MATH325: Discrete Math 2
Assignment 7

The questions involve working with series and power series. Here's a power series:

$$\sum_{n=0}^{\infty} x^n$$

and here's another

$$\sum_{n=0}^{\infty} \frac{1}{n^2} x^n$$

In general a power series looks like this:

$$\sum_{n=0}^{\infty} a_n x^n$$

As you can see a power series is like a polynomial except that it can "go over forever".

In case you don't remember "coefficients" of a polynomial ... The coefficient of x^2 of $1 + 3x + 5x^2 + 7x^3$ is 5. In general if you're looking at a power series

$$\sum a_n x^n$$

the coefficient of x^n, or we say that n–th coefficient, is a^n.

Here are some basic facts.

1. $\sum_{i \in I} a_i + \sum_{i \in I} b_i = \sum_{i \in I} (a_i + b_i)$
2. $c \sum_{i \in I} a_i = \sum_{i \in I} c a_i$
3. $c \sum_{i \in I} a_i + d \sum_{i \in I} b_i = \sum_{i \in I} (c a_i + d b_i)$

Etc. When a_i looks like $a_n x^n$ (we change the i to n) of course you get the following corresponding facts:

1. $\sum_{n \in I} a_n x^n + \sum_{n \in I} b_n x^n = \sum_{n \in I} (a_n + b_n) x^n$
2. $c \sum_{n \in I} a_i = \sum_{n \in I} c a_n x^n$
3. $c \sum_{n \in I} a_n x^n + d \sum_{n \in I} b_n x^n = \sum_{n \in I} (c a_n + d b_n) x^n$

Etc.

Here's the geometric series formula again:

$$\sum_{n=0} x^n = \frac{1}{1-x}$$

The left-hand side is numerically the same as the right-hand side if $|x| < 1$. The left is a power series and the right is what we call a closed form because it does not involve a sum (or a loop if you like). The expression on the right is also called a rational expression because it is a fraction of polynomials 1 and $1 - x$. The geometric series formula is therefore a tool for moving between two worlds: the world of power series and the world of rational expressions.

Q1. [READING COEFFICIENTS]

(a) What is the coefficient of x^5 in

$$\sum_{n=2}^{\infty} \left(n^2 \binom{n}{2} \right) x^n$$

Simplify your answer.

(b) What is the coefficient of x^1 in the above power series?

SOLUTION.

(a) The coefficient of x^5 is

$$5^2 \binom{5}{2} = 25 \cdot \frac{5 \cdot 4}{1 \cdot 2} = 25 \cdot 10 = 250$$

ANSWER: $\boxed{250}$ □

(b) The series begins with x^2, i.e., the x^1–term is $0x^1$.

ANSWER: $\boxed{0}$ □

Q2. [READING COEFFICIENTS]

(a) What is the coefficient of x^5 in

$$\sum_{n=2}^{\infty}\left(\frac{1}{n}+\binom{n+2}{n}\right)x^{2n}$$

Simplify your answer.

(b) What is the coefficient of x^{100} in the above power series?

SOLUTION.

$$\sum_{n=2}^{\infty}\left(\frac{1}{n}+\binom{n+2}{n}\right)x^{2n}$$

$$=\sum_{n=2}^{\infty}\left(\frac{1}{n}+\binom{n+2}{n}\right)(x^2)^n$$

(a) The coefficient of x^5 is 0 because x can only be raised to even powers because n is multiplied by 2.

ANSWER: $\boxed{0}$

(b) The coefficient of x^{100} is when $n = 50$ so $\left(\frac{1}{n}+\binom{n+2}{n}\right)$
$=\left(\frac{1}{50}+\binom{50+2}{50}\right) = 1326$

ANSWER: $\boxed{1326}$

Q3. [ADDING POWER SERIES]

(a) Rewrite the following as a *single* series:

$$\sum_{n=0}^{\infty} \frac{1}{n} x^n + \sum_{n=0}^{\infty} \frac{2n}{1+n} x^n$$

(i.e. rewrite it in the form $\sum_n a_n x^n$)

(b) What is the coefficient of x^3 of the power series in (a)?

(c) What is the coefficient of x^n of the power series in (a)?

SOLUTION.

