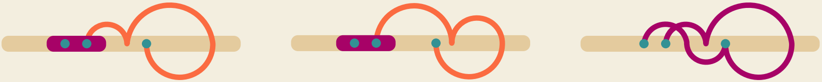
Taking a Hard Look at
Generalized Coloring Numbers



@blairdsullivan
University of Utah

AMS Spring 2023
Southeastern Sectional

Mar 18 2023
Georgia Tech

Special thanks to
Felix Reidl (Birkbeck College, London)
for original artwork and collaboration
on scientific communication

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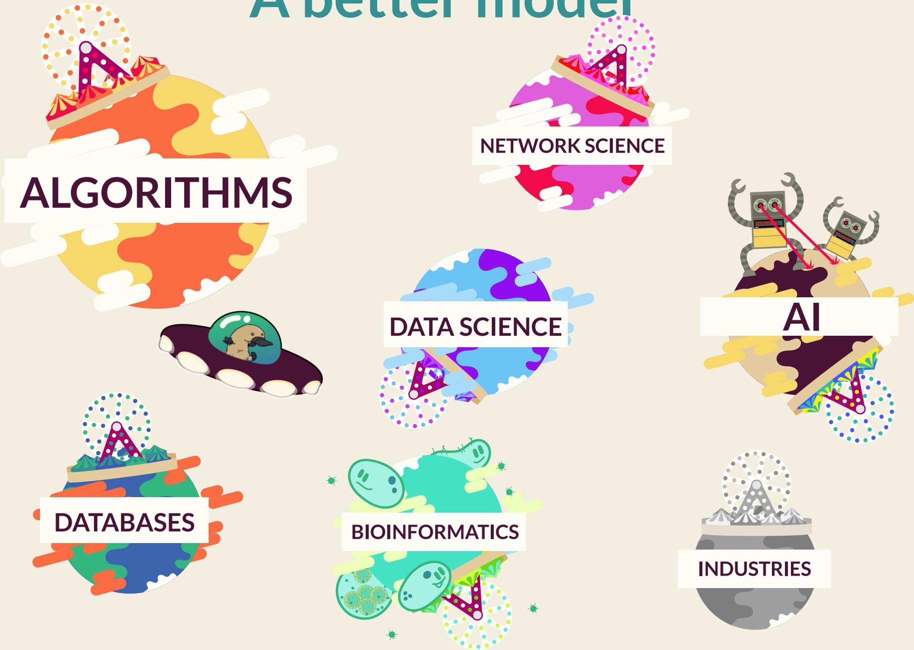
Part I

Sparse classes

A false dichotomy

A better model



Parameterized algorithms

	Classical view	Parameterized view
Bad	$O(n^{\log n}), O(2^n), \dots$ Not polynomial-time	$O(n^{f(k)})$ Slice-wise polynomial time
Good	$O(n^c)$ Polynomial-time ("efficient")	$O(f(k)n^c)$ Fixed-parameter tractable

[Dow99 Param. Complex.]

Choosing a class

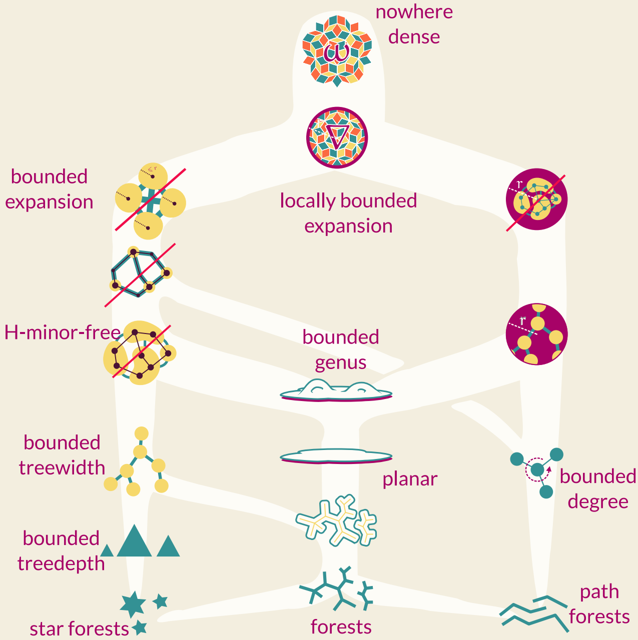
Larger classes

Less

Structure

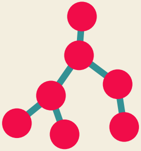
More

Algorithmic tractability

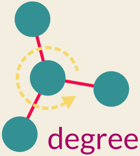


Can We Use Sparse Structure?

genus g



treewidth t

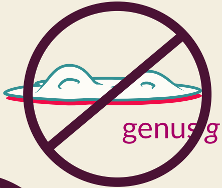


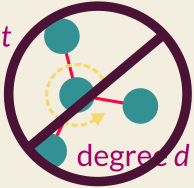
degree d

[Bul19 Alg] [Cyg19 Param. Alg]

Evidence says...

