# Day 2, Session 1: Logs/Exponentiation

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# Motivation

Generally, our goal in a statistical analysis is to assess an association between two variables (e.g., smoking and lung cancer). We can do this by investigating whether a summary measure (e.g., mean, median) of our outcome (e.g., lung cancer) is unequal between two groups differing in our predictor of interest (e.g., smoking).

There are two simple ways to tell if two numbers are unequal:

- their difference is not equal to 0, or
- their ratio is not equal to 1.

We choose between these based on a variety of criteria, which fall into two general categories: (1) adequately address the scientific question, and (2) gain desirable statistical properties.

# Motivation

Differences:

- are generally easier to understand, and
- are better for describing the scientific importance of many comparisons
    - You probably always want \$1,000,000 more than me, even if I have \$10,000,000 (a ratio of 1.1)

Ratios work well when working with small numbers (disclaimer: these numbers are probably only correct to the order of magnitude, but get the point across); for example:

- In the US, 60–64 year old current or former smokers have a probability of 0.00296 of being diagnosed with lung cancer during the next year
- In the US, 60–64 year old never smokers have a probability of 0.000148 of being diagnosed with lung cancer during the next year
- Difference in incidence rates: 0.002812; ratio of incidence rates: 20!

# Motivation

Sometimes, the scientific mechanism dictates that ratios are more generalizeable:

- Interventions or risk factors that affect a rate over time (e.g., HIV incidence)
- Biochemical processes (e.g., rates of absorption, where the rate is proportional to drug concentration)

When ratios are scientifically preferred, we can use the logarithm of the ratio to get back to comparing differences (more on this later).

# Common variables

Some variables are almost always log transformed:

- Acidity/alkalinity of an aqueous solution: measured as hydrogen ion concentration, but pH usually reported $[-\log_{10}(\text{H ion conc.})]$

- Concentrations of antibodies or mRNA: these differ by orders of magnitude across people, and within people over time

Properties of exponentiation and logarithms come in handy throughout statistics and data analysis; a solid understanding of the basics goes a long way.

# Example: gender bias in salary (from Scott Emerson, MD PhD)

We want to investigate the differences in salaries between male and female faculty members at the University of Washington for the years 1976–1995. The main question is does discrimination in salaries exist in 1995, based on a retrospective cohort study of 1,597 faculty members at the University of Washington.

The average monthly salary for female faculty in 1995 was \$5,396.91; for male faculty, the average monthly salary in 1995 was \$6,731.64.

There are a variety of potential confounding factors that we will consider: start year at UW, year of degree, field of study, highest degree, administrative duties, rank.

# Example: gender bias in salary

For our statistical analysis, we need to decide which is more meaningful: reporting differences in mean salary, or ratios of geometric mean salary.

Since salaries are expected to be somewhat comparable, perhaps with small differences, comparing salary on a multiplicative scale (i.e., ratios of geometric means) makes sense. Also, we expect the difference between people who make $1,000 and $2,000 per month is more similar to the difference between $10,000 and $20,000 than the difference between $2,000 and $3,000. This means we have to log transform the outcome!

It also turns out that transforming the salary may give us more statistical precision, if the geometric mean is the correct comparison to make.

# Exponentiation

Exponentiation corresponds to repeated multiplication, and is:

- the second in the order of operations! (P**E**MDAS)
- composed of two numbers: a base, $b$, and an exponent, $n$
  - represented as $b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ times}}$

Positive exponents multiply the base $b$ a number of times given by $n$; negative exponents multiply the reciprocal of the base $b$ $n$ times. For example, $2^2 = 2 \times 2$, and $2^{-2} = \left(\frac{1}{2}\right)^2 = \frac{1}{2} \times \frac{1}{2}$.