(a)

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n} x^n + \sum_{n=0}^{\infty} \frac{2n}{1+n} x^n \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n} + \frac{2n}{1+n}\right) x^n \\ &= \sum_{n=0}^{\infty} \left(\frac{2n^2+n+1}{n^2+n}\right) x^n\end{aligned}$$

ANSWER: $\boxed{\sum_{n=0}^{\infty} (\frac{2n^2+n+1}{n^2+n}) x^n}$

(b)When x^3 then $n = 3$, $\frac{2n^2+n+1}{n^2+n}$
$= \frac{2 \cdot 3^2+3+1}{3^2+3} = 1$
ANSWER: $\boxed{\frac{11}{6}}$

(c) For all n's $\frac{2n^2+n+1}{n^2+n}$ get common denominator.
ANSWER: $\boxed{\frac{2n^2+n+1}{n^2+n}}$

Q4. [ADDING LINEAR COMBINATION OF POWER SERIES]

(a) Rewrite the following as a single series:

$$7\sum_{n=0}^{\infty}\frac{1}{n+1}x^n + 8\sum_{n=0}^{\infty}\frac{2n}{n^2+1}x^n$$

(b) What is the coefficient of x^3 of the power series in (a)?

(c) What is the coefficient of x^n of the power series in (a)?

SOLUTION.

(a)

$$\begin{aligned}
7\sum_{n=0}^{\infty}\frac{1}{n+1}x^n + 8\sum_{n=0}^{\infty}\frac{2n}{n^2+1}x^n & \\
&= \sum_{n=0}^{\infty}\frac{7}{n+1}+\frac{16n}{n^2+1}x^n \\
&= \sum_{n=0}^{\infty}\frac{7n^2+7+16n^2+16n}{n^3+n^2+n+1}x^n \\
&= \sum_{n=0}^{\infty}\frac{23n^2+16n+7}{n^3+n^2+n+1}x^n
\end{aligned}$$

ANSWER: $\boxed{\sum_{n=0}^{\infty}\frac{23n^2+16n+7}{n^3+n^2+n+1}x^n}$

(b) $\frac{23\cdot 3^2+16\cdot 3+7}{3^3+3^2+3+1}$

ANSWER: $\boxed{\frac{262}{40}}$

(c) ANSWER: $\boxed{\frac{23n^2+16n+7}{n^3+n^2+n+1}}$

Q5. [SUBSTITUTION OF EXPONENT]

(a) Rewrite the following series so that the x-term is x^n instead of x^{n-5}.

$$\sum_{n=7}^{\infty} \frac{2^n}{1+3^{n+1}} x^{n-5}$$

(b) What is the coefficient of x^0?

(c) What is the coefficient of x^{10}?

SOLUTION.

(a)

$$\begin{aligned}
&\sum_{n=7}^{\infty} \frac{2^n}{1+3^{n+1}} x^{n-5} \\
&\quad = \sum_{m=2}^{\infty} \frac{2^{m+5}}{1+3^{m+5+1}} x^m && \text{(let } m = n-5 \text{, so } n = m+5\text{)} \\
&\quad = \sum_{n=2}^{\infty} \frac{2^{n+5}}{1+3^{n+6}} x^n && \text{(change m back to n)}
\end{aligned}$$

ANSWER: $\boxed{\sum_{n=2}^{\infty} \frac{2^{n+5}}{1+3^{n+6}} x^n}$

(b) The series doesn't start until n = 2, so when n = 0 the sum is zero ANSWER: $\boxed{0}$

(c)

$$n = 10$$
$$\frac{2^{n+5}}{1+3^{n+6}}$$

ANSWER: $\boxed{\frac{32768}{43046722}}$

Q6. [MONOMIAL MULTIPLE OF POWER SERIES]

(a) Rewrite the following series as a power series in x^n terms.

$$x^3 \sum_{n=2}^{\infty} \binom{n+5}{n} x^{n-2}$$

(b) What is the coefficient of x^1?

(c) What is the coefficient of x^5?

SOLUTION.

