MATH325: Discrete Math 2
Assignment 9

This assignment involves converting rational functions to power series and partial fractions.

The two most important formulas for converting rational functions to power series are

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

and

$$\left(\frac{1}{1-x}\right)^k = \sum_{n=0}^{\infty} \binom{k+n-1}{n} x^n$$

(Well ... actually the second formula include the first ... $k = 1$ in the second gives you the first.) In order to use both, remember that x is a variable. For instance you can think of the first formula as:

$$\frac{1}{1-\text{BLAH}} = \sum_{n=0}^{\infty} \text{BLAH}^n$$

So for instance this is also true

$$\frac{1}{1-12345x} = \sum_{n=0}^{\infty} (12345x)^n = \sum_{n=0}^{\infty} 12345^n x^n$$

Remember that if you want to test the equality above with a specific value for x, you can only use a value for x such that $|12345x| < 1$, i.e., $|x| < 1/12345$. If you go outside this range, the power series will very likely blow up in your face. Also,

$$\left(\frac{1}{1-123x^{567}}\right)^{999} = \sum_{n=0}^{\infty} \binom{999+n-1}{n} (123x^{567})^n = \sum_{n=0}^{\infty} \binom{999+n-1}{n} 123^n x^{567n}$$

Note however that the two 1's must be 1's:

$$\frac{\underline{1}}{\underline{1}-x} = \sum_{n=0}^{\infty} x^n$$

As shown in class if they are not 1's, then you just ... well ... make them 1:

$$\frac{111}{222-333x} = 111\frac{1}{222-333x} = \frac{111}{222}\frac{1}{1-333x/222}$$

In questions where you are asked to read a coefficient, if the value is too big, you can simply tidy up the expression and leave it as it is, i.e., you need not evaluate the expression to get a value. You can leave huge powers, binomial coefficients, etc. alone.

Q1. (a) Rewrite

$$\frac{1}{1-2x}$$

as a power series.

(b) What is the coefficient of x^{1000}?

SOLUTION.

(a)

$$\begin{aligned}\frac{1}{1-2x}& \\ &= \sum_{n=0}^{\infty}(2x)^n \\ &= \sum_{n=0}^{\infty} 2^n x^n\end{aligned}$$

ANSWER: $\boxed{\sum_{n=0}^{\infty} 2^n x^n}$

(b) $x^{1000} = 2^{1000}$

ANSWER: $\boxed{2^{1000}}$

Q2. (a) Rewrite

$$\frac{1}{5x-3}$$

as a power series.

[HINT: Remember to make the above like $\frac{1}{1-\text{BLAH}}$.]

(b) What is the coefficient of x^{1000}?

SOLUTION.

(a)

$$\begin{aligned}
&\frac{1}{5-3x} \\
&= \frac{1}{-3x+5} \\
&= \frac{1}{-3}\frac{1}{1-\frac{5x}{3}} \\
&= \frac{1}{-3}\sum_{n=0}^{\infty}\left(\frac{5x}{3}\right)^n \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{-3}\times\frac{5^n}{3^n}\right)x^n \\
&= \sum_{n=0}^{\infty}\frac{5^n}{-3^{n+1}}x^n
\end{aligned}$$

ANSWER: $\boxed{\sum_{n=0}^{\infty}\frac{5^n}{-3^{n+1}}x^n}$

(b) $x^{1000} = \frac{5^{1000}}{-3^{1000+1}} = \frac{5^{1000}}{-3^{1001}}$

ANSWER: $\boxed{\frac{5^{1000}}{-3^{1001}}}$

Q3. (a) Rewrite

$$\left(\frac{1}{2x-3}\right)^5$$

as a power series.

(b) What is the coefficient of x^3?

SOLUTION.

