#3. $\underset{K\times 1}{Y} = \underset{K\times k}{X}\underset{k\times 1}{\beta} + \underset{K\times 1}{\varepsilon}$

① unrestricted OLS, $\hat{\beta}$

$$\hat{\beta} = (X'X)^{-1}X'Y = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'\varepsilon = \beta + (X'X)^{-1}X'\varepsilon$$

② Restricted OLS, $\tilde{\beta}$

$$\min_{\beta}\ [Y - X\beta]'[Y - X\beta] \quad \text{s.t} \quad \underset{r\times k}{C}\underset{k\times 1}{\beta} = \underset{r\times 1}{r}$$

$$\mathcal{L} = [Y - X\beta]'[Y - X\beta] - \lambda[r - C\beta]$$

$$= Y'Y - X\beta'Y - Y'X\beta + \beta'X'X\beta - \lambda(r - C\beta)$$

$$\frac{\partial \mathcal{L}}{\partial \beta} = -2X'Y + 2X'X\tilde{\beta} + C'\lambda = 0 \ \cdots ① , \quad \frac{\partial \mathcal{L}}{\partial \lambda} = C\tilde{\beta} - r = 0 \ \cdots ②$$

↓ Multiplying $C(X'X)^{-1}$

$$-2C(X'X)^{-1}X'Y + 2C(X'X)^{-1}X'X\tilde{\beta} + C(X'X)^{-1}C'\lambda = 0.$$

Then, $C(X'X)^{-1}C'\lambda = 2C\hat{\beta} - 2C\tilde{\beta} = 2C\hat{\beta} - 2r \quad \therefore \lambda = -2\left(C(X'X)^{-1}C'\right)^{-1}(r - C\hat{\beta})$

↓ put λ into ①

$$2(X'X)\tilde{\beta} = 2X'Y - C'\lambda = 2X'Y + 2C'\left(C(X'X)^{-1}C'\right)^{-1}(r - C\hat{\beta}) \ \cdots ③$$

$$\tilde{\beta} = (X'X)^{-1}X'Y + (X'X)^{-1}C'\left(C(X'X)^{-1}C'\right)^{-1}(r - C\hat{\beta})$$

$$\tilde{\beta} = \hat{\beta} + (X'X)^{-1}C'\left(C(X'X)^{-1}C'\right)^{-1}(r - C\hat{\beta}) \ \cdots ④$$

To justify F-test. We need to get $SSR_u - SSR_r$ and SSR_u

- WTS: get $SSR_u - SSR_r$ and SSR_u

From ③, $(X'X)\tilde{\beta} = X'Y + C'\left(C(X'X)^{-1}C'\right)^{-1}(r - C\hat{\beta})$

↓ Multiplying $X(X'X)^{-1}$

$$X\tilde{\beta} = X(X'X)^{-1}X'Y + X(X'X)^{-1}C'\left(C(X'X)^{-1}C'\right)^{-1}(r - C\hat{\beta})$$

$$\mathbb{X}\tilde{\beta} = \mathbb{X}\hat{\beta} + \mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta})$$

Then,

$$\mathbb{Y} - \mathbb{X}\tilde{\beta} = \mathbb{Y} - \mathbb{X}\hat{\beta} - \mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta})$$

Let $\mathbb{Y} - \mathbb{X}\tilde{\beta} = \hat{u}$ and $\mathbb{Y} - \mathbb{X}\hat{\beta} = \hat{\varepsilon}$

Then

$$\hat{u}'\hat{u} = [\hat{\varepsilon}' - (r - C\hat{\beta})'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}C(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'][\hat{\varepsilon} - \mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta})]$$

$$= \hat{\varepsilon}'\hat{\varepsilon} - \hat{\varepsilon}'\mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta}) - (r - C\hat{\beta})'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}C(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\hat{\varepsilon}$$

$$+ (r - C\hat{\beta})'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}C(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta})$$

$$= \hat{\varepsilon}'\hat{\varepsilon} - \underset{\sim}{\hat{\varepsilon}'\mathbb{X}}(\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta}) - (r - C\hat{\beta})'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}$$

$$\cdot C(\mathbb{X}'\mathbb{X})^{-1}\underset{\sim}{\mathbb{X}'\hat{\varepsilon}} + (r - C\hat{\beta})'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta})$$

Then, by $\mathbb{X}'\hat{\varepsilon} = 0$,

$$\hat{u}'\hat{u} - \hat{\varepsilon}'\hat{\varepsilon} = (r - C\hat{\beta})'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta})$$

Thus, $SSR_u - SSR_r = \hat{u}'\hat{u} - \hat{\varepsilon}'\hat{\varepsilon} = (r - C\hat{\beta})'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta})$

and $SSR_u = \hat{\varepsilon}'\hat{\varepsilon}$.