(a)

$$\begin{aligned}
& x^3 \sum_{n=2}^{\infty} \binom{n+5}{n} x^{n-2} \\
&= \sum_{n=2}^{\infty} \binom{n+5}{n} \times x^3 \times x^{n-2} \\
&= \sum_{n=2}^{\infty} \binom{n+5}{n} x^{n-2+3} \\
&= \sum_{n=2}^{\infty} \binom{n+5}{n} x^{n+1} \\
&= \sum_{m=3}^{\infty} \binom{m-1+5}{m-1} x^{m} && \text{(let } m = n+1\text{, so } n = m-1\text{)} \\
&= \sum_{n=3}^{\infty} \binom{n+4}{n-1} x^{n} && \text{(change m back to n)}
\end{aligned}$$

ANSWER: $\boxed{\sum_{n=3}^{\infty} \binom{n+4}{n-1} x^{n}}$

(b)When x^1, $n = 1$

$$\begin{aligned}&\binom{n+4}{n-1}\\ &= \binom{1+4}{1-1}\\ &= \binom{5}{0}\\ &= 0\end{aligned}$$

ANSWER: $\boxed{0}$

(c)When x^5, $n = 5$

$$\begin{aligned}&\binom{n+4}{n-1}\\ &= \binom{5+4}{5-1}\\ &= \binom{9}{4}\\ &= \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4}\\ &= 126\end{aligned}$$

ANSWER: $\boxed{126}$

Q7. [POLYNOMIAL MULTIPLE OF POWER SERIES]

(a) Rewrite the following series as a power series in x^n term.

$$(1+2x)\sum_{n=2}^{\infty}\frac{1+(-1)^n 2^n}{3^{n+1}}x^n$$

For this power series, you may want to express the power series by providing several case-by-case expressions for the coefficients. Here's an example:

$$\sum_{n=0}^{\infty} a_n x^n$$

where

$$a_n = \begin{cases} 0 & \text{if } n = 0, 1, 2 \\ \frac{1}{2^n} & \text{if } n = 3, ..., 100 \\ \frac{1}{2^n} + n^2 & \text{if } n > 100 \end{cases}$$

(b) What is the coefficient of x^1?

(c) What is the coefficient of x^5?

SOLUTION.

(a)

$$\begin{aligned}
&(1+2x)\sum_{n=2}^{\infty}\frac{1+(-1)^n 2^n}{3^{n+1}}x^n \\
&\qquad = 1\cdot\sum_{n=2}^{\infty}\frac{1+(-1)^n 2^n}{3^{n+1}}x^n + 2x\cdot\sum_{n=2}^{\infty}\frac{1+(-1)^n 2^n}{3^{n+1}}x^n \\
&\qquad = \sum_{n=2}^{\infty}\frac{1+(-1)^n 2^n}{3^{n+1}}x^n + \sum_{n=2}^{\infty}2\frac{1+(-1)^n 2^n}{3^{n+1}}x^{n+1} \qquad (1)
\end{aligned}$$

Rewriting the second term on the right so that we have x^n terms instead of x^{n+1} terms, we obtain

$$\begin{aligned}
\sum_{n=2}^{\infty} 2\frac{1+(-1)^n 2^n}{3^{n+1}} x^{n+1} &= \sum_{n=2}^{\infty} 2\frac{1+(-1)^n 2^n}{3^{n+1}} x^{n+1} && \text{(let } m = n+1\text{)} \\
&= \sum_{m=3}^{\infty} 2\frac{1+(-1)^{m-1} 2^{m-1}}{3^{m}} x^{m} \\
&= \sum_{n=3}^{\infty} 2\frac{1+(-1)^{n-1} 2^{n-1}}{3^{n}} x^{n} && \text{(change m back to n)} \qquad (2)
\end{aligned}$$

Substituting (2) into (1) we obtain

$$(1+2x)\sum_{n=2}^{\infty}\frac{1+(-1)^n2^n}{3^{n+1}}x^n$$

$$=\sum_{n=2}^{\infty}\frac{1+(-1)^n2^n}{3^{n+1}}x^n+\sum_{n=3}^{\infty}2\frac{1+(-1)^{n-1}2^{n-1}}{3^n}x^n$$

$$=\sum_{n=2}^{\infty}(\frac{1}{3})\frac{1+(-1)^n2^n}{3^n}x^n+\sum_{n=3}^{\infty}(\frac{2}{1+(-1)2})\frac{1+(-1)^n2^n}{3^n}x^n$$

