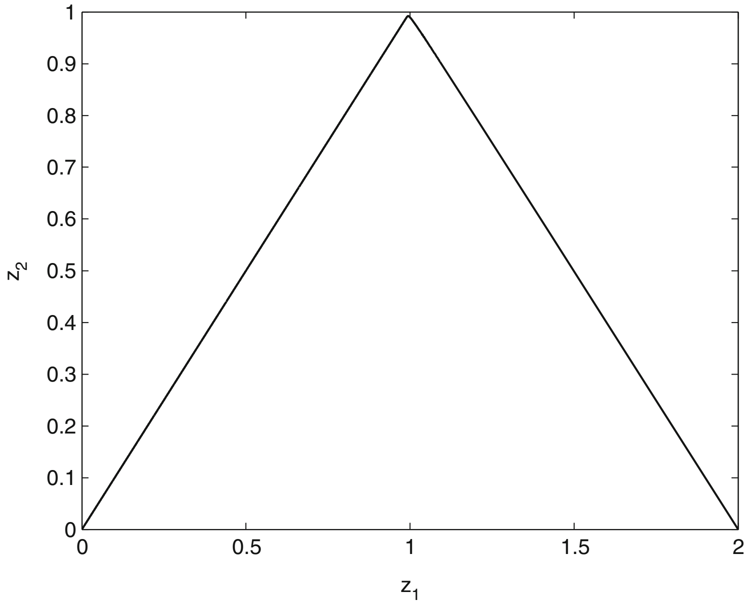

Fig. 3.13 The curve $\mathbf{z}$

While $\lim_{t\to\infty} \mathbf{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the sequence $\{\mathbf{x}_{\text{app}}(0 + ih)\}_{i\in\mathbb{N}}$ converges to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ iff

$$|1 - h\lambda_1| < 1 \quad \text{and} \quad |1 - h\lambda_2| < 1,$$

where $\lambda_1 = 1$ and $\lambda_2 = 2$ are the eigenvalues of $\mathbf{N}$. Hence, the sequence $\{\mathbf{x}_{\text{app}}(0 + ih)\}_{i\in\mathbb{N}}$ converges to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ iff $0 < h < 1$.

The second example is given by

$$\dot{\mathbf{z}}(t) = -\begin{pmatrix} 500.5 & -499.5 \\ -499.5 & 500.5 \end{pmatrix} \mathbf{z}(t) \;\; (= -\mathbf{M}\mathbf{z}(t))\,, \quad \mathbf{z}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

with the unique solution

$$\mathbf{z} : [0, \infty) \to \mathbb{R}^2, \quad t \mapsto \begin{pmatrix} e^{-t} + e^{-1000t} \\ e^{-t} - e^{-1000t} \end{pmatrix} \quad \text{(see Fig. 3.13).}$$

Using the Euler method with constant step size $h > 0$, the sequence

$$\{\mathbf{z}_{\text{app}}(0 + ih)\}_{i\in\mathbb{N}}$$

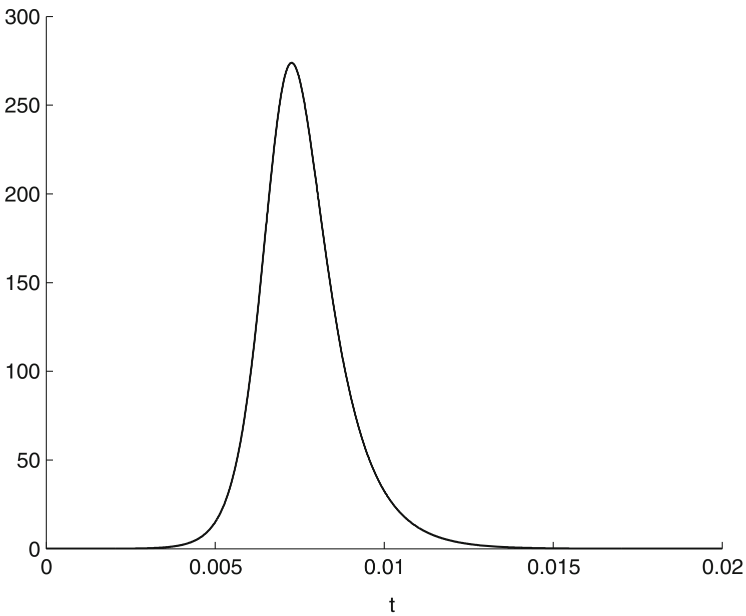

Fig. 3.14 Curvature of $\mathbf{z}$

converges to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ iff $h \leq 0.002$, since $\lambda_1 = 1$ and $\lambda_2 = 1000$ are the eigenvalues of $\mathbf{M}$. This follows from the curvature of $\mathbf{z}$ (Fig. 3.14) with maximum value $\kappa(\hat{t}) \approx 275$ at $\hat{t} \approx 0.008$.

In order to avoid too small step sizes, one replaces

$$\int_{\bar{t}}^{\bar{t}+h} \mathbf{f}(\mathbf{x}(t), t)dt \quad \text{with} \quad h\mathbf{f}(\mathbf{x}_{\text{app}}(\bar{t}+h), \bar{t}+h)$$

in the integral form

$$\mathbf{x}(\bar{t}+h) = \mathbf{x}_{\text{app}}(\bar{t}) + \int_{\bar{t}}^{\bar{t}+h} \mathbf{f}(\mathbf{x}(t), t)dt$$

of the initial value problem

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), t), \quad \mathbf{x}(\bar{t}) = \mathbf{x}_{\text{app}}(\bar{t}),$$

which leads to the implicit Euler method with step size $h > 0$:

$$\mathbf{x}_{\text{app}}(\bar{t} + h) = \mathbf{x}_{\text{app}}(\bar{t}) + h\mathbf{f}(\mathbf{x}_{\text{app}}(\bar{t} + h), \bar{t} + h)$$

as approximation of $\mathbf{x}(\bar{t} + h)$. For our second example

$$\dot{\mathbf{z}}(t) = -\begin{pmatrix} 500.5 & -499.5 \\ -499.5 & 500.5 \end{pmatrix} \mathbf{z}(t) \ \left(= -\mathbf{Mz}(t)\right), \quad \mathbf{z}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

we obtain

$$\mathbf{z}_{\text{app}}(0 + ih) = \mathbf{z}_{\text{app}}(0 + (i-1)h) - h\mathbf{M}\mathbf{z}_{\text{app}}(0 + ih)$$

or explicitly

$$\begin{aligned} \mathbf{z}_{\text{app}}(0 + ih) &= (\mathbf{I}_2 + h\mathbf{M})^{-1}\mathbf{z}_{\text{app}}(0 + (i-1)h) = \\ &= (\mathbf{I}_2 + h\mathbf{M})^{-i} \begin{pmatrix} 2 \\ 0 \end{pmatrix}. \end{aligned}$$

Now, the sequence $\{\mathbf{z}_{\text{app}}(0 + ih)\}_{i\in\mathbb{N}}$ converges to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ iff

$$|1 + 1h| > 1 \quad \text{and} \quad |1 + 1000h| > 1,$$

and hence for each $h > 0$. Unfortunately, the implicit Euler method requires to solve

$$\mathbf{x}_{\text{app}}(\bar{t} + h) = \mathbf{x}_{\text{app}}(\bar{t}) + h\mathbf{f}(\mathbf{x}_{\text{app}}(\bar{t} + h), \bar{t} + h)$$

or equivalently

$$\mathbf{x}_{\text{app}}(\bar{t} + h) - h\mathbf{f}(\mathbf{x}_{\text{app}}(\bar{t} + h), \bar{t} + h) - \mathbf{x}_{\text{app}}(\bar{t}) = \mathbf{0},$$

which is a system of nonlinear equations in general. Considering the function

$$\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{w} \mapsto \mathbf{w} - h\mathbf{f}(\mathbf{w}, \bar{t} + h) - \mathbf{x}_{\text{app}}(\bar{t}),$$

the linearization of $\mathbf{F}$ at $\mathbf{x}_{\text{app}}(\bar{t})$ is given by

$$\mathbf{LF} : \mathbb{R}^n \to \mathbb{R}^n, \quad \mathbf{w} \mapsto -h\mathbf{f}(\mathbf{x}_{\text{app}}(\bar{t}), \bar{t} + h) + (\mathbf{I}_n - h\mathbf{J}_{\mathbf{x}}(\mathbf{f})(\mathbf{x}_{\text{app}}(\bar{t}), \bar{t}+h))\,(\mathbf{z}-\mathbf{x}_{\text{app}}(\bar{t})),$$

where $\mathbf{I}_n$ denotes the n-dimensional identity matrix and $\mathbf{J}_{\mathbf{x}}(\mathbf{f})(\mathbf{x}_{\text{app}}(\bar{t}), \bar{t}+h)$ the Jacobian of f with respect to $\mathbf{x}$ at $(\mathbf{x}_{\text{app}}(\bar{t}), \bar{t}+h)$. The equation

$$\mathbf{x}_{\text{app}}(\bar{t}+h) - h\mathbf{f}(\mathbf{x}_{\text{app}}(\bar{t}+h), \bar{t}+h) - \mathbf{x}_{\text{app}}(\bar{t}) = \mathbf{0}$$

is equivalent to

$$\mathbf{F}(\mathbf{x}_{\text{app}}(\bar{t}+h)) = \mathbf{0}.$$

Replacing $\mathbf{F}$ by $\mathbf{LF}$ leads to

$$-h\mathbf{f}(\mathbf{x}_{\text{app}}(\bar{t}), \bar{t}+h) + (\mathbf{I}_n - h\mathbf{J}_{\mathbf{x}}(\mathbf{f})(\mathbf{x}_{\text{app}}(\bar{t}), \bar{t}+h))\,(\mathbf{x}_{\text{app}}(\bar{t}+h) - \mathbf{x}_{\text{app}}(\bar{t})) = \mathbf{0}$$

or equivalently

$$\mathbf{x}_{\text{app}}(\bar{t}+h) = \mathbf{x}_{\text{app}}(\bar{t}) + \left(\frac{1}{h}\mathbf{I}_n - \mathbf{J}_{\mathbf{x}}(\mathbf{f})(\mathbf{x}_{\text{app}}(\bar{t}), \bar{t}+h)\right)^{-1} \mathbf{f}(\mathbf{x}_{\text{app}}(\bar{t}), \bar{t}+h)$$

for suitable $h > 0$ (small enough such that

$$\left(\frac{1}{h}\mathbf{I}_n - \mathbf{J}_{\mathbf{x}}(\mathbf{f})(\mathbf{x}_{\text{app}}(\bar{t}), \bar{t}+h)\right)$$

is regular). This method is called semi-implicit Euler method, which guarantees useful step sizes without solving systems of nonlinear equations. Now, we come back to our stochastic differential equation (in integral form)

$$\mathbf{X}_t(\omega) = \begin{pmatrix} U_0 \\ \dot{U}_0 \end{pmatrix} + \begin{pmatrix} \int\limits_{[0,t]} (X_2)_\tau(\omega) d\lambda(\tau) \\ \int\limits_{[0,t]} \left(-\frac{1}{LC}(X_1)_\tau(\omega) - \frac{R}{L}(X_2)_\tau(\omega) + \frac{1}{LC}u(\tau)\right) d\lambda(\tau) \end{pmatrix} +$$

$$+ \begin{pmatrix} 0 \\ \sigma_U \left(\int\limits_0^t \frac{1}{LC} d(B_1)_\tau\right)(\omega) - \sigma_R \left(\int\limits_0^t \frac{(X_2)_\tau}{L} d(B_2)_\tau\right)(\omega) \end{pmatrix}, \quad t \in [0, \infty).$$

We are going to compute an approximation of a path of this equation using the implicit Euler method and we are going to compare this approximation with the solution of the initial value problem

$$LC\ddot{U}_C(t) + RC\dot{U}_C(t) + U_C(t) = U(t), \quad t \geq 0, \ U_C(0) = U_0, \ \dot{U}_C(0) = \dot{U}_0.$$

For $0 \leq t \leq 2\pi$, this comparison is documented in Fig. 3.15 with $U_c(0) = 1$, $\dot{U}_C(0) = 0$, $U = \sin$, $R = 1k\Omega$, $L = 0.1H$, $C = 1\mu F$, $\sigma_U = 0.01$, and $\sigma_R = 0.0004$. In Fig. 3.16, the time interval $0 \leq t \leq 0.008$ is chosen. Since the paths of the solution

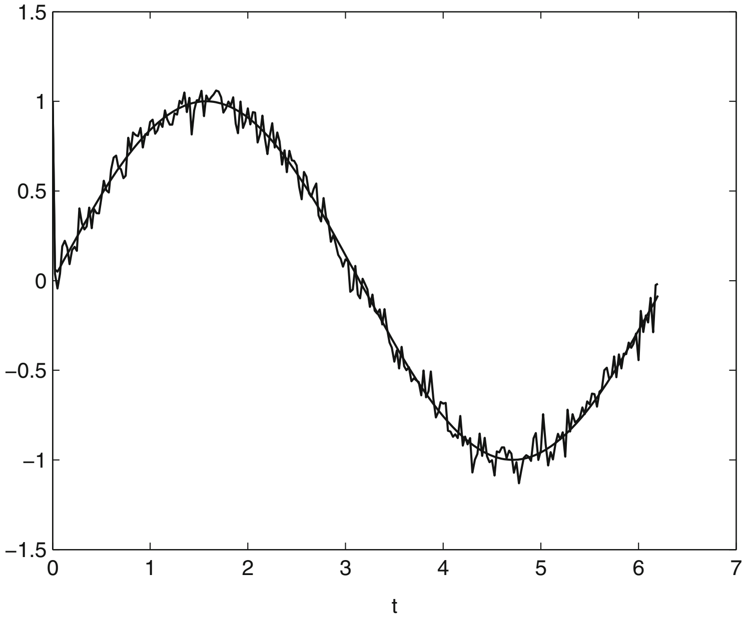

Fig. 3.15 $U_c, X_t(\omega), 0 \le t \le 2\pi$

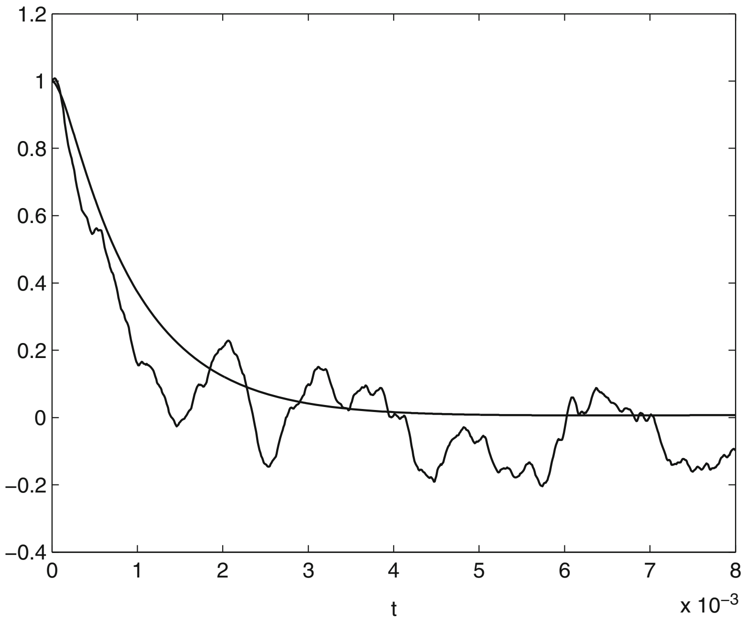

Fig. 3.16 $U_c, X_t(\omega), 0 \le t \le 0.008$

of the above stochastic differential equation are continuous, the Lebesgue integrals can be computed using Riemann integration and we are able to replace

$$\int_{[\bar{t},\bar{t}+h]} (X_2)_\tau d\lambda(\tau) \quad \text{with} \quad h(X_2)_{\bar{t}+h}(\omega),$$

and

$$\int_{[\bar{t},\bar{t}+h]} \left(-\frac{1}{LC}(X_1)_\tau(\omega) - \frac{R}{L}(X_2)_\tau(\omega) + \frac{1}{LC}u(\tau)\right) d\lambda(\tau)$$

with

$$h\left(-\frac{1}{LC}(X_1)_{\bar{t}+h}(\omega) - \frac{R}{L}(X_2)_{\bar{t}+h}(\omega) + \frac{1}{LC}u(\bar{t}+h)\right).$$