From ⊕,

$$\tilde{\beta} - \hat{\beta} = (\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta}).$$

$$(\mathbb{X}'\mathbb{X})(\tilde{\beta} - \hat{\beta}) = C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta}).$$

$$\downarrow (\tilde{\beta} - \hat{\beta})' = (r - C\hat{\beta})'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}C(\mathbb{X}'\mathbb{X})^{-1}$$

$$(\tilde{\beta} - \hat{\beta})'(\mathbb{X}'\mathbb{X})(\tilde{\beta} - \hat{\beta}) = (r - C\hat{\beta})'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}C(\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta})$$

$$= (r - C\hat{\beta})(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta})$$

Thus, $SSR_u - SSR_r = (\tilde{\beta} - \hat{\beta})'(X'X)(\tilde{\beta} - \hat{\beta})$

$$\frac{(SSR_u - SSR_r)/r}{SSR_u / N-K} = \frac{(\tilde{\beta} - \hat{\beta})' X'X (\tilde{\beta} - \hat{\beta})/r}{\hat{\varepsilon}'\hat{\varepsilon} / N-K} \quad \cdots \textcircled{6} \atop \cdots \textcircled{5}$$

From ⑤,

$$\hat{\varepsilon}'\hat{\varepsilon} = (Y - X\hat{\beta})'(Y - X\hat{\beta}) = (Y - X(X'X)^{-1}X'Y)'(Y - X(X'X)^{-1}X'Y)$$

$$= [(I - X(X'X)^{-1}X')Y]'(I - X(X'X)^{-1}X')Y$$

$\downarrow$ $M := I_N - X(X'X)^{-1}X'$

$$= [M(X\beta + \varepsilon)]'[M(X\beta + \varepsilon)] = \varepsilon' M' M \varepsilon = \varepsilon' M \varepsilon.$$

$$E(\hat{\varepsilon}'\hat{\varepsilon}) = E(tr(\hat{\varepsilon}'\hat{\varepsilon})) = E(tr(\hat{\varepsilon}\hat{\varepsilon}')) = E(tr(M\varepsilon\varepsilon' M'))$$

$$= tr(M E(\varepsilon\varepsilon') M') = \sigma^2 tr(MM')$$

Note $MM' = I_N - X(X'X)^{-1}X'$ Thus $tr(MM') = N-K$

$$= \sigma^2 (N-K)$$

Thus, $\hat{\varepsilon}'\hat{\varepsilon} \overset{a}{\sim} \chi^2_{N-K} \cdot (N-K)$ Therefore. $\hat{\varepsilon}'\hat{\varepsilon}/N-K \overset{a}{\sim} \chi^2_{N-K}$

From ⑥

$$(\tilde{\beta} - \hat{\beta})'(X'X)(\tilde{\beta} - \hat{\beta})/r$$

$$= \underbrace{\sqrt{N}(\tilde{\beta} - \hat{\beta})'}_{ⓐ} \underbrace{\left(\frac{1}{N} X'X\right)}_{ⓑ} \sqrt{N}(\tilde{\beta} - \hat{\beta})/r \quad \cdots\cdots \textcircled{7}$$

First of all, ⓑ : $\frac{1}{N} X'X \xrightarrow{a.s} E(x_i x_i')$ by SLLN.

For checking ⑥, we need to show $E(\tilde{\beta}-\beta)$ and $Var(\tilde{\beta})$.

