Econ 520 HW 2

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(1) Bowl I containts 3 red chips and 7 blue chips. Bowl II contains 6 red chips and 4 blue chips. A bowl is slected at random, and the 1 chip is drawn from this bowl. What is the probability that the chip drawn is red?

Let R denote the event that a red chip is drawn and:

Let I denote the event that bowl one is chosen and II denote event that bowl II is chosen. I assume that:

$$P(I) = P(II) = 0.5$$

If this is true, what bowl the chip came from technically does not matter. There are 3 red chips in 7 blue chips in bowl I and 6 red chips and 4 blue chips in bowl II. So, the total amount of red chips is 9 and the total amount of blue is 11. Therefore, the probability of drawing a red chip is $P(R) = \frac{9}{9+11} = \frac{9}{20} = 0.45$.

Alternatively, I could have factored in the probability of randomly selecting a bowl as well, making a weighted average of probabilities such that:

$$\frac{1}{2}\frac{3}{3+7} + \frac{1}{2}\frac{6}{6+4} = 0.45$$

Given that the chip is red, what is conditional probability it came from bowl II?

$$P(II|R) = \frac{P(II \cap R)}{P(R)} = \frac{.5(6/10)}{0.45} = \frac{2}{3}$$

This makes sense because bowl I has 3 red chips and bowl 2 has 6 red chips. There are 9 chips total, and bowl II has 2/3 of the population of red chips.

(2) Two dice are thrown and three events are defined as follows: A means "odd face with first die"; B means "odd face with second die"; and C means "odd sum" (one face is odd, one face is even). If each of the 36 sample points has a probaility of $\frac{1}{36}$ then:

(a) Show that the events are pairwise independent

$$P(A) \cap P(B) \cap P(C) = 0$$

Two or more events are independent if their intersection is the product of the probabilities such that $P(A \cap B) = P(A)P(B)$, which is equivalent to showing: $P(A|B) = P(A)$

$$P(A|B) = P(B|A) * P(A)/P(B) = \frac{1}{2} * \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2} = P(A)$$

$$P(B|C) = P(C|B) * P(B)/P(C) = \frac{1}{2} * \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2} = P(B)$$

$$P(C|A) = P(A|C) * P(C)/P(A) = \frac{1}{2} * \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1}{2} = P(C)$$

The probability of strictly getting an odd face on the first die and and odd face on a second die and having one face odd and the other even cannot happen even conceptually.

(b) Show that the events are not jointly independent

$$P(A)P(B)P(C) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

(3) Two telegraph signals are sent. Assume that the two signals were sent independently, and that the (marginal) probability of a dot being sent is $\frac{3}{7}$ and the (marginal) probability of a dash being sent is $\frac{4}{7}$. The probability that a dash sent is read as a dot is $\frac{1}{3}$ and the probability that a dot is read as a dash is $\frac{1}{4}$. What is the probability for all four combinations to have been sent if two dots are received?

Let A denote event that a dot was sent. Let A' denote event that a dash was sent. Let B denote event that a dot was received. Let B' denote event that a dash was received.

$$P(A) = \frac{3}{7}$$

$$P(A') = \frac{4}{7}$$

$$P(B|A') = \frac{1}{3} \implies P(B'|A') = \frac{2}{3}$$

$$P(B'|A) = \frac{1}{4} \implies P(B|A) = \frac{3}{4}$$

$$P(B) = P(B|A)P(A) + P(B|A')P(A') = \frac{3}{4}\frac{3}{7} + \frac{1}{3}\frac{4}{7} = \frac{43}{84}$$

Case I: Two dots sent given two dots received

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{\frac{3}{4}\frac{3}{7}}{\frac{43}{84}} = \frac{9}{28}\frac{84}{43} = \frac{27}{43} = 0.627907$$

Transmitted dots/dashes are independent of one another, so the probability that two dots are sent given two dots are recieved is:

$$P(A|B)P(A|B) = \left(\frac{27}{43}\right)^2 = 0.3942672$$

Case II: One dot sent first and dash sent second given two dots received

$$P(A'|B) = \frac{P(B|A')P(A')}{P(B)} = \frac{\frac{1}{3}\frac{4}{7}}{\frac{43}{84}} = \frac{16}{43}$$

$$P(A|B)P(A'|B) = \left(\frac{27}{43}\right)\left(\frac{16}{43}\right) = \frac{432}{1849}$$

