

Taking a Hard Look at Generalized Coloring Numbers



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*Special thanks to
Felix Reidl (Birkbeck College, London)
for original artwork and collaboration
on scientific communication*



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with additional funding from DARPA, ARO, and NIH.*

Part I

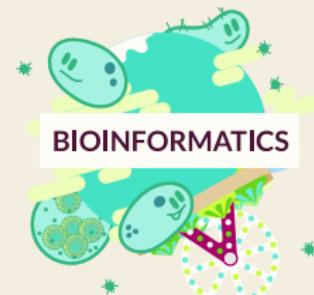
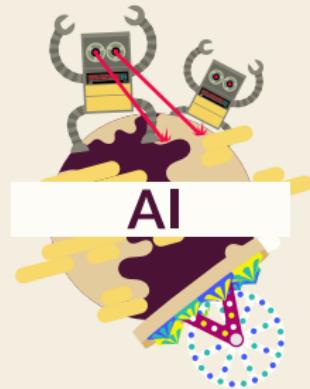
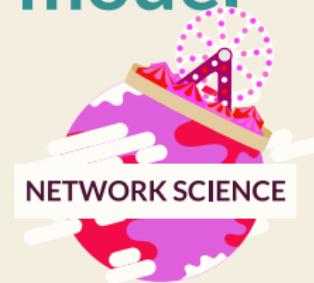
Sparse classes



A false dichotomy



A better model



Parameterized algorithms

Good
Bad

Classical view

$O(n^{\log n})$, $O(2^n)$, ...

Not polynomial-time

$O(n^c)$

Polynomial-time
("efficient")



Parameterized view

$O(n^{f(k)})$

Slice-wise
polynomial time

$O(f(k)n^c)$

Fixed-parameter
tractable



[Dow99 Param. Complex.]

Choosing a class

Larger classes

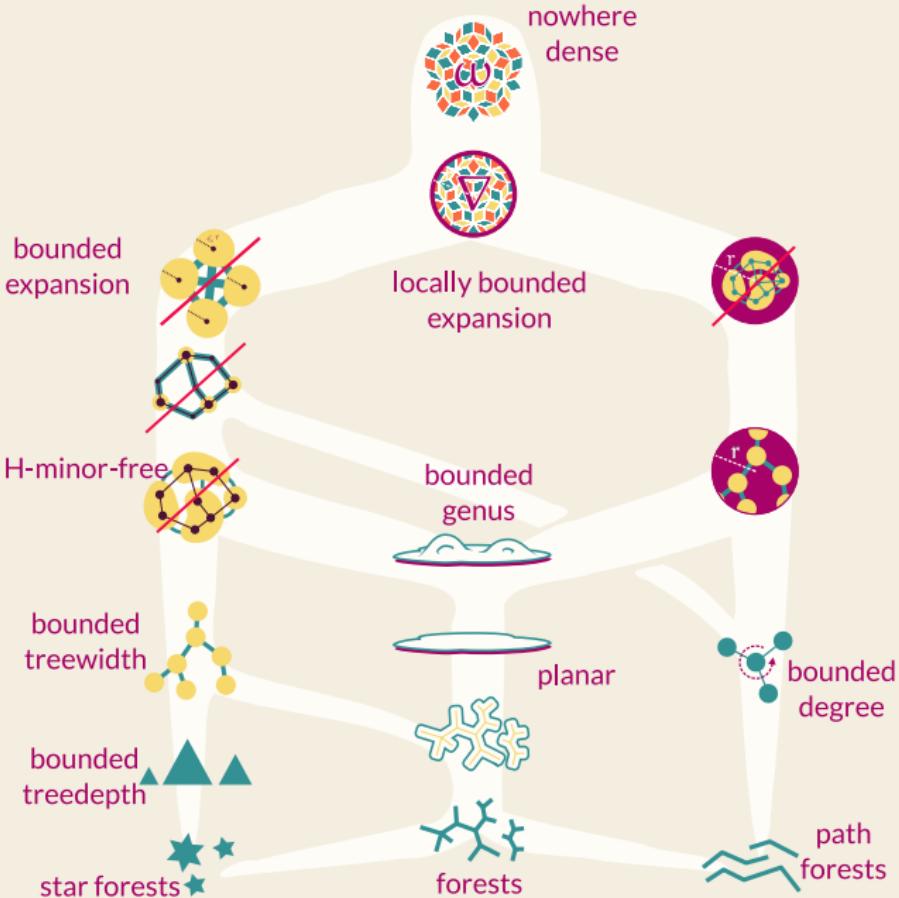


Less
Structure

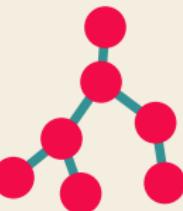
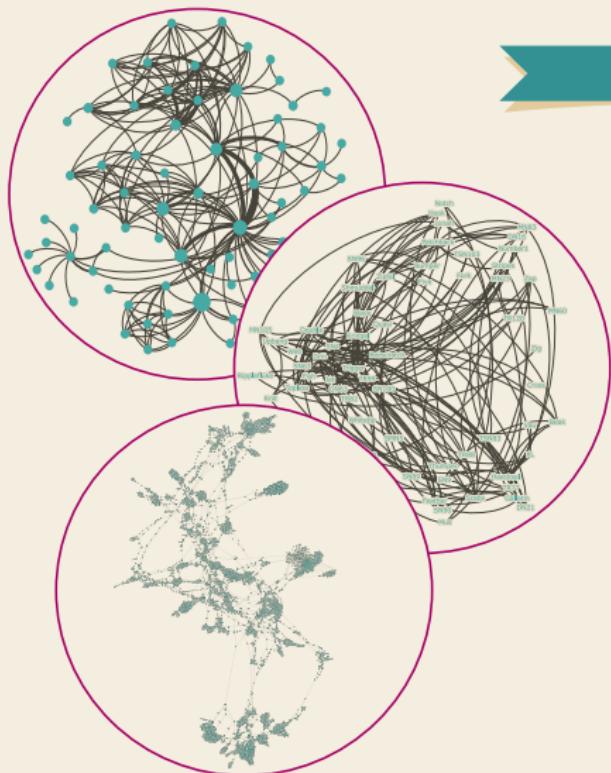
More



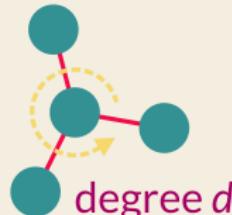
Algorithmic
tractability



Can We Use Sparse Structure?



treewidth t

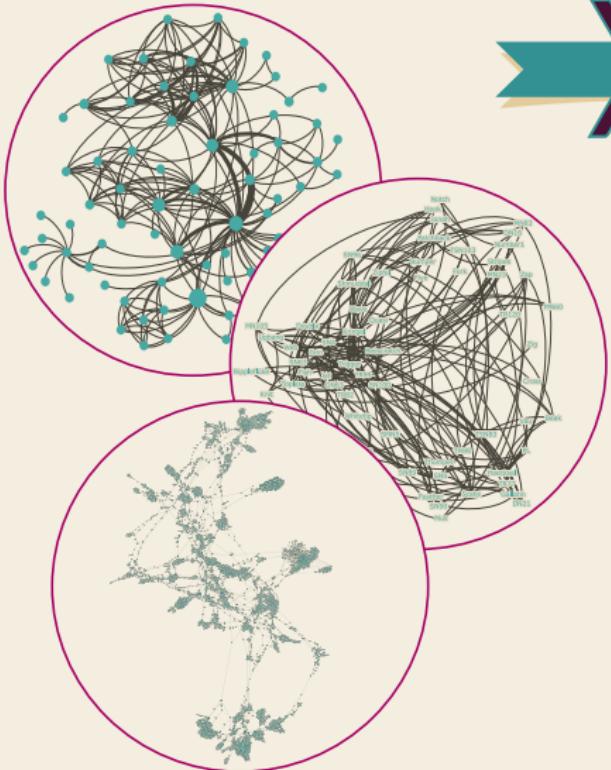


degree d

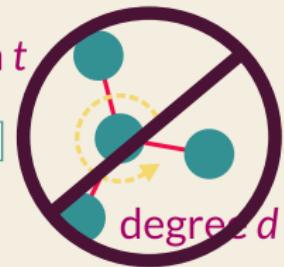
⋮

[Bul19 Alg] [Cyg19 Param. Alg]

Evidence says...



[Gas21 BIBM]
[Boe20 Env Plan B]



[Mar22 Alg Mol Bio]
[Man19 ICDT]
[Adc13 ICDM]*

⋮

The Goldilocks Zone

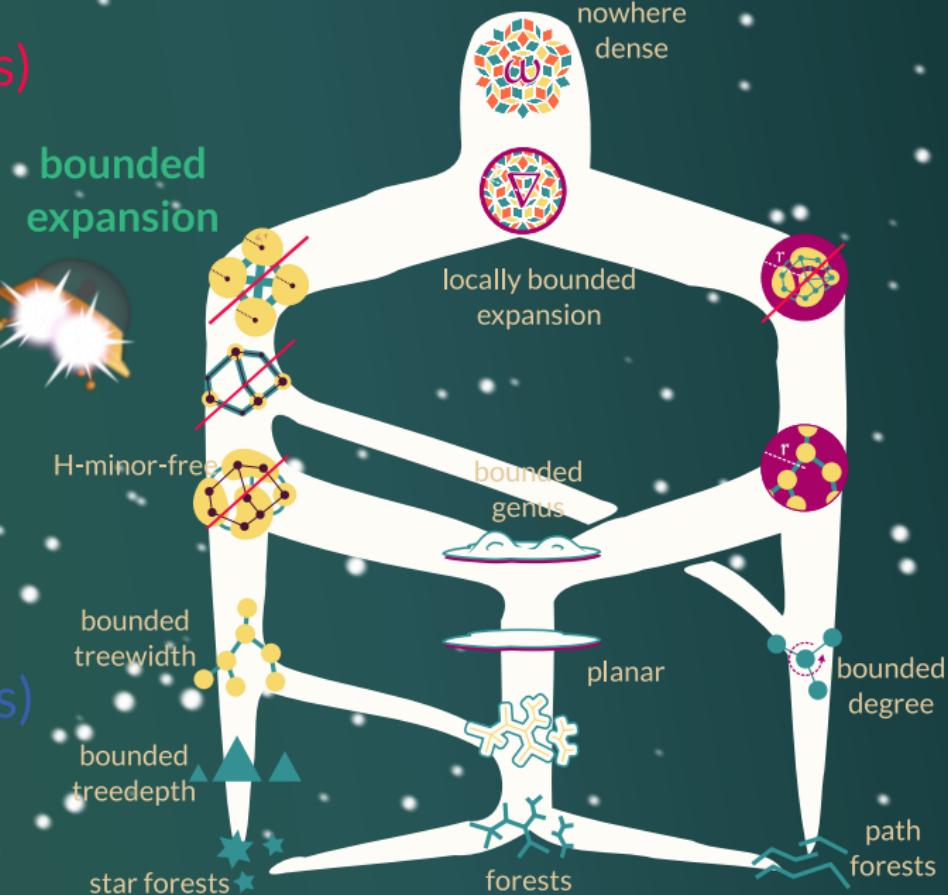
Too big!
(few algorithms)



