MATH325: Discrete Math 2
Assignment 6

Questions are taken from Rosen, Discrete Mathematics and Applications, 6th edition. When I write "Exercise 5.1.2", I mean "Exercise 2 of section 5.1".

The questions are on the Inclusion-Exclusion Principle.

Q1. Exercise 7.5.1.

SOLUTION.

Let

$$A_1 = \text{set of 12 elements}$$
$$A_2 = \text{set of 18 elements}$$

a) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 0 elements. Then,

$$\begin{aligned} &|A_1 \cup A_2| \\ &\qquad = (|A_1| + |A_2|) \\ &\qquad\quad - (|A_1 \cap A_2|) \\ &\qquad = 18 + 12 - 0 = 30 \end{aligned}$$

ANSWER: $\boxed{30}$

b) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 1 elements. Then,

$$\begin{aligned} &|A_1 \cup A_2| \\ &\qquad = (|A_1| + |A_2|) \\ &\qquad\quad - (|A_1 \cap A_2|) \\ &\qquad = 18 + 12 - 1 = 29 \end{aligned}$$

ANSWER: $\boxed{29}$

c) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 6 elements. Then,

$$\begin{aligned} &|A_1 \cup A_2| \\ &\qquad = (|A_1| + |A_2|) \\ &\qquad\quad - (|A_1 \cap A_2|) \\ &\qquad = 18 + 12 - 6 = 24 \end{aligned}$$

ANSWER: $\boxed{24}$

d) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 12 elements. Then,

$$\begin{aligned} |A_1 \cup A_2| & \\ &= (|A_1| + |A_2|) \\ &\quad - (|A_1 \cap A_2|) \\ &= 18 + 12 - 12 = 18 \end{aligned}$$

ANSWER: $\boxed{18}$

Q2. Exercise 7.5.2.

SOLUTION. Let

$$A = \text{set of 345 students taken calculus}$$
$$B = \text{set of 212 students taken discrete math}$$

a) By the inclusion-exclusion principle, if $|A \cap B|$ is the set of 188 students taken both calculus and discrete math. Then,

$$\begin{aligned} &|A \cup B| \\ &\quad = (|A| + |B|) \\ &\qquad - (|A \cap B|) \\ &\quad = 345 + 212 - 188 = 369 \end{aligned}$$

ANSWER: $\boxed{369}$

Q3. Exercise 7.5.3. SOLUTION PROVIDED.

SOLUTION.

Let

$$\begin{aligned} A =&\ \text{set of households with at least one television set} \\ B =&\ \text{set of households with telephone service} \end{aligned}$$

Let X be the set of all households. We are given:

$$\frac{|A|}{|X|} = 0.96, \quad \frac{|B|}{|X|} = 0.98, \quad \frac{|A \cap B|}{|X|} = 0.95$$

By the inclusion-exclusion principle,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

we have

$$\begin{aligned} \frac{|A \cup B|}{|X|} &= \frac{|A|}{|X|} + \frac{|B|}{|X|} - \frac{|A \cap B|}{|X|} \\ &= 0.96 + 0.98 - 0.95 \\ &= 0.99 \end{aligned}$$

From

$$|X| = |\overline{A \cup B}| + |A \cup B|$$

we have

$$\begin{aligned} 1 &= \frac{|\overline{A \cup B}|}{|X|} + \frac{|A \cup B|}{|X|} \\ &= \frac{|\overline{A \cup B}|}{|X|} + 0.99 \end{aligned}$$

and hence

$$\frac{|\overline{A \cup B}|}{|X|} = 1 - 0.99 = 0.01$$

Therefore the percentage of households with neither telephone service nor a television set is 1%.

ANSWER: $\boxed{1\%}$

Q4. Exercise 7.5.4.

SOLUTION.

Let

$$A = \text{set of people going to by by a modem (650,000)}$$
$$B = \text{set of people going to by by a software package (1,250,000)}$$

By the inclusion-exclusion principle, if $|A \cup B|$ is the set of people going to by by a modem or a software package (1,450,000). Then,

$$\begin{aligned}
|A \cap B| & \\
&= (|A| + |B|) \\
&\quad - (|A \cup B|) \\
&= 650,000 + 1,250,000 - 1,450,000 = 450,000
\end{aligned}$$

ANSWER: $\boxed{450,000}$

Q5. Exercise 7.5.5.

SOLUTION.

