MATH325: Discrete Math 2
Assignment 7

Questions are taken from Rosen, Discrete Mathematics and Applications, 6th edition. When I write "Exercise 5.1.2", I mean "Exercise 2 of section 5.1".

The questions are on the Inclusion-Exclusion Principle.

Q1. Exercise 7.6.1.

SOLUTION.

Let the set $|U|$ be the universe consisting of all the apples there are 100.
Let the set $|A|$ be the apples with worms there are 20 elements in this set.
Let the set $|B|$ be the apples with bruises there are 15 elements in this set.
Let the set $|A \cap B|$ be the apples with worms and bruises there are 10 elements in this set.

We need to find the set $|\overline{A \cup B}|$ be the apples without worms and bruises.
So,

$$
\begin{aligned}
|\overline{A \cup B}| &= \\
&= |U| \\
&- (|A| + |B|) \\
&+ (|A \cap B|)
\end{aligned}
$$

Altogether, $100 - (20 + 15) + 10 = 75$

ANSWER: $\boxed{75}$

Q2. Exercise 7.6.2.

SOLUTION.

Let the set $|U|$ be the universe consisting of all the applicants there are 1000.
Let the set $|A|$ be the applicants with altitude sickness there are 450 elements in this set.
Let the set $|B|$ be the applicants with poor-shape there are 622 elements in this set.
Let the set $|C|$ be the applicants with allergies there are 30 elements in this set.
Let the set $|A \cap B|$ be the applicants with altitude sickness and poor-shape there are 111 elements in this set.
Let the set $|A \cap C|$ be the applicants with altitude sickness and allergies there are 14 elements in this set.
Let the set $|B \cap C|$ be the applicants with poor-shape and allergies there are 18 elements in this set.
Let the set $|A \cap B \cap C|$ be the applicants with altitude sickness, poor-shape, and allergies there are 9 elements in this set.

We need to find the set $|\overline{A \cup B}|$ be the....
So,

$$
\begin{aligned}
|\overline{A \cup B}| = & \\
& = |U| \\
& - (|A| + |B| + |C|) \\
& + (|A \cap B| + |A \cap C| + |B \cap C|) \\
& - (|A \cap B \cap C|)
\end{aligned}
$$

Altogether, $1000 - (450 + 622 + 30) + (111 + 14 + 18) - 9 = 32$

ANSWER: $\boxed{32}$

Q3. Exercise 7.6.3. SOLUTION PROVIDED.

SOLUTION.

Let $U = \{(x_1, x_2, x_3 \mid x_i \text{ are all non-negative integers such that}, x_1 + x_2 + x_3 = 13\}$. We are interested in the set of solution (x_1, x_2, x_3) such that

$$\begin{aligned}
&x_1 + x_2 + x_3 = 13 \\
&0 \leq x_1 \leq 5,\ \ 0 \leq x_2 \leq 5,\ \ 0 \leq x_3 \leq 5 \\
&x_1, x_2, x_3 \in \mathbb{Z}
\end{aligned}$$

Define the sets

$$\begin{aligned}
A &= \{(x_1, x_2, x_3) \in U \mid 6 \leq x_1\} \\
B &= \{(x_1, x_2, x_3) \in U \mid 6 \leq x_2\} \\
C &= \{(x_1, x_2, x_3) \in U \mid 6 \leq x_3\}
\end{aligned}$$

Then an element (x_1, x_2, x_3) is in $\overline{A \cup B \cup C}$ exactly when x_1, x_2, x_3 are non-negative integers such that $x_1 + x_2 + x_3 = 13$ and

$$x_1 \leq 5,\ \ x_2, \leq 5,\ \ x_3 \leq 5$$

Hence the required number is $|\overline{A \cup B \cup C}|$. By the inclusion-exclusion principle

$$\begin{aligned}
|\overline{A \cup B \cup C}| = |U| & \\
&- (|A| + |B| + |C|) \\
&+ (|A \cap B| + |A \cap C| + |B \cap C|) \\
&- (|A \cap B \cap C|)
\end{aligned}$$

We now compute the terms that appear on the right.