From ④

- $E(\tilde{\beta}) = E\left(\hat{\beta} + (\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\hat{\beta}\right)$

$$= \beta + (\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(r - C\,E(\hat{\beta}))$$

$$= \beta + (\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}(\underbrace{r - C\beta}_{=0}) = \beta.$$

Therefore, $E(\tilde{\beta}) = \beta$ i.e., $E(\tilde{\beta}-\beta) = \beta - \beta = 0$.

$$\tilde{\beta} - \beta = \underbrace{\tilde{\beta} - \hat{\beta}} + \underbrace{\hat{\beta} - \beta} = (\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}C(r - C\hat{\beta}) + (\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon$$

Note $r - C\hat{\beta} = C\beta - C\hat{\beta} = -C(\hat{\beta}-\beta) = -C(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon$

$$= -(\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}C(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon + (\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon$$

$$= [\underbrace{I - (\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}C}_{:=\tilde{M}}](\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon$$

$$= \tilde{M}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon$$

- $Var(\tilde{\beta}-\beta) = Var(\tilde{M}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon)$

$$= E[\tilde{M}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon - E(\tilde{M}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon)][\tilde{M}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon - E(\tilde{M}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon)]'$$

$$= E(\tilde{M}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\varepsilon\varepsilon'\mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}\tilde{M}')$$

$$= \sigma^2 E(\tilde{M}(\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{X}(\mathbb{X}'\mathbb{X})^{-1}\tilde{M}')$$

$$= \sigma^2 \tilde{M}\tilde{M}'(\mathbb{X}'\mathbb{X})^{-1} = \sigma^2\tilde{M}(\mathbb{X}'\mathbb{X})^{-1} \quad (\because \tilde{M}\tilde{M}' = \tilde{M})$$

$$= \sigma^2(\mathbb{X}'\mathbb{X})^{-1} - \sigma^2(\mathbb{X}'\mathbb{X})^{-1}C'(C(\mathbb{X}'\mathbb{X})^{-1}C')^{-1}C(\mathbb{X}'\mathbb{X})^{-1}$$

Therefore, $\sqrt{N}(\tilde{\beta}-\hat{\beta}) \xrightarrow{d} N(E(\tilde{\beta}-\hat{\beta}), Var(\tilde{\beta}-\hat{\beta}))$

$$= N(0, \underbrace{\sigma^2 \tilde{M}(X'X)^{-1}}_{:= \underset{r\times r}{V}})$$

where $\tilde{M} = \underset{r\times r}{I} - (X'X)^{-1}C'(C(X'X)^{-1}C')^{-1}C$

Thus, from ⑰

$$\underset{\downarrow d \; N(0,V)}{\sqrt{N}(\tilde{\beta}-\hat{\beta})'} \; \underset{\downarrow a.s \; E(x_i x_i')}{\left(\frac{1}{N}X'X\right)} \; \underset{\downarrow d \; N(0,V)}{\sqrt{N}(\tilde{\beta}-\hat{\beta})} / r \overset{a}{\sim} \chi^2_{(r)}/r$$

In conclusion,

$$\frac{(SSR_u - SSR_r)/r}{SSR_u / N-k} = \frac{(r-C\hat{\beta})'(C(X'X)^{-1}C')^{-1}(r-C\hat{\beta})/r}{\hat{\varepsilon}'\hat{\varepsilon}/N-k}$$

$$= \frac{(\tilde{\beta}-\hat{\beta})(X'X)(\tilde{\beta}-\hat{\beta})/r}{\hat{\varepsilon}'\hat{\varepsilon}/N-k} \quad \begin{matrix} \overset{a}{\sim} \chi^2_{(r)}/r \\ \overset{a}{\sim} \chi^2_{(N-k)}/N-k \end{matrix}$$

$$\sim F(r, N-k). \qquad //$$

#4 (a) $y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + u_i$

x_{3i}, u_i : correlated x_{2i}, x_{3i} : independent x_{2i}, u_i : independent.

