

Fundamentals of Geometry Project 2

Coloring Gaussian Curvature on Quartic Triangular Bézier Patches From Smooth Approximations of Irregular Closed Blender Meshes

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Project Description:

The goal of this project was to color Gaussian curvature on quartic triangular Bézier Patches. These patches were generated from smooth approximations of arbitrary closed meshes, created in the open source modeling program Blender. The refinement process follows the algorithm detailed by Loop [4].

Spline Construction:

The spline construction is split into three stages, each of which is briefly detailed and illustrated:

1. Mesh Refinement
2. Quad-Net Construction
3. Quartic Triangular Bézier Patch Construction

The output of this algorithm is a collection of quartic triangular Bézier patches.

Mesh Refinement

The refined meshes are a “smoothing” of the the inputted closed meshes. The meshes resulting from this step have the property that each vertex in the mesh has exactly 4 edges incident on it. As the process is computationally expensive for complex meshes (as with e edges and v vertices $\mathcal{O}(ev)$), simpler meshes are included for illustration.

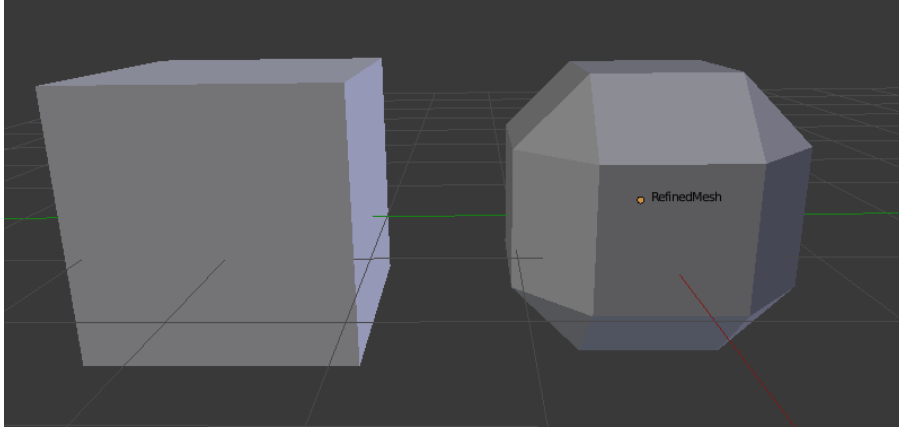


Figure 1: A cube (left) and the resulting refined mesh \mathcal{M}^1 (right).

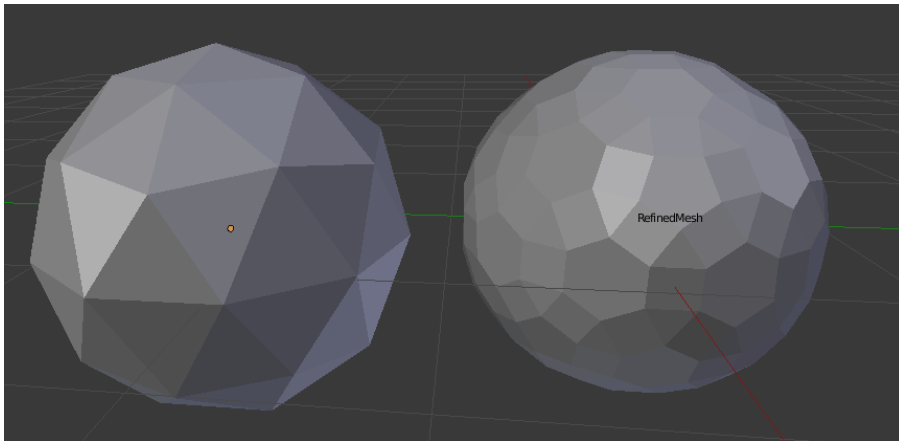


Figure 2: An icosphere (left) and the resulting refined mesh \mathcal{M}^1 (right).