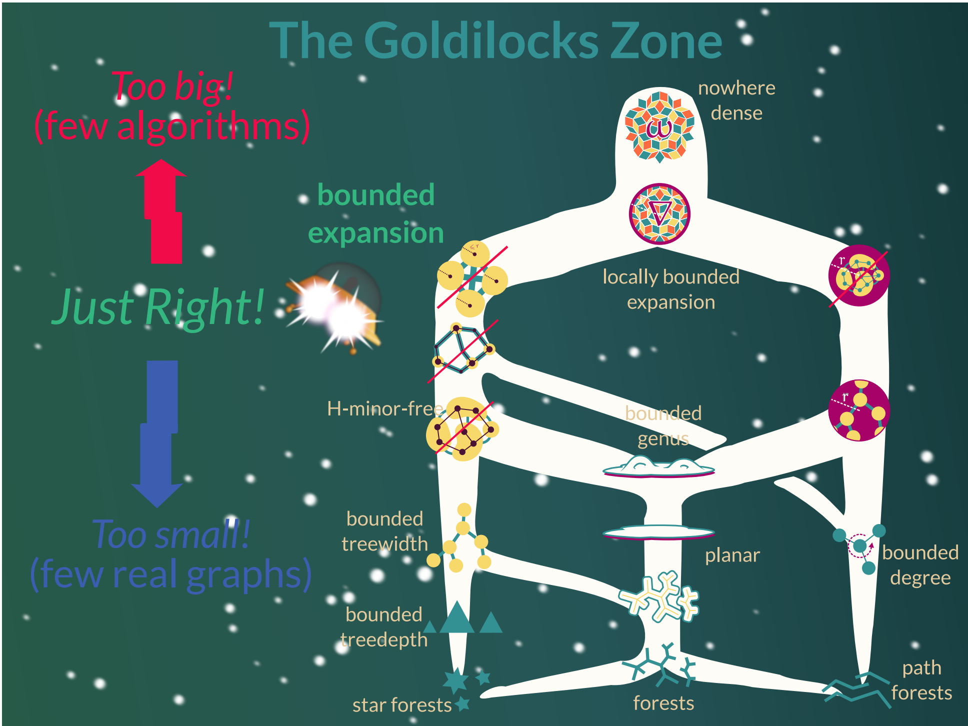
[Gas21 BIBM]
[Boe20 Env Plan B]





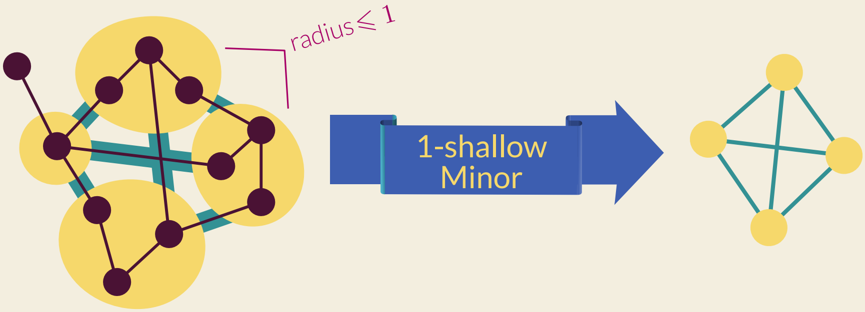
[Mar22 Alg Mol Bio]
[Man19 ICDT]
[Adc13 ICDM]*

⋮

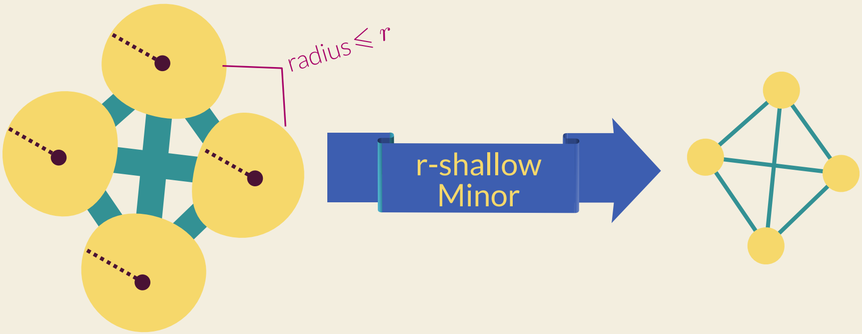


The Goldilocks Zone
Too big!
(few algorithms)
Just Right!
Too small!
(few real graphs)
bounded
expansion
nowhere
dense
locally bounded
expansion
H-minor-free
bounded
genus
bounded
treewidth
planar
bounded
degree
bounded
treedepth
star forests
forests
path
forests

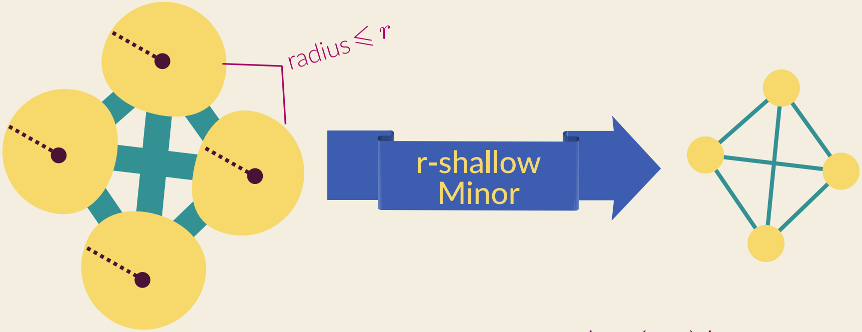
Shallow minors & bounded expansion



Shallow minors & bounded expansion



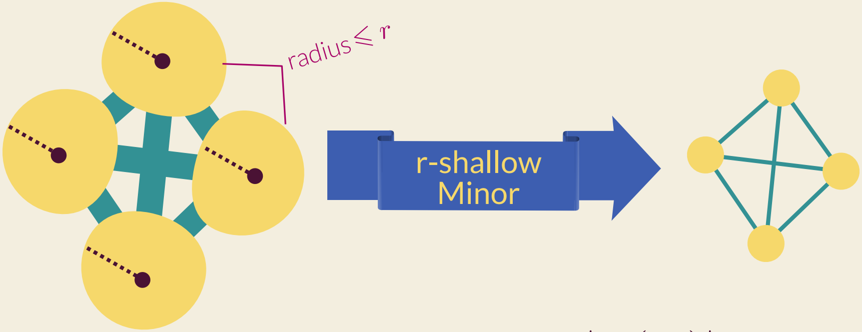
Shallow minors & bounded expansion



$$\nabla_r(G) = \max_{H \preccurlyeq_r G} \frac{|E(H)|}{|V(H)|}$$

A graph class has bounded expansion iff it is ∇_r-bounded.