$\bar{y} = \beta_1 + \beta_2 \bar{x}_2 + \beta_3 x_3.$

$y_i - \bar{y} = \beta_2 (x_{2i} - \bar{x}_2) + \beta_3 (x_{3i} - \bar{x}_3) + u_i$

$$\begin{pmatrix} y_1 - \bar{y} \\ \vdots \\ y_K - \bar{y} \end{pmatrix} = \beta_2 \begin{pmatrix} x_{21} - \bar{x}_2 \\ \vdots \\ x_{2K} - \bar{x}_2 \end{pmatrix} + \beta_3 \begin{pmatrix} x_{31} - \bar{x}_3 \\ \vdots \\ x_{3K} - \bar{x}_3 \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_K \end{pmatrix}$$

$$\underset{K\times 1}{\mathbb{Y}} = \beta_2 \cdot \underset{K\times 1}{\mathbb{X}_2} + \beta_3 \underset{K\times 1}{\mathbb{X}_3} + \mathbb{U}.$$

$$= \underbrace{\begin{bmatrix} \underset{K\times 1}{\mathbb{X}_2} & \underset{K\times 1}{\mathbb{X}_3} \end{bmatrix}}_{K\times 2} \begin{bmatrix} \beta_2 \\ \beta_3 \end{bmatrix} + \mathbb{U}. = \underset{K\times 2}{\mathbb{X}} \underset{2\times 1}{\beta} + \underset{K\times 1}{\mathbb{U}}$$

$$\hat{\beta} = (\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Y} = \left[\begin{bmatrix} \mathbb{X}_2' \\ \mathbb{X}_3' \end{bmatrix} [\mathbb{X}_2 \ \ \mathbb{X}_3]\right]^{-1} \begin{bmatrix} \mathbb{X}_2' \\ \mathbb{X}_3' \end{bmatrix} \mathbb{Y} = \begin{bmatrix} \mathbb{X}_2'\mathbb{X}_2 & \mathbb{X}_2'\mathbb{X}_3 \\ \mathbb{X}_3'\mathbb{X}_2 & \mathbb{X}_3'\mathbb{X}_3 \end{bmatrix}^{-1} \begin{bmatrix} \mathbb{X}_2' \\ \mathbb{X}_3' \end{bmatrix} \mathbb{Y}$$

$$= \begin{pmatrix} (\mathbb{X}_2'\mathbb{X}_2)^{-1} & 0 \\ 0 & (\mathbb{X}_3'\mathbb{X}_3)^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{X}_2' \\ \mathbb{X}_3' \end{pmatrix} \mathbb{Y} = \begin{pmatrix} (\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{Y} \\ (\mathbb{X}_3'\mathbb{X}_3)^{-1}\mathbb{X}_3'\mathbb{Y} \end{pmatrix}.$$

Thus, $\hat{\beta}_2 = (\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{Y} = \left[\sum_{i=1}^{K} (x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2)'\right]^{-1} \sum_{i=1}^{K} (x_{2i} - \bar{x}_2)(y_i - \bar{y}).$

① Consistency : $\hat{\beta}_2 \xrightarrow{P} \beta_2.$

$$Pr(|\hat{\beta}_2 - \beta_2| \geq \varepsilon) \leq \frac{E(\hat{\beta}_2 - \beta_2)^2}{\varepsilon^2}$$

$$E(\hat{\beta}_2 - \beta_2)^2 = E(\hat{\beta}_2^2 - \hat{\beta}_2\beta_2 - \beta_2\hat{\beta}_2 + \beta_2^2)$$

Note $E(\hat{\beta}_2) = E((\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{Y}) = E((\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'(\beta_2\mathbb{X}_2 + \beta_3\mathbb{X}_3 + \mathbb{U}))$

$= E((\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{X}_2\beta_2 + (\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{X}_3\beta_3 + (\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{U})$

$= \beta_2$ (By x_{2i}, x_{3i} independent and x_{2i}, u_i independent)

$$= E(\hat{\beta}_2^2) - \beta_2 E(\hat{\beta}_2) - \beta_2 E(\hat{\beta}_2) + E(\beta_2^2) = E(\hat{\beta}_2^2) - \beta_2^2$$

$$E(\hat{\beta}_2^2) - \beta_2^2 = E[\{(X_2'X_2)^{-1} X_2' Y\}^2] - \beta_2^2$$

$$= E[\{(X_2'X_2)^{-1} X_2' (X_2\beta_2 + X_3\beta_3 + U)\}^2] - \beta_2^2$$

$$= E[(\beta_2 + (X_2'X_2)^{-1} X_2' X_3 \beta_2 + (X_2'X_2)^{-1} X_2' U)^2] - \beta_2^2$$

