ECON 520 Homework 9

Due date: November 2, 2018

1. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with finite mean (assume the mean is not equal to zero) and finite variance, and let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Let $Z_n = 1/(\overline{X}_n)$. Provide a large-sample approximation to the distribution of Z_n (using appropriate normalization to ensure a nondegenerate distribution).

2. Let $X_1 \ldots X_n$ be iid with pdf $f(x|\theta) = \theta x^{\theta-1}, 0 \leq x \leq 1, 0 < \theta < \infty$.

   (a) Find the MLE of θ, and show that its variance $\to 0$ as $n \to \infty$.

   (b) Find the method of moments estimator of θ.

   (For part (a), you do not need to compute the variance of the MLE).

3. Suppose $Y_1, \ldots, Y_n$ are IID discrete random variables with

$$\begin{aligned} P(Y_i = 0) &= \theta_0, \\ P(Y_i = 1) &= \theta_1, \\ P(Y_i = 2) &= \theta_2, \end{aligned}$$

   where the parameter vector $\theta = (\theta_0, \theta_1, \theta_2)$ satisfies: $\theta_j \geq 0$ and $\sum_{j=0}^2 \theta_j = 1$.

   (a) Calculate $E[Y_i]$ and $E[Y_i^2]$, and use the results to derive a method of moments estimator for the parameters (θ_1, θ_2).

   (b) Show that the maximum likelihood estimator for $\theta = (\theta_0, \theta_1, \theta_2)$ is

$$\begin{aligned} \hat{\theta}_0 &= \frac{1}{n}\sum_i 1(Y_i = 0), \\ \hat{\theta}_1 &= \frac{1}{n}\sum_i 1(Y_i = 1), \\ \hat{\theta}_2 &= \frac{1}{n}\sum_i 1(Y_i = 2). \end{aligned}$$

   (Hint: be sure to impose the constraint that $\theta_0 + \theta_1 + \theta_2 = 1$.)

4. Suppose that $X_1, X_2, \ldots, X_n$ are IID with probability density function

$$f(x;\theta) = \theta x^{\theta-1} \quad 0 \leq x \leq 1,$$

   where the parameter $\theta > 0$.

   (a) Derive the log likelihood function and show that it depends on the data $x_1, \ldots, x_n$ only through $\sum_{i=1}^n \log x_i$.

   (b) Derive the maximum likelihood estimator for θ.

   (c) Derive the method of moments estimator for θ.

   (d) Now suppose that we observe data where $n = 30$, $\sum_{i=1}^n x_i = 20$, and $\sum_{i=1}^n \log x_i = -13.67$. Calculate the maximum likelihood and method of moments estimates.

5. Suppose that $X_1, \ldots, X_n$ are IID Uniform on $[\theta, \theta+1]$. Derive the MLE. (Note: it is better here to work directly with the likelihood function, not the log likelihood.) Is the MLE unique?