

# *Taking a Hard Look at Generalized Coloring Numbers*



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Georgia Tech

*Special thanks to  
Felix Reidl (Birkbeck College, London)  
for original artwork and collaboration  
on scientific communication*



*My work in this talk was primarily supported by the Gordon & Betty Moore Foundation,  
with additional funding from DARPA, ARO, and NIH.*

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## *Part I*

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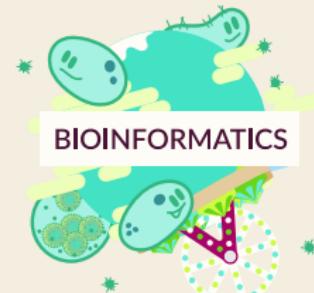
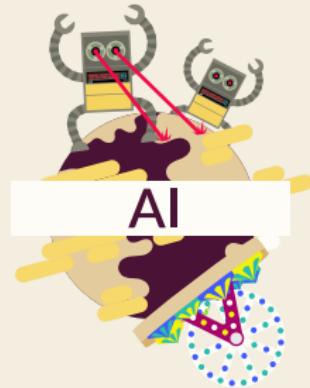
# Sparse classes



# A false dichotomy



# A better model



# Parameterized algorithms

Good  
Bad

Classical view

$O(n^{\log n})$ ,  $O(2^n)$ , ...

Not polynomial-time

$O(n^c)$

Polynomial-time  
("efficient")



Parameterized view

$O(n^{f(k)})$

Slice-wise  
polynomial time

$O(f(k)n^c)$

Fixed-parameter  
tractable



[Dow99 Param. Complex.]

# Choosing a class

Larger classes

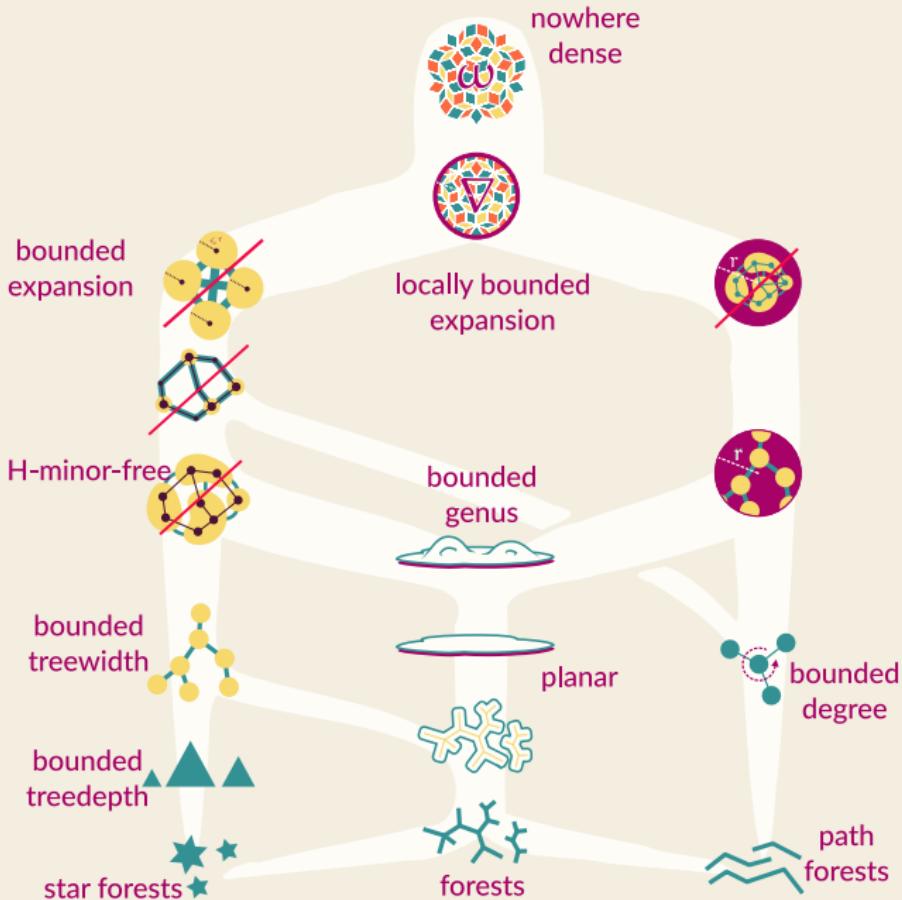


Less Structure

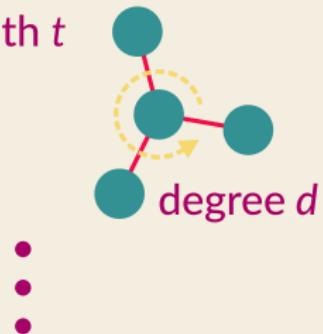
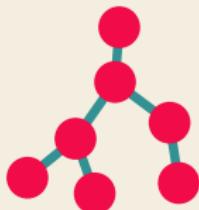
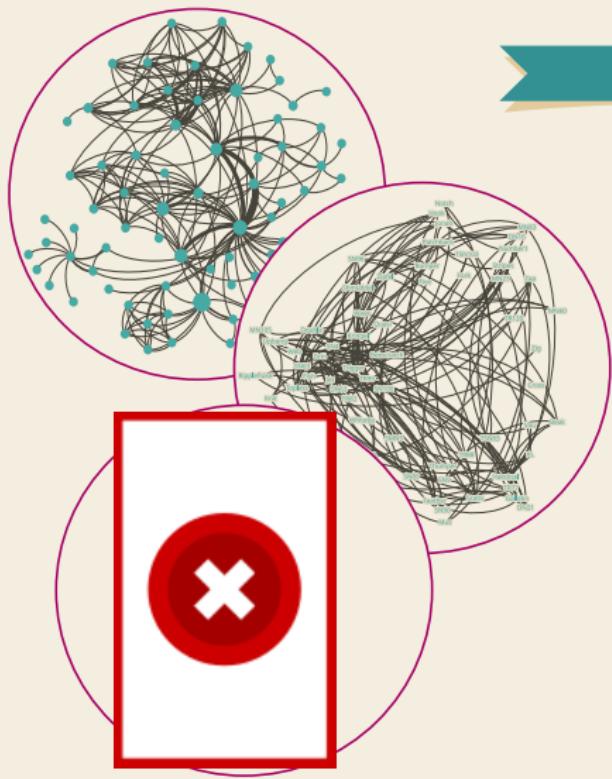
More



Algorithmic tractability



# Can We Use Sparse Structure?

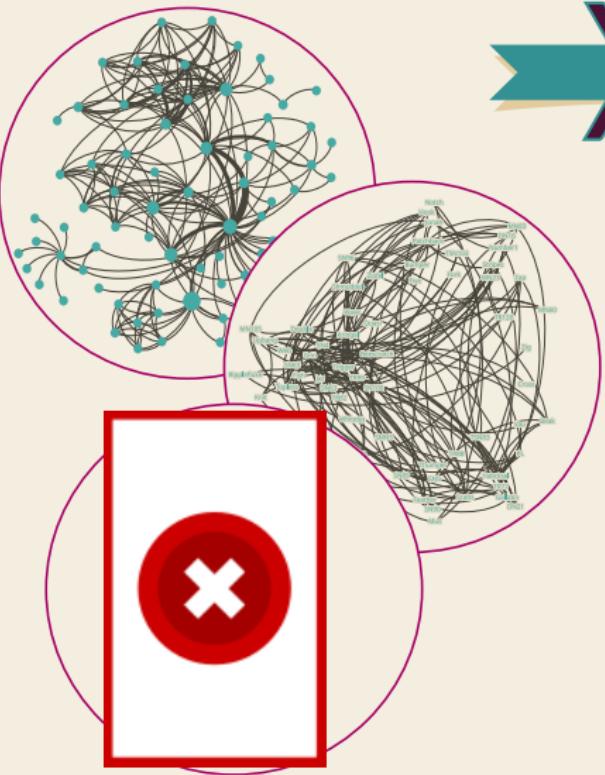


[Bul19 Alg] [Cyg19 Param. Alg]

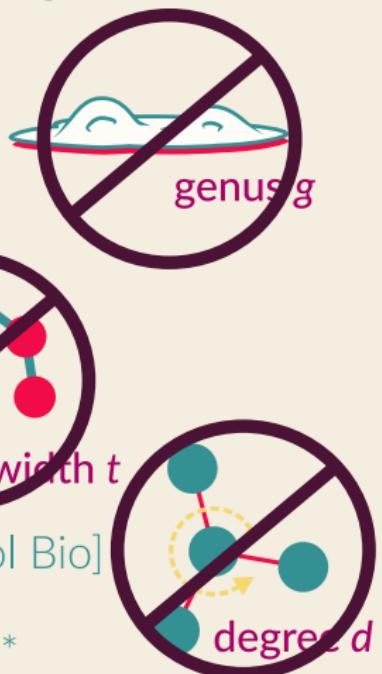
# Evidence says...

