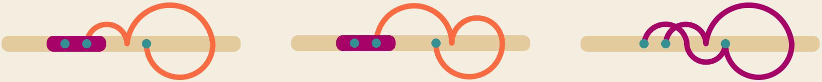
Taking a Hard Look at
Generalized Coloring Numbers



@blairdsullivan
University of Utah

AMS SE Sectional

Mar 18 2023

Special thanks to
Felix Reidl (Birkbeck College, London)
for original artwork and collaboration
on scientific communication

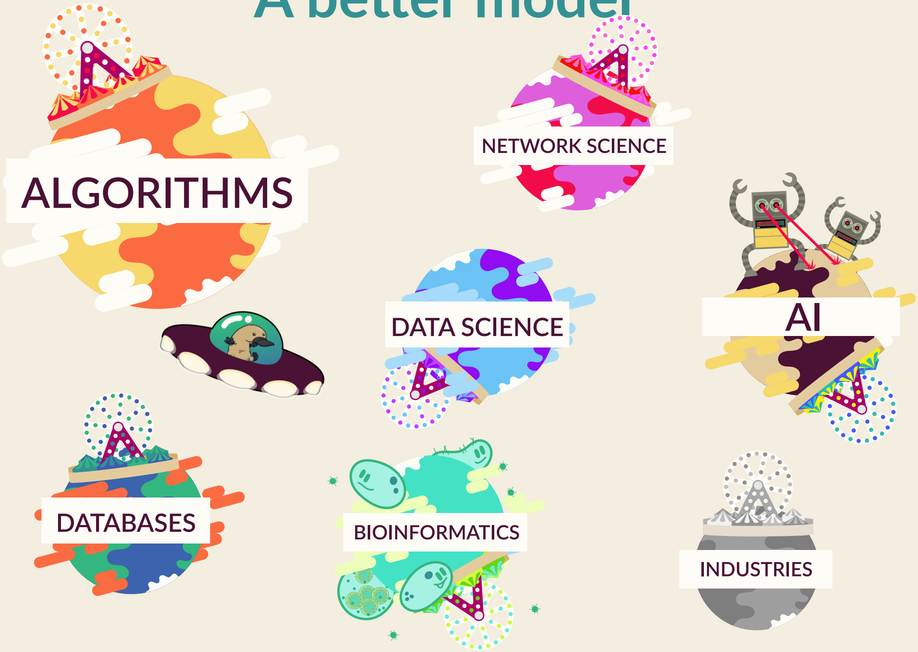
My work in this talk was primarily supported by the Gordon & Betty Moore Foundation,
with additional funding from DARPA, ARO, and NIH..

Part I

Sparse classes

A false dichotomy

A better model



Parameterized algorithms

	Classical view	Parameterized view
Bad	$O(n^{\log n}), O(2^n), \dots$ Not polynomial-time	$O(n^{f(k)})$ Slice-wise polynomial time
Good	$O(n^c)$ Polynomial-time ("efficient")	$O(f(k)n^c)$ Fixed-parameter tractable

Choosing a class

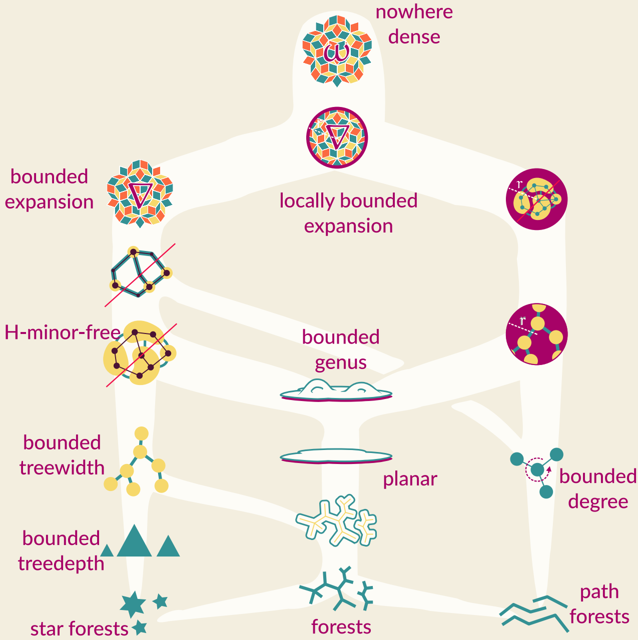
Larger classes

Less

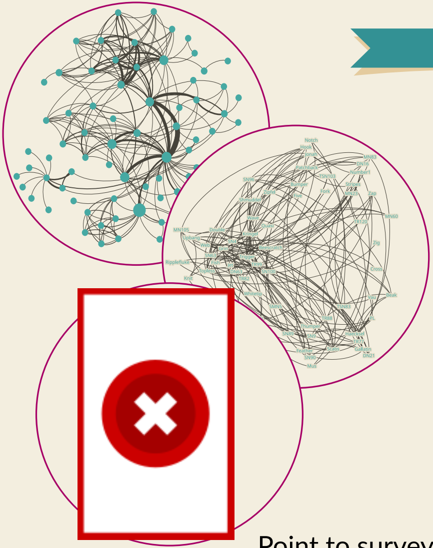
Structure

More

Algorithmic tractability



Can We Use Sparse Structure?



