MATH325: Discrete Math 2
Assignment 10

This assignment involves counting solutions to linear systems using power series or generating functions.

The goal is to study the number of solutions to the following system:

$$\begin{cases} a+b+c=n \\ a,b,c\in\mathbb{Z} \\ 0\le a, 2\le b, 0<c, c \text{ odd} \end{cases}$$

for different values of $n \ge 0$. Let a_n be the number of solutions to the above system. For instance a_{1000} is the number of solutions to the above when n is 1000.

To be more specific, a_0 is the number of solutions to

$$\begin{cases} a+b+c=0 \\ a,b,c\in\mathbb{Z} \\ 0\le a, 2\le b, 0<c, c \text{ odd} \end{cases}$$

In this case, we can compute the value of a_0 quickly: $a_0 = 0$. In fact, since $b \ge 2$ and $c \ge 1$, if $a+b+c=n$ has a solution, then $a+b+c \ge 3$. We conclude immediately that

$$a_0 = a_1 = a_2 = 0$$

Clearly $a_3 = 1$ since the only solution to

$$\begin{cases} a+b+c=0 \\ a,b,c\in\mathbb{Z} \\ 0\le a, 2\le b, 0<c, c \text{ odd} \end{cases}$$

is

$$a=0, b=2, c=1$$

It's easy to see that $a_4 = 2$:

$$4 = 1+2+1 = 0+3+1$$

Before going on, you should compute a_5 and a_6 to get a feel for the problem. (You can either do this by hand or write a program to this.) The values of a_n for $n = 0,1,2,\ldots,6$ is useful for checking against the general result that you will be deriving.

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

You will see that we do not need to guess a formula for a_n and then prove by mathematical induction. Instead we will use the method of power series to compute a formula for a_n.

Q1. Write down $f(x)$, the product of 3 power series such that the coefficient of x^n of this product is a_n. You need not simplify. For instance the following is a product of power series (which is not the answer to this question!)

$$(2 + 2x + 2x^2 + \cdots)(0 + 1x + 2x^2 + \cdots)(1 + 2x + 3x^2 + \cdots)$$

You should write down explicitly at least 3 nonzero terms for each power series.

SOLUTION.

$$f(x) = \left(x^0 + x^1 + x^2 + \cdots\right)\left(x^2 + x^3 + x^4 + \cdots\right)\left(x^1 + x^3 + x^5 + \cdots\right)$$

Q2 Rewrite the product of power series in (a) as a rational function. The denominator should be factorized (for instance $(1-x^2) = (1-x)(1+x)$), with similar terms collected together. For instance the following is a rational function (which is not the right answer to this question!)

$$f(x) = \frac{x}{(2-3x)^3(1+5x)^6}$$

Note that all the $(1+5x)$ factors (there are 6 of them) are collected together. Do *not* write for instance

$$f(x) = \frac{x}{(2-3x)^3(1+5x)^2(1+5x)^4}$$

SOLUTION.

$$f(x) = \left(x^0 + x^1 + x^2 + \cdots\right) x^2 \left(x^0 + x^1 + x^2 + \cdots\right) x^1 \left(x^0 + x^2 + x^2 + \cdots\right)$$

$$= x^2 \left(\frac{1}{1-x}\right)^2 x \frac{1}{1-x^2}$$

$$= x^3 \frac{1}{1-x} \frac{1}{1-x} \frac{1}{1-x} \frac{1}{1+x}$$

$$= x^3 \left(\frac{1}{1-x}\right)^3 \frac{1}{1+x}$$

$$= \frac{x^3}{(1-x)^3(1+x)}$$

Q3. Using the theory of partial fractions, rewrite the rational expression as a linear sum of simpler rational functions of the form $\frac{1}{p(x)^k}$ where $p(x)$ is a polynomial of degree at most 2. For instance the following is a linear sum of such rational functions:

$$f(x) = 5\frac{1}{(1-x)^3} + \frac{3}{4}\frac{1}{(2+x+3x^2)^5}$$

(of course this is not the answer!)

[Hint: For this problem, the $p(x)$'s are in fact linear, i.e. degree 1. Also, there are 4 terms in this sum.]