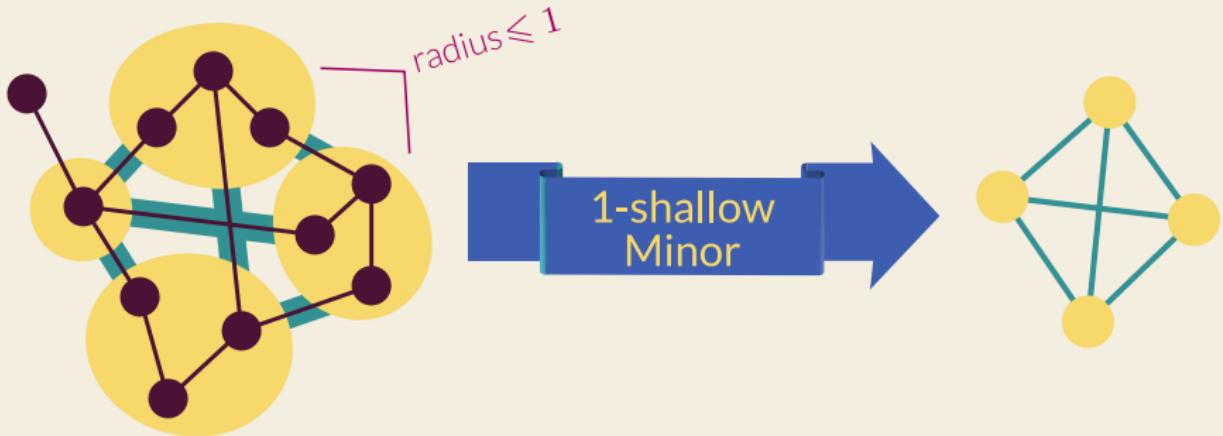
Just Right!



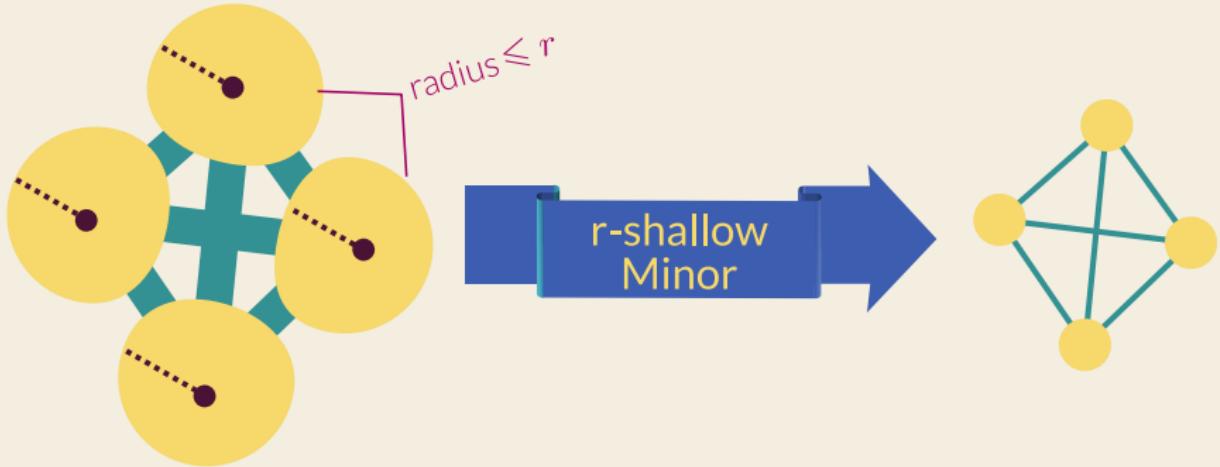
Too small!
(few real graphs)



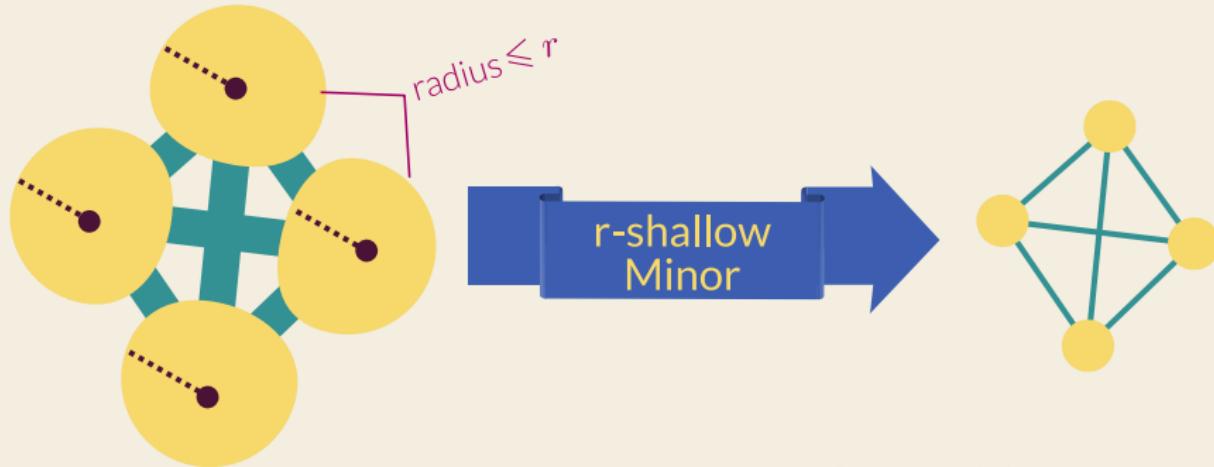
What is bounded expansion?



It's not just for radius 1...



Sparse Shallow Minors == Bounded Expansion

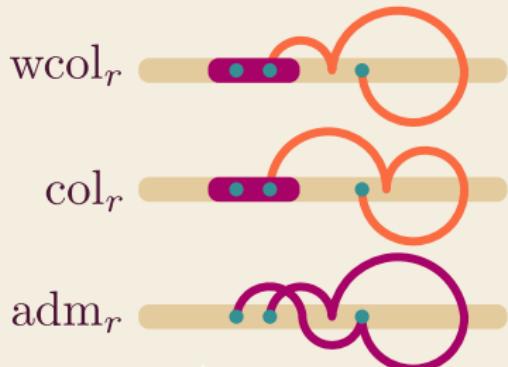


$$\nabla_r(G) = \max_{H \preccurlyeq_r G} \frac{|E(H)|}{|V(H)|}$$

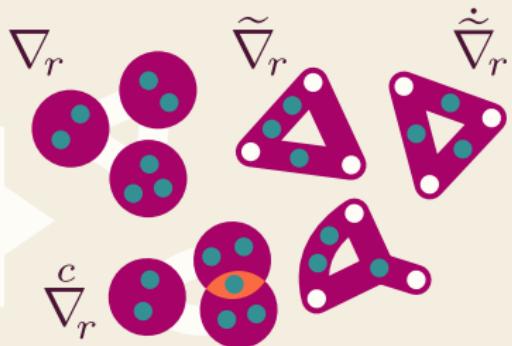


A graph class has bounded expansion iff it is ∇_r -bounded.