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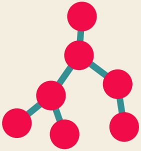
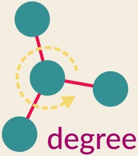


treewidth t

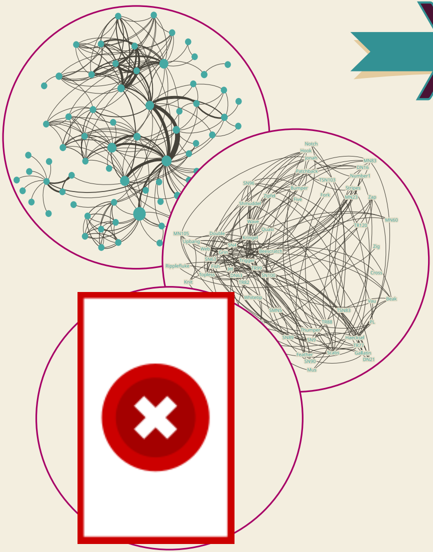


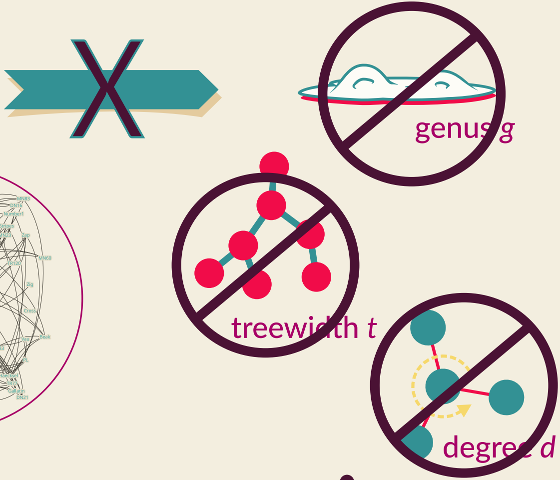
degree d

⋮

Point to survey in bioinformatics, other applications

Evidence says...





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The sparse class hierarchy

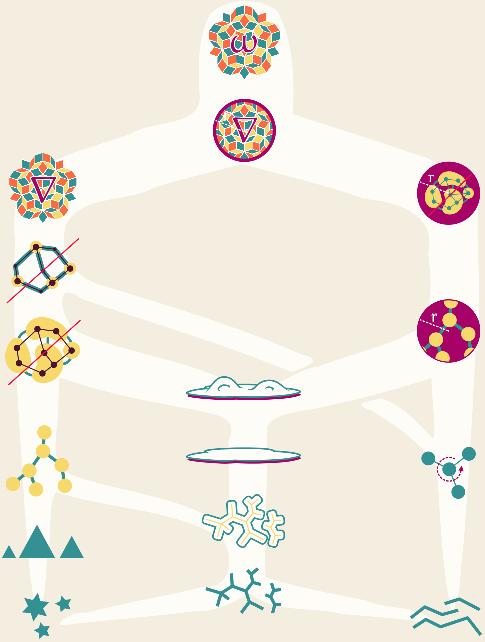
Larger classes

Less

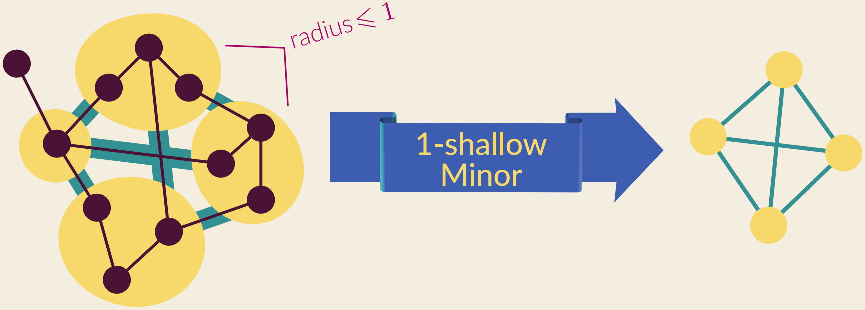
Structure

More

Algorithmic tractability



Shallow minors & bounded expansion



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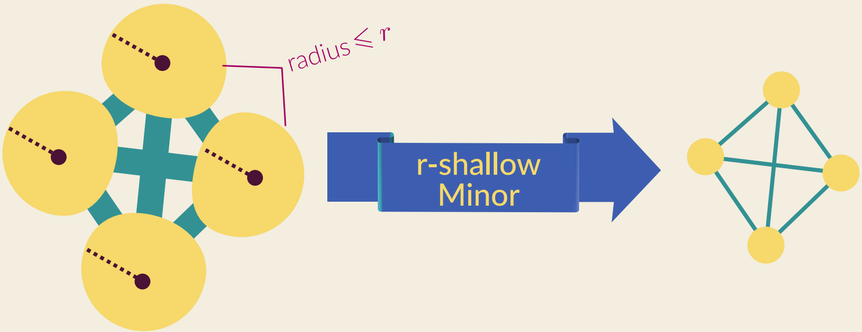
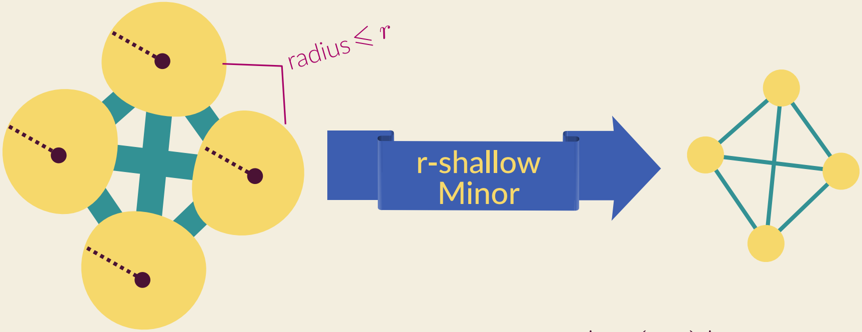


Shallow minors & bounded expansion

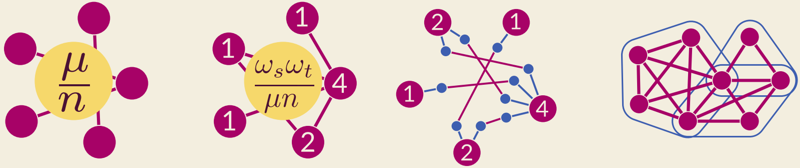


$$\nabla_r(G) = \max_{H \preccurlyeq_r G} \frac{|E(H)|}{|V(H)|}$$

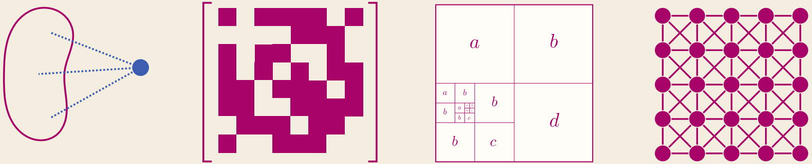
A graph class has bounded expansion iff it is ∇_r-bounded.