Note that $|U|$ is the number of permutations of 13 0's and 2 1's, i.e.

$$|U| = \frac{(13+2)!}{13!2!} = \binom{15}{2}$$

$|A|$ is the number of solutions to

$$\begin{aligned}
&x_1 + x_2 + x_3 = 13 \\
&6 \leq x_1,\ \ 0 \leq x_2,\ \ 0 \leq x_3 \\
&x_1, x_2, x_3 \in \mathbb{Z}
\end{aligned}$$

which is the same as the number of solutions to

$$\begin{aligned}&x_1' + x_2' + x_3' = 13 - 6 = 7\\&0 \le x_1',\ \ 0 \le x_2',\ \ 0 \le x_3'\\&x_1', x_2', x_3' \in \mathbb{Z}\end{aligned}$$

which is the number of permutations of 7 0's and 2 1's, i.e.

$$|A| = \frac{(7+2)!}{7!2!} = \binom{9}{2}$$

Note that $|A| = |B| = |C|$.

Next, note that $|A \cap B|$ is the number of solutions to

$$\begin{aligned}&x_1 + x_2 + x_3 = 13\\&6 \le x_1,\ \ 6 \le x_2,\ \ 0 \le x_3\\&x_1, x_2, x_3 \in \mathbb{Z}\end{aligned}$$

which is the same as the number of solutions to

$$\begin{aligned}&x_1' + x_2' + x_3' = 13 - 6 - 6 = 1\\&0 \le x_1',\ \ 0 \le x_2',\ \ 0 \le x_3'\\&x_1', x_2', x_3' \in \mathbb{Z}\end{aligned}$$

which is the number of permutations of 1 0's and 2 1's, i.e.

$$|A \cup B| = \frac{(1+2)!}{1!2!} = \binom{3}{1}$$

Note that $|A \cap B| = |A \cap C| = |B \cap C|$.

Finally $|A \cap B \cap C|$ is the number of solutions to

$$\begin{aligned}&x_1 + x_2 + x_3 = 13\\&6 \le x_1,\ \ 6 \le x_2,\ \ 6 \le x_3\\&x_1, x_2, x_3 \in \mathbb{Z}\end{aligned}$$

which is clearly impossible since if x_i are all at least 6, then $x_1 + x_2 + x_3$ is at least 18. In other words $|A \cap B \cap C| = 0$.

Altogether we have

$$\begin{aligned}
|\overline{A \cup B \cup C}| &= |U| \\
&\quad - (|A| + |B| + |C|) \\
&\quad + (|A \cap B| + |A \cap C| + |B \cap C|) \\
&\quad - (|A \cap B \cap C|) \\
&= \binom{15}{2} - \left(\binom{9}{2} + \binom{9}{2} + \binom{9}{2}\right) + \left(\binom{3}{1} + \binom{3}{1} + \binom{3}{1}\right) - 0 \\
&= \frac{15 \cdot 14}{1 \cdot 2} - 3 \cdot \frac{9 \cdot 8}{1 \cdot 2} + 3 \cdot 3 - 0 \\
&= 6
\end{aligned}$$

ANSWER: $\boxed{6}$

Q4. Exercise 7.6.4.

SOLUTION.

Let $U = \{(x_1, x_2, x_3, x_4 \mid x_i$ are all non-negative integers such that, $x_1 + x_2 + x_3 + x_4 = 17\}$. We are interested in the set of solution (x_1, x_2, x_3, x_4) such that

$$\begin{aligned}&x_1 + x_2 + x_3 + x_4 = 17\\&0 \leq x_1 \leq 3,\ \ 0 \leq x_2 \leq 4,\ \ 0 \leq x_3 \leq 5,\ \ 0 \leq x_4 \leq 8\\&x_1, x_2, x_3, x_4 \in \mathbb{Z}\end{aligned}$$

Define the sets

$$\begin{aligned}A &= \{(x_1, x_2, x_3, x_4) \in U \mid 3 \leq x_1\}\\B &= \{(x_1, x_2, x_3, x_4) \in U \mid 4 \leq x_2\}\\C &= \{(x_1, x_2, x_3, x_4) \in U \mid 5 \leq x_3\}\\D &= \{(x_1, x_2, x_3, x_4) \in U \mid 8 \leq x_4\}\end{aligned}$$