$$= E[\beta_2^2 + \{(X_2'X_2)^{-1} X_2' U\}^2] - \beta_2^2 = E\{(X_2'X_2)^{-1} X_2' U)^2\}.$$

$$= E\{(X_2'X_2)^{-1} X_2' U U' X_2 (X_2'X_2)^{-1}\}$$

$E(u_i^2)=\sigma^2$

$$= \sigma^2 E\{(X_2'X_2)^{-1}\} = \sigma^2 \left[\sum_{i=1}^{k} (x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2)'\right]^{-1}$$

$$= \frac{1}{N} \sum_{i=1}^{N} u_i^2 \cdot \left[\sum_{i=1}^{k} (x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2')\right]^{-1}$$

Therefore, $\Pr(|\hat{\beta}_2 - \beta_2| \geq \varepsilon) \leq \dfrac{E(\hat{\beta}_2 - \beta_2)^2}{\varepsilon^2} = \dfrac{\sum_{i=1}^{N} u_i^2 \left[\sum_{i=1}^{k} (x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2')\right]^{-1}}{N \varepsilon^2} \to 0$

$\therefore \hat{\beta}_2$ is consistent.

② $$\sqrt{N}(\hat{\beta}_2 - \beta_2) = \sqrt{N}\left[(X_2'X_2)^{-1} X_2' (\beta_2 X_2 + \beta_3 X_3 + U) - \beta_2\right]$$

$$= \sqrt{N}\left[(X_2'X_2^{-1}) X_2' X_3 \beta_3 + (X_2'X_2)^{-1} X_2' U\right] = \sqrt{N} (X_2'X_2)^{-1} X_2' U$$

$$= \left(\frac{1}{N} \sum_{i=1}^{N} (x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2)'\right)^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (x_{2i} - \bar{x}_2) u_i$$

Let $\frac{1}{N} \sum_{i=1}^{N} (x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2)' := S_{x_2}$

$$\xrightarrow{d} S_{x_2}^{-1} \cdot N(0, \sigma^2 S_{x_2}) \sim N(0, \sigma^2 S_{x_2}^{-1})$$

by slutsky and CLT.

#4. (b) $y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + u_i$

$= \beta_1 + \beta_2 x_{2i} + v_i$ where $v_i = \beta_3 x_{3i} + u_i$

x_{2i}, v_i independent but x_{3i}, u_i : correlated

$$\left.\begin{array}{l} y_i = \beta_1 + \beta_2 x_{2i} + v_i \\ \bar{y} = \beta_1 + \beta_2 \bar{x}_2 \end{array}\right] \quad y_i - \bar{y} = \beta_2(x_{2i} - \bar{x}_2) + v_i \quad \text{for } i = 1, \dots, K$$

In the same manner, $\mathbb{Y} = \beta_2 \mathbb{X}_2 + \mathbb{V}$ where $\mathbb{V} = \begin{pmatrix} v_1 \\ \vdots \\ v_K \end{pmatrix}$

$$\hat{\beta}_2 = (\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2\mathbb{Y} = \left[\sum_{i=1}^{K}(x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2)'\right]^{-1}\sum_{i=1}^{K}(x_{2i} - \bar{x}_2)(y_i - \bar{y})$$

① consistency : $\hat{\beta}_2 \xrightarrow{p} \beta_2$

$$E(\hat{\beta}_2) = E\left((\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{Y}\right) = E\left[(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'(\beta_2\mathbb{X}_2 + \mathbb{V})\right]$$

$$= E\left[(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{X}_2\beta_2 + (\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{V}\right] = \beta_2$$

