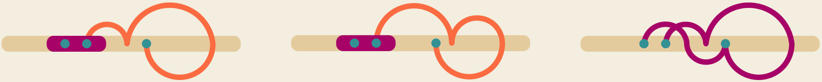
Taking a Hard Look at ***Generalized Coloring Numbers***



@blairdsullivan
University of Utah

AMS SE Sectional

Mar 18 2023

Special thanks to
Felix Reidl (Birkbeck College, London)
for original artwork and collaboration
on scientific communication

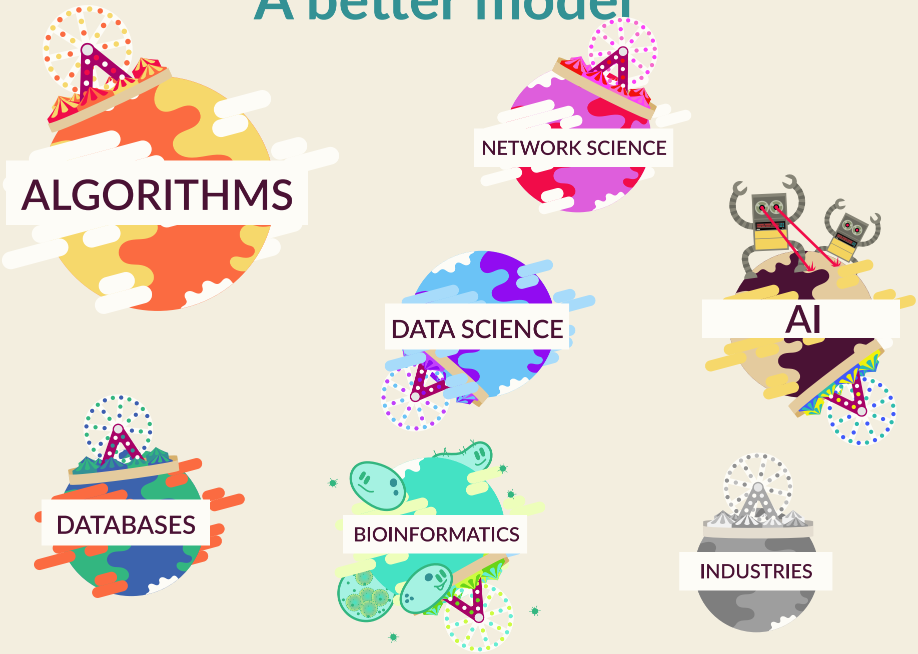
My work in this talk was primarily supported by the Gordon & Betty Moore Foundation, with additional funding from DARPA, ARO, and NIH..

Part I

Sparse classes

A false dichotomy

A better model



Parameterized algorithms

	Classical view	Parameterized view
Bad	$O(n^{\log n}), O(2^n), \ldots$ Not polynomial-time	$O(n^{f(k)})$ Slice-wise polynomial time
Good	$O(n^c)$ Polynomial-time ("efficient")	$O(f(k)n^c)$ Fixed-parameter tractable

