Review

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STABLE NON-GAUSSIAN RANDOM PROCESSES

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1. INTRODUCTION

Stable random variables and random processes are to infinite variance random variables what Gaussian random variables and random processes are to random variables with finite variance. For example, the Central Limit Theorem tells us that we can approximate the distribution of a sum of finite variance random variables by an appropriate Gaussian process; likewise, the distribution of a sum of infinite variance random variables can be approximated by a stable distribution. A distribution F is said to be stable if, for independent random variables $X_1, \ldots, X_n$ with common distribution F, there exist constants $\{b_n\}$ and α such that

$$n^{-1/\alpha}(X_1 + \ldots + X_n) - b_n$$

has distribution F (for all n). It can be shown that $0 < \alpha \leq 2$, where $\alpha = 2$ corresponds to a Gaussian distribution while, if $\alpha < 2$, the X_i's have infinite variance.

Research in the area of stable random variables and random processes has often been regarded as "exotic" by many statisticians and econometricians. Such research was often viewed as interesting from an academic point of view but of little use from an "applied" point of view. Moreover, the literature on stable random processes was widely scattered and no good reference existed beyond graduate textbooks in probability, which typically only gave the most basic introductions to the topic. (A notable exception here is Feller, 1971.)

In recent years, however, more attention has been given to the possibility that certain phenomena (e.g., stock returns, telephone call lengths, insurance claims) can be better modeled by distributions with heavier tails than those traditionally used; this naturally leads to the consideration of distributions with infinite variance. Moreover, many time series appear to exhibit "discontinuities" (e.g., large jumps) and, thus, may be more adequately modeled by time-series models whose increments have infinite variance than by models with finite variance increments. Perhaps for this reason, we have recently seen the publication of several monographs dealing with stable random

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processes—for example, Zolotarev (1986), Christoph and Wolf (1992), Janicki and Weron (1993), and Samorodnitsky and Taqqu (1994) (hereafter S&T), which we are reviewing here.

What distinguishes this monograph from the others is its approach. Traditionally, researchers have used analytic approaches (e.g., characteristic functions) to describe probability distributions and describe convergence in distribution results. Throughout, S&T emphasize the probabilistic properties (such as the tail behavior) of the random variables and processes. One consequence of this is that explicit representations of stable random variables and processes can be given (rather than, e.g., simply giving the characteristic function). For example, in Chapter 1 of S&T, a fairly simple representation of a strictly stable random variable is proved. Suppose that $E_1, E_2, \ldots$ are independent exponential random variables with mean 1 and $W_1, W_2, \ldots$ are independent and identically distributed (i.i.d.) random variables that are also independent of the E_i's; define $\Gamma_i = E_1 + \ldots + E_i$ (for $i \geq 1$), which are the arrival times of a Poisson process with unit rate. Then, if the infinite series

$$\sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} W_i \tag{1}$$

converges almost surely (for $0 < \alpha < 2$), the limit is a strictly stable random variable with index α; in fact, any strictly stable random variable can be represented by a random variable of the form given in (1). (Representations such as (1) are sometimes called *Lepage representations.*) The main advantage of the probabilistic approach is that it develops the reader's intuition more quickly than any analytic approach; for example, probabilistic representations allow a better understanding of sample path properties of stable processes.

The coverage of S&T is truly remarkable. Nowhere else can one find so much about univariate and multivariate stable distributions and stable processes. Several examples of stable processes are given (including generalizations of Brownian motion and the Ornstein–Uhlenbeck process) and information on how to simulate sample paths of stable processes is provided. Chapter 7, which deals with self-similar processes, is particularly impressive. Self-similar processes have, of course, been much written about (including in the popular press), although much of the literature is nonmathematical. S&T present a very thorough and rigorous presentation, which is, nonetheless, very clear given the technicality of the presentation. Chapter 14, which contains historical notes on Chapters 1–13, provides the reader with additional references to explore certain topics in more depth.