# Properties of exponents

- If we multiply two numbers with the same base, we add their exponents
  - $10^3 \times 10^2 = 10^5$; $10^3 \times 10^{-2} = 10^1$
- If we raise an exponentiated number to a power, we multiply the exponents; $(b^n)^m = b^{n \times m}$
- The exponent can be a fraction (like $\frac{1}{2}$), which gives us the root of the base (if the base is positive)
  - $4^{1/2} = \sqrt{4} = 2$; $81^{1/4} = \sqrt[4]{81} = 3$
- Multiplying different bases: first manipulate the exponent so the bases are equal, then add exponents
  - $2^3 \times 4^5 = 2^3 \times (2^2)^5 = 2^3 \times 2^{10} = 2^{13}$
- For any $b, c \neq 0$: $b^0 = 1$, $(b \times c)^n = b^n \times c^n$

# Exponential function

- An important constant: $e$, approximately 2.718
- Useful as a base for powers
- Define $\exp(x) = e^x$ as the exponential function

# Exercise: exponents and the exponential function

1. What is the result of $x^2$ multiplied by $x^3$?
2. $(x^{-2})^4 = ?$
3. $\exp(x - y) = ?$

# Solutions: exponents and the exponential function

1. $x^2 \times x^3 = x^5$, since we add the exponents when we multiply
2. $(x^{-2})^4 = x^{-2\times 4} = x^{-8}$
3. $\exp(x-y) = e^{x-y} = e^x \times e^{-y} = e^x / e^y = \exp(x)/\exp(y)$

# Logarithms

Logarithms (logs) transform multiplication into addition. This leads to many of their mathematical properties.

Before calculators, to multiply two large numbers (e.g., 1234 and 4747), you would:

1. choose a common base (e.g., 10)
2. convert each number into exponentiated form with this base (e.g., $10^{3.091315}$ and $10^{3.676419}$) using a table of logarithms (usually base 10)
3. add the exponents (e.g., $3.091315 + 3.676419 = 6.767734$)
4. convert back to un-exponentiated form (e.g., $10^{6.767734} = 5857798$)

The logarithm base 10 of a number is just the exponent of the number expressed as a power of 10; the logarithm base 10 of 100 is 2, because $10^2 = 100$.

# Logs: definition

More generally, we can define the logarithm base $k$ of a number $x$, written $\log_k(x)$. If $\log_k(x) = y$, then $k^y = x$.

Common convention in early math courses:

- "log(x)" is $\log_{10}(x)$, and
- "ln(x)" is the *natural log* $\log_e(x)$.

In many scientific applications, "log(x)" is $\log_e(x)$!

- This is also true in most biostatistics courses and software

Some basic properties of logarithms:

- undefined for $x \leq 0$
- increasing: as $x$ increases, $\log_k(x)$ increases
- $\log_k(k) = 1$

# Changing bases

Using different bases for logarithms is similar to measuring length in different units (e.g., inches, centimeters). No matter what base you use, $\log(1) = 0$.

This implies that we can convert between bases! This is often useful in science: if you transform a variable using log base $e$, you can change the base to 10 for a (potentially) more interpretable answer.

We can find the base $k$ logarithm of any number using the most common bases ($e$ and 10): $\log_k(x) = \frac{\log_e(x)}{\log_e(k)} = \frac{\log_{10}(x)}{\log_{10}(k)}$.

# Logs: identities

- Multiplication: $\log_b(xy) = \log_b(x) + \log_b(y)$
- Division: for $y \neq 0$, $\log_b(x/y) = \log_b(x) - \log_b(y)$
- Powers: $\log_b(x^p) = p \log_b(x)$
- Roots: for $p \neq 0$, $\log_b(x^{1/p}) = \log_b(x)/p$
- Inverse function: $\log_b(b^x) = x \log_b(b) = x$

# $\exp(\cdot)$ and $\log(\cdot)$

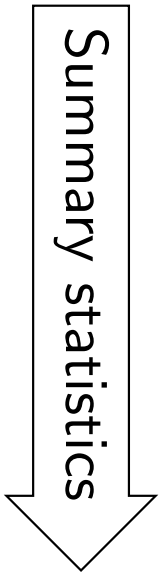
- Recall $\exp(x) = e^x$
- Natural log: $\log(x) = \log_e(x)$
- So $x = \log[\exp(x)]$! And $x = \exp[\log(x)]$!