[Gas21 BIBM]

[Boe20 Env Plan B]



[Mar22 Alg Mol Bio]  
[Man19 ICDT]  
[Adc13 ICDM]\*



# The Goldilocks Zone

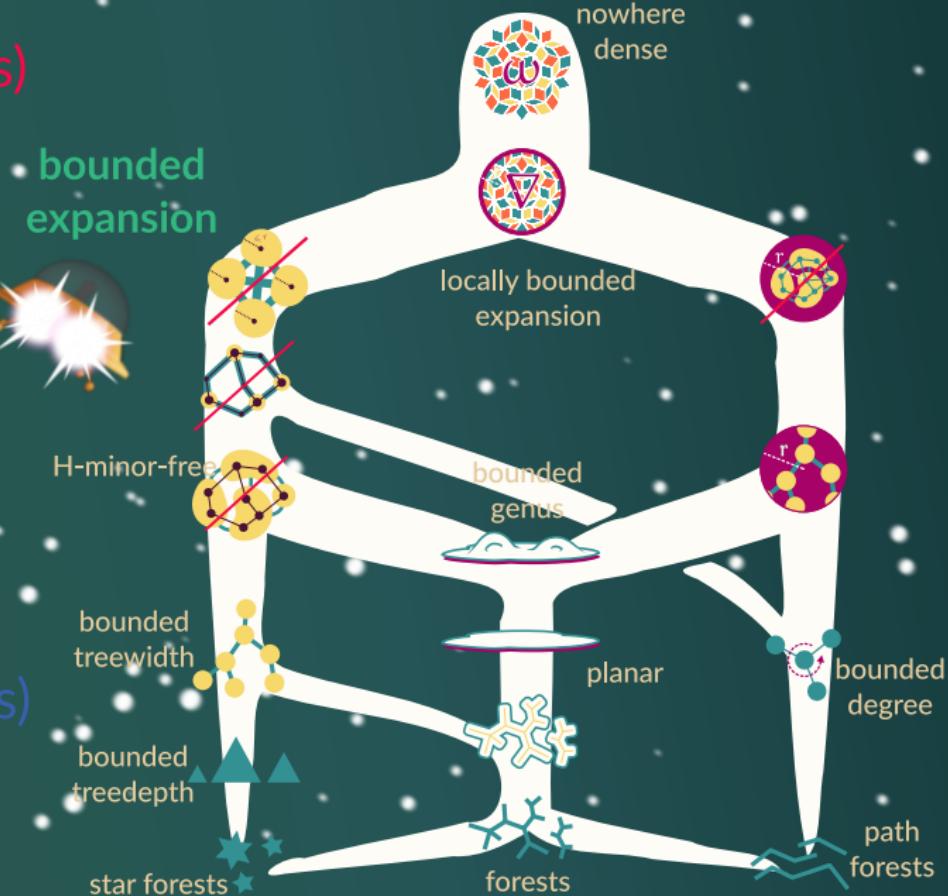
Too big!  
(few algorithms)



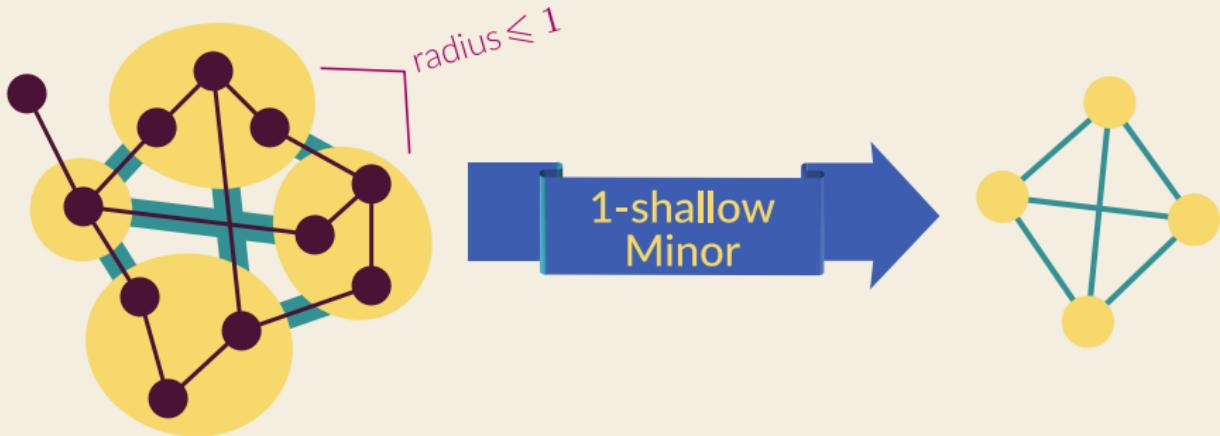
Just Right!



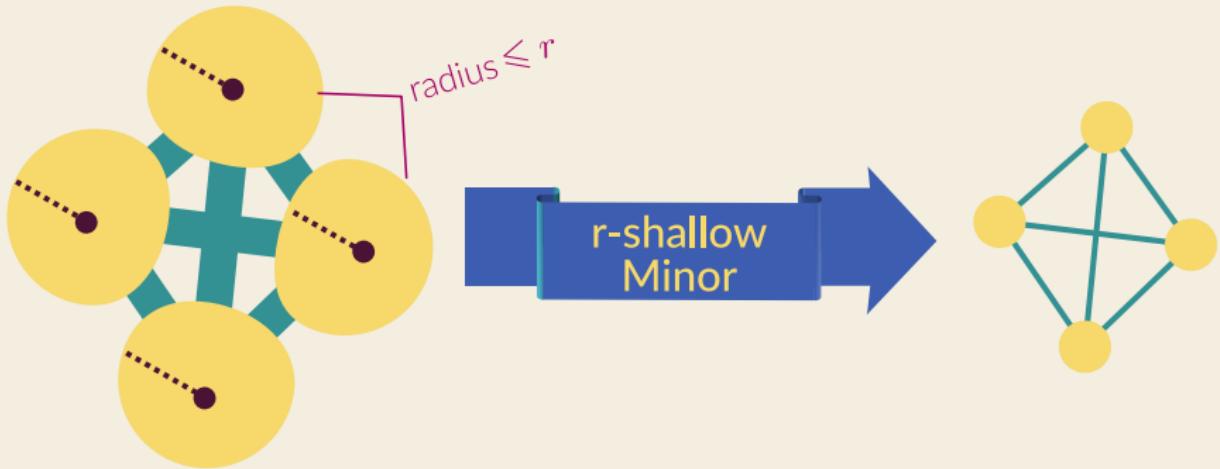
Too small!  
(few real graphs)



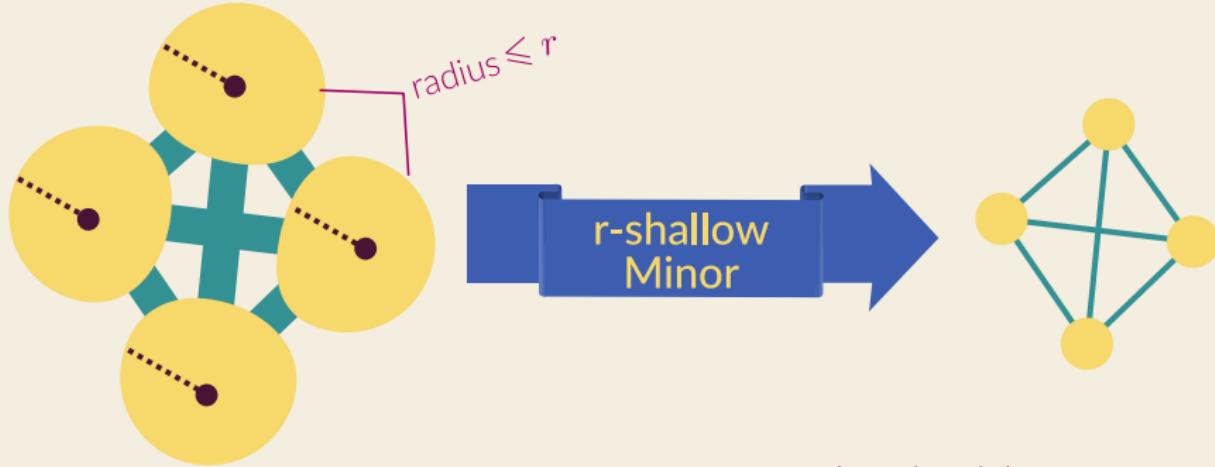
# What is bounded expansion?



# It's not just for radius 1...



# Sparse Shallow Minors == Bounded Expansion

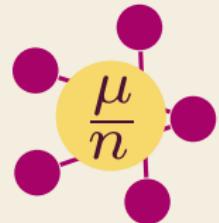
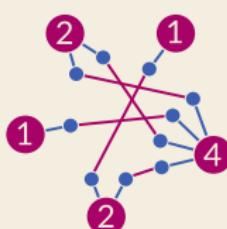
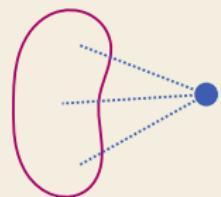
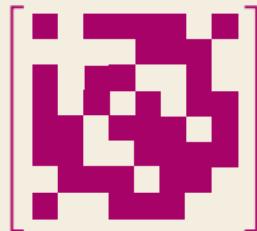
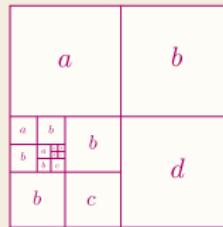
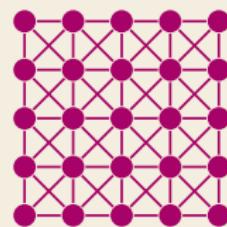


$$\nabla_r(G) = \max_{H \preccurlyeq_r G} \frac{|E(H)|}{|V(H)|}$$

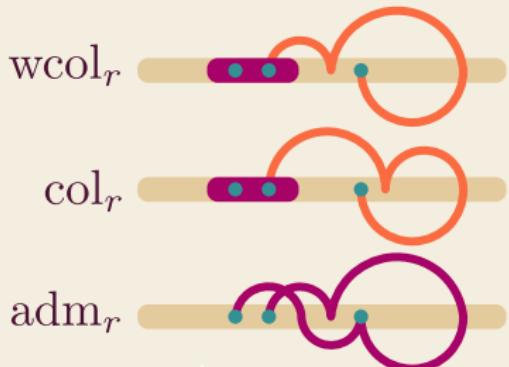


A graph class has bounded expansion iff it is  $\nabla_r$ -bounded.