Choosing a class

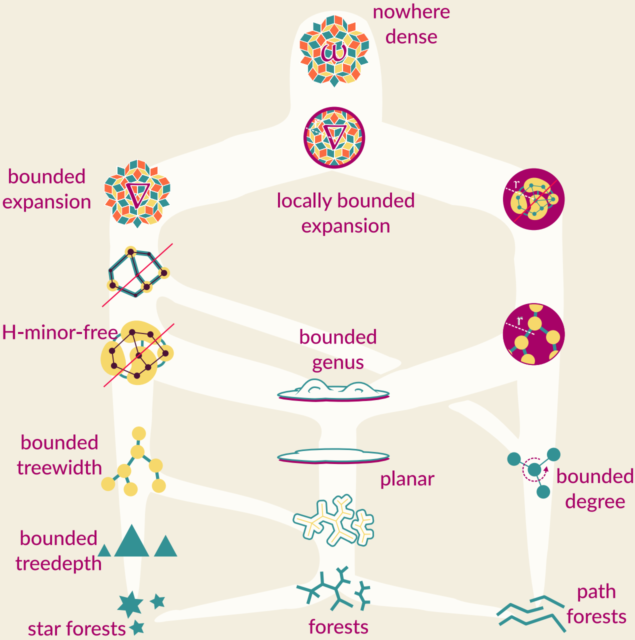
Larger classes

Less

Structure

More

Algorithmic tractability



The sparse class hierarchy

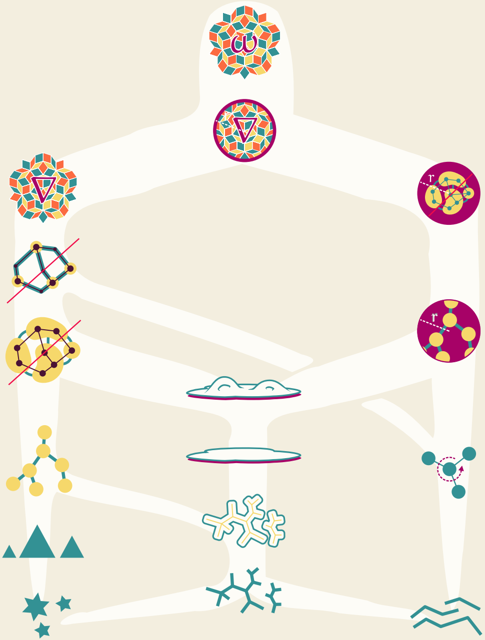
Larger classes

Less

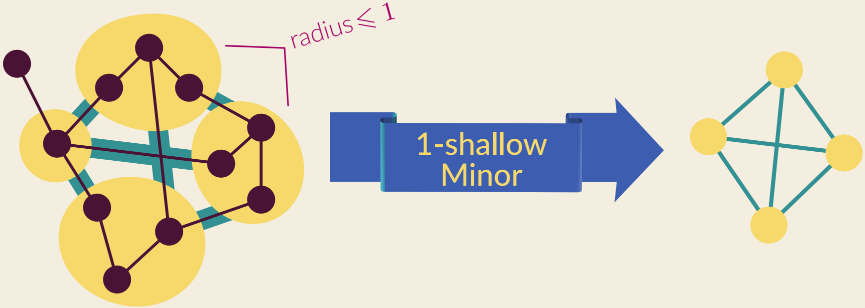
Structure

More

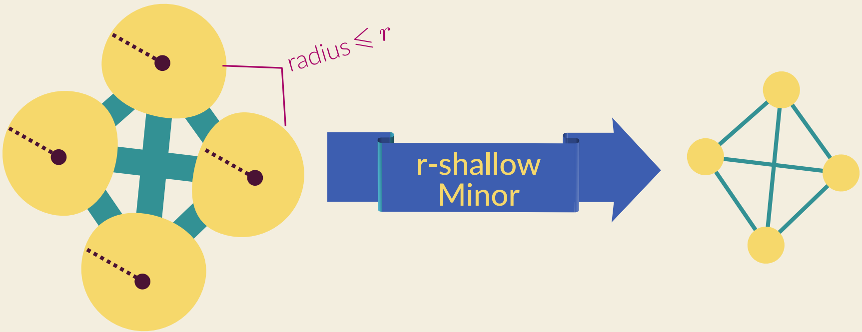
Algorithmic tractability



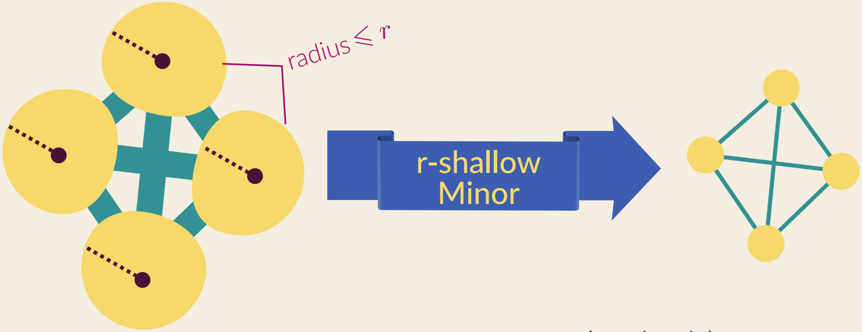
Shallow minors & bounded expansion



Shallow minors & bounded expansion



Shallow minors & bounded expansion

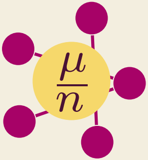


$$\nabla_r(G) = \max_{H \preccurlyeq_r G} \frac{|E(H)|}{|V(H)|}$$

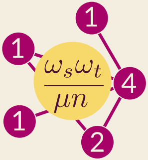
A graph class has bounded expansion iff it is ∇_r-bounded.

Random model sparsity

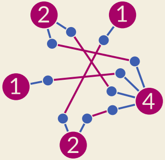
$G(n, \frac{\mu}{n})$



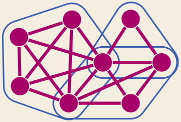
$G^{\mathrm{CL}}(D_n)$



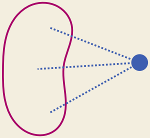
$G^{\mathrm{CF}}(D_n)$



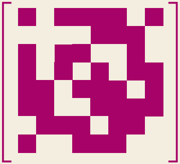
$G^{\mathrm{RIG}}(n, \alpha, \beta, \gamma)$



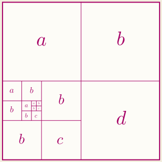
$G^{\mathrm{BA}}(n, n_0, k)$



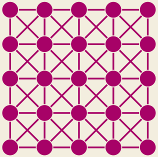
$G^{\mathrm{SGK}}(k, \alpha, \ldots, \gamma)$



