ECON 520 Homework 13

Due date: November 28, 2018

1. Let $X_1, \ldots, X_{10}$ be a random sample of size 10 from a normal distribution with mean zero (known), and variance σ^2. We want to test the null hypothesis $H_0 : \sigma^2 = 1$ against the alternative hypothesis $H_a : \sigma^2 > 1$. Consider a test that rejects the null hypothesis if $\sum_i x_i^2 > C$, where C is a constant. Find the constant C so that the test has significance level .05. (Hint: use a table of the chi-square distribution.)

2. Suppose we want to test whether a coin is fair (that is, the probability of heads is $p = .5$). We toss the coin 1000 times, and record the number of heads. Let T denote the number of heads divided by 1000. Consider a test that rejects the null hypothesis that $p = .5$ if $T > c$.

   (a) Write down a formula for $P(T > c|p = .5)$.

   (b) Give a large-sample approximate formula for $P(T > c|p = .5)$ based on the central limit theorem.

   (c) Using the large-sample approximation, calculate the appropriate value c for a test at the 0.05 percent level. You can use the fact that if $Z \sim N(0, 1)$, then $P(Z > 1.645) = 0.05$. Would your test reject the null hypothesis if we observed 560 heads and 440 tails?

   (d) Explain how to construct an approximate 95% confidence interval for p based on the statistic T. Give a general formula, and also provide a specific interval based on observing 560 heads.

3. Let $X_i, i = 1, ..., n$ be a random sample from a population with mean μ and variance σ^2. Let $\omega_i, i = 1, ..., n$ be a deterministic sequence of weights. Consider the statistic

$$T(X) = \sum_{i=1}^{n} \omega_i X_i$$

   (a) What is the condition on the ω_i weights needed to ensure that $E[T(X)] = \mu$?

   (b) Compute $Var(T(X))$ under this condition.

   (c) Think of the case where the weighted sample mean $T(X)$ is more precise than the equally weighted one (recall that $\bar{X} = n^{-1} \sum_{n=1}^{n} X_i$ so in this cae all the weights are equal to $1/n$).

4. Suppose that $X_i \sim N(\mu, 1)$ and $Y_i \mid X_i \sim N(\beta X_i, 1)$. We have a random sample of n observations on the vector (X_i, Y_i).

   (a) What are the maximum likelihood estimators of μ and β?

   (b) Suppose that we have an improper prior for the two parameters $p(\mu, \beta) \propto 1$. Calculate the (marginal) posterior distribution of β.

(c) Suppose the following are available from a random sample of 100 observations on (X_i, Y_i):

$$\sum_{i=1}^{100} X_i = 111 \quad \sum_{i=1}^{100} X_i^2 = 199 \quad \sum_{i=1}^{100} Y_i = 217 \quad \sum_{i=1}^{100} Y_i^2 = 878 \quad \sum_{i=1}^{100} (X_i Y_i) = 391$$

Which (if any) of these numbers can be dropped if you only wish to report sufficient statistics? Justify your reasoning.

(d) Provide an exact 95% confidence interval for μ.

(e) Suppose that we reparameterize the model so that $X_i \sim N(log(\gamma), 1)$ and $Y_i \mid X_i \sim N(log(\theta)X_i, 1)$. Let $\hat{\gamma}$ be the MLE for γ. Derive its large sample distribution, and using the data above obtain an asymptotic 95% confidence interval.

5. Let $X_1 ... X_n$ be iid Bernoulli(p). Show that the variance of $\bar{X}$ attains the CR Lower Bound.