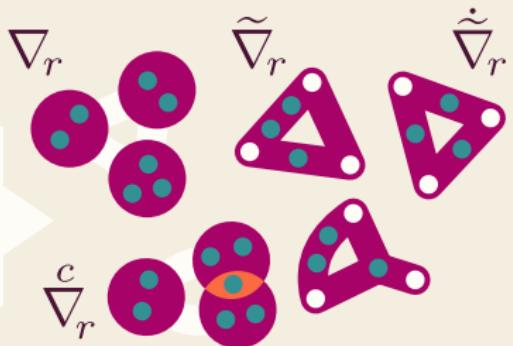
# Random model sparsity

 $G(n, \frac{\mu}{n})$  $G^{\text{CL}}(D_n)$  $G^{\text{CF}}(D_n)$  $G^{\text{RIG}}(n, \alpha, \beta, \gamma)$  $G^{\text{BA}}(n, n_0, k)$  $G^{\text{SGK}}(k, \alpha, \dots, \gamma)$  $G^{\text{RMAT}}(k, m, a, b, c)$  $G^{\text{KL}}(n, p, q, \gamma)$ 

# Bounded expansion



Density  
of shallow  
minors



$\nu_r$

Size of r-reachable  
sets in ordering



Normalized number of  
traces r-neighbourhoods  
leave in any subset

$\Delta^-(\vec{G}_r)$



In-degree of  
r-step (d)tf-  
augmentation

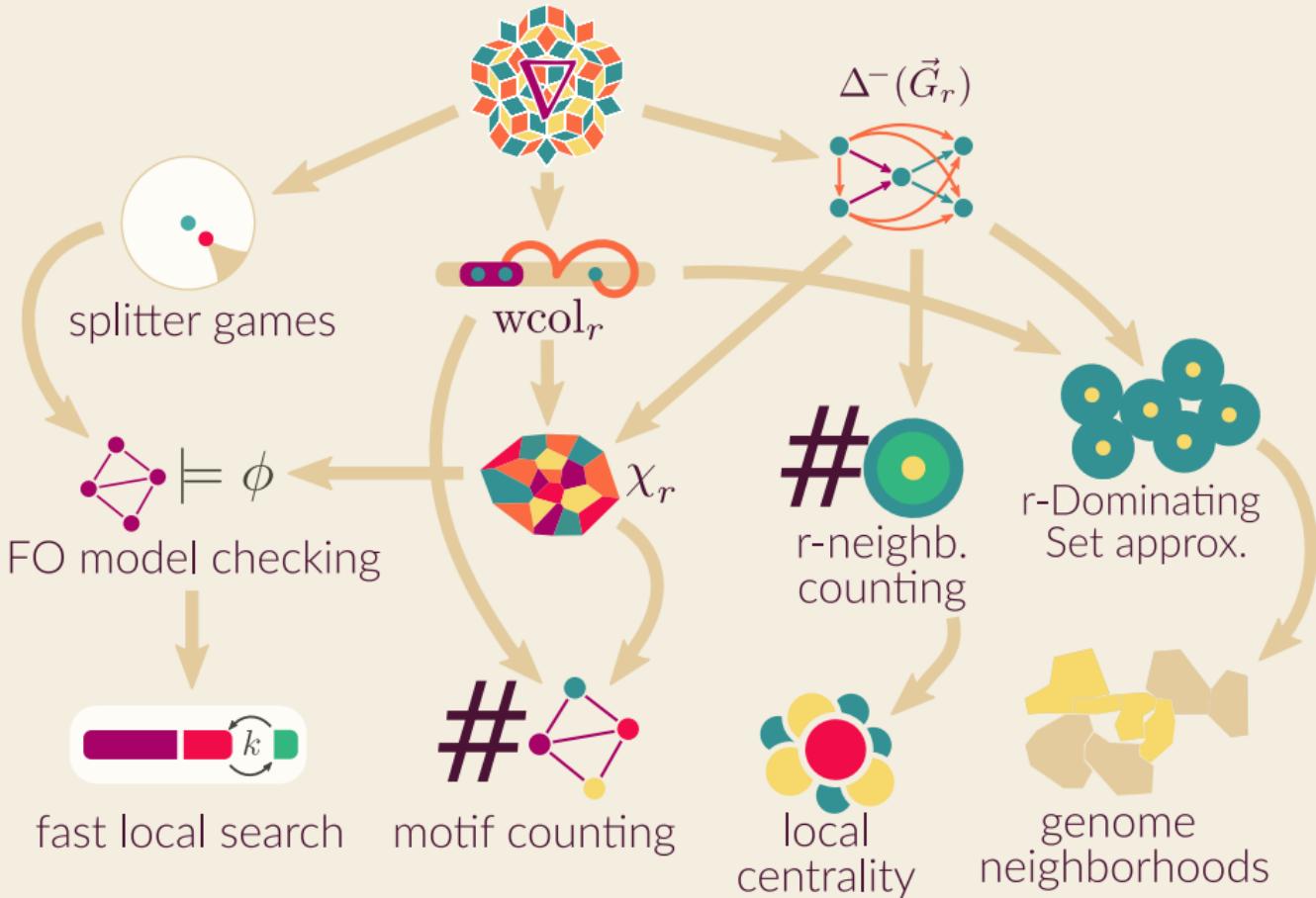
Number of colours  
in r-treedepth  
colouring

$\chi_r$



[Neš12 Sparsity]

# Applications & Algorithms



# Index/Query Metagenomes



de-Bruijn  
graphs



BCALM 2



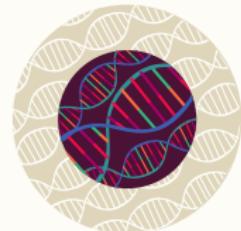
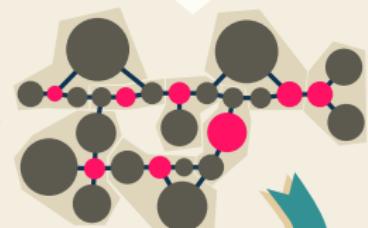
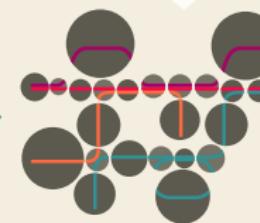
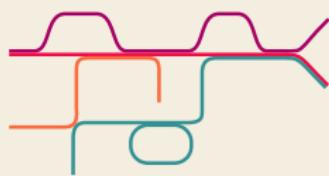
contracted  
DBGs



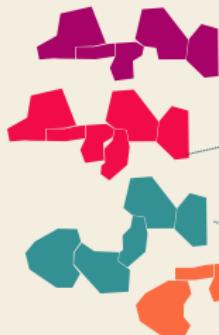
Dvořák's  
Algorithm



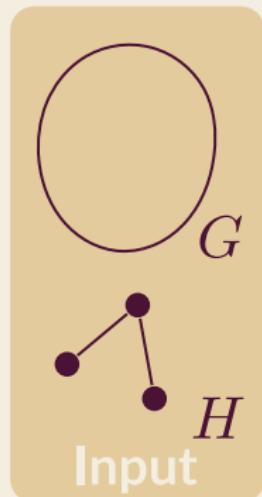
Domset



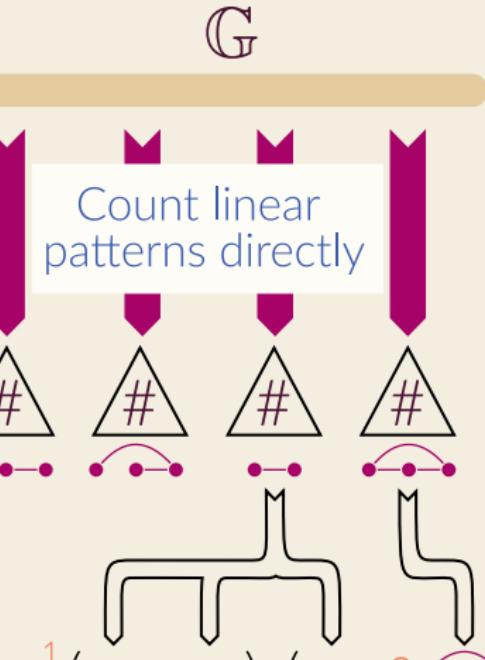
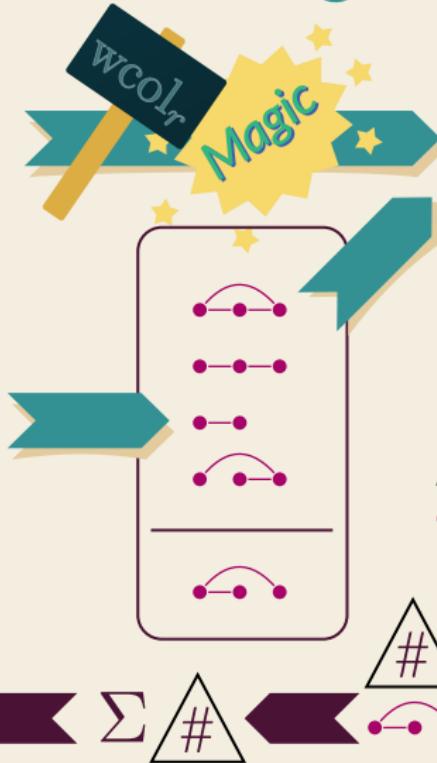
Genome  
neighbourhoods