As with any specialized monograph, there is a lot of what some might regard as "esoterica." Nonetheless, the well-motivated reader will find some real gems in these sections. For example, Chapter 10 contains some interesting material on the continuity and oscillations of sample paths of stable processes. For example, it is well known that the so-called α-stable Lévy motion (the infinite variance analogue of Brownian motion) is a pure jump process and, hence, has discontinuous sample paths. (Brownian motion has continuous sample paths.) This raises the question of what conditions are needed for a stable process to have continuous sample paths. S&T give necessary (and, in some cases, sufficient) conditions for a certain class of stable processes (integrals with respect to a stable measure) to have continuous sample paths.

If there is a weakness in S&T, it is in the handling of two issues of great interest to both statisticians and econometricians: weak convergence of sums of random variables to stable laws and statistical modeling of infinite variance data. We will discuss these in the following two sections.

2. WEAK CONVERGENCE TO STABLE LAWS

There are two approaches for showing weak convergence of a sum of i.i.d. random variables. The first approach, which is analytical, involves showing that the sequence of characteristic functions converges to the characteristic function of a stable random variable. The second approach is probabilistic; we obtain a representation for the extreme summands and show that these extremes converge to their corresponding terms in the Lepage representation.

To illustrate these methods, we will assume throughout this section that $X_1, X_2, X_3, \ldots$ is an i.i.d. sequence of symmetric (around 0) random variables with

$$P(|X_i| > x) = x^{-\alpha}L(x), \tag{2}$$

where $L(x)$ is a slowly varying function and $0 < \alpha < 2$. Let $\{a_n\}$ be a sequence of constants satisfying the condition

$$\lim_{x\to\infty} nP(|X_i| > a_n x) = x^{-\alpha}$$

and define

$$S_n = a_n^{-1} \sum_{i=1}^{n} X_i. \tag{3}$$

(We are assuming that the X_i's are symmetric for ease of exposition; it is fairly straightforward to generalize to the nonsymmetric case.)

From the independence of the X_i's, it follows that

$$\begin{aligned} E[\exp(itS_n)] &= (E[\exp(ita_n^{-1}X_i)])^n \\ &= \left(\int_{-\infty}^{\infty} \exp(ita_n^{-1}x)F(dx)\right)^n \\ &= \left(1 + \frac{1}{n}\int_{-\infty}^{\infty} n(\exp(ity) - 1)F(a_n\,dy)\right)^n, \end{aligned}$$

where $F(x) = P(X_i \le x)$ is the distribution function of the X_i's. Because $nP(|X_i| > a_n x) \to x^{-\alpha}$ and X_i's are symmetric, we can replace $nF(a_n dy)$ by $(\alpha/2)|y|^{-\alpha-1}\,dy$ in the limit as $n \to \infty$ and the imaginary part of the integral is exactly 0; thus,

$$E[\exp(itS_n)] \to \exp\left[-\frac{\alpha}{2}\int_{-\infty}^{\infty}(1-\cos(ty))|y|^{-\alpha-1}\,dy\right] = \exp[-c(\alpha)|t|^{\alpha}],$$

which is the characteristic function of a symmetric stable random variable.

The probabilistic approach is due to Lepage, Woodroofe, and Zinn (1981) (although, ironically, this paper is not cited in S&T). Given $X_1, \ldots, X_n$, define $Y_i = |X_i|$ and order the Y_i's from largest to smallest:

$$Y_{1n} \ge Y_{2n} \ge \ldots \ge Y_{nn}.$$

Let X_{in} be the X_i corresponding to Y_{in}, and define δ_{in} so that $X_{in} = \delta_{in} Y_{in}$. Clearly now, the δ_{in}'s are i.i.d. random variables with $P(\delta_{in} = \pm 1) = \frac{1}{2}$; moreover, the δ_{in}'s are independent of the Y_{in}'s. (Similar results follow in an asymptotic sense in the general case.) Defining S_n as in (3), it follows that