$$=\sum_{n=3}^{\infty}(\frac{1}{3})\frac{1+(-1)^n2^n}{3^n}x^n+(\frac{1}{3})\frac{1+(-1)^22^2}{3^2}x^2$$
$$+\sum_{n=3}^{\infty}(\frac{2}{1+(-1)2})\frac{1+(-1)^n2^n}{3^n}x^n$$

$$=\sum_{n=3}^{\infty}(\frac{1}{3})\frac{1+(-1)^n2^n}{3^n}x^n+\sum_{n=3}^{\infty}-2\frac{1+(-1)^n2^n}{3^n}x^n+\frac{5}{27}x^2$$

$$=\sum_{n=3}^{\infty}(\frac{1}{3})\frac{1+(-1)^n2^n}{3^n}+-2\frac{1+(-1)^n2^n}{3^n}x^n+\frac{5}{27}x^2$$

$$=\sum_{n=3}^{\infty}\frac{-2}{3}\quad\frac{1+(-1)^n2^n}{3^n}x^n+\frac{5}{27}x^2$$

The power series is

$$a_n=\begin{cases}\frac{-2}{3} & \text{if } n=0\\ \frac{2}{9} & \text{if } n=1\\ \frac{-10}{27} & \text{if } n=2\\ \frac{-2}{3}\quad\frac{1+(-1)^n2^n}{3^n} & \text{if } n>=3\end{cases}$$

ANSWER: $\boxed{\sum_{n=3}^{\infty}\frac{-2}{3}\quad\frac{1+(-1)^n2^n}{3^n}x^n+\frac{5}{27}x^2}$

(b) Let $n = 1$ and $\frac{-2}{3}$ $\frac{1+(-1)^n 2^n}{3^n}$, so $\frac{-2}{3}$ $\frac{1+(-1)^1 2^1}{3^1}$ ANSWER: $\boxed{\dfrac{2}{9}}$

(c) Let $n = 5$ and $\frac{-2}{3}$ $\frac{1+(-1)^n 2^n}{3^n}$, so $\frac{-2}{3}$ $\frac{1+(-1)^5 2^5}{3^5}$ ANSWER: $\boxed{\dfrac{-1}{7}}$

Q8. [POLYNOMIAL MULTIPLE OF POWER SERIES]

(a) Rewrite the following series as a power series in x^n term.

$$(1 - 6x + 3x^2)\sum_{n=2}^{\infty}\binom{n+4}{n+2}x^n$$

(b) What is the coefficient of x^0?

(c) What is the coefficient of x^{10}?

SOLUTION.

(a)

$$
\begin{aligned}
&(1-6x+3x^2)\sum_{n=2}^{\infty}\binom{n+4}{n+2}x^n \\
&\qquad = \sum_{n=2}^{\infty}\binom{n+4}{n+2}x^n \\
&\qquad - \sum_{n=2}^{\infty}\binom{n+4}{n+2}(-6x)x^n \\
&\qquad + \sum_{n=2}^{\infty}\binom{n+4}{n+2}(3x^2)x^n \\
&\qquad = \sum_{n=2}^{\infty}\binom{n+4}{n+2}x^n \\
&\qquad + 6\sum_{n=2}^{\infty}\binom{n+4}{n+2}x^{n+1} \\
&\qquad + 3\sum_{n=2}^{\infty}\binom{n+4}{n+2}x^{n+2} \\
&\qquad = \sum_{n=2}^{\infty}\binom{n+4}{n+2}x^n \\
&\qquad + 6\sum_{n=3}^{\infty}\binom{n+3}{n+1}x^n \\
&\qquad + 3\sum_{n=4}^{\infty}\binom{n+2}{n}x^n \\
&\qquad = \sum_{n=4}^{\infty}\binom{n+4}{n+2}x^n + \binom{2+4}{2+2}x^2 + \binom{3+4}{3+2}x^3 \\
&\qquad + 6\sum_{n=4}^{\infty}\binom{n+3}{n+1}x^n + \binom{3+3}{3+1}x^3 \\
&\qquad + 3\sum_{n=4}^{\infty}\binom{n+2}{n}x^n
\end{aligned}
$$

$$= \sum_{n=4}^{\infty} \binom{n+4}{n+2} + 6\binom{n+3}{n+1} + 3\binom{n+2}{n} x^n$$
$$+ \binom{6}{4}x^2 + \binom{7}{5}x^3 + \binom{6}{4}x^3$$

$$= \sum_{n=4}^{\infty} \binom{n+4}{n+2} + 6\binom{n+3}{n+1} + 3\binom{n+2}{n} x^n + 15x^2 + 21x^3 + 15x^3$$