Case III: One dash sent first and one dot sent second given two dots are received

This will be the same as Case II:

$$P(A'|B)P(A|B) = \left(\frac{16}{43}\right)\left(\frac{27}{43}\right) = \frac{432}{1849}$$

Case IV: Two dashes sent given two dots are received

$$P(A'|B)P(A'|B) = \left(\frac{16}{43}\right)\left(\frac{16}{43}\right) = \frac{256}{1849}$$

(4) Bowl I contains 6 red chips and 4 blue chips. Five of these chips are selected at random and without replacement are put in bowl II, which was originally empty. One chip is then at random drawn from bowl II. Find the conditional probability that 2 red chips and 3 blue chips were transferred from bowl I to bowl II given that a blue chip is drawn from bowl II.

Let the sample space for five chips in bowl II be:

$$S = \{(BBBBR), (BBBRR), (BBRRR), (BRRRR), (RRRRR)\}$$

Let the random variable X be assigned to each outcome:

Outcome	Random Variable X	P(X = x)
BBBBR	x = 1	$\frac{6}{252}$
BBBRR	x = 2	$\frac{60}{252}$
BBRRR	x = 3	$\frac{120}{252}$
BRRRR	x = 4	$\frac{60}{252}$
RRRRR	x = 5	$\frac{6}{252}$

Each Probability P(X=x) comes from the fact that:

$$P(x=1) = \frac{\binom{4}{4}\binom{6}{1}}{\binom{10}{5}} = \frac{6}{252}$$

$$P(x=2) = \frac{\binom{4}{3}\binom{6}{2}}{\binom{10}{5}} = \frac{60}{252}$$

$$P(x=3) = \frac{\binom{4}{2}\binom{6}{3}}{\binom{10}{5}} = \frac{120}{252}$$

$$P(x=4) = \frac{\binom{4}{1}\binom{6}{4}}{\binom{10}{5}} = \frac{60}{252}$$

$$P(x=5) = \frac{\binom{4}{0}\binom{6}{5}}{\binom{10}{5}} = \frac{6}{252}$$

So, we seek to find the probability $P(x=2)$ conditional on the fact that a blue chip is drawn from bowl II. Let the event that such happens as B. Hence we want:

$$P(X=2|B) = \frac{P(x=2 \cap B)}{P(B)},$$

Now, we invoke the intercession of St. Thomas Bayes to aid us in our cause. Notice the above is the same as summing up the probability of drawing a blue chip partitioned about each random variable x. In other words it is the denominator of Bayes' Theorem such that:

$$P(B) = P(B|x=1)P(x=1) + P(B|x=2)P(x=2) + ... + P(B|x=5)P(x=5)$$

$$P(B) = \frac{4}{5}\frac{6}{252} + \frac{3}{5}\frac{60}{252} + \frac{2}{5}\frac{120}{252} + \frac{1}{5}\frac{60}{252} + \frac{0}{5}\frac{6}{252} = \frac{2}{5}$$

$$P(x=2|B) = \frac{P(B|x=2)P(x=2)}{P(B)} = \frac{P(x=2 \cap B)}{P(B)}$$

$$P(B|x=2) = \frac{\binom{3}{1}}{\binom{5}{1}} = \frac{3}{5}$$

$$P(x=2) = \frac{60}{252}$$

$$P(x=2|B) = \frac{\frac{3}{5}\frac{60}{252}}{\frac{2}{5}} = \frac{5}{14} = 0.3571429$$

(5) Two litters of a particular rodent species have been born, one with two brown-haired and one gray-haired (litter 1), and the other with three brown-haired and two gray-haired (litter 2). We select a litter at random and then select an offspring at random from the selected litter.

(a) What is the probability that an animal chosen has brown hair?