# Counting Subgraphs



Output



Compute composite pattern counts

[github.com/theoryinpractice/mandoline](https://github.com/theoryinpractice/mandoline)

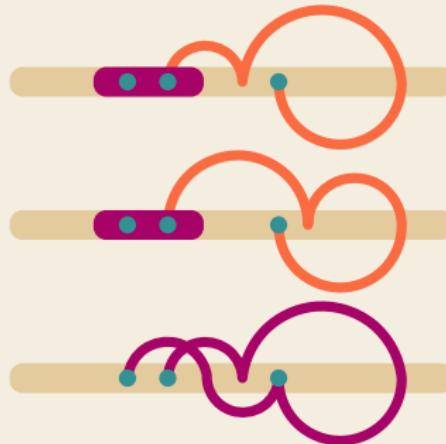
[Rei23 Algorithmica]\*

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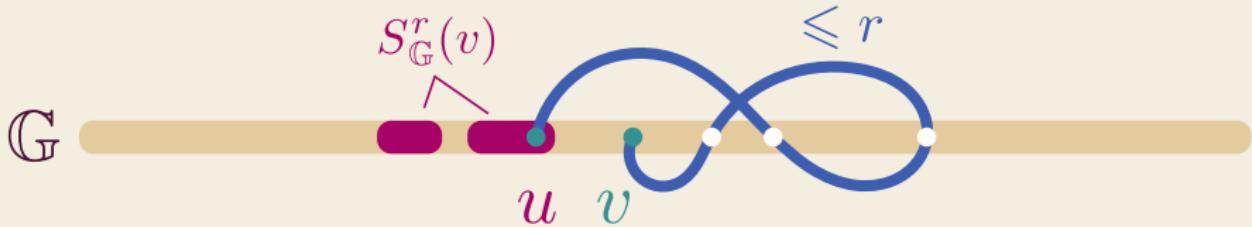
## *Part II*

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# Generalized coloring numbers

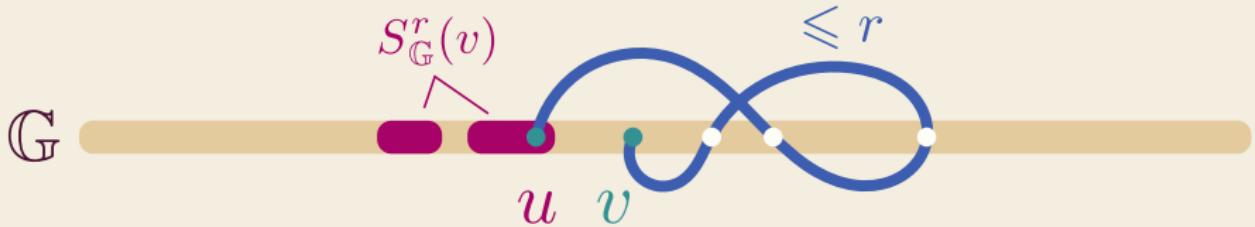


# Strong coloring & bounded expansion



$u$  is strongly  $r$ -reachable from  $v$  if there exists a path from  $v$  to  $u$  of length at most  $r$  such that all interior vertices lie right of  $v$ .

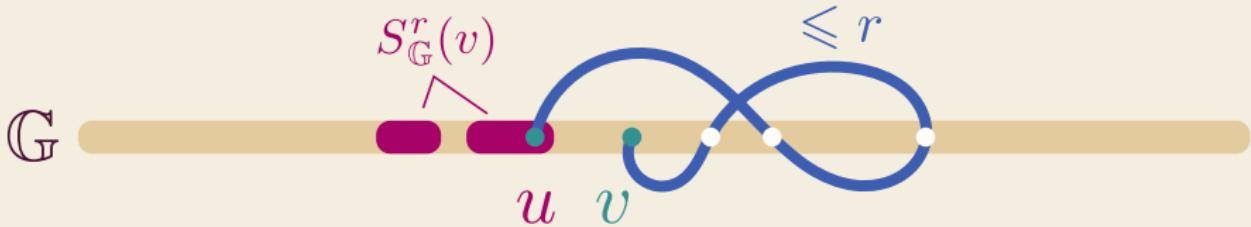
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$$\text{col}_r(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} |S_{\mathbb{G}}^r(v)|$$

# Strong coloring & bounded expansion



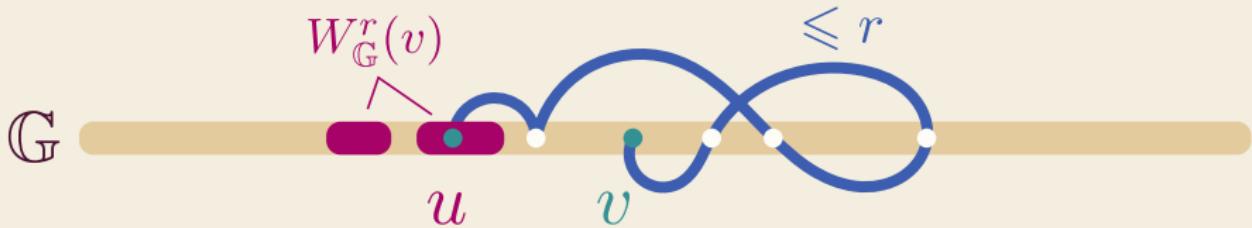
$u$  is **strongly  $r$ -reachable** from  $v$  if there exists a path from  $v$  to  $u$  of length at most  $r$  such that all interior vertices lie right of  $v$ .

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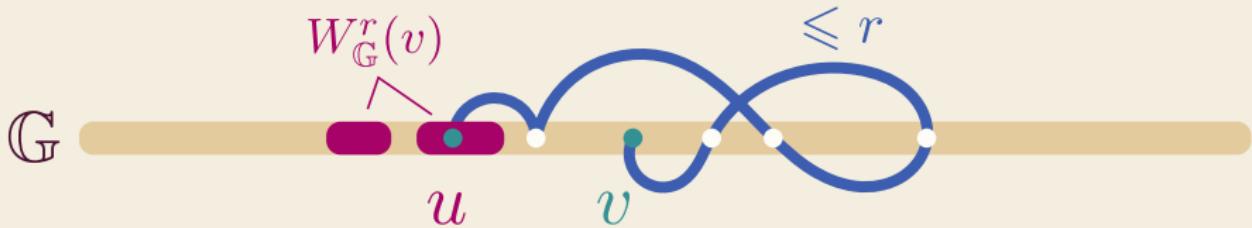
A graph class has bounded expansion iff it is  $\text{col}_r$ -bounded.

# Weak coloring & bounded expansion



$u$  is weakly  $r$ -reachable from  $v$  if there exists a path from  $v$  to  $u$  of length at most  $r$  such that  $u$  is the path's leftmost vertex.

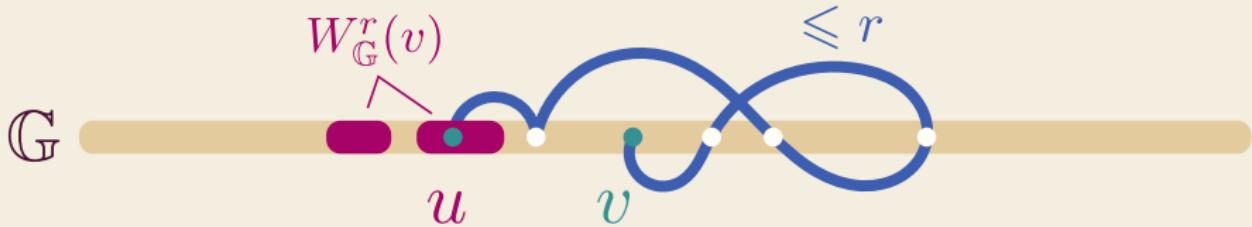
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$$\text{wcol}_r(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} |W_{\mathbb{G}}^r(v)|$$

# Weak coloring & bounded expansion



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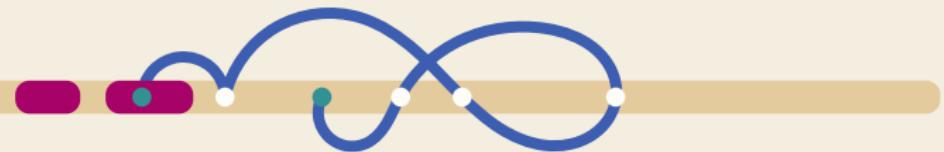
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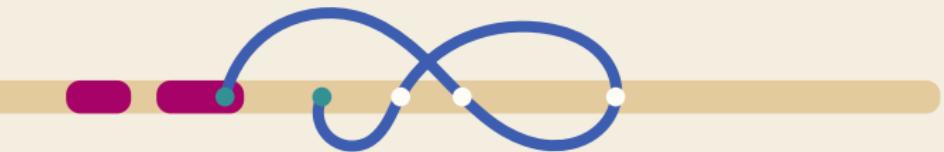
A graph class has bounded expansion iff it is  $\text{wcol}_r$ -bounded.