ANSWER: $\boxed{\sum_{n=4}^{\infty} \binom{n+4}{n+2} + 6\binom{n+3}{n+1} + 3\binom{n+2}{n} x^n + 15x^2 + 36x^3}$

(b) ANSWER: $\boxed{0}$

(c) Let $n = 10$, then

$$= \binom{10+4}{10+2} + 6\binom{10+3}{10+1} + 3\binom{10+2}{10} \qquad = \binom{14}{12} + 6\binom{13}{11} + 3\binom{12}{10}$$

ANSWER: $\boxed{667}$

Q9. [POLYNOMIAL COMBINATION OF POWER SERIES]

(a) Rewrite the following series as a power series in x^n term.

$$(1-6x)\sum_{n=0}^{\infty}\binom{n+4}{2}x^n + \left(1+\frac{1}{2}x\right)\sum_{n=0}^{\infty}n^2x^n$$

(b) What is the coefficient of x^1?

(c) What is the coefficient of x^{10}?

SOLUTION.

(a)

$$(1-6x)\sum_{n=0}^{\infty}\binom{n+4}{2}x^n + \left(1+\frac{1}{2}x\right)\sum_{n=0}^{\infty}n^2x^n$$

$$= \sum_{n=0}^{\infty}(1-6x)\binom{n+4}{2}x^n + \sum_{n=0}^{\infty}\left(1+\frac{1}{2}x\right)n^2x^n$$

$$= \sum_{n=0}^{\infty}(1-6x)\binom{n+4}{2} + \left(1+\frac{1}{2}x\right)n^2x^n$$

$$= \sum_{n=0}^{\infty}\binom{n+4}{2} - 6x\binom{n+4}{2} + n^2 + n^2\frac{1}{2}xx^n$$

$$= \sum_{n=0}^{\infty}\binom{n+4}{2}x^n - \sum_{n=0}^{\infty}6x\binom{n+4}{2}x^n$$
$$+ \sum_{n=0}^{\infty}n^2x^n + \sum_{n=0}^{\infty}n^2\frac{1}{2}xx^n$$

$$= \sum_{n=0}^{\infty}\binom{n+4}{2}x^n - \sum_{n=0}^{\infty}6\binom{n+4}{2}x^{n+1}$$
$$+ \sum_{n=0}^{\infty}n^2x^n + \sum_{n=0}^{\infty}n^2\frac{1}{2}x^{n+1}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \binom{n+4}{2} x^n + \binom{0+4}{2} x^0 \\
&- \sum_{n=1}^{\infty} 6 \binom{n+3}{2} x^n \\
&+ \sum_{n=1}^{\infty} n^2 x^n + 0^2 x^0 \\
&+ \sum_{n=1}^{\infty} \left(n - 1^2\right) \frac{1}{2} x^n
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \binom{n+4}{2} x^n + \binom{4}{2} \\
&- \sum_{n=1}^{\infty} 6 \binom{n+3}{2} x^n \\
&+ \sum_{n=1}^{\infty} n^2 x^n \\
&+ \sum_{n=1}^{\infty} \left(n - 1^2\right) \frac{1}{2} x^n
\end{aligned}
$$

$$
= \sum_{n=1}^{\infty} \left(\binom{n+4}{2} - 6\binom{n+3}{2} + n^2 + \left(n - 1^2\right) \frac{1}{2} \right) x^n + 6
$$

ANSWER: $\boxed{\sum_{n=1}^{\infty} \left(\binom{n+4}{2} - 6\binom{n+3}{2} + n^2 + \left(n - 1^2\right) \frac{1}{2} \right) x^n + 6}$

(b) Let $n = 1$

$$
\begin{aligned}
&= \binom{n+4}{2} - 6\binom{n+3}{2} + n^2 + \left(n - 1^2\right) \frac{1}{2} \\
&= \binom{5}{2} - 6\binom{4}{2} + 1^2 + \left(0^2\right) \frac{1}{2}
\end{aligned}
$$