Let

$$\begin{aligned} A_1 &= \text{set of 100 elements} \\ A_2 &= \text{set of 100 elements} \\ A_3 &= \text{set of 100 elements} \end{aligned}$$

a) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 0 elements, $|A_1 \cap A_3|$ is the set of 0 elements, $|A_2 \cap A_3|$ is the set of 0 elements, $|A_1 \cap A_2 \cap A_3|$ is the set of 0 elements. Then,

$$\begin{aligned} &|A_1 \cup A_2| \\ &\quad = (|A_1| + |A_2| + |A_3|) \\ &\qquad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\ &\qquad + (|A_1 \cap A_2 \cap A_3|) \\ &\quad = (100 + 100 + 100) - (0 + 0 + 0) + (0) = 300 \end{aligned}$$

ANSWER: $\boxed{300}$

b) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 50 elements, $|A_1 \cap A_3|$ is the set of 50 elements, $|A_2 \cap A_3|$ is the set of 50 elements, $|A_1 \cap A_2 \cap A_3|$ is the set of 0 elements. Then,

$$\begin{aligned} &|A_1 \cup A_2| \\ &\quad = (|A_1| + |A_2| + |A_3|) \\ &\qquad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\ &\qquad + (|A_1 \cap A_2 \cap A_3|) \\ &\quad = (100 + 100 + 100) - (50 + 50 + 50) + (0) = 150 \end{aligned}$$

ANSWER: $\boxed{150}$

c) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 50 elements, $|A_1 \cap A_3|$ is the set of 50 elements, $|A_2 \cap A_3|$ is the set of 50 elements, $|A_1 \cap A_2 \cap A_3|$ is the set of 25 elements.

Then,

$$\begin{aligned}
&|A_1 \cup A_2| \\
&\quad = (|A_1| + |A_2| + |A_3|) \\
&\quad\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\
&\quad\quad + (|A_1 \cap A_2 \cap A_3|) \\
&\quad = (100 + 100 + 100) - (50 + 50 + 50) + (25) = 175
\end{aligned}$$

ANSWER: $\boxed{175}$

d) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 100 elements, $|A_1 \cap A_3|$ is the set of 100 elements, $|A_2 \cap A_3|$ is the set of 100 elements, $|A_1 \cap A_2 \cap A_3|$ is the set of 100 elements. Then,

$$\begin{aligned}
&|A_1 \cup A_2| \\
&\quad = (|A_1| + |A_2| + |A_3|) \\
&\quad\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\
&\quad\quad + (|A_1 \cap A_2 \cap A_3|) \\
&\quad = (100 + 100 + 100) - (0 + 0 + 0) + (0) = 100
\end{aligned}$$

ANSWER: $\boxed{100}$

Q6. Exercise 7.5.6.

SOLUTION.

Let

$$\begin{aligned}A_1 &= \text{set of 100 elements}\\A_2 &= \text{set of 1000 elements}\\A_3 &= \text{set of 10000 elements}\end{aligned}$$

a) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 100 elements, $|A_1 \cap A_3|$ is the set of 100 elements, $|A_2 \cap A_3|$ is the set of 100 elements, $|A_1 \cap A_2 \cap A_3|$ is the set of 100 elements. Then,

$$\begin{aligned}&|A_1 \cup A_2|\\&\quad= (|A_1| + |A_2| + |A_3|)\\&\qquad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|)\\&\qquad + (|A_1 \cap A_2 \cap A_3|)\\&\quad= (100 + 1000 + 10000) - (100 + 100 + 100) + (100) = 10900\end{aligned}$$

ANSWER: $\boxed{10,900}$

b) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 0 elements, $|A_1 \cap A_3|$ is the set of 0 elements, $|A_2 \cap A_3|$ is the set of 0 elements, $|A_1 \cap A_2 \cap A_3|$ is the set of 0 elements. Then,

$$\begin{aligned}&|A_1 \cup A_2|\\&\quad= (|A_1| + |A_2| + |A_3|)\\&\qquad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|)\\&\qquad + (|A_1 \cap A_2 \cap A_3|)\\&\quad= (100 + 1000 + 10000) - (0 + 0 + 0) + (0) = 11100\end{aligned}$$

ANSWER: $\boxed{11100}$

c) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 2 elements, $|A_1 \cap A_3|$ is the set of 2 elements, $|A_2 \cap A_3|$ is the set of 2 elements, $|A_1 \cap A_2 \cap A_3|$ is the set of 1 elements.