Then an element (x_1, x_2, x_3, x_4) is in $\overline{A \cup B \cup C \cup D}$ exactly when x_1, x_2, x_3, x_4 are non-negative integers such that $x_1 + x_2 + x_3, x_4 = 17$ and

$$x_1 \leq 3,\ \ x_2, \leq 4,\ \ x_3 \leq 5,\ \ x_4 \leq 8$$

Hence the required number is $|\overline{A \cup B \cup C \cup D}|$. By the inclusion-exclusion principle

$$\begin{aligned}|\overline{A \cup B \cup C \cup D}| = |U|&\\&- (|A| + |B| + |C| + |D|)\\&+ (|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|)\\&- (|A \cap B \cap C| + |A \cap C \cap D| + |B \cap C \cap D|)\\&+ (|A \cap B \cap C \cap D|)\end{aligned}$$

We now compute the terms that appear on the right.

Note that $|U|$ is the number of permutations of 17 0's and 3 1's, i.e.

$$|U| = \frac{(17+3)!}{17!3!} = \binom{20}{3}$$

$|A|$ is the number of solutions to

$$\begin{aligned}&x_1 + x_2 + x_3 + x_4 = 17\\&4 \leq x_1,\ \ 0 \leq x_2,\ \ 0 \leq x_3,\ \ 0 \leq x_4\\&x_1, x_2, x_3, x_4 \in \mathbb{Z}\end{aligned}$$

which is the same as the number of solutions to

$$\begin{aligned}
&x_1' + x_2' + x_3' = 17 - 4 = 13 \\
&0 \leq x_1', \;\; 0 \leq x_2', \;\; 0 \leq x_3', \;\; 0 \leq x_4' \\
&x_1', x_2', x_3', x_4' \in \mathbb{Z}
\end{aligned}$$

which is the number of permutations of 13 0's and 3 1's, i.e.

$$|A| = \frac{(13+3)!}{13!3!} = \binom{16}{3}$$

Note that

$$\begin{aligned}
|A| &= \binom{16}{3} \\
|B| &= \binom{15}{3} \\
|C| &= \binom{14}{3} \\
|D| &= \binom{11}{3}
\end{aligned}$$

Next, note that $|A \cap B|$ is the number of solutions to

$$\begin{aligned}
&x_1 + x_2 + x_3 + x_4 = 17 \\
&4 \leq x_1, \;\; 5 \leq x_2, \;\; 0 \leq x_3, \;\; 0 \leq x_4 \\
&x_1, x_2, x_3, x_4 \in \mathbb{Z}
\end{aligned}$$

which is the same as the number of solutions to

$$\begin{aligned}
&x_1' + x_2' + x_3' = 17 - 4 - 5 = 8 \\
&0 \leq x_1', \;\; 0 \leq x_2', \;\; 0 \leq x_3', \;\; 0 \leq x_4' \\
&x_1', x_2', x_3', x_4' \in \mathbb{Z}
\end{aligned}$$

which is the number of permutations of 8 0's and 3 1's, i.e.

$$|A \cup B| = \frac{(8+3)!}{8!3!} = \binom{11}{3}$$

Note that $|A \cap B| = |A \cap C| = |A \cap D| = |B \cap C| = |B \cap D| = |C \cap D|$. Note that

$$|A \cap B| = \binom{11}{3}$$
$$|A \cap C| = \binom{10}{3}$$
$$|A \cap D| = \binom{7}{3}$$
$$|B \cap C| = \binom{9}{3}$$
$$|B \cap D| = \binom{6}{3}$$
$$|C \cap D| = \binom{5}{3}$$

Next, note that $|A \cap B \cap C|$ is the number of solutions to

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 = 17 \\ &4 \leq x_1,\ \ 5 \leq x_2,\ \ 6 \leq x_3,\ \ 0 \leq x_4 \\ &x_1, x_2, x_3, x_4 \in \mathbb{Z} \end{aligned}$$

which is the same as the number of solutions to

$$\begin{aligned} &x_1' + x_2' + x_3' = 17 - 4 - 5 - 6 = 2 \\ &0 \leq x_1',\ \ 0 \leq x_2',\ \ 0 \leq x_3',\ \ 0 \leq x_4' \\ &x_1', x_2', x_3', x_4' \in \mathbb{Z} \end{aligned}$$

which is the number of permutations of 2 0's and 3 1's, i.e.