Bounded expansion



Density
of shallow
minors

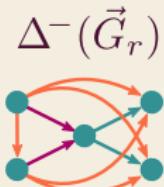


ν_r

Size of r-reachable
sets in ordering

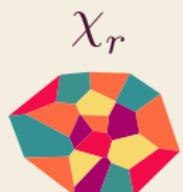


Normalized number of
traces r-neighbourhoods
leave in any subset



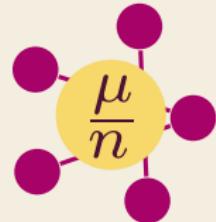
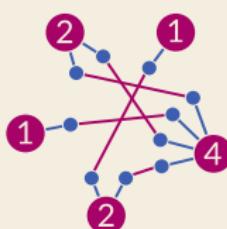
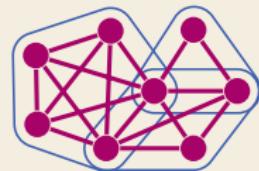
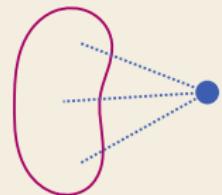
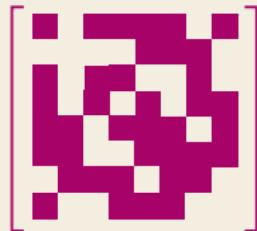
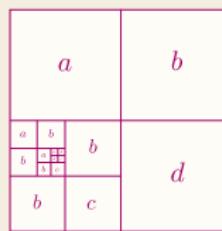
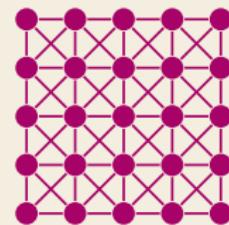
In-degree of
r-step (d)tf-
augmentation

Number of colours
in r-treedepth
colouring

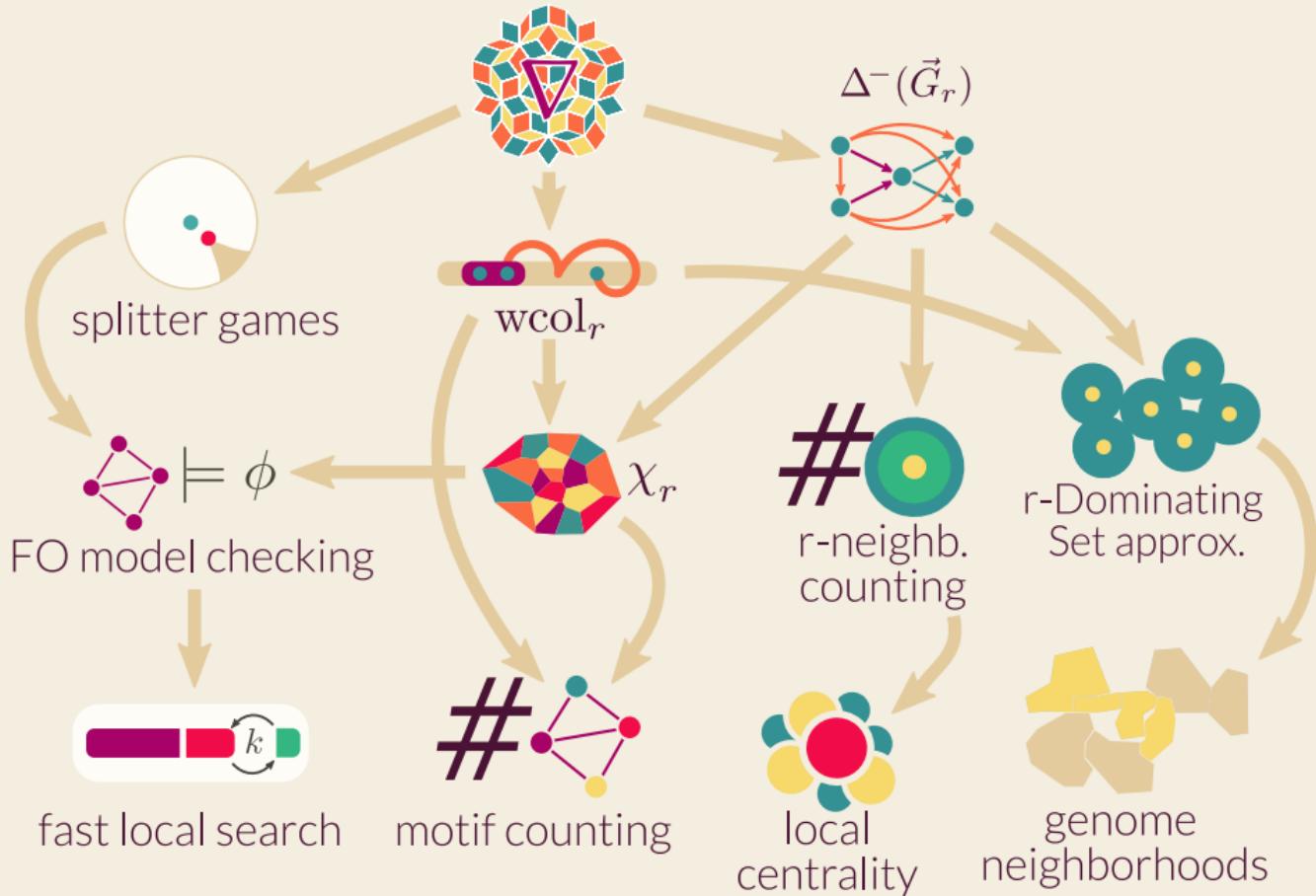


[Neš12 Sparsity]

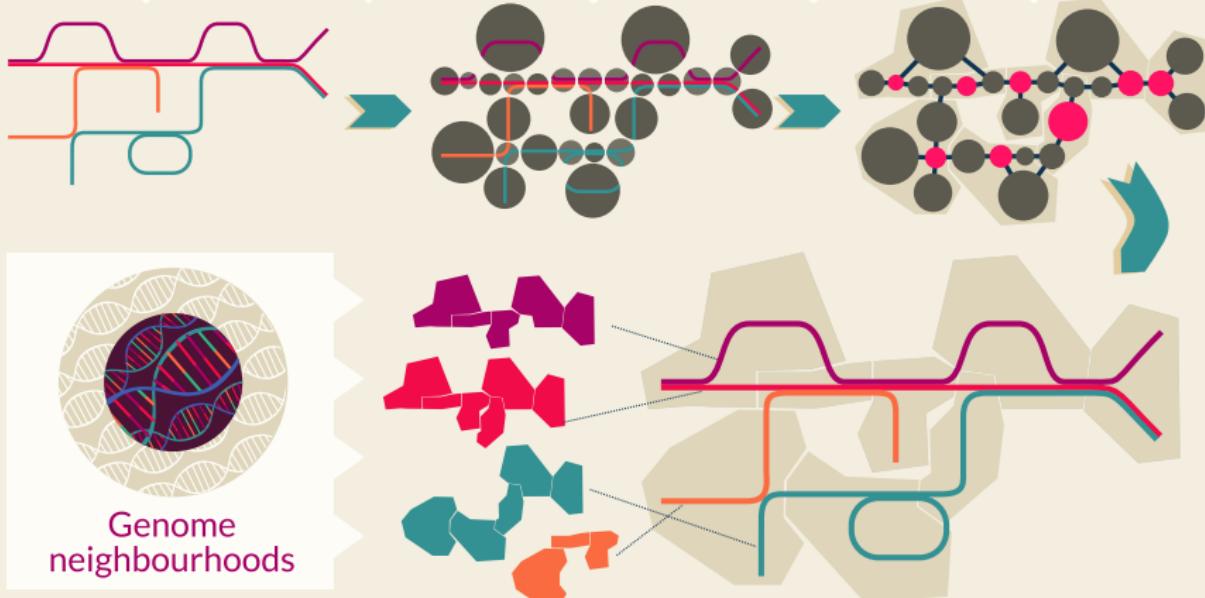
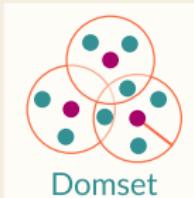
Random model sparsity

 $G(n, \frac{\mu}{n})$  $G^{\text{CL}}(D_n)$  $G^{\text{CF}}(D_n)$  $G^{\text{RIG}}(n, \alpha, \beta, \gamma)$  $G^{\text{BA}}(n, n_0, k)$  $G^{\text{SGK}}(k, \alpha, \dots, \gamma)$  $G^{\text{RMAT}}(k, m, a, b, c)$  $G^{\text{KL}}(n, p, q, \gamma)$ 

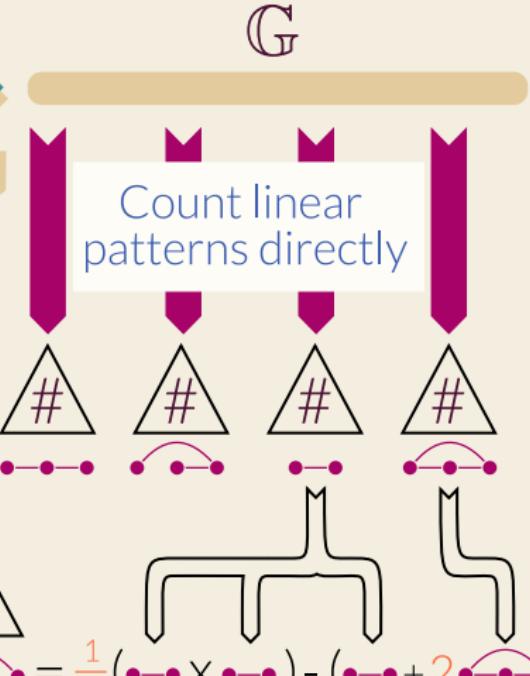
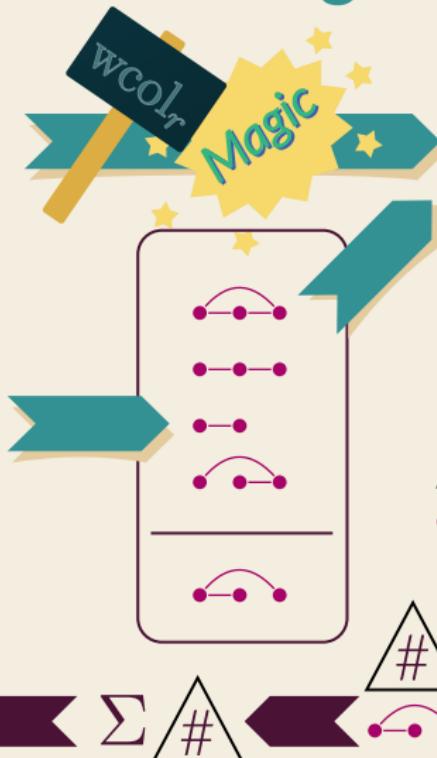
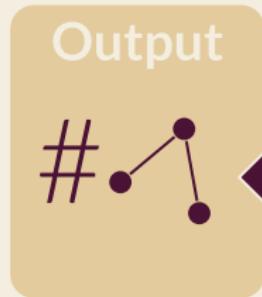
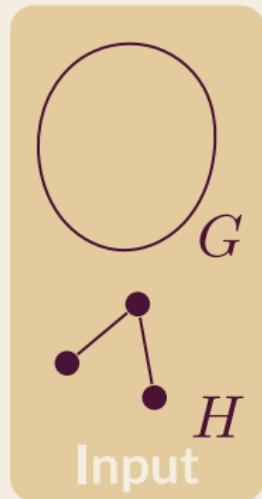
Applications & Algorithms