The approximation of Itô integrals has to be done very carefully. Just as a quick reminder:
Let $(Z_t)_{t\in[0,\infty)}$ be a stochastic process of the following type:
For each $T > 0$, there exists a partition

$$0 = t_0 < t_1 < \ldots < t_{k_T} = T, \quad k_T \in \mathbb{N}$$

such that

$$Z_t : \Omega \to \mathbb{R}, \quad \omega \mapsto \sum_{j=0}^{k_T-1} Z_j'(\omega) \cdot I_{[t_j,t_{j+1})}(t), \quad t \in [0,\infty),$$

where

$$I_{[t_j,t_{j+1})} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} 1 & \text{for } t \in [t_j, t_{j+1}) \\ 0 & \text{for } t \notin [t_j, t_{j+1}) \end{cases}, \quad j \in \{0, \ldots, k_T - 1\}.$$

Furthermore, assume that for each $j \in \{0, \ldots, k_T - 1\}$ the random variable

$$Z_j' : \Omega \to \mathbb{R}$$

is $\mathcal{F}_{t_j}^B$-$\mathcal{B}$-measurable and bounded, then we define

$$\int_0^T Z_t dB_t := \sum_{j=0}^{k_T-1} Z_j' \frac{B_{t_{j+1}} - B_{t_j}}{t_{j+1} - t_j} \cdot (t_{j+1} - t_j) = \sum_{j=0}^{k_T-1} Z_j' \cdot (B_{t_{j+1}} - B_{t_j}).$$

It is very important that Z_j' is assumed to be $\mathcal{F}_{t_j}^B$-$\mathcal{B}$-measurable. As a consequence, we are not able to approximate the integral

$$\left(\int_{\bar{t}}^{\bar{t}+h} \frac{(X_2)_\tau}{L} d(B_2)_\tau\right)(\omega)$$

by

$$((B_2)_{(\bar{t}+h)}(\omega) - (B_2)_{\bar{t}}(\omega))\frac{(X_2)_{\bar{t}+h}}{L}(\omega)$$

using the implicit Euler method, since the random variable

$$\frac{(X_2)_{\bar{t}+h}}{L}$$

is not $\mathcal{F}_{\bar{t}}^B$-$\mathcal{B}$-measurable. Therefore, we approximate

$$\left(\int_{\bar{t}}^{\bar{t}+h} \frac{(X_2)_\tau}{L} d(B_2)_\tau\right)(\omega)$$

by

$$((B_2)_{(\bar{t}+h)}(\omega) - (B_2)_{\bar{t}}(\omega))\frac{(X_2)_{\bar{t}}}{L}(\omega),$$

where we substitute

$$((B_2)_{(\bar{t}+h)}(\omega) - (B_2)_{\bar{t}}(\omega))$$

by a realization $r_2^{\bar{t}+h}$ of a $\mathcal{N}(0, h)$ Gaussian distributed random variable.

If one postulates that the random variable Z_j' is $\mathcal{F}^B_{\frac{t_{j+1}+t_j}{2}}$-$\mathcal{B}$-measurable in the definition of stochastic integration, then we would obtain a new stochastic integral, the so-called **Fisk-Stratonovich integral** with properties, which differing essentially from the properties of Itô integrals. The computation of the integral

$$\left(\int_{\bar{t}}^{\bar{t}+h} \frac{1}{LC} d(B_1)_\tau\right)(\omega) = \frac{((B_1)_{(\bar{t}+h)}(\omega) - (B_1)_{\bar{t}}(\omega))}{LC}$$

is given by a realization $r_1^{\bar{t}+h}$ of a $\mathcal{N}(0,h)$ Gaussian distributed random variable, which is stochastically independent from

$$(B_2)_{(\bar{t}+h)} - (B_2)_{\bar{t}}.$$

The real numbers $r_1^{\bar{t}+h}, r_2^{\bar{t}+h}$ are computed by a pseudo random generator. Since the increments of a Brownian Motion are stochastically independent, the real numbers $r_1^{\bar{t}+h}, r_2^{\bar{t}+h}$ can be computed independently of the corresponding realizations $r_1^{\bar{t}}, r_2^{\bar{t}}$ computed in the previous step. Finally, we obtain a system of linear equations with the variables $x_1 := (X_1^{\text{app}})_{\bar{t}+h}(\omega)$ and $x_2 := (X_2^{\text{app}})_{\bar{t}+h}(\omega)$:

$$\begin{aligned} x_1 &= (X_1^{\text{app}})_{\bar{t}}(\omega) + hx_2 \\ x_2 &= (X_2^{\text{app}})_{\bar{t}}(\omega) + h\left(-\frac{1}{LC}x_1 - \frac{R}{L}x_2 + \frac{1}{LC}u(\bar{t}+h)\right) + \\ &\qquad + \sigma_u\frac{r_1^{\bar{t}+h}}{LC} - \sigma_R(X_2^{\text{app}})_{\bar{t}}(\omega)\frac{r_2^{\bar{t}+h}}{L}. \end{aligned}$$

At the end of this chapter, we investigate stochastic differential equations with a Gaussian white noise process and with a Poissonian white noise process (stochastically independent from the Gaussian white noise process). For this purpose, we need a q-dimensional compensated Poisson process $(\hat{\mathbf{P}}_t)_{t\in[0,\infty)}$, which for $\nu_1, \ldots, \nu_q > 0$ is given by

$$\hat{\mathbf{P}}_t : \Omega \to \{\kappa_1 - \nu_1 t;\ \kappa_1 \in \mathbb{N}_0\} \times \ldots \times \{\kappa_q - \nu_q t;\ \kappa_q \in \mathbb{N}_0\}, \quad t \in [0,\infty),$$

where

$$\mathbb{P}_{\hat{\mathbf{P}}_t}(\{\kappa_1 - \nu_1 t, \ldots, \kappa_q - \nu_q t\}) = \prod_{j=1}^{q} \frac{1}{e^{\nu_j t}} \frac{(\nu_j t)^{\kappa_j}}{\kappa_j!} \quad \text{for} \quad t > 0,\ \kappa_1, \ldots, \kappa_q \in \mathbb{N}_0,$$

and where

$$\mathbb{P}_{\hat{\mathbf{P}}_0}(\{0, \ldots, 0\}) = 1.$$

In addition to the functions

$$\mathbf{h} : \mathbb{R}^k \times [0,b] \to \mathbb{R}^k$$

and

$$\mathbf{G} : \mathbb{R}^k \times [0,b] \to \mathbb{R}^{k,n},$$

we use a further function

$$\mathbf{H} : \mathbb{R}^k \times [0, b] \to \mathbb{R}^{k,q}$$

and investigate for $t \in [0, b]$ the stochastic differential equation

$$\mathbf{X}_t(\omega) = \mathbf{X}^0(\omega) + \begin{pmatrix} \int\limits_{[0,t]} \mathbf{h}_1(\mathbf{X}_\tau(\omega), \tau) d\lambda(\tau) \\ \vdots \\ \int\limits_{[0,t]} \mathbf{h}_k(\mathbf{X}_\tau(\omega), \tau) d\lambda(\tau) \end{pmatrix} +$$

$$+ \begin{pmatrix} \left(\int\limits_0^t \mathbf{G}_{1,1}(\mathbf{X}_\tau, \tau) d(B_1)_\tau\right)(\omega) + \ldots + \left(\int\limits_0^t \mathbf{G}_{1,n}(\mathbf{X}_\tau, \tau) d(B_n)_\tau\right)(\omega) \\ \vdots \\ \left(\int\limits_0^t \mathbf{G}_{k,1}(\mathbf{X}_\tau, \tau) d(B_1)_\tau\right)(\omega) + \ldots + \left(\int\limits_0^t \mathbf{G}_{k,n}(\mathbf{X}_\tau, \tau) d(B_n)_\tau\right)(\omega) \end{pmatrix} +$$

$$+ \begin{pmatrix} \left(\int\limits_0^t \mathbf{H}_{1,1}(\mathbf{X}_\tau, \tau) d(\hat{P}_1)_\tau\right)(\omega) + \ldots + \left(\int\limits_0^t \mathbf{H}_{1,q}(\mathbf{X}_\tau, \tau) d(\hat{P}_q)_\tau\right)(\omega) \\ \vdots \\ \left(\int\limits_0^t \mathbf{H}_{k,1}(\mathbf{X}_\tau, \tau) d(\hat{P}_1)_\tau\right)(\omega) + \ldots + \left(\int\limits_0^t \mathbf{H}_{k,q}(\mathbf{X}_\tau, \tau) d(\hat{P}_q)_\tau\right)(\omega) \end{pmatrix}.$$

This equation is denoted by

$$d\mathbf{X}_t = \mathbf{h}(\mathbf{X}_t, t)dt + \mathbf{G}(\mathbf{X}_t, t)d\mathbf{B}_t + \mathbf{H}(\mathbf{X}_t, t)d\hat{\mathbf{P}}_t, \quad \mathbf{X}_0 = \mathbf{X}^0, \quad t \in [0, b].$$

Existence and uniqueness of the solution of this stochastic differential equation can be proven analogously to the proof above, if $\mathbf{h}$, $\mathbf{G}$, and $\mathbf{H}$ are Lipschitz continuous with common Lipschitz constant. But now, we have to deal with the set $\mathfrak{Y}$ of all stochastic processes $(\mathbf{Y}_t)_{t\in[0,b]}$ given by

$$\mathbf{Y}_t : \Omega \to \mathbb{R}^k, \quad t \in [0, b]$$

with the following properties

(i)

$$\mathbf{Y} : [0, b] \times \Omega \to \mathbb{R}^k, \quad (t, \omega) \mapsto \mathbf{Y}_t(\omega)$$

is $(\mathcal{B}_{[0,b]} \times \mathcal{S})$-$\mathcal{B}^k$-measurable,

(ii) all paths of $(\mathbf{Y}_t)_{t\in[0,b]}$ are continuous from the right with existing limit from the left for each $t \in [0, b]$,

(iii) $\sup\limits_{0\le t\le b} \left\{\mathscr{E}\left(\|\mathbf{Y}_t\|_F^2\right)\right\} < \infty.$

Problems and Solutions

Problems

1. For $t \in [0, \infty)$, prove that

$$\int_0^t B_\tau dB_\tau = \frac{1}{2}\left(B_t^2 - t\right) \quad (\mathbb{P}\text{-})\text{almost surely.}$$

2. Prove that

$$\int_0^T t dB_t = TB_T - \int_0^T B_t dt$$

holds ($\mathbb{P}$-)almost surely.

(Hint: $t_{j+1}B_{t_{j+1}} - t_j B_{t_j} = t_j(B_{t_{j+1}} - B_{t_j}) + B_{t_{j+1}}(t_{j+1} - t_j)$)

3. Let $(V, \|\bullet\|)$ be a seminormed space. Show that

$$\|x+y+z\|^2 \le 3\left(\|x\|^2 + \|y\|^2 + \|z\|^2\right) \quad \text{for all} \quad x, y, z \in V,$$

and

$$\|x+y\|^2 \le 2\left(\|x\|^2 + \|y\|^2\right) \quad \text{for all} \quad x, y \in V.$$

4. Let $(\hat{P}_t)_{t\in[0,\infty)}$ be a compensated Poisson process with

$$\mathscr{V}(\hat{P}_t) = \nu t, \quad \nu > 0, \quad t \in [0, \infty),$$

and let $\left(\mathcal{F}_t^B\right)_{t\in[0,\infty)}$ be its natural filtration. Furthermore, let $(Z_t)_{t\in[0,\infty)}$ be a stochastic process with the property:

There exists a partition

$$0 = t_0 < t_1 < \ldots < t_{k_T} = T, \quad k_T \in \mathbb{N},$$

with

$$Z_t : \Omega \to \mathbb{R}, \quad \omega \mapsto \sum_{j=0}^{k_T-1} Z_j'(\omega) \cdot I_{[t_j,t_{j+1})}(t), \quad t \in [0, \infty),$$

where

$$I_{[t_j,t_{j+1})} : \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} 1 & \text{for } t \in [t_j, t_{j+1}) \\ 0 & \text{for } t \notin [t_j, t_{j+1}) \end{cases}, \quad j \in \{0, \ldots, k_T - 1\}.$$

In addition, we assume that the random variable

$$Z'_j : \Omega \to \mathbb{R}$$

is $\mathcal{F}^B_{t_j}$-$\mathcal{B}$-measurable and bounded for each $j \in \{0, \ldots, k_T - 1\}$.

Using

$$\int_0^T Z_t d\hat{P}_t := \sum_{j=0}^{k_T-1} Z'_j \cdot (\hat{P}_{t_{j+1}} - \hat{P}_{t_j}),$$

prove that

$$\mathscr{E}\left(\left(\int_0^T Z_t d\hat{P}_t\right)^2\right) = \nu \mathscr{E}\left(\int_0^T Z_t^2 dt\right),$$

where

$$\int_0^T Z_t^2 dt : \Omega \to \mathbb{R}, \quad \omega \mapsto \int_0^T (Z_t(\omega))^2 dt.$$

5. Prove the existence and uniqueness of a solution of the stochastic differential equation

$$dX_t = \ln\left(1 + X_t^2\right) dt + \sin(X_t) dB_t, \quad X_0 = X^0 \in \mathbb{R}, \quad t \in [0, \infty).$$

Solutions

1. Choose

$$\begin{aligned} X_0 : \Omega &\to \mathbb{R}, & \omega &\mapsto 0, \\ U : [0,\infty) \times \Omega &\to \mathbb{R}, & (t,\omega) &\mapsto 0, \\ V : [0,\infty) \times \Omega &\to \mathbb{R}, & (t,\omega) &\mapsto 1, \\ g : [0,T] \times \mathbb{R} &\to \mathbb{R}, & (x_1, x_2) &\mapsto \frac{x_2^2 - x_1}{2}, \end{aligned}$$

then the proof is given by Itô's lemma.