ANSWER: $\boxed{-25}$

(c) Let $n = 10$

$$
\begin{aligned}
&= \binom{n+4}{2} - 6\binom{n+3}{2} + n^2 + \left(n - 1^2\right)\frac{1}{2} \\
&= \binom{14}{2} - 6\binom{13}{2} + 10^2 + \left(9^2\right)\frac{1}{2}
\end{aligned}
$$

ANSWER: $\boxed{699}$

Q10. [GENERATING POWER SERIES WITH GEOMETRIC SERIES]

[SOLUTION PROVIDED]

(a) Using the geometric series formula, generate a rational expression for the power series

$$\sum_{n=0}^{\infty} 2^n x^n$$

(b) What is the value of

$$\sum_{n=0}^{\infty} 2^n \left(\frac{1}{3}\right)^n$$

SOLUTION.

(a)

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^n x^n &= \sum_{n=0}^{\infty} (2x)^n \\
&= \frac{1}{1-(2x)} \qquad\qquad \text{by geometric series formula} \\
&= \frac{1}{1-2x}
\end{aligned}$$

ANSWER: $\boxed{\frac{1}{1-2x}}$ □

(b) With $x = 1/3$ in the above we have

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^n (1/3)^n &= \frac{1}{1-2(1/3)} \\
&= \frac{1}{1/3} \\
&= 3
\end{aligned}$$

ANSWER: $\boxed{3}$ □

Note. It is a smart thing to check your computations with a program, i.e., write a simple program in your favorite programming language such as:

```
def f(x):
    x = float(x)
    sum = 0
    for n in range(1000):
        term = 2.0**n * x**n
        sum += term
    return sum

def g(x):
    x = float(x)
    return 1/(1 - 2 * x)

for i in range(0, 10):
    x = i / 100.0
    print f(x), g(x), f(x) - g(x)
```

to check that the power series and rational expression evaluates to (approximately) the same value. I say "approximately" because there will be rounding errors when working with floating point numbers. Here's my output:

```
1.0 1.0 0.0
1.02040816327 1.02040816327 -2.22044604925e-16
1.04166666667 1.04166666667 2.22044604925e-16
1.06382978723 1.06382978723 2.22044604925e-16
1.08695652174 1.08695652174 2.22044604925e-16
1.11111111111 1.11111111111 0.0
1.13636363636 1.13636363636 2.22044604925e-16
1.16279069767 1.16279069767 4.4408920985e-16
1.19047619048 1.19047619048 2.22044604925e-16
1.21951219512 1.21951219512 -4.4408920985e-16
```

Note that you should only use "small" values of `x`. How small? Well, if you look closely at the application of the geometric series formula:

$$\sum_{n=0}^{\infty} (2x)^n = \frac{1}{1-(2x)}$$

and you check your notes, you know that you should only apply the geomtric series formula in this case for

$$|2x| < 1$$

which means that

$$|x| < 1/2 = 0.5$$

Q11. [GENERATING POWER SERIES WITH GEOMETRIC SERIES]

Using the geometric series formula, generate a rational expression for the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{3^n} x^n$$

SOLUTION.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{3^n} x^n &= \sum_{n=1}^{\infty} \left(\frac{-2x}{3}\right)^n \\
&= \sum_{n=0}^{\infty} \left(\frac{-2x}{3}\right)^n - \left(\frac{-2x}{3}\right)^0 \\
&= \sum_{n=0}^{\infty} \left(\frac{-2x}{3}\right)^n - 1 \\
&= \frac{1}{1+\frac{2x}{3}} - 1 \\
&= \frac{3}{3+2x} - 1 \\
&= \frac{3-3-2x}{3+2x} \\
&= \frac{-2x}{3+2x}
\end{aligned}$$

ANSWER: $\boxed{\frac{-2x}{3+2x}}$

Q12. [GENERATING POWER SERIES WITH GEOMETRIC SERIES]

Using the geometric series formula, generate a rational expression for the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}} x^n$$

[HINT: See next page for spoiler hint.]