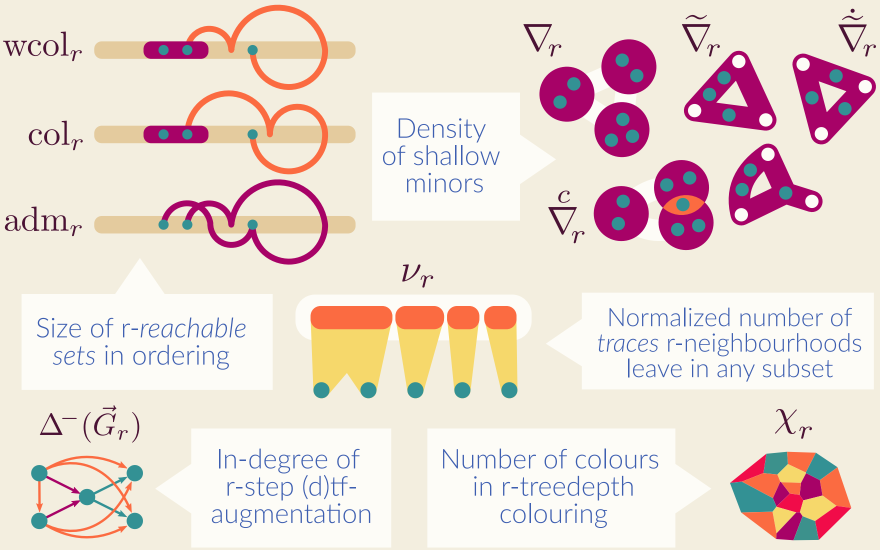
$G^{\mathrm{RMAT}}(k, m, a, b, c)$



$G^{\mathrm{KL}}(n, p, q, \gamma)$

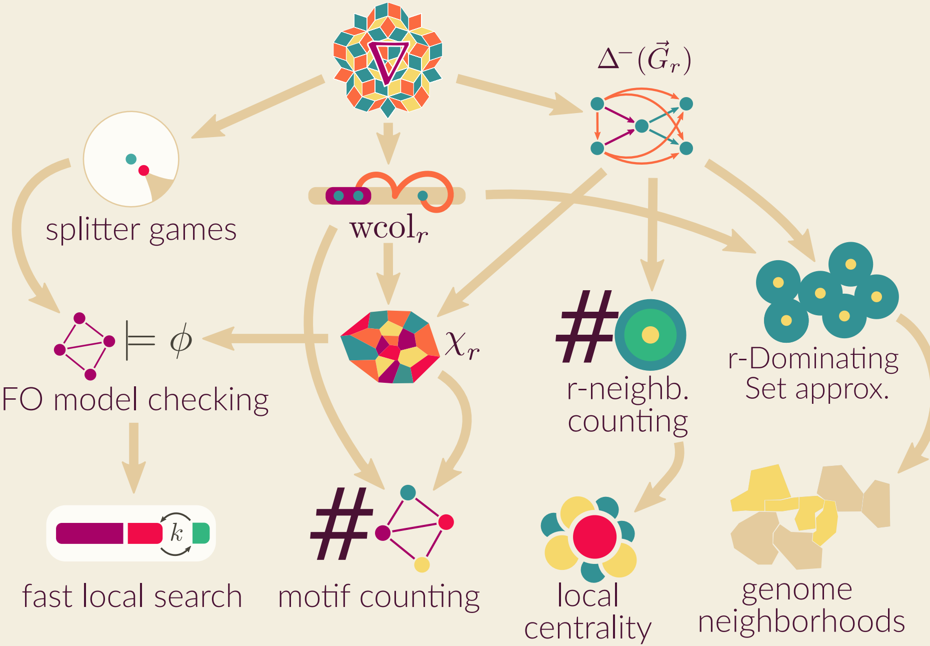


Bounded expansion

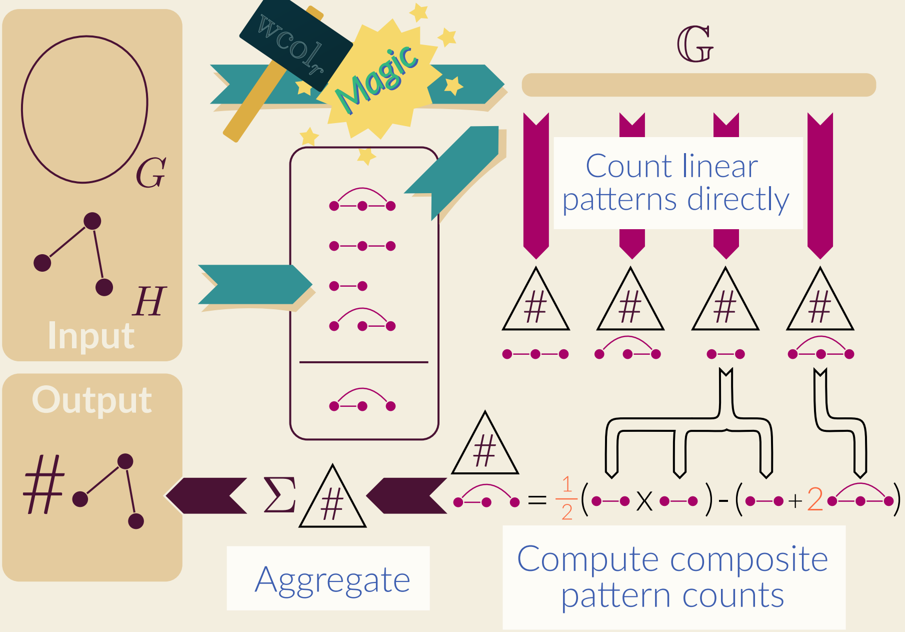


Nešetřil J, Ossona de Mendez P. **Sparsity**.
Algorithms and Combinatorics. 2012;28.