2. Choose $\{t_0^i, \ldots, t_{k_i}^i\}_{i\in\mathbb{N}}$ such that

$$0 = t_0^i < t_1^i < \ldots < t_{k_i}^i = T, \quad i \in \mathbb{N},\ k_i \in \mathbb{N},$$

and

$$\lim_{i\to\infty} \max\{t_j^i - t_{j-1}^i;\, j = 1, \ldots, k_i\} = 0.$$

We obtain:

$$TB_T = \sum_{j=0}^{k_i-1}(t_{j+1}^i B_{t_{j+1}^i} - t_j^i B_{t_j^i}) = \sum_{j=0}^{k_i-1} t_j^i (B_{t_{j+1}^i} - B_{t_j^i}) + \sum_{j=0}^{k_i-1} B_{t_{j+1}^i}(t_{j+1}^i - t_j^i).$$

For $i \to \infty$ we get convergence in the mean square

$$\sum_{j=0}^{k_i-1} t_j^i (B_{t_{j+1}^i} - B_{t_j^i}) \quad \longrightarrow \quad \int_0^T t dB_t.$$

For $i \to \infty$ we get pathwise convergence

$$\sum_{j=0}^{k_i-1} B_{t_{j+1}^i}(\omega)(t_{j+1}^i - t_j^i) \quad \longrightarrow \quad \int_0^T B_t(\omega) dt.$$

3. Let $x, y, z \in V$:

$$\begin{aligned}
(\|x\| - \|y\|)^2 + (\|x\| - \|z\|)^2 + (\|y\| - \|z\|)^2 \;&\geq\; 0 \\
&\Longrightarrow \\
2\|x\|\,\|y\| + 2\|x\|\,\|z\| + 2\|y\|\,\|z\| \;&\leq\; 2\|x\|^2 + 2\|y\|^2 + 2\|z\|^2 \\
&\Longrightarrow \\
\|x\|^2 + \|y\|^2 + \|z\|^2 + 2\|x\|\,\|y\| + 2\|x\|\,\|z\| + 2\|y\|\,\|z\| \;&\leq\; 3\|x\|^2 + 3\|y\|^2 + 3\|z\|^2 \\
&\Longrightarrow \\
(\|x\| + \|y\| + \|z\|)^2 \;&\leq\; 3\left(\|x\|^2 + \|y\|^2 + \|z\|^2\right) \\
&\Longrightarrow \\
(\|x + y + z\|)^2 \;&\leq\; 3\left(\|x\|^2 + \|y\|^2 + \|z\|^2\right).
\end{aligned}$$

3

Let $x, y \in V$:

$$\begin{array}{rcl} 0 & \leq & (\|x\| - \|y\|)^2 \\ & \Longrightarrow & \\ 0 & \leq & \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \\ & \Longrightarrow & \\ \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| & \leq & 2\|x\|^2 + 2\|y\|^2 \\ & \Longrightarrow & \\ (\|x\| + \|y\|)^2 & \leq & 2\left(\|x\|^2 + \|y\|^2\right) \\ & \Longrightarrow & \\ (\|x+y\|)^2 & \leq & 2\left(\|x\|^2 + \|y\|^2\right). \end{array}$$

4.

$$\mathscr{E}\left(\left(\int_0^T Z_t d\hat{P}_t\right)^2\right) = \mathscr{E}\left(\left(\sum_{j=0}^{k_T-1} Z_j' \cdot (\hat{P}_{t_{j+1}} - \hat{P}_{t_j})\right)^2\right) =$$

$$= \sum_{\substack{i,j=0 \\ i\neq j}}^{k_T-1} \mathscr{E}\left(Z_i' \cdot (\hat{P}_{t_{i+1}} - \hat{P}_{t_i}) \cdot Z_j' \cdot (\hat{P}_{t_{j+1}} - \hat{P}_{t_j})\right) +$$

$$+ \mathscr{E}\left(\sum_{j=0}^{k_T-1} \left(Z_j' \cdot (\hat{P}_{t_{j+1}} - \hat{P}_{t_j})\right)^2\right) =$$

$$= \sum_{j=0}^{k_T-1} \mathscr{E}\left(\left(Z_j' \cdot (\hat{P}_{t_{j+1}} - \hat{P}_{t_j})\right)^2\right) = \sum_{j=0}^{k_T-1} \mathscr{E}\left((Z_j')^2\right) \cdot \nu(t_{j+1} - t_j) =$$

$$= \mathscr{E}\left(\sum_{j=0}^{k_T-1} (Z_j')^2 \cdot \nu(t_{j+1} - t_j)\right) = \nu\mathscr{E}\left(\int_0^T Z_t^2 dt\right).$$

We used the fact that Z_j' is $\mathcal{F}_{t_j}^{\hat{P}}$-$\mathcal{B}$-measurable and hence, that Z_j' and $(\hat{P}_{t_{j+1}} - \hat{P}_{t_j})$ are stochastically independent.

5. We prove the Lipschitz continuity of

$$h : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \ln\left(1 + x^2\right)$$

and

$$g : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sin(x)$$

with a common Lipschitz constant. Choosing $x, y \in \mathbb{R}$, from Taylor expansion one gets

$$h(x) = h(y) + h'(\xi)(x - y) \quad \text{with} \quad \xi \in \begin{cases} [x, y] & \text{for } x \leq y \\ [y, x] & \text{for } x > y \end{cases}$$

and

$$g(x) = g(y) + g'(\eta)(x - y) \quad \text{with} \quad \eta \in \begin{cases} [x, y] & \text{for } x \leq y \\ [y, x] & \text{for } x > y \end{cases}.$$

It follows

$$|h(x) - h(y)| + |g(x) - g(y)| = \left(\left| \frac{2\xi}{1 + \xi^2} \right| + |\cos(\eta)| \right) |x - y| \leq 2|x - y|,$$

since

$$\left| \frac{2\xi}{1 + \xi^2} \right| \leq 1 \quad \text{and} \quad |\cos(\eta)| \leq 1 \quad \text{for all} \quad \xi, \eta \in \mathbb{R}.$$

Generalized Random Fields

Stefan Schäffler

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4.1 Basics

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let $(H, \mathcal{H}, \mu)$ be a measure space. With

$$\mathcal{H}_\mu := \{M \in \mathcal{H};\ \mu(M) < \infty\}$$

we call a stochastic process $(X_M)_{M\in\mathcal{H}_\mu}$ defined by random variables

$$X_M : \Omega \to \mathbb{R}$$

a **set indicated random field**. The stochastic process $(X_M)_{M\in\mathcal{H}_\mu}$ is called a **μ-noise**, if the following conditions are fulfilled:

(i) $\mathscr{E}(X_M) = 0 \quad$ for all $M \in \mathcal{H}_\mu$,

(ii) $\mathscr{E}\left((X_M)^2\right) = \mu(M)\ (= \mathscr{V}(X_M)) \quad$ for all $M \in \mathcal{H}_\mu$,

(iii) if $M_1, M_2 \in \mathcal{H}_\mu$ and $M_1 \cap M_2 = \emptyset$, then it follows:

$$X_{M_1\cup M_2} = X_{M_1} + X_{M_2} \quad (\mathbb{P}\text{-})\text{almost surely},$$

(iv) if $M_1, M_2 \in \mathcal{H}_\mu$ and $M_1 \cap M_2 = \emptyset$, then it follows:

$$\mathscr{E}(X_{M_1} \cdot X_{M_2}) = 0\ (= \mathscr{C}(X_{M_1}, X_{M_2})).$$

Moreover, if we assume that the random variables X_M are jointly Gaussian distributed, then $(X_M)_{M\in\mathcal{H}_\mu}$ is called a **Gaussian μ-noise**. A Gaussian μ-noise exists, if we choose

$$\mathscr{C}(X_{M_1}, X_{M_2}) = \mu(M_1 \cap M_2) \quad \text{for all} \quad M_1, M_2 \in \mathcal{H}_\mu,$$

for instance (see [AdTay07]).

Now, we choose $H = \mathbb{R}$, $\mathcal{H} = \mathcal{B}$, and $\mu = \lambda$ (Lebesgue-Borel measure). With the covariances

$$\mathscr{C}(X_{M_1}, X_{M_2}) := \lambda(M_1 \cap M_2) \quad \text{for all} \quad M_1, M_2 \in \mathcal{B}_\lambda,$$

we obtain a Gaussian λ-noise $(X_M)_{M\in\mathcal{B}_\lambda}$. Fixing

$$B_t := X_{[0,t]} \quad \text{for all} \quad t \in [0, \infty)$$

leads to a Brownian Motion $(B_t)_{t\in[0,\infty)}$. Condition (iv) assures the stochastical independence of the increments of $(B_t)_{t\in[0,\infty)}$, for instance.

With $n \in \mathbb{N}$, $H = \mathbb{R}^n$, $\mathcal{H} = \mathcal{B}^n$, $\mu = \lambda^n$ (n-dim. Lebesgue-Borel measure), and with

$$\mathscr{C}(X_{M_1}, X_{M_2}) := \lambda^n(M_1 \cap M_2) \quad \text{for all} \quad M_1, M_2 \in \mathcal{B}^n_{\lambda^n},$$

we obtain a Gaussian λ^n-noise $(X_M)_{M\in\mathcal{B}^n_{\lambda^n}}$. Using

$$[\mathbf{s}, \mathbf{t}] := [s_1, t_1] \times \ldots \times [s_n, t_n] \quad \text{for all} \quad s_i \leq t_i,\ i \in \{1, \ldots, n\},$$

(analogously for half-open and open intervals), we consider the random variables

$$B_{\mathbf{t}} := X_{[\mathbf{0},\mathbf{t}]}, \quad \mathbf{t} \in [0, \infty)^n.$$

The stochstic process $(B_{\mathbf{t}})_{\mathbf{t}\in[0,\infty)^n}$ is called an **n-dimensional Brownian sheet**. It follows:

$$\mathscr{C}(B_{\mathbf{s}}, B_{\mathbf{t}}) = \prod_{i=1}^{n} \min(s_i, t_i) \quad \text{for all} \quad \mathbf{s}, \mathbf{t} \in [0, \infty)^n.$$

The increments of an n-dimensional Brownian sheet are stochastically independent. Hence, for

$$0 \leq t_i^1 < t_i^2 < \ldots < t_i^k, \quad k \in \mathbb{N}, \quad i = 1, \ldots, n,$$

we obtain that

$$B_{\mathbf{t}^k} - B_{\mathbf{t}^{k-1}}, \ldots, B_{\mathbf{t}^2} - B_{\mathbf{t}^1}$$

are stochastically independent (see Problem 2.). Furthermore, we are able to choose a modification of $(B_{\mathbf{t}})_{\mathbf{t}\in[0,\infty)^n}$ such that all paths are continuous (see [AdTay07]).

A stochastic process with an index set $I \subseteq \mathbb{R}^n$, $n > 1$, is called a **random field**. Each discrete white noise process is a μ-noise (using an appropriate measure space $(H, \mathcal{H}, \mu)$) (see Problem 1.). Furthermore, each Lévy martingale $(L_t)_{t\in[0,\infty)}$ with $\mathscr{V}(L_1) < \infty$ is a $(\mathscr{V}(L_1) \cdot \lambda)$-noise, where $H = \mathbb{R}$, $\mathcal{H} = \mathcal{B}$, and $\mu = \mathscr{V}(L_1) \cdot \lambda$. A proof relies on the representation of Lévy martingales by characteristic functions and is beyond the scope of this book.

The most important application of random fields deals with image processing (see [Wink95] and [BreLor11], for instance). In this book, we are going to introduce image proceeding by an example. Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let $B \in \mathcal{P}\left(\mathbb{N}^2\right)$ be a nonempty finite subset of $\mathbb{N}^2$. Each element $b \in B$ represents a pixel of an image. The color of such a pixel is given by the color mixing of the three primary colors Red, Green, and Blue (RGB colors); each primary color is assigned an intensity coded by a natural number z with $0 \leq z \leq 255$. Therefore, each pixel is described by a triple

$$(z_R, z_G, z_B) \in \{0, 1, 2, \ldots, 255\}^3.$$

If $z_R = 0$, then the color of the pixel is mixed only with the primary colors Green and Blue, for instance. The value $z_R = 255$ indicates that the share of the primary color Red is maximal for this pixel. The triple $(255, 255, 255)$ represents a white pixel and the triple $(0, 0, 0)$ represents a black pixel. Since any shade of the color Grey is represented by $z_R = z_G = z_B$, each pixel of a black-and-white image is represented by a single natural number $0 \leq z \leq 255$. Hence, we are able to depict a pixel in $2^{24} \approx 16,78 \cdot 10^6$ different colors using one byte for each primary color. A color image is given by a mapping

$$f : B \to \{0, 1, \ldots, 255\}^3,$$

whereas a black-and-white image is given by a mapping

$$g : B \to \{0, 1, \ldots, 255\}.$$

Figure 4.1 is described by a mapping

$$g : \{1, 2, \ldots, 1423\} \times \{1, 2, \ldots, 1044\} \to \{0, 1, \ldots, 255\},$$

for instance.