$$S_n = \sum_{i=1}^{n} \delta_{in}(a_n^{-1} Y_{in}).$$

Let $G(x) = P(Y_i > x)$ and define $G^{-1}(y) = \inf\{x : G(x) \le y\}$ to be its left-continuous inverse. $G^{-1}(t)$ is a decreasing function and $G^{-1}(U)$ will have the same distribution as any Y_i if U has a uniform distribution on [0,1]. Let E_1, E_2, ... be i.i.d. Exponential random variables with mean 1 (as earlier) and define

$$\Gamma_k = E_1 + \ldots + E_k.$$

Because Γ_i/Γ_{n+1} has the same distribution as the ith order statistic from an i.i.d. sample of n Uniform random variables on [0,1], we have that

$$a_n^{-1} Y_{in} =_d a_n^{-1} G^{-1}(\Gamma_i/\Gamma_{n+1}).$$

Noting that $\Gamma_{n+1}/n \to_{a.s.} 1$ by the Strong Law of Large Numbers, it follows that

$$a_n^{-1} G^{-1}(\Gamma_i/\Gamma_{n+1}) \to_{a.s.} \Gamma_i^{-1/\alpha}$$

and so $a_n^{-1} Y_{in} \to_d \Gamma_i^{-1/\alpha}$; it also follows that

$$a_n^{-1}(Y_{1n}, Y_{2n}, \ldots, Y_{mn}) \to_d (\Gamma_1^{-1/\alpha}, \ldots, \Gamma_m^{-1/\alpha})$$

for any fixed m. These results now suggest that

$$S_n \to_d \sum_{k=1}^{\infty} \delta_k \Gamma_k^{-1/\alpha},$$

where the δ_k's are i.i.d. random variables with $P(\delta = \pm 1) = \frac{1}{2}$, which are independent of the Γ_k's. The limiting random variable is simply a special case of the random variable in (1), which is strictly stable with index α. This probabilistic approach is made rigorous (and extended to the general case) by Lepage et al. (1981). This approach, while it may seem more complicated technically, more readily reveals the contribution that the extreme values make to the asymptotic distribution. This contrasts sharply with the case where the limiting distribution is Gaussian where typically no single summand contributes more than a negligible amount to the variance of the sum.

Suppose that $\{Z_i\}$ is not an i.i.d. sequence but rather a linear process with

$$Z_i = \sum_{j=0}^{\infty} c_j X_{i-j},$$

where the c_j's are constants and the X_i's are i.i.d. symmetric random variables satisfying (2). (Summability conditions on the c_j's are required so that the infinite series will converge.) Then, similar results can be obtained as in the i.i.d. case. In particular,

$$S_n^* = a_n^{-1} \sum_{i=1}^{n} Z_i \to_d S_\alpha^*,$$

where

$$S_\alpha^* =_d \left(\sum_{j=0}^{\infty} c_j \right) S_\alpha \quad \text{and} \quad a_n^{-1} \sum_{i=1}^{n} X_i \to_d S_\alpha.$$

This result was proved by Davis and Resnick (1985) and can be easily extended to the case of linear processes with nonsymmetric X_i's. An alternative (and simpler) proof is given in Phillips and Solo (1992), which uses the so-called Beveridge–Nelson decomposition (Beveridge and Nelson, 1981).

We can also consider the asymptotic behavior of partial sums. Again, let $\{X_i\}$ be a sequence of i.i.d. symmetric random variables satisfying (2), and define

$$S_n(t) = a_n^{-1} \sum_{i=1}^{[nt]} X_i.$$

It is easy to verify that the finite-dimensional distributions of $S_n(\cdot)$ converge weakly to those of a so-called α-stable Lévy motion $S_\alpha(\cdot)$ (to use the terminology of S&T). ($S_\alpha(t)$ has a stable distribution with index α at each t and stationary independent increments.)