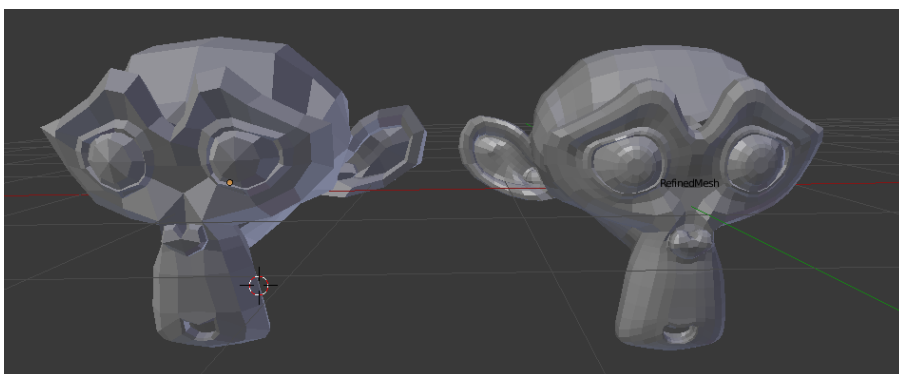


Figure 3: A monkey head (nicknamed 'Suzanne' left) and the resulting refined mesh \mathcal{M}^1 (right).

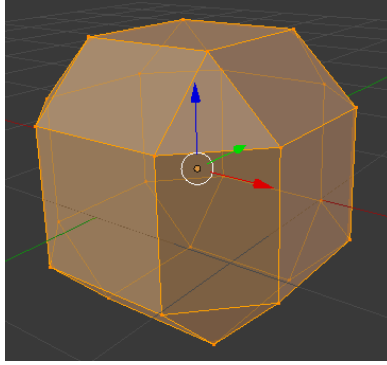


Figure 4: A closer view of the refined cube in Blender's edit mode.

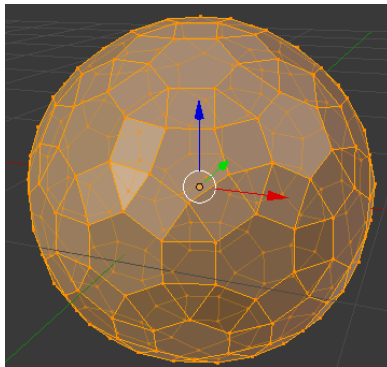


Figure 5: A closer view of the refined icosphere in Blender's edit mode.

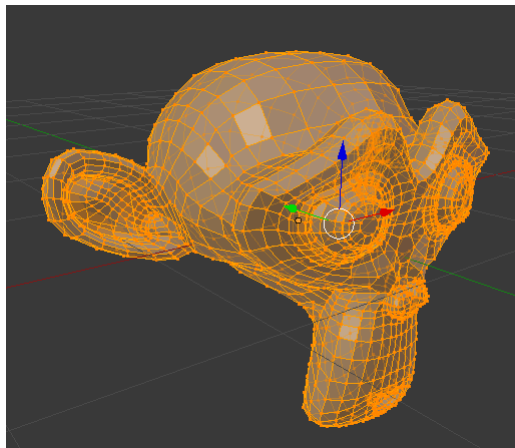


Figure 6: A closer view of the refined monkey in Blender's edit mode.

Quad-Net Construction

Quad-nets are sets of 16 points that surround each vertex. These points form nets that when “stitched” together cover the entire refined mesh \mathcal{M}^1 . Ultimately, the quad-nets serve as an intermediary phase between the mesh refinement and the generation of quartic triangular Bèzier surface patches.