Applications & Algorithms



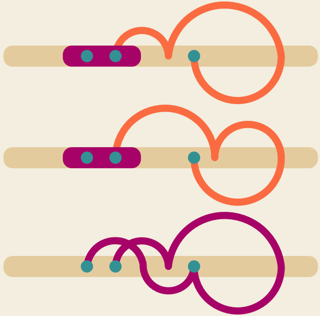
Subgraph counting using wcol_r



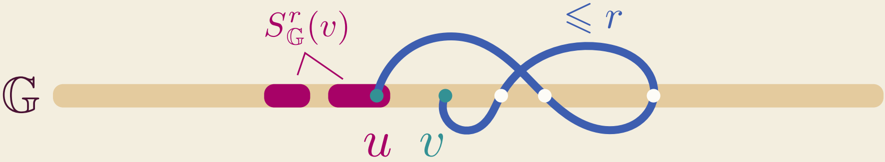
SpaceGraphCats

Part II

Generalized coloring numbers

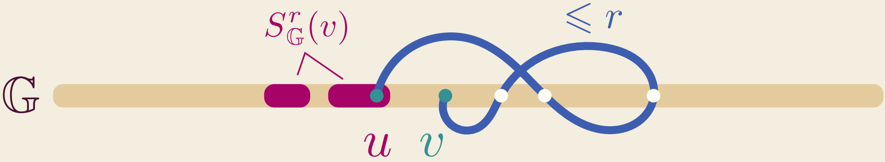


Strong coloring & bounded expansion



u is **strongly r-reachable** from v if there exists a path from v to u of length at most r such that all interior vertices lie right of v.

Strong coloring & bounded expansion



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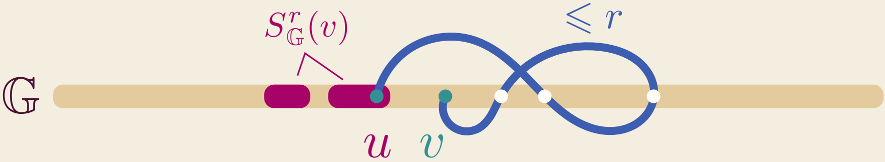
$$\mathrm{col}_r(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} |S^r_{\mathbb{G}}(v)|$$

Strong coloring & bounded expansion



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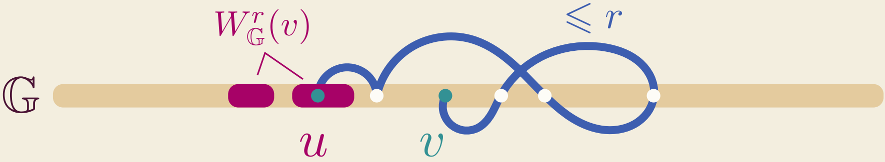
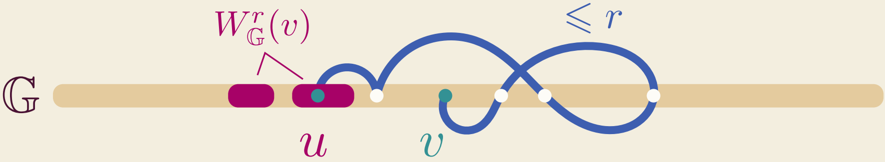
A graph class has bounded expansion iff it is col_r-bounded.

Weak coloring & bounded expansion



u is **weakly r-reachable** from v if there exists a path from v to u of length at most r such that u is the path's leftmost vertex.

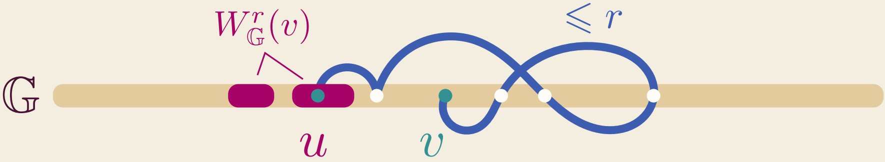
Weak coloring & bounded expansion



u is **weakly r-reachable** from v if there exists a path from v to u of length at most r such that u is the path's leftmost vertex.

$$\mathrm{wcol}_r(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} |W^r_{\mathbb{G}}(v)|$$

Weak coloring & bounded expansion



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$$\mathrm{wcol}_r(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} |W^r_{\mathbb{G}}(v)|$$

A graph class has bounded expansion iff it is wcol_r-bounded.

