The 12-fold Way: A Conceptual Grouping of Proofs

Blaine Pack

1 Introductory remarks

The Twelvefold Way is a collection of counting methods that are packaged together due to a surface similarity which, nevertheless, produces surprisingly different results. This paper is meant to be a companion to a presentation given on May 5th 2021, and so we will give only a cursory look at the basic concepts of the 12FW, instead focusing on argumments for why each equation does, in fact, count what it purports to. First off, remember that the 12FW deals with the metaphor of balls sorted into boxes, but can also be seen as counting possible functions f where $f : N \to X$, where $|N| = n$ and $|X| = x$. The columns of the following diagram are seperated by restrictions on f, while the rows are seperated by whether or not the balls and or boxes are 'labelled', that is whether there is a difference between them. So, with all that, here is the chart in question:

I propose that there are a few logical groupings of these equations based on similarities in the equations and therefore in their conceptual basis that would more easily introduce the concepts to a the basic learner. While I won't endeavor to demonstrate every concept here in, I will be noting many of them as far as when they are introduced for the first time. The sections that follow each deal with a 'family' of these functions.

	No restriction on f()	f() is injective	f() is surjective
N and X are distinct	1.x^n	2.$(x)_n$	3.$x!S(n,x)$
N is ind. X is dis.	4.$\left(\!\binom{x}{n}\!\right)$	5.$\binom{x}{n}$	6.$\left(\!\!\binom{x}{n-x}\!\!\right)$
N is dis X is ind	7. $\sum_{i=1}^{x} S(n,i)$	8. 1 if $n \leq x$ o if $n > x$	9. S(n,x)
N and X are indistinct	10. $\sum_{i=1}^{x} p_i(n)$	11. 1 if $n \leq x$ o if $n > x$	12.$p_x(b)$

2 Equations 8 and 11 and the nature of Injective Functions

A cursory examination of the chart given above will immediately show that equations 8 and 11 are the exact same, and are in fact the simplest equations to understand and calculate. Why is this so? It is due to the nature of injective functions. In our metaphor, if an injective f means that each ball must have it's own box. In more strictly mathematical terms, a function $f : N \to X$ is injective iff, for each i and j in N, $f(i) \neq f(j)$ unless $i = j$. So what happens when X is indistinct? It doesn't matter which box each ball goes into, so long as each has a box. Whether the balls are distinct or not, therefore, does not matter, because there either are or are not enough boxes, and ordering is irrelevant!

3 Equations 1 and 2: A New Idea in Familiar Clothes

Here I assume a familiarity with the exponential function, such as might be expected in an student being introduced to Combinatorics. But what is it doing here? Well, think about a specific item in our group N (a specific ball). We know from our spot in the table that this item is distinct as are the boxes we might place it in. We also know there is no restriction on how we place our balls, so for each item there are x possible places for it to be. So if n=1, the possible placements are just x. but if n is 2, we still have the same x placements, but now for each of those x placements for the first ball there are x places that the second could go, leaving a total of x^2 possible placements. This reasoning follows for each of the n balls, giving us x^n.

For equation 2, we may need to introduce the idea of the floor function, but once it is introduced it's easy to see how this one works. Since each of our n balls needs it's own box, the first ball can still be in any of the x boxes, but wherever it goes the second has one less place it can be place, thus x-1, and so on to construct $(x)_n$. A clever instructor may decide to first derive the function, and only then reveal the notation.

4 Equation 4,5 and 6: The Binomial Function

Here is where our theoretical students will finally be introduced to the binomial function. I'd start with equation 5, using it to introduce the basic binomial. In our metaphor, it's worth noting that since n is indistinct it does not matter what order x is chosen in, so long as n of them are chosen. Similar to equation 2, the proof is just the definition of the binomial. Then we get to equation 4, which introduces multichoosing. Here the point to get across is that, since each n can be assigned to any x, we end up with a list of n elements of x where a single element can appear multiple times, which is what multichoosing means.

For equation 6, we finally get to a surjective function. Here let's remember that surjective means that each x needs to be assigned to at least one n, or rather that we cant have an empty box. So, following from number 4, we know that x of the n balls are assigned one to each box, and we only need to assign the remaing n-x balls.

5 Equation s 10 and 12: Intro to Recurrences

Here is where we can explain the concept of recurrence, and with it we can introduce the partition function, $p_x(n)$. Let's start with equation 12. For a group of indistinct balls we can represent them as a line of stars, and instead of 'boxes', we place a line after the first few stars, representing however many balls are in the first box. We follow this through till the end , making sure that no two lines are placed next to eachother, since that would represent an empty box. Now imagine we had a function that gave us the number of possible arrangements, that is, the exact answer we're looking for. Let's call int $p_x(n)$. Now consider the last partition. Either that partition has one member, or more than one. If it only has one, then the remaining $n-1$ balls are in $x-1$ boxes, so they can be counted by $p_{x-1}(n-1)$. If there are more than one ball in the final box, then there are $p_x(n-1)$ possible ways to arrange the first n-1 balls. So now we can show that the recurrence $p_x(n) = p_{x-1}(n-1) + p_x(n-1)$. And then we can explain recurrence relations, give the base cases, and go on to equation 10, which no longer has the restriction of filling all the boxes, and so we can see that it will be the sum of partitions of n into any number up to x boxes, as given.

6 Equations 9, 7 and 3: Recurring Recurrences

Now we are armed with the knowledge to tackle the Stirling numbers. I'll start with equation 9, since it is just the Stirling numbers. At this point we can use a very similar argument, asking how many ways can the last ball be placed? If all the boxes have balls already there are x choices, but if one of them doesn't, there is only one choice. So we get $S(n,x) = xS(n-1,x) + S(n-1,x-1)$ as a recurrence that will satisfy our needs. Just as we did for 10 above, 7 is just the sum of all the possible surjective counts of any number of boxes up to x. For 3, we already know how many ways we can fill x boxes if they aren't labeled (it's given in equation 9) so now we should ask how many different ways can we label those boxes? the answer is x! We can actually show this with equation 2, if we treat labels as balls and boxes as boxes, and know that the number of labels = the number of boxes, and know that each label needs to go with one box, we can see that we just get $(x)_x$. but this is just $1 * 2 * 3 ... * x = x!$. And that is just about the best way I can think of to finish up the proofs of the 12 fold way.

7 In Conclusion

I see the 12 fold way as serving two possible purposes, first as a sort of "cheat sheet" for combinatorics problems, and second as a teaching aid. As a cheat sheet, it's a very clever way to take a really broad metaphor and break it down into useful chunks, and I imagine it has saved a good deal of time for many provers, from undergrad assignments to published papers, but it certainly doesn't allow you to just mindlessly plug numbers in, since it can be quite tricky to see exactly what's what in a given problem. That said, the metaphor is so powerful that I suspect I'll find some reason to reach for it in the future, assuming any sort of complex counting comes up.

But it is as a teaching tool where I believe the 12 fold way really shines. Even though by the time I started this project I already knew about all the basic concepts discussed, it helped me to more intuitively understand injection, surjection and bijection, concepts which I've understood for at least 2 years, but until now I would have had to think pretty hard to remember exactly which was which. It also highlights fascinating connections between apparently distinct equations. Well worth the study.