Shallow minors & bounded expansion



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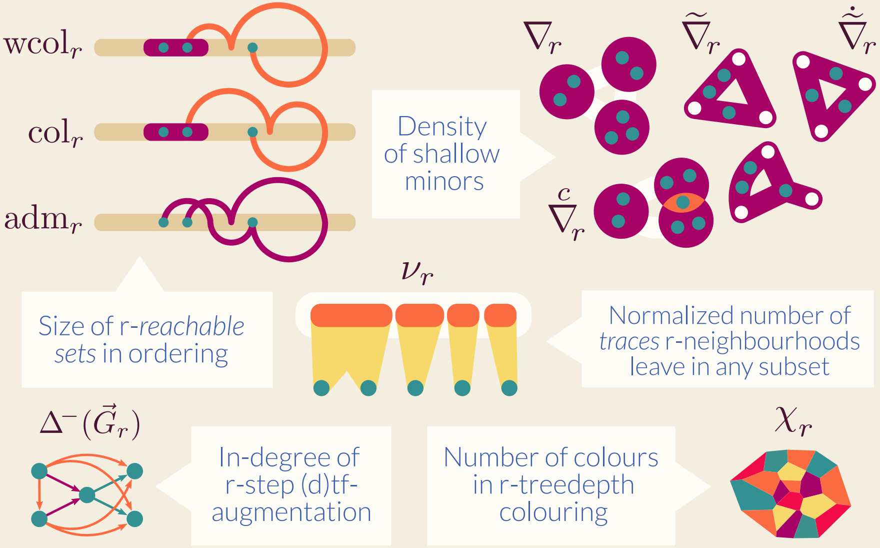
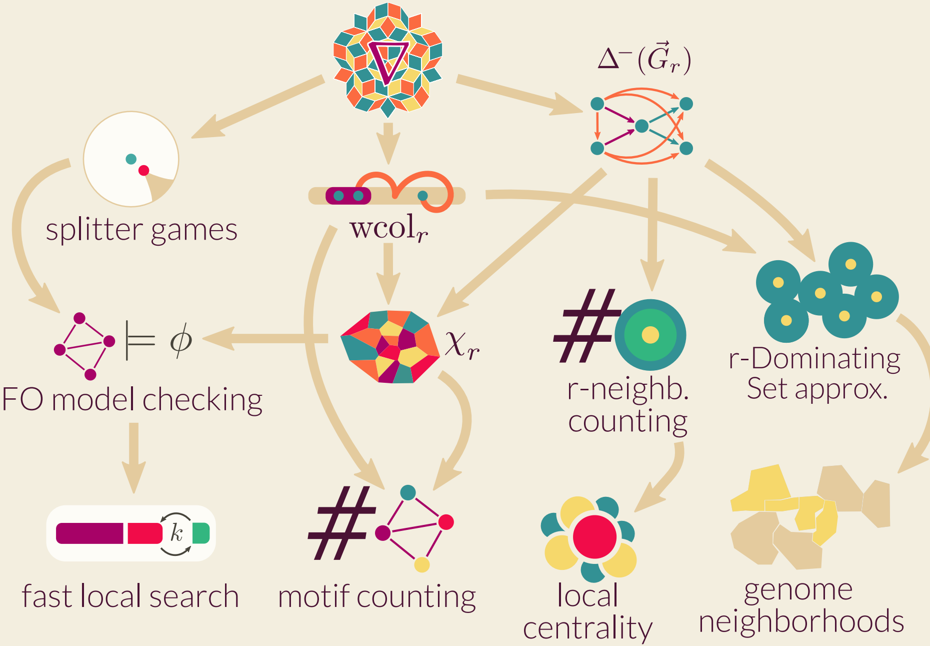
Bounded expansion

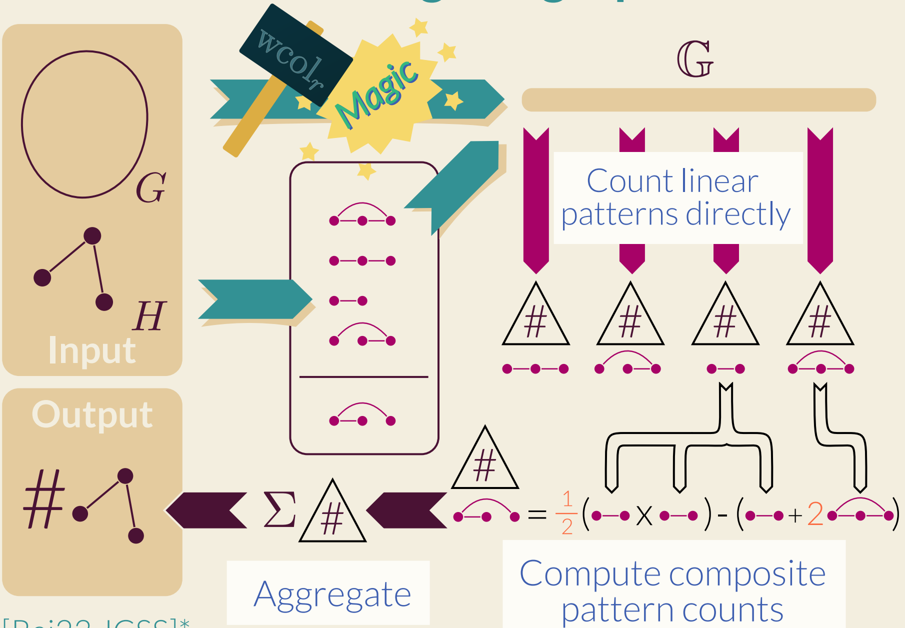


[Neš12 Sparsity]