# Weak & strong coloring

$\mathbb{G}$



$\mathbb{G}$



$$\text{col}_r(\mathbb{G}) \leq \text{wcol}_r(\mathbb{G}) \leq r(\text{col}_r(\mathbb{G}) - 1)^r + 1$$

# Origin story

Introduced by Kierstead & Yang, generalizing

**Def.**  $\text{col}(\mathbb{G}, x) := |N[x] \cap \{y \leqslant_{\mathbb{G}} x\}|$

$$\text{col}(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} \text{col}(\mathbb{G}, v)$$


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We can greedily **color** a graph with such an ordering:

$$\chi(G) \leqslant \chi^\ell(G) \leqslant \text{col}(G)$$



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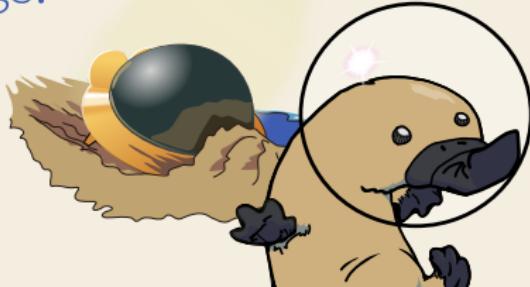
We can greedily **color** a graph with such an ordering:

$$\chi(G) \leqslant \chi^\ell(G) \leqslant \text{col}(G)$$

Further connections to colorings:

$$\text{col}_2(G) = \chi_{\text{acyclic}}(G)$$

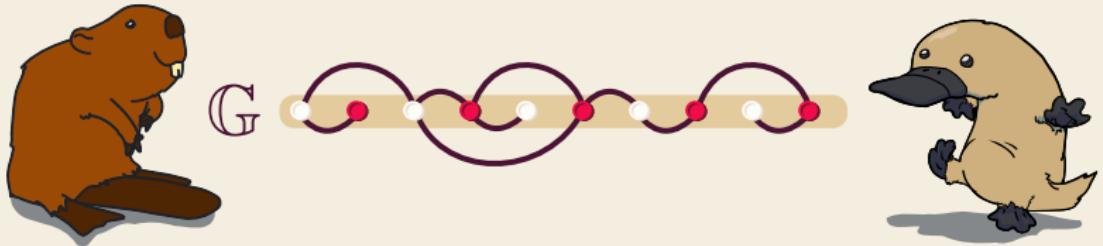
$$\text{wcol}_2(G) = \chi_{\text{star}}(G)$$



# It's all fun and games...

## r-Ordering Game

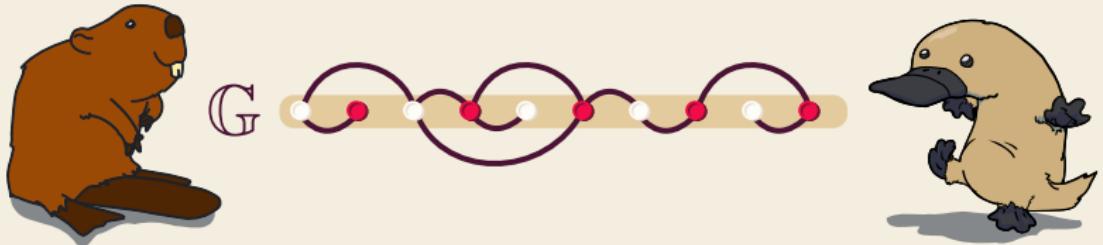
- Alice & Bob play on a graph  $G$  by alternating choosing vertices ( $n$  turns)
- Creates an order  $\mathbb{G}$ , score is  $\text{col}_r(\mathbb{G})$
- Alice plays first and wants to minimize score; Bob wants to maximize



# It's all fun and games...

## r-Ordering Game

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- Alice plays first and wants to minimize score; Bob wants to maximize



$\text{gcol}_r(G) :=$  Lowest score Alice  
can always achieve

# To planarity and beyond!

A (non-comprehensive) selection of recent work

Planar graphs have  $\text{wcol}_2(G) \leq 23$

[Alm22 DM]

Koebe orders of planar graphs satisfy

$$\text{wcol}_r(\mathbb{G}) = O(r^4 \ln r)$$

[Ned22 Arxiv]

Graphs of treewidth  $t$  have

$$\text{wcol}_r(G) = \Theta(r^{t-1} \log r)$$

[Jor22 Elec J Comb]



# To planarity and beyond!

A (non-comprehensive) selection of recent work

Graph powers:  $\text{col}_r(G^p)$  can be bounded in terms of  $\text{wcol}_r(G)$  and  $\Delta(G)$ .

[Kie20 DM]

If  $G$  has genus  $g$   $\begin{cases} \text{col}_r(G) \leq (2g+3)(2r+1) \\ \text{wcol}_r(G) \leq (2g + \binom{2r+2}{2})(2r+1) \end{cases}$

If  $G$  excludes a minor  $\begin{cases} \text{col}_r(G) \leq \binom{t-1}{2}(2r+1) \\ \text{wcol}_r(G) \leq O(r^{t-1}) \end{cases}$

[van17 EuJC]



# ~~One Ring to Rule Them All~~ Order

?

Do orders witnessing low  $\text{col}_r$  also have small  $\text{col}_{r'}$ ?

Not if low/small mean optimal!

For every  $r \neq r'$   $\exists G$  such that every ordering  $\mathbb{G} \in \Pi(G)$  is non-optimal for  $\text{col}_r$  or  $\text{col}_{r'}$ .

**But:** known bounds use one order for all  $r$ !



# ~~One Ring to Rule Them All~~ Order

?

If  $\text{col}_r(G) \leq c(r)$ , does  $\exists c^*$  such that some  $\mathbb{G}$  has  $\text{col}_r(\mathbb{G}) \leq c^*(r)$  for all  $r$  ?

**Thm.** (van Heuvel, Kierstead)

For all  $G$  there exists  $\mathbb{G} \in \Pi(G)$  with

$$\text{col}_r(\mathbb{G}) \leq (2^r + 1)(\text{col}_{2r}(G))^{4r}$$

for all  $r$ .

[van21 EuJC]



# Twins: twice the fun, half the sleep

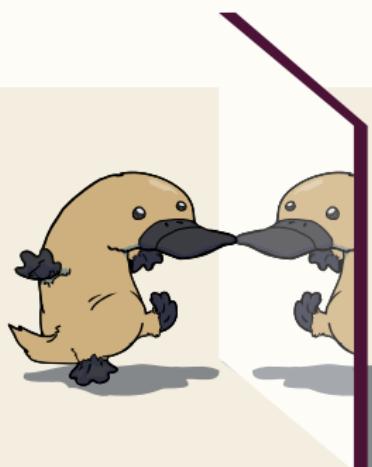
**Twinwidth** is a hot new graph parameter generalizing an invariant of permutation classes.

[Bon20 FOCS][Gui14 SODA]

**Thm.** (Bonnet et al.)

If  $\text{tww}(G) \leq t$  and  $K_{s,s} \not\subseteq G$ , then there ex.  $f_r$  s.t.  
for all  $r$ .  $\text{col}_r(G) \leq f_r(s,t)$

[Bon21 SODA]



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$$\text{col}_r(G) \leq f_r(s, t)$$

for all  $r$ .

