MATH325: Discrete Math 2
Assignment 12

For the questions below, I expect justification of your answers. For the derivation of rational functions, I expect a derivation and not a quote of a theorem that translate a recurrence relation to a rational function without proof. For the derivation of closed form from rational function, only basic formulas should be used. Partial fractions, if necessary, must be derived. Formulas for rewriting rational functions to power series is limited to the formulas:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

and

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{k+n-1}{n} x^n$$

Q1. The goal of Q1 is to compute a closed form for a_n ($n = 0, 1, 2, \ldots$) where

$$\begin{aligned} a_0 &= 5 \\ a_n &= 7a_{n-1} \quad \text{if } n > 0 \end{aligned}$$

Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence a_n ($n = 0, 1, 2, \ldots$) where

$$\begin{aligned} a_0 &= 5 \\ a_n &= 7a_{n-1} \quad \text{if } n > 0 \end{aligned}$$

(a) Derive a rational function for $a(x)$.

SOLUTION.

$$\begin{aligned} a(x) &= a_0 x^0 + \sum_{n=1}^{\infty} a_n x^n \\ &= 5 + \sum_{n=1}^{\infty} 7a_{n-1} x^n \\ &= 5 + 7x^1 \sum_{n=1}^{\infty} a_{n-1} x^{n-1} \\ &= 5 + 7x^1 \sum_{n=0}^{\infty} a_n x^n \\ &= 5 + 7x^1 a(x) \\ \therefore \quad (1 - 7x^1)a(x) &= 5 \\ \therefore \quad a(x) &= \frac{5}{1 - 7x^1} \end{aligned}$$

ANSWER: $\boxed{a(x) = \dfrac{5}{1 - 7x}}$

(b) Find a closed form for a_n using the rational function in (a). (You should check your closed form against several a_n's from the recursive definition of a_n ($n = 0, 1, 2, \ldots$) for at least $n = 0, 1, 2, 3, 4$.

SOLUTION.

$$\begin{aligned} a(x) &= \frac{5}{1-7x} \\ &= 5\frac{1}{1-7x} \\ &= \sum_{n=0}^{\infty}(5\cdot 7^n)x^n \end{aligned}$$

Hence

$$a_n = 5\cdot 7^n$$

for $n \geq 0$.

ANSWER: $\boxed{a_n = 5\cdot 7^n \text{ for } n \geq 0}$

Q2. The goal of Q2 is to compute a closed form for b_n $(n = 0, 1, 2, \ldots)$ where

$$\begin{aligned} b_0 &= 5 \\ b_1 &= 3 \\ b_n &= 2b_{n-1} - b_{n-2} \quad \text{if } n > 1 \end{aligned}$$

Let $b(x) = \sum_{n=0}^{\infty} b_n x^n$ be the generating function of the sequence b_n $(n = 0, 1, 2, \ldots)$ where

$$\begin{aligned} b_0 &= 5 \\ b_1 &= 3 \\ b_n &= 2b_{n-1} - b_{n-2} \quad \text{if } n > 1 \end{aligned}$$

(a) Derive a rational function for $b(x)$.

SOLUTION.

$$\begin{aligned} b(x) &= b_0 x^0 + b_1 x^1 + \sum_{n=2}^{\infty} b_n x^n \\ &= 5 + 3x + \sum_{n=2}^{\infty} (2b_{n-1} - b_{n-2}) x^n \\ &= 5 + 3x + 2\sum_{n=2}^{\infty} b_{n-1} x^n - \sum_{n=2}^{\infty} b_{n-2} x^n \\ &= 5 + 3x + 2x\sum_{n=2}^{\infty} b_{n-1} x^{n-1} - x^2 \sum_{n=2}^{\infty} b_{n-2}) x^{n-2} \\ &= 5 + 3x + 2x\sum_{n=1}^{\infty} b_n x^n - x^2 \sum_{n=0}^{\infty} b_n) x^n \\ &= 5 + 3x + 2x \left(\sum_{n=1}^{\infty} b_n x^n - b_0 x^0 \right) - x^2 b(x) \\ &= 5 + 3x + 2x\,(b(x) - 5) - x^2 b(x) \\ &= 5 + 3x + 2x b(x) - 10x - x^2 b(x) \\ \therefore \quad b(x) - 2x b(x) + x^2 b(x) &= 5 + 3x - 10x \\ \therefore \quad (1 - 2x + x^2) b(x) &= 5 - 7x \\ \therefore \quad b(x) &= \frac{5 - 7x}{1 - 2x + x^2} \end{aligned}$$

ANSWER: $\boxed{\dfrac{5-7x}{1-2x+x^2}}$

(b) Derive a closed form for b_n. (You should check your closed form against several b_n's from the recursive definition of b_n ($n = 0, 1, 2, \dots$) for at least $n = 0, 1, 2, 3, 4$.)

SOLUTION.

$$\begin{aligned}
b(x) &= \frac{5-7x}{1-2x+x^2} \\
&= (5-7x)\frac{1}{1-2x+x^2} \\
&= (5-7x)\frac{1}{(1-x)^2} \\
&= (5-7x)\sum_{n=0}^{\infty}\binom{2+n-1}{n}x^n \\
&= (5-7x)\sum_{n=0}^{\infty}(n+1)x^n \\
&= \sum_{n=0}^{\infty}(5n+5)x^n - \sum_{n=0}^{\infty}7(n+1)x^{n+1} \\
&= \sum_{n=0}^{\infty}(5n+5)x^n - \sum_{n=1}^{\infty}7nx^n \\
&= \sum_{n=0}^{\infty}(5n+5)x^n - \sum_{n=0}^{\infty}7nx^n - 7(0x^0) \\
&= \sum_{n=0}^{\infty}(5n+5-7n)x^n \\
&= \sum_{n=0}^{\infty}(-2n+5)x^n
\end{aligned}$$

Hence

$$b_n = -2n + 5$$

for $n \geq 0$.

