520 11/16 TA

(Hw 12 ⇒ Next Wednesday.
(Hw 13 ⇒

#1. $X_1, \dots, X_n$: independent r.v with $B(2,p)$

(a) pdf : $f(x_i|p) = \binom{2}{x_i} p^{x_i}(1-p)^{2-x_i}$ $x_i = 1, 2, 0$

$$\mathcal{L} = \prod_{i=1}^{n} \binom{2}{x_i} p^{x_i}(1-p)^{2-x_i} \Rightarrow \ln L = \sum_{i=1}^{n}\{\ln\binom{2}{x_i} + x_i \ln p + (2-x_i)\ln(1-p)\}$$

$$\frac{\partial \ln L}{\partial p} = \sum_{i=1}^{n}\left(\frac{x_i}{p} - \frac{2-x_i}{1-p}\right) = 0 \Rightarrow \sum_i x_i = 2np \Rightarrow \hat{p} = \frac{1}{2}\cdot\frac{1}{n}\sum_{i=1}^{n} x_i$$

(b) Minimum variance unbiased estimator.

$E(X_i) = 2p$

$Var(X_i) = 2p(1-p)$

$$E(\hat{p}) = E\left(\tfrac{1}{2}\bar{X}\right) = \tfrac{1}{2}E\left[\tfrac{1}{n}\Sigma x_i\right] = \tfrac{1}{2n}\left[\Sigma x_i\right] = \tfrac{n}{2n}\cdot E[X_i] = p.$$

therefore, this is unbiased.

CR - Bound

$$\ln f(x_i;p) = \ln\binom{2}{x_i} + x \log p + (2-x)\log(1-p)$$

$$\frac{\partial \log f(x_i;p)}{\partial p} = \frac{x}{p} - \frac{2-x}{1-p} = \frac{x-2p}{p(1-p)}$$ ⇐ Score fn

↓

$$\left(\frac{\partial \ln f(x_i;p)}{\partial p}\right)^2 = \frac{(x-2p)^2}{p^2(1-p)^2} \longrightarrow E\left(\frac{\partial \ln f(x_i;p)}{\partial p}\right)^2 = \frac{1}{p^2(1-p)^2}E[(x-2p)^2]$$

$$= \frac{Var(X_i)}{p^2(1-p)^2} = \frac{2p(1-p)}{p^2(1-p)^2} = \underbrace{\frac{2}{p(1-p)}}_{I} \longrightarrow \therefore CR = \frac{p(1-p)}{2n}$$

$n \cdot I^{-1}$

$$Var(\hat{p}_{MLE}) = Var\left(\tfrac{1}{2}\bar{X}\right) = \tfrac{1}{4}Var\bar{X} = \tfrac{1}{4}Var\left(\tfrac{1}{n}\Sigma X_i\right) = \tfrac{1}{4n^2}Var(\Sigma X_i) = \tfrac{n}{4n^2}\cdot Var(X_i)$$

$$= \frac{1}{4n}\cdot 2p(1-p) = \frac{p(1-p)}{2n}$$

(c) $\hat{p} = \frac{1}{2}\bar{x} = \frac{1}{2} \cdot \frac{40}{100} = \frac{1}{5}$.

#2 (a) $\ln L = (\ln\lambda)\sum_i x_i - n\lambda - \sum_i \ln x_i!$)

FOC: $\frac{\partial \ln L}{\partial \lambda} = \frac{\sum_i x_i}{\lambda} - n = 0 \quad \therefore \hat{\lambda} = \bar{x}$

Since x_i is non-negative and $\lambda > 0$.

$\sqrt{n}(\hat{\lambda} - \lambda) \xrightarrow{d} N(0, I(\lambda)^{-1})$

$\ln f(x;\lambda) = x\log\lambda - \lambda - \log x!$

$\frac{\partial \log f(x|\lambda)}{\partial \lambda} = \frac{x}{\lambda} - 1$, which has mean zero. ($\because E(\cdot) = 0$)

$E\left(\frac{\partial \log f(x|\lambda)}{\partial \lambda}\right) = 0$

$I(\lambda) = Var\left(\frac{x}{\lambda} - 1\right) = Var\left(\frac{x}{\lambda}\right) = \frac{1}{\lambda^2}Var(x) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

Thus, $\sqrt{n}(\hat{\lambda} - \lambda) \xrightarrow{d} N(0, \lambda)$. //

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if $E\left(\frac{\partial f(x;\lambda)}{\partial \lambda}\right) = 0$, then $I(\lambda) = Var\left(\frac{\partial \ln f(x|\lambda)}{\partial \lambda}\right)$