[Bon21 SODA]

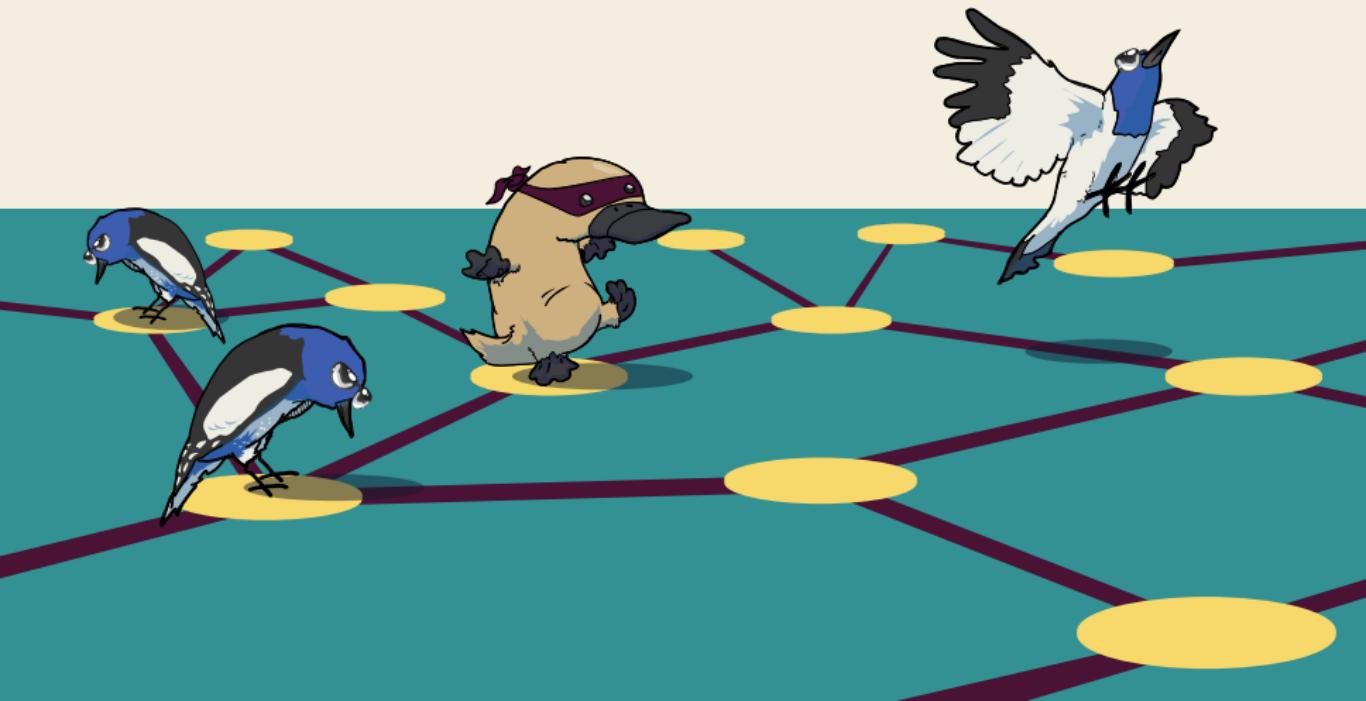
**Thm.** (Dreier et al.)

There ex.  $G$  with  $\text{col}_r(G) \geq (t - 4)^r s$ , and

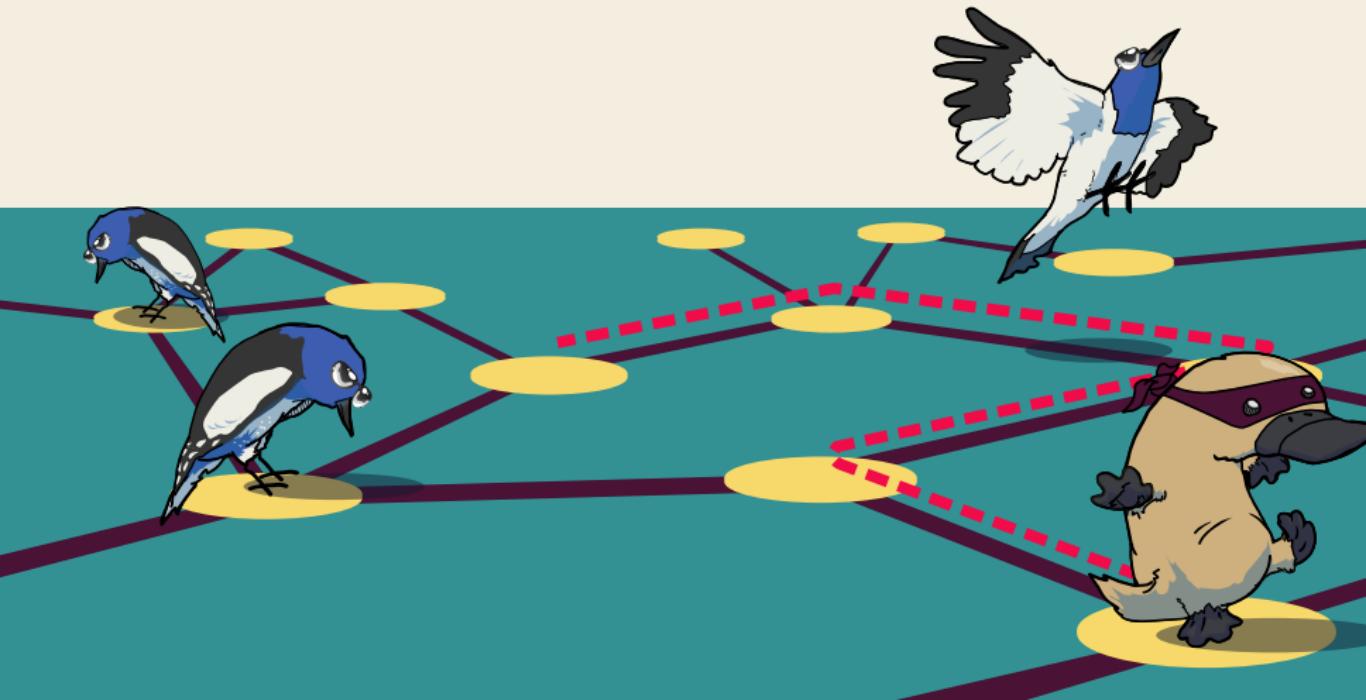
$$\overbrace{K_{s,s} \not\subseteq G}^{\text{tww}(G) \leq t} \quad f_r(s, t) \leq (t^r + 3)s$$

[Dre22 DM]

