MATH325: Discrete Math 2
Assignment 5

Questions are taken from Rosen, Discrete Mathematics and Applications, 6th edition. When I write "Exercise 5.1.2", I mean "Exercise 2 of section 5.1".

Q1. Exercise 5.1.35.

SOLUTION.

(a) The task of creating one-to-one functions from the set 1,2,...n, where n is a positive integer, to the set 0, 1 can be broken into cases. (a function f has an inverse $f^{-1}f(a) = b$ and $f^{-1}(b) = a$) The range cardinality is 2 then the n must be 2 or less (by piegion hole principle).

There are two cases $n = 1$ and $n = 2$:

If $n = 1$ then there are two posibilities, (1, 0) or (1, 1).

If $n = 2$ then there are two posibilities, ((1, 0) and (2, 1)) or ((1, 1) and (2, 0)).

So depends on n the are either one or two.

ANSWER: $\boxed{\text{1 or 2}}$

(b) The task of creating functions from the set 1,2,...n, where n is a positive integer, to the set 0, 1, that assigns 0 to both 1 and n. 1 and n have a decided places their are $n - 2$ values in the domain that can go to 2 possible range values either a 0 or a 1.

By the multiplication principle their are 2^{n-2} functions.

ANSWER: $\boxed{2^{n-2}}$

(c) The task of creating functions from the set 1,2,...n, where n is a positive integer, to the set 0, 1, that assigns 1 to to exactly one positive integer less than n. (n must be bigger than 1 or else there are no positive numbers less than n).

So once the fate of one number is decided to go to 1 and all the other numbers less than n go to 0. So there are n-1 values to be chosen to go to 1, then the fait of n must be decided which can go to either 0 or 1. By the multiplication principle the are $2 \times (n - 1)$ choices.

ANSWER: $\boxed{2 \times (n - 1)}$

Q2. Exercise 5.1.38.

SOLUTION. In order to find the number of subset of a set with 100 elements with more than one element is the total number of subsets minus the number of subsets with only one element and the empty set.

The total number of subsets is for each element the element has two choices either the element is or is not in the subset, so 2^{100}.

The number of subset with one element is equal the number of elements in the set, so 100.e

Altogether $2^{100} - 100 - 1 \approx 1.268 \times 10^{30}$

ANSWER: $\boxed{1.268 \times 10^{30}}$

Q3. Exercise 5.1.40.

SOLUTION.

There are ten people (including bride and groom) in a group choosing six of those people to be in a photograph,the number of ways to arrage those 6 people

a) If the bride must be in the picture the number of ways to choose 5 people from 9 people $\frac{9!}{(9-5)!} = 15,120$

ANSWER: $\boxed{15,120}$

b) If the bride and groom must be in the picture the number of ways to choose 4 people from 8 people $\frac{8!}{(8-4)!} = 1,680$

ANSWER: $\boxed{1,680}$

c) If the bride or groom must be in the picture but not both.

The number of ways to choose is like choosing 6 people without restriction for choosing people minus the restriction that both bride and groom are in the picture and the restriction that neither the bride nor groom are in the picture.

So $\frac{10!}{(10-6)!} - \frac{8!}{(8-4)!} - \frac{8!}{(8-6)!} = 129,360$

ANSWER: $\boxed{129,360}$

Q4. Exercise 5.1.41. SOLUTION PROVIDED.

SOLUTION.

(a) Let T be the task of arranging the 6 people in a row so that the bride is next to the groom. Performing T is the same as performing the following in sequence. Suppose the bride is B and the groom is G and the rest are X_1, X_2, X_3, X_4. Create a new symbol BG.

1. Task T_1: Permute the symbols X_1, X_2, X_3, X_4, BG (There are 5! ways of doing this).
2. Task T_2: From the permutation from T_1, replace BG with B, G and G, B to create two permutations.

By the multiplication principle, there are

$$5!2 = 240$$

possible permutations created.

ANSWER: $\boxed{240}$

(b) There are altogether 6! permutations without any restriction. From (a), there are 5!2 permutations where the bride and groom are next to each other. Therefore there are

$$6! - 5!2 = 480$$

permutations where the bride and groom are not next to each other.

ANSWER: $\boxed{480}$

(c) Let X be the set of all permutations. Note that $|X| = 6!$. Let A be the set of permutations where the bride is to the left of the groom. Let B be the set of permutations where the bride is to the right of the groom. By the addition principle:

$$|X| = |A| + |B|$$

Now we note that there is an obvious function from A to B: If you are given a permutation in A, by swapping the bride and groom, you have a permutation in B. It is easy to see that the function has an inverse function, i.e., itself. and hence it is 1–1 and onto. Therefore $|A| = |B|$. Hence from $|X| = |A| + |B|$ we have

$$|X| = |A| + |B| = 2|A|$$

Therefore

$$|A| = |X|/2 = 6!/2 = 720/2 = 360$$

Therefore there are 360 permutations where the bride is to the left of the groom.