$$|A \cup B| = \frac{(2+3)!}{2!3!} = \binom{5}{3}$$

Note that $|A \cap B \cap C| = |A \cap B \cap D| = |A \cap C \cap D| = |B \cap C \cap D|$. Note that

$$|A \cap B \cap C| = \binom{5}{3}$$

Finally $|A \cap B \cap C \cap D|$ is the number of solutions to

$$\begin{aligned}&x_1 + x_2 + x_3 + x_4 = 17\\ &4 \leq x_1,\;\; 5 \leq x_2,\;\; 6 \leq x_3,\;\; 9 \leq x_4\\ &x_1, x_2, x_3, x_4 \in \mathbb{Z}\end{aligned}$$

which is clearly impossible because the sum of x_i is greater than 17, In other words $|A \cap B \cap C \cap D| = 0$.

Altogether we have

$$\begin{aligned}|\overline{A \cup B \cup C \cup D}|&\\ &= |U|\\ &\quad - (|A| + |B| + |C| + |D|)\\ &\quad + (|A \cap B| + |A \cap C| + |A \cap B| + |B \cap C| + |B \cap D| + |C \cap D\\ &\quad - (|A \cap B \cap C| + |A \cap C \cap D| + |B \cap C \cap D|)\\ &\quad + (|A \cap B \cap C \cap D|)\\ &= \binom{20}{3}\\ -\left(\binom{16}{3} + \binom{15}{3} + \binom{14}{3} + \binom{11}{3}\right)&\\ +\left(\binom{11}{3} + \binom{10}{3} + \binom{7}{3}\right)&\\ +\left(\binom{9}{3} + \binom{6}{3} + \binom{5}{3}\right)&\\ -\left(\binom{5}{3} + 0 + 0\right)&\\ +0&\\ &= 1140\\ -(560 + 455 + 364 + 165)&\\ +(165 + 120 + 35 + 84 + 20 + 10)&\\ -(10 + 0 + 0 + 0)&\\ +0&\\ &= 20\end{aligned}$$

ANSWER: $\boxed{20}$

Q5. Exercise 7.6.5.

SOLUTION.

$|U| = 199$ This is the universe of 199 numbers.

$|A|$ This is the numbers divisible by 2. $=$

$$\left\lfloor \frac{199}{2} \right\rfloor = 99$$

$|B|$ This is the numbers divisible by 3. $=$

$$\left\lfloor \frac{199}{3} \right\rfloor = 66$$

$|C|$ This is the numbers divisible by 5. $=$

$$\left\lfloor \frac{199}{5} \right\rfloor = 39$$

$|D|$ This is the numbers divisible by 7. $=$

$$\left\lfloor \frac{199}{7} \right\rfloor = 28$$

$|E|$ This is the numbers divisible by 11. $=$

$$\left\lfloor \frac{199}{11} \right\rfloor = 18$$

$|F|$ This is the numbers divisible by 13. $=$

$$\left\lfloor \frac{199}{13} \right\rfloor = 15$$

By the principle of inclusion and exclusion we see that...

The number of the combined indivdual sets is

$$\begin{aligned} \left\lfloor \frac{199}{2} \right\rfloor + \left\lfloor \frac{199}{3} \right\rfloor + \left\lfloor \frac{199}{5} \right\rfloor + \left\lfloor \frac{199}{7} \right\rfloor + \left\lfloor \frac{199}{11} \right\rfloor + \left\lfloor \frac{199}{13} \right\rfloor \\ = 99 + 66 + 39 + 28 + 18 + 15 \\ = 265 \end{aligned}$$

