Solution For Assignment 1

Bogdan Dumitriu

September 17, 2004

PROOF asgn1

[**A1:**] $(\exists i : 0 \le i < n : a[i] < 0)$
[**A2:**] $(\forall j : 0 \le j < i : a[j] = 0)$
[**A3:**] $a[i] > i$
[**G1:**] $(\exists j : i \le j < n : a[j] < 0)$

BEGIN _____

1 {See proof (by contradiction) sp1}

   $n \ge 0$

   PROOF sp1

   [**A1:**] $n < 0$
   [**G1:**] F

   BEGIN _____

      1 {Quantification over empty domain, justified by A1}
         $(\exists i : 0 \le i < n : a[i] < 0) = F$
      2 {Contradiction between 1 and asgn1.A1}
         F

   END _____

2 {See proof (by contradiction) sp2}

   $i < n$

   PROOF sp2

   [**A1:**] $i \ge n$
   [**G1:**] F

   BEGIN _____

      1 {Negate $\exists$ on A2}
         $\neg(\exists j : 0 \le j < i : a[j] \ne 0)$
      2 {Domain merging on 1, justified by asgn1.1 and A1}
         $\neg(\exists j : 0 \le j < n \vee n \le j < i : a[j] \ne 0)$
      3 {Domain split on 2}
         $\neg((\exists j : 0 \le j < n : a[j] \ne 0) \vee (\exists j : n \le j < i : a[j] \ne 0))$

4 {De Morgan on 3}

$\neg(\exists j : 0 \le j < n : a[j] \ne 0) \land \neg(\exists j : n \le j < i : a[j] \ne 0)$

5 {$\land$-Elimination on 4}

$\neg(\exists j : 0 \le j < n : a[j] \ne 0)$

6 {Rename bound variable of asgn1.A1}

$(\exists j : 0 \le j < n : a[j] < 0)$

7 {Trivial, from 6}

$(\exists j : 0 \le j < n : a[j] \ne 0)$

7 {Contradiction between 5 and 7}

F

END ─────────────────────────────────

4 {see proof sp3}

$i < 0 \Rightarrow (\exists j : i \le j < n : a[j] < 0)$

PROOF sp3

[A1:] $i < 0$

[G1:] $(\exists j : i \le j < n : a[j] < 0)$

BEGIN ─────────────────────────────────

1 {$\exists$-Elimination on asgn1.A1}

[SOME k]

$0 \le k < n \land a[k] < 0$

2 {$\land$-Elimination on 1}

$0 \le k < n$

3 {$\land$-Elimination on 1}

$a[k] < 0$

4 {Trivial from A1 and 2}

$i \le k < n$

5 {Conjunction on 3 and 4}

$i \le k < n \land a[k] < 0$

6 {$\exists$-Introduction on 5}

$(\exists k : i \le k < n : a[k] < 0)$

7 {Rename bound variable of 6}

$(\exists j : i \le j < n : a[j] < 0)$

END ─────────────────────────────────

5 {See proof sp4}

$i \ge 0 \Rightarrow (\exists j : i \le j < n : a[j] < 0)$

PROOF sp4

[A1:] $i \ge 0$

[G1:] $(\exists j : i \le j < n : a[j] < 0)$

BEGIN ─────────────────────────────────

1 {Negate ∀ on asgn1.A1}

  $\neg(\forall i : 0 \le i < n : a[i] \ge 0)$

2 {Trivial, from asgn1.2}

  $i \le n$

3 {Rename bound variable of 1}

  $\neg(\forall j : 0 \le j < n : a[j] \ge 0)$

4 {Domain merging on 3, justified by A1 and 2}

  $\neg(\forall j : 0 \le j < i \lor i \le j < n : a[j] \ge 0)$

5 {Domain split on 4}

  $\neg((\forall j : 0 \le j < i : a[j] \ge 0) \land (\forall j : i \le j < n : a[j] \ge 0))$

6 {De Morgan on 5}

  $\neg(\forall j : 0 \le j < i : a[j] \ge 0) \lor \neg(\forall j : i \le j < n : a[j] \ge 0)$

7 {Trivial, from asgn1.A2}

  $(\forall j : 0 \le j < i : a[j] \ge 0)$

8 {Rewrite 6 with 7}

  $F \lor \neg(\forall j : i \le j < n : a[j] \ge 0)$

9 {Basic equalities of boolean connectors on 8}

  $\neg(\forall j : i \le j < n : a[j] \ge 0)$

10 {Negate ∀ on 9}

  $(\exists j : i \le j < n : a[j] < 0)$

END _____

6 {Excluded middle}

  $i < 0 \lor i \ge 0$

7 {Case split on 6, 4 and 5}

  $(\exists j : i \le j < n : a[j] < 0)$

END _____