A histogram of different shades of Grey in Fig. 4.1 is shown in Fig. 4.2. The transmission of bits using a noisy channel can cause errors as we discussed in ▶ Sect. 2.4. Figure 4.3 shows Diogenes the Cynic, where bits are flipped at random, which decreases or reduces the shade of Grey of a pixel. A histogram of different shades of Grey in Fig. 4.3 is shown in Fig. 4.4. Hence, the shades of Grey in Fig. 4.3 are given by a realization of a random field $(\hat{G}_{(p,q)})_{(p,q)\in B}$ with

$$\hat{G}_{(p,q)} : \Omega \to \{0, 1, \ldots, 255\}.$$

Fig. 4.1 Jean-Léon Jérome: Diogenes the Cynic. Oil painting. By courtesy of: The Walters Art Museum, Baltimore

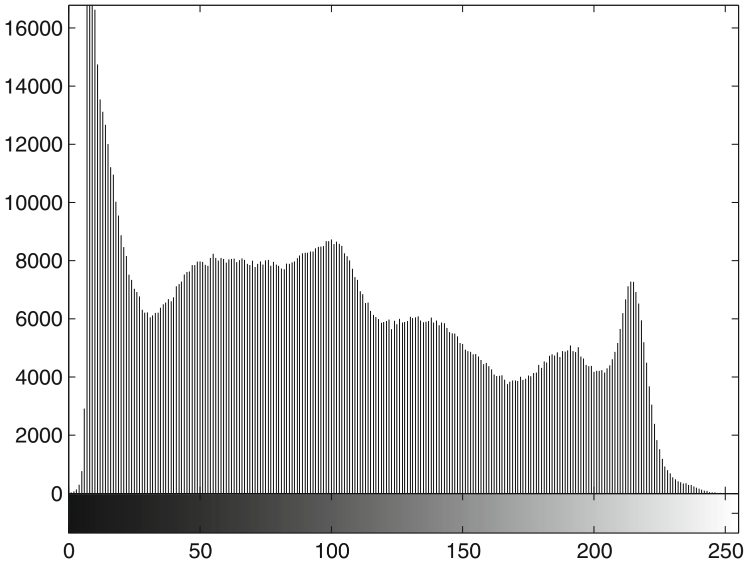

Fig. 4.2 Histogram of different shades of Grey in Fig. 4.1

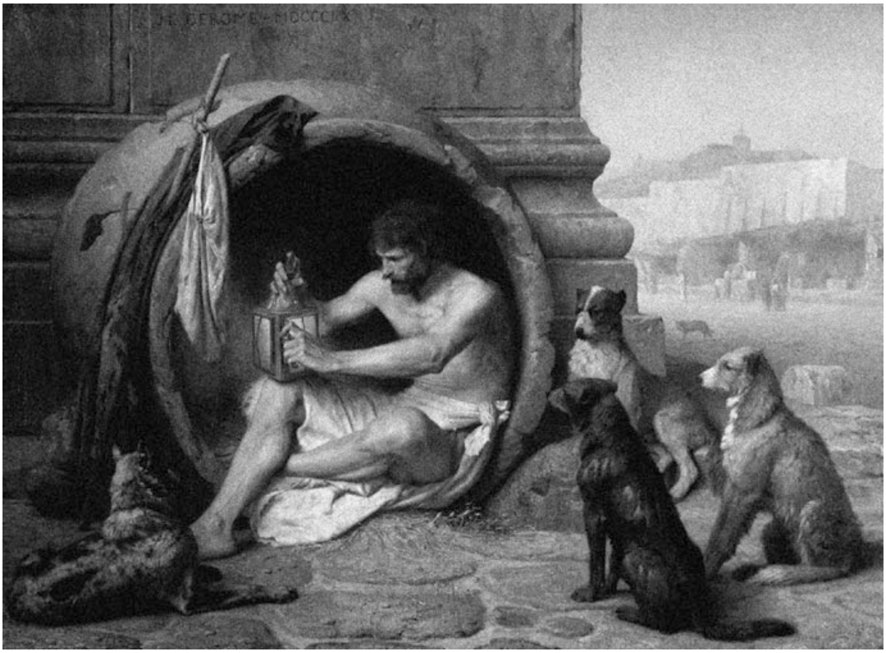

Fig. 4.3 Noisy Diogenes the Cynic. Jean-Léon Jérome: Diogenes the Cynic. Oil painting. By courtesy of: The Walters Art Museum, Baltimore

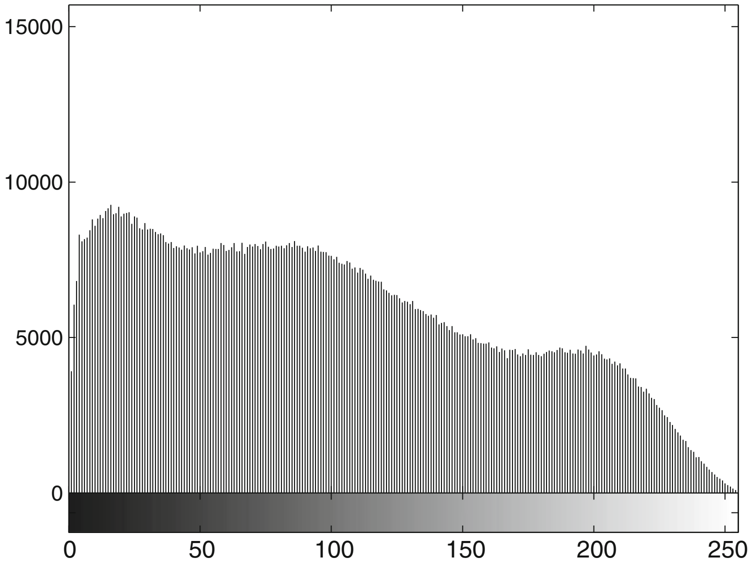

Fig. 4.4 Histogram of different shades of Grey in Fig. 4.3

■ **Fig. 4.5** Soft-focused Diogenes the Cynic. Jean-Léon Jérome: Diogenes the Cynic. Oil painting. By courtesy of: The Walters Art Museum, Baltimore

In order to reduce noise effects, it is common to apply a **scrim diffusor** to the image. In this case, the shade of Grey of a pixel (p, q) is computed by the average of shades of Grey of neighbouring pixel, where the weighting of a pixel decreases exponentially with the distance to the pixel (p, q). This is done mathematically by a convolution, i.e. the integration of a random field $(\hat{G}_{(p,q)})_{(p,q)\in B}$. ■ Figures 4.5 and 4.6 show the results of such a scrim diffusor. The noise effects are reduced, but also the contrast of the image.

In mathematical image processing, images are modelled very often using subsets of $\mathbb{R}^2$ (areas, for instance). The function values between known pixels are computed by interpolation. It is important to know that a lot of methods for editing an image (like soft-focus effect and edge detection) are based on convolutions and therefore on integration. Hence, we have to generalize the Wiener integration theory.

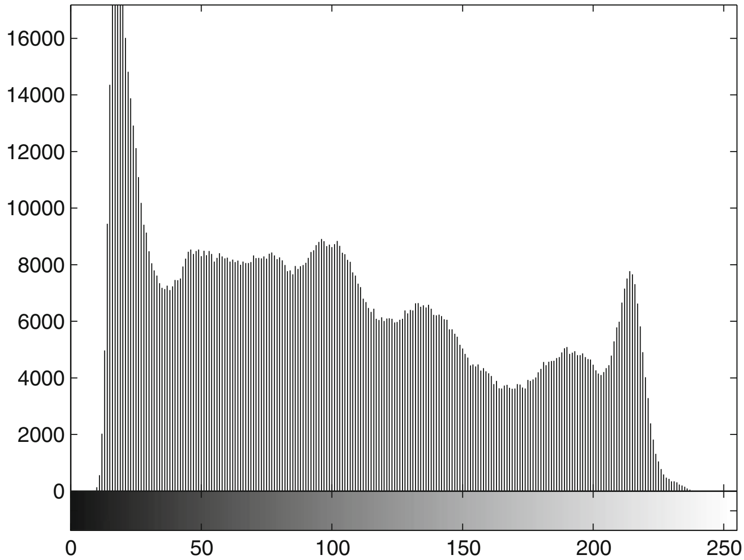

Fig. 4.6 Histogram of different shades of Grey in Fig. 4.5

4.2 Wiener Integration Using a μ-Noise

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space, $(H, \mathcal{H}, \mu)$ be a measure space and let $(X_M)_{M \in \mathcal{H}_\mu}$ be a μ-noise. Using $A \in \mathcal{H}_\mu$, a $\mathcal{H}$-$\mathcal{B}$-measurable function $e : A \to \mathbb{R}$ is called a **simple function on A**, if f takes only a finite number of different function values. An important class of simple functions on A is given by **indicator functions on A**

$$I_B^A : A \to \mathbb{R}, \quad a \mapsto \begin{cases} 1 \text{ if } a \in B \\ 0 \text{ elsewhere} \end{cases},$$

which indicate, whether a is an element of B or not. Simple functions on A can be represented by indicator functions on A:

Let $e : A \to \mathbb{R}$ be a simple function on $A \in \mathcal{H}_\mu$, then there exists a natural number n, pairwise disjoint sets $A_1, \ldots, A_n \in \mathcal{H}_\mu$, and real numbers $\alpha_1, \ldots, \alpha_n$ such that

$$e = \sum_{i=1}^{n} \alpha_i I_{A_i}^A, \quad \bigcup_{i=1}^{n} A_i = A.$$

The representation of a simple function e by indicator functions is called a **standard form** of e. If all real numbers α_i are pairwise different and if $A_i \neq \emptyset$, $1 \leq i \leq n$, then the standard form is called a **shortest standard form**. A shortest standard form of a simple function on A is unique. Given a simple function

$$e = \sum_{i=1}^{n} \alpha_i I_{A_i}^{A}, \quad \bigcup_{i=1}^{n} A_i = A,$$

we are going to define the Wiener integral

$$\int_A e dX_M := \sum_{i=1}^{n} \alpha_i X_{A_i}.$$

This integral is independent of the standard form used for e (see Problem 3.). We obtain:

(i) $\mathscr{E}\left(\int_A e\, dX_M\right) = 0,$

(ii) $\mathscr{V}\left(\int_A e\, dX_M\right) = \sum_{i=1}^{n} \alpha_i^2 \mu(A_i) = \int_A e^2 d\mu < \infty.$

Let

$$f : A \to \mathbb{R}$$

be a bounded, $\mathcal{H}$-$\mathcal{B}$-measurable function, then the sequence $\{e_n\}_{n\in\mathbb{N}}$ of simple functions on A given by

$$\begin{aligned} e_n : A \to \mathbb{R}, \quad x \mapsto & \sum_{k=1}^{n2^n} \frac{k-1}{2^n} I^A_{\{a\in A;\, \frac{k-1}{2^n} \le f(a) < \frac{k}{2^n}\}}(x) + \\ & + n I^A_{\{a\in A; f(a)\ge n\}}(x) + \\ & + \sum_{k=1}^{n2^n} -\frac{k-1}{2^n} I^A_{\{a\in A;\, -\frac{k-1}{2^n} \ge f(a) > -\frac{k}{2^n}\}}(x) + \\ & + (-n) I_{\{a\in A; f(a)\le -n\}}(x). \end{aligned}$$

converges pointwise to f. Hence, the sequence

$$\left\{ \int_A e_n\, dX_M \right\}_{n\in\mathbb{N}}$$

is a Cauchy sequence with elements in the complete seminormed space $\mathscr{L}_2((\Omega, \mathcal{S}, \mathbb{P}), \mathbb{R})$. Hence,

$$\left\{ \int_A e_n\, dX_M \right\}_{n\in\mathbb{N}}$$

converges in the mean square. Let Y and Z be two limits of this Cauchy sequence, then we obtain

$$\|Y - Z\|^2_{\mathscr{L}_2} = \int (Y - Z)^2 d\mathbb{P} = 0,$$

and hence,

$$Y = Z \quad (\mathbb{P}\text{-})\text{almost surely.}$$

Defining

$$\int_A f\, dX_M := Y \quad \textbf{(Wiener integral)},$$

we get

(i) $\mathscr{E}\left(\int_A f\, dX_M\right) = 0,$

(ii) $\mathscr{V}\left(\int_A f\, dX_M\right) = \int_A f^2 d\mu < \infty.$

Now, it is necessary to show that $\int_A f\, dX_M$ is well defined, which we leave to the reader (see the prove that Itô integration is well defined).

Using a Gaussian λ-noise $(X_M)_{M\in\mathcal{B}_\lambda}$ with

$$\mathscr{C}(X_{M_1}, X_{M_2}) := \lambda(M_1 \cap M_2) \quad \text{for all} \quad M_1, M_2 \in \mathcal{B}_\lambda,$$

and using $A = [0, T]$, $T \in (0, \infty)$, then we obtain Wiener integration defined in ► Sect. 2.4 (see Problem 4.).

In ► Sect. 2.4, we have defined Wiener integration for step functions

$$f_i : [0, T] \to \mathbb{R}, \quad t \mapsto \sum_{j=0}^{k_i-1} f(t_j^i) I_{[t_j^i, t_{j+1}^i)}(t), \quad i \in \mathbb{N},$$

in the following way:

$$\left(\int_0^T f_i(t) dB_t\right)(\omega) := \int_0^T r_i(t, \omega) dt =$$

$$= \sum_{j=0}^{k_i-1} f(t_j^i) \frac{B_{t_{j+1}^i}(\omega) - B_{t_j^i}(\omega)}{t_{j+1}^i - t_j^i} (t_{j+1}^i - t_j^i) =$$

$$= \sum_{j=0}^{k_i-1} f(t_j^i) \left(B_{t_{j+1}^i}(\omega) - B_{t_j^i}(\omega)\right), \quad \omega \in \Omega,$$

where we have interpreted the increments

$$\frac{B_{t_1^i} - B_{t_0^i}}{t_1^i - t_0^i}, \dots, \frac{B_{t_{j+1}^i} - B_{t_j^i}}{t_{j+1}^i - t_j^i}$$

as an approximation of a Gaussian white noise process. Now, we consider an n-dimensional Brownian sheet $(B_{\mathbf{t}})_{\mathbf{t}\in[0,\infty)^n}$, a set of pairwise disjoint intervals

$$\left(\mathbf{s}^1, \mathbf{t}^1\right], \dots, \left(\mathbf{s}^k, \mathbf{t}^k\right] \subset \mathbb{R}^n, \quad k \in \mathbb{N},$$

and a function

$$f : \underbrace{\bigcup_{i=1}^{k} \left(\mathbf{s}^i, \mathbf{t}^i\right]}_{=:A} \to \mathbb{R}, \quad \mathbf{x} \mapsto \sum_{i=1}^{k} f_i I^A_{(\mathbf{s}^i, \mathbf{t}^i]}(\mathbf{x}), \quad f_1, \dots, f_k \in \mathbb{R}.$$

Wiener integration gives

$$\int_A f dB_{\mathbf{t}} := \sum_{i=1}^{k} f_i (B_{\mathbf{t}^i} - B_{\mathbf{s}^i}) = \sum_{i=1}^{k} f_i \frac{B_{\mathbf{t}^i} - B_{\mathbf{s}^i}}{\prod\limits_{m=1}^{n} \left(\mathbf{t}^i_m - \mathbf{s}^i_m\right)} \prod_{m=1}^{n} \left(\mathbf{t}^i_m - \mathbf{s}^i_m\right) =$$

$$= \sum_{i=1}^{k} \int_{\mathbf{s}^i_1}^{\mathbf{t}^i_1} \dots \int_{\mathbf{s}^i_n}^{\mathbf{t}^i_n} f_i \frac{B_{\mathbf{t}^i}(\omega) - B_{\mathbf{s}^i}(\omega)}{\prod\limits_{m=1}^{n} \left(\mathbf{t}^i_m - \mathbf{s}^i_m\right)} d\mathbf{x}.$$

As done in ▶ Sect. 2.4, we interpret the increments

$$\frac{B_{\mathbf{t}^1} - B_{\mathbf{s}^1}}{\prod\limits_{m=1}^{n} \left(\mathbf{t}^1_m - \mathbf{s}^1_m\right)}, \dots, \frac{B_{\mathbf{t}^k} - B_{\mathbf{s}^k}}{\prod\limits_{m=1}^{n} \left(\mathbf{t}^k_m - \mathbf{s}^k_m\right)}$$

as an approximation of a noise process, which we investigate in the following section.