The sum of the combination of 2 sets is

$$\begin{aligned}
&\left\lfloor\frac{199}{2\cdot 3}\right\rfloor+\left\lfloor\frac{199}{2\cdot 5}\right\rfloor+\left\lfloor\frac{199}{2\cdot 7}\right\rfloor+\left\lfloor\frac{199}{2\cdot 11}\right\rfloor+\left\lfloor\frac{199}{2\cdot 13}\right\rfloor+\\
&\left\lfloor\frac{199}{3\cdot 5}\right\rfloor+\left\lfloor\frac{199}{3\cdot 7}\right\rfloor+\left\lfloor\frac{199}{3\cdot 11}\right\rfloor+\left\lfloor\frac{199}{3\cdot 13}\right\rfloor+\left\lfloor\frac{199}{5\cdot 7}\right\rfloor+\\
&\left\lfloor\frac{199}{5\cdot 11}\right\rfloor+\left\lfloor\frac{199}{5\cdot 13}\right\rfloor+\left\lfloor\frac{199}{7\cdot 11}\right\rfloor+\left\lfloor\frac{199}{7\cdot 13}\right\rfloor+\left\lfloor\frac{199}{11\cdot 13}\right\rfloor\\
&=33+19+14+9+7+13+9+6+5+5+3+3+2+2+1\\
&=131
\end{aligned}$$

The number of the combination of 3 sets (not exceeding the sum of 200) is

$$\begin{aligned}
&\left\lfloor\frac{199}{2\cdot 3\cdot 5}\right\rfloor+\left\lfloor\frac{199}{2\cdot 3\cdot 7}\right\rfloor+\left\lfloor\frac{199}{2\cdot 3\cdot 11}\right\rfloor+\left\lfloor\frac{199}{2\cdot 3\cdot 13}\right\rfloor+\\
&\left\lfloor\frac{199}{2\cdot 5\cdot 7}\right\rfloor+\left\lfloor\frac{199}{2\cdot 5\cdot 11}\right\rfloor+\left\lfloor\frac{199}{2\cdot 5\cdot 13}\right\rfloor+\left\lfloor\frac{199}{2\cdot 7\cdot 11}\right\rfloor+\\
&\left\lfloor\frac{199}{2\cdot 7\cdot 13}\right\rfloor+\left\lfloor\frac{199}{3\cdot 5\cdot 7}\right\rfloor+\left\lfloor\frac{199}{3\cdot 5\cdot 11}\right\rfloor+\left\lfloor\frac{199}{3\cdot 5\cdot 13}\right\rfloor+\\
&=6+4+3+2+2+1+1+1+1+1+1+1\\
&=24
\end{aligned}$$

Any combinations beyond 3 sets is irrelevant because the smallest is $2\cdot 3\cdot 5\cdot 7=210$ so there are no numbers less than 200 in these sets they are all empty.

Altogether the universe minus the single sets plus the intersection of two sets minus the intersection of three sets plus the set of prime numbers we removed $2,3,7,11,13$ there are 6 so, $199-265+131-24+6=46$

ANSWER: $\boxed{46}$

Q6. Exercise 7.6.6.

SOLUTION.

$|U| = 99$ This is the universe of 99 numbers.

$|A|$ This is the numbers divisible by 2. =

$$\left\lfloor \frac{99}{2^2} \right\rfloor = 24$$

$|B|$ This is the numbers divisible by 3. =

$$\left\lfloor \frac{99}{3^2} \right\rfloor = 11$$

$|C|$ This is the numbers divisible by 5. =

$$\left\lfloor \frac{99}{5^2} \right\rfloor = 3$$

$|D|$ This is the numbers divisible by 7. =

$$\left\lfloor \frac{99}{7^2} \right\rfloor = 2$$

By the principle of inclusion and exclusion we see that...

The number of the combined indivdual sets is

$$\begin{aligned}\left\lfloor \frac{99}{2^2} \right\rfloor + \left\lfloor \frac{99}{3^2} \right\rfloor + \left\lfloor \frac{99}{5^2} \right\rfloor + \left\lfloor \frac{99}{7^2} \right\rfloor \\ = 24 + 11 + 3 + 2 \\ = 40\end{aligned}$$

The sum of the combination of 2 sets (not exceeding 100) is

$$\begin{aligned}\left\lfloor \frac{99}{2^2 \cdot 3^2} \right\rfloor \\ = 2\end{aligned}$$

Any combinations beyond 3 sets is irrelevant because the smallest is $2 \cdot 3 \cdot 5 \cdot 7 = 210$ so there are no numbers less than 100 in these sets they are all empty.