Applications & Algorithms



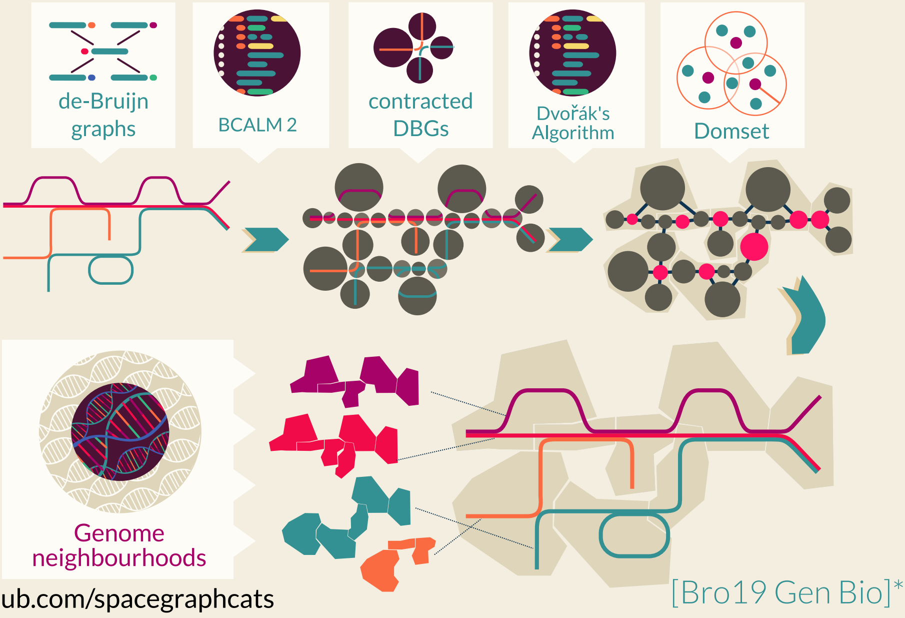
Counting Subgraphs



[Rei23 JCSS]*

github.com/theoryinpractice/mandoline

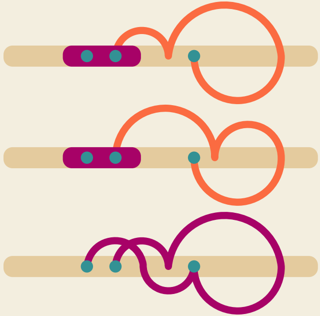
Index/Query Metagenomes



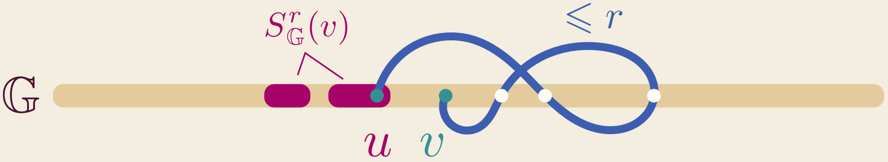
github.com/spacegraphcats

Part II

Generalized coloring numbers

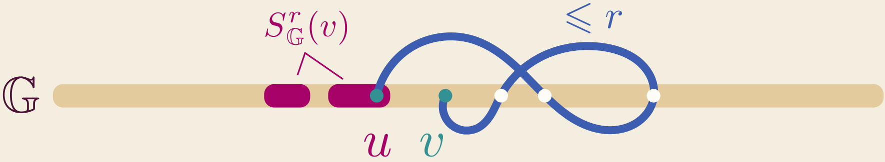


Strong coloring & bounded expansion



u is **strongly r-reachable** from v if there exists a path from v to u of length at most r such that all interior vertices lie right of v.

Strong coloring & bounded expansion



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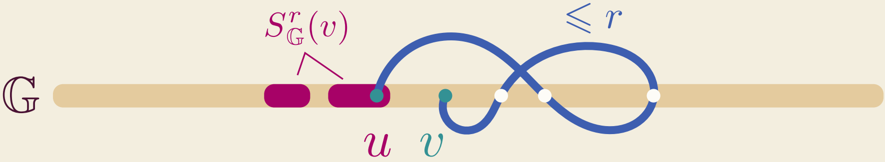
$$\mathrm{col}_r(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} |S^r_{\mathbb{G}}(v)|$$

Strong coloring & bounded expansion



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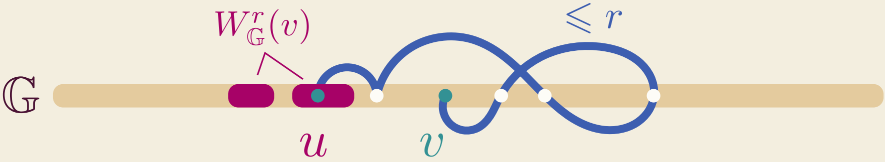
A graph class has bounded expansion iff it is col_r-bounded.

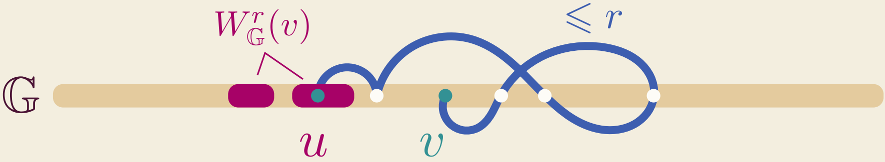
[Neš08 EuJC]

Weak coloring & bounded expansion



u is **weakly r-reachable** from v if there exists a path from v to u of length at most r such that u is the path's leftmost vertex.