4.3 Gaussian White Noise Process Defined on $[0, \infty)^n$

Let $n \in \mathbb{N}$, $n > 1$, let $(B_{\mathbf{t}})_{\mathbf{t}\in[0,\infty)^n}$ be a Brownian sheet and let

$$B_\varphi : \Omega \to \mathbb{R}, \quad \omega \mapsto \int_0^\infty \dots \int_0^\infty B_{\mathbf{t}}(\omega)\varphi(\mathbf{t}) d\mathbf{t} \quad \text{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R})$$

be the corresponding generalized stochastic process, then the covariance functional is given by

$$\mathscr{C}(\varphi, \psi) = \int\limits_{[0,\infty)^n} \int\limits_{[0,\infty)^n} \prod_{i=1}^{n} \min(t_i, s_i)\varphi(\mathbf{t})\psi(\mathbf{s})d\mathbf{t}d\mathbf{s} \quad \text{for all} \quad \varphi, \psi \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}).$$

Investigating the stochastic process $\left(\frac{\partial B_\varphi}{\partial t_1}\right)_{\varphi \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R})}$, we obtain analogously to the one-dimensional case:

$$\frac{\partial B_\varphi}{\partial t_1} = -\int\limits_{[0,\infty)^n} B_{\mathbf{t}} \frac{\partial \varphi}{\partial t_1}(\mathbf{t})d\mathbf{t}$$

with covariance functional

$$\mathscr{C}_1(\varphi, \psi) = \mathscr{C}\left(\frac{\partial \varphi}{\partial t_1}, \frac{\partial \psi}{\partial t_1}\right), \qquad \varphi, \psi \in \mathfrak{D}(\mathbb{R}^n, \mathbb{R}).$$

Using Theorem 2.4, we get

$$\mathscr{C}_1(\varphi, \psi) = \int\limits_{[0,\infty)^n} \int\limits_{[0,\infty)^n} \prod_{i=1}^{n} \min(t_i, s_i) \frac{\partial \varphi}{\partial t_1}(\mathbf{t}) \frac{\partial \psi}{\partial s_1}(\mathbf{s})d\mathbf{t}d\mathbf{s} =$$

$$= \int\limits_{[0,\infty)^{n-1}} \int\limits_{[0,\infty)^{n-1}} \prod_{i=2}^{n} \min(t_i, s_i) \left(\int_0^\infty \int_0^\infty \min(t_1, s_1) \frac{\partial \varphi}{\partial t_1}(\mathbf{t}) \frac{\partial \psi}{\partial s_1}(\mathbf{s})dt_1 ds_1 \right) d\hat{\mathbf{t}}d\hat{\mathbf{s}} =$$

$$= \int\limits_{[0,\infty)^{n-1}} \int\limits_{[0,\infty)^{n-1}} \prod_{i=2}^{n} \min(t_i, s_i) \left(\int_0^\infty \varphi(t_1, \hat{\mathbf{t}})\psi(t_1, \hat{\mathbf{s}})dt_1 \right) d\hat{\mathbf{t}}d\hat{\mathbf{s}},$$

where $\hat{\mathbf{t}} = \{t_2, \ldots, t_n\}$ and $\hat{\mathbf{s}} = \{s_2, \ldots, s_n\}$. Hence, the covariance functional $\mathscr{C}_{1,\ldots,n}$ of $\left(\frac{\partial^n B_\varphi}{\partial t_1, \ldots, \partial_{t_n}}\right)_{\varphi \in \mathfrak{D}(\mathbb{R}^n,\mathbb{R})}$ is given by

$$\mathscr{C}_{1,\ldots,n}(\varphi, \psi) = \int\limits_{[0,\infty)^n} \varphi(\mathbf{t})\psi(\mathbf{t})d\mathbf{t}.$$

Assume the existence of a function $c : [0, \infty)^n \times [0, \infty)^n \to \mathbb{R}$ such that

$$\mathscr{C}_{1,\ldots,n}(\varphi, \psi) = \int\limits_{[0,\infty)^n} \int\limits_{[0,\infty)^n} c(\mathbf{t}, \mathbf{s})\varphi(\mathbf{t})\psi(\mathbf{s})d\mathbf{t}d\mathbf{s}.$$

Hence, $\mathscr{C}_{1,\ldots,n}(\varphi,\psi)$ would be the representation of the covariance function c of

$$\left(\frac{\partial^n B_\varphi}{\partial t_1,\ldots,\partial_{t_n}}\right)_{\varphi\in\mathfrak{D}(\mathbb{R}^n,\mathbb{R})}$$

as generalized function.

From

$$\begin{aligned}\mathscr{C}_{1,\ldots,n}(\varphi,\psi) &= \int\limits_{[0,\infty)^n}\varphi(\mathbf{t})\psi(\mathbf{t})d\mathbf{t} = \int\limits_{[0,\infty)^n}\int\limits_{[0,\infty)^n} c(\mathbf{t},\mathbf{s})\varphi(\mathbf{t})\psi(\mathbf{s})d\mathbf{t}d\mathbf{s}\\ &= \int\limits_{[0,\infty)^n}\varphi(\mathbf{t})\int\limits_{[0,\infty)^n} c(\mathbf{t},\mathbf{s})\psi(\mathbf{s})d\mathbf{s}d\mathbf{t}\quad\text{for all}\quad \varphi,\psi\in\mathfrak{D}(\mathbb{R}^n,\mathbb{R})\end{aligned}$$

we obtain the following property of c:

$$\int\limits_{[0,\infty)^n} c(\mathbf{t},\mathbf{s})\psi(\mathbf{s})d\mathbf{s} = \psi(\mathbf{t})\quad\text{for all}\quad \psi\in\mathfrak{D}(\mathbb{R}^n,\mathbb{R}),\ t\in[0,\infty)^n.$$

This equation describes the Dirac distribution

$$\delta_{\mathbf{t}}:\mathfrak{D}(\mathbb{R}^n,\mathbb{R})\to\mathbb{R},\quad \psi\mapsto\psi(\mathbf{t}),\quad \mathbf{t}\in[0,\infty)^n.$$

4.4 Partial Differential Equations and Green's Function

In this section, we consider an example of a stochastic partial differential equation and its solution using Green's function. To this end, we choose $d=2$,

$$K_{\mathbf{0},1} = \left\{(x,y)\in\mathbb{R}^2;\ x^2+y^2<1\right\},$$

and a continuous function $f: K_{\mathbf{0},1}\to\mathbb{R}$.

In a first step, we want to compute a solution u of the boundary value problem (Poisson's equation)

$$\begin{aligned}&\frac{\partial^2 u}{\partial x^2}(\xi,\eta) + \frac{\partial^2 u}{\partial y^2}(\xi,\eta) = f(\xi,\eta)\quad\text{for all}\quad(\xi,\eta)\in K_{\mathbf{0},1},\\ &\quad u(\xi,\eta)=0\quad\text{for all}\quad(\xi,\eta)\in\partial K_{\mathbf{0},1}\left(=\left\{(x,y)\in\mathbb{R}^2;\ x^2+y^2=1\right\}\right).\end{aligned}$$

Therefore, we consider the equation

$$\frac{\partial^2 u}{\partial x^2}(\xi,\eta) + \frac{\partial^2 u}{\partial y^2}(\xi,\eta) = f(\xi,\eta)\quad\text{for all}\quad(\xi,\eta)\in K_{\mathbf{0},1}.$$

Using $f = \delta_{\mathbf{0}}$ (Dirac distribution) and using the fact that a partial derivative of a generalized function F is given by

$$\frac{\partial F}{\partial x_i} : \mathfrak{D}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto -F\left(\frac{\partial \varphi}{\partial x_i}\right), \quad i = 1, \ldots, n,$$

we can reformulate our boundary value problem in terms of generalized functions:

$$F\left(\frac{\partial^2 \varphi}{\partial x^2}\right) + F\left(\frac{\partial^2 \varphi}{\partial y^2}\right) = \varphi(\mathbf{0}), \quad \varphi \in \mathfrak{D}(\mathbb{R}^2, \mathbb{R}).$$

With the locally integrable function

$$N : \mathbb{R}^2 \setminus \{\mathbf{0}\} \to \mathbb{R}, \quad (x, y) \mapsto \frac{\ln\left(\sqrt{x^2 + y^2}\right)}{2\pi},$$

it is possible to show (see [LarTho08]) that

$$F_N : \mathfrak{D}(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}, \quad \varphi \mapsto \int_{\mathbb{R}^2} N(x, y)\varphi(x, y)dxdy$$

is a solution of

$$F\left(\frac{\partial^2 \varphi}{\partial x^2}\right) + F\left(\frac{\partial^2 \varphi}{\partial y^2}\right) = \varphi(\mathbf{0}), \quad \varphi \in \mathfrak{D}(\mathbb{R}^2, \mathbb{R}).$$

The function N is called a **fundamental solution** of

$$\frac{\partial^2 u}{\partial x^2}(\xi, \eta) + \frac{\partial^2 u}{\partial y^2}(\xi, \eta) = f(\xi, \eta) \quad \text{for all} \quad (\xi, \eta) \in K_{\mathbf{0},1}$$

(see ▫ Fig. 4.7). Using this fundamental solution N, we compute for each $(s, t) \in K_{\mathbf{0},1}$ a solution of the boundary value problem

$$\frac{\partial^2 u_{(s,t)}}{\partial x^2}(\xi, \eta) + \frac{\partial^2 u_{(s,t)}}{\partial y^2}(\xi, \eta) = 0 \quad \text{for all} \quad (\xi, \eta) \in K_{\mathbf{0},1},$$
$$u_{(s,t)}(\xi, \eta) = N((\xi, \eta) - (s, t)) \quad \text{for all} \quad (\xi, \eta) \in \partial K_{\mathbf{0},1},$$

which gives us Green's function :

$$G_{(\xi,\eta)} : K_{\mathbf{0},1} \setminus \{(\xi, \eta)\} \to \mathbb{R}, \quad (s, t) \mapsto N((\xi, \eta) - (s, t)) - u_{(s,t)}(\xi, \eta) =$$
$$= \frac{\ln\left(\sqrt{(\xi - s)^2 + (\eta - t)^2}\right)}{2\pi} -$$
$$- \frac{\ln\left(\sqrt{\left(\frac{\xi}{\sqrt{\xi^2+\eta^2}} - \sqrt{\xi^2 + \eta^2}s\right)^2 + \left(\frac{\eta}{\sqrt{\xi^2+\eta^2}} - \sqrt{\xi^2 + \eta^2}t\right)^2}\right)}{2\pi}, \quad (\xi, \eta) \in K_{\mathbf{0},1}.$$

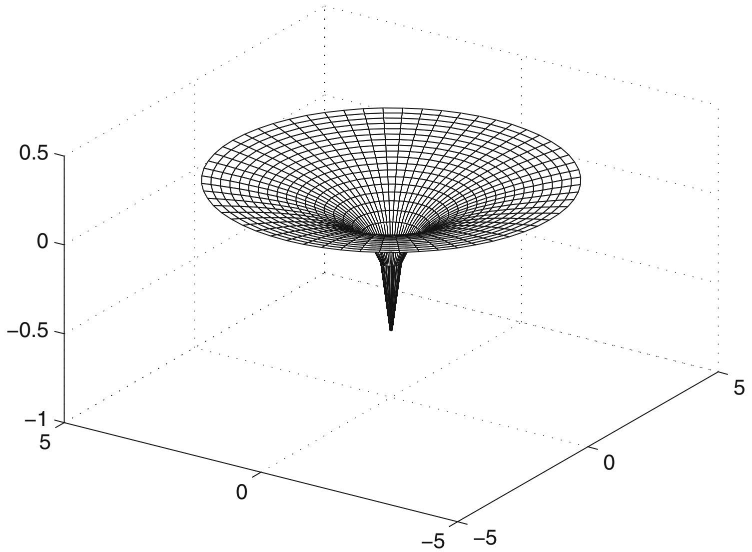

Fig. 4.7 Fundamental solution for Poisson's equation

Hence, a solution of Poisson's equation

$$\frac{\partial^2 u}{\partial x^2}(\xi,\eta) + \frac{\partial^2 u}{\partial y^2}(\xi,\eta) = f(\xi,\eta) \quad \text{for all} \quad (\xi,\eta) \in K_{\mathbf{0},1},$$

$$u(\xi,\eta) = 0 \quad \text{for all} \quad (\xi,\eta) \in \partial K_{\mathbf{0},1} \left(= \left\{(x,y) \in \mathbb{R}^2;\ x^2+y^2=1\right\}\right)$$

is given by

$$u : K_{\mathbf{0},1} \to \mathbb{R}, \quad (\xi,\eta) \mapsto \int_{K_{\mathbf{0},1}} G_{(\xi,\eta)}(s,t) f(s,t)\,ds dt.$$

Figure 4.8 shows u for $f : K_{\mathbf{0},1} \to \mathbb{R},\ (x,y) \mapsto xy$.

Let $(P_M)_{M\in\mathcal{B}^2_{\lambda^2}}$ be a Gaussian $(\sigma^2\cdot\lambda^2)$-noise, where $\sigma > 0$. Using

$$\mathcal{K}_{\mathbf{0},1} := \{M \cap K_{\mathbf{0},1};\ M \in \mathcal{B}^2_{\lambda^2}\},$$

we consider the stochastic process $(P_M)_{M\in\mathcal{K}_{\mathbf{0},1}}$. If we assume, that the function f (more accurately: the generalized function F_f) is additively disturbed by a Gaussian white noise process $(W_\varphi)_{\varphi\in\mathfrak{D}(\mathbb{R}^2,\mathbb{R})}$ on $K_{\mathbf{0},1}$ defined by

$$\mathscr{E}(W_\varphi) = 0 \quad \text{for all} \quad \varphi \in \mathfrak{D}(\mathbb{R}^2,\mathbb{R})$$

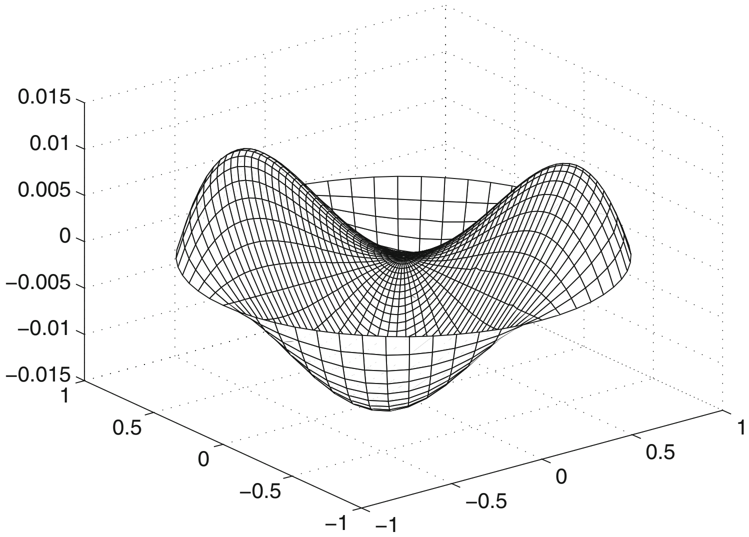

Fig. 4.8 Solution u for Poisson's equation

and

$$\mathscr{C}(\varphi, \psi) = \sigma^2 \int_{K_{\mathbf{0},1}} \varphi(\mathbf{t})\psi(\mathbf{t})d\mathbf{t} \quad \text{for all} \quad \varphi, \psi \in \mathfrak{D}(\mathbb{R}^2, \mathbb{R}),$$

then we obtain a solution U of Poisson's equation by

$$U : K_{\mathbf{0},1} \times \Omega \to \mathbb{R}, \quad (\xi, \eta, \omega) \mapsto \int_{K_{\mathbf{0},1}} G_{(\xi,\eta)}(s,t)f(s,t)dsdt + \left(\int_{K_{\mathbf{0},1}} G_{(\xi,\eta)}\, dP_M\right)(\omega).$$

Problems and Solutions

Problems

1. Let $(w_i)_{i\in\mathbb{Z}}$ be a discrete white noise process, which is defined by

(i) $\mathscr{E}(w_i) = 0, \quad i \in \mathbb{Z}$,

(ii) $\mathscr{C}(w_i, w_j) = \sigma^2\delta_{ij}$ (Kronecker delta), $\quad i, j \in \mathbb{Z}, \ \sigma^2 > 0$.

