Permutations

A permutation of a set of distinct elements is an ordered sequence or arrangement of these elements.

Example. Three students $A, B,$ and C stand in line for a picture. The possible ways $A, B,$ and C can stand in the line are:

The number of ways is:

Factorials

Let n be a nonnegative integer.

$$0! = 1.$$

If $n > 1$, $\quad n! = n \cdot (n-1)! = n \cdot (n-1) \cdot \dots \cdot 1$

An ordered arrangement of r elements out of n distinct elements is called an r-permutation.

Note: $0 \leq r \leq n$ and $n \geq 0$.

Notation: $P(n.r)$.

$$P(n, 0) = 1$$

$$P(n, n) = n!$$

For the students in a line problem, we counted 3-permutations of 3 distinct entities.

$$P(3,3) = 3 \cdot 2 \cdot 1 = 3! = 6.$$

Example

3.1: If the number of available students is five and the line should have three students, then the number of ways to form a line is

Theorem 3.1. If n is a positive integer and r is an integer, $1 \le r \le n$, then there are

$$P(n,r) = n \cdot (n-1) \cdot \dots \cdot (n-r+1)$$

r-permutations of a set with n distinct elements.

Proof: Use the product rule.

Example 3.4. The number of ways to select a first-prize, a second-prize, and a third-prize winner from 100 different people in a contest is

Corollary 3.1. If n and r are integers with $0 \le r \le n$, then

$$P(n,r) = \frac{n!}{(n-r)!}.$$

Proof: Consider $r = 0$ and $r > 1$. Expand the factorials and simplify.

Problem 3.2. Let $S = \{a, b, c, d, e, f, g\}$. What is the number of permutations of S.

Example 3.7. How many permutations of the letters $ABCDEFGH$ contain the string ABC.

Problem 3.25. How many ways are there for four men and five women stand in a line so that

a. all men stand together?

b. all women stand together?

A **circular r-permutation of n** people is a seating of r of these n people around a circular table, where seatings are considered the same if they can be obtained from each other by rotating the table.

3.43. What is the number of circular r-permutations of n people?

Combinations

How many different committees of three students can be formed from a group of four?
[Note: unless otherwise indicated, the committee member selection order is irrelevant. Only membership matters.]

An r-combination of the elements of a set is an unordered selection of r elements from the set.

Notation: $C(n, r)$ or $\binom{n}{k}$.

Also, called a binomial coefficient.

Theorem 3.2. $\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$.

Proof: Use the Division Rule on the number of ordered selections.

Example 3.11. The number of five-card poker hands from a standard deck of 52 cards is:

The number of 47-card poker hands from a standard deck of cards is:

Corollary 3.2. $\binom{n}{r} = \binom{n}{n-r}$.

Prove it using algebraic manipulation and combinatorial arguments.

Problems

3.23. How many ways are there to place eight men and five women in a line so that no two women stand next to each other?

3.37. Count the number of bit strings that contain exactly eight 0s and ten 1s if every 0 must be followed by a 1?

3.35-3.36. A department has 10 men and 15 women. How many ways are there to form a committee of six members if it must have

a. equal number of men and women?

b. more women than men?

3.11. How many bit strings of length 10 contain

a. exactly four 1s?

b. at most four 1s?

c. at least four 1s?

d. an equal number of 0s and 1s?

R3.1. Ten distinct paintings are to be allocated to k office rooms so that (i) no room gets more than one painting and (ii) there are either no paintings left or no empty rooms left. Find the number of ways to do this if (a) $k = 14$ and (b) $k = 6$.

R 3.1′. Repeat the above problem with ten identical posters instead of paintings.

3.13. Count the number of ways n men and n women can form a line if men and women must alternate.

3.19. A coin is flipped 10 times. Each flip results in a head or a tail. How many possible outcomes

a. are there in total?

b. contain exactly two heads?

c. contain at most three tails?

d. contain the same number heads and tails?

3.27. One hundred tickets, numbered $1, 2, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes including a grand prize are awarded. How many ways are there to award the prizes if

a. there are no restrictions?

b. the person holding ticket 47 wins the grand prize?

c. the person holding ticket 47 wins one of the prizes?

d. the person holding ticket 47 does not win a prize?

e. the people holding tickets 19 and 47 win prizes?

i. the grand prize winner is a person holding ticket 19, 47, 73, or 97?

Binomial Coefficients	Binomial Expressions
Enumerate and count all 4-bit strings. Count 4-bit strings that have exactly $0, 1, 2, 3,$ or 4 1s.	

Theorem 4.1 Binomial Theorem

Let x and y be variables and let n be a nonnegative integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof: Use a combinatorial proof.

Example 4.2. What is the expansion of $(x+y)^4$?

4.3. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

4.4 What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Corollary 4.1. Let n be a nonnegative integer. Then,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof. Can be proved using the binomial theorem or combinatorial arguments.

Corollary 4.2. Let n be a nonnegative integer. Then,

$$\sum_{k=0}^{n} (-1)^n \binom{n}{k} = 0.$$

Proof. Use the binomial theorem.

Pascal's Identity and Triangle

Theorem 4.2. Pascal's Identity.

Let n and k be positive integers with $n \geq k$. Then,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof. You can use combinatorial argument or algebraic manipulation of the formulas for binomial coefficients.

Pascal's Triangle

Revisit the Password Counting Example 1.16.

The password for a computer account can be 6, 7, or 8 characters in length. A character can be a letter (case is ignored) or a digit. The password must contain at least one digit. What is the number of ways a valid password can be formed?

Vandermonde's Identity

Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Corollary 3.4. If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Problem 4.27. Show that if n and k are integers with $1 \le k \le n$, then

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Using a combinatorial argument.