Index/Query Metagenomes



Counting Subgraphs



Aggregate

[Rei23 Algorithmica]*

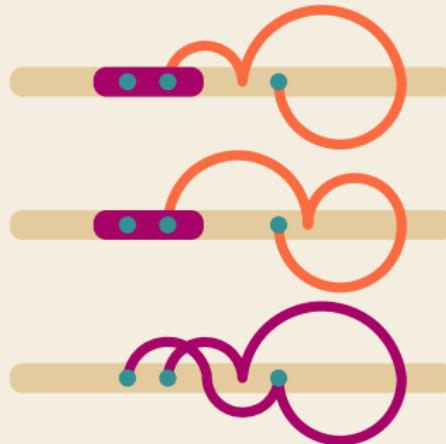
Compute composite pattern counts

github.com/theoryinpractice/mandoline

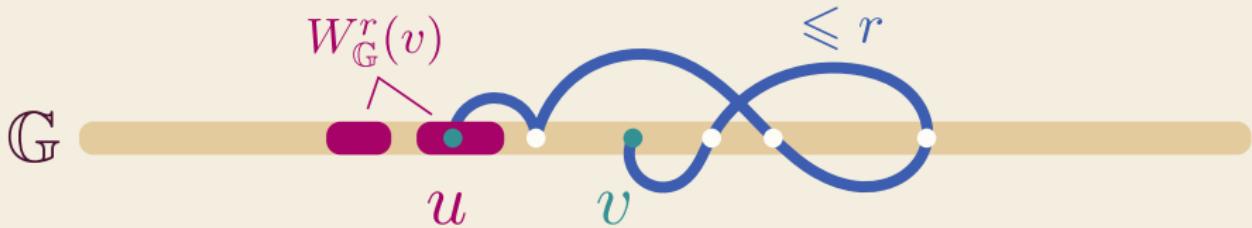
$$\sum \triangle \# = \frac{1}{2} (\cdot \cdot \cdot \times \cdot \cdot \cdot) - (\cdot \cdot \cdot + 2 \cdot \cdot \cdot)$$

Part II

Generalized coloring numbers

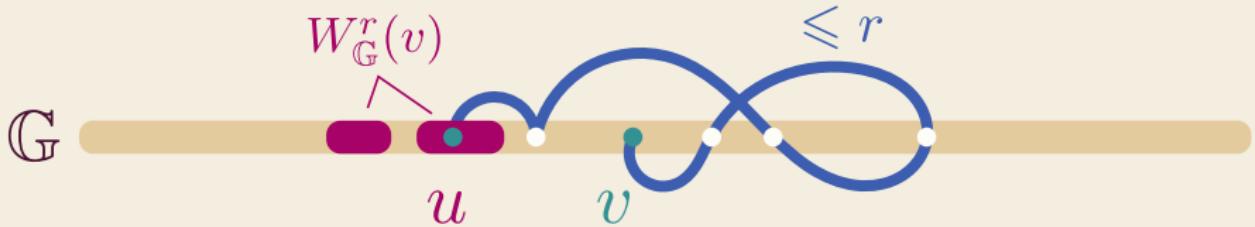


Weak coloring & bounded expansion



u is weakly r -reachable from v if there exists a path from v to u of length at most r such that u is the path's leftmost vertex.

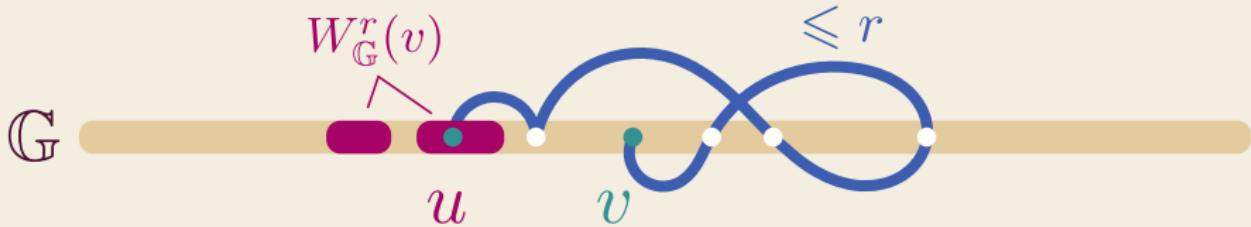
Weak coloring & bounded expansion



u is **weakly r -reachable** from v if there exists a path from v to u of length at most r such that u is the path's leftmost vertex.

$$\text{wcol}_r(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} |W_{\mathbb{G}}^r(v)|$$

Weak coloring & bounded expansion



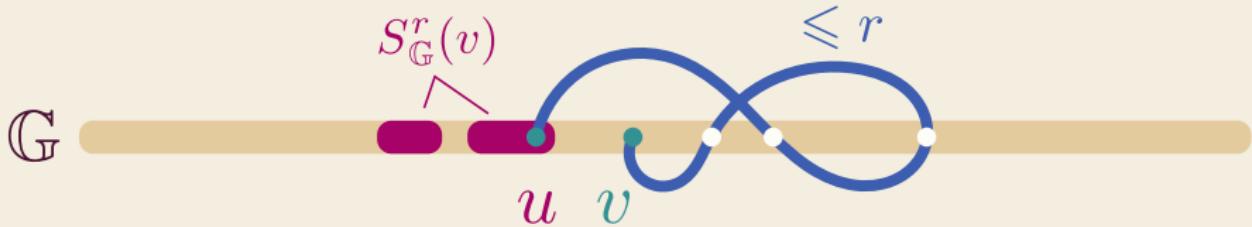
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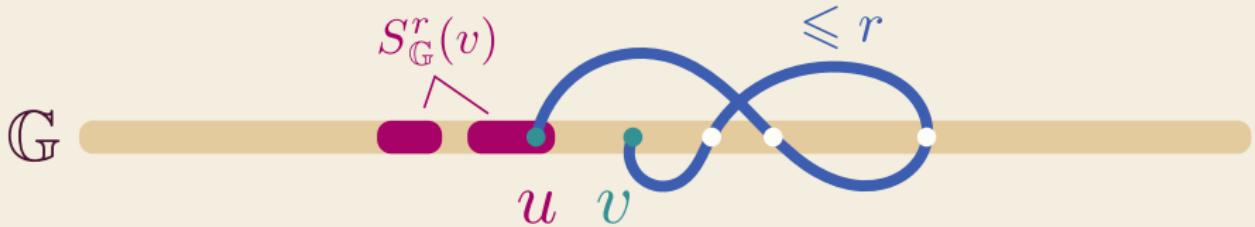
A graph class has bounded expansion iff it is wcol_r -bounded.

Strong coloring & bounded expansion



u is strongly r -reachable from v if there exists a path from v to u of length at most r such that all interior vertices lie right of v .

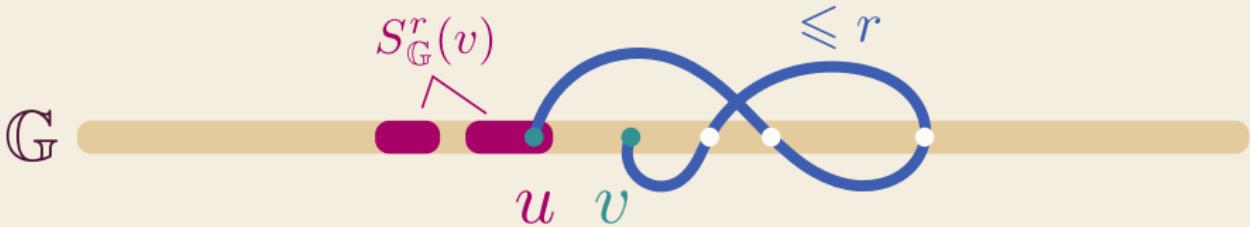
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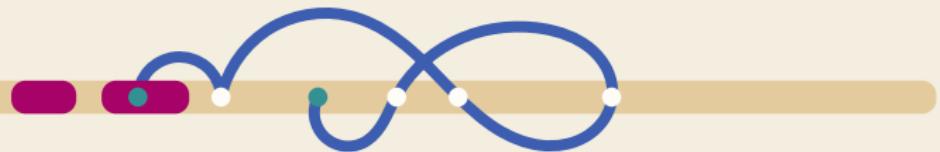
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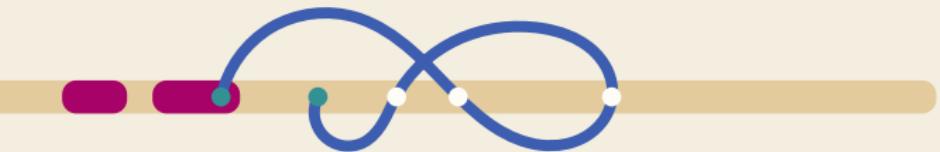
A graph class has bounded expansion iff it is col_r -bounded.