Then,

$$
\begin{aligned}
|A_1 \cup A_2| & \\
&= (|A_1| + |A_2| + |A_3|) \\
&\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\
&\quad + (|A_1 \cap A_2 \cap A_3|) \\
&= (100 + 1000 + 10000) - (2 + 2 + 2) + (1) = 11095
\end{aligned}
$$

ANSWER: $\boxed{11095}$

Q7. Exercise 7.5.7. SOLUTION PROVIDED.

SOLUTION.

Let

$$\begin{aligned}
U &= \text{set of CS students} \\
P &= \text{set of CS students who took Pascal} \\
F &= \text{set of CS students who took Fortran} \\
C &= \text{set of CS students who took C}
\end{aligned}$$

We are given the following facts

$$\begin{aligned}
|U| &= 2504 \\
|P| &= 1876 \\
|F| &= 999 \\
|C| &= 345 \\
|P \cap F| &= 876 \\
|F \cap C| &= 231 \\
|P \cap C| &= 290 \\
|P \cap F \cap C| &= 189
\end{aligned}$$

The required number is $|\overline{P \cap F \cap C}|$ and by the inclusion-exclusion principle,

$$\begin{aligned}
|\overline{P \cap F \cap C}| &= |U| \\
&\quad - (|P| + |F| + |C|) \\
&\quad + (|P \cap F| + |F \cap C| + |P \cap C|) \\
&\quad - (|P \cap F \cap C|) \\
&= 2504 \\
&\quad - (1876 + 999 + 345) \\
&\quad + (876 + 231 + 290) \\
&\quad - (189) \\
&= 492
\end{aligned}$$

ANSWER: $\boxed{492}$

Q8. Exercise 7.5.8.

SOLUTION.

Let

$U =$ is the universe iin this case it is 270 college students$S$ $=$ set of 64 students that like brussels spr
$B =$ set of 94 students that like broccoli
$C =$ set of 58 students that like cauliflower

a) By the inclusion-exclusion principle, if $|S \cap B|$ is the set of 26 students that like brussels sprouts and broccoli, $|S \cap C|$ is the set of 28 students that like brussels sprouts and cauliflower, $|B \cap C|$ is the set of 22 students that like broccoli and cauliflower, $|S \cap B \cap C|$ is the set of 14 students that like all the vegtables. Then,

$$\begin{aligned}
&|\overline{S \cup B \cup C}| \\
&\quad = |U| \\
&\qquad - (|S| + |B| + |C|) \\
&\qquad + (|S \cap B| + |S \cap C| + |B \cap C|) \\
&\qquad - (|S \cap B \cap C|) \\
&\quad = (270) - (64 + 94 + 58) + (26 + 28 + 22) - (14) = 116
\end{aligned}$$

ANSWER: $\boxed{116}$

Q9. Exercise 7.5.9.

SOLUTION.

Let

$$
\begin{aligned}
C &= \text{set of 507 students taking calculus} \\
M &= \text{set of 64 students taking discrete} \\
S &= \text{set of 94 students taking data stuctures} \\
P &= \text{set of 58 students taking programming languages}
\end{aligned}
$$

a) By the inclusion-exclusion principle, if $|C \cap M|$ is the set of 0 students that take calculus and discrete, $|C \cap S|$ is the set of 14 students that take calculus and data stuctures, $|C \cap P|$ is the set of 213 students that take calculus and programming languages, $|M \cap S|$ is the set of 211 students that take discrete and data stuctures, $|M \cap P|$ is the set of 43 students that take discrete and programming languages, $|S \cap P|$ is the set of 0 students that take data stuctures and programming languages, $|C \cap M \cap S|$ is the set of 0 students that take calculus and discrete and data stuctures, $|C \cap M \cap P|$ is the set of 0 students that take calculus and discrete and programming languages, $|M \cap S \cap P|$ is the set of 0 students that take discrete and and programming languages, $|C \cap S \cap P|$ is the set of 0 students that take calculus and data stuctures and programming languages, $|C \cap M \cap S \cap P|$ is the set of 0 students that take all four classes. Then,

$$
\begin{aligned}
&|C \cup M \cup S \cup P| \\
&\qquad = (|C| + |M| + |S| + |P|) \\
&\qquad\quad - (|C \cap M| + |C \cap S| + |C \cap P| + |M \cap S| + |M \cap P| + |S \cap P|) \\
&\qquad\quad + (|C \cap M \cap S| + |C \cap M \cap P| + |M \cap S \cap P| + |C \cap S \cap P|) \\
&\qquad\quad - (|C \cap M \cap S \cap P|) \\
&\qquad = (507 + 292 + 312 + 344) - (0 + 14 + 213 + 211 + 43 + 0) \\
&\qquad - (0 + 0 + 0 + 0) + 0 = 974
\end{aligned}
$$

ANSWER: $\boxed{974}$

Q10. Exercise 7.5.10.