Let B denote event that rodent is brown and G denote event that rodent is gray. Also, Let the random variable X be assigned to each litter, taking on values of $x = 1$ and $x = 2$ for litter 1 and 2 respectively.

$$P(B) = P(B|x=1)P(x=1) + P(B|x=2)P(x=2) = \frac{2}{3}\frac{1}{2} + \frac{3}{5}\frac{1}{2} = \frac{19}{30}$$

(b) Give that a brown-haried offsping was selected, what is the probability that the sampling was from litter 1?

$$P(x=1|B) = \frac{P(B|x=1)P(x=1)}{P(B)} = \frac{\frac{2}{3}\frac{1}{2}}{\frac{19}{30}} = \frac{10}{19}$$

(6) Consider a fair coin tossed until a tail appears. O each toss, head or tail are equally likely and each toss is an independent event.

(a) What is the sample space?

$$S = \{...H^{(n-1)}T^x, H^nT^x\}$$

Where n is the number of tosses until $x = 1$ indicating that a Tail has appears on the n^{th} toss. $x = 0$ otherwise.

(b) Calculate the probability that the tail appears on the fourth toss

This is a geometric distribution. The probability of getting a tail on the 4th toss:

$$P(X = k) = (1 - p)^{k-1}p$$

$$P(X = 4) = \left(1 - \frac{1}{2}\right)^{4-1}\left(\frac{1}{2}\right) = 0.0625$$

OR: $P(H)P(H)P(H)P(T) = \left(\frac{1}{4}\right)^4 = \frac{1}{16} = 0.0625$

(c) Calculate the probability that it takes x tosses for the tail to appear

$$P(X = x) = (1 - 0.5)^{x-1}0.5 = (0.5)^x$$

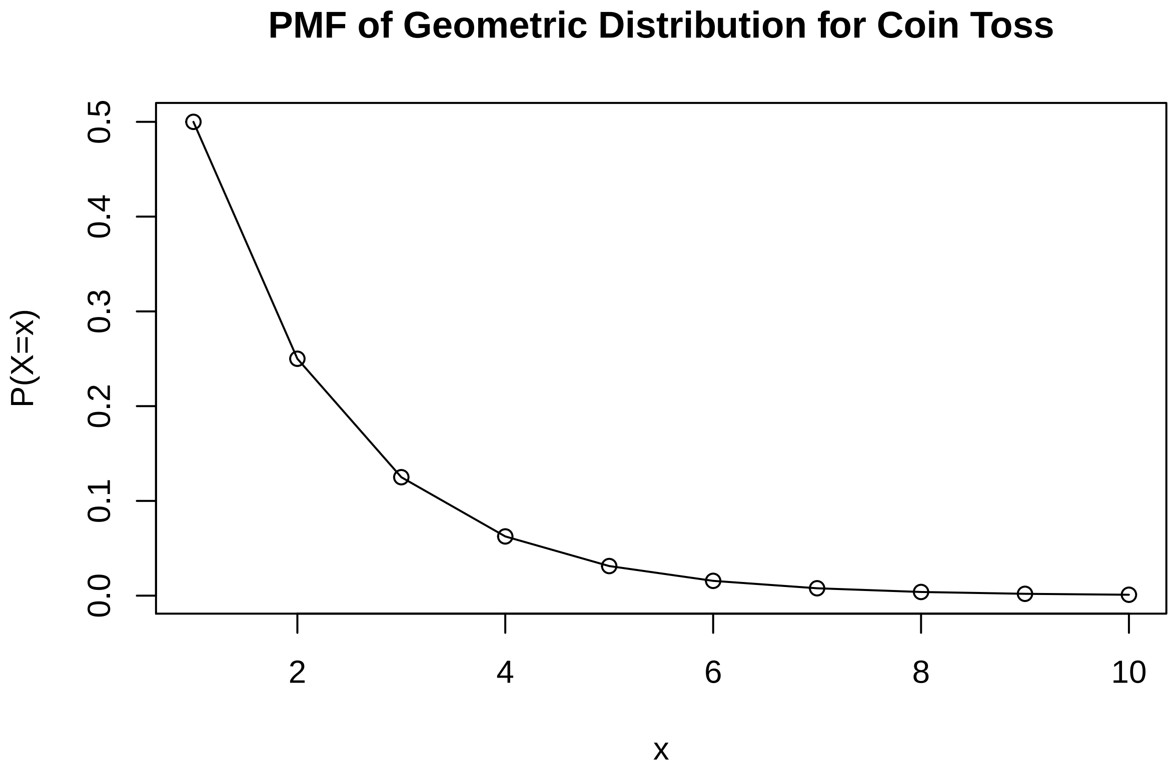
(d) Graph the probability mass function (pmf) of X = number of tosses to get a tail to appear.

```r
geop <- function(x, p1, p2){
  y = (1 - p1)^(x-1)*p2
  return(y)
}

k = seq(1,10)

probs = geop(k,0.5,0.5)

plot(k, probs,
     type='l',
     main = 'PMF of Geometric Distribution for Coin Toss',
     ylab = 'P(X=x)',
     xlab = 'x'
)
points(k, probs)
```