Weak & strong coloring

\mathbb{G}



\mathbb{G}



$$\text{col}_r(\mathbb{G}) \leq \text{wcol}_r(\mathbb{G}) \leq r(\text{col}_r(\mathbb{G}) - 1)^r + 1$$

Origin story

Introduced by Kierstead & Yang, generalizing

Def. $\text{col}(\mathbb{G}, x) := |N[x] \cap \{y \leqslant_{\mathbb{G}} x\}|$

$$\text{col}(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} \text{col}(\mathbb{G}, v)$$


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We can greedily color a graph with such an ordering:

$$\chi(G) \leqslant \chi^\ell(G) \leqslant \text{col}(G)$$



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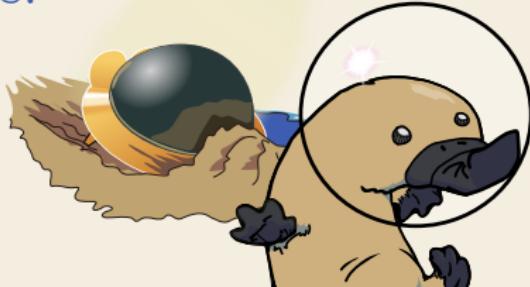
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Further connections to colorings:

$$\text{col}_2(G) = \chi_{\text{acyclic}}(G)$$

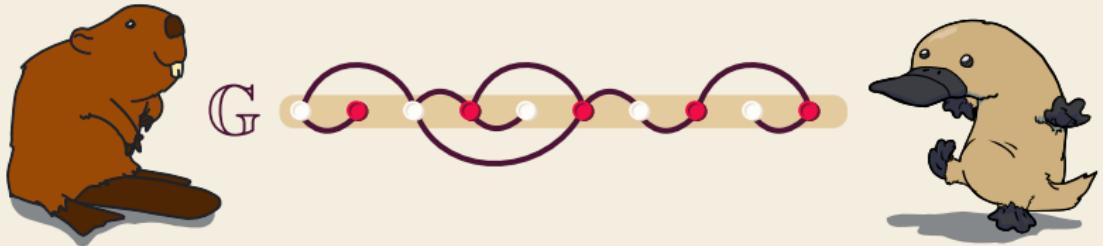
$$\text{wcol}_2(G) = \chi_{\text{star}}(G)$$



It's all fun and games...

r-Ordering Game

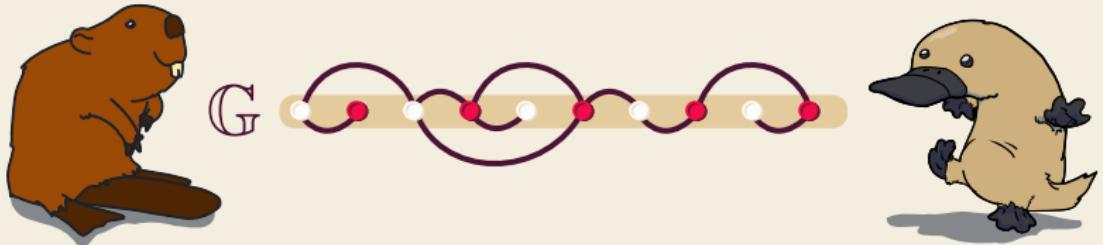
- Alice & Bob play on a graph G by alternating choosing vertices (n turns)
- Creates an order \mathbb{G} , score is $\text{col}_r(\mathbb{G})$
- Alice plays first and wants to minimize score; Bob wants to maximize



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r-Ordering Game

- Alice & Bob play on a graph G by alternating choosing vertices (n turns)
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$\text{gcol}_r(G) :=$ Lowest score Alice
can always achieve

To planarity and beyond!

A (non-comprehensive) selection of recent work

Planar graphs have $\text{wcol}_2(G) \leq 23$

[Alm22 DM]

Koebe orders of planar graphs satisfy

$$\text{wcol}_r(\mathbb{G}) = O(r^4 \ln r)$$

[Ned22 Arxiv]

Graphs of treewidth t have

$$\text{wcol}_r(G) = \Theta(r^{t-1} \log r)$$

[Jor22 Elec J Comb]



To planarity and beyond!

A (non-comprehensive) selection of recent work

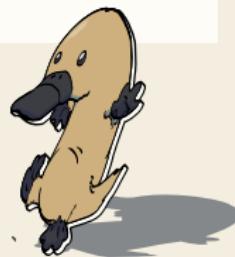
Graph powers: $\text{col}_r(G^p)$ can be bounded in terms of $\text{wcol}_r(G)$ and $\Delta(G)$,

[Kie20 DM]

If G has genus g $\begin{cases} \text{col}_r(G) \leq (2g + 3)(2r + 1) \\ \text{wcol}_r(G) \leq (2g + \binom{2r+2}{2})(2r + 1) \end{cases}$

If G excludes a minor $\begin{cases} \text{col}_r(G) \leq \binom{t-1}{2}(2r + 1) \\ \text{wcol}_r(G) \leq O(r^{t-1}) \end{cases}$

[van17 EuJC]



~~One Ring to Rule Them All~~ Order

?

Do orders witnessing low col_r also have small $\text{col}_{r'}$?

Not if low/small mean optimal!

For every $r \neq r'$ $\exists G$ such that every ordering $\mathbb{G} \in \Pi(G)$ is non-optimal for col_r or $\text{col}_{r'}$.

But: known bounds use one order for all r !



~~One Ring to Rule Them All~~ Order

?

If $\text{col}_r(G) \leq c(r)$, does $\exists c^*$ such that some \mathbb{G} has $\text{col}_r(\mathbb{G}) \leq c^*(r)$ for all r ?

Thm. (van Heuvel, Kierstead)

For all G there exists $\mathbb{G} \in \Pi(G)$ with

$$\text{col}_r(\mathbb{G}) \leq (2^r + 1)(\text{col}_{2r}(G))^{4r}$$

for all r :

[van21 EuJC]



Twins: twice the fun, half the sleep

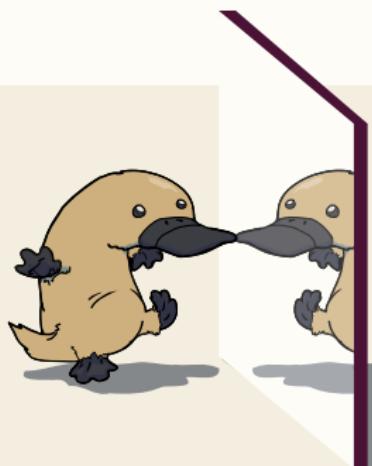
Twinwidth is a hot new graph parameter generalizing an invariant of permutation classes.

[Bon20 FOCS][Gui14 SODA]

Thm. (Bonnet et al.)

If $\text{tww}(G) \leq t$ and $K_{s,s} \not\subseteq G$, then there ex. f_r s.t.
for all r . $\text{col}_r(G) \leq f_r(s, t)$

[Bon21 SODA]



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$$\text{col}_r(G) \leq f_r(s, t)$$

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[Bon21 SODA]

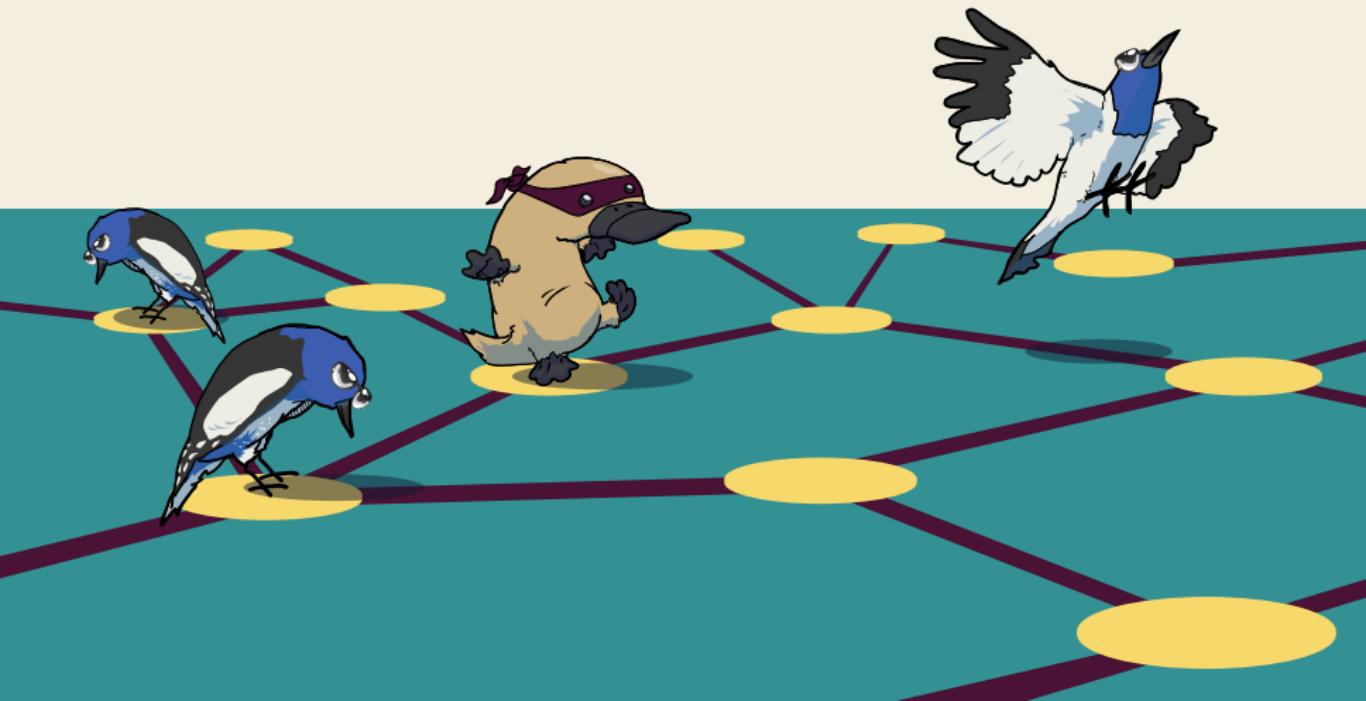
Thm. (Dreier et al.)