SOLUTION.

$$\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}} x^n$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty} \left(\frac{1}{3} \times \frac{2^n}{3^n} \right) x^n \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{3} \times \frac{2^n}{3^n} \right) x^n \\
&- \left(\frac{1}{3} \times \frac{2^0}{3^0} \right) x^0 \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n - \frac{1}{3} \\
&= \frac{1}{3} \sum_{n=0}^{\infty} \frac{2x}{3}^n - \frac{1}{3} \\
&= \frac{1}{3} \times \frac{1}{1 - \frac{2x}{3}} - \frac{1}{3} \\
&= \frac{1}{3} \times \frac{3}{3 - 2x} - \frac{1}{3} \\
&= \frac{3}{9 - 6x} - \frac{1}{3} \\
&= \frac{9}{27 - 18x} - \frac{9 - 6x}{27 - 18x} \\
&= \frac{9 - 9 + 6x}{27 - 18x} \\
&= \frac{6x}{27 - 18x}
\end{aligned}$$

ANSWER: $\boxed{\dfrac{6x}{27 - 18x}}$

WARNING ... INCOMING SPOILER!!! ...

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You probably want to remove a 3 from the denominator of the coefficient outside the summation.

Q13. [GENERATING POWER SERIES WITH GEOMETRIC SERIES]

Using the geometric series formula, generate a rational expression for the power series

$$\sum_{n=0}^{\infty} \frac{1-2^{n+1}}{3^n} x^n$$

[HINT: See the spoiler on next page.]

SOLUTION.

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1-2^{n+1}}{3^n} x^n \\
&\qquad = \sum_{n=0}^{\infty} \frac{1}{3^n} x^n - \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^n} x^n \\
&\qquad = \sum_{n=0}^{\infty} \frac{1}{3^n} x^n - 2\sum_{n=0}^{\infty} \frac{2^n}{3^n} x^n \\
&\qquad = \sum_{n=0}^{\infty} \frac{x}{3}^{n} - 2\sum_{n=0}^{\infty} \left(\frac{2x}{3}\right)^n \\
&\qquad = \frac{1}{1-\frac{x}{3}} - 2\frac{1}{1-\frac{2x}{3}} \\
&\qquad = \frac{3}{3-x} - 2\frac{3}{3-2x}
\end{aligned}$$

ANSWER: $\boxed{\frac{3}{3-x} - 2\frac{3}{3-2x}}$

WARNING ... SPOILERS COMING ...

WARNING ... SPOILERS COMING ...

WARNING ... SPOILERS COMING ...

[HINT FOR Q13: Rewrite the given power series as *two* power series and you'll see the geometric series a lot easier.]

Q14. [GENERATING POWER SERIES WITH GEOMETRIC SERIES]

Using the geometric series formula, generate a rational expression for the power series

$$\sum_{n=1}^{\infty} \frac{2^n - 3^{n+1}}{4^{n+2}} x^n$$

SOLUTION.

$$\sum_{n=1}^{\infty} \frac{2^n - 3^{n+1}}{4^{n+2}} x^n$$

$$= \sum_{n=1}^{\infty} \frac{2^n}{4^{n+2}} x^n - \sum_{n=1}^{\infty} \frac{3^{n+1}}{4^{n+2}} x^n$$

$$= \frac{1}{4^2} \sum_{n=1}^{\infty} \frac{2^n}{4^n} x^n - \frac{3}{4^2} \sum_{n=1}^{\infty} \frac{3^n}{4^n} x^n$$

$$= \frac{1}{4^2} \sum_{n=1}^{\infty} \left(\frac{2x}{4}\right)^n - \frac{3}{4^2} \sum_{n=1}^{\infty} \left(\frac{3x}{4}\right)^n$$

$$= \frac{1}{4^2} \left(\sum_{n=0}^{\infty} \left(\frac{2x}{4}\right)^n - \left(\frac{2x}{4}\right)^0 \right) - \frac{3}{4^2} \left(\sum_{n=0}^{\infty} \left(\frac{3x}{4}\right)^n - \left(\frac{3x}{4}\right)^0 \right)$$

$$= \left(\frac{1}{4^2} \times \left(\frac{1}{1 - \frac{2x}{4}} - 1 \right) \right) - \left(\frac{3}{4^2} \times \left(\frac{1}{1 - \frac{3x}{4}} - 1 \right) \right)$$

$$= \left(\frac{1}{4^2} \times \left(\frac{4}{4 - 2x} - 1 \right) \right) - \left(\frac{3}{4^2} \times \left(\frac{4}{4 - 3x} - 1 \right) \right)$$