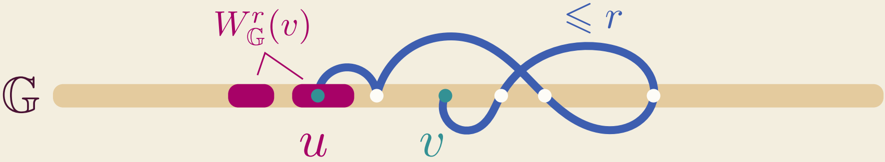
Weak coloring & bounded expansion



u is **weakly r-reachable** from v if there exists a path from v to u of length at most r such that u is the path's leftmost vertex.

$$\mathrm{wcol}_r(G) := \min_{\mathbb{G}\in\Pi(G)} \max_{v\in G} |W^r_{\mathbb{G}}(v)|$$

Weak coloring & bounded expansion



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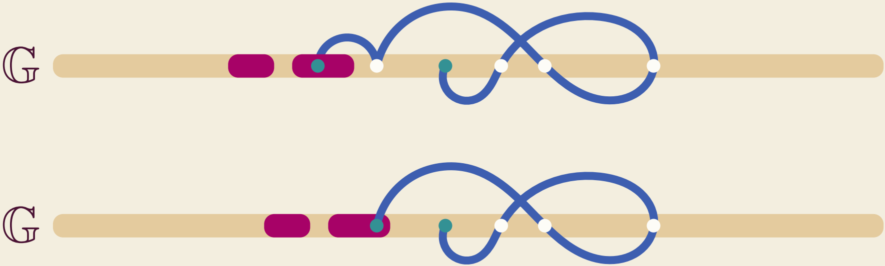
A graph class has bounded expansion iff it is $\mathbf{wcol}_r$-bounded.

[Neš08 EuJC]

Weak & strong coloring



$$\mathrm{col}_r(\mathbb{G}) \leqslant \mathrm{wcol}_r(\mathbb{G}) \leqslant r(\mathrm{col}_r(\mathbb{G}) - 1)^r + 1$$

Origin story

Introduced by Kierstead & Yang, generalizing

Def. $$\mathrm{col}(\mathbb{G}, x) := |N[x] \cap \{y \leqslant_{\mathbb{G}} x\}|$$

$$\mathrm{col}(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} \mathrm{col}(\mathbb{G}, v)$$

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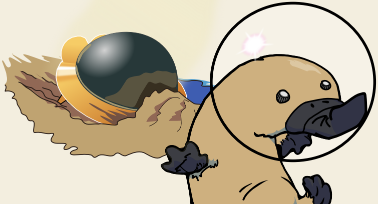
We can greedily color a graph with such an ordering:

$$\chi(G) \leqslant \chi^{\ell}(G) \leqslant \operatorname{col}(G)$$



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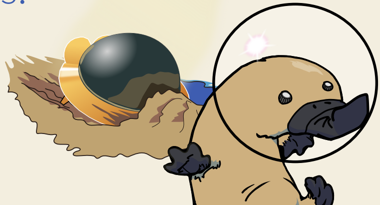
Further connections to colorings:

$$\operatorname{col}_2(G) = \chi_{\text{acyclic}}(G)$$
$$\operatorname{wcol}_2(G) = \chi_{\text{star}}(G)$$

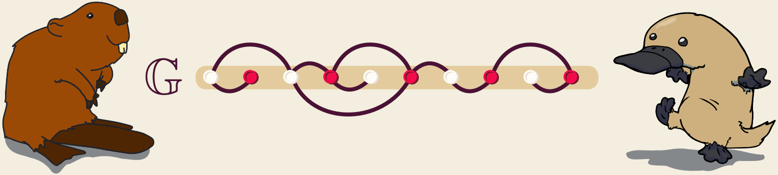


[Kie03 Order]

It's all fun and games...

r-Ordering Game

- Alice & Bob play on a graph G by alternating choosing vertices (n turns)
- Creates an order $\mathbb{G}$, score is $\mathrm{col}_r(\mathbb{G})$
- Alices plays first and wants to minimize score; Bob wants to maximize



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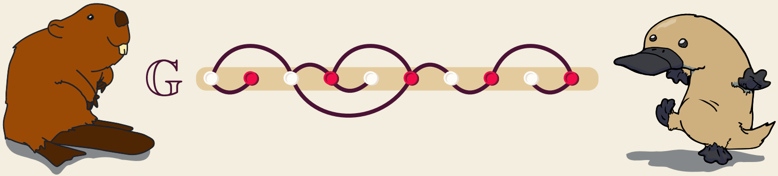


$\mathrm{gcol}_r(G) :=$ Lowest score Alice can always achieve

To planarity and beyond!

A (non-comprehensive) selection of recent work

Planar graphs have $\mathbf{wcol}_2(G) \leqslant 23$

[Alm22 DM]

Koebe orders of planar graphs satisfy

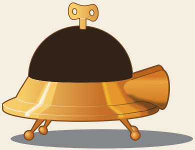
$$\mathbf{wcol}_r(\mathbb{G}) = O(r^4 \ln r)$$

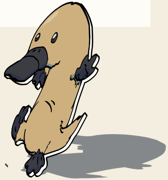
[Ned22 Arxiv]

Graphs of treewidth t have

$$\mathbf{wcol}_r(G) = \Theta(r^{t-1} \log r)$$

[Jor22 Elec J Comb]





To planarity and beyond!

A (non-comprehensive) selection of recent work

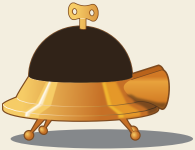
Graph powers: $\mathbf{col}_r(G^p)$ can be bounded in terms of $\mathbf{wcol}_r(G)$ and $\Delta(G)$

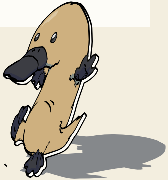
[Kie20 DM]

If G has genus g $\begin{cases} \mathrm{col}_r(G) & \leq (2g+3)(2r+1) \\ \mathrm{wcol}_r(G) & \leq (2g + \binom{2r+2}{2})(2r+1) \end{cases}$

If G excludes a minor $\begin{cases} \mathrm{col}_r(G) & \leq \binom{t-1}{2}(2r+1) \\ \mathrm{wcol}_r(G) & \leq O(r^{t-1}) \end{cases}$

[van17 EuJC]





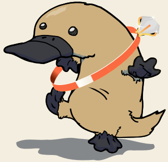
One ~~Ring~~ *Order* to Rule Them All

? Do orders witnessing low col_r also have small $\mathrm{col}_{r'}$?

Not if low/small mean optimal!

For every $r \neq r'$ $\exists G$ such that every ordering $\mathbb{G} \in \Pi(G)$ is non-optimal for col_r or $\mathrm{col}_{r'}$.