Check $\boxed{\hat{\lambda} \overset{a}{\sim} N(\lambda, \frac{\lambda}{n})}$

#2 (b). WTS : $E(\hat{\lambda}) = E(\bar{X}) = \frac{1}{n} E[\sum x_i] = \frac{n}{n} E[x_i] = \lambda$

· WTS : $\lim_{n\to\infty} P(|\hat{\lambda}-\lambda| > \varepsilon) = 0$

$$\lim_{n\to\infty} P(|\hat{\lambda}-\lambda| > \varepsilon) \le \lim_{n\to\infty} \frac{Var(\hat{\lambda})}{\varepsilon^2} = \lim_{n\to\infty} \frac{\frac{\lambda}{n}}{\varepsilon^2} = 0$$

(c) $\theta = e^{-\lambda}$

By ☆ invariance property of MLE. $\hat{\theta} = e^{-\hat{\lambda}} = e^{-\bar{X}}$

$$\sqrt{n}(\bar{X}-\mu) \xrightarrow{d} N(0, \sigma^2)$$

By delta method, $g'(\lambda) = -e^{-\lambda}$ $(g'(\lambda))^2 = e^{-2\lambda}$

$$\sqrt{n}(e^{-\bar{X}} - e^{-\lambda}) \xrightarrow{d} N(0, \lambda e^{-2\lambda})$$

#3. $X_1, \dots, X_n$

ⓐ By CLT. $\sqrt{n}(\bar{X}-\mu) \xrightarrow{d} N(0, \sigma^2)$

$g'(\bar{X}) = \left.\frac{\partial e^{\bar{X}}}{\partial \bar{X}}\right|_{\bar{X}=\mu} = e^{\mu}$

$g'(\mu)^2 = e^{2\mu}$

By delta method, $g(\bar{X}) = e^{\bar{X}}$ $g'(\bar{X}) = e^{\mu}$

$$\sqrt{n}(e^{\bar{X}} - e^{\mu}) \xrightarrow{d} N(0, e^{2\mu}\sigma^2)$$

#4. $f_X(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}$ $\quad 0 \le x < \infty$

$\alpha, \beta > 0$

$E(X) = \alpha\beta$

$E(X^2) = \alpha\beta^2 + \alpha^2\beta^2$ $\quad (\because Var(X^2) = \alpha\beta^2)$

$\alpha\beta = \bar{X}$ $\quad \Rightarrow \alpha = \frac{1}{\beta}\bar{X}$

$\alpha\beta^2 + \alpha^2\beta^2 = \overline{X^2}$ $\quad \Rightarrow \frac{1}{\beta}\bar{X}\cdot\beta^2 + (\frac{\bar{X}}{\beta})^2 \cdot \beta^2 = \overline{X^2}$

$\hat{\beta}_{MM} = \frac{\overline{X^2} - (\bar{X})^2}{\bar{X}}$, $\quad \hat{\alpha}_{MM} = \frac{1}{\beta}\bar{X} = \frac{(\bar{X})^2}{\overline{X^2} - (\bar{X})^2}$

#5 (a) key: θ is parameter, $\underline{f(\theta|x) \propto f(x|\theta)\cdot f(\theta)}$

$f(x|p) = \binom{k}{x} p^x (1-p)^{k-x}$

$f(p) = p^{\alpha-1}(1-p)^{\beta-1} \cdot \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$

✓ $f(p|x) \propto \boxed{p^x (1-p)^{k-x} p^{\alpha-1} (1-p)^{\beta-1}.}$ ⇐ conjugate prior

추2) $f(x|\lambda) = \lambda e^{-\lambda x}$ $\quad x \ge 0, \lambda > 0$

$\hat{\lambda} = \frac{1}{\bar{x}}$

Consider reparameterization $\lambda = \frac{1}{\mu}$. find $\hat{\mu}_{MLE}$

By invariance property, $\hat{\mu}_{MLE} = \bar{X}$ is $\hat{\mu}$, unbiased.

$E(\hat{\mu}) = E[\bar{X}] = \frac{1}{n} E[X_1 + \cdots + X_n] = \mu.$

☆ $E(\hat{\lambda}) = E\left[\frac{1}{\bar{X}}\right] > \frac{1}{E(\bar{X})}$, $\hat{\lambda}$ is biased.

strictly convex.

By $\boxed{\text{Jensen's ineq.}}$