ANSWER: $\boxed{360}$

Q5. Exercise 5.1.42.

SOLUTION.

The task T of finding bit strings of length 7 either begining with 00 or ending with 111 is the task T_1 plus task T_2 minus task T_3.

Where task T_1 is finding the strings starting with 00 is like choosing from a 0 or a 1 to fill 5 spaces, so 2^5.

Where task T_1 is finding the strings ending with 111 is like choosing from a 0 or a 1 to fill 4 spaces, so 2^4.

Where task T_3 is finding the overlap (finding the strings starting with 00 and ending with 111) is like choosing from a 0 or a 1 to fill 2 spaces, so 2^2.

Altogether $2^5 + 2^4 - 2^2 = 44$

ANSWER: $\boxed{44}$

Q6. Exercise 5.1.43.

SOLUTION.

The task T of finding bit strings of length 10 either begining with 000 or ending with 00 is the task T_1 plus task T_2 minus task T_3.

Where task T_1 is finding the strings starting with 000 is like choosing from a 0 or a 1 to fill 5 spaces, so 2^7.

Where task T_1 is finding the strings ending with 00 is like choosing from a 0 or a 1 to fill 4 spaces, so 2^8.

Where task T_3 is finding the overlap (finding the strings starting with 000 and ending with 00) is like choosing from a 0 or a 1 to fill 2 spaces, so 2^5.

Altogether $2^7 + 2^8 - 2^5 = 352$

ANSWER: $\boxed{352}$

Q7. Exercise 5.1.44.

SOLUTION.

The task T is filling 10 spaces with either 5 consecutive 0's or 5 consecutive 1's. Task T is T_1 plus T_2 minus the overlap.

T_1 is the task of filling 10 spaces with 5 consecutive 0's. So there are 6 subtasks.

$$\underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{\qquad}\ \ \underline{\qquad}\ \ \underline{\qquad}\ \ \underline{\qquad}\ \ \underline{\qquad}$$

There are 5 spaces to be filled with either a 0 or a 1.

$$\underline{1}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{\qquad}\ \ \underline{\qquad}\ \ \underline{\qquad}\ \ \underline{\qquad}$$

There are 4 spaces to be filled with either a 0 or a 1.

$$\underline{\qquad}\ \ \underline{1}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{\qquad}\ \ \underline{\qquad}\ \ \underline{\qquad}$$

There are 4 spaces to be filled with either a 0 or a 1.

$$\underline{\qquad}\ \ \underline{\qquad}\ \ \underline{1}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{\qquad}\ \ \underline{\qquad}$$

There are 4 spaces to be filled with either a 0 or a 1.

$$\underline{\qquad}\ \ \underline{\qquad}\ \ \underline{\qquad}\ \ \underline{1}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{\qquad}$$

There are 4 spaces to be filled with either a 0 or a 1.

$$\underline{\qquad}\ \ \underline{\qquad}\ \ \underline{\qquad}\ \ \underline{\qquad}\ \ \underline{1}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}\ \ \underline{0}$$

There are 4 spaces to be filled with either a 0 or a 1.

So $2^5 + 2^4 + 2^4 + 2^4 + 2^4 + 2^4 = 96$

WLOG T_1 is the task of filling 10 spaces with 5 consecutive 1's is the same as filling 10 spaces with 5 consecutive 0's. So this like $2^5 + 2^4 + 2^4 + 2^4 + 2^4 + 2^4 = 96$.

The overlap is 1111100000 and 0000011111 so just 2.

Altogether $2 \times (2^5 + 2^4 + 2^4 + 2^4 + 2^4 + 2^4) - 2 = 190$

ANSWER: $\boxed{190}$

Q8. Exercise 5.1.45. NOT GRADED.

SOLUTION.

Q9. Exercise 5.1.46.

SOLUTION.

If there are 38 computer science majors and 23 math majors and 7 joint majors then the number of students is set of cs major plus math major minus the overlap of joint majors is $38 + 23 - 7 = 54$

ANSWER: $\boxed{54}$

Q10. Exercise 5.1.47.

SOLUTION.

To find the number of positive integers not over 100 divisible by 4 or 6 is the number divisible by 6 plus the number divisible by 4 minus the overlap.

$$\left\lfloor \frac{100}{4} \right\rfloor + \left\lfloor \frac{100}{6} \right\rfloor - \left\lfloor \frac{100}{12} \right\rfloor = 33$$

ANSWER: $\boxed{33}$

Q11. Exercise 5.1.48.

SOLUTION.

The task T is the number of different initials a person can have if their initials range from length 2 to 5. So T is $T_1 + T_2 + T_3 + T_4$.

Where T_1 the task of finding initials length 2 so there are 26 choices, so 26^2.