ANSWER: $\boxed{b_n = -2n + 5 \text{ for } n \geq 0}$

Q3. The goal of Q3 is to compute a closed form for c_n $(n = 0, 1, 2, \ldots)$ where

$$\begin{aligned} c_0 &= 2 \\ c_1 &= 1 \\ c_n &= 5c_{n-1} - 6c_{n-2} \quad \text{if } n > 1 \end{aligned}$$

Let $c(x) = \sum_{n=0}^{\infty} c_n x^n$ be the generating function of the sequence c_n $(n = 0, 1, 2, \ldots)$ which satisfies the following:

$$\begin{aligned} c_0 &= 2 \\ c_1 &= 1 \\ c_n &= 5c_{n-1} - 6c_{n-2} \quad \text{if } n > 1 \end{aligned}$$

(a) Derive a rational function for $c(x)$.

SOLUTION.

$$\begin{aligned} c(x) &= c_0x^0 + c_1x^1 + \sum_{n=2}^{\infty} c_n x^n \\ &= 2 + x + \sum_{n=2}^{\infty}(5c_{n-1} - 6c_{n-2})x^n \\ &= 2 + x + 5\sum_{n=2}^{\infty} c_{n-1}x^n - 6\sum_{n=2}^{\infty} c_{n-2}x^n \\ &= 2 + x + 5x\sum_{n=2}^{\infty} c_{n-1}x^{n-1} - 6x^2\sum_{n=2}^{\infty} c_{n-2}x^{n-2} \\ &= 2 + x + 5x\sum_{n=1}^{\infty} c_n x^n - 6x^2\sum_{n=0}^{\infty} c_n x^n \\ &= 2 + x + 5x\left(\sum_{n=0}^{\infty} c_n x^n - c_0x^0\right) - 6x^2c(x) \\ &= 2 + x + 5x\,(c(x) - 2) - 6x^2c(x) \\ &= 2 + x + 5xc(x) - 10x - 6x^2c(x) \\ \therefore \quad (1 - 5x + 6x^2)c(x) &= 2 + x - 10x \\ \therefore \quad c(x) &= \frac{2 + x - 10x}{1 - 5x + 6x^2} \end{aligned}$$

ANSWER: $\boxed{c(x) = \dfrac{2 + x - 10x}{1 - 5x + 6x^2}}$

(b) Derive a closed form for c_n. (You should check your closed form against several b_n's from the recursive definition of b_n ($n = 0, 1, 2, \ldots$) for at least $n = 0, 1, 2, 3, 4$.)

[NOTE: For this rational function, you will need to use the theory of partial functions.]

SOLUTION.

By the theory of partial fractions, we have

$$\begin{aligned}\frac{1}{(2x-1)(3x-1)} &= \frac{A}{(2x-1)} + \frac{B}{(3x-1)} \\ 1 &= A(3x-1) + B(2x-1)\end{aligned}$$

Let $x = \frac{1}{3}$, then $B = -3$. Let $x = \frac{1}{2}$, then $A = 2$.

$$\frac{1}{(2x-1)(3x-1)} = \frac{2}{(2x-1)} + \frac{-3}{(3x-1)}$$

Therefore

$$
\begin{aligned}
c(x) &= \frac{2+x-10x}{1-5x+6x^2} \\
&= (2+x-10x)\frac{1}{1-5x+6x^2} \\
&= (2-9x)\frac{1}{(2x-1)(3x-1)} \\
&= (2-9x)\frac{2}{(2x-1)} + \frac{-3}{(3x-1)} \\
&= (2-9x) - 2\frac{1}{(1-2x)} + 3\frac{1}{(1-3x)} \\
&= (2-9x) - 2\sum_{n=0}^{\infty} 2x^n + 3\sum_{n=0}^{\infty} 3x^n \\
&= (2-9x)\sum_{n=0}^{\infty} -2^{n+1}x^n + \sum_{n=0}^{\infty} 3^{n+1}x^n \\
&= (2-9x)\sum_{n=0}^{\infty}(-2^{n+1}+3^{n+1})x^n \\
&= 2\sum_{n=0}^{\infty}(-2^{n+1}+3^{n+1})x^n - 9x\sum_{n=0}^{\infty}(-2^{n+1}+3^{n+1})x^n \\
&= \sum_{n=0}^{\infty} 2(-2^{n+1}+3^{n+1})x^n - \sum_{n=0}^{\infty} 9(-2^{n+1}+3^{n+1})x^{n+1} \\
&= \sum_{n=0}^{\infty} 2(-2^{n+1}+3^{n+1})x^n - \sum_{n=1}^{\infty} 9(-2^{n}+3^{n})x^{n} \\
&= 2(-2^{0+1}+3^{0+1})x^0 + \sum_{n=1}^{\infty}(2(-2^{n+1}+3^{n+1}) - 9(-2^n+3^n))x^n \\
&= 2 + \sum_{n=1}^{\infty}(5\cdot 2^n + 3^{n+1})x^n
\end{aligned}
$$

Hence

$$c_n = 5\cdot 2^n + 3^{n+1}$$

for $n \geq 0$.

ANSWER: $\boxed{c_n = 5\cdot 2^n + 3^{n+1} \text{ for } n \geq 0}$