Origin story

Introduced by Kierstead & Yang, generalizing

Def.

$$\operatorname{col}(\mathbb{G}, x) := |N[x] \cap \{y \leqslant_{\mathbb{G}} x\}|$$

$$\operatorname{col}(G) := \min_{\mathbb{G} \in \Pi(G)} \max_{v \in G} \operatorname{col}(\mathbb{G}, v)$$

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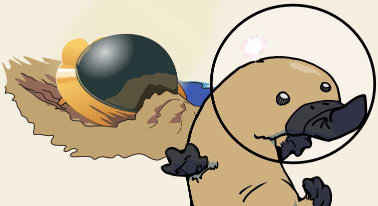
We can greedily color a graph with such an ordering:

$$\chi(G) \leqslant \chi^{\ell}(G) \leqslant \mathrm{col}(G)$$



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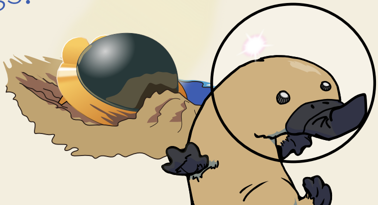
Further connections to colorings:

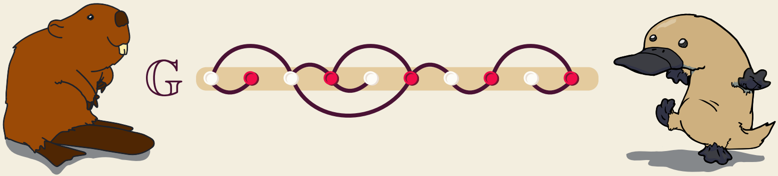
$$\mathrm{col}_2(G) = \chi_{\mathrm{acyclic}}(G)$$
$$\mathrm{wcol}_2(G) = \chi_{\mathrm{star}}(G)$$



It's all fun and games...

r-Ordering Game

- Alice & Bob play on a graph G by alternating choosing vertices (n turns)
- Creates an order $\mathbb{G}$, score is $\mathrm{col}_r(\mathbb{G})$
- Alices plays first and wants to minimize score; Bob wants to maximize



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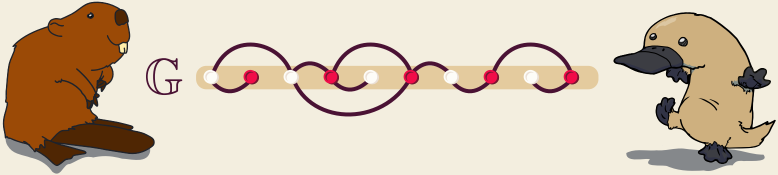
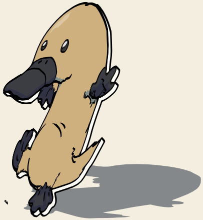


$\mathrm{gcol}_r(G) :=$ Lowest score Alice can always achieve

Planarity and beyond

TEXT

- BULLET



Uniform orderings

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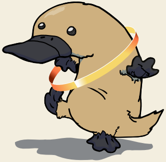
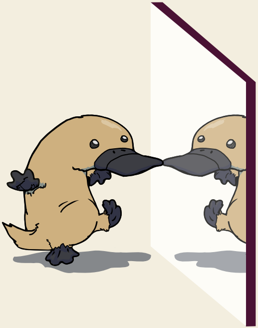
- BULLET