So, $E(\hat{\beta}_2 - \beta_2)^2 = E(\hat{\beta}_2^2 - \hat{\beta}_2\beta_2 - \beta_2\hat{\beta}_2 + \beta_2^2) = E(\hat{\beta}_2^2) - \beta_2^2$

$$E(\hat{\beta}_2^2) - \beta_2^2 = E\left[\{(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{Y}\}^2\right] - \beta_2^2$$

$$= E\left[\{(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'(\beta_2\mathbb{X}_2 + \mathbb{V})\}^2\right] - \beta_2^2$$

$$= E\left[(\beta_2 + (\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{V})^2\right] - \beta_2^2$$

$$= E\left\{\beta_2^2 + 2\beta_2(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{V} + [(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{V}]^2\right] - \beta_2^2$$

$$= E\left[(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{V}\right]^2 = E\left\{(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{V}\mathbb{V}'\mathbb{X}_2(\mathbb{X}_2'\mathbb{X}_2)^{-1'}\right\}$$

$$= \sum_{i=1}^{K}(x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2)'\, \underset{\sim}{E}\left[v_i^2(x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2)\right]\sum_{i=1}^{K}(x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2)'$$

$$E(v_i^2) = E(\beta_3 x_{3i} + u_i - \beta_3\bar{x}_3 - \bar{u})^2 = E(\beta_3(x_{3i} - \bar{x}_3) + u_i)^2$$

$$= E\left(\beta_3^2(x_{3i} - \bar{x}_3)^2 + 2\beta_3 u_i(x_{3i} - \bar{x}_3) + u_i^2\right)$$

$= \beta_3^2 E(x_{3i} - \bar{x}_3)^2 + E(u_i^2) \quad (\because E(u_i) = 0)$

$= \beta_3^2 \cdot \frac{1}{N}\sum_{i=1}^{N}(x_{3i} - \bar{x}_3)^2 + \frac{1}{N}\sum_{i=1}^{N} u_i^2 = \frac{1}{N} \cdot$ constant #. $\quad \geq E(u_i^2)$

Therefore, $Pr(|\tilde{\beta}_2 - \beta_2| \geq \varepsilon) \leq \frac{E(\tilde{\beta}_2^2 - \beta_2^2)}{\varepsilon^2} = \frac{\text{constant \#}}{N\varepsilon^2} \longrightarrow 0.$

$\therefore \tilde{\beta}_2$ is consistent.

③ $\sqrt{N}(\hat{\beta}_2 - \beta_2) = \sqrt{N}\left[(X_2'X_2)^{-1}X_2'(\beta_2 X_2 + V) - \beta_2\right]$

$= \sqrt{N}\left[\cancel{(X_2'X_2)^{-1}X_2'X_2\beta_2} + (X_2'X_2)^{-1}X_2'V - \cancel{\beta_2}\right]$

$= \sqrt{N}(X_2'X_2)^{-1}X_2'V.$

$= \left[\frac{1}{N}\sum_{i=1}^{N}(x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2)'\right]^{-1} \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(x_{2i} - \bar{x}_2)V_i$

$\downarrow$ Let $\frac{1}{N}\sum_{i=1}^{N}(x_{2i} - \bar{x}_2)(x_{2i} - \bar{x}_2) := S_{x_2}$

$\xrightarrow{d} \quad S_{x_2}^{-1} \cdot N(0, E(V_i^2) \cdot S_{x_2}) = N(0, E(V_i)^2 S_{x_2}^{-1})$

by slutsky and CLT.

(c) $\quad \hat{\beta}_2$ is more efficient than $\tilde{\beta}_2$.

Because each variance of their asymptotic Normal distribution is $\sigma^2 S_{x_2}^{-1}$ for $\hat{\beta}_2$ and $E(V_i)^2 S_{x_2}^{-1}$ and $\sigma^2 \leq E(V_i)^2$ as we show $\sigma^2 \leq E(V_i^2)$ at (b).

#4. (d) Since $\hat{\beta}_2$ and $\tilde{\beta}_2$ are consistent,

x_{2i} is independent with (x_{3i}, u_i)

However, from $E(v_i^2)$,

$$E(v_i^2) = \beta_3^2 \cdot \frac{1}{N} \sum_{i=1}^{N} (x_{3i} - \bar{x}_3)^2 + E(u_i^2)$$

Thus, $E(v_i^2) \geqq E(u_i^2)$, which shows that

x_{2i} is independent u_i but x_{3i} and v_i may be correlated. //

#6. $y_i^S = \beta_1^S + \beta_2^S P_i + u_i^S$

$y_i^D = \beta_1^D + \beta_2^D P_i + u_i^D$

(a) $E(y_i^S | P_i) = \beta_1^S + \beta_2^S P_i \qquad E(y_i^D | P_i) = \beta_1^D + \beta_2^D P_i$