SOLUTION.

Let

$$\begin{aligned}
U &= \text{is the universe iin this case it is 100 numbers} \\
A &= \text{set of 20 numbers divisable by 5 } (\left\lfloor \frac{100}{5} \right\rfloor) \\
B &= \text{set of 14 numbers divisable by 7 } (\left\lfloor \frac{100}{7} \right\rfloor)
\end{aligned}$$

By the inclusion-exclusion principle, if $|A \cap B|$ is the set of 2 numbers divisible by 5×7 $(\lfloor \frac{100}{5\times 7} \rfloor)$, Then,

$$\begin{aligned}
&|\overline{A \cup B}| \\
&\qquad = |U| \\
&\qquad\quad - (|A| + |B|) \\
&\qquad\quad + (|A \cap B|) \\
&\qquad = (100) - (20 + 14) + (2) = 68
\end{aligned}$$

ANSWER: $\boxed{68}$

Q11. Exercise 7.5.11.

SOLUTION.

Let

$$U = \text{is the universe iin this case it is 100 numbers}$$
$$A = \text{set of 50 numbers divisable by 2 } \left(\left\lfloor \frac{100}{2} \right\rfloor\right)$$
$$B = \text{set of 10 numbers that are the square of an integer not exceeding 100}$$

By the inclusion-exclusion principle, if $|A \cap B|$ is the set of 5 numbers divisible by 2 and are the square of an integer not exceeding 100, Then,

$$\begin{aligned}
&|\overline{A \cup B}| \\
&\quad = |U| \\
&\quad\quad - (|A| + |B|) \\
&\quad\quad + (|A \cap B|) \\
&\quad = (100) - (50 + 10) + (5) = 55
\end{aligned}$$

ANSWER: $\boxed{55}$

Q12. Exercise 7.5.12.

SOLUTION.

Let

$$
\begin{aligned}
U &= \text{is the universe iin this case it is 100 numbers} \\
A &= \text{set of 31 numbers that are the square of an integer not exceeding 1000} \\
B &= \text{set of 10 numbers that are the cube of an integer not exceeding 1000}
\end{aligned}
$$

By the inclusion-exclusion principle, if $|A \cap B|$ is the set of 2 numbers are the square and the cube of an integer not exceeding 1000, Then,

$$
\begin{aligned}
&|\overline{A \cup B}| \\
&\qquad = |U| \\
&\qquad\quad - (|A| + |B|) \\
&\qquad\quad + (|A \cap B|) \\
&\qquad = (100) - (31 + 10) + (2) = 61
\end{aligned}
$$

ANSWER: $\boxed{61}$

Q13. Exercise 7.5.13.

SOLUTION.

Let us pretend that the 6 consecutive 0's are one thing so now we are filling three spots instead of 8 spots (gluing technique).

Assume the 6 consecutive 0's are at the begining so there are 2^2 possibilities for the reamining spots.
Assume the 6 consecutive 0's are in the middle so there are 2 possibilities for the reamining spot (one must be the first thing).
Assume the 6 consecutive 0's are in the middle so there are 2 possibilities for the reamining spot (one must be the first thing).
Assume the 6 consecutive 0's are at the end so there are 2 possibilities for the reamining spot (one must be in one of the first two spots).

Altogether there are $2^2 + 2 + 2 = 8$

ANSWER: $\boxed{8}$

Q14. Exercise 7.5.14

SOLUTION.

The number of permutations without rat, fish, or bird, is the total number of permutaions of the alpabet 26!

minus those with fish (26 - 4 + 1)! = 23!, those with bird (26 - 4 + 1)! = 23!, and those with rat (26 - 3 + 1)! = 24!

plus those with rat and fish (26 - 7 + 2)! = 21!, those with fish and bird 0, and rat and bird 0

minus those with all three 0

Altogether 26! - (23! + 23! + 24!) + 21! - 0 = 4.02619×10^{26}

ANSWER: $\boxed{4.02619 \times 10^{26}}$

Q15. Exercise 7.5.15. SOLUTION PROVIDED.

SOLUTION.