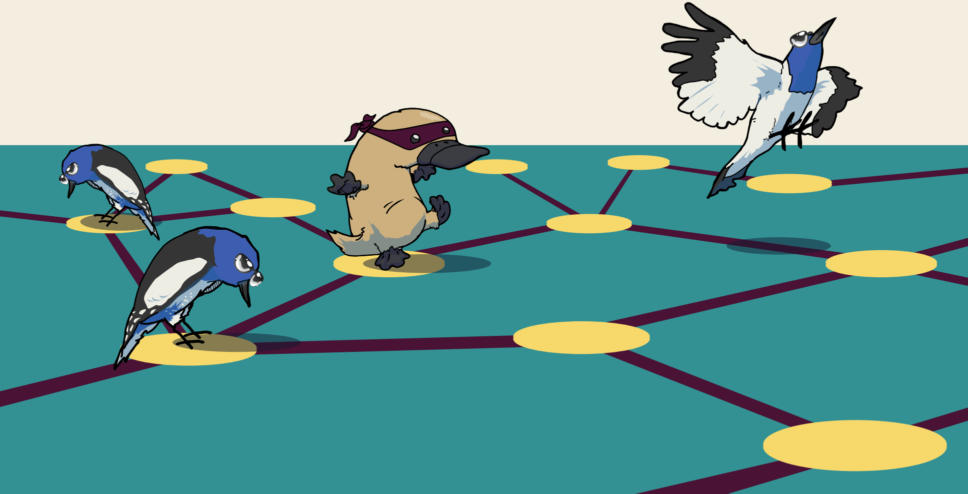
Twinwidth

TEXT

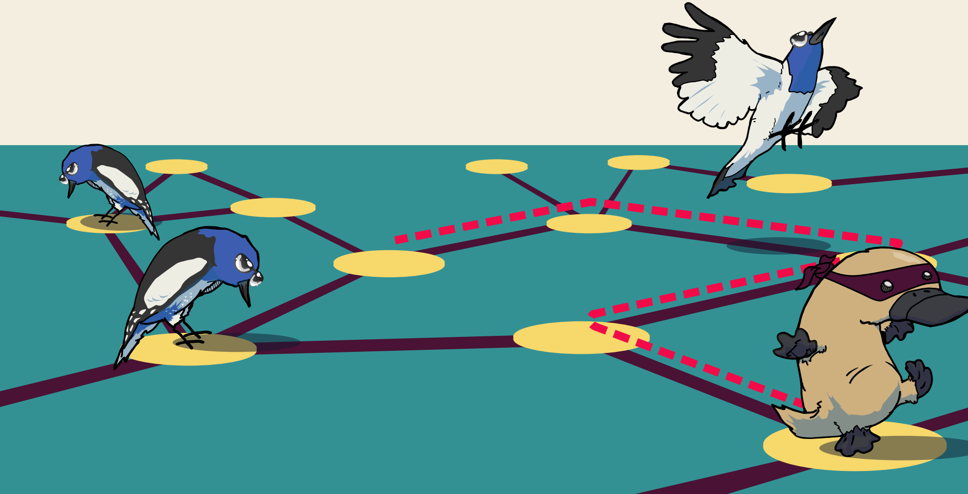
- BULLET



Cops & robbers

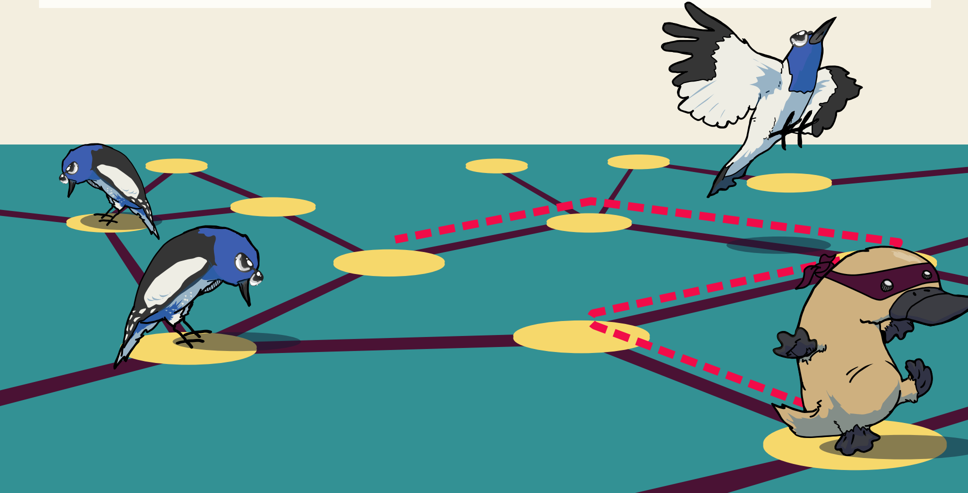


Cops & robbers

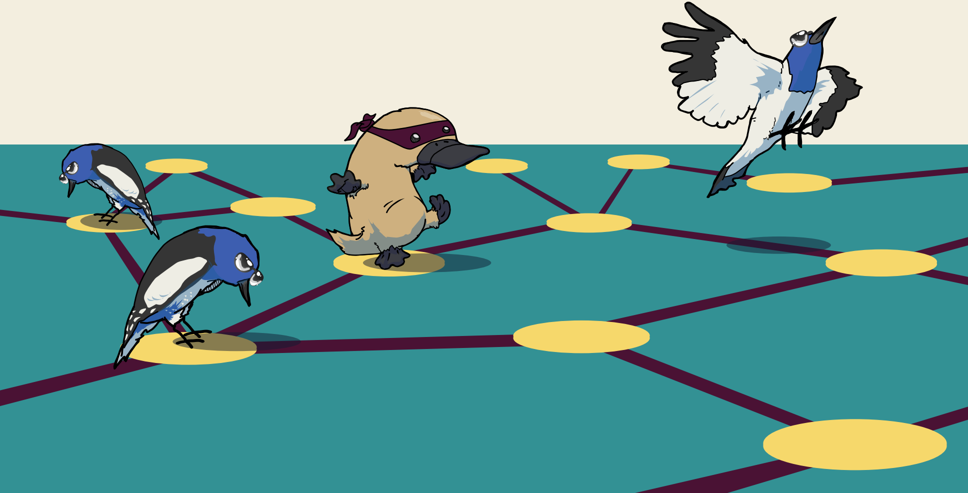


Cops & robbers

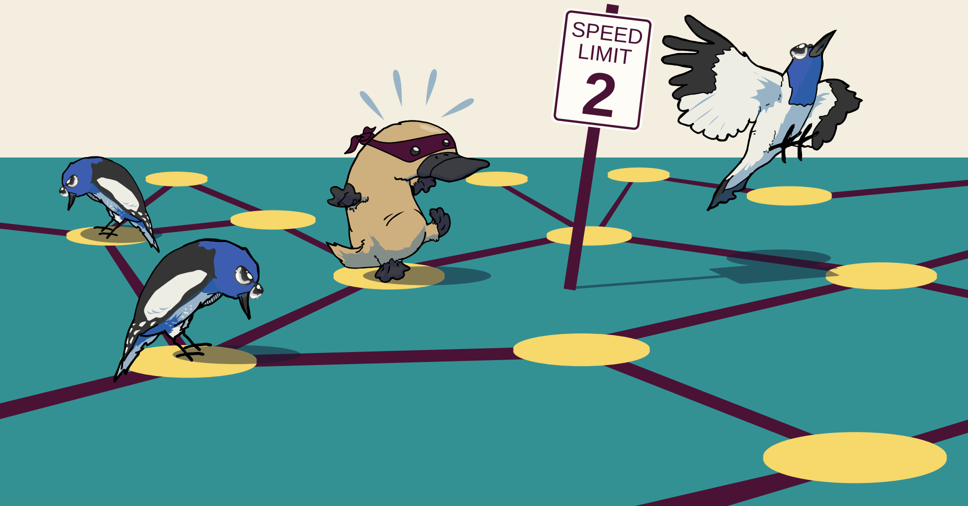
$k+1$ cops can catch the robber $\iff$ $\mathrm{tw}(G) = k$



Cops & robbers

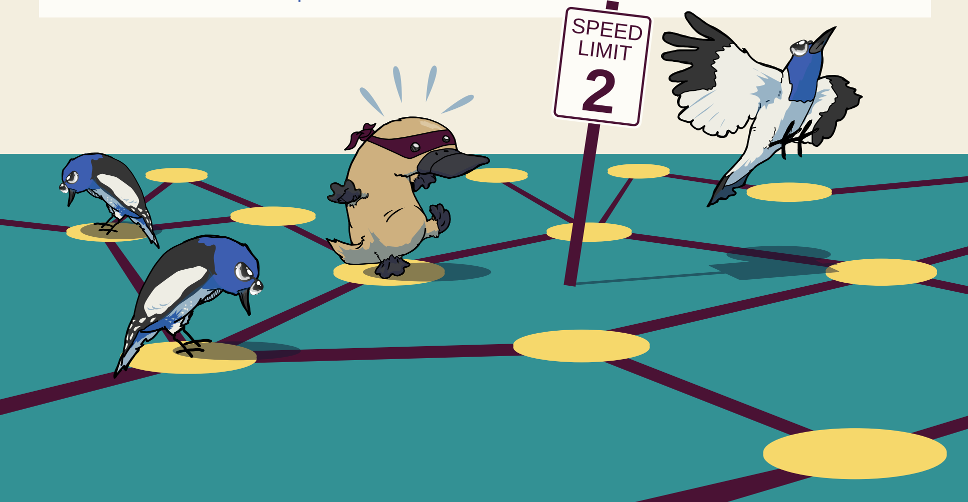


Cops & robbers



Cops & robbers

$k+1$ cops can catch a robber with maximum speed r $\iff$ $\mathrm{copw}_r(G) = k$



Cops & robbers

$k+1$ cops can catch a robber with maximum speed r $\iff$ $\operatorname{copw}_r(G) = k$

Thm. (Toruńczyk)

$$\operatorname{adm}_r(G) + 1 \leqslant \operatorname{copw}_r(G) \leqslant \operatorname{wcol}_{2r}(G) + 1$$

