③ ⓐ $E[T(X)] = E[\sum_{i=1}^{n} w_i X_i] = \sum_{i=1}^{n} w_i E(X_i)$

$= \mu \sum_{i=1}^{n} w_i$ so we require

that $\sum_{i=1}^{n} w_i = 1$

ⓑ $var(T(X)) = E[T(X) - \mu]^2$

$= E\left[\sum_{i=1}^{n} w_i X_i - \mu \sum_{i=1}^{n} w_i\right]^2$

$= E[\sum_{i=1}^{n} w_i (X_i - \mu)]^2$

By independence $= \sum_{i=1}^{n} w_i^2 E(X_i - \mu)^2$

$= \sigma^2 \sum_{i=1}^{n} w_i^2$

ⓒ consider the case that X_i, $i = 1 \ldots n$ be a random sample from a population with mean μ and variance σ_i^2 where $\sigma_i^2 \neq \sigma_j^2$ for $i \neq j$ (i.e. heteroskedasticity)

(i) using equally weighted sample mean,

$T(x) = \frac{1}{n} \sum_{i=1}^{n} X_i$

$$E(T(X)) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu$$

$$\text{var}(T(X)) = \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2$$

② using the weighted sample mean,

$$T(X) = \sum_{i=1}^{n} w_i X_i \quad \text{where } w_i = \frac{1/\sigma_i^2}{\sum_{j=1}^{n} 1/\sigma_j^2}$$

$$E(T(X)) = \sum_{i=1}^{n} w_i E(X_i) = \mu \quad \left(\text{since } \sum_{i=1}^{n} w_i = 1\right)$$

$$\text{var}(T(X)) = \text{var}\left(\sum_{i=1}^{n} w_i X_i\right) = \sum_{i=1}^{n} w_i^2\,\text{var}(X_i)$$

$$= \sum_{i=1}^{n}\frac{1}{\sigma_i^4}\sigma_i^2\frac{1}{\left(\sum_{j=1}^{n} 1/\sigma_j^2\right)^2} = \frac{1}{\sum_{j=1}^{n} 1/\sigma_j^2}$$

$$= \frac{1}{\sum_{j=1}^{n}\frac{1}{\sigma_j^2}}$$

comparing $\text{var}^{①}(T(X)) - \text{var}^{②}(T(X))$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma_i^2 - \frac{1}{\sum_{j=1}^{n} 1/\sigma_j^2} \geq 0$$

so the weighted sample mean is more precise than equal weights

(b) we are given $p(\mu,\beta) \propto 1$

let X, Y denote the full set of data

$$p(\mu,\beta \mid X,Y) \propto \mathcal{L}(\mu,\beta)\, p(\mu,\beta)$$

$$\propto \exp\left(-\frac{1}{2}\sum_i (x_i-\mu)^2\right) \exp\left(-\frac{1}{2}\sum_i (y_i-\beta x_i)^2\right)$$

this factors into terms involving β and μ separately so that μ is independent of β conditional on X, Y

$$p(\beta \mid X,Y) \propto \exp\left(-\frac{1}{2}\sum_i (y_i-\beta x_i)^2\right)$$

$$= \exp\left(-\frac{1}{2}\sum_i (y_i^2 - 2\beta y_i x_i + \beta^2 x_i^2)\right)$$

→ complete the square ←

$$\propto \exp\left[-\frac{1}{2}\left(\sum_i x_i^2\right)\left(\beta - \frac{\sum_i y_i x_i}{\sum_i x_i^2}\right)^2\right]$$

$$\beta \mid X,Y \sim N\left(\hat{\beta}, \left(\sum_i x_i^2\right)^{-1}\right)$$