Random model sparsity

$G(n, \frac{\mu}{n})$ $G^{\mathrm{CL}}(D_n)$ $G^{\mathrm{CF}}(D_n)$ $G^{\mathrm{RIG}}(n, \alpha, \beta, \gamma)$

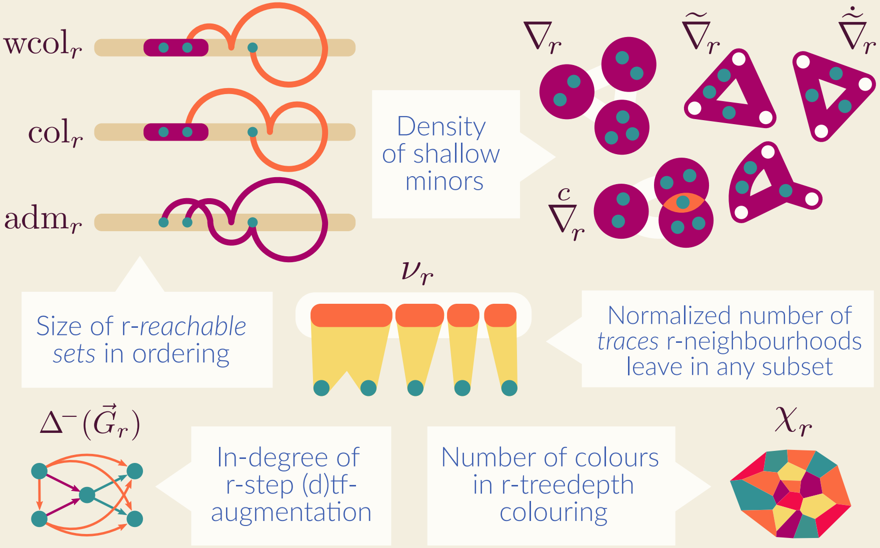


$G^{\mathrm{BA}}(n, n_0, k)$ $G^{\mathrm{SGK}}(k, \alpha, \ldots, \gamma)$ $G^{\mathrm{RMAT}}(k, m, a, b, c)$ $G^{\mathrm{KL}}(n, p, q, \gamma)$