Altogether the universe minus the single sets plus the intersection of two sets minus the intersection of three sets plus the one because one is a square free number. $99 - 40 + 2 = 61$

ANSWER: $\boxed{61}$

Q7. Exercise 7.6.7.

SOLUTION.

[HINT: Let $U = \{x \in \mathbb{Z} \mid 2 \leq x \leq 10000\}$; the integer 1 is excluded from U since 1 is a second power, third power, fourther power, etc. Define $A_1 = \{x \in U \mid x \text{ is a second power}\}$, $A_2 = \{x \in U \mid x \text{ is a third power}\}$, $A_3 = \{x \in U \mid x \text{ is a fourth power}\}$, etc. You hope that these sets and their intersections are easier to count. If this is the case, then the inclusion-exclusion principle can be used. We only need to do the primes. For instance in the case of A_1, we have $A_1 = \{2^2, 3^2, ..., 99^2\}$. Therefore $|A_1| = 98$.]

The $|U|$ universe is 9998 numbers from 2 - 9999

$$\begin{aligned}
A_1 &= 2^2, 3^2, ..., 99^2 = 98 \\
A_2 &= 2^3, 3^3, ..., 21^3 = 20 \\
A_3 &= 2^5, 3^5, ..., 6^5 = 5 \\
A_4 &= 2^7, 3^7 = 2 \\
A_5 &= 2^{11} = 1 \\
A_6 &= 2^{13} = 1 \\
|A_1| + |A_2| + |A_3| + |A_4| + |A_5| + |A_6| & \\
&= 98 + 20 + 5 + 2 + 1 + 1 \\
&= 127
\end{aligned}$$

The deduct the over lap

$$\begin{aligned}
A_1 &= 2^6, 3^6, ..., 4^6 = 3 \\
A_2 &= 2^{10} = 1 \\
|A_1 \cap A_2| + |A_1 \cap A_3| & \\
&= 3 + 1 \\
&= 4
\end{aligned}$$

Altogether $|U| - (|A| + |B|) + (|A_1 \cap A_2| + |A_1 \cap A_3|) = 9998 - 127 + 4 = 9875$

ANSWER: $\boxed{9875}$

Q8. Exercise 7.6.8.

SOLUTION.

Let the set of 7 be $X = \{x_1, x_2, \ldots, x_7\}$ and the set of 5 be $Y = \{y_1, y_2, \ldots, y_5\}$. Each distribution is a function $f : X \to Y$. The fact that each urn gets at least one ball is the same as saying the function f is onto.

By Theorem 1

$$\begin{aligned}
&= 5^7 - C(5,1)4^7 + C(5,2)3^7 - C(5,3)2^7 + C(5,4)1^7 \\
&= 5^7 - (5 \cdot 4^7) + (10 \cdot 3^7) - (10 \cdot 2^7) + (5 \cdot 1^7) \\
&= 78125 + 81920 + 21870 - 1280 + 5 \\
&= 16800
\end{aligned}$$

ANSWER: $\boxed{16800}$

Q9. Exercise 7.6.9.

SOLUTION.

[HINT: Let the 6 toys be $T = \{t_1, t_2, \ldots, t_6\}$ and the three children be $C = \{c_1, c_2, c_3\}$. Each distribution is a function $f : T \to C$. The fact that each child gets at least one toy is the same as saying the function f is onto.]

Let the 6 toys be $T = \{t_1, t_2, \ldots, t_6\}$ and the three children be $C = \{c_1, c_2, c_3\}$.

By Theorem 1

$$\begin{aligned} 3^6 - C(3,1)2^6 + C(3,2)1^6 \\ &= 540 \end{aligned}$$

ANSWER: $\boxed{540}$

Q10. Exercise 7.6.10.

SOLUTION.

Let the 8 balls be $B = \{b_1, b_2, \ldots, b_8\}$ and the 3 urns be $U = \{u_1, u_2, u_3\}$. Each distribution is a function $f : B \to U$. The fact that each urn gets at least one ball is the same as saying the function f is onto.