Where T_2 the task of finding initials length 3 so there are 26 choices, so 26^3.

Where T_3 the task of finding initials length 4 so there are 26 choices, so 26^4.

Where T_4 the task of finding initials length 5 so there are 26 choices, so 26^5.

Altogether $26^2 + 26^3 + 26^4 + 26^5 = 12, 356, 604$

ANSWER: $\boxed{12, 356, 604}$

Q12. Exercise 5.1.49.

SOLUTION.

a) The number of passwords (with length 8 to 12) is like finding passwords of length 8 + length 9 + length 10 + length 11 + length 12.

Finding passwords of length 8 means there are 68 possibilites for each place so 68^8

Finding passwords of length 9 means there are 68 possibilites for each place so 68^9

Finding passwords of length 10 means there are 68 possibilites for each place so 68^{10}

Finding passwords of length 11 means there are 68 possibilites for each place so 68^{11}

Finding passwords of length 12 means there are 68 possibilites for each place so 68^{12}

Altogether $68^8 + 68^9 + 68^{10} + 68^{11} + 68^{12} \approx 9.9 \times 10^{21}$

ANSWER: $\boxed{9.9 \times 10^{21}}$

b) The number of passwords with at least one occurance of one of the special characters is like finding the total number of passwords (part a) minus those without an occurance of one of the 6 special characters.

Finding passwords of length 8 means there are 62 possibilites for each place so 62^8

Finding passwords of length 9 means there are 62 possibilites for each place so 62^9

Finding passwords of length 10 means there are 62 possibilites for each place so 62^{10}

Finding passwords of length 11 means there are 62 possibilites for each place so 62^{11}

Finding passwords of length 12 means there are 62 possibilites for each place so 62^{12}

Altogether $62^8 + 62^9 + 62^{10} + 62^{11} + 62^{12} \approx 3.2 \times 10^{21}$

So $9.9 \times 10^{21} - 3.2 \times 10^{21} \approx 6.6 \times 10^{21}$

ANSWER: $\boxed{6.6 \times 10^{21}}$

c) The number of years to find all possible passwords is taking (part a) and dividing by the number of seconds in a minute times number of minutes in an hour times the number of hours in a day times the number of days in a year. $\frac{9.9\times 10^{21}}{60\cdot 60\cdot 24\cdot 365.2425} \approx 314,374$.

ANSWER: $\boxed{314,374\text{years.}}$

Q13. Exercise 5.1.53.

SOLUTION.

The task T of arranging 4 letters a, b, c, d without b following a, is T_1 minus task T_2.

Where T_1 is finding the total number of strings so there are 4 possibilities for the first position, 3 for the next position, 2 for the third position, and one for the fourth. So by multiplication principle there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ possiblities.

Where T_2 is finding the total number of strings without b followig a so treat ab as a unit and that leaves 3 spots to fill;3 possibilities for the first position, 2 for the next position, and one choice for the final postition. So by multiplication principle there are $3 \cdot 2 \cdot 1 = 6$ possiblities.

Altogether there are $T = T_1 - T_2 = 24 - 6 = 18$ possiblities.

ANSWER: $\boxed{18}$

Q14. Exercise 5.1.61.

SOLUTION.

To form a diagonal is the same as selecting two points which are not adjacent and joining the two points with a line. Therefore the number of diagonals must be the number of ways to select two non-adjacent points from the n points.

Let T be the task of writing down two non-adjacent points from the n points. Then T is the same as performing T_1 and T_2 in sequence where T_1 is choosing a point from the n possible points and T_2 is the task of selecting a point from the n points which is not adjacent to the one selected by T_1. Note that there are n ways to perform T_1. Suppose the point selected by T_1 is p. After T_1 is performed, there are $n-3$ points left which are both not p and not adjacent to p. Therefore there are $n-3$ ways to perform T_2. By the multiplication principle, there are $n(n-3)$ ways to perform T. Note that this is possible only when $n > 3$. For $n \leq 3$, there are 0 such permutations.

Therefore there are $n(n-3)$ ways of writing down such permutations for $n \geq 3$; 0 otherwise.

Now let T_3 be the task of selecting two non-adjacent points from the given n points. If we perform T_3 and then T_4, the task of permuting the selected points, we would have generated all permutations of non-adjacent points. Let the number of ways to perform T_3 be N. The number of ways to perform T_4 is 2. Therefore by the multiplication principle, we have

$$N \cdot 2 = n(n-3)$$

if $n \geq 3$; 0 otherwise. Therefore the number of ways to select two non-adjacent points is

$$N = \frac{n(n-3)}{2}$$

if $n \geq 3$ and 0 otherwise.