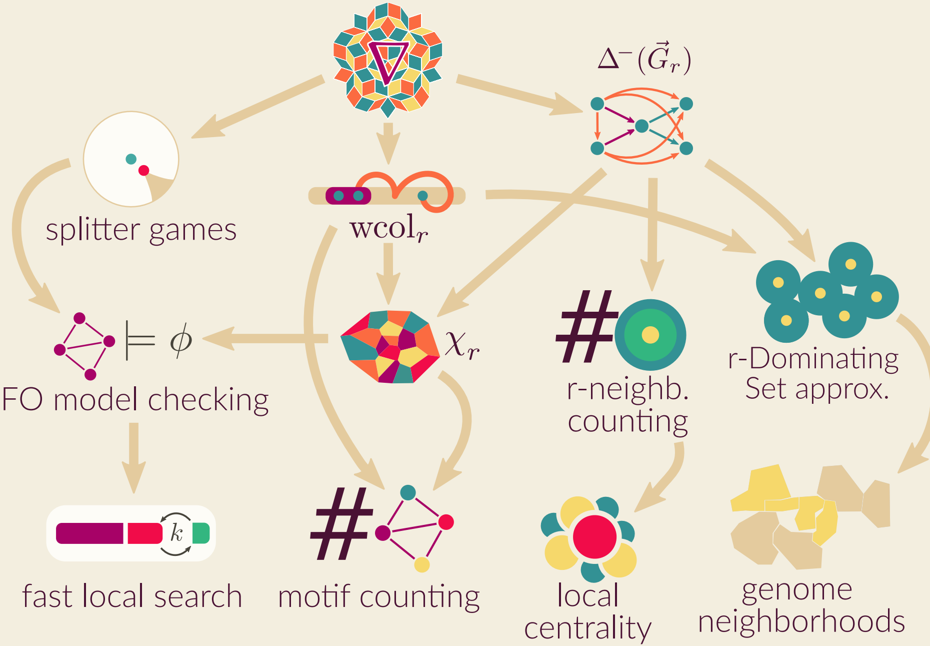
[Dem19 JCSS]*

Bounded expansion

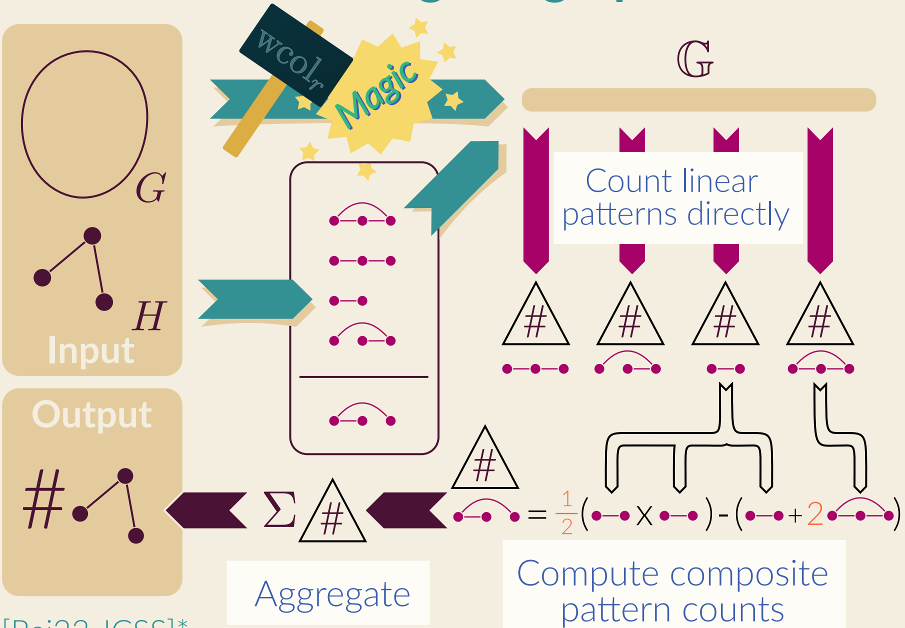


[Neš12 Sparsity]

Applications & Algorithms



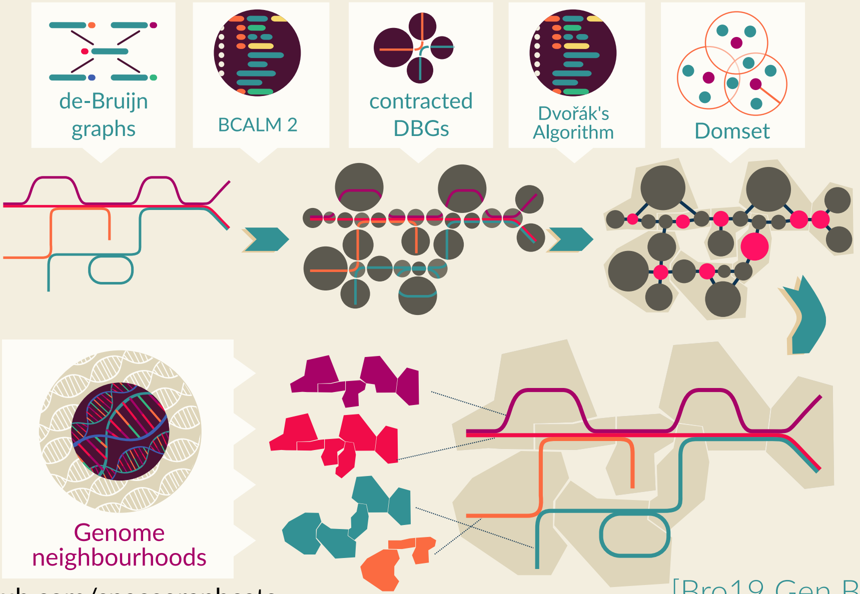
Counting Subgraphs



[Rei23 JCSS]*

github.com/theoryinpractice/mandoline

Index/Query Metagenomes

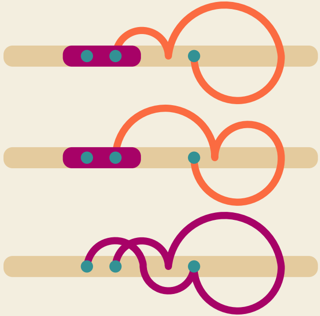


github.com/spacegraphcats

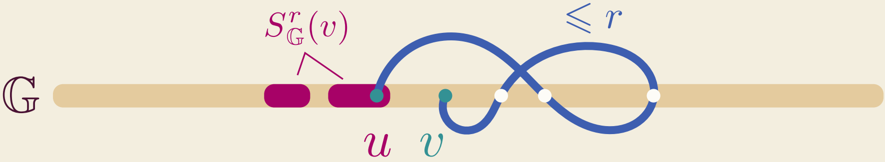
[Bro19 Gen Bio]*

Part II

Generalized coloring numbers

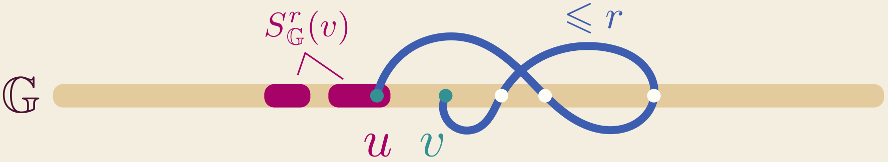


Strong coloring & bounded expansion



u is **strongly r-reachable** from v if there exists a path from v to u of length at most r such that all interior vertices lie right of v.

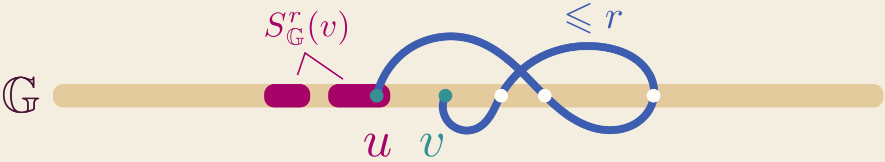
Strong coloring & bounded expansion



u is **strongly r-reachable** from v if there exists a path from v to u of length at most r such that all interior vertices lie right of v.

$$\mathrm{col}_r(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} |S^r_{\mathbb{G}}(v)|$$

Strong coloring & bounded expansion



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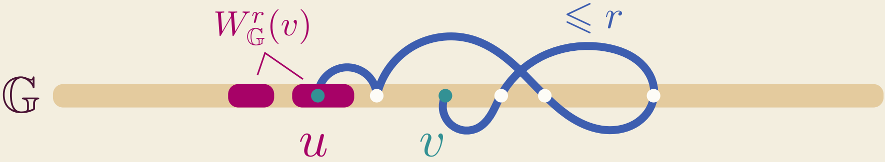
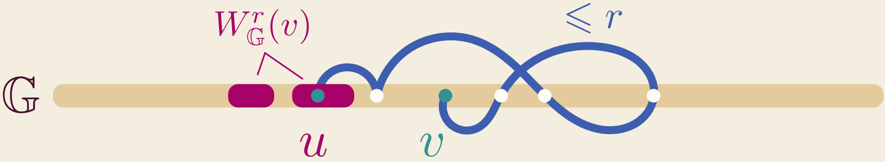
A graph class has bounded expansion iff it is col_r-bounded.

