Day 2, Session 1: Logs/Exponentiation

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Motivation

Generally, our goal in a statistical analysis is to assess an association between two variables (e.g., smoking and lung cancer). We can do this by investigating whether a summary measure (e.g., mean, median) of our outcome (e.g., lung cancer) is unequal between two groups differing in our predictor of interest (e.g., smoking).

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There are two simple ways to tell if two numbers are unequal:

- their difference is not equal to 0, or
- their ratio is not equal to 1.

We choose between these based on a variety of criteria, which fall into two general categories: (1) adequately address the scientific question, and (2) gain desirable statistical properties.

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- In the US, 60–64 year old current or former smokers have a probability of 0.00296 of being diagnosed with lung cancer during the next year
- In the US, 60–64 year old never smokers have a probability of 0.000148 of being diagnosed with lung cancer during the next year

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- In the US, 60–64 year old never smokers have a probability of 0.000148 of being diagnosed with lung cancer during the next year
- Difference in incidence rates: 0.002812; ratio of incidence rates: 20!

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When ratios are scientifically preferred, we can use the logarithm of the ratio to get back to comparing differences (more on this later).

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Properties of exponentiation and logarithms come in handy throughout statistics and data analysis; a solid understanding of the basics goes a long way.

Example: gender bias in salary (from Scott Emerson, MD PhD)

We want to investigate the differences in salaries between male and female faculty members at the University of Washington for the years 1976–1995. The main question is does discrimination in salaries exist in 1995, based on a retrospective cohort study of 1,597 faculty members at the University of Washington.

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The average monthly salary for female faculty in 1995 was $5,396.91; for male faculty, the average monthly salary in 1995 was $6,731.64.

There are a variety of potential confounding factors that we will consider: start year at UW, year of degree, field of study, highest degree, administrative duties, rank.

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It also turns out that transforming the salary may give us more statistical precision, if the geometric mean is the correct comparison to make.

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Positive exponents multiply the base b a number of times given by n; negative exponents multiply the reciprocal of the base b a total of n times. For example, $2^2 = 2 \times 2$, and $2^{-2} = \left(\frac{1}{2}\right)^2 = \frac{1}{2} \times \frac{1}{2}$.

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  - $2^3 \times 4^5 = 2^3 \times (2^2)^5 = 2^3 \times 2^{10} = 2^{13}$
- For any $b, c \neq 0$: $b^0 = 1$, $(b \times c)^n = b^n \times c^n$

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- Define $\exp(x) = e^x$ as the exponential function

Exercise: exponents and the exponential function

1. What is the result of x^2 multiplied by x^3?

2. $(x^{-2})^4 = ?$

3. $\exp(x - y) = ?$

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3. $\exp(x - y) = e^{x-y} = e^x \times e^{-y} = e^x/e^y = \exp(x)/\exp(y)$

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The logarithm base 10 of a number is just the exponent of the number expressed as a power of 10; the logarithm base 10 of 100 is 2, because $10^2 = 100$.

Logs: definition

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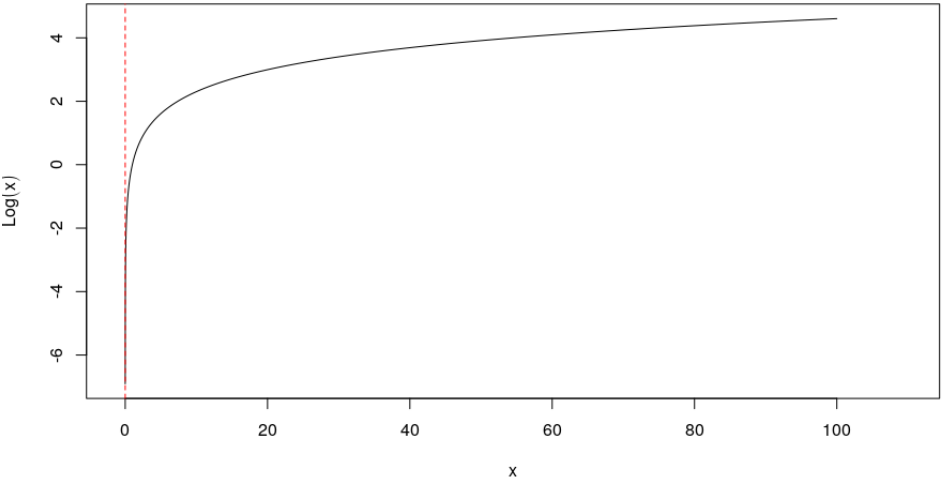

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Equivalently, to find out the average difference in FEV for a 10% increase in height, we find

$$\log_{1.1}(height) = \frac{\log_e(height)}{\log_e(1.1)}$$

and re-compute the average based on this new variable.

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- Inverse function: $\log_b(b^x) = x \log_b(b) = x$

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- So $x = \log[\exp(x)]$! And $x = \exp[\log(x)]$!

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Example: weights

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Example: $Z = (Z_1, Z_1, \ldots, Z_n)$,

where $Z_i = \log(X_i)$ for $i = 1, 2, \ldots, n$

Summary statistics

mean of X
median of X
SD of X

Summary statistics

mean of Z
median of Z
SD of Z

Real world vs Log world

Real world (real numbers)

Example: weights

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mean of X

mean of Z

exp (?)

median of X

median of Z

SD of X

SD of Z

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mean of X

geometric mean of X

median of X

SD of X

exp

exp

mean of Z

median of Z

SD of Z

$\exp(\cdot)$, $\log(\cdot)$, and summary statistics

It turns out, as we saw on the previous slides, that for a summary statistic function f (e.g., $f(\cdot) = \text{mean}(\cdot)$), $\exp\{f(\log x)\}$ is not, in general, equal to $f(x)$.

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The standard deviation has no meaningful interpretation when back-transformed—typically, if we need to report a standard deviation, we report it on the log scale.

Exercise: logarithms

1. $\log(xy) = ?$
2. $\log(x/y) = ?$
3. $\log\{\exp(2x)\} = ?$
4. $\exp\{\log(x^2)\} = ?$

Solutions: logarithms

1. $\log(xy) = \log(x) + \log(y)$
2. $\log(x/y) = \log(x) - \log(y)$
3. $\log\{\exp(2x)\} = 2x$
4. $\exp\{\log(x^2)\} = \exp\{2\log(x)\} = e^{2\log(x)} = \{e^{\log(x)}\}^2 = x^2$

Example: gender bias in salary

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Based on a 95% confidence interval, this difference in geometric mean salary would not be surprising if the true difference in monthly geometric mean salary were between 8.87% and 4.12% less for women compared to men. A two-sided p-value of < 0.0001 indicates that we reject the null hypothesis of no association between sex and monthly salary in 1995, in groups with similar degrees, field, administrative duties, starting year, and year of degree.

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  - $\exp\{\text{SD}(\log x)\}$ doesn't make sense!