# Cops & robbers



# Cops & robbers

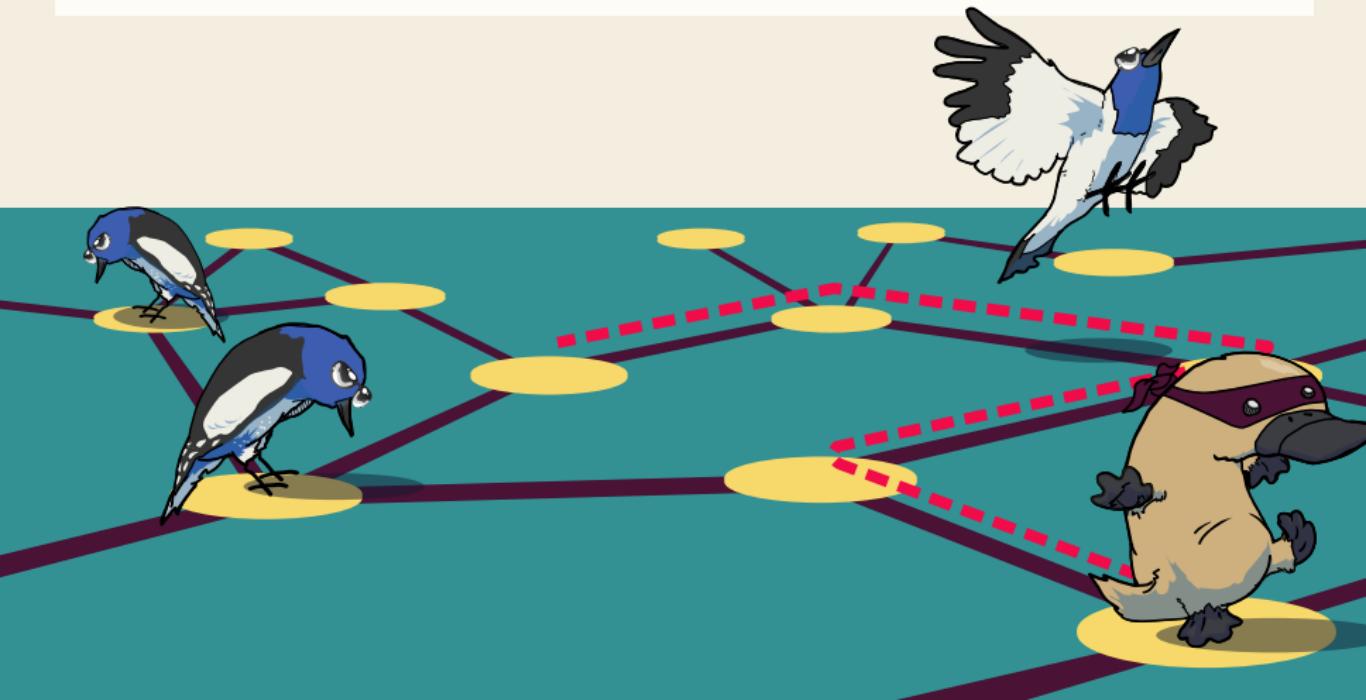


# Cops & robbers

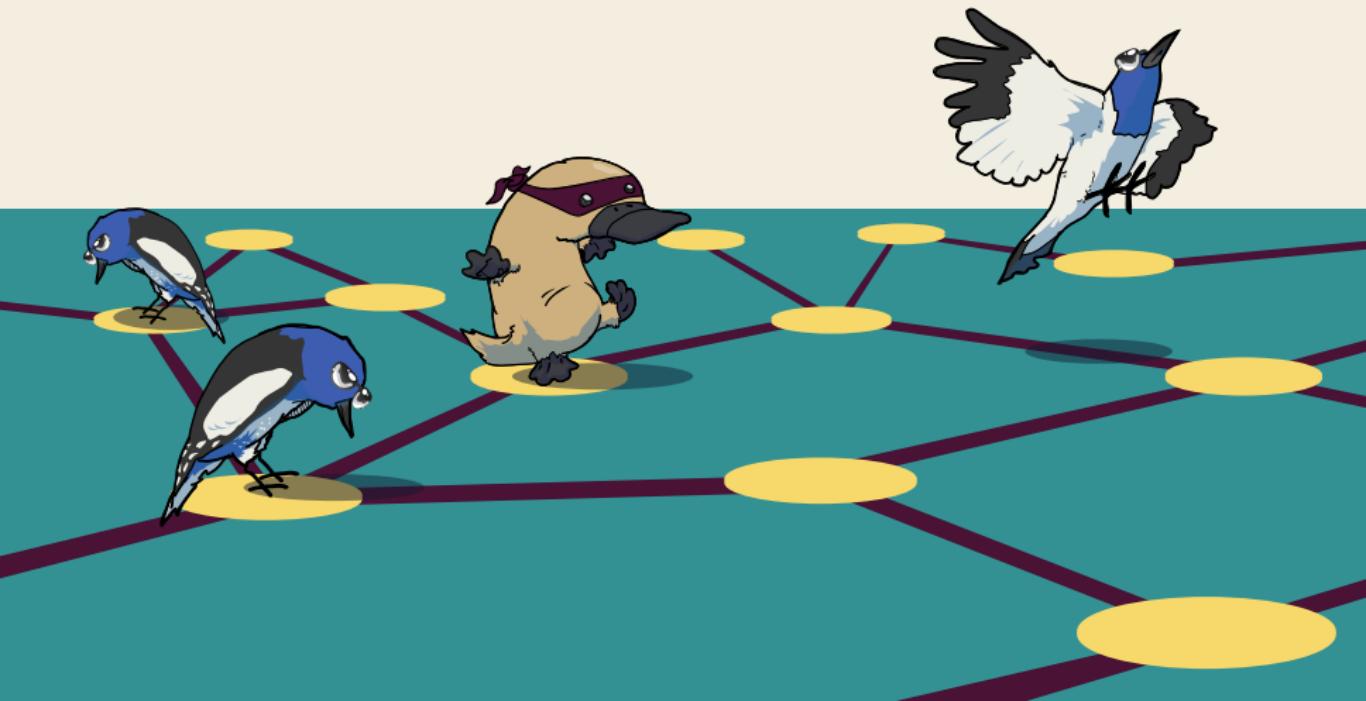
$k + 1$  cops can  
catch the robber



$\text{tw}(G) = k$



# Cops & robbers



# Cops & robbers



# Cops & robbers

$k + 1$  cops can  
catch a robber with  
maximum speed  $r$



$$\text{copw}_r(G) = k$$



# Cops & robbers

$k + 1$  cops can  
catch a robber with  
maximum speed  $r$     $\iff$    $\text{copw}_r(G) = k$



Thm. (Toruńczyk)

$$\text{adm}_r(G) + 1 \leq \text{copw}_r(G) \leq \text{wcol}_{2r}(G) + 1$$

[Tor23 ArXiv]



A graph class has bounded  
expansion iff it is  $\text{copw}_r$ -bounded.

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## *Part III*

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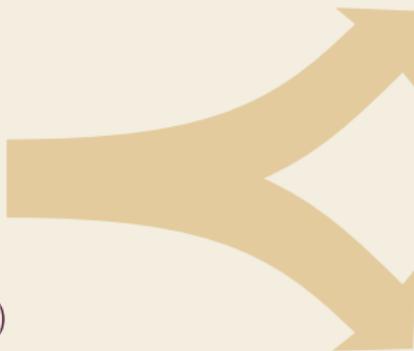
# Algorithmic questions



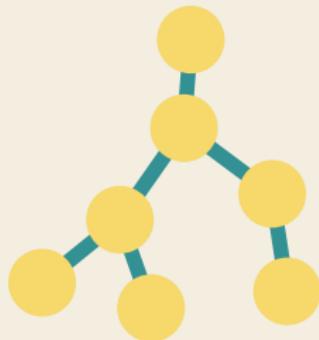
# 'Limits' of coloring numbers



Degeneracy  
 $\text{col}_1(G) = \text{wcol}_1(G)$

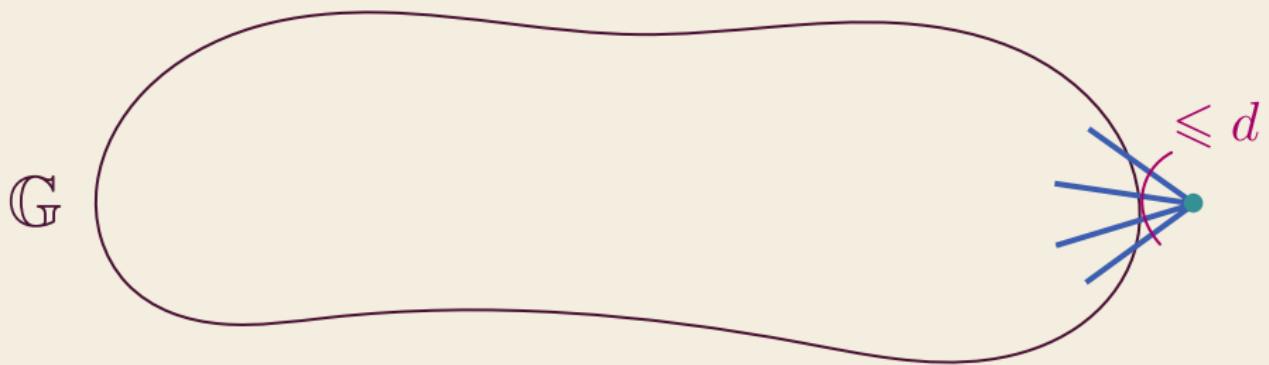


$$\text{wcol}_{\infty}(G) = \text{td}(G)$$

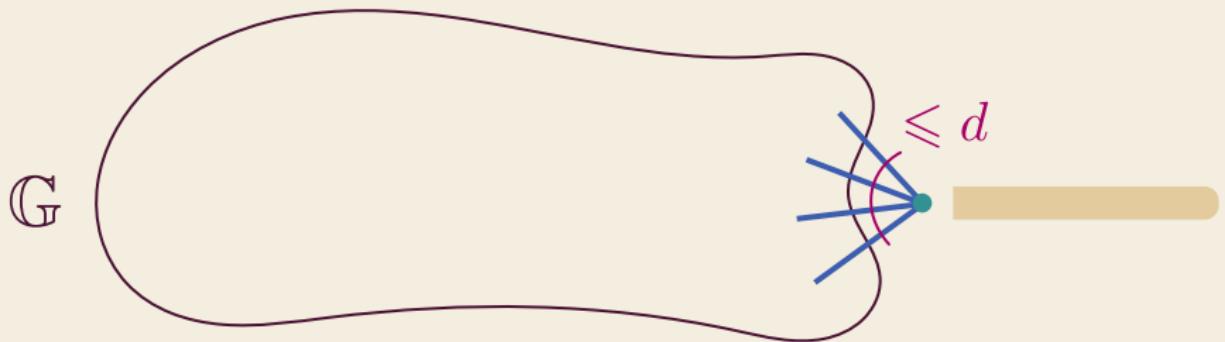


$$\text{col}_{\infty}(G) = \text{tw}(G)$$

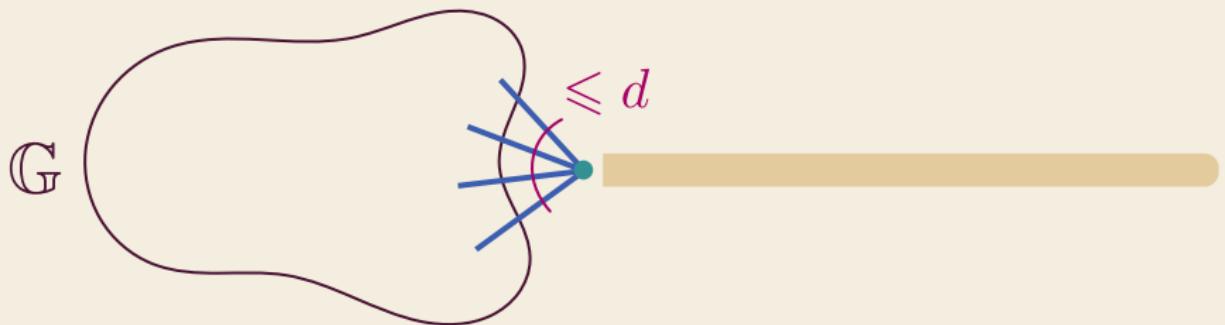
# Computing degeneracy ordering



# Computing degeneracy ordering



# Computing degeneracy ordering



# Computing degeneracy ordering

$\mathbb{G}$

$$\forall v \quad |N(v) \cap \{u \leq_{\mathbb{G}} v\}| \leq d$$

# No free lunch

**Thm.** (Breen-McKay, Lavallee, **S**)

Deciding whether  $\text{wcol}_r(G) \leq k$  or  $\text{col}_r(G) \leq k$   
is NP-complete even for  $r = 2$ .

[Bre23 EuJC]\*

Anything beyond degeneracy is  
hard to compute exactly.