Weak coloring & bounded expansion



u is **weakly r-reachable** from v if there exists a path from v to u of length at most r such that u is the path's leftmost vertex.

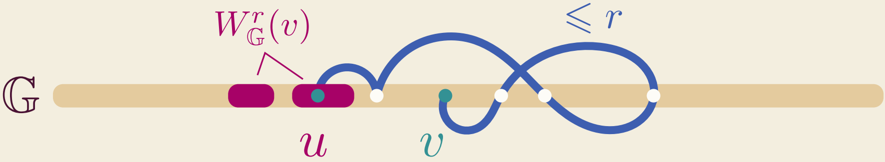
Weak coloring & bounded expansion



u is **weakly r-reachable** from v if there exists a path from v to u of length at most r such that u is the path's leftmost vertex.

$$\mathrm{wcol}_r(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} |W^r_{\mathbb{G}}(v)|$$

Weak coloring & bounded expansion



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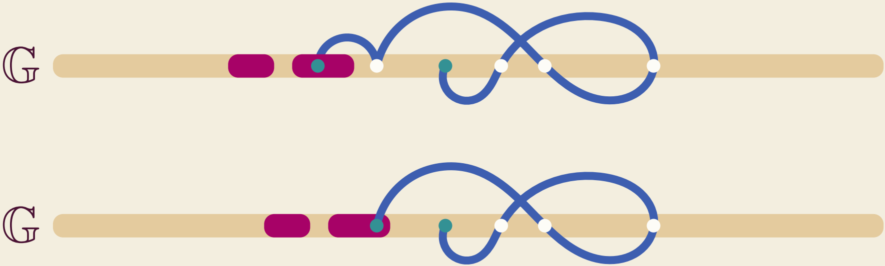
A graph class has bounded expansion iff it is wcol_r-bounded.

Weak & strong coloring



$$\mathrm{col}_r(\mathbb{G}) \leqslant \mathrm{wcol}_r(\mathbb{G}) \leqslant r(\mathrm{col}_r(\mathbb{G}) - 1)^r + 1$$

Origin story

Introduced by Kierstead & Yang, generalizing

Def. $$\mathrm{col}(\mathbb{G}, x) := |N[x] \cap \{y \leqslant_{\mathbb{G}} x\}|$$

$$\mathrm{col}(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} \mathrm{col}(\mathbb{G}, v)$$

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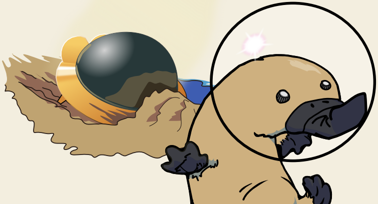
We can greedily color a graph with such an ordering:

$$\chi(G) \leqslant \chi^{\ell}(G) \leqslant \mathrm{col}(G)$$



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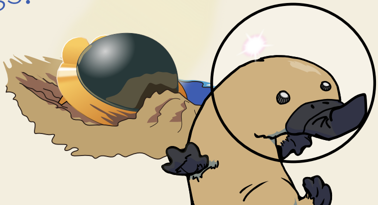
Further connections to colorings:

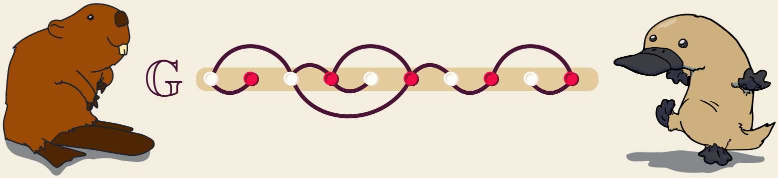
$$\mathrm{col}_2(G) = \chi_{\mathrm{acyclic}}(G)$$
$$\mathrm{wcol}_2(G) = \chi_{\mathrm{star}}(G)$$



It's all fun and games...

r-Ordering Game

- Alice & Bob play on a graph G by alternating choosing vertices (n turns)
- Creates an order $\mathbb{G}$, score is $\mathrm{col}_r(\mathbb{G})$
- Alices plays first and wants to minimize score; Bob wants to maximize



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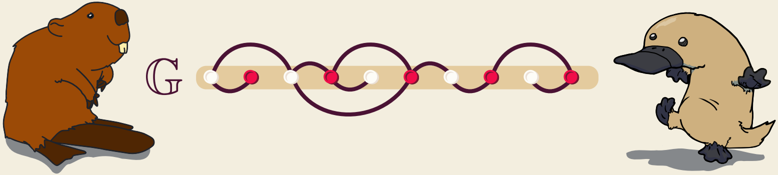


$\mathrm{gcol}_r(G) :=$ Lowest score Alice can always achieve

To planarity and beyond!