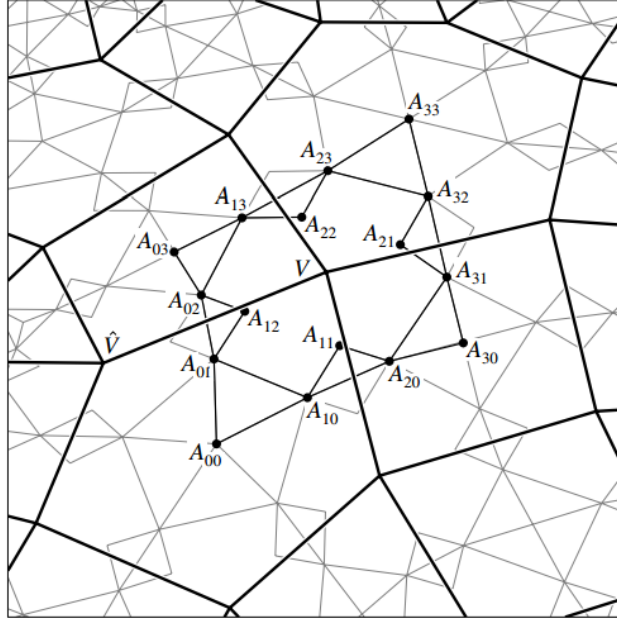


Figure 7: The quadnet corresponding to vertex V on the refined mesh \mathcal{M}^1 [4].

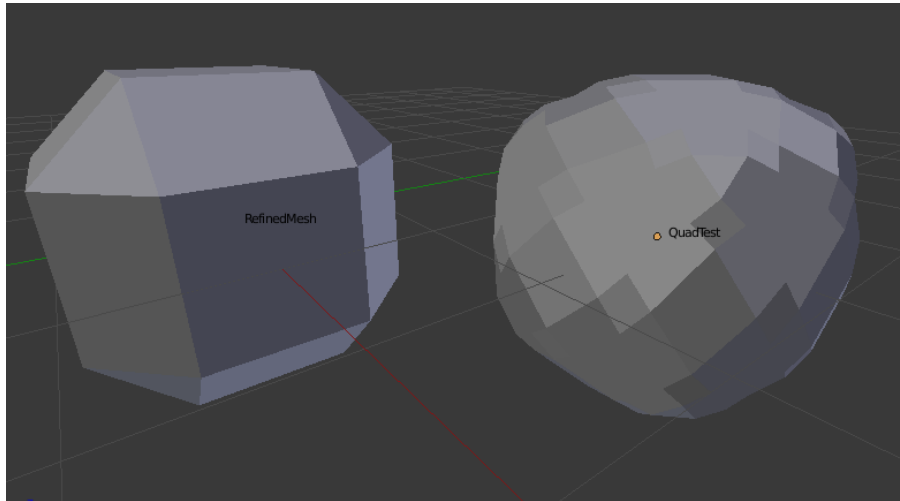


Figure 8: Quad-nets (right) corresponding to the cube’s \mathcal{M}^1 (left).

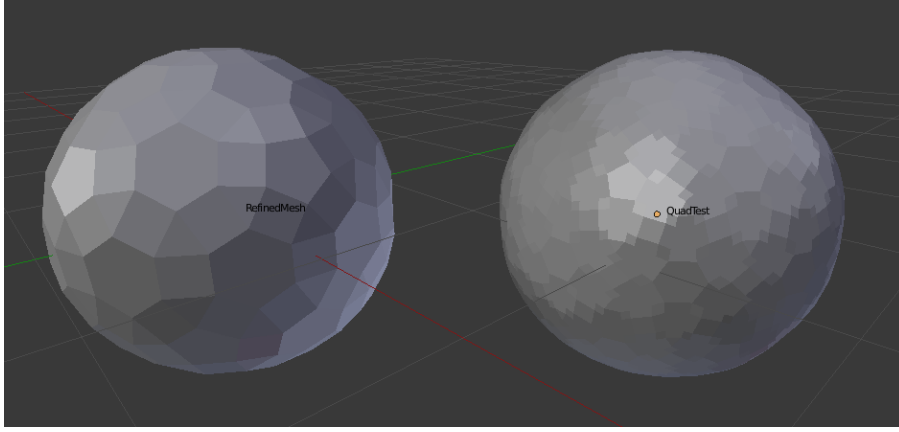


Figure 9: Quad-nets (right) corresponding to the icosphere's \mathcal{M}^1 (left).

As previously stated, the implementation is contingent on the fact that the \mathcal{M}^0 is a closed mesh. Upon closer examination one finds that the monkey head 'Suzanne' is an open mesh. Although the smoothing process succeeds, quad-net resolution fails for this open surface.

