Day 1, Session 1: Order of operations and negative numbers

Brian D. Williamson

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Outline for Session 1

- Order of operations
- Negative numbers
- Fractions
- Algebra
- Graphs

Evaluating expressions

- Example expression: $3(1+2)+5$
- How do we evaluate the above expression? In other words:
    - Which terms to we compute first?
    - Are there rules for evaluating expressions?

Order of operations

- Rules for evaluating expressions:
  1. Parentheses
  2. Exponents
  3. Multiplication and division
  4. Addition and subtraction

- A handy memory device: PEMDAS — Please Excuse My Dear Aunt Sally

Example: order of operations in action!

- Example from slide 2: $3(1+2)+5$
- This notation is equivalent to $3 \times (1+2)+5$
- Apply PEMDAS:

Step	Current Expression
1. Parentheses: add 1 and 2	$3(3)+5$
2. Exponents: none	$3(3)+5$
3. Multiplication: multiply 3 and 3	$9+5$
4. Division: none	$9+5$
5. Addition: add 9 and 5	14
6. Subtraction: none	

- The final answer is 14!

Example: order of operations with exponents!

- Expression: $\dfrac{(2^2+5)^2}{3\times 3}+5$

- Apply PEMDAS:

	Current Expression
1. Parentheses: 2^2+5. Need to apply PEMDAS again!	$\frac{(2^2+5)^2}{3\times3}+5$
1.1 Parentheses: none	$\frac{(2^2+5)^2}{3\times3}+5$
1.2 Exponents: $2^2=4$	$\frac{(4+5)^2}{3\times3}+5$
1.3 Multiplication/division: none	$\frac{(4+5)^2}{3\times3}+5$
1.4 Addition/subtraction: $4+5=9$	$\frac{(9)^2}{3\times3}+5$
2. Exponents: $9^2=81$	$\frac{81}{3\times3}+5$
3. Multiplication: $3\times3=9$	$\frac{81}{9}+5$
4. Division: $81/9=9$	$9+5$
5. Addition/subtraction: $9+5=14!$	

Order of operations: nesting

- Earlier, we needed to apply PEMDAS a second time, within the evaluation of the parentheses
- This is common!
- Apply PEMDAS as many times as necessary within each sub-expression, like $(2^2 + 5)$ in the previous example

Exercise: order of operations

- Try to work out the following examples by yourself or in pairs:
  1. $(5 \times 6) + 4$
  2. $5(4 - 2)^2$
  3. $[(2 + 1)^2 + 1]^2$

Note on exercises

- Solutions for the exercises are usually in the slides
- However, please attempt them first without looking at the solution!

Solution: order of operations

1. $(5 \times 6) + 4 = 34$. PEMDAS:
   - Parentheses: 5×6. Nested PEMDAS:
     - Multiplication: $5 \times 6 = 30$
   - Addition: $30 + 4 = 34$

2. $5(4 - 2)^2 = 20$. PEMDAS:
   - Parentheses: $4 - 2$. Nested PEMDAS:
     - Subtraction: $4 - 2 = 2$
   - Exponents: $2^2 = 4$
   - Multiplication: $5 \times 4 = 20$

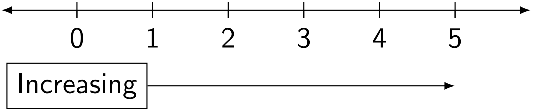
3. $[(2 + 1)^2 + 1]^2 = 100$. PEMDAS:
   - Parentheses: $(2 + 1)^2 + 1$. Nested PEMDAS:
     - Parentheses: $2 + 1$. Nested PEMDAS — Addition: $2 + 1 = 3$
     - Exponent: $3^2 = 9$
     - Addition: $9 + 1 = 10$
   - Exponent: $10^2 = 100$

Negative numbers: what are they?

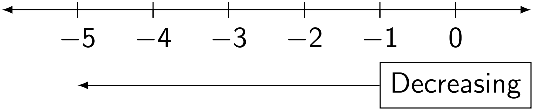
- Ways to think about negative numbers:
  - A positive number subtracted from zero
  - Opposites of positive numbers: $-4 + 4 = 0$
  - Movement left on the number line

The number line

- Movement on the number line:
  - Right (positive numbers):



  - Left (negative numbers):



- Example: move left 3, starting at 100?
  - Subtract 3
  - Add negative 3

Properties of negative numbers

- Represent opposites of positive numbers, or movement left on the number line
- Subtraction $=$ adding a negative number
- The product of two negatives is a positive
- Movement left on the numberline $\rightarrow$ smaller numbers