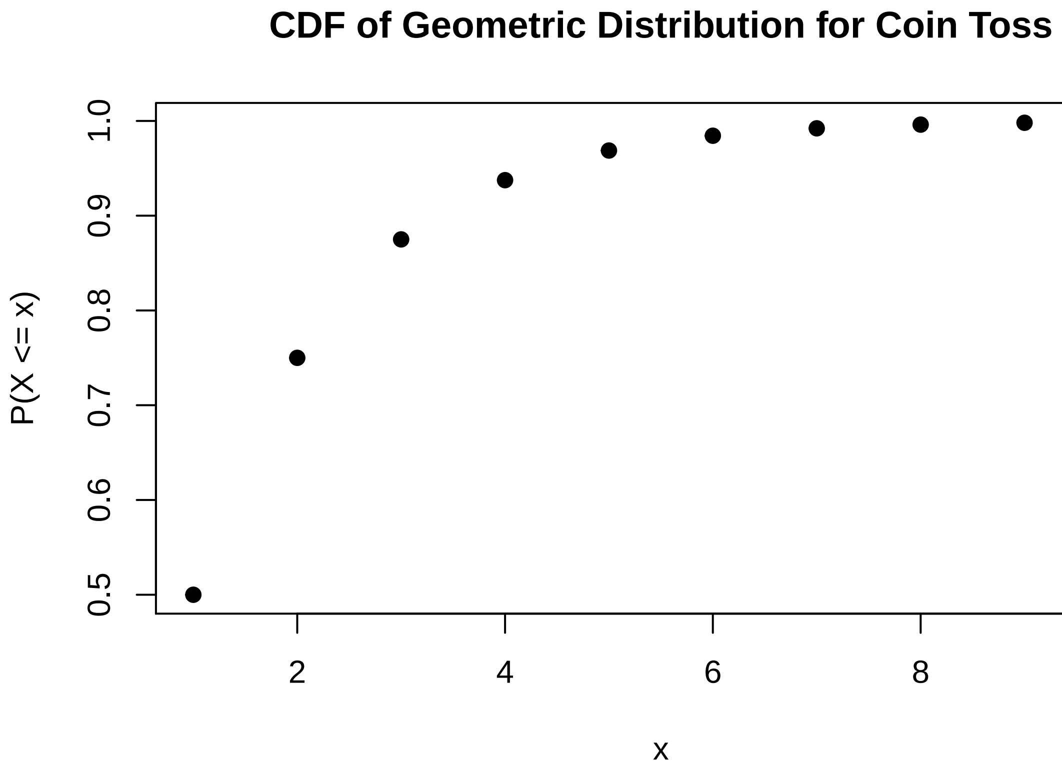


(e) Graph the cumulative density function of X.

```
k = seq(1,10)

sums = cumsum(geop(k,0.5,0.5))

plot(k, sums,
     main = 'CDF of Geometric Distribution for Coin Toss',
     ylab = 'P(X <= x)',
     xlab = 'x',
     pch=19
)
```


CDF of Geometric Distribution for Coin Toss
P(X <= x)
1.0
0.9
0.8
0.7
0.6
0.5
2
4
6
8
10
x