ECON 520 Homework 7

Due date: October 10, 2018

1. Let X be a random variable with moment generating function $M_X(t)$, $-h < t < h$. Prove that
$$P(X \geq a) \leq \exp(-at) \cdot M_X(t), \quad 0 < t < h,$$
and
$$P(X \leq a) \leq \exp(-at) \cdot M_X(t), \quad -h < t < 0.$$
Hint: Use Markov's inequality.

2. Let X be a discrete random variable with $P(X = -1) = 1/8$, $P(X = 0) = 6/8$ and $P(X = 1) = 1/8$. Calculate the bound on $P(|X - \mu_X| \geq k \cdot \sigma_X)$ for $k = 2$ using Chebyshev's inequality. Compare this to the actual probability $P(|X - \mu_X| \geq k \cdot \sigma_X)$. This shows that Chebyshev's inequality can be a sharp bound for some k for some random variables.

3. We say that a sequence of random variables $X_1, \dots, X_n$ *converges in quadratic mean* to a random variable iff
$$\lim_{n\to\infty} E[(X_n - X)^2] = 0.$$
Show the following version of the WLLN: if $X_1, \dots, X_n$ are iid random variables with mean μ and variance $\sigma^2 < \infty$, and $\bar{X}_n = n^{-1} \sum_{i=1}^n X_i$ is the sample mean, then $\bar{X}_n$ converges in quadratic mean to μ.

4. (a) Show that convergence in quadratic mean implies convergence in probability.

   (b) The converse is not true, i.e., convergence in probability does not imply convergence in quadratic mean. For this, consider the following counter-example: for any $i, 1 \leq i \leq n$, let
$$X_i = \begin{cases} 0, & \text{with probability } 1 - \frac{1}{i} \\ i, & \text{with probability } \frac{1}{i} \end{cases}$$
   Show that $X_i \xrightarrow{p} 0$ (note that 0 is not the expected value of X_i). Show that, however, X_i does not converge to either 0 or 1 in quadratic mean.

5. Consider the sequence of discrete random variables $X_n, n = 1, 2, \dots$, where

x_n	$\Pr(X_n = x_n)$
$-n$	$\frac{1}{2n}$
0	$1 - \frac{1}{n}$
n	$\frac{1}{2n}$

   (a) Show that X_n converges to zero in probability.
   (b) Does X_n converge to zero in quadratic mean?
   (c) Does X_n converge to zero almost surely?