At the equilibrium $y_i = y_i^S = y_i^D$,

$\beta_1^S + \beta_2^S P_i + u_i^S = \beta_1^D + \beta_2^D P_i + u_i^D$

Thus,

$$P_i = \frac{\beta_1^S - \beta_1^D}{\beta_2^S - \beta_2^D} + \frac{u_i^D - u_i^S}{\beta_2^S - \beta_2^D}$$

For the existence of P_i, $\beta_2^S \neq \beta_2^D$

(b) When $y_i = y_i^S = y_i^D$,

① $y_i = \beta_1^S + \beta_2^S P_i + u_i^S \rightarrow \lambda y_i = \lambda\beta_1^S + \lambda\beta_2^S P_i + \lambda u_i^S$

② $y_i = \beta_1^D + \beta_2^D P_i + u_i^D \rightarrow (1-\lambda) y_i = (1-\lambda)\beta_1^D + (1-\lambda)\beta_2^D P_i + (1-\lambda) u_i^D$

$$y_i = (\lambda\beta_1^S + (1-\lambda)\beta_1^D) + (\lambda\beta_2^S + (1-\lambda)\beta_2^D) P_i + (\lambda u_i^S + (1-\lambda) u_i^D)$$

i.e., $y_i = \alpha + \beta P_i + \varepsilon_i$ where $\alpha = \lambda\beta_1^S + (1-\lambda)\beta_1^D$

$\beta = \lambda_2\beta_2^S + (1-\lambda)\beta_2^D$

$\varepsilon_i = \lambda u_i^S + (1-\lambda) u_i^D$

Then, $\bar{y} = \alpha + \beta\bar{P}$ so $y_i - \bar{y} = \beta(P_i - \bar{P}) + \varepsilon_i$

$$\hat{\beta} = \frac{\sum_{i=1}^n (P_i - \bar{P})(y_i - \bar{y})}{\sum_{i=1}^n (P_i - \bar{P})^2} = \frac{\sum_{i=1}^n (P_i - \bar{P})[\beta(P_i - \bar{P} + \varepsilon_i]}{\sum_{i=1}^n (P_i - \bar{P})^2} = \beta + \frac{\sum_{i=1}^n (P_i - \bar{P})\varepsilon_i}{\sum_{i=1}^n (P_i - \bar{P})^2}$$

$$\operatorname{plim} \hat{\beta} = \operatorname*{plim}_{n\to\infty} \beta + \frac{\sum_{i=1}^{n}(P_i-\bar{P})\varepsilon_i}{\sum_{i=1}^{n}(P_i-\bar{P})^2} = \operatorname*{plim}_{n\to\infty} \beta + \frac{\frac{1}{n}\sum_{i=1}^{n}P_i\cdot\varepsilon_i - \bar{P}\cdot\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i}{\frac{1}{n}\sum_{i=1}^{P}(P_i-\bar{P})^2}$$

$$= \beta + \frac{\operatorname{cov}(P_i,\varepsilon_i)}{\operatorname{Var}(P_i)}$$

(c) From ②, $y_i = \beta_1^S + \beta_2^S P_i + u_i^S$

$\bar{y} = \beta_1^S + \beta_2^S \bar{P}$.

$$\hat{\beta}_2^S = \frac{\sum_{i=1}^{n}(P_i-\bar{P})(y_i-\bar{y})}{\sum_{i=1}^{n}(P_i-\bar{P})^2} = \beta_2^S + \frac{\frac{1}{n}\sum_{i=1}^{n}(P_i-\bar{P})u_i^S}{\frac{1}{n}\sum_{i=1}^{n}(P_i-\bar{P})^2}$$

$$\operatorname*{plim}_{n\to\infty} \hat{\beta}_2^S = \beta_2^S + \frac{\operatorname{cov}(P_i,u_i^S)}{\operatorname{Var}(P_i}$$

For the equality $\operatorname{plim} \hat{\beta}_2^S = \beta_2^S$, $\operatorname{cov}(P_i, u_i^S) = 0$.

i.e., P_i and u_i^S should be No correlated. //