The stochastic processes $\{S_n(\cdot)\}$ and $S_\alpha(\cdot)$ are random elements of the function space $D[0,1]$, which consists of real-valued right-continuous functions on [0,1] with left-hand limits. The usual topology for $D[0,1]$ is the so-called J_1 topology due to Skorokhod (1956). A function $g: D[0,1] \to \mathrm{R}$ is continuous in the J_1 topology if $g(f_n) \to g(f)$ whenever there exists a sequence of strictly increasing functions $\{\lambda_n(\cdot)\}$ mapping [0,1] onto [0,1] such that

$$\sup_{0\le t\le 1} |\lambda_n(t) - t| \to 0 \quad \text{and} \quad \sup_{0\le t\le 1} |f_n(\lambda_n(t)) - f(t)| \to 0.$$

Weak convergence of random elements of $D[0,1]$ endowed with the J_1 topology is equivalent to finite-dimensional convergence plus an "equicontinuity" (or "tightness") condition (see Pollard, 1984). It was shown by Skorokhod (1956) that $S_n(\cdot) \to_d S_\alpha(\cdot)$ in the J_1 topology

Suppose now that $\{Z_i\}$ is the linear process

$$Z_i = \sum_{j=0}^{\infty} c_j X_{i-j},$$

where $\{X_i\}$ is an i.i.d. sequence of symmetric random variables. We again define the partial sum process

$$S_n^*(t) = a_n^{-1} \sum_{i=1}^{[nt]} Z_i.$$

It is reasonably easy to verify (e.g., by using the results of Davis and Resnick, 1985) that the finite-dimensional distributions of $S_n^*(\cdot)$ converge weakly to those of an α-stable Lévy motion

$$S_\alpha^*(t) =_d \left(\sum_{j=0}^{\infty} c_j \right) S_\alpha(t),$$

where $S_\alpha(\cdot)$ is the weak limit of $a_n^{-1} \sum_{i=1}^{[nt]} X_i$. However, for a non-i.i.d. linear process, $S_n^*(\cdot)$ does not converge weakly to $S_\alpha^*(\cdot)$ in $D[0,1]$ endowed with the J_1 topology. This point is illustrated in a very interesting paper by Avram and Taqqu (1992). Consider the following example from Avram and Taqqu: Let $Z_i = X_i - X_{i-1}$, which implies that

$$S_n^*(t) = a_n^{-1}(X_{[nt]} - X_0).$$

For each t, $S_n^*(t) \to_p 0$, and so if $\{S_n^*(\cdot)\}$ converges weakly then the limit must be 0. However,

$$\begin{aligned} V_n = \sup_{0\le t\le 1} |S_n^*(t)| &= \max_{1\le i\le n} a_n^{-1} |X_i - X_0| \\ &= \max_{1\le i\le n} a_n^{-1} |X_i| + o_p(1) \\ &\to_d V, \end{aligned}$$

where the limiting random variable V is strictly positive. Because $\sup_{0\le t\le 1} |f(t)|$ is a continuous function on $D[0,1]$ (endowed with the J_1 topology), this shows that $S_n^*(\cdot)$ cannot converge weakly to 0 because, if it did, V_n would also converge weakly to 0.

What happens to the partial sum process $S_n^*(\cdot)$ when the summands are a non-i.i.d. linear process? When the summands are i.i.d., large jumps in $S_n(\cdot)$ (caused by large $|X_i|$) are isolated in the sense that the probability of two or more such large jumps occurring in a small interval is very small. However, when the summands are a linear process, the situation is much different. A large value of $|Z_i|$ is typically caused by a large $|X_i|$ and so $|Z_{i+1}|, |Z_{i+2}|, \ldots$ will also be large. Thus, rather than having isolated jumps (as in the i.i.d. case), $S_n^*(\cdot)$ tends to consist of a series of "staircases," each of which degenerates to a single jump as $n \to \infty$.