There ex. G with $\text{col}_r(G) \geq (t - 4)^r s$, and

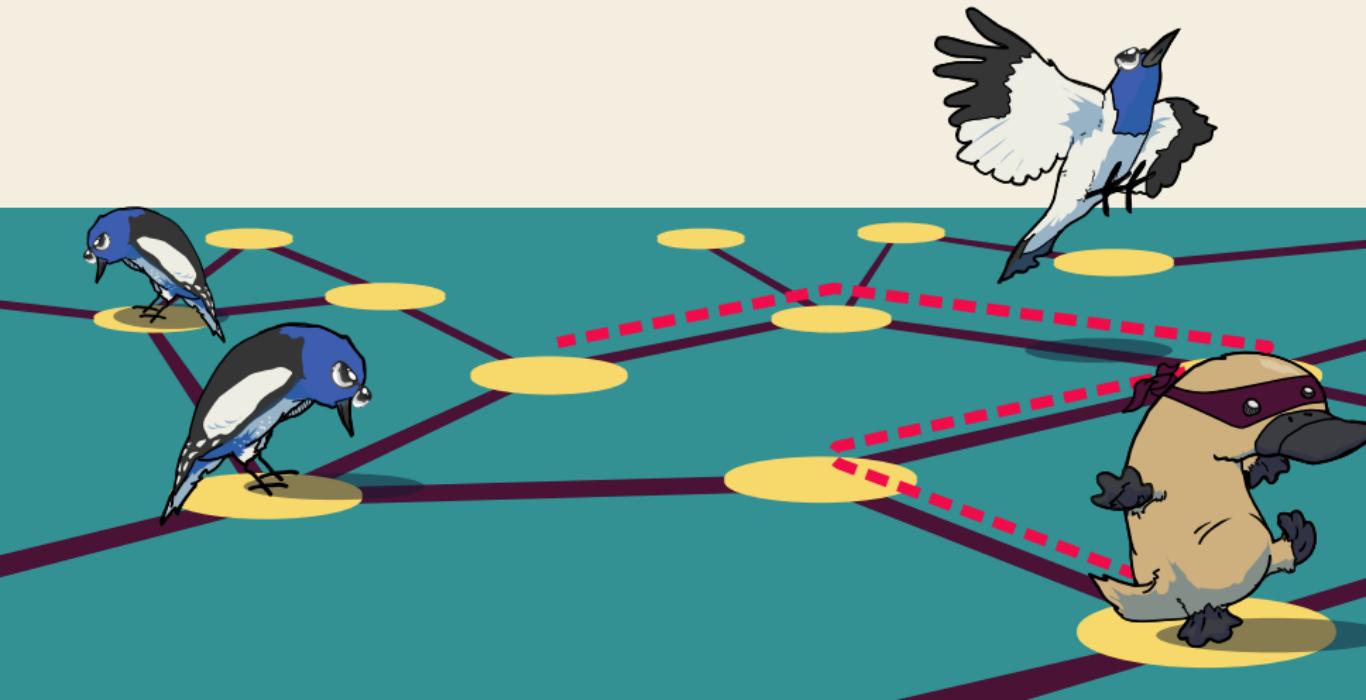
$$\overbrace{\text{tww}(G) \leq t \quad K_{s,s} \not\subseteq G}^{\text{tww}(G) \leq t} \quad f_r(s, t) \leq (t^r + 3)s$$

[Dre22 DM]