A (non-comprehensive) selection of recent work

Planar graphs have $\mathrm{wcol}_2(G) \leqslant 23$

[Alm22 DM]

Koebe orders of planar graphs satisfy

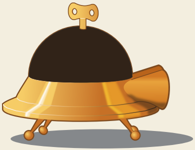
$$\mathrm{wcol}_r(\mathbb{G}) = O(r^4 \ln r)$$

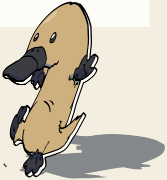
[Ned22 Arxiv]

Graphs of treewidth t have

$$\mathrm{wcol}_r(G) = \Theta(r^{t-1} \log r)$$

[Jor22 Elec J Comb]





To planarity and beyond!

A (non-comprehensive) selection of recent work

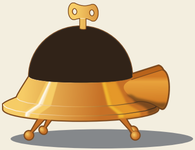
Graph powers: $\mathrm{col}_r(G^p)$ can be bounded in terms of $\mathrm{wcol}_r(G)$ and $\Delta(G)$.

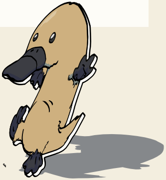
[Kie20 DM]

If G has genus g $\begin{cases} \mathrm{col}_r(G) & \leq (2g+3)(2r+1) \\ \mathrm{wcol}_r(G) & \leq \left(2g + \binom{2r+2}{2}\right)(2r+1) \end{cases}$

If G excludes a minor $\begin{cases} \mathrm{col}_r(G) & \leq \binom{t-1}{2}(2r+1) \\ \mathrm{wcol}_r(G) & \leq O(r^{t-1}) \end{cases}$

[van17 EuJC]





One ~~Ring~~ *Order* to Rule Them All

? Do orders witnessing low col_r also have small $\mathrm{col}_{r'}$?

Not if low/small mean optimal!

For every $r \neq r'$ $\exists G$ such that every ordering $\mathbb{G} \in \Pi(G)$ is non-optimal for col_r or $\mathrm{col}_{r'}$.

But: known bounds use one order for all r!

One ~~Ring~~ Order to Rule Them All

? If $\mathrm{col}_r(G) \leqslant c(r)$, does $\exists c^*$ such that some $\mathbb{G}$ has $\mathrm{col}_r(\mathbb{G}) \leqslant c^*(r)$ for all r ?

Thm. (van Heuvel, Kierstead)
For all G there exists $\mathbb{G} \in \Pi(G)$ with

$$\mathrm{col}_r(\mathbb{G}) \leqslant (2^r + 1)(\mathrm{col}_{2r}(G))^{4r}$$

for all r.

[vanH21 EuJC]

Twins: twice the fun, half the sleep

Twinwidth is a hot new graph parameter generalizing an invariant of permutation classes.

[Bon20 FOCS][Gui14 SODA]

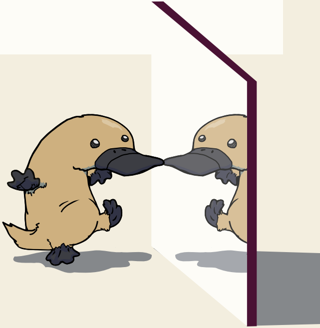
Thm. (Bonnet *et al.*)

If $\mathrm{tww}(G) \leqslant t$ and $K_{s,s} \not\subseteq G$, then there ex. f_r s.t.

$$\mathrm{col}_r(G) \leqslant f_r(s,t)$$

for all r.

[Bon21 SODA]



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$$\operatorname{col}_r(G) \leqslant f_r(s,t)$$

for all r.

[Bon21 SODA]

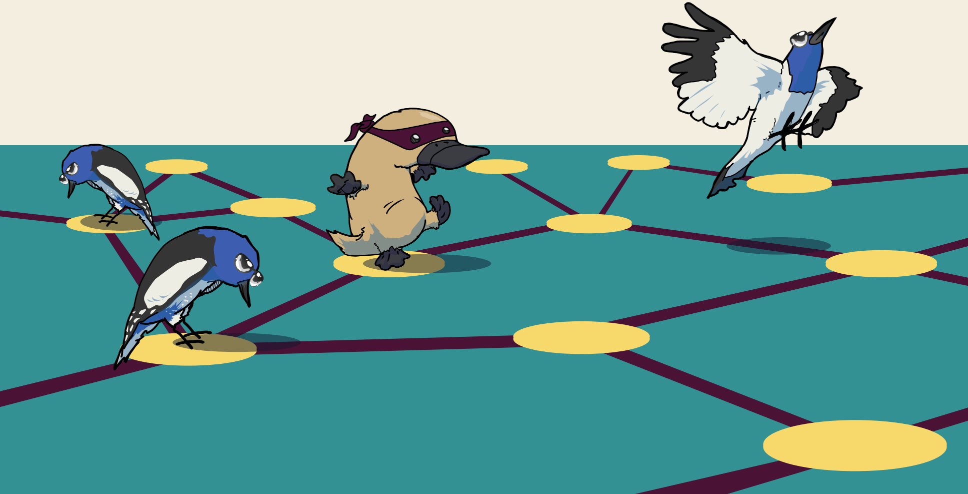
Thm. (Dreier *et al.*)
There ex. G with $\operatorname{col}_r(G) \geqslant (t-4)^r s$, and