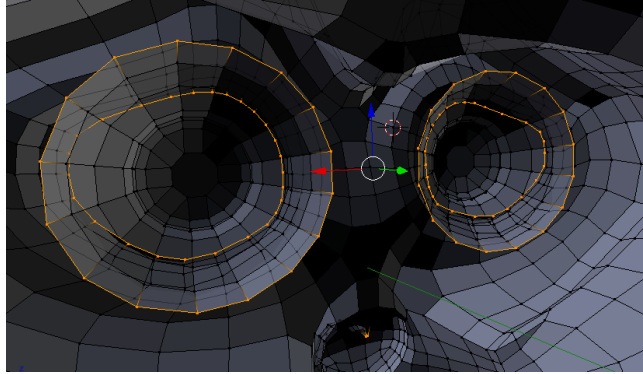


Figure 10: Inside of 'Suzanne'. Upon closer examination one recognizes the mesh is open.

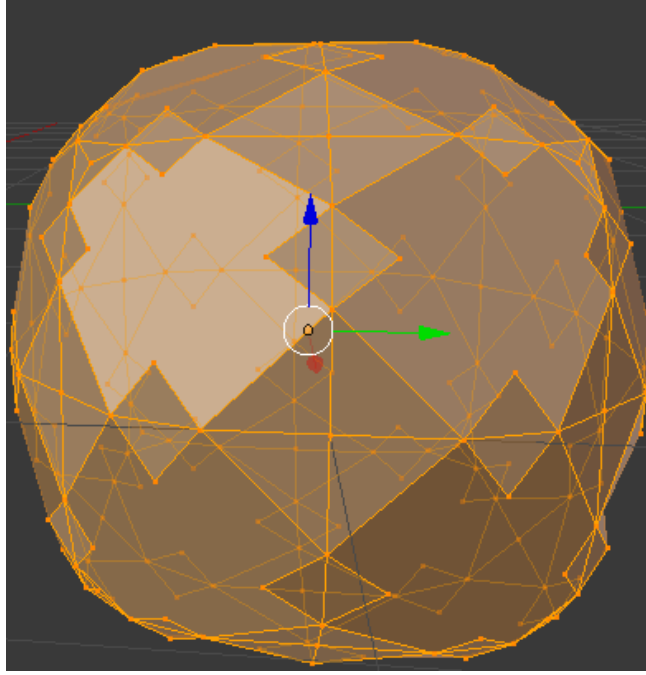


Figure 11: Quad-nets corresponding to the cube's \mathcal{M}^1 .

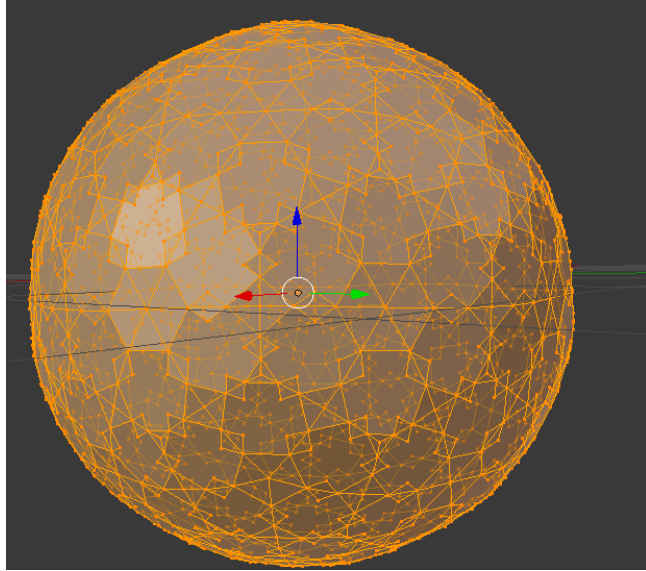


Figure 12: Quad-nets corresponding to the icosphere's \mathcal{M}^1 .

Quartic Triangular Bèzier Patch Construction

The final step of the construction creates a collection of triangular patch control points, from which unique equations for quartic triangular Bèzier patches are calculated. From these patches the Gaussian curvature is calculated and colored.