Cops & robbers



Cops & robbers

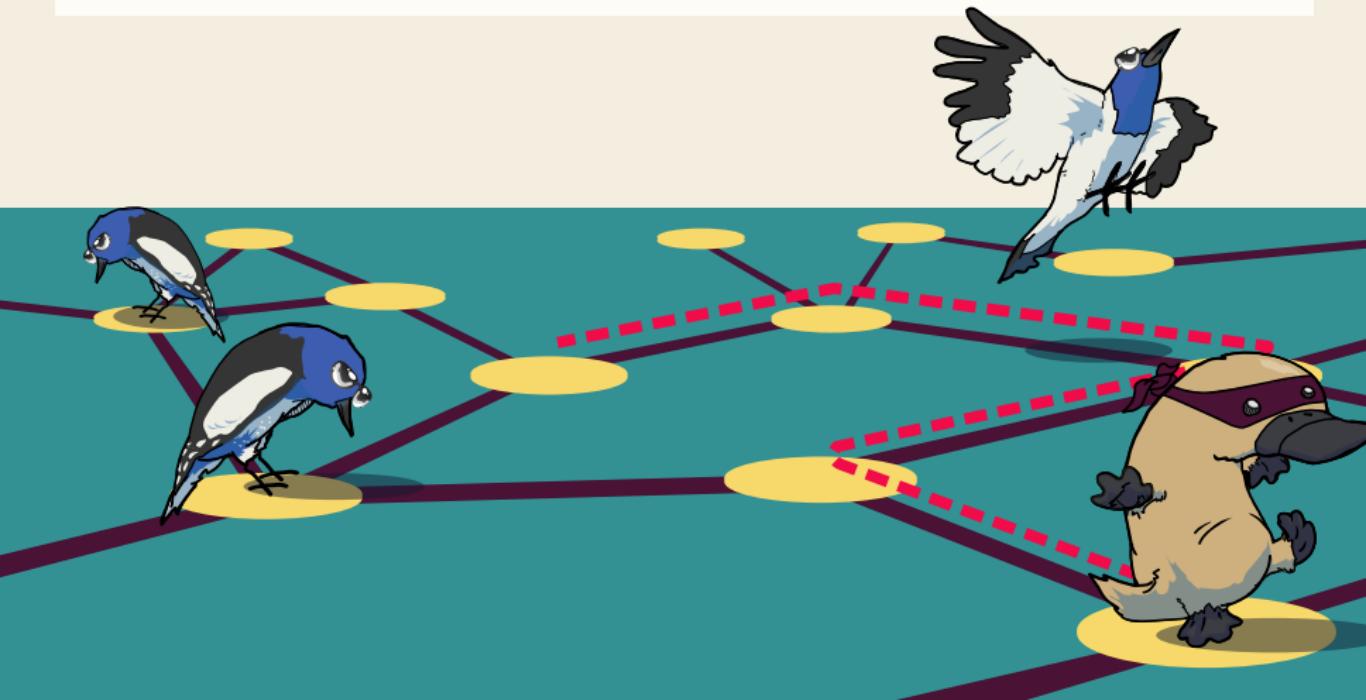


Cops & robbers

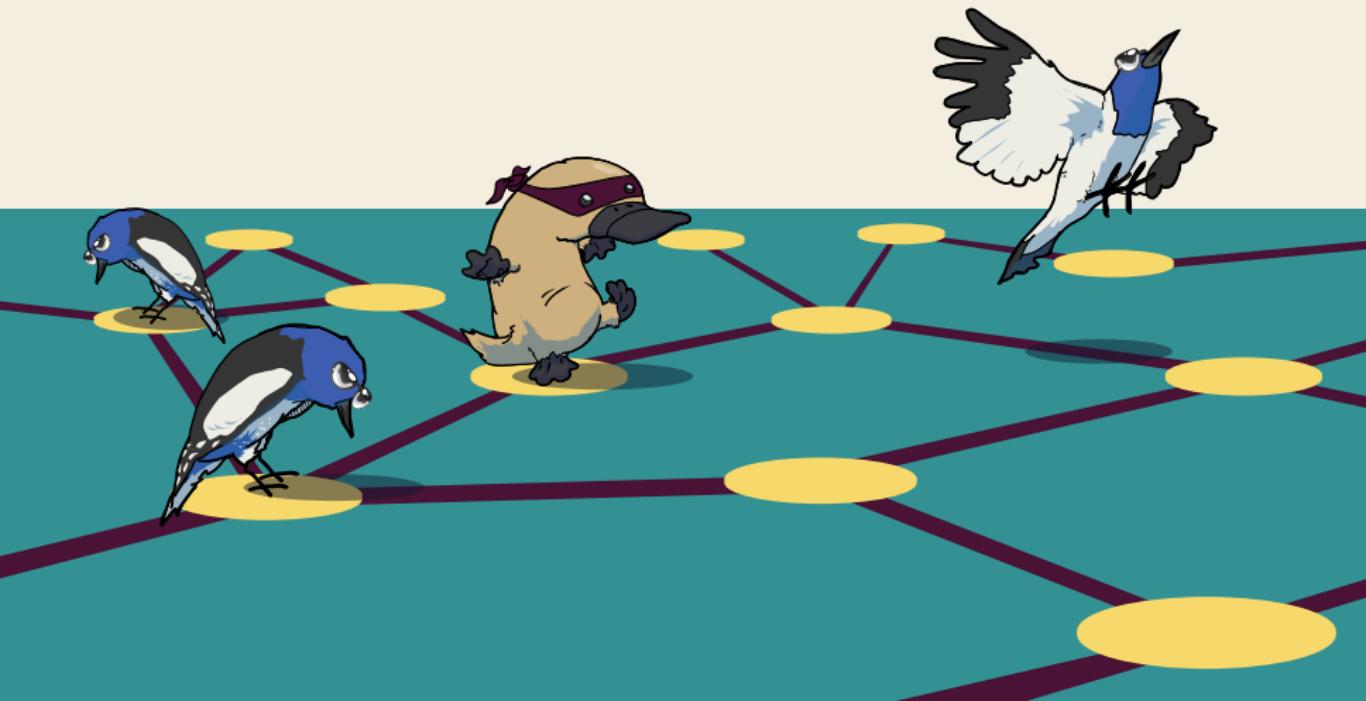
$k + 1$ cops can
catch the robber



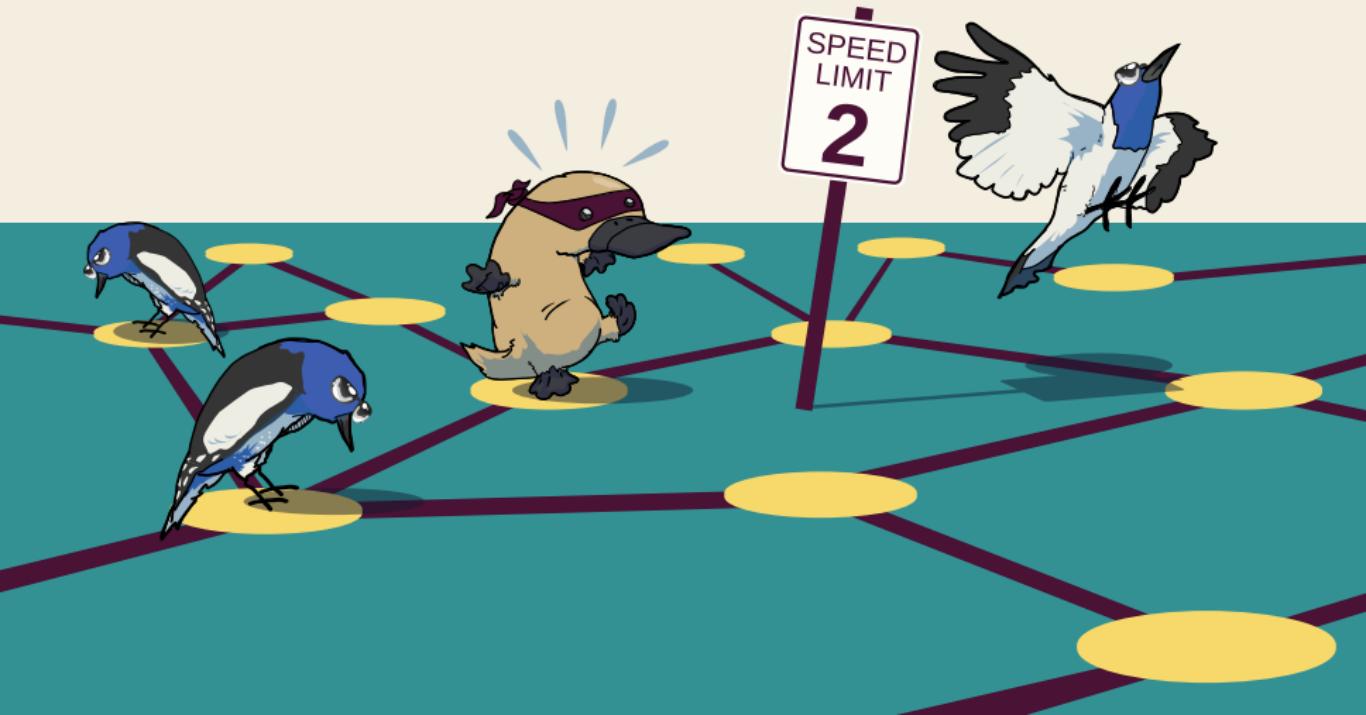
$\text{tw}(G) = k$



Cops & robbers



Cops & robbers



Cops & robbers

$k + 1$ cops can
catch a robber with
maximum speed r



$$\text{copw}_r(G) = k$$



Cops & robbers

$k + 1$ cops can
catch a robber with
maximum speed r

$$\iff \text{copw}_r(G) = k$$



Thm. (Toruńczyk)

$$\text{adm}_r(G) + 1 \leq \text{copw}_r(G) \leq \text{wcol}_{2r}(G) + 1$$

[Tor23 ArXiv]



A graph class has bounded
expansion iff it is copw_r -bounded.

Part III

Algorithmic questions

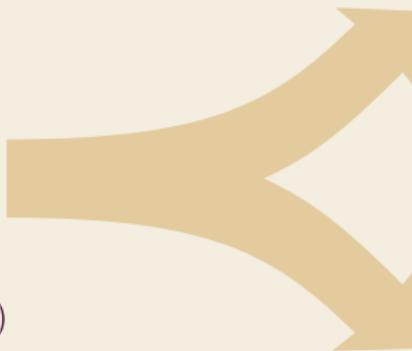


'Limits' of coloring numbers

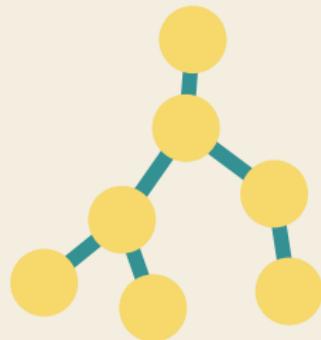


Degeneracy

$$\text{col}_1(G) = \text{wcol}_1(G)$$

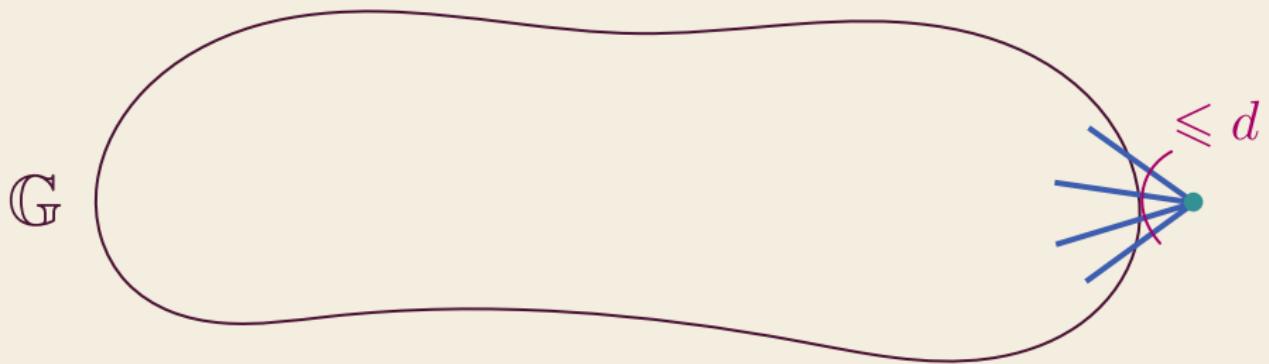


$$\text{wcol}_{\infty}(G) = \text{td}(G)$$

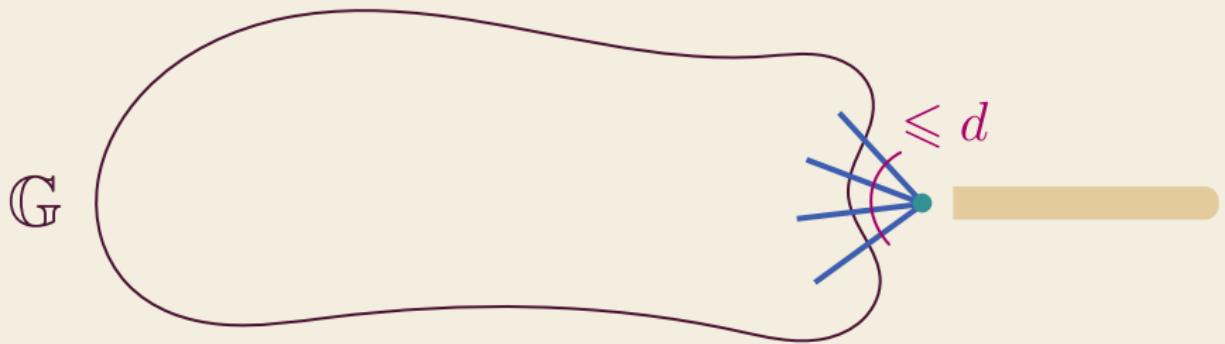


$$\text{col}_{\infty}(G) = \text{tw}(G)$$

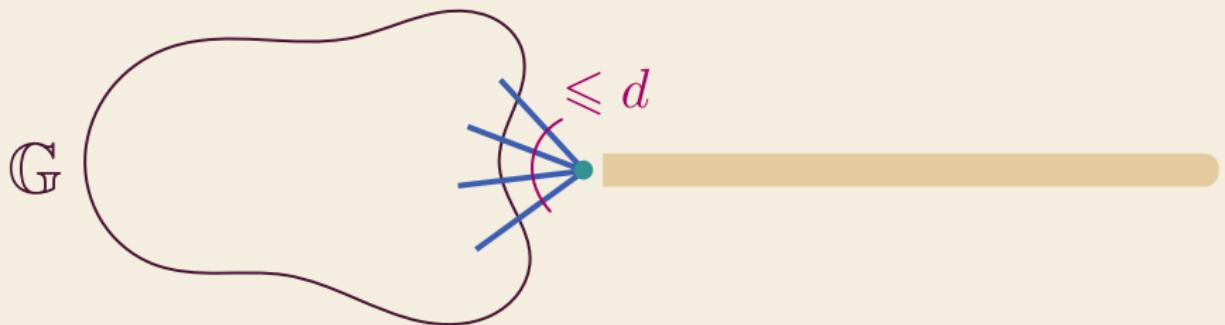
Computing degeneracy ordering



Computing degeneracy ordering



Computing degeneracy ordering



Computing degeneracy ordering

\mathbb{G}

$$\forall v \quad |N(v) \cap \{u \leq_{\mathbb{G}} v\}| \leq d$$

No free lunch

Thm. (Breen-McKay, Lavallee, **S**)

Deciding whether $\text{wcol}_r(G) \leq k$ or $\text{col}_r(G) \leq k$
is NP-complete even for $r = 2$

[Bre23 EuJC]*

Anything beyond degeneracy is
hard to compute exactly.

Maybe cheap lunch?