Prove that $(w_i)_{i\in\mathbb{Z}}$ is a μ-noise.

2. Let $n \in \mathbb{N}$ and let $(B_\mathbf{t})_{\mathbf{t}\in[0,\infty)^n}$ be an n-dimensional Brownian sheet. Furthermore, let

$$0 \le t_i^1 < t_i^2 < \ldots < t_i^k, \quad k \in \mathbb{N}, \quad i = 1, \ldots, n.$$

Prove that

$$B_{\mathbf{t}^k} - B_{\mathbf{t}^{k-1}}, \ldots, B_{\mathbf{t}^2} - B_{\mathbf{t}^1}$$

are stochastically independent.

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3. Let $(X_M)_{M\in\mathcal{H}_\mu}$ be a set indicated random field and let $e : A \to \mathbb{R}$ be a simple function on $A \in \mathcal{H}_\mu$. Prove that

$$\int_A e\, dX_M$$

is well defined.

4. Let $(X_M)_{M\in\mathcal{B}_\lambda}$ be a λ-noise such that

$$\mathscr{C}(X_{M_1}, X_{M_2}) := \lambda(M_1 \cap M_2) \quad \text{for all} \quad M_1, M_2 \in \mathcal{B}_\lambda,$$

and choose $A = [0, T]$, $T \in (0, \infty)$. Furthermore, let $f : [0, T] \to \mathbb{R}$ be a continuous function. Prove that:

$$\int_{[0,T]} f\, dX_\lambda = \int_0^T f(t) dB_t \quad (\mathbb{P}\text{-})\text{almost surely.}$$

5. Let $n \in \mathbb{N}$ and let $(B_\mathbf{t})_{\mathbf{t}\in[0,\infty)^n}$ be an n-dimensional Brownian sheet. Prove that

$$B_\mathbf{t} = 0 \quad (\mathbb{P}\text{-})\text{almost surely,}$$

if there exists an $i \in \{1, \ldots, n\}$ with $t_i = 0$.

Solutions

1. Choose $H = \mathbb{Z}$, $\mathcal{H} = \mathcal{P}(\mathbb{Z})$ (power set of $\mathbb{Z}$), and

$$\mu : \mathcal{P}(\mathbb{Z}) \to [0, \infty], \quad A \mapsto \sigma^2 |A|.$$

With $(X_A)_{\{A\subset\mathbb{Z};\, |A|<\infty\}}$, the proof is completed using

$$w_k = X_{\{k\}}, \quad k \in \mathbb{Z}.$$

2. Since

$$B_{\mathbf{t}^i} - B_{\mathbf{t}^{i-1}} = X_{(\mathbf{t}^{i-1}, \mathbf{t}^i]} \quad (\mathbb{P}\text{-})\text{almost surely}$$

and since the covariance matrix of

$$X_{(\mathbf{t^{k-1}},\mathbf{t^k}]}, \ldots, X_{(\mathbf{t^2},\mathbf{t^1}]}$$

is a diagonal matrix, the proof is complete.

3. Let

$$e = \sum_{i=1}^{n} \alpha_i I_{A_i}^A, \quad \bigcup_{i=1}^{n} A_i = A,$$

$$e = \sum_{i=1}^{k} \gamma_i I_{G_i}^A, \quad \bigcup_{i=1}^{k} G_i = A,$$

be two standard forms of e, then we obtain

$$\int_A e dX_M = \sum_{i=1}^{n} \alpha_i X_{A_i} = \sum_{\substack{i=1,\ldots,n \\ j=1,\ldots,k}} \alpha_i X_{A_i \cap B_j} = \sum_{\substack{i=1,\ldots,n \\ j=1,\ldots,k}} \gamma_i X_{A_i \cap B_j} = \sum_{i=1}^{k} \gamma_i X_{B_i}.$$

4. Since $f : [0, T] \to \mathbb{R}$ is continuous, f is bounded. Now, we choose a sequence of partitions $\{t_0^i, \ldots, t_{k_i}^i\}_{i \in \mathbb{N}}$ with

$$0 = t_0^i < t_1^i < \ldots < t_{k_i}^i = T, \quad i \in \mathbb{N}, \ k_i \in \mathbb{N},$$

and with

$$\lim_{i \to \infty} \max\{t_j^i - t_{j-1}^i; j = 1, \ldots, k_i\} = 0.$$

For each $i \in \mathbb{N}$, we consider

$$e_i : [0, T] \to \mathbb{R}, \quad t \mapsto f(0) I_{\{0\}}^{[0,T]}(t) + \sum_{j=0}^{k_i - 1} f\left(t_j^i\right) I_{(t_j^i, t_{j+1}^i]}^{[0,T]}(t).$$

Since f is continuous, the sequence $\{e_i\}_{i \in \mathbb{N}}$ converges pointwise to f and we obtain:

$$\int_{[0,T]} f\, dX_M = f(0) X_{\{0\}} + \lim_{i \to \infty} \left(\sum_{j=0}^{k_i - 1} f\left(t_j^i\right) X_{(t_j^i, t_{j+1}^i]} \right) \quad \text{(in the mean square)}.$$

From

$$B_t := X_{[0,t]} \quad \text{for all} \quad t \in [0, \infty)$$

and

$$B_t - B_s = X_{(s,t]} \quad (\mathbb{P}\text{-})\text{almost surely}$$

follows $X_{\{0\}} = 0$ ($\mathbb{P}$-)almost surely.

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5. If there exists an $i \in \{1, \ldots, n\}$ with $t_i = 0$, then we obtain

$$\mathscr{V}(B_\mathbf{t}) = \lambda^n([\mathbf{0}, \mathbf{t}]) = \prod_{k=1}^{n} t_k = 0.$$

Appendix A
A Short Course in Probability Theory

Stefan Schäffler

S. Schäffler, *Generalized Stochastic Processes*, Compact Textbooks in Mathematics,
https://doi.org/10.1007/978-3-319-78768-8

Consider a nonempty set Ω. With $\mathcal{P}(\Omega)$ we denote the power set of Ω consisting of all subsets of Ω. Using $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, we expand the algebraic structure of $\mathbb{R}$ to $\bar{\mathbb{R}}$ by

$$a + (\pm\infty) = (\pm\infty) + a = (\pm\infty) + (\pm\infty) = (\pm\infty), \quad +\infty - (-\infty) = +\infty,$$

$$a \cdot (\pm\infty) = (\pm\infty) \cdot a = \begin{cases} (\pm\infty), & \text{for } a > 0 \\ 0, & \text{for } a = 0, \\ (\mp\infty), & \text{for } a < 0 \end{cases}$$

$$(\pm\infty) \cdot (\pm\infty) = +\infty, \quad (\pm\infty) \cdot (\mp\infty) = -\infty, \quad \frac{a}{\pm\infty} = 0$$

for all $a \in \mathbb{R}$. With $-\infty < a$ and $a < \infty$ for all $a \in \mathbb{R}$, $(\bar{\mathbb{R}}, \leq)$ remains an ordered set, but not an ordered field.

Definition A.1 ((σ-Finite) Measure)

Let $\mathcal{F}$ be a family of subsets of Ω with $\emptyset \in \mathcal{F}$, then a function

$$\mu : \mathcal{F} \to \bar{\mathbb{R}}$$

is called a **measure** on $\mathcal{F}$, if the following conditions are fulfilled:

(M1) $\mu(A) \geq 0$ *for all* $A \in \mathcal{F}$,
(M2) $\mu(\emptyset) = 0$,

(continued)

Definition A.1 (continued)

(M3) *For each sequence* $\{A_i\}_{i\in\mathbb{N}}$ *of pairwise disjoint sets with* $A_i \in \mathcal{F}$, $i \in \mathbb{N}$, *and with* $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i) \quad (\sigma\text{-additivity})$$

holds.

Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence with $B_i \subseteq B_{i+1}$, $B_i \in \mathcal{F}$, and with $\bigcup_{i=1}^{\infty} B_i = \Omega$. If $\mu(B_i) < \infty$ for all $i \in \mathbb{N}$, then μ is called **σ-finite**. ◁

If one is only interested in measures on power sets, these measures have limited properties and are not useful for practical applications. Therefore, we consider the following special class of families of subsets of Ω.

Definition A.2 (σ-Field)

A family of subsets of Ω is called a **σ-field** on Ω, if the following conditions are fulfilled:

(S1) $\Omega \in \mathcal{S}$,

(S2) *If* $A \in \mathcal{S}$, *then* $A^c := \Omega \setminus A \in \mathcal{S}$,

(S3) *If* $A_i \in \mathcal{S}$, $i \in \mathbb{N}$, *then* $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$.

◁

Theorem and Definition A.3 (Generated σ-Field)

Let I be any nonempty set and let $\mathcal{S}_i$ be a σ-field on Ω for all $i \in I$, then $\bigcap_{i\in I} \mathcal{S}_i$ is again a σ-field on Ω. Assume $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ and let Σ be the set of all σ-fields on Ω with

$$\mathcal{S} \in \Sigma \iff \mathcal{F} \subseteq \mathcal{S},$$

then

$$\sigma(\mathcal{F}) := \bigcap_{\mathcal{S}\in\Sigma} \mathcal{S}$$

is called ***σ-field generated by $\mathcal{F}$***. ◁

For $\Omega = \mathbb{R}^n$, $n \in \mathbb{N}$, we consider the σ-field

$$\mathcal{B}^n = \sigma(\{([a_1, b_1) \times \ldots \times [a_n, b_n)) \cap \mathbb{R}^n;\ -\infty \le a_i \le b_i \le \infty,\ i = 1, \ldots, n\}),$$

where $[a_1, b_1) \times \ldots \times [a_n, b_n) := \emptyset$, if $a_j \geq b_j$ for at least one $j \in \{1, \ldots, n\}$. With

$$\lambda(([a_1, b_1) \times \ldots \times [a_n, b_n)) \cap \mathbb{R}^n) := \begin{cases} \prod_{i=1}^{n}(b_i - a_i) & \text{if } b_i > a_i,\ i = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

we obtain a unique measure on $\mathcal{B}^n$. This measure is called **Lebesgue-Borel measure** and is again denoted by λ. The σ-field $\mathcal{B}^n$ is called **Borel σ-field on** $\mathbb{R}^n$. Although

$$\mathcal{B}^n \neq \mathcal{P}(\mathbb{R}^n),$$

all important subsets of $\mathbb{R}^n$ (e.g, all open, closed, and compact subsets) are elements of $\mathcal{B}^n$.
Let μ be a measure defined on a σ-field $\mathcal{S}$ on Ω, then each set $A \in \mathcal{S}$ with $\mu(A) = 0$ is called a **μ-null set**. It is obvious that $\mu(B) = 0$ for each subset $B \subseteq A$ of a μ-null set with $B \in \mathcal{S}$. On the other hand, it is not guaranteed that

$$B \in \mathcal{S} \quad \text{for each} \quad B \subseteq A.$$

A σ-field $\mathcal{S}$ on Ω equipped with a measure $\mu : \mathcal{S} \to \bar{\mathbb{R}}$ is called **complete**, if each subset of a μ-null set is an element of $\mathcal{S}$. The σ-field

$$\mathcal{S}_0 := \{A \cup N;\ A \in \mathcal{S},\ N \text{ subset of a } \mu\text{-null set}\}$$

is called **μ-completion of $\mathcal{S}$** with the appropriate measure

$$\mu_0 : \mathcal{S}_0 \to \bar{\mathbb{R}}, \quad (A \cup N) \mapsto \mu(A).$$

Let $\mathcal{S}$ be a σ-field on Ω. The pair $(\Omega, \mathcal{S})$ is called a **measurable space**. With a measure μ on $\mathcal{S}$ the triple $(\Omega, \mathcal{S}, \mu)$ is called a **measure space**.

Definition A.4 (Measurable Mapping)

Let $(\Omega_1, \mathcal{S}_1)$ and $(\Omega_2, \mathcal{S}_2)$ be two measurable spaces. A mapping

$$T : \Omega_1 \to \Omega_2 \quad \text{with} \quad T^{-1}(A') := \{x \in \Omega_1; T(x) \in A'\} \in \mathcal{S}_1 \text{ for all } A' \in \mathcal{S}_2$$

is called **$\mathcal{S}_1$-$\mathcal{S}_2$-measurable**. ◁

Consider two measurable spaces $(\Omega_1, \mathcal{S}_1)$ and $(\Omega_2, \mathcal{S}_2)$ with $\mathcal{S}_2 = \sigma(\mathcal{F})$. A mapping $T : \Omega_1 \to \Omega_2$ is $\mathcal{S}_1$-$\mathcal{S}_2$-measurable, iff $T^{-1}(A') \in \mathcal{S}_1$ for all $A' \in \mathcal{F}$.
Let $(\Omega_1, \mathcal{S}_1, \mu_1)$ be a measure space, $(\Omega_2, \mathcal{S}_2)$ be a measurable space, and let $T : \Omega_1 \to \Omega_2$ be $\mathcal{S}_1$-$\mathcal{S}_2$-measurable, then we are able to equip $(\Omega_2, \mathcal{S}_2)$ with a measure μ_2 transformed from $(\Omega_1, \mathcal{S}_1, \mu_1)$ via T by

$$\mu_2 : \mathcal{S}_2 \to \bar{\mathbb{R}}, \quad A' \mapsto \mu_1\left(T^{-1}(A')\right), \quad A' \in \mathcal{S}_2.$$

This measure is called **image measure** or **pushforward measure** of T.
Based on a measurable space $(\Omega, \mathcal{S})$ a $\mathcal{S}$-$\mathcal{B}$-measurable function $e : \Omega \to \mathbb{R}$ is called a **simple function**, if $|e(\Omega)| < \infty$, where $|e(\Omega)|$ denotes the cardinality of $e(\Omega)$. An important class of simple functions is given by indicator functions

$$I_A : \Omega \to \mathbb{R}, \quad \omega \mapsto \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ otherwise} \end{cases}, \quad A \in \mathcal{S}.$$

An indicator function I_A indicates whether $\omega \in A$ or not. For each simple function $e : \Omega \to \mathbb{R}$, there exists a natural number n, pairwise disjoint sets $A_1, \ldots, A_n \in \mathcal{S}$ and real numbers $\alpha_1, \ldots, \alpha_n$ with:

$$e = \sum_{i=1}^{n} \alpha_i I_{A_i}, \quad \sum_{i=1}^{n} A_i = \Omega.$$

Definition A.5 ((μ-)Integral)

Consider nonnegative simple functions

$$e : \Omega \to \mathbb{R}_0^+, \quad e = \sum_{i=1}^{n} \alpha_i I_{A_i}, \quad \alpha_i \geq 0, \quad i = 1, \ldots, n,$$

defined on a measure space $(\Omega, \mathcal{S}, \mu)$, then we define the **($\mu-$)integral** for this functions by

$$\int e \, d\mu := \int_{\Omega} e \, d\mu := \sum_{i=1}^{n} \alpha_i \cdot \mu(A_i).$$

◁

It is important to recognize that this integral does not depend on any representation of e by $A_1, \ldots, A_n \in \mathcal{S}$ and $\alpha_1, \ldots, \alpha_n$. Let E be the set of all nonnegative simple functions on $(\Omega, \mathcal{S}, \mu)$, then it is easy to see, that

- $\int I_A \, d\mu = \mu(A)$ for all $A \in \mathcal{S}$.
- $\int (\alpha e) d\mu = \alpha \int e \, d\mu$ for all $e \in E$, $\alpha \in \mathbb{R}_0^+$.
- $\int (u + v) d\mu = \int u \, d\mu + \int v \, d\mu$ for all $u, v \in E$.
- If $u(\omega) \leq v(\omega)$ for all $\omega \in \Omega$, then $\int u \, d\mu \leq \int v \, d\mu$, $u, v \in E$.

