520 TA 11/2 <HW9>

#1 < large sample approximation>

$E(X_i) = \mu \neq 0$ and $Var(X_i) = \sigma^2$.

By CLT : $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$

$Z_n = \frac{1}{\bar{X}_n}$ Then, $g(x) := x^{-1}$, $g'(x) = -x^{-2}$

By Delta method. $\sqrt{n}(g(\bar{X}_n) - g(\mu)) \xrightarrow{d} N(0, g'(\mu)^2 \sigma^2)$

So $\sqrt{n}\left(\frac{1}{\bar{X}_n} - \frac{1}{\mu}\right) \xrightarrow{d} N\left(0, \frac{\sigma^2}{\mu^4}\right)$

#3. $Y_1, \ldots, Y_n$, iid
$$\begin{cases} P(Y_i = 0) = \theta_0 \\ P(Y_i = 1) = \theta_1 \\ P(Y_i = 2) = \theta_2 \end{cases}$$

(a) $E(Y_i) = \theta_0 \cdot 0 + \theta_1 \cdot 1 + \theta_2 \cdot 2 = \theta_1 + 2\theta_2 = \bar{Y} = \frac{1}{n}\sum_i Y_i$

$E(Y_i^2) = \theta_0 \cdot 0^2 + \theta_1 \cdot 1^2 + \theta_2 \cdot 2^2 = \theta_1 + 4\theta_2 = \overline{Y^2} = \frac{1}{n}\sum_i Y_i^2$

$$\therefore \begin{cases} \hat{\theta}_{1MM} = 2\bar{Y} - \overline{Y^2} = \frac{2}{n}\sum_i Y_i - \frac{1}{n}\sum_i Y_i^2 \\ \hat{\theta}_{2MM} = \frac{1}{2}(\overline{Y^2} - \bar{Y}) = \frac{1}{2}\left(\frac{1}{n}\sum_i Y_i^2 - \frac{1}{n}\sum_i Y_i\right) \end{cases}$$

same as $f(y_i;\theta)$

(b) $f(y_i|\theta) = \theta_0^{1(y_i=0)}\theta_1^{1(y_i=1)}\theta_2^{1(y_i=2)}$ ← indicator function.

⇓

$L(\theta) = f(y_1, \ldots, y_n|\theta) = \prod_{i=1}^{n} \theta_0^{1(y_i=0)}\theta_1^{1(y_i=1)}\theta_2^{1(y_i=2)}$

$= \theta_0^{\sum_i 1(y_i=0)}\theta_1^{\sum_i 1(y_i=1)}\theta_2^{\sum_i 1(y_i=2)}$

$\ln L(\theta) = \sum_i 1(y_i=0)\ln\theta_0 + \sum_i 1(y_i=1)\ln\theta_1 + \sum_i 1(y_i=2)\ln\theta_2$.

#3 (continue)

$$\max_{\theta_0,\theta_1,\theta_2,\lambda} \ln L \quad \text{s.t} \quad \theta_0+\theta_1+\theta_2=1$$

$$\mathcal{L} = \ln L + \lambda(1-\theta_0-\theta_1-\theta_2)$$

F.O.C

$$\frac{\partial \mathcal{L}}{\partial \theta_0} = \frac{\sum_i 1(y_i=0)}{\theta_0} - \lambda = 0 \qquad \hat{\theta}_0 = \frac{\sum_i 1(y_i=0)}{\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_1} = \frac{\sum_i 1(y_i=1)}{\theta_1} - \lambda = 0 \qquad \hat{\theta}_1 = \frac{\sum_i 1(y_i=1)}{\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial \theta_2} = \frac{\sum_i 1(y_i=2)}{\theta_2} - \lambda = 0 \qquad \hat{\theta}_2 = \frac{\sum_i 1(y_i=2)}{\lambda}$$

$$\frac{1}{\lambda}\left[\sum_i 1(y_i=0) + \sum_i 1(y_i=1) + \sum_i 1(y_i=2)\right] = 1 \implies \frac{n}{\lambda}=1, \quad \lambda=n$$

Therefore, $\underline{\hat{\theta}_0 = \frac{1}{n}\sum_i 1(y_i=0), \quad \hat{\theta}_1 = \frac{1}{n}\sum_i 1(y_i=1), \quad \hat{\theta}_2 = \frac{1}{n}\sum_i 1(y_i=2)}$

#2 = #4 $\quad f(x;\theta) = \theta x^{\theta-1}, \quad 0<x<1$

(a) $L = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$

$\Downarrow$

$$\ln L = \sum_{i=1}^{n} \ln(\theta x_i^{\theta-1}) = \sum_{i=1}^{n}[\ln\theta + (\theta-1)\ln x_i] = n\ln\theta + (\theta-1)\sum_i \ln x_i$$

(b) $\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + \sum_i \ln x = 0 \qquad \hat{\theta}_{MLE} = \frac{-n}{\sum_i \ln x_i}$

(c) $E(X_i) = \int_0^1 x \cdot \theta x^{\theta-1}\,dx = \frac{\theta}{\theta+1} = \frac{1}{n}\sum_i X_i = \bar{X} \rightarrow \hat{\theta}_{MM} = \frac{\bar{X}}{1-\bar{X}} = \frac{\Sigma X_i}{n-\Sigma X_i}$

(d) $\hat{\theta}_{MLE} \ \frac{-30}{-13.67} = 2.19, \quad \hat{\theta}_{MM} = 2$

#2. (a) "Variance of θ_{MLE}"

$Y_i = -\ln x_i \sim \text{exponential}\left(\frac{1}{\theta}\right)$

so, $-\sum_i \ln x_i \sim \text{gamma}\left(n, \frac{1}{\theta}\right)$. Thus $\hat{\theta} = \frac{n}{T}$

where $T \sim \text{gamma}\left(n, \frac{1}{\theta}\right)$

$$E\left(\frac{1}{T}\right) = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t} \cdot t^{n-1} e^{-\theta t}\, dt = \frac{\theta}{n-1}$$

$$E\left(\frac{1}{T^2}\right) = \frac{\theta^n}{\Gamma(n)} \int_0^\infty \frac{1}{t^2} t^{n-1} e^{-\theta t}\, dt = \frac{\theta^2}{(n-1)(n-2)}$$

$$\therefore Var(\hat{\theta}) = \frac{n^2}{(n-1)^2(n-2)} \theta^2 \to 0 \quad \text{as } n \to \infty$$

#5. $f(x|\theta) = \begin{cases} 1, & \theta \leq x \leq \theta+1 \\ 0, & o/w \end{cases} = \begin{cases} 1, & x-1 \leq \theta \leq x \\ 0, & o/w \end{cases}$

$$L = \prod_{i=1}^{n} f(x_i|\theta) = \begin{cases} 1 & \text{if } x_i - 1 \leq \theta \leq x_i \\ 0 & o/w \end{cases} = \begin{cases} 1 & \text{if } x_{max} - 1 \leq \theta \leq x_{min} \\ 0 & o/w. \end{cases}$$

→ NOT unique