

MATH325: Discrete Math 2
Assignment 10

This assignment involves counting solutions to linear systems using power series or generating functions.

The goal is to study the number of solutions to the following system:

$$\begin{cases} a + b + c = n \\ a, b, c \in \mathbb{Z} \\ 0 \leq a, 2 \leq b, 0 < c, c \text{ odd} \end{cases}$$

for different values of $n \geq 0$. Let a_n be the number of solutions to the above system. For instance a_{1000} is the number of solutions to the above when n is 1000.

To be more specific, a_0 is the number of solutions to

$$\begin{cases} a + b + c = 0 \\ a, b, c \in \mathbb{Z} \\ 0 \leq a, 2 \leq b, 0 < c, c \text{ odd} \end{cases}$$

In this case, we can compute the value of a_0 quickly: $a_0 = 0$. In fact, since $b \geq 2$ and $c \geq 1$, if $a + b + c = n$ has a solution, then $a + b + c \geq 3$. We conclude immediately that

$$a_0 = a_1 = a_2 = 0$$

Clearly $a_3 = 1$ since the only solution to

$$\begin{cases} a + b + c = 0 \\ a, b, c \in \mathbb{Z} \\ 0 \leq a, 2 \leq b, 0 < c, c \text{ odd} \end{cases}$$

is

$$a = 0, b = 2, c = 1$$

It's easy to see that $a_4 = 2$:

$$4 = 1 + 2 + 1 = 0 + 3 + 1$$

Before going on, you should compute a_5 and a_6 to get a feel for the problem. (You can either do this by hand or write a program to this.) The values of a_n for $n = 0, 1, 2, \dots, 6$ is useful for checking against the general result that you will be deriving.

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

You will see that we do not need to guess a formula for a_n and then prove by mathematical induction. Instead we will use the method of power series to compute a formula for a_n .

Q1. Write down $f(x)$, the product of 3 power series such that the coefficient of x^n of this product is a_n . You need not simplify. For instance the following is a product of power series (which is not the answer to this question!)

$$(2 + 2x + 2x^2 + \cdots)(0 + 1x + 2x^2 + \cdots)(1 + 2x + 3x^2 + \cdots)$$

You should write down explicitly at least 3 nonzero terms for each power series.

SOLUTION.

Q2 Rewrite the product of power series in (a) as a rational function. The denominator should be factorized (for instance $(1 - x^2) = (1 - x)(1 + x)$), with similar terms collected together. For instance the following is a rational function (which is not the right answer to this question!)

$$f(x) = \frac{x}{(2 - 3x)^3(1 + 5x)^6}$$

Note that all the $(1 + 5x)$ factors (there are 6 of them) are collected together. Do *not* write for instance

$$f(x) = \frac{x}{(2 - 3x)^3(1 + 5x)^2(1 + 5x)^4}$$

SOLUTION.

Q3. Using the theory of partial fractions, rewrite the rational expression as a linear sum of simpler rational functions of the form $\frac{1}{p(x)^k}$ where $p(x)$ is a polynomial of degree at most 2. For instance the following is a linear sum of such rational functions:

$$f(x) = 5\frac{1}{(1-x)^3} + \frac{3}{4}\frac{1}{(2+x+3x^2)^5}$$

(of course this is not the answer!)

[Hint: For this problem, the $p(x)$'s are in fact linear, i.e. degree 1. Also, there are 4 terms in this sum.]

SOLUTION.

Q4. Using Q3, rewrite $f(x)$ as a power series with the coefficient of x^n in terms of n . The following is an example (which is of course not the right answer!)

$$f(x) = \sum_{n=0}^{\infty} \frac{2+n^2}{1+n} x^n$$

Simplify the coefficients of x^n so that the binomial coefficients (if any does) does not occur. For instance you know that

$$\binom{100}{3} = \frac{100 \cdot 99 \cdot 98}{1 \cdot 2 \cdot 3}$$

and likewise

$$\binom{n}{3} = \frac{(n-1) \cdot (n-2) \cdot (n-3)}{1 \cdot 2 \cdot 3}$$

etc. You are strongly advised to write some simple programs to check your computation.
SOLUTION.

Q5. What is the coefficient x^n from Q4 in terms of n Recall that the coefficient of x^n for the power series of $f(x)$ is a_n . In other words what is the formula for a_n in terms of n ?

[It's a good idea now to check the formula against the values of a_n for $n = 0, 1, 2, \dots, 6$ that you have computed earlier.]

SOLUTION.

Q6. (a) How many solutions are there to

$$\begin{cases} a + b + c = 1000 \\ a, b, c \in \mathbb{Z} \\ 0 \leq a, 2 \leq b, 0 < c, c \text{ odd} \end{cases}$$

(b) How many solutions are there to

$$\begin{cases} a + b + c = 1001 \\ a, b, c \in \mathbb{Z} \\ 0 \leq a, 2 \leq b, 0 < c, c \text{ odd} \end{cases}$$

SOLUTION.