# Real world vs Log world

## Real world (real numbers)

Example: weights

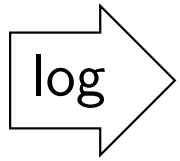
$X = (X_1, X_2, \ldots, X_n)$




mean of $X$

median of $X$

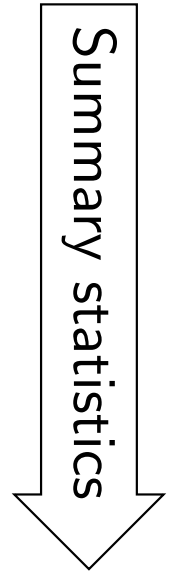
SD of $X$




## Log world (log numbers)

Example: $Z = (Z_1, Z_1, \ldots, Z_n)$,

where $Z_i = \log(X_i)$ for $i = 1, 2, \ldots, n$




mean of $Z$

median of $Z$

SD of $Z$

# Real world vs Log world

Real world (real numbers)

Example: weights

$X = (X_1, X_2, \ldots, X_n)$

log

Log world (log numbers)

Example: $Z = (Z_1, Z_1, \ldots, Z_n)$,

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Summary

Summary

mean of $X$

median of $X$

SD of $X$

exp (?)

mean of $Z$

median of $Z$

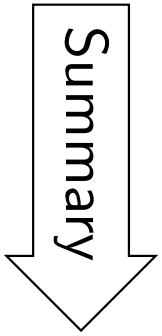
SD of $Z$

# Real world vs Log world

## Real world (real numbers)

Example: weights

$X = (X_1, X_2, \ldots, X_n)$

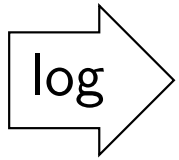


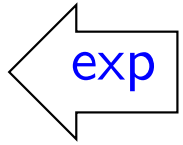

mean of $X$

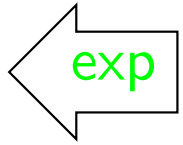
geometric mean of $X$

median of $X$

SD of $X$




## Log world (log numbers)

Example: $Z = (Z_1, Z_1, \ldots, Z_n)$,

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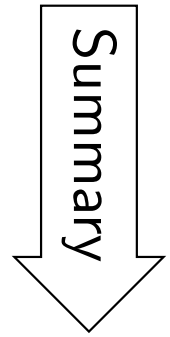


mean of $Z$

median of $Z$

SD of $Z$

# $\exp(\cdot)$, $\log(\cdot)$, and summary statistics

It turns out, as we saw on the previous slides, that for a summary statistic function $f$ (e.g., $f(\cdot) = \text{mean}(\cdot)$), $\exp\{f(\log x)\}$ is not, in general, equal to $f(x)$.

Both exp and log preserve order; thus the median (the middle value) stays the median after being back-transformed.

In contrast, the mean of a log-transformed variable, when back-transformed, is the geometric mean of the original variable. Rather than talking about a difference in means, we talk about a ratio of geometric means.

The standard deviation has no meaningful interpretation when back-transformed—typically, if we need to report a standard deviation, we report it on the log scale.

# Exercise: logarithms

1. $\log(xy) = ?$
2. $\log(x/y) = ?$
3. $\log\{\exp(2x)\} = ?$
4. $\exp\{\log(x^2)\} = ?$

# Solutions: logarithms

1. $\log(xy) = \log(x) + \log(y)$
2. $\log(x/y) = \log(x) - \log(y)$
3. $\log\{\exp(2x)\} = 2x$
4. $\exp\{\log(x^2)\} = \exp\{2\log(x)\} = e^{2\log(x)} = \{e^{\log(x)}\}^2 = x^2$

# Example: gender bias in salary

After running a linear regression analysis on the log-transformed monthly salaries, we find that (adjusted for confounders) the average difference in log monthly salary between women and men in 1995 is $-0.067$, with women having the lower average salary.

After exponentiating, we estimate that geometric mean monthly salary for females in 1995 is 6.53% less than the geometric mean monthly salary for men in 1995 in groups with the same degree, field, administrative duties, starting year and year of degree.

Based on a 95% confidence interval, this difference in geometric mean salary would not be surprising if the true difference in monthly geometric mean salary were between 8.87% and 4.12% less for women compared to men. A two-sided p-value of $< 0.0001$ indicates that we reject the null hypothesis of no association between sex and monthly salary in 1995, in groups with similar degrees, field, administrative duties, starting year, and year of degree.

# Summary

- The comparison that makes the most scientific sense is often ratios
- Logarithms are the typical way to compare ratios
- Exponentiation: can create terms of higher order (larger exponent) than linear terms (exponent 1)
- Logarithms: turn multiplication into addition, using a base
- exp back-transforms log; however:
  - $\exp\{\text{mean}(\log x)\}$ is the geometric mean of $x$
  - $\exp\{\text{median}(\log x)\}$ is the median of $x$
  - $\exp\{\text{SD}(\log x)\}$ doesn't make sense!