$\operatorname{tww}(G) \leqslant t$
$K_{s,s} \not\subseteq G$

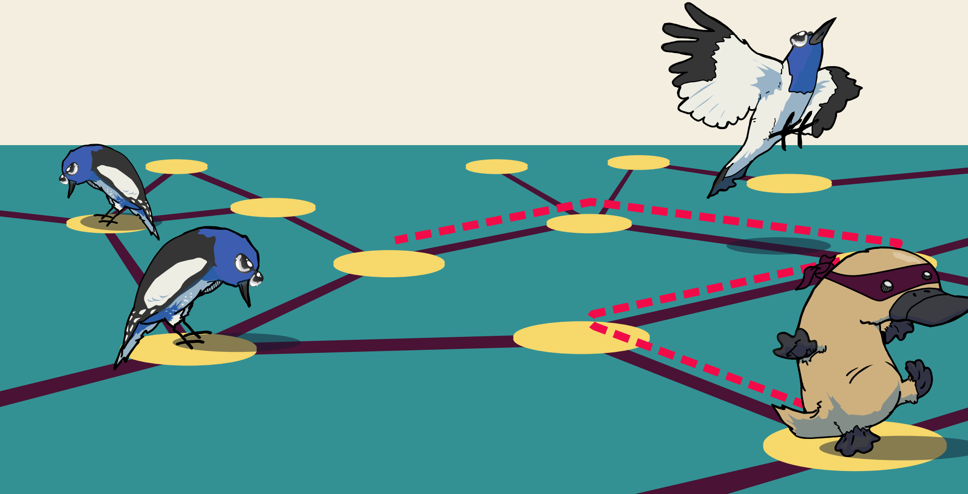
$$f_r(s,t) \leqslant (t^r + 3)s$$

[Dre22 DM]

Cops & robbers

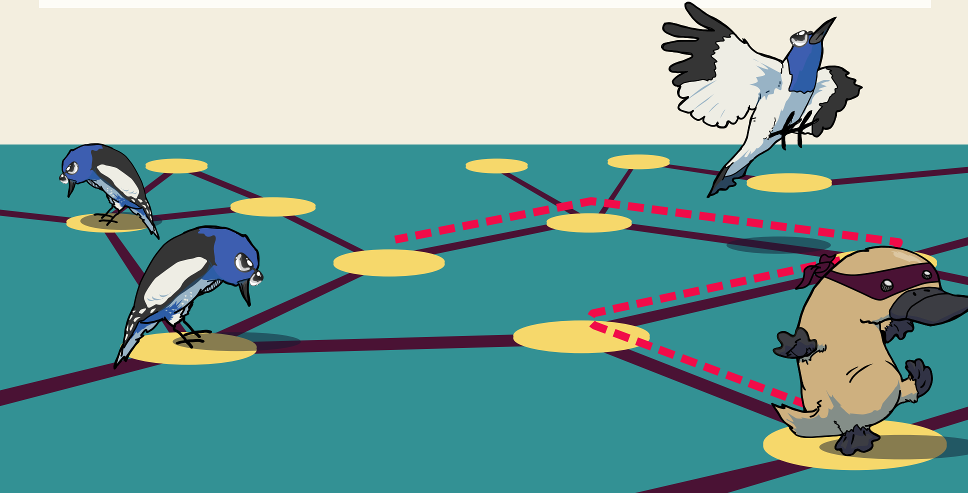


Cops & robbers



Cops & robbers

$k+1$ cops can catch the robber $\iff$ $\mathrm{tw}(G) = k$



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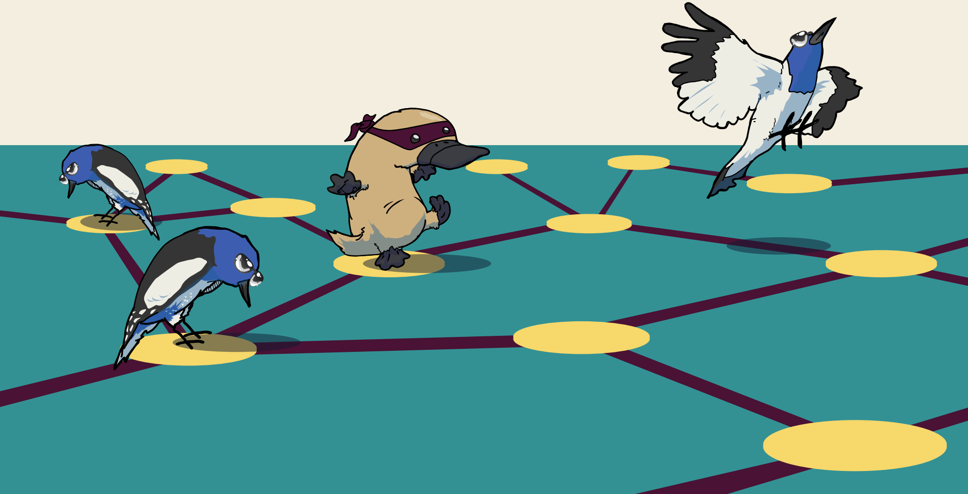
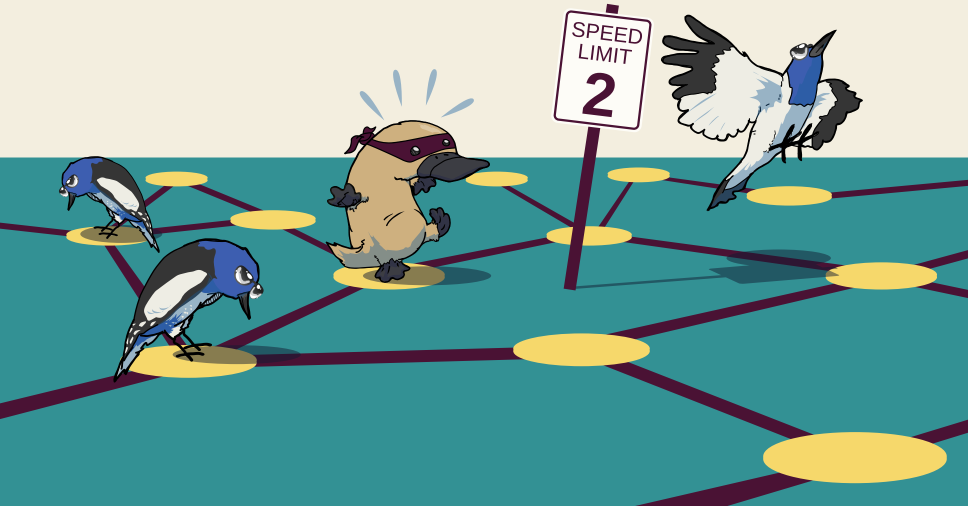