[Tor23 Flip-width]

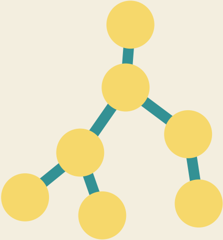
A graph class has bounded expansion iff it is copw_r-bounded.

'Limits' of coloring numbers

Degeneracy

$\mathrm{col}_1(G) = \mathrm{wcol}_1(G)$

$\mathrm{wcol}_\infty(G) = \mathrm{td}(G)$



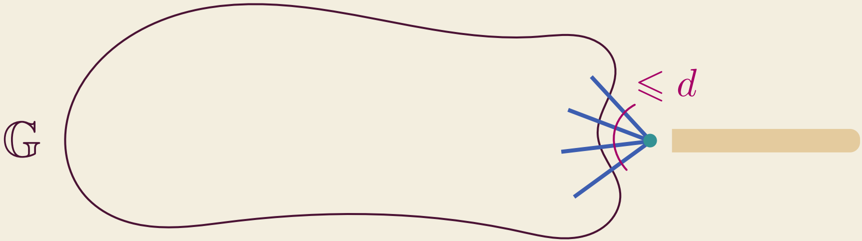
$\mathrm{col}_\infty(G) = \mathrm{tw}(G)$

Part III

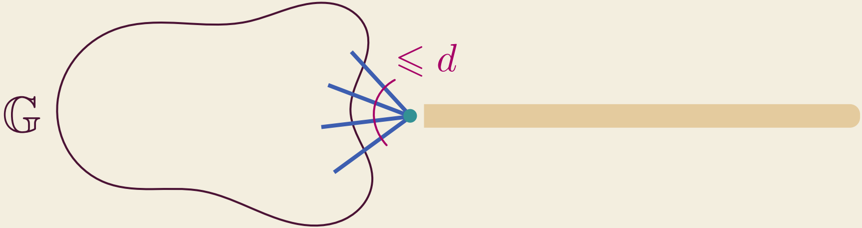
Algorithmic question

Computing degeneracy ordering

Computing degeneracy ordering



Computing degeneracy ordering



Computing degeneracy ordering

$\mathbb{G}$

No free lunch

Thm. (Breen-McKay, Lavallee, **S**)
Deciding wether $\mathrm{wcol}_r(G) \leqslant k$ or $\mathrm{col}_r(G) \leqslant k$ is NP-complete even for $r = 2$.