ANSWER: $\boxed{\left(\frac{1}{4^2} \times \left(\frac{4}{4 - 2x} - 1 \right) \right) - \left(\frac{3}{4^2} \times \left(\frac{4}{4 - 3x} - 1 \right) \right)}$

Q15. [GENERATING POWER SERIES WITH GEOMETRIC SERIES]

Using the geometric series formula, generate a rational expression for the power series

$$1+\sum_{n=1}^{\infty}\frac{2^n}{3^{n+2}}x^n$$

SOLUTION.

$$\begin{aligned}
&1+\sum_{n=1}^{\infty}\frac{2^n}{3^{n+2}}x^n\\
&\qquad =1+\sum_{n=1}^{\infty}\frac{1}{3^2}\times\frac{2^n}{3^n}x^n\\
&\qquad =1+\frac{1}{3^2}\sum_{n=1}^{\infty}\frac{2x}{3}^n\\
&\qquad =1+\frac{1}{3^2}\left(\sum_{n=0}^{\infty}\left(\frac{2x}{3}\right)^n-\left(\frac{2x}{3}\right)^0\right)\\
&\qquad =1+\frac{1}{3^2}\left(\frac{1}{1-\frac{2x}{3}}-1\right)\\
&\qquad =1+\frac{1}{3^2}\left(\frac{3}{3-2x}-1\right)
\end{aligned}$$

ANSWER: $\boxed{1+\frac{1}{3^2}\left(\frac{3}{3-2x}-1\right)}$

Q16. [GENERATING POWER SERIES WITH GEOMETRIC SERIES]

Using the geometric series formula, generate a rational expression for the power series

$$1 + x \sum_{n=2}^{\infty} \frac{1 + (-1)^n 2^n}{3^{n+2}} x^{2n}$$

SOLUTION.

$$1 + x\sum_{n=2}^{\infty} \frac{1+(-1)^n 2^n}{3^{n+2}} x^{2n}$$

$$= 1 + \frac{x}{3^2}\sum_{n=2}^{\infty} \frac{1+(-1)^n 2^n}{3^n} x^{2n}$$

$$= 1 + \frac{x}{3^2}\left(\sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{3^n} x^{2n} + \sum_{n=2}^{\infty} \frac{1}{3^n} x^{2n}\right)$$

$$= 1 + \frac{x}{3^2}\left(\sum_{n=2}^{\infty} \left(\frac{-2x^2}{3}\right)^n + \sum_{n=2}^{\infty} \left(\frac{x^2}{3}\right)^n\right)$$

$$= 1 + \frac{x}{3^2}\left(\sum_{n=0}^{\infty} \left(\frac{-2x^2}{3}\right)^n - \left(\frac{-2x^2}{3}\right)^0 - \left(\frac{-2x^2}{3}\right)^1\right)$$
$$+ \frac{x}{3^2}\left(\sum_{n=0}^{\infty} \left(\frac{x^2}{3}\right)^n - \left(\frac{x^2}{3}\right)^0 - \left(\frac{x^2}{3}\right)^1\right)$$

$$= 1 + \frac{x}{3^2}\left(\frac{1}{1 - -2x^2 3} - 1 + \frac{2x^2}{3}\right)$$
$$+ \frac{x}{3^2}\left(\frac{1}{1 - \frac{x^2}{3}} - 1 - \frac{x^2}{3}\right)$$

$$= 1 + \frac{x}{3^2}\left(\frac{3}{3 + 2x^2} - 1 + \frac{2x^2}{3}\right)$$
$$+ \frac{x}{3^2}\left(\frac{3}{3 - x^2} - 1 - \frac{x^2}{3}\right)$$

ANSWER: $\boxed{1 + \frac{x}{3^2}\left(\frac{3}{3 + 2x^2} - 1 + \frac{2x^2}{3}\right) + \frac{x}{3^2}\left(\frac{3}{3 - x^2} - 1 - \frac{x^2}{3}\right)}$