But: known bounds use one order for all r!



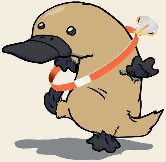
One ~~Ring~~ *Order* to Rule Them All

?

If $\mathrm{col}_r(G) \leqslant c(r)$, does $\exists c^*$ such that some $\mathbb{G}$ has $\mathrm{col}_r(\mathbb{G}) \leqslant c^*(r)$ for all r?

Thm. (van Heuvel, Kierstead)
For all G there exists $\mathbb{G} \in \Pi(G)$ with
$$\mathrm{col}_r(\mathbb{G}) \leqslant (2^r + 1)(\mathrm{col}_{2r}(G))^{4r}$$
for all r.

[van21 EuJC]



Twins: twice the fun, half the sleep

Twinwidth is a hot new graph parameter generalizing an invariant of permutation classes.

[Bon20 FOCS][Gui14 SODA]

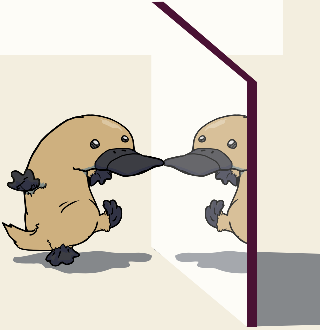
Thm. (Bonnet *et al.*)

If $\mathrm{tww}(G) \leqslant t$ and $K_{s,s} \not\subseteq G$, then there ex. f_r s.t.

$$\mathrm{col}_r(G) \leqslant f_r(s,t)$$

for all r.

[Bon21 SODA]



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$$\operatorname{col}_r(G) \leqslant f_r(s,t)$$

for all r.

[Bon21 SODA]

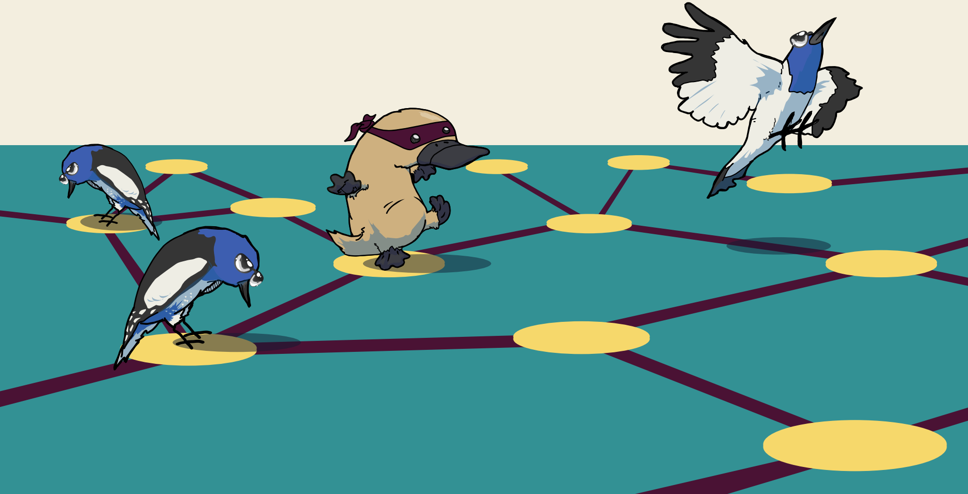
Thm. (Dreier *et al.*)
There ex. G with $\operatorname{col}_r(G) \geqslant (t-4)^r s$, and

$\operatorname{tww}(G) \leqslant t$
$K_{s,s} \not\subseteq G$

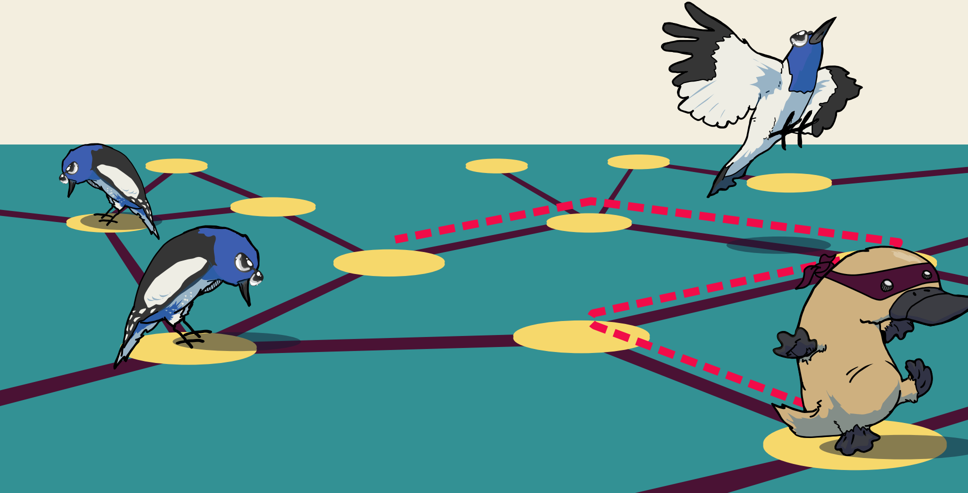
$$f_r(s,t) \leqslant (t^r+3)s$$

[Dre22 DM]