ANSWER: $\boxed{0 \text{ if } n \leq 3 \text{ and } \dfrac{n(n-3)}{2} \text{ otherwise}}$

The remaining questions involve the Pigeonhole Principle and combinatorial proofs. You should attempt *all* questions on the Pigeonhole Principle on your own.

Q15. Exercise 5.2.2.

SOLUTION.

If there are 30 students then the first 26 students may have distinct first letters, one for each letter of the alphabet, but the 27th student must choose a letter of the alphabet that has been assigned to one of the first 26 students. Therefore by the pigeonhole principle at least two of the 30 students will have same first letter to their name.

ANSWER: $\boxed{\text{At least two have same last name.}}$

Q16. Exercise 5.2.4. NOT GRADED.

SOLUTION.

The following involves combinatorial proofs of identities. A comobinatorial proof involved viewing a counting problem in two different ways. I have already given some examples in class. Here's another example:

$$\binom{n}{r} = \binom{n}{n-r}$$

The first thing to do is of course to play around with the above identity by checking that it's true with some values of n and r.

Here's an algebric proof.

Algebraic Proof. By definition

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{1}$$

Therefore

$$\begin{aligned}
\binom{n}{n-r} &= \frac{n!}{(n-r)!(n-(n-r))!} \\
&= \frac{n!}{(n-r)!r!} && \text{because } n-(n-r)=r \\
&= \frac{n!}{r!(n-r)!} && \text{because } (n-r)!r! = r!(n-r)! \\
&= \binom{n}{r} && \text{by (1)}
\end{aligned}$$

□

(This identity is so easy that the algebraic proof is not too complicated.)

Here's a combinatorial proof:

Combinatorial Proof. Given n distinct objects and $0 \le r \le n$, the number of ways to form sets of r objects from the collection of n objects is the same as the number of ways select $n-r$ objects *not* to be include in the sets to form. Therefore

$$\binom{n}{r} = \binom{n}{n-r}$$

□

Read the above carefully and make sure you understand it. In case you don't see it let me give you more details.

Suppose T_1 is the task of forming sets X of r objects from the n distinct objects. The task T_1 can be achieved in a different way. We perform the following in sequence:

1. Task T_{21}: Form sets Y of $n-r$ objects from the given n objects.
2. Task T_{22}: Form sets X of r objects from the objects *not* in Y.

For instance suppose the given n objects are A, B, C, D, E and $r = 2$. One way of forming a set of $r = 2$ objects is $\{B, E\}$. This is the method T_1. This set can also be forms using the other method (i.e. T_{21} followed by T_{22}) like this:

1. Task T_{21}: Form sets $Y = \{A, C, D\}$.
2. Task T_{22}: Form sets X to be the objects not in Y, i.e., $X = \{B, E\}$.

Essentially forming the number of sets of 2 elements is the same as the number of sets of 3 elements.

Many complex combintorial identities can be proven with shorter combinatorial proofs than algebraic proofs.

Here's one that I proved in class:

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

You recall this was proven in the following way. The left hand side says

$$\binom{n}{r}$$

which is the number of ways to select r objects from n objects (all distinct). Now this can be counted in a different way. Suppose among the n objects you have a special one, call it X. The task of selecting r objects from the n is the same as performing either task T_1, the task of selecting r objects so that X is included, or performing the task T_2 where X is not included. For the case where X is included, the number of ways would be the same as the number of ways to select $r-1$ from $n-1$ objects since X is already included. This gives us

$$\binom{n-1}{r-1}$$

And in the case where X is not included, then the task needs to select r objects from $n-1$ (since X is not included). Therefore

$$\binom{n-1}{r}$$

By the addition principle,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

Here's an important hint: There's an addition in the above identity. This is a hint that the addition principle is involved.

There's another identity that I talked about in class:

$$r\binom{n}{r} = n\binom{n-1}{r-1}$$

Recall how I proved this:

The left hand side can be interpreted as the number of ways to form a committee of r members from n, and selecting a chairperson among the r members in committee.

The right hand side can be interpreted as selecting a person among the n to be the chairperson, and then selecting $r-1$ persons from the remaining $n-1$ to join this chairperson to complete the committee.

Here's another important hint: The multiplication principle must be involved since multiplication appears in the above. Why? Note that the above can be rewritten as

$$\binom{r}{1} \times \binom{n}{r} = \binom{n}{1} \times \binom{n-1}{r-1}$$

So remember this: Rewrite everything into the n-choose-r symbol, and then look for addition and/or multiplication.

Q17. [NOT GRADED]

For each of the following identities, either prove it by providing a combinatorial proof or disprove it by construct the simplest possible counterexample.

(a)

$$\binom{3n}{2n}\binom{2n}{n}\binom{n}{1} = \binom{3n}{1}$$

(b) Suppose n is odd, say $n = 2k + 1$, then

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots + \binom{n}{n}$$

(c)

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

[HINT: At least one of the above is a lie.]

SOLUTION.