Consider the σ-field

$$\bar{\mathcal{B}} := \{A \cup B;\ A \in \mathcal{B}, B \subseteq \{-\infty, \infty\}\}$$

on $\bar{\mathbb{R}}$, let $(\Omega, \mathcal{S})$ be a measurable space, and let $f : \Omega \to \bar{\mathbb{R}}_0^+$ be a nonnegative $\mathcal{S}$-$\bar{\mathcal{B}}$-measurable function, then there exists a pointwise monotonic increasing sequence $\{e_n\}_{n\in\mathbb{N}}$ of nonnegative $\mathcal{S}$-$\bar{\mathcal{B}}$-measurable simple functions $e_n : \Omega \to \mathbb{R}_0^+$, $n \in \mathbb{N}$, which converges pointwise to f. Therefore, we are able to define the (μ-)integral for nonnegative $\mathcal{S}$-$\bar{\mathcal{B}}$-measurable functions f via

$$\int f\,d\mu := \int_{\Omega} f\,d\mu := \lim_{n\to\infty} \int e_n\,d\mu.$$

Notice that the (μ-)integral is well-defined. Based on a $\mathcal{S}$-$\bar{\mathcal{B}}$-measurable function $f : \Omega \to \bar{\mathbb{R}}$, the function

$$f^+ : \Omega \to \bar{\mathbb{R}}_0^+, \quad \omega \mapsto \begin{cases} f(\omega) & \text{if } f(\omega) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is called *positive part* of f and the function

$$f^- : \Omega \to \bar{\mathbb{R}}_0^+, \quad \omega \mapsto \begin{cases} -f(\omega) & \text{if } f(\omega) \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

is called *negative part* of f with the following properties (see. Figs. A.1, A.2 and A.3)

- $f^+(\omega) \geq 0, f^-(\omega) \geq 0$ for all $\omega \in \Omega$.
- f^+ and f^- are $\mathcal{S}$-$\bar{\mathcal{B}}$-measurable.
- $f = f^+ - f^-$.

Using the positive part and the negative part of f, we define

$$\int f\,d\mu := \int_{\Omega} f\,d\mu := \int f^+\,d\mu - \int f^-\,d\mu,$$

if

$$\int f^+\,d\mu < \infty \quad \text{or} \quad \int f^-\,d\mu < \infty.$$

Furthermore, we define

$$\int_A f\,d\mu := \int f \cdot I_A\,d\mu.$$

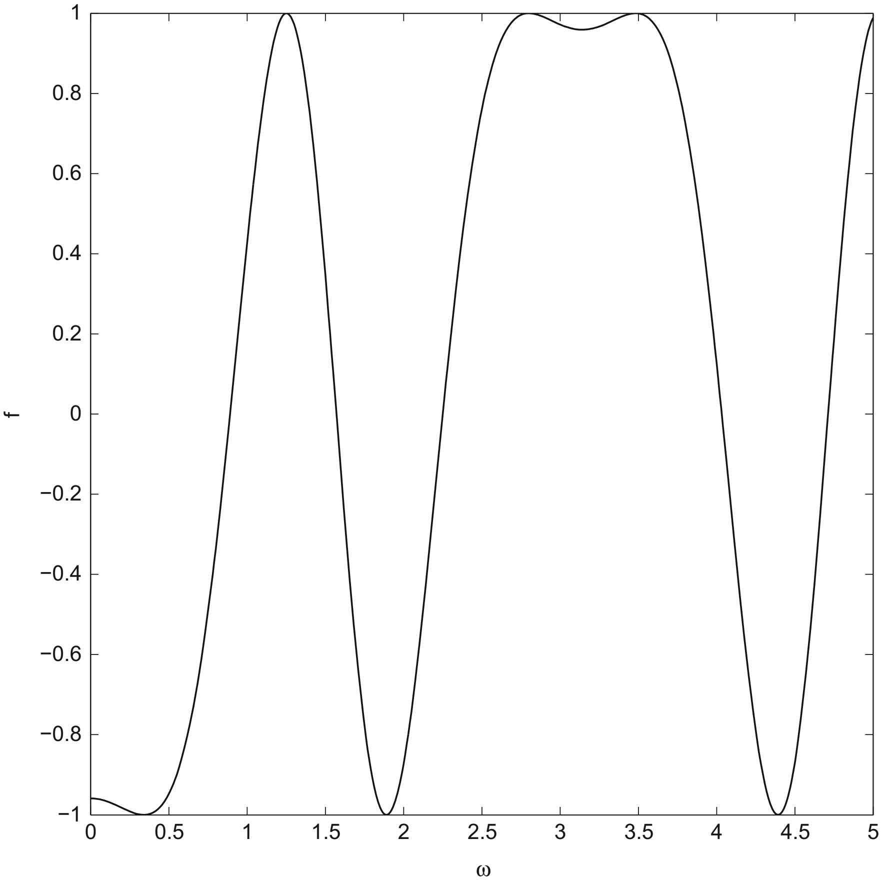

Fig. A.1 Function f

Definition A.6 (Probability Space, Probability)

A measure space $(\Omega, \mathcal{S}, \mathbb{P})$ with $\mathbb{P}(\Omega) = 1$ is called **probability space** with probability measure $\mathbb{P}$. For all sets $A \in \mathcal{S}$, the real number $\mathbb{P}(A)$ is called **probability of** A. ◁

Definition A.7 ((Real) Random Variable, Distribution)

Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let $(\Omega', \mathcal{S}')$ be a measurable space, then a $\mathcal{S}$-$\mathcal{S}'$-measurable function $X: \Omega \to \Omega'$ is called a **random variable**. If $\Omega' = \mathbb{R}^n$, $n \in \mathbb{N}$, and $\mathcal{S}' = \mathcal{B}^n$, then X is said to be an **n-dimensional real random variable**. The image measure $\mathbb{P}'$ on $\mathcal{S}'$ is called **distribution** of X and is denoted by $\mathbb{P}_X$. ◁

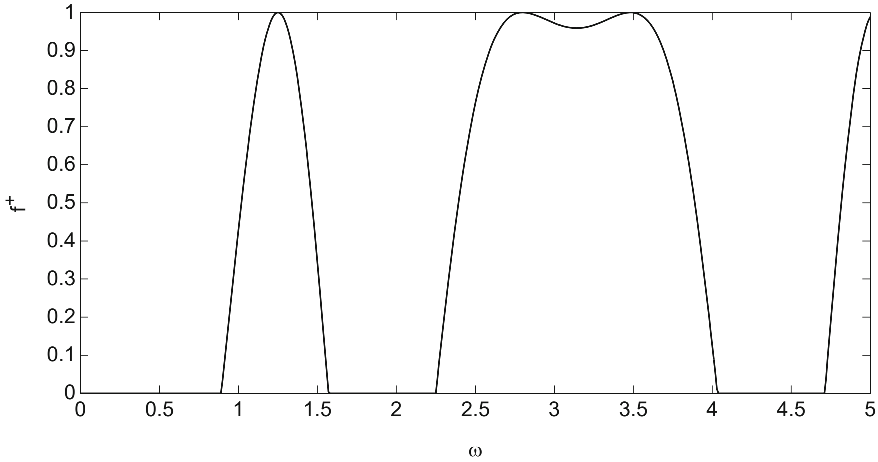

Fig. A.2 Function f^+

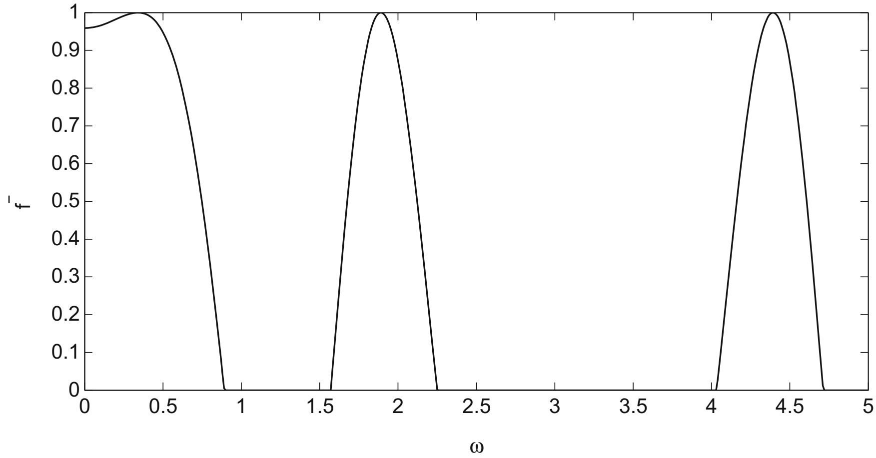

Fig. A.3 Function f^-

Consider a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ and a random variable $X : \ \Omega \to \bar{\mathbb{R}}$ with

$$\int X^+ \, d\mathbb{P} < \infty \quad \text{or} \quad \int X^- \, d\mathbb{P} < \infty,$$

then

$$\mathscr{E}(X) := \int X \, d\mathbb{P}$$

is called **expectation of X**. The **variance** of a random variable $X : \Omega \to \bar{\mathbb{R}}$ with finite expectation $\mathscr{E}(X)$ is given by

$$\mathscr{V}(X) := \int (X - \mathscr{E}(X))^2 \, d\mathbb{P}.$$

Assume that $B \in \mathcal{S}$ and $\mathbb{P}(B) > 0$, then we are able to define another probability measure $\mathbb{P}(\bullet|B)$ (**conditional probability**) on $\mathcal{S}$ by

$$\mathbb{P}(\bullet|B) : \mathcal{S} \to [0, 1], \quad A \mapsto \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Definition A.8 (Density)

Consider a measure space $(\Omega, \mathcal{S}, \mu)$ and a $\mathcal{S}$-$\bar{\mathcal{B}}$-measurable function

$$f : \Omega \to \bar{\mathbb{R}}$$

such that

- $f(\omega) \geq 0 \quad$ for all $\quad \omega \in \Omega$,
- $\int f \, d\mu = 1$,

then we obtain a probability measure

$$\mathbb{P} : \mathcal{S} \to [0, 1], \quad A \mapsto \int_A f \, d\mu.$$

The function f is called **density of $\mathbb{P}$ with respect to μ**. ◁

For the special case $(\mathbb{R}^n, \mathcal{B}^n, \mathbb{P})$, $n \in \mathbb{N}$, the probability measure $\mathbb{P}$ can be expressed by a so-called **distribution function**

$$F : \mathbb{R}^n \to [0, 1], \ (x_1, \ldots, x_n)^\top \mapsto \mathbb{P}\left((-\infty, x_1) \times \ldots \times (-\infty, x_n)\right)$$

The distribution function of an image measure $\mathbb{P}_X$ of an m-dimensional real-valued random variable $X : \mathbb{R}^n \to \mathbb{R}^m$, $m \in \mathbb{N}$, is called **distribution function of X**. The most important class of distribution functions of this type is given by

$$F_{\nu_{\mathbf{e},\boldsymbol{\Sigma}}} : \mathbb{R}^m \to [0, 1]$$

$$\mathbf{x} \mapsto \int\limits_{(-\infty, x_1)} \ldots \int\limits_{(-\infty, x_m)} \frac{1}{\sqrt{(2\pi)^m \det(\boldsymbol{\Sigma})}} \cdot \exp\left(-\frac{(\mathbf{x}-\mathbf{e})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x}-\mathbf{e})}{2}\right) d\mathbf{x}$$

for each $\mathbf{e} \in \mathbb{R}^m$ and a positive definite matrix $\boldsymbol{\Sigma} = (\sigma_{i,j}) \in \mathbb{R}^{m,m}$. A random variable X of this type is called $\mathcal{N}(\mathbf{e}, \boldsymbol{\Sigma})$ **Gaussian distributed**. Its image measure is given by the Lebesgue density

$$\nu_{\mathbf{e},\boldsymbol{\Sigma}} : \mathbb{R}^m \to \mathbb{R}, \quad \mathbf{x} \mapsto \frac{1}{\sqrt{(2\pi)^m \det(\boldsymbol{\Sigma})}} \cdot \exp\left(-\frac{(\mathbf{x}-\mathbf{e})^\top \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\mathbf{e})}{2}\right)$$

with

$$e_i = \mathscr{E}(X_i), \quad i = 1, \ldots, m$$

and

$$\sigma_{i,j} = \mathscr{E}((X_i - e_i)(X_j - e_j)), \quad i, j = 1, \ldots, m.$$

The matrix $\boldsymbol{\Sigma}$ is called **covariance matrix**.
Based on a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ we have defined a probability measure $\mathbb{P}(\bullet|B)$ (conditional probability) on $\mathcal{S}$ by

$$\mathbb{P}(\bullet|B) : \mathcal{S} \to [0, 1], \quad A \mapsto \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

under the assumption that $\mathbb{P}(B) > 0$. The fact that

$$\mathbb{P}(A|B) = \mathbb{P}(A) \quad \Longleftrightarrow \quad \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

leads to the definition

$$A, B \in \mathcal{S} \quad \textbf{stochastically independent} \quad :\Longleftrightarrow \quad \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Elements

$$\{A_i \in \mathcal{S};\ i \in I\},\ I \neq \emptyset,$$

of $\mathcal{S}$ are called **stochastically independent**, iff

$$\mathbb{P}\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} \mathbb{P}(A_j) \quad \text{for all subsets} \quad J \subseteq I \quad \text{with} \quad 0 < |J| < \infty.$$

A family

$$\{\mathcal{F}_i \subseteq \mathcal{S};\ i \in I\}, \quad I \neq \emptyset$$

of subsets of $\mathcal{S}$ is called **stochastically independent**, iff

$$\mathbb{P}\left(\bigcap_{j\in J} A_j\right) = \prod_{j\in J} \mathbb{P}(A_j)$$

for all $A_j \in \mathcal{F}_j, j \in J$, and for all $J \subseteq I$ with $0 < |J| < \infty$. Let $(\Omega', \mathcal{S}')$ be a measurable space and let $X : \Omega \to \Omega'$ be a random variable. With $\mathcal{F}$ we denote the set of all σ-fields on Ω such that

$$X \text{ is } \mathcal{C}\text{-}\mathcal{S}'\text{-measurable, iff } \mathcal{C} \in \mathcal{F}.$$

