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Final Exam, Econ 520 (Fall 2017)

December 14

Instructions: This is a closed book exam. You have 120 minutes for the exam. Answer as many questions as possible. Partial answers get partial credit. Please write legibly. *Good luck!*

$\alpha \; \beta$

Problem 1.[35pt] Suppose that X is distributed as a Gamma(2, θ) random variable, i.e., 2

$$f_X(x|\theta) = \frac{1}{\Gamma(2)\theta^2} x e^{-x/\theta} \qquad e^{-x} \cdot \qquad 2 \cdot \beta$$

for $x > 0$ and $\theta > 0$. We have a sample of n independent identically distributed (iid) observations, $X_1, \ldots, X_n$. The mean of a Gamma(α, β) distribution is $\alpha\beta$ and variance is $\alpha\beta^2$.

(a) (*5 pt*) Find the method of moments estimator $\hat{\theta}_{MM}$ (based on the first moment) for θ.

(b) (*5 pt*) Is $\hat{\theta}_{MM}$ an unbiased estimator of θ?

(c) (*5 pt*) Find the MLE $\hat{\theta}_{MLE}$ for θ.

(d) (*5 pt*) Is $\hat{\theta}_{MLE}$ an unbiased estimator of θ?

(e) (*10 pt*) Compare the variance of $\hat{\theta}_{MLE}$ and $\hat{\theta}_{MM}$ with the Cramer-Rao lower bound and comment on the efficiency of your estimators.

(f) (*5 pt*) Are the estimators (weakly) consistent? [Hint: Use Chebyshev's inequality.] $P(|x| > \varepsilon) \leq \frac{E(x)^2}{\varepsilon^2}$

Problem 2.[15 pt] Suppose $f_n(x;\theta)$ is such that the following interchange of integration and differentiation is justified: $\frac{d^2}{d\theta d\theta^\top}\int_{\mathbb{R}^n}\frac{\partial}{\partial\theta}f_n(x;\theta)dx = \int_{\mathbb{R}^n}\frac{\partial^2}{\partial\theta\partial\theta^\top}f_n(x;\theta)dx$. Prove that the Fisher information matrix satisfies

$$H_n(\theta) = -\mathrm{E}_\theta\left(\frac{d^2}{d\theta d\theta^\top}\ell_n(\theta)\right).$$

$\frac{d^2}{d\theta d\theta^\top}$ Var $\frac{d}{d\theta}$

$\theta^{-\frac{1}{2}}$

Problem 3.[30pt] Let $X_1, \ldots, X_n$ be a random sample from a continuous distribution

$$f_X(x|\theta) = \frac{1}{\sqrt{\pi\theta}} x^{-1} \exp\left[-\frac{1}{\theta}(\log x)^2\right] 1(x > 0),$$

where $\theta \in \Theta = (0, \infty)$ is an unknown parameter, and $1(\cdot)$ is the indicator function.

(a) (*5 pt*) Use the fact that $\mathrm{E}_\theta(X_i) = \exp(\theta/4)$ to derive a method of moments estimator $\hat{\theta}_{MM}$ of θ.

(b) (*5 pt*) Is $\hat{\theta}_{MM}$ an unbiased estimator of θ?

(c) (*5 pt*) Find the log likelihood function and show that the maximum likelihood estimator of θ is given by

$$\hat{\theta}_{MLE} = \frac{2}{n} \sum_{i=1}^{n} (\log X_i)^2.$$

$\sigma^2 = \theta/2$

$3 \cdot \theta^2/4$

(d) (*5 pt*) Show that $\hat{\theta}_{MLE}$ is a sufficient statistic for θ.

(e) (*5 pt*) Find a best unbiased estimator (i.e., uniform minimum variance unbiased estimator) of θ and compute its variance. [Hint: Use the facts that $Y = \log X \sim N(0, \theta/2)$ and (about the normal distribution) that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} r^4 \exp\left(-\frac{1}{2\sigma^2} r^2\right) dr = 3\sigma^4$$

$2\sigma^2 = \theta$

$3\sigma^4 =$

for any $\sigma^2 > 0$.]

(f) (*5 pt*) Compute the Cramér-Rao lower bound (on the variance of unbiased estimators of θ). Is this bound attained by the estimator from (e)?

Problem 4.[20 pt] Suppose that $X_i \sim N(\theta, 1)$. We have a random sample of n observations of X_i.

(a) (*5 pt*) Provide an asymptotic 95% confidence interval for θ. [Hint: $\hat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i$.]

(b) (*10 pt*) Suppose that we reparameterize the model so that $\omega = e^\theta$. Let $\hat{\omega}$ be the MLE for ω. Derive its large sample distribution and obtain an asymptotic 95% confidence interval.

(c) (*5 pt*) Now suppose $n = 2000$ and $\hat{\theta} = 17$. Provide an exact 95% confidence interval for θ and ω.

$\boxed{\cancel{-5}}$ $\boxed{-10}$ $\boxed{-5}$ $\boxed{-15}$

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