Cops & robbers

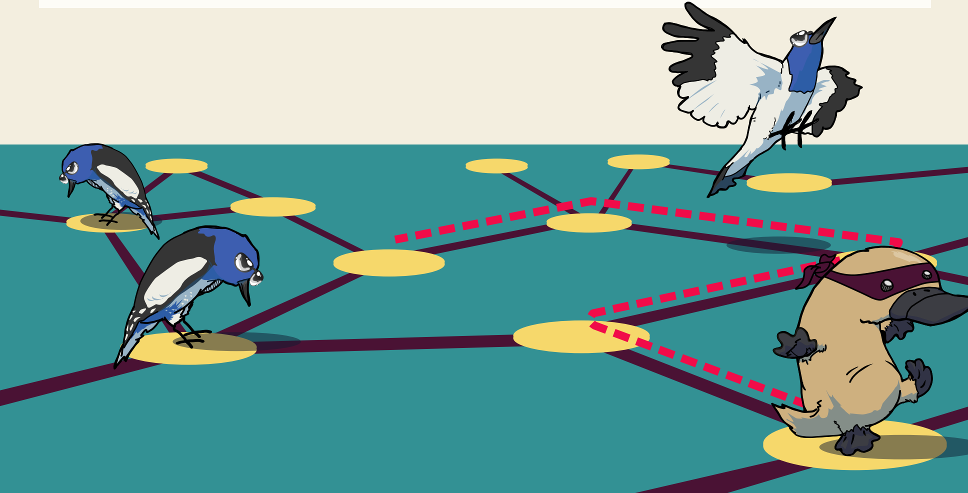


Cops & robbers

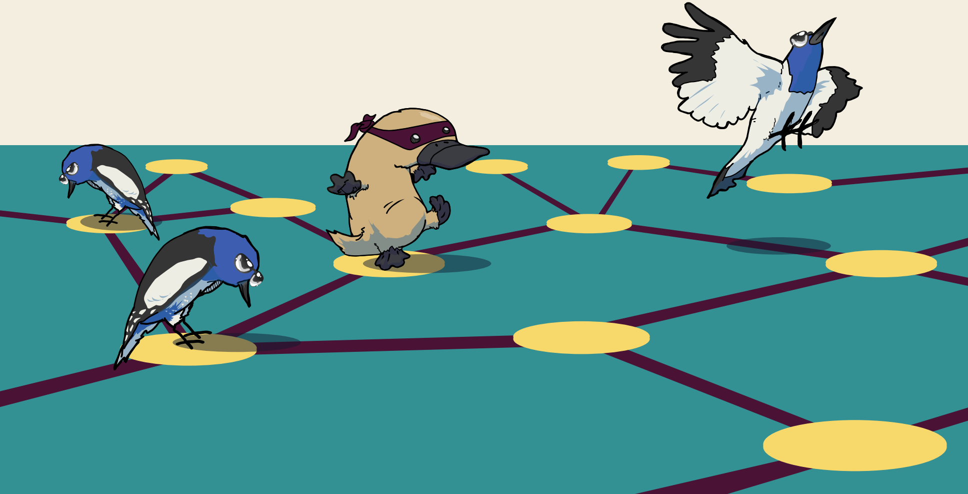


Cops & robbers

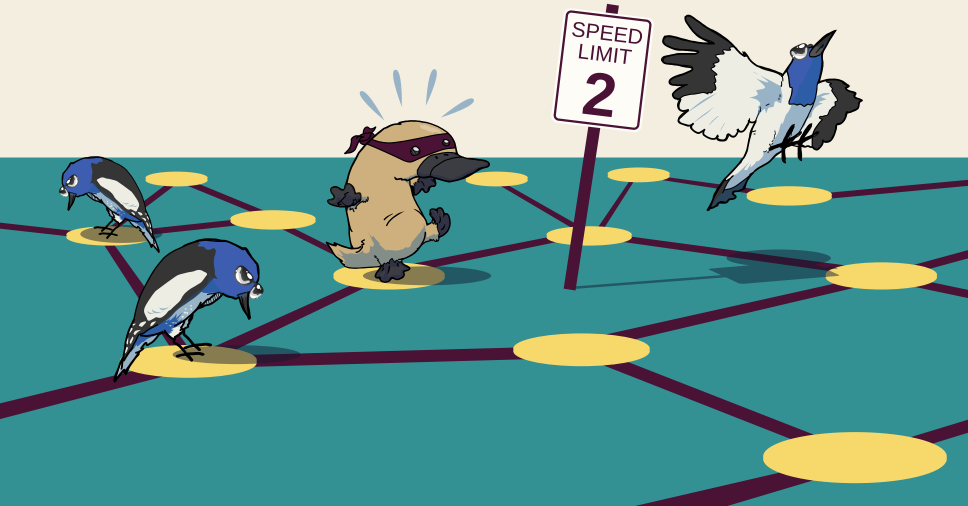
$k+1$ cops can catch the robber $\iff$ $\mathrm{tw}(G) = k$



Cops & robbers

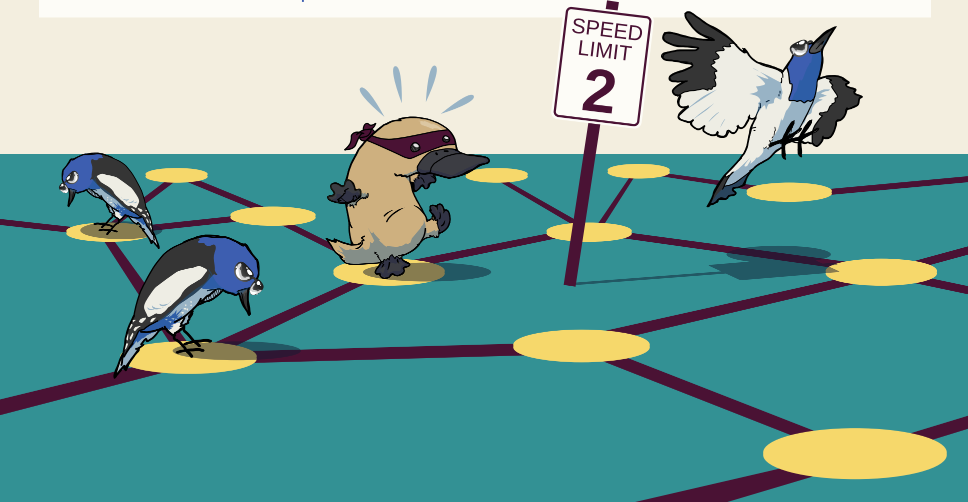


Cops & robbers



Cops & robbers

$k+1$ cops can catch a robber with maximum speed r $\iff$ $\mathrm{copw}_r(G) = k$



Cops & robbers

$k+1$ cops can catch a robber with maximum speed r $\iff$ $\mathrm{copw}_r(G) = k$

Thm. (Toruńczyk)

$$\mathrm{adm}_r(G) + 1 \leqslant \mathrm{copw}_r(G) \leqslant \mathrm{wcol}_{2r}(G) + 1$$

[Tor23 Flip-width]

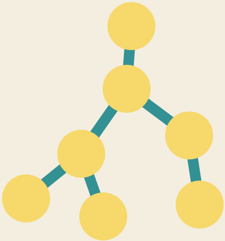
A graph class has bounded expansion iff it is copw_r-bounded.