(a)

$$\begin{aligned}
&\left(\frac{1}{2x-3}\right)^5 \\
&= \left(\frac{1}{-3+2x}\right)^5 \\
&= \left(\frac{1}{-3}\right)^5 \times \left(\frac{1}{1-\frac{2x}{3}}\right)^5 \\
&= \frac{1}{-3^5}\sum_{n=0}^{\infty}\binom{5+n-1}{n}\left(\frac{2x}{3}\right)^n \\
&= \sum_{n=0}^{\infty}\binom{4+n}{n}\frac{2^n}{-3^{n+5}}x^n
\end{aligned}$$

ANSWER: $\boxed{\sum_{n=0}^{\infty}\binom{4+n}{n}\frac{2^n}{-3^{n+5}}x^n}$

(b) $x^3 = \binom{4+3}{3}\frac{2^3}{-3^{3+5}} = \binom{7}{3}\frac{2^3}{-3^8}$

ANSWER: $\boxed{\binom{7}{3}\frac{2^3}{-3^8}}$

Q4. (a) Rewrite

$$\left(\frac{1}{1-4x^2}\right)^5$$

as a power series.

(b) What is the coefficient of x^{1000}?

(c) What is the coefficient of x^{1001}?

SOLUTION.

(a)

$$\begin{aligned}
&\left(\frac{1}{1-4x^2}\right)^5 \\
&\quad = \sum_{n=0}^{\infty} \binom{5+n-1}{n} (4x^2)^n \\
&\quad = \sum_{n=0}^{\infty} \binom{4+n}{n} (4^n)\, x^{2n}
\end{aligned}$$

ANSWER: $\boxed{\sum_{n=0}^{\infty} \binom{4+n}{n} (4^n)\, x^{2n}}$

(b) x^{1000} so $n = 500$

ANSWER: $\boxed{\binom{504}{500} 4^{500}}$

(c) x^{1001} so n is never odd in the summation

ANSWER: $\boxed{0}$

Q5. (a) Solve for A and B where

$$\frac{1}{(1-2x)(3x-1)} = \frac{A}{1-2x} + \frac{B}{3x-1}$$

(b) Rewrite

$$f(x) = \frac{1}{(1-2x)(3x-1)}$$

as a power series.

(c) What is the coefficient of x^{1000} of $f(x)$?

SOLUTION.

(a) When we multiply

$$\frac{1}{(1-2x)(3x-1)} = \frac{A}{1-2x} + \frac{B}{3x-1} \tag{1}$$

with $(1-2x)(3x-1)$, we get

$$1 = A(3x-1) + B(1-2x) \tag{2}$$

Substituting $x = \frac{1}{2}$ into (2), we get

$$\begin{aligned}
1 &= A(\frac{3}{2}-1) + B(1-\frac{2}{2}) \\
1 &= A(\frac{1}{2}) + B(0) \\
1 &= A(\frac{1}{2}) \\
\therefore \quad A &= 2
\end{aligned} \tag{3}$$

Substituting $x = \frac{1}{3}$ into (2), we get

$$\begin{aligned}
1 &= A(\frac{3}{3}-1) + B(1-\frac{2}{3}) \\
1 &= A(0) + B(\frac{1}{3}) \\
1 &= B(\frac{1}{3}) \\
\therefore \quad B &= 3
\end{aligned} \tag{4}$$

ANSWER: $\boxed{A = 2, \qquad B = 3}$

(b) Substituting (3) and (4) into (1) we get

$$\begin{aligned}
f(x) &= \frac{1}{(1-2x)(3x-1)} \\
&= 2\frac{1}{1-2x} + 3\frac{1}{3x-1} \\
&= 2\frac{1}{1-2x} + -3\frac{1}{1-3x} \\
&= \sum_{n=0}^{\infty} 2(2x)^n + \sum_{n=0}^{\infty} -3(3x)^n \\
&= \sum_{n=0}^{\infty} 2^{n+1}x^n + \sum_{n=0}^{\infty} -3^{n+1}x^n \\
&= \sum_{n=0}^{\infty} \left(2^{n+1} - 3^{n+1}\right) x^n
\end{aligned}$$

ANSWER: $\boxed{\sum_{n=0}^{\infty} \left(2^{n+1} - 3^{n+1}\right) x^n}$

(c) $x^{1000} = 2^{1000+1} - 3^{1000+1} = 2^{1001} - 3^{1001}$

ANSWER: $\boxed{2^{1001} - 3^{1001}}$