Quartic triangular Bèzier patches are calculated using the equation ($n = 4$):

$$\sigma(s, t, u) = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \frac{n!}{i!j!k!} s^i t^j u^k \alpha^i \beta^j \gamma^k \quad s + t + u = 1$$

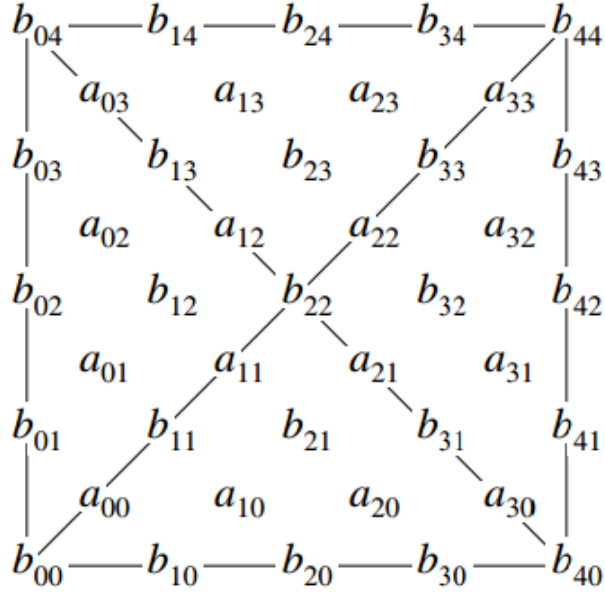


Figure 13: The 4 triangular Bèzier patches on a quad-net [4].

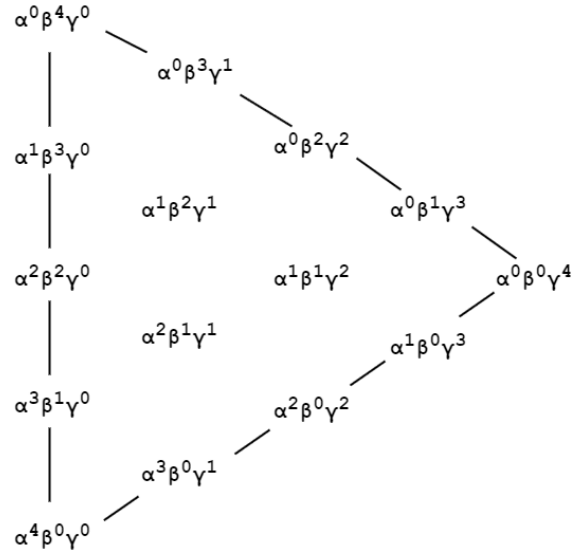


Figure 14: Application of the aforementioned formula to the left triangular Bèzier patch in Figure 13.

One notes that the above formula is equivalently expressed in the usual u, v parametrization as:

$$\sigma(u, v) = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \frac{n!}{i!j!k!} (1-u-v)^i u^j v^k \alpha^i \beta^j \gamma^k \quad 0 \leq u, v \quad u+v \leq 1$$

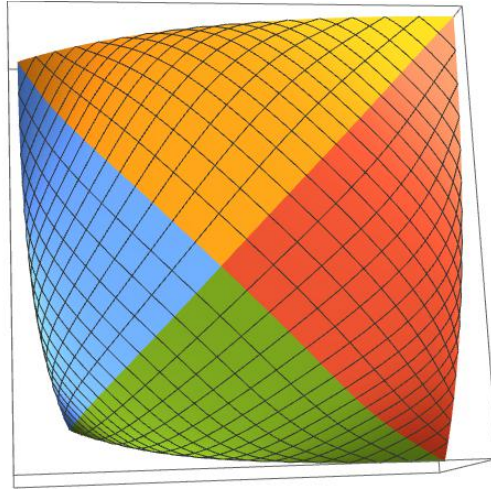


Figure 15: Mathematica Plot of Quartic Triangular Bèzier patches from a cubic Blender mesh.

Coloring Quartic Bèzier Patches:

Once the Quartic Triangular Bèzier patches are resolved into $\sigma(u, v)$ using sympy, the patches are colored with a UV map based on Gaussian curvature. Namely, Gaussian curvature is calculated using the first and second fundamental form coefficients

$$K = \frac{LN - M^2}{EG - F^2}.$$

Positive Gaussian curvature is colored in greens, negative Gaussian curvature in reds, and zero Gaussian curvature in blue.