Cops & robbers



Cops & robbers

$k+1$ cops can catch a robber with maximum speed r $\iff$ $\mathrm{copw}_r(G) = k$

Cops & robbers

$k+1$ cops can catch a robber with maximum speed r $\iff$ $\mathrm{copw}_r(G) = k$

Thm. (Toruńczyk)

$$\mathrm{adm}_r(G) + 1 \leqslant \mathrm{copw}_r(G) \leqslant \mathrm{wcol}_{2r}(G) + 1$$

[Tor23 Flip-width]

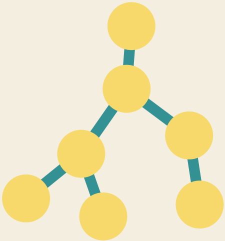
A graph class has bounded expansion iff it is copw_r-bounded.

'Limits' of coloring numbers

Degeneracy

$\mathrm{col}_1(G) = \mathrm{wcol}_1(G)$

$\mathrm{wcol}_\infty(G) = \mathrm{td}(G)$



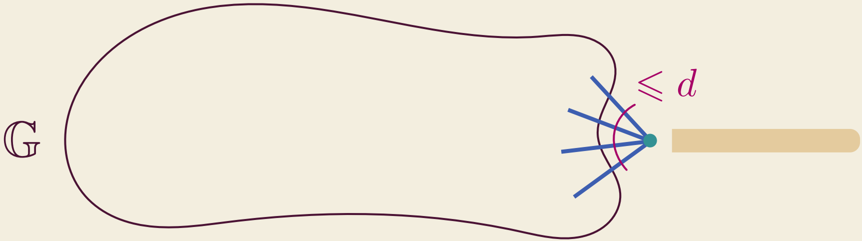
$\mathrm{col}_\infty(G) = \mathrm{tw}(G)$

Part III

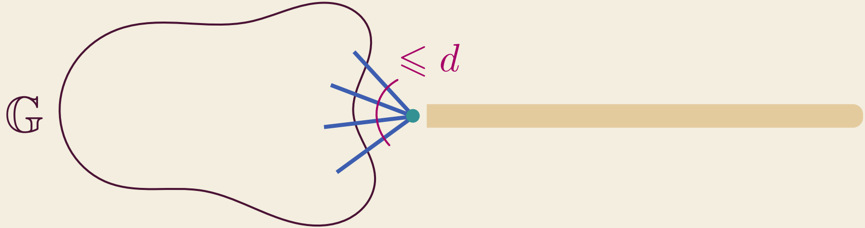
Algorithmic questions

Computing degeneracy ordering

Computing degeneracy ordering



Computing degeneracy ordering



Computing degeneracy ordering

$\mathbb{G}$

No free lunch

Thm. (Breen-McKay, Lavallee, **S**)
Deciding wether $\mathrm{wcol}_r(G) \leqslant k$ or $\mathrm{col}_r(G) \leqslant k$ is NP-complete even for $r = 2$.

[Bre23 EuJC (in press)]*

Anything beyond degeneracy is hard to compute exactly.

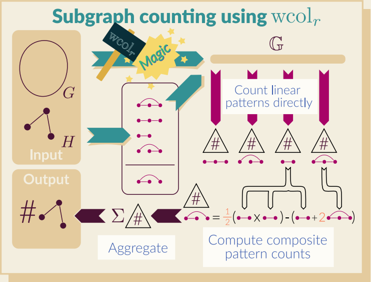
Maybe cheap lunch?

Thm. (Dvořák)

$$\mathrm{adm}_r(\mathbb{G}) \leqslant k \implies \begin{aligned} \mathrm{col}_r(\mathbb{G}) &\leqslant (k-1)^r + 1 \\ \mathrm{wcol}_r(\mathbb{G}) &\leqslant (r^2 k)^r \end{aligned}$$

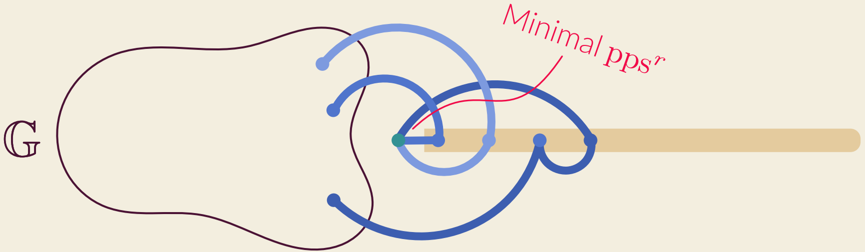
[TODO] [TODO]

Approximation works just fine here! (though slower)



Estimating path-packings

Estimate pp^r by packing *shortest* paths!



Thm. (Breen-McKay, Lavallee, **S**)

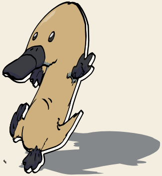
$$\boxed{\begin{array}{c}\mathrm{pps}^r_{\mathbb{G}}(v) \leqslant k \\ \text{for all } v \in G\end{array}} \implies \begin{array}{l}\mathrm{col}_r(\mathbb{G}) \leqslant k(k-1)^{r-1} \\ \mathrm{wcol}_r(\mathbb{G}) \leqslant \dfrac{k^{r+1}-1}{k-1}\end{array}$$

[Bre23 EuJC (in press)]*

Open Problems

TEXT

TEXT



TEXT

THANKS

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