Let

$$\begin{aligned}
U &= \text{set of permutations of the 10 digits} \\
A &= \text{set of permutations in } U \text{ that begin with 987} \\
B &= \text{set of permutations in } U \text{ that contain 45 in the 5th,6th positions} \\
C &= \text{set of permutations in } U \text{ that end with 123}
\end{aligned}$$

The required number is $|A \cup B \cup C|$. By the inclusion-exclusion principle

$$\begin{aligned}
|A \cup B \cup C| = {} & |A| + |B| + |C| \\
& - (|A \cap B| + |A \cap C| + |B \cap C|) \\
& + (|A \cap B \cap C|)
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
|A| &= 7! \\
|B| &= 8! \\
|C| &= 7! \\
|A \cap B| &= 5! \\
|A \cap C| &= 4! \\
|B \cap C| &= 5! \\
|A \cap B \cap C| &= 2!
\end{aligned}$$

Hence

$$\begin{aligned}
|A \cup B \cup C| &= 7! + 8! + 7! \\
& \quad - (5! + 4! + 5!) \\
& \quad + (2!) \\
&= 50138
\end{aligned}$$

ANSWER: $\boxed{50138}$

Q16. Exercise 7.5.16.

SOLUTION.

Let

$$\begin{aligned}A_1 &= \text{set of 100 elements}\\ A_2 &= \text{set of 100 elements}\\ A_3 &= \text{set of 100 elements}\\ A_4 &= \text{set of 100 elements}\end{aligned}$$

a) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 50 elements, $|A_1 \cap A_3|$ is the set of 50 elements, $|A_1 \cap A_4|$ is the set of 50 elements, $|A_2 \cap A_3|$ is the set of 50 elements, $|A_2 \cap A_4|$ is the set of 50 elements, $|A_3 \cap A_3|$ is the set of 50 elements, $|A_1 \cap A_2 \cap A_3|$ is the set of 25 elements, $|A_1 \cap A_2 \cap A_4|$ is the set of 25 elements, $|A_1 \cap A_3 \cap A_4|$ is the set of 25 elements, $|A_2 \cap A_3 \cap A_4|$ is the set of 25 elements, $|A_1 \cap A_2 \cap A_3 \cap A_4|$ is the set of 5 elements. Then,

$$\begin{aligned}&|A_1 \cup A_2 \cup A_3 \cup A_4|\\ &\quad = (|A_1| + |A_2| + |A_3| + |A_4|)\\ &\qquad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|)\\ &\qquad + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4|)\\ &\qquad - (|A_1 \cap A_2 \cap A_3 \cap A_4|)\\ &\quad = (100 + 100 + 100 + 100) - (50 + 50 + 50 + 50 + 50 + 50)\\ &\quad + (25 + 25 + 25 + 25) - 5 = 195\end{aligned}$$

ANSWER: $\boxed{195}$

Q17. Exercise 7.5.17.

Let

$$\begin{aligned} A_1 &= \text{set of 50 elements} \\ A_2 &= \text{set of 60 elements} \\ A_3 &= \text{set of 70 elements} \\ A_4 &= \text{set of 80 elements} \end{aligned}$$

a) By the inclusion-exclusion principle, if $|A_1 \cap A_2|$ is the set of 5 elements, $|A_1 \cap A_3|$ is the set of 5 elements, $|A_1 \cap A_4|$ is the set of 5 elements, $|A_2 \cap A_3|$ is the set of 5 elements, $|A_2 \cap A_4|$ is the set of 5 elements, $|A_3 \cap A_3|$ is the set of 5 elements, $|A_1 \cap A_2 \cap A_3|$ is the set of 1 elements, $|A_1 \cap A_2 \cap A_4|$ is the set of 1 elements, $|A_1 \cap A_3 \cap A_4|$ is the set of 1 elements, $|A_2 \cap A_3 \cap A_4|$ is the set of 1 elements, $|A_1 \cap A_2 \cap A_3 \cap A_4|$ is the set of 0 elements. Then,

$$\begin{aligned} &|A_1 \cup A_2 \cup A_3 \cup A_4| \\ &= (|A_1| + |A_2| + |A_3| + |A_4|) \\ &\quad - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\ &\quad + (|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \\ &\quad - (|A_1 \cap A_2 \cap A_3 \cap A_4|) \\ &= (50 + 60 + 70 + 80) - (5 + 5 + 5 + 5 + 5 + 5) + (1 + 1 + 1 + 1) - 0 = 234 \end{aligned}$$

ANSWER: $\boxed{234}$