3) b) $g(\bar{X}) = \log(\bar{X})$ HW 11

$$\left.\frac{\partial \log(\bar{X})}{\partial \bar{X}}\right|_{X=\mu} = \frac{1}{\mu}$$

By delta method,

$$\sqrt{n}(\log \bar{x} - \log \mu) \xrightarrow{d} N(0, \sigma^2/\mu^2)$$

c) $g(\bar{X}) = \bar{X}^3$

$$\left.\frac{\partial \bar{X}^3}{\partial \bar{X}}\right|_{\bar{X}=\mu} = 3\mu^2$$

$$\sqrt{n}(\bar{X}^3 - \mu^3) \xrightarrow{d} N(0, 9\mu^4\sigma^2)$$

5) b) $f(x_i|\theta) = \frac{1}{\theta} 1(0 \leq x_i \leq \theta)$

$f(x_1 \ldots x_n|\theta) = \frac{1}{\theta^N} 1(\min(x_1 \ldots x_n) \geq 0) 1(\max(x_1 \ldots x_n) \leq \theta)$

$f(\theta) = ba^b \frac{1}{\theta^{b+1}} 1(\theta \geq a)$

so $f(\theta|x_1 \ldots x_n) \propto \frac{1}{\theta^{b+1}} \cdot \frac{1}{\theta^N} 1(\theta \geq a) 1(\max(x_1 \ldots x_n) \leq \theta)$

$$= \frac{1}{\theta^{b+N+1}} 1(\theta \geq \max(a, x_1 \ldots x_n))$$

Note: I used indicators for this problem but you don't have to

(c) $f(x_i|\lambda) = \lambda e^{-\lambda x_i}$

$f(x_1 \dots x_n|\lambda) = \prod_i f(x_i|\lambda) = \lambda^N e^{-\lambda \sum_i x_i}$

$f(\lambda) \propto \lambda^{\alpha-1} e^{-\lambda/\gamma}$

so $f(\lambda|x_1 \dots x_N) \propto \lambda^{\alpha-1} \lambda^N e^{-\lambda \sum_i x_i} e^{-\lambda/\gamma}$

$= \lambda^{\alpha+N-1} e^{-\lambda(\frac{1}{\gamma} + \sum_i x_i)}$

HW 10

(4) (a) $EY = \int_0^\theta 2y^2 \theta^{-2} dy = 2\theta^{-2}\left[y^3/3\right]_0^\theta = \frac{2\theta}{3}$

MME $\Rightarrow \frac{2\theta}{3} = \bar{y}$ and $\hat{\theta}_{MME} = \frac{3\bar{y}}{2}$

(b) $E(Y^2) = \int_0^\theta 2y^3 \theta^{-2} dy = 2\theta^{2}\left[y^4/4\right]_0^\theta = \frac{\theta^2}{2}$

$var(y) = E[Y^2] - [EY]^2 = \frac{\theta^2}{2} - \frac{4}{9}\theta^2 = \frac{\theta^2}{18}$

so $var(\bar{y}) = var(\frac{1}{n}\sum_i y_i) = \frac{1}{n^2} var(\sum_i y_i)$

$= \frac{1}{n^2}[var(y_1) + var(y_2) + \dots + var(y_n)]$

$= \frac{1}{n^2} \cdot n\, var(y_i) = \frac{\theta^2}{18n}$ $\quad var(\theta) = var(\frac{3}{2}\bar{y}) = \frac{\theta^2}{8n}$

(c) $\log f(y;\theta) = \log(2y) - 2\log\theta$

$$\frac{\partial \log f_Y(y;\theta)}{\partial\theta} = \frac{-2}{\theta}$$

$$I(\theta) = nE\left[\left(\frac{\partial \log(f(y;\theta))}{\partial\theta}\right)^2\right] = nE\left[\left(\frac{-2}{\theta}\right)^2\right] = \frac{4n}{\theta^2}$$

(d) $I(\theta)^{-1}$ is $\frac{\theta^2}{4n} > V(\hat{\theta}_{MME}) = \frac{\theta^2}{8n}$

The method of moments estimator is <u>efficient</u> compared to what is suggested as a lower bound, <u>BUT</u> we can't pass differentiation through integrals because they depend on the parameter so the results are not valid.