Images/animations produced:

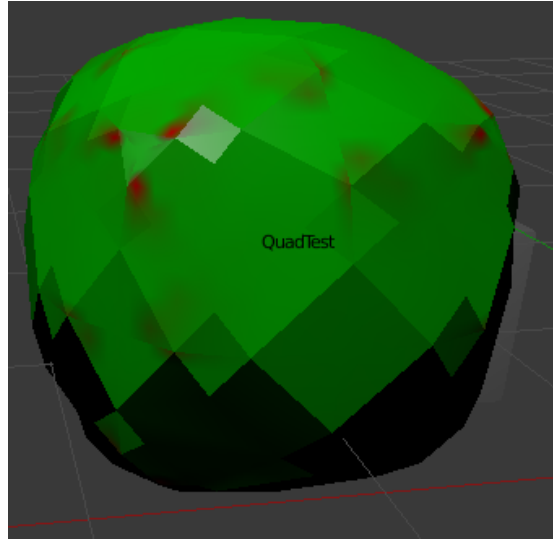


Figure 16: UV coloring of patches from a cubic Blender mesh.

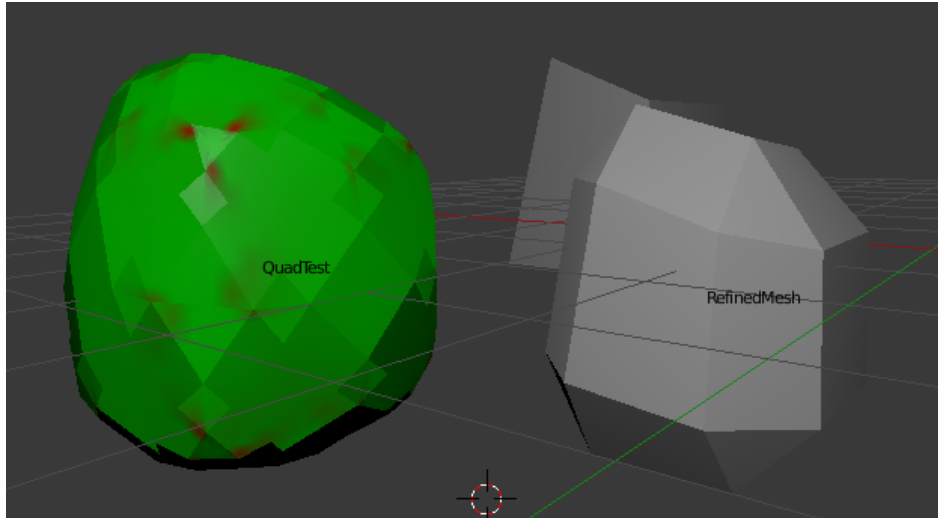


Figure 17: UV coloring of patches from an irregular Blender mesh.

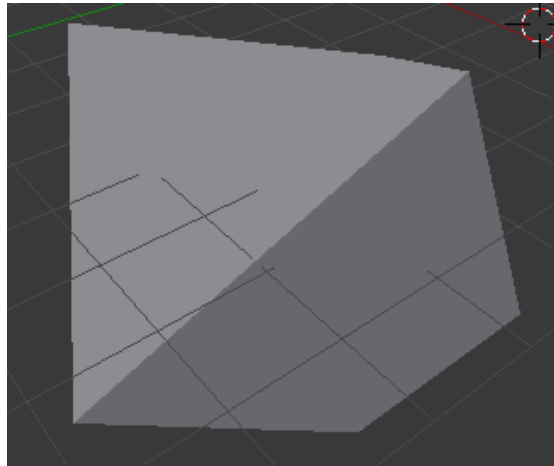


Figure 18: An starting irregular mesh.

