EE402 - Discrete Time Systems **Spring 2018**

Lecture 16

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State-Feedback & Pole Placement

Given a discrete-time state-evolution equation

$$x[k+1] = Gx[k] + Hu[k]$$

If direct measurements of all of the states of the system (e.g. $y[k] = x[k]$) are available, one of the most popular control methods is the linear state feedback control,

$$u[k] = -Kx[k]$$

which can be thought as a generalization of P controller to the vector form. Under this control law, without any reference signal, the system becomes an autonomous system

$$\begin{aligned} x[k+1] &= Gx[k] + H(-Kx[k]) \\ x[k+1] &= (G - HK)\, x[k] \end{aligned}$$

The system matrix of this new autonomous system is $\hat{G} = G - HK$. The important question is how we need to choose K?. Note that

$$\begin{aligned} K &\in \mathbb{R}^n \quad \text{Single} - \text{Input} \\ K &\in \mathbb{R}^{n \times p} \quad \text{Multi} - \text{Input} \end{aligned}$$

As in all of the control design techniques, the most critical criterion is stability, thus we want all of the eigenvalues to be strictly inside the unit circle. However, we know that there could be different requirements on the poles/eigenvalues of the system.

The fundamental principle of "pole-placement" design is that we first define a desired closed-loop eigenvalue set $\mathcal{E}^* = \{\lambda_1^*, \ \cdots \ , \ \lambda_1^*\}$, and then if possible we choose K^* such that the closed-loop eigenvalues match the desired ones.

The necessary and sufficient condition on arbitrary pole-placement is that the system should be fully reachable.

In Pole-Placement, the first step is computing the desired characteristic polynomial.

$$\begin{aligned} \mathcal{E}^* &= \{\lambda_1^*, \ \cdots \ , \ \lambda_n^*\} \\ p^*(z) &= (z - \lambda_1^*) \cdots (z - \lambda_n^*) \\ &= z^n + a_1^* z^{n-1} + \cdots + a_{n-1}^* z + a_n^* \end{aligned}$$

Then we tune K such that

$$\det\left(zI - (G - HK)\right) = p^*(z)$$

Direct Design of State-Feedback Gain

If n is small, the most efficient method could be the direct design.

Example: Consider the following DT system

$$x[k+1] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]$$

Design a state-feedback rule such that poles are located at $\lambda_{1,2} = 0$ (Dead-beat gain)

Solution: Desired characteristic equation can be computed as

$$p^*(z) = z^2$$

Let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, then the characteristic equation of $\hat{G}$ can be computed as

$$\begin{aligned} \det\left(zI - (G - HK)\right) &= \det\left(\begin{bmatrix} z-1+k_1 & k_2 \\ k_1 & z-2+k_2 \end{bmatrix}\right) \\ &= z^2 + z(k_1 + k_2 - 3) + (2 - 2k_1 - k_2) \end{aligned}$$

If we match the equations

$$\begin{aligned} z^2 + z(k_1 + k_2 - 3) + (2 - 2k_1 - k_2) &= z^2 \\ k_1 + k_2 &= 3 \\ 2k_1 + k_2 &= 2 \\ k_1 &= -1 \\ k_2 &= 4 \end{aligned}$$

Thus $K = \begin{bmatrix} -1 & 4 \end{bmatrix}$. Now let's compute $\hat{G}^k$

$$\begin{aligned} \hat{G} &= \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \\ \hat{G}^2 &= \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &\vdots \end{aligned}$$

It can be seen that closed-loop system rejects all initial condition perturbations in 2 steps.

Design of State-Feedback Gain Using Reachable Canonical Form

Let's assume that the state-space representation is in reachable canonical form and we have access to all states of this form

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u[k]$$

Let $K = \begin{bmatrix} k_n & \cdots & k_1 \end{bmatrix}$, then autonomous system takes the form

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} x[k] - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_n & \cdots & k_1 \end{bmatrix} x[k]$$

$$x[k+1] = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -(a_n + k_n) & -(a_{n-1} + k_{n-1}) & -(a_{n-2} + k_{n-2}) & \cdots & -(a_1 + k_1) \end{bmatrix} x[k]$$

Let $p^*(z) = z^2 + a_1^* z + \cdots + a_{n-1}^* z + a_n^*$, then K can be computed as

$$K = \begin{bmatrix} (a_n^* - a_n) & \cdots & (a_1^* - a_1) \end{bmatrix}$$

However, what if the system is not in Reachable canonical form. We can find a transformation which finds the Reachable canonical representation.

The reachability matrix of a state-space representation is given as

$$M = \begin{bmatrix} H \mid GH \mid \cdots \mid G^{n-1}H \end{bmatrix}$$

Let's define a transformation matrix T as follows:

$$T = MW \quad , \quad x[k] = T\hat{x}[k]$$
$$\hat{x}[k+1] = \left[T^{-1}GT\right]\hat{x}[k] + T^{-1}Hu[k]$$

where

$$W = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & & \vdots \\ a_1 & 1 & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix}$$

Then it is given that

$$T^{-1}GT = \hat{G} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$T^{-1}H = \hat{H} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Let's compute $T\hat{H}$

$$
\begin{aligned}
T\hat{H} = MW\hat{H} &= M \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & \iddots & & \vdots \\ a_1 & 1 & & & \\ 1 & 0 & \cdots & & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \\
&= \left[\begin{array}{c|c|c|c} H & GH & \cdots & G^{n-1}H \end{array} \right] \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \\
&= H
\end{aligned}
$$

A similar approach (but longer) can be used to show that $T^{-1}GT = \hat{G}$. We know how to design a state-feedback gain $\hat{K}$ for the Reachable canonical form. Given $\hat{K}$, $u[k]$ is given as

$$
\begin{aligned}
u[k] &= -\hat{K}\hat{x}[k] \\
&= -\hat{K}T^{-1}\hat{x}[k] \\
K &= \hat{K}T^{-1}
\end{aligned}
$$

Example 2: Consider the following DT system

$$
x[k+1] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]
$$

Design a state-feedback rule using the Reachable canonical form approach, such that poles are located at $\lambda_{1,2} = 0$ (Dead-beat gain)

Solution: Characteristic equation of G can be derived as

$$
\det\left(\begin{bmatrix} z-1 & 0 \\ 0 & z-2 \end{bmatrix} \right) = z^2 - 3z + 2
$$

The Reachability matrix can be computed as

$$
M = \left[\begin{array}{c|c} H & GH \end{array} \right] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}
$$

The matrix W can be computed as

$$
W = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}
$$

Transformation matrix, T and its inverse T^{-1} can be computed as

$$
\begin{aligned}
T = MW &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \\
T^{-1} &= \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}
\end{aligned}
$$

Given that desired characteristic polynomial is $p^*(z) = z^2$, $\hat{K}$ of reachable canonical from can be computed as

$$\begin{aligned} \hat{K} &= \begin{bmatrix} -a_2 & -a_1 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 3 \end{bmatrix} \end{aligned}$$

Finally K can be computed as

$$\begin{aligned} K = \hat{K}T^{-1} &= \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 4 \end{bmatrix} \end{aligned}$$

As expected this is the same result with the one found in Example 1 (Direct-Method).