Negative fractions

- First, positive fractions: for the same numerator, a larger denominator makes a smaller number, e.g. $1/4 < 1/2$
- Negatives are opposites: think of zero as a mirror
- So for the same numerator (a negative number), a larger denominator makes a less negative number, e.g. -$1/2 <$ -$1/4$

Example: two negatives make a positive

- Expression: -1×-1
- Answer: 1!
- Why?
  - -1 is a negative number
  - Negative numbers mean opposites; the opposite of -1 is 1

Example: ordering negative numbers

- Expression: $-3 \_\_\_ -2$
- Answer: $-3 < -2$
- Why?
    - Negative numbers are left motion on number line! -3 is further left than -2

Exercise: negative numbers

- Try to work out the following examples by yourself or in pairs:
    1. $-5.2-(-11.3)$
    2. $-5-6$
    3. $(-1)\times(-5)+(-3)$
    4. $-\frac{3}{7}-\left(-\frac{1}{4}\right)$

Solution: negative numbers

1. $-5.2 - (-11.3) = 6.1$
   - $-(-11.3) = -1 \times (-11.3) = 11.3$
   - $-5.2 + 11.3 = 11.3 - 5.2 = 6.1$

2. $-5 - 6 = -11$

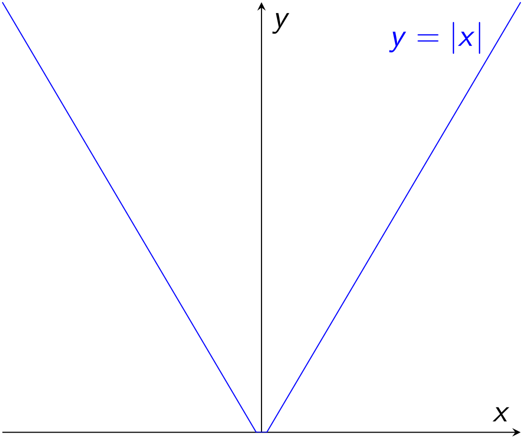
3. $(-1) \times (-5) + (-3) = 2$
   - $-1 \times (-5) = 5$
   - $5 + (-3) = 5 - 3 = 2$

4. $-\frac{3}{7} - \left(-\frac{1}{4}\right) = -5/28 \approx -0.17$
   - $-\frac{3}{7} - \left(-\frac{1}{4}\right) = \frac{3}{7} + \frac{1}{4}$
   - $-\frac{3}{7} + \frac{1}{4} = -12/28 + 7/28 = -5/28$

Related concepts: absolute value

- Magnitudes: how "large" is a number, with no direction
  - Examples: speed (how fast an object is moving), length

- Symbol for absolute value is $|\cdot|$



Example: absolute value of a positive number

- Expression: $|4|$
- Answer is 4! Positive numbers already measure size, with no direction

Example: absolute value of a negative number

- Expression: $|-4|$
- Answer is 4!
- Why?
  - Negatives are opposites of positives
  - Absolute value has no direction
  - 4 and -4 are equally far away from zero

Related concepts: negative numbers and inequalities

- Expression from before: $-3 < -2$
- What happens if we multiply both sides by -1?
- Negatives are opposites: signs change and inequality flips, yielding $2 < 3$

Back to negative fractions

- Recall that for a fixed negative numerator, a larger denominator means a less negative number

- Absolute value makes this more clear!

- Example: -1/2, -1/4
    1. $|-1/2| = 1/2$, $|-1/4| = 1/4$
    2. We already know that $1/4 < 1/2$
    3. Multiply both sides by -1, yielding (with inequality rules) $-1/2 < -1/4$

Exercise: absolute value, negative numbers

- Try to work out the following examples by yourself or in pairs:
  1. $|-5|$ and $|5|$
  2. Is $|-5| < 4$?
  3. Is $-15 > -14$?
  4. Is $-(3+1) \times 5 < -(4+1) \times 3$?

Solution: absolute value, negative numbers

1. $|-5| = |5| = 5$

2. $|-5| = 5$, and $5 > 4$; answer is no

3. -15 is further from 0 than -14; also, $14 < 15$. Hence $-15 < -14$

4. Two ways to solve this:
   - $-(3+1) \times 5 = -1 \times (4) \times 5 = -20$, and $-(4+1) \times 3 = -1 \times (5) \times 3 = -15$. So $-20 < -15$
   - If $-(3+1) \times 5 < -(4+1) \times 3$, then $(3+1) \times 5 > (4+1) \times 3$. But this means $20 > 15$, which is true!

Summary

- The order of operations is a recipe for solving expressions
- A handy memory tool is PEMDAS: Parentheses, Exponents, Multiplication/Division, Addition/Subtraction
- Negative numbers decrease as we move away from zero (left on the number line)
- Absolute value measures the magnitude of a number — how far away from zero is it?