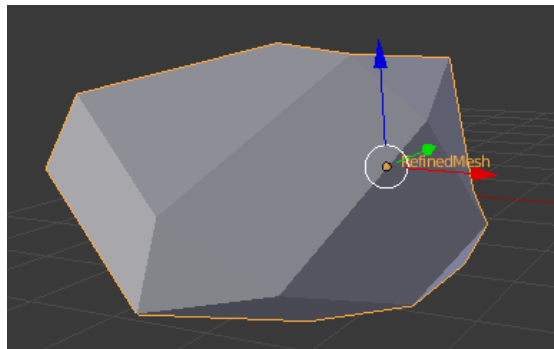


Figure 19: Refined irregular mesh from Figure 18.

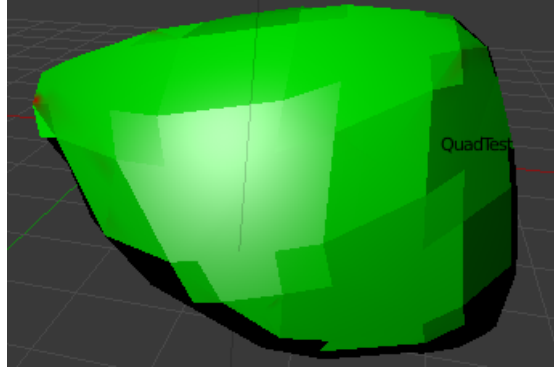


Figure 20: Colored irregular mesh for Figure 18.

Findings:

As the computation time is extremely substantial for meshes consisting of more than a few vertices, simpler meshes, consisting of a small number of vertices were used for inclusion in this report. As the refinement process itself was more involved than anticipated, the included results are limited to those above.

Relevant concepts/equations/formulas covered in class:

1. The unit normal to a surface patch σ used to calculate the Gaussian curvature.

$$\vec{n} = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

2. The first fundamental form coefficient used to calculate the Gaussian curvature.

$$E = \sigma_u \cdot \sigma_u$$

3. The first fundamental form coefficient used to calculate the Gaussian curvature.

$$F = \sigma_u \cdot \sigma_v$$

4. The first fundamental form coefficient used to calculate the Gaussian curvature.

$$G = \sigma_v \cdot \sigma_v$$

5. The second fundamental form coefficient used to calculate the Gaussian curvature.

$$L = \sigma_{uu} \cdot \vec{n}$$

6. The second fundamental form coefficient used to calculate the Gaussian curvature.

$$M = \sigma_{uv} \cdot \vec{n}$$

7. The second fundamental form coefficient used to calculate the Gaussian curvature.

$$N = \sigma_{vv} \cdot \vec{n}$$

8. The Gaussian curvature.

$$K = \frac{LN - M^2}{EG - F^2}$$

9. The formulation of quartic Bèzier patches for control points $\alpha^i, \beta^j, \gamma^k$

$$\sigma(u, v) = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \frac{n!}{i!j!k!} (1-u-v)^i u^j v^k \alpha^i \beta^j \gamma^k \quad 0 \leq u, v \quad u+v \leq 1$$

10. For each quartic patch this is in the form:

$$\begin{aligned} \sigma(u, v) = & b_{00}(1-u-v)^4 + 4b_{01}(1-u-v)^3u + 6b_{02}(1-u-v)^2u^2 + 4b_{03}(1-u-v)u^3 \\ & + b_{04}u^4 + 4a_{03}u^3v + 6b_{13}u^2v^2 + 4a_{12}uv^3 + b_{22}v^4 + 4a_{11}(1-u-v)v^3 \\ & + 6b_{11}(1-u-v)^2v^2 + 4a_{00}(1-u-v)^3v + 12a_{02}(1-u-v)u^2v \\ & + 12a_{01}(1-u-v)^2uv + 12b_{12}(1-u-v)uv^2 \end{aligned}$$

where $0 \leq u, v \quad u+v \leq 1$

Software:

Blender, Python 3.5.1, Mathematica, Draw.io

References

- [1] *Blender*, 2016. <https://www.blender.org>.
- [2] *Draw.io*, 2016. <https://www.draw.io/>.
- [3] *Python 3.5.1*, 2016. <https://www.python.org/downloads/release/python-351/>.
- [4] Charles Loop. Smooth spline surfaces over irregular meshes. In *Proceedings of the 21st annual conference on Computer graphics and interactive techniques*, pages 303–310. ACM, 1994.