By Theorem 1

$$\begin{aligned} 3^8 - C(3,1)2^8 + C(3,2)1^8 & \\ &= 6561 - 768 + 3 \\ &= 5796 \end{aligned}$$

ANSWER: $\boxed{5796}$

Q11. Exercise 7.6.11.

SOLUTION.

Let the 7 jobs be $J = \{j_1, j_2, \ldots, j_7\}$ and the four employees be $E = \{e_1, e_2, e_3, e_4\}$. Each distribution is a function $f : J \to E$. The fact that each employee gets at least one job is the same as saying the function f is onto.

Let us look at all the onto functions, but... Only a fourth of these meet the requirment that the best employee has the best job.

By Theorem 1

$$\begin{aligned}4^7 - C(4,1)3^7 + C(4,2)2^7 - C(4,3)1^7 \\ &= 8400/2 \\ &= 2100\end{aligned}$$

ANSWER: $\boxed{2100}$

Q12. Exercise 7.6.12.

SOLUTION.

According to Theorem 2 the derangement of a set with four elements is
Theorem 2:

$$\begin{aligned} D_4 &= 4!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}) \\ &= 24(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}) \\ &= 9 \end{aligned}$$

ANSWER: $\boxed{9}$

Q13. Exercise 7.6.13.

SOLUTION.

According to Theorem 2 the derangement of a set with seven elements is
Theorem 2:

$$
\begin{aligned}
D_7 &= 7!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}) \\
&= 5040 - 5040 + 2520 - 840 + 210 - 42 + 7 - 1 \\
&= 1854
\end{aligned}
$$

ANSWER: $\boxed{1854}$

Q14. Exercise 7.6.14

SOLUTION.

This the same as a derangement. According to Theorem 2 the derangement of a set with 10 elements is Theorem 2:

$$\begin{aligned}
D_{10} &= 10!(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} - \frac{1}{9!} + \frac{1}{10!}) \\
&= (10! - \frac{10!}{1!} + \frac{10!}{2!} - \frac{10!}{3!} + \frac{10!}{4!} - \frac{10!}{5!} + \frac{10!}{6!} - \frac{10!}{7!} + \frac{10!}{8!} - \frac{10!}{9!} + \frac{10!}{10!}) \\
&= 10! - 10! + \frac{3628800}{2} - \frac{3628800}{6} + \frac{3628800}{24} - \frac{3628800}{120} + \frac{3628800}{720} \\
&\qquad - \frac{3628800}{5040} + \frac{3628800}{40320} - \frac{3628800}{362880} + \frac{3628800}{3628800} \\
&= 0 + 1814400 - 604800 + 151200 - 30240 + 5040 - 720 + 90 - 10 + 1 \\
&= 1334961
\end{aligned}$$

ANSWER: $\boxed{1334961}$

Q15. Exercise 7.6.15. NOT GRADED.

SOLUTION.

Q16. Exercise 7.6.16.

SOLUTION.

This the same as a derangement of n people no person can recieve there origional position the second time around.
According to Theorem 2 the derangement of a set with n elements is
Theorem 2:

$$D_n = n!(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^n \frac{1}{n!})$$

Q17. Exercise 7.6.17.

SOLUTION.

[HINT: This is not an difficult as you think. Think of the opposite condition “digit 2 is in the original position”.]

There are 10 permutations altogether, now we need to pay attention to only 5 positions the even ones and deduct the possibility that these appear in the same place. Now we need to consider the strings that 1 of the five even positions is left in its origional spot then we add back the situations when two of the even numbers are left in their origional spot and so on... The principle of inclusion exclusion.

The number of ways to arrage the digits so that no even is left in its origional spot is

$$
\begin{aligned}
&= 10! - c(5,1)9! + c(5,2)8! - c(5,3)7! + c(5,4)6! - c(5,5)5! \\
&= 10! - 5 \cdot 9! + 10 \cdot 8! - 10 \cdot 7! + 5 \cdot 6! - 5! \\
&= 2,170,680
\end{aligned}
$$

ANSWER: $\boxed{2,170,680}$