Despite the lack of weak convergence in $D[0,1]$ with the J_1 topology, it is often possible to prove weak convergence of $g(S_n^*)$ to $g(S_\alpha^*)$ directly for many real-valued functions g. This is particularly true for functions g, which are integrals. For example, suppose as before that $Z_i = X_i - X_{i-1}$, and let

$$g(S_n^*) = \int_0^1 S_n^*(t)\,dt.$$

We then have

$$\begin{aligned}\int_0^1 S_n^*(t)\,dt &= \frac{1}{n}\sum_{i=0}^{n-1} S_n^*(i/n)\\ &= \frac{1}{n}\,a_n^{-1}\sum_{i=1}^{n-1}(X_i - X_0)\\ &= \frac{1}{n}\,a_n^{-1}\sum_{i=1}^{n-1}X_i - a_n^{-1}X_0\\ &= \frac{1}{n}\,S_n(1-1/n) - a_n^{-1}X_0\\ &\to_p 0\end{aligned}$$

because both $S_n(1-1/n)$ and X_0 are $O_p(1)$. This is what we would expect intuitively because the finite-dimensional limit of S_n^* is 0. See Phillips (1990) and Knight (1991) for applications of these asymptotic results to estimation in econometric models.

Avram and Taqqu (1992) also discussed conditions under which $S_n^*(\cdot) \to_d S^*(\cdot)$ in $D[0,1]$ endowed with the weaker M_1 topology. They showed that a necessary (but not sufficient) condition for such convergence is $c_j \geq 0$ for all $j \geq 0$ (or $c_j \leq 0$ for all j). Essentially, this condition will guarantee that the steps in the preceding "staircase" go in the same direction.

3. STATISTICAL ANALYSIS

At present, there is fairly extensive literature on the behavior of classical statistical estimation methods when finite variance noise is replaced by infinite variance noise in, for example, linear regression and linear time-series models (see, e.g., Kanter and Steiger, 1974; Davis, Knight, and Liu, 1992; Mikosch, Gadrich, Klüppelberg, and Adler, 1995).

However, much work needs to be done to develop methodology specifically designed for the analysis of infinite variance data and of data from systems driven by infinite variance noise. One possible reason for this lack of specific methodology for data analysis is the reservation regarding the existence of infinite variance data; in fact, the very possibility seems to be troubling to some. A rather unfortunate attitude in this regard is often summarized by a statement to the effect that "infinite variance models are unnecessary since all data are really bounded." (The same argument could, of course, be applied to the Gaussian distribution.) One problem in the identification of infinite variance models is the fact that large amounts of data are needed to confidently identify such models. (For example, the Hill estimator (Hill, 1975) of the tail probability parameter α is extremely unreliable for small sample sizes.) Moreover, any type of likelihood procedure (including Bayesian procedures) is difficult; for example, closed-form densities for stable distributions do not exist in general.

Nonetheless, in many fields, there do seem to exist data that are well modeled by processes with infinite variance. For example, Mandelbrot (1963, 1967) and Fama (1965) long ago indicated that stock and exchange rate returns seem to exhibit infinite variance behavior; a more recent paper that addresses this issue is that by Mantegna and Stanley (1995). A dissenting opinion can be found in Lau, Lau, and Wingender (1990).

Recent work by Walter Willinger and co-workers at Bell Communications Research (see, e.g., Beran, Sherman, Taqqu, and Willinger, 1995; Duffy, McIntosh, Rosenstein, and Willinger, 1994; Willinger, Taqqu, Leland, and Wilson, 1995; Willinger, Taqqu, Sherman, and Wilson, 1995) indicates that infinite variance and self-similarity are present in telecommunications data. For example, lengths of telephone calls are well modeled by an infinite variance distribution; this is in sharp contrast with the commonly held assumption that such data are exponentially distributed. This heavy-tailed behavior has very significant implications for the design of telecommunications networks.

4. CONCLUSION

S&T is definitely not for the faint of heart. In their preface, the authors state that readers should have had at least a first-year graduate course in probability; however, probably more maturity is needed to derive maximum benefit from this monograph. Beyond that, S&T is a pleasure to read. Should a statistician or econometrician read S&T? The answer is an unqualified "yes," if he or she is seriously interested in studying stable random processes in depth. Even someone with only a casual interest in stable random processes would certainly benefit from skimming at least the first three chapters of S&T.

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