[Bre21 Arxiv]

Anything beyond degeneracy is hard to compute exactly.

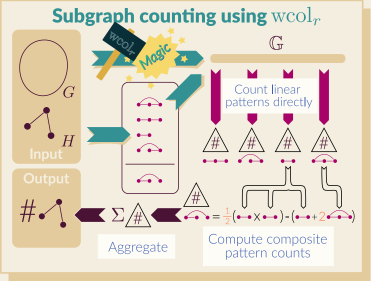
Maybe cheap lunch?

Thm. (Dvořák)

$$\mathrm{adm}_r(\mathbb{G}) \leqslant k \implies \begin{aligned} \mathrm{col}_r(\mathbb{G}) &\leqslant k(k-1)^{r-1} \\ \mathrm{wcol}_r(\mathbb{G}) &\leqslant \frac{k^{r+1}-1}{k-1} \end{aligned}$$

[TODO]

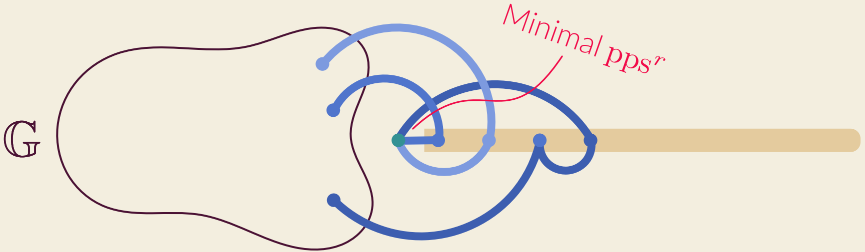
Approximation works just fine here! (though slower)



fo

Estimating path-packings

Estimate pp^r by packing *shortest* paths!



Thm. (Breen-McKay, Lavallee, **S**)

$\mathrm{pps}^r_{\mathbb{G}}(v) \leqslant k$ for all $v \in G$ $\Longrightarrow$

$$\mathrm{col}_r(\mathbb{G}) \leqslant k(k-1)^{r-1}$$

$$\mathrm{wcol}_r(\mathbb{G}) \leqslant \frac{k^{r+1}-1}{k-1}$$

[Bre21 Arxiv]

Estimating path-packings

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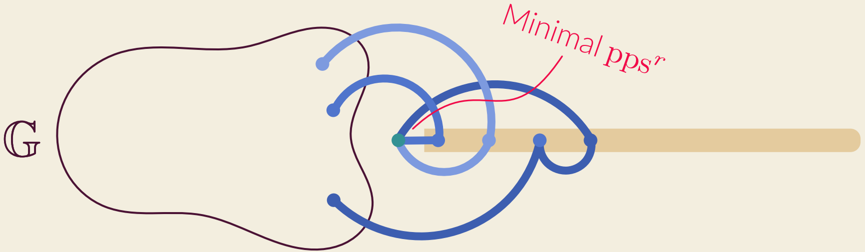

Thm. (Breen-McKay, Lavallee, **S**)

$$\boxed{\begin{gathered}\mathrm{pps}^r_{\mathbb{G}}(v) \leqslant k \\ \text{for all } v \in G\end{gathered}} \implies \begin{aligned} \mathrm{col}_r(\mathbb{G}) &\leqslant k(k-1)^{r-1} \\ \mathrm{wcol}_r(\mathbb{G}) &\leqslant \frac{k^{r+1}-1}{k-1}\end{aligned}$$

[Bre21 Arxiv]

Open questions & future work

TEXT

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TEXT

THANKS

[Adc13 ICDM] Aaron B. Adcock, Blair D. Sullivan, and Michael W. Mahoney. "Tree-Like Structure in Large Social and Information Networks". In: *2013 IEEE 13th International Conference on Data Mining*. 2013, pp. 1–10. DOI: 10.1109/ICDM.2013.77.

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[Bre21 Arxiv] Michael Breen-McKay, Brian Lavallee, and Blair D. Sullivan. *Hardness of the Generalized Coloring Numbers*. In Press, European Journal of Combinatorics, March 2023. 2021. DOI: 10.48550/ARXIV.2112.10562. URL: https://arxiv.org/abs/2112.10562.

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[Tor23 Flip-width] Symon Toruńczyk. “Flip-width: Cops and Robber on dense graphs”. In: arXiv:2302.00352 (Feb. 2023). arXiv:2302.00352 [cs, math]. URL: http://arxiv.org/abs/2302.00352.