'Limits' of coloring numbers

Degeneracy

$$\mathrm{col}_1(G) = \mathrm{wcol}_1(G)$$

$$\mathrm{wcol}_\infty(G) = \mathrm{td}(G)$$



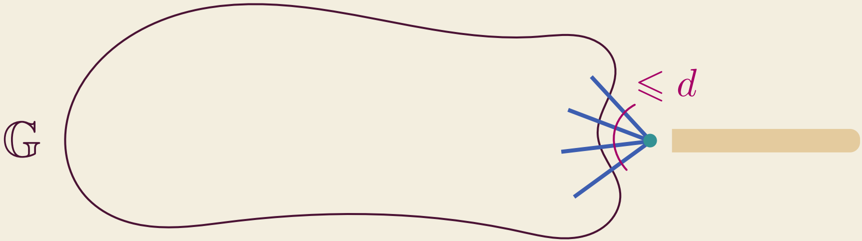
$$\mathrm{col}_\infty(G) = \mathrm{tw}(G)$$

Part III

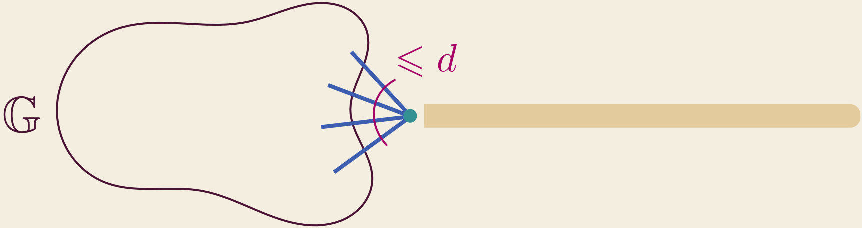
Algorithmic questions

Computing degeneracy ordering

Computing degeneracy ordering



Computing degeneracy ordering



Computing degeneracy ordering

$\mathbb{G}$

$$\forall v \quad |N(v) \cap \{u \leq_{\mathbb{G}} v\}| \leq d$$

No free lunch

Thm. (Breen-McKay, Lavallee, **S**)
Deciding wether $\mathrm{wcol}_r(G) \leqslant k$ or $\mathrm{col}_r(G) \leqslant k$ is NP-complete even for $r = 2$

[Bre23 EuJC (in press)]*

Anything beyond degeneracy is hard to compute exactly.

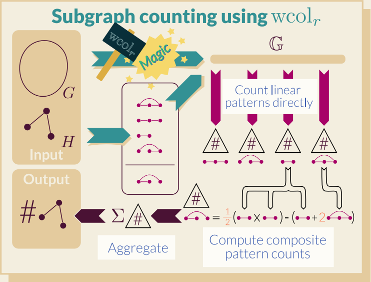
Maybe cheap lunch?

Thm. (Dvořák)

$$\mathrm{adm}_r(\mathbb{G}) \leqslant k \implies \begin{array}{r}\mathrm{col}_r(\mathbb{G}) \leqslant (k-1)^r + 1 \\ \mathrm{wcol}_r(\mathbb{G}) \leqslant (r^2k)^r\end{array}$$

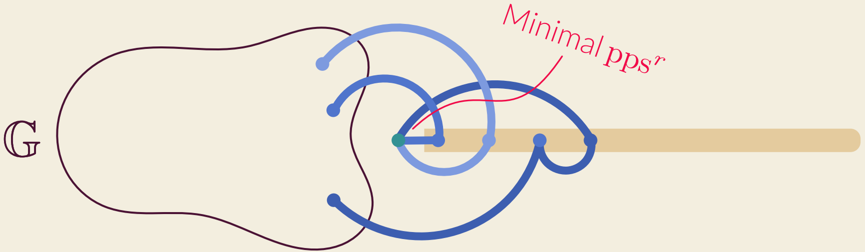
[Dvo22 JGT][Dvo13 EuJC]

Approximation works just fine here! (though slower)



Estimating path-packings

Estimate pp^r by packing *shortest* paths!



Thm. (Breen-McKay, Lavallee, **S**)

$$\boxed{\mathrm{pps}^r_{\mathbb{G}}(v) \leqslant k \text{ for all } v \in G} \implies \begin{aligned} \mathrm{col}_r(\mathbb{G}) &\leqslant k(k-1)^{r-1} \\ \mathrm{wcol}_r(\mathbb{G}) &\leqslant \frac{k^{r+1}-1}{k-1} \end{aligned}$$

[Bre23 EuJC (in press)]*

Some Open Problems

Is there a constant-factor approximation algorithm for $\mathrm{wcol}_r(G)$?

Open even if $r = 2$

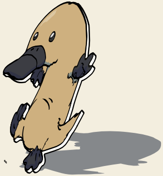
What are asymptotics of max $\mathrm{wcol}_r(G)$?

Known: $\Omega(r^2 \log r)$ and $O(r^3)$

Conjecture [Jor22 Elec J Comb]

Conjecture [Alm22 DM] $\mathrm{wcol}_2(G) \leq 18$

planar



Does there exist a polynomial $P(\cdot, \cdot)$ s.t. $\forall G\, \forall r\ \mathrm{col}_r \leq P(r, \nabla_r(G))$?

Known: yes for admissibility, no for wcol_r

degree of P must be ind. of r

THANKS

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