The set

$$\sigma(X) := \bigcap_{\mathcal{C}\in\mathcal{F}} \mathcal{C}$$

is called **σ-field generated by X**. It is the smallest σ-field of all σ-fields $\mathcal{A}$ on Ω such that X is $\mathcal{A}$-$\mathcal{S}'$-measurable. The stochastical independence of a family

$$\{X_i : \ \Omega \to \Omega'; \ i \in I\}, \quad I \neq \emptyset$$

of random variables is defined by the stochastical independence of

$$\{\sigma(X_i); \ i \in I\}.$$

A sequence of random variables is a sequence of functions; therefore, different concepts of convergence do exist, as in real calculus, too. Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space, let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of real-valued random variables

$$X_i : \ \Omega \to \mathbb{R}, \quad i \in \mathbb{N},$$

and let $X : \ \Omega \to \mathbb{R}$ be a real-valued random variable, then we define for $r \in \mathbb{R}$, $r > 0$:

$$L^r\text{-}\lim_{i\to\infty} X_i = X \quad :\Longleftrightarrow \quad \lim_{i\to\infty} \int |X - X_i|^r \, d\mathbb{P} = 0,$$

where

$$\int |X_i|^r \, d\mathbb{P} < \infty \ \text{ for all } \ i \in \mathbb{N} \ \text{ and } \ \int |X|^r \, d\mathbb{P} < \infty$$

is assumed.
The sequence $\{X_i\}_{i\in\mathbb{N}}$ converges in **$\mathbb{P}$-measure to X** iff

$$\lim_{i\to\infty} \mathbb{P}\left(\{\omega \in \Omega; \ |X_i(\omega) - X(\omega)| < \epsilon\}\right) = 1$$

for all $\epsilon > 0$, and it converges ($\mathbb{P}$-)**almost surely** iff

$$\mathbb{P}\left(\left\{\omega \in \Omega;\ \lim_{i\to\infty} X_i(\omega) = X(\omega)\right\}\right) = 1.$$

Convergency in distribution means that

$$\lim_{i\to\infty} \int f\, d\mathbb{P}_{X_i} = \int f\, d\mathbb{P}_X$$

holds for all continuous bounded functions $f : \mathbb{R} \to \mathbb{R}$.

Theorem A.9 (Dominated Convergence Theorem)
Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables

$$X_n : \Omega \to \bar{\mathbb{R}}, \quad n \in \mathbb{N},$$

such that

$$\int X_n^2\, d\mathbb{P} < \infty \quad \textit{for all} \quad n \in \mathbb{N}$$

and such that $\{X_n\}_{n\in\mathbb{N}}$ converges ($\mathbb{P}$-)almost surely.
Let

$$g : \Omega \to \bar{\mathbb{R}}$$

be a random variable such that

$$|X_n(\omega)| \le g(\omega) \quad \textit{for all} \quad \omega \in \Omega$$

and let

$$\int g^2\, d\mathbb{P} < \infty.$$

Then there exists a random variable

$$X : \Omega \to \mathbb{R}$$

(continued)

Theorem A.9 (continued)
such that

(i) $\int X^2 \, d\mathbb{P} < \infty$,

(ii) $\lim\limits_{n\to\infty} \int (X - X_n)^2 \, d\mathbb{P} = 0$,

(iii) $X_n \longrightarrow X$ $(\mathbb{P}\text{-})$*almost surely.*

◁

A **stochastic process** is a parameterized collection $(X_i)_{i\in I}$, $I \neq \emptyset$, of random variables

$$X_i : \Omega \to \Omega'$$

based on a probability space $(\Omega, \mathcal{S}, \mathbb{P})$ and a measurable space $(\Omega', \mathcal{S}')$. A function

$$X_\bullet(\omega) : I \to \Omega', \quad i \mapsto X_i(\omega)$$

is called a **path** of $(X_i)_{i\in I}$ for each $\omega \in \Omega$. Choosing $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in I$, we can define a probability measure on $(\mathbb{R}^k, \mathcal{B}^k)$ via

$$\mathbb{P}_{X_{i_1,\ldots,i_m}} : \mathcal{B}^k \to [0,1],$$
$$(A_1, \ldots, A_k) \mapsto \mathbb{P}(\{\omega \in \Omega;\, X_{t_1}(\omega) \in A_1 \wedge \ldots \wedge X_{t_k}(\omega) \in A_k\}).$$

Such a probability measure is called a **finite-dimensional distribution of $\mathbf{(X_t)_{i\in I}}$**.
Let $(\Omega, \mathcal{S}, \mathbb{P})$ be a probability space and let $(\mathbb{R}, \mathcal{B})$ be a measurable space. For two one-dimensional real random variables

$$X, Y : \Omega \to \mathbb{R},$$

the real number

$$\mathscr{C}(X, Y) := \int (X - \mathscr{E}(X))(Y - \mathscr{E}(Y)) d\mathbb{P}$$

is called the **covariance** of X and Y. The covariance $\mathscr{C}(X, Y)$ exists under the assumption that the variances of X and Y exist. A stochastic process $(X_i)_{i\in I}$ with

$$X_i : \Omega \to \mathbb{R}, \quad i \in I$$

and with a commutative composition

$$+ : I \times I \to I$$

is called a **weakly stationary stochastic process**, if expectation and variance exist for all random variables X_i, $i \in I$, and if the following conditions are fulfilled:

(i) $\mathscr{E}(X_i) = m,\ i \in I,$
(equal expectation for all random variables)

(ii) $\mathscr{C}(X_i, X_{i+h}) = \mathscr{C}(X_j, X_{j+h}),\ i, j, h \in I,$
(covariance of X_i and X_{i+h} depends only on h)

(iii) $\mathscr{V}(X_i) = \sigma^2,\ i \in I.$
(equal variance for all random variables)

Appendix B Spectral Theory of Stochastic Processes

Stefan Schäffler

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Let $(\varepsilon_i)_{i\in I}$ be a weakly stationary stochastic process. If $I = [0, \infty)$, then the function

$$\gamma_\varepsilon : \mathbb{R} \to \mathbb{R}, \quad h \mapsto \begin{cases} \mathscr{C}(\varepsilon_0, \varepsilon_h) & \text{for } h \geq 0 \\ \mathscr{C}(\varepsilon_0, \varepsilon_{-h}) & \text{for } h < 0 \end{cases},$$

is called **covariance function** of $(\varepsilon_i)_{i\in[0,\infty)}$. If $I = \mathbb{R}$, then

$$\gamma_\varepsilon : \mathbb{R} \to \mathbb{R}, \quad h \mapsto \mathscr{C}(\varepsilon_0, \varepsilon_h),$$

is called **covariance function** of $(\varepsilon_i)_{i\in\mathbb{R}}$. From the theorem of Wiener-Khintchine we know the existence of a function

$$S : \mathbb{R} \to \mathbb{R}$$

such that

(i) there exists a unique measure $\mu_S : \mathcal{B} \to [0, \infty]$ with

$$\mu_S((-\infty, b)) = S(b) - \lim_{a\to-\infty} S(a) \quad \text{for all} \quad b \in \mathbb{R},$$

(ii) $\gamma_\varepsilon(h) = \int e_h d\mu_S, \quad$ where $\quad e_h : \mathbb{R} \to \mathbb{R}, f \mapsto e^{i2\pi fh}, \quad h \in \mathbb{R}$.

Conversely, given a monotonic increasing function $S : \mathbb{R} \to \mathbb{R}$, which is continuous from the right and which has the properties

(i) $\lim_{f\to\infty} S(f) - \lim_{f\to-\infty} S(f) < \infty$,

(ii) $S(f) + S(-f) = 2S(0)$ for all $f \in \mathbb{R}$, then there exists a weakly stationary stochastic process $(\varepsilon_i)_{i\in I}$ with $I = [0, \infty)$ or $I = \mathbb{R}$ and with covariance function

$$\gamma_\varepsilon : \mathbb{R} \to \mathbb{R}, \quad h \mapsto \int e_h d\mu_S.$$

The function S is called **spectral function** and the induced measure μ_S is called **spectral measure**. If there exists a function $s : \mathbb{R} \to \mathbb{R}$ such that

$$\gamma_\varepsilon(h) = \int e_h d\mu_S = \int (s \cdot e_h) d\lambda \quad \text{for all} \quad h \in \mathbb{R},$$

then the function s is called **spectral density function**. The set $\text{supp}(s)$ is called **spectrum** of $(\varepsilon_i)_{i\in I}$. From the symmetry of γ_ε we get

$$\gamma_\varepsilon(h) = \frac{1}{2}\left(\gamma_\varepsilon(h) + \gamma_\varepsilon(-h)\right).$$

Therefore, if γ_ε has a spectral density function s, we obtain

$$\gamma_\varepsilon(h) = \frac{1}{2}\left(\int (s \cdot e_h) d\lambda + \int \frac{s}{e_h} d\lambda\right) = \int (s \cdot \cos_h) d\lambda \quad \text{for all} \quad h \in \mathbb{R},$$

where

$$\cos_h : \mathbb{R} \to \mathbb{R}, \quad f \mapsto \cos(2\pi f h), \quad h \in \mathbb{R}.$$

Furthermore, if

$$\int |\gamma_\varepsilon| d\lambda < \infty,$$

then s is given by

$$s(f) = \int \frac{\gamma_\varepsilon}{e_f} d\lambda \quad \text{for all} \quad f \in \mathbb{R},$$

where

$$e_f : \mathbb{R} \to \mathbb{R}, \quad h \mapsto e^{i2\pi f h}, \quad f \in \mathbb{R}.$$

Considering

$$\gamma_\varepsilon : \mathbb{R} \to \mathbb{R}, \quad h \mapsto c e^{-\alpha|h|}, \quad \alpha, c > 0,$$

as an example, we obtain

$$s(f) = \int_{-\infty}^{\infty} ce^{-\alpha|h|}e^{-i2\pi fh}dh = c\int_{-\infty}^{\infty} e^{-\alpha|h|}e^{-i2\pi fh}dh =$$

$$= c\int_{-\infty}^{0} e^{h(\alpha - i2\pi f)}dh + c\int_{0}^{\infty} e^{h(-\alpha - i2\pi f)}dh =$$

$$= c\frac{1}{\alpha - i2\pi f} + c\frac{1}{\alpha + i2\pi f} = c\frac{2\alpha}{\alpha^2 + 4\pi^2 f^2} \quad \text{for all} \quad f \in \mathbb{R}.$$

Figure B.1 shows γ_ε for $\alpha = c = 1$ and Fig. B.2 shows the corresponding spectral density function.

Since

$$\gamma_\varepsilon(0) = \lim_{t\to\infty} \frac{1}{t} \int_{[0,t]} \mathscr{E}\left((\varepsilon_\bullet - \mathscr{E}(\varepsilon_\bullet))^2\right) d\lambda \quad \text{for} \quad I = [0, \infty)$$

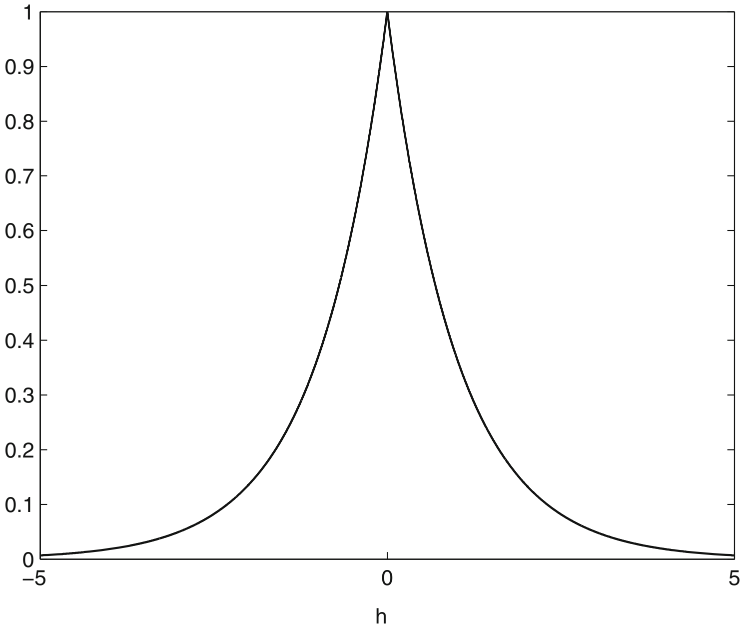

Fig. B.1 $\gamma_\varepsilon, \alpha = c = 1$

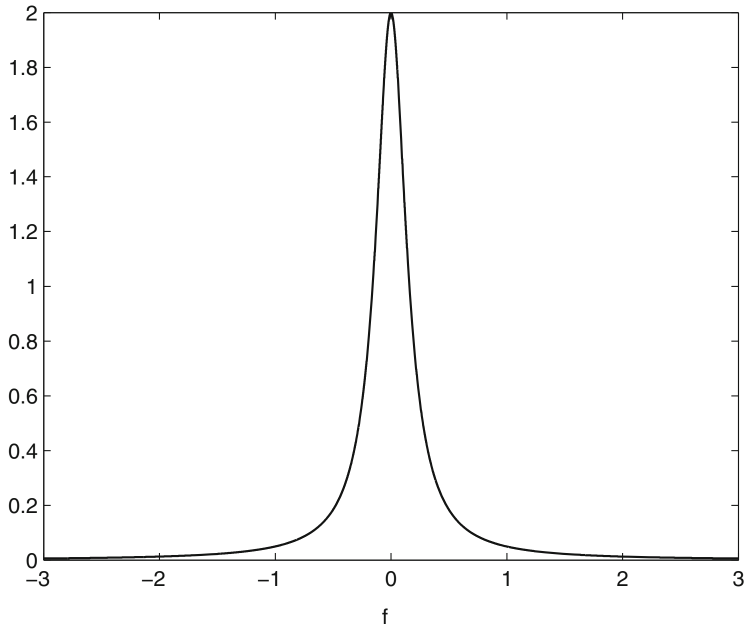

Fig. B.2 Spectral density function s, $\alpha = c = 1$

or

$$\gamma_\varepsilon(0) = \lim_{t\to\infty} \frac{1}{2t} \int\limits_{[-t,t]} \mathscr{E}\left((\varepsilon_\bullet - \mathscr{E}(\varepsilon_\bullet))^2\right) d\lambda \quad \text{for} \quad I = \mathbb{R},$$

where

$$\varepsilon_\bullet : I \times \Omega \to \mathbb{R}, \quad (i, \omega) \mapsto \varepsilon_i(\omega),$$

the nonnegative real number $\gamma_\varepsilon(0)$ represents the mean total power of $(\varepsilon_i)_{i\in I}$. Since

$$\gamma_\varepsilon(0) = \int s d\lambda,$$

the spectral density function s is also called **power density function**.

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