SOLUTION.

$$1 = \frac{1}{(1-x)^3(1+x)}$$

$$= \frac{A}{1-x}\frac{B}{(1-x)^2}\frac{C}{(1-x)^3}\frac{D}{1+x}$$

$$\text{Note: } A = \frac{1}{8}\, B = \frac{1}{4}\, C = \frac{1}{2}\, D = \frac{1}{8}$$

$$1 = \frac{\frac{1}{8}}{1-x}\frac{\frac{1}{4}}{(1-x)^2}\frac{\frac{1}{2}}{(1-x)^3}\frac{\frac{1}{8}}{1+x}$$

$$f(x) = x^3\frac{\frac{1}{8}}{1-x}\frac{\frac{1}{4}}{(1-x)^2}\frac{\frac{1}{2}}{(1-x)^3}\frac{\frac{1}{8}}{1+x}$$

Q4. Using Q3, rewrite $f(x)$ as a power series with the coefficient of x^n in terms of n. The following is an example (which is of course not the right answer!)

$$f(x) = \sum_{n=0}^{\infty} \frac{2+n^2}{1+n} x^n$$

Simplify the coefficients of x^n so that the binomial coefficients (if any does) does not occur. For instance you know that

$$\binom{100}{3} = \frac{100 \cdot 99 \cdot 98}{1 \cdot 2 \cdot 3}$$

and likewise

$$\binom{n}{3} = \frac{(n-1)\cdot(n-2)\cdot(n-3)}{1\cdot 2\cdot 3}$$

etc. You are strongly advised to write some simple programs to check your computation.
SOLUTION.

$$\begin{aligned}
f(x) &= \frac{x^3}{(1-x)^3(1+x)} \\
&= x^3 \frac{\frac{1}{8}}{1-x} \frac{\frac{1}{4}}{(1-x)^2} \frac{\frac{1}{2}}{(1-x)^3} \frac{\frac{1}{8}}{1+x} \\
&= x^3 \left(\sum_{n=0}^{\infty} \left(\frac{1}{8}\right) x^n + \sum_{n=0}^{\infty} \left(\frac{1}{4}\binom{2+n-1}{n}\right) x^n + \sum_{n=0}^{\infty} \left(\frac{1}{2}\binom{3+n-1}{n}\right) x^n + \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{8}\right) x^n \right) \\
&= x^3 \sum_{n=0}^{\infty} \left(\left(\frac{1}{8}\right) + \left(\frac{1}{4}\binom{2+n-1}{n}\right) + \left(\frac{1}{2}\binom{3+n-1}{n}\right) + \left(\frac{(-1)^n}{8}\right) \right) x^n \\
&= x^3 \sum_{n=0}^{\infty} \left(2n^2 + 8n + 7 + (-1)^n\right) x^n \\
&= \sum_{n=3}^{\infty} \left(\frac{1}{8} \left(2n^2 - 4n + 1 + (-1)^{n-3}\right) \right) x^n
\end{aligned}$$

Q5. What is the coefficient x^n from Q4 in terms of n Recall that the coefficient of x^n for the power series of $f(x)$ is a_n. In other words what is the formula for a_n in terms of n?

[It's a good idea now to check the formula against the values of a_n for $n = 0, 1, 2, \ldots, 6$ that you have computed earlier.]

SOLUTION.

$$\left(\frac{1}{8}\left(2n^2 - 4n + 1 + (-1)^{n-3}\right)\right)$$

Q6. (a) How many solutions are there to

$$\begin{cases} a+b+c=1000 \\ a,b,c\in\mathbb{Z} \\ 0\le a, 2\le b, 0<c, c \text{ odd} \end{cases}$$

(b) How many solutions are there to

$$\begin{cases} a+b+c=1001 \\ a,b,c\in\mathbb{Z} \\ 0\le a, 2\le b, 0<c, c \text{ odd} \end{cases}$$

SOLUTION.

a)

$$\left(\frac{1}{8}\left(2(1000)^2-4000+1+(-1)^{(1000-3)}\right)\right)=249,500$$

b)

$$\left(\frac{1}{8}\left(2(1001)^2-4004+1+(-1)^{(1001-3)}\right)\right)=250,000$$