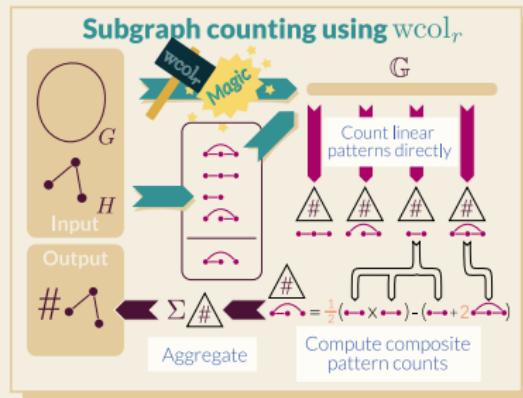
Thm. (Dvořák)

$$\text{adm}_r(\mathbb{G}) \leq k \implies$$

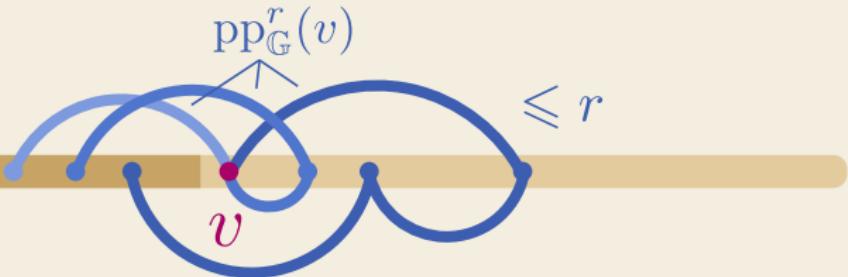
$$\begin{aligned}\text{col}_r(\mathbb{G}) &\leq (k-1)^r + 1 \\ \text{wcol}_r(\mathbb{G}) &\leq (r^2 k)^r\end{aligned}$$

[Dvo22 JGT][Dvo13 EuJC]

Approximation
works just fine here!
(though slower)



Admissibility



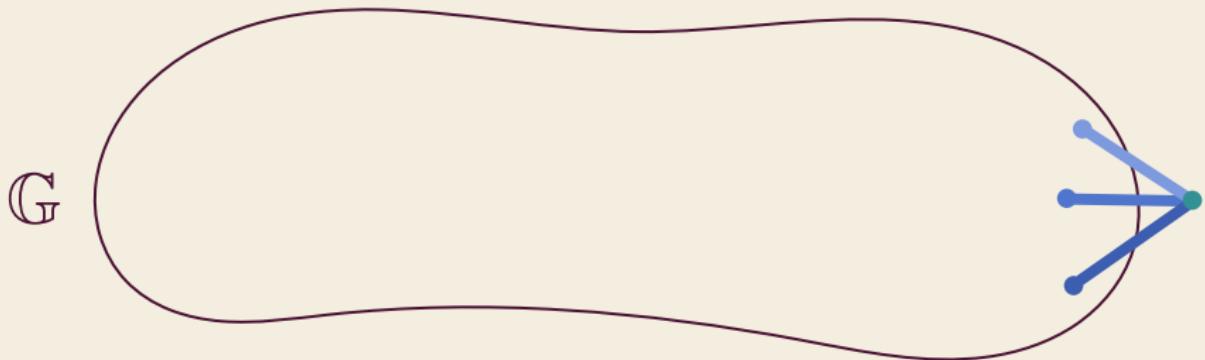
An r -path packing at v is a set of disjoint paths of length $\leqslant r$ which start at v and end left of v in G . $\text{pp}_G^r(v)$ is the maximum size of such a set at v .

$$\text{adm}_r(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} \text{pp}_{\mathbb{G}}^r(v)$$



A graph class has bounded expansion iff it is adm_r -bounded.

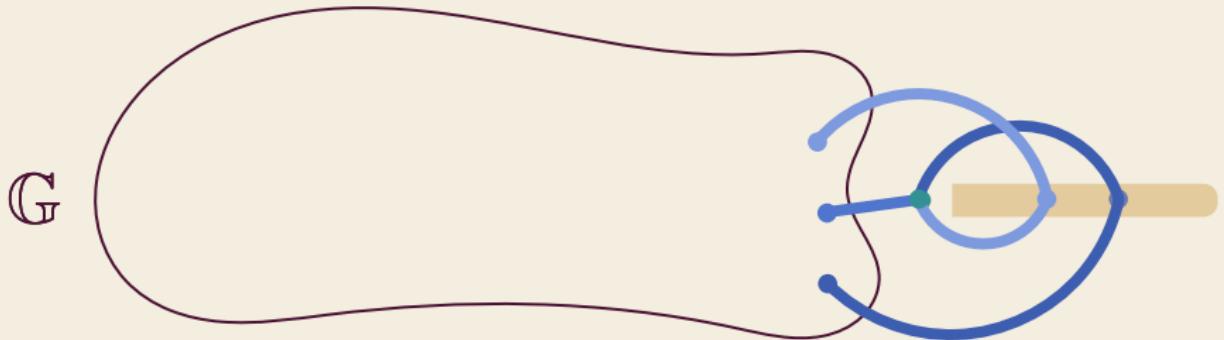
Approximating orderings



Select vertex which minimizes pp^r at each step:

Resulting order minimizes $\text{adm}_r(G)$

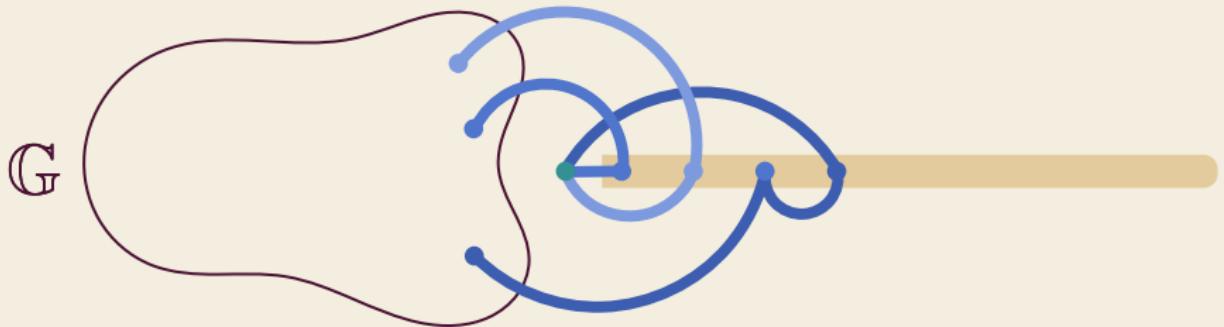
Approximating orderings



Select vertex which minimizes pp^r at each step:

Resulting order minimizes $\text{adm}_r(G)$

Approximating orderings



Select vertex which minimizes pp^r at each step:

Resulting order minimizes $\text{adm}_r(G)$

Approximating orderings

G

Select vertex which minimizes pp^r at each step:

Resulting order minimizes $\text{adm}_r(G)$

Approximating orderings

G

Select vertex which minimizes pp^r at each step:

Resulting order minimizes $\text{adm}_r(G)$

⇒ Approximates wcol_r and col_r

Approximating orderings

G

Select vertex which minimizes pri^r at each step:

Resulting ordering minimizes $\text{adm}_r(G)$

⇒ Approximates val_r and col_r



Factoring in Approximation

Thm. (Dvořák)

$$\text{adm}_r(\mathbb{G}) \leq k \implies \begin{aligned} \text{col}_r(\mathbb{G}) &\leq (k-1)^r + 1 \\ \text{wcol}_r(\mathbb{G}) &\leq (r^2 k)^r \end{aligned}$$

approximate
 adm_r ↓

[Dvo22 JGT][Dvo13 EuJC]

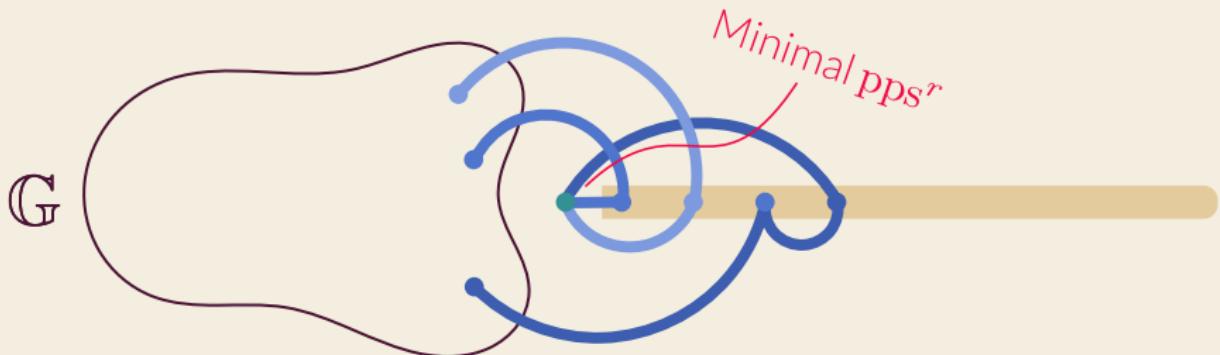
Thm. (Dvořák)

There is an $\frac{r(rk-1)^{r-1}}{(rk)^{r-1}}$ -approximation for $\frac{\text{col}_r}{\text{wcol}_r}$

[Dvo13 EuJC]

Estimating path-packings

Estimate pp^r by packing shortest paths!



Thm. (Breen-McKay, Lavallee, S)

$$\begin{array}{c} \text{pps}_{\mathbb{G}}^r(v) \leq k \\ \text{for all } v \in G \end{array} \implies \begin{array}{c} \text{col}_r(\mathbb{G}) \leq k(k-1)^{r-1} \\ \text{wcol}_r(\mathbb{G}) \leq \frac{k^{r+1}-1}{k-1} \end{array}$$

Some Open Problems



Is there a constant-factor approximation algorithm for $\text{wcol}_r(G)$?

Open even if $r = 2$

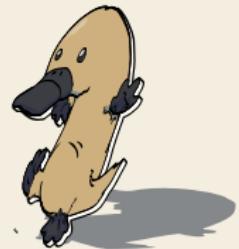
What are asymptotics of $\max \text{wcol}_r(G)$?

Known: $\Omega(r^2 \log r)$ and $O(r^3)$

Conjecture
[Jor22 Elec J Comb]

Conjecture
[Alm22 DM] $\text{wcol}_2(G) \leq 18$

planar



Does there exist a polynomial $P(\cdot, \cdot)$
s.t. $\forall G \forall r \text{ col}_r \leq P(r, \nabla_r(G))$?

Known: yes for admissibility, no for wcol_r



degree of P must be ind. of r

THANKS



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