# Maybe cheap lunch?

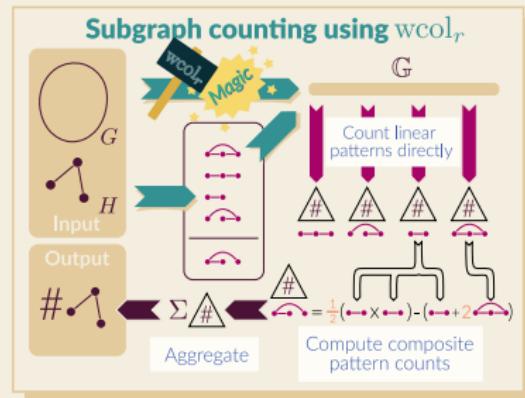
Thm. (Dvořák)

$$\text{adm}_r(\mathbb{G}) \leq k \implies$$

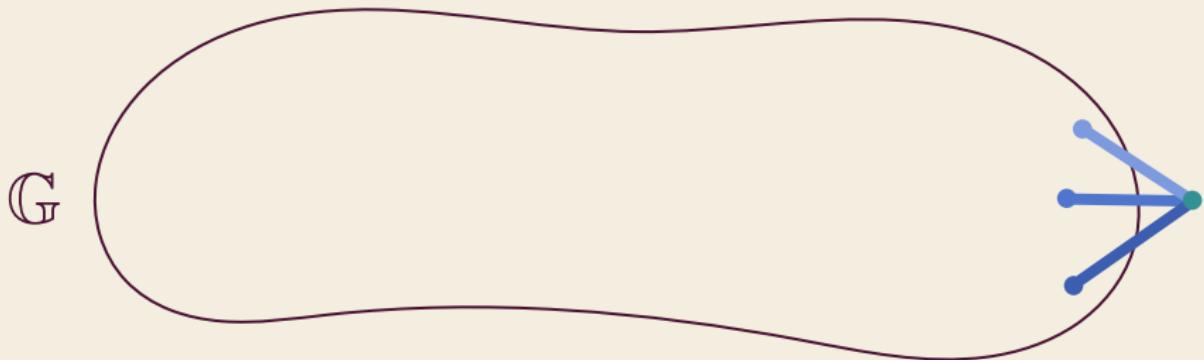
$$\begin{aligned}\text{col}_r(\mathbb{G}) &\leq (k-1)^r + 1 \\ \text{wcol}_r(\mathbb{G}) &\leq (r^2 k)^r\end{aligned}$$

[Dvo22 JGT] [Dvo13 EuJC]

Approximation  
works just fine here!  
(though slower)



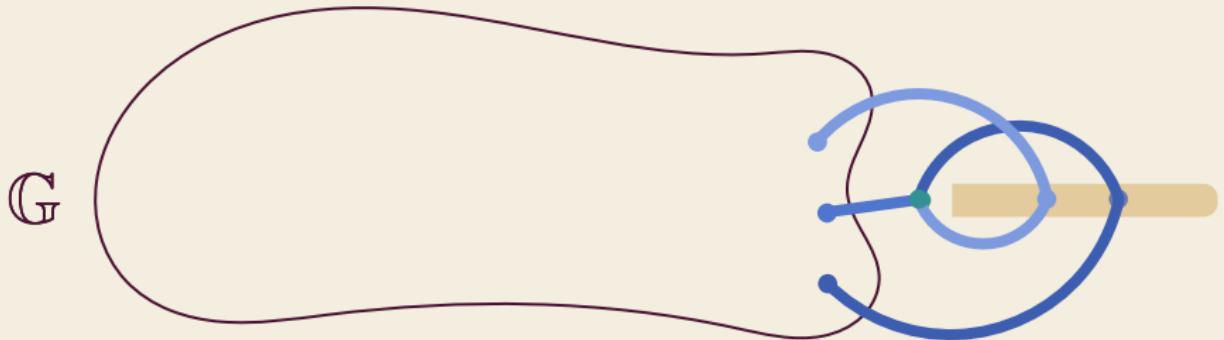
# Approximating orderings



Select vertex which minimizes  $\text{pp}^r$  at each step:

Resulting order minimizes  $\text{adm}_r(G)$

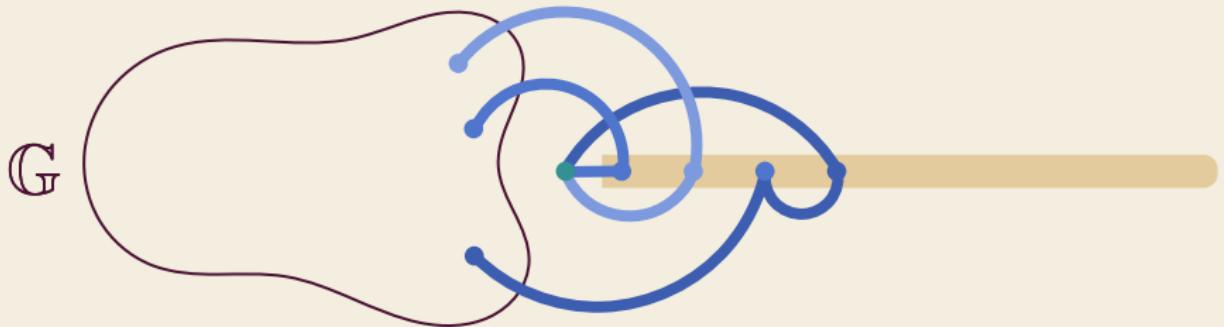
# Approximating orderings



Select vertex which minimizes  $\text{pp}^r$  at each step:

Resulting order minimizes  $\text{adm}_r(G)$

# Approximating orderings



Select vertex which minimizes  $\text{pp}^r$  at each step:

Resulting order minimizes  $\text{adm}_r(G)$

# Approximating orderings

G

Select vertex which minimizes  $\text{pp}^r$  at each step:

Resulting order minimizes  $\text{adm}_r(G)$

# Approximating orderings

G

Select vertex which minimizes  $\text{pp}^r$  at each step:

Resulting order minimizes  $\text{adm}_r(G)$

⇒ Approximates  $\text{wcol}_r$  and  $\text{col}_r$

# Approximating orderings

G

Select vertex which minimizes  $\text{pri}^r$  at each step:

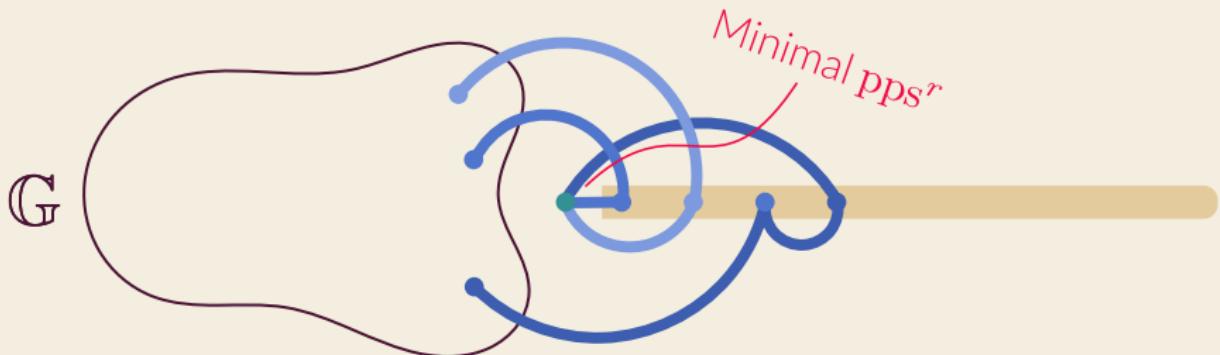
Resulting ordering minimizes  $\text{adm}_r(G)$

⇒ Approximates  $\text{val}_r$  and  $\text{col}_r$



# Estimating path-packings

Estimate  $\text{pp}^r$  by packing shortest paths!



**Thm.** (Breen-McKay, Lavallee, **S**)

$$\begin{aligned} \text{pps}_{\mathbb{G}}^r(v) &\leq k \\ \text{for all } v \in G \end{aligned}$$

$$\implies$$

$$\begin{aligned} \text{col}_r(\mathbb{G}) &\leq k(k-1)^{r-1} \\ \text{wcol}_r(\mathbb{G}) &\leq \frac{k^{r+1}-1}{k-1} \end{aligned}$$

# Some Open Problems



Is there a constant-factor approximation algorithm for  $\text{wcol}_r(G)$  ?

Open even if  $r = 2$

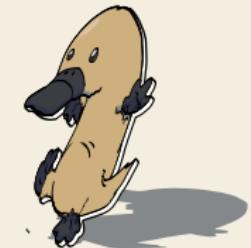
What are asymptotics of  $\text{maxwcol}_r(G)$

Known:  $\Omega(r^2 \log r)$  and  $O(r^3)$

Conjecture  
[Jor22 Elec J Comb]

Conjecture  
[Alm22 DM]  $\text{wcol}_2(G) \leq 18$

?



Does there exist a polynomial  $P(\cdot, \cdot)$   
s.t.  $\forall G \forall r \text{ col}_r \leq P(r, \nabla_r(G))$  ?

Known: yes for admissibility, no for  $\text{wcol}_r$

degree of  $P$  must be ind. of  $r$

# THANKS



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