1

a

$$
\begin{aligned}
E\begin{pmatrix} u_i \\ x_{2i}u_i \\ z_iu_i \end{pmatrix} &= \begin{pmatrix} E\left[u_i\right] \\ E\left[x_{2i}u_i\right] \\ E\left[z_iu_i\right] \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ E\left[\left(x_i^* + v_{2i}\right)u_i\right] \\ E\left[z_iu_i\right] \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ E\left[x_i^*u_i\right] + E\left[v_{2i}u_i\right] \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
E\begin{pmatrix} 1 & x_i & z_i \\ x_{2i} & x_{2i}x_i & x_{2i}z_i \\ z_i & z_ix_i & z_i^2 \end{pmatrix} &= \begin{pmatrix} 1 & E\left[x_i\right] & E\left[z_i\right] \\ E\left[x_{2i}\right] & E\left[x_{2i}x_i\right] & E\left[x_{2i}z_i\right] \\ E\left[z_i\right] & E\left[z_ix_i\right] & E\left[z_i^2\right] \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & E\left[\left(x_i^* + v_{2i}\right)\left(x_i^* + v_i\right)\right] & E\left[\left(x_i^* + v_{2i}\right)z_i\right] \\ 0 & E\left[z_i\left(x_i^* + v_i\right)\right] & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}.
\end{aligned}
$$

which is invertible.

b The first condition is important because it is THE moment condition that we use for identification. The second condition is important because it is necessary condition for us to recover estimates for each β uniquely and separately.

f

i. Note that $P_Z = Z\left(Z'Z\right)^{-1}Z'$.

$$
\begin{aligned}
\hat{\beta}_{2SLS} &= \left(X'P_ZX\right)^{-1}X'P_ZY \\
&= \left(X'Z\left(Z'Z\right)^{-1}Z'X\right)^{-1}X'Z\left(Z'Z\right)^{-1}Z'Y.
\end{aligned}
$$

ii. X is $N \times K$. When the number of the instrumental variables and the number of regressors in the model are the same, then $J = K$, and Z is also $N \times K$. Therefore $X'Z$ and $Z'X$ are both $K \times K$. If the relevance condition is statisfied, which is $E\left[X'Z\right] \neq 0$, then $X'Z$ and $Z'X$ are invertible with probability equal to 1 as N goes to infinity.

iii. It does not because when $J > K$, then $X'Z$ is $K \times J$ that is not square matrix, and of course is not invertible.

iv. When $J = K$, the results in (ii) implies $(X'Z)^{-1}$ and $(Z'X)^{-1}$ exist. Use this result, we have

$$\begin{aligned}
\hat{\beta}_{2SLS} &= (X'P_Z X)^{-1} X'P_Z Y \\
&= \left(X'Z\,(Z'Z)^{-1}\,Z'X\right)^{-1} X'Z\,(Z'Z)^{-1}\,Z'Y \\
&= (Z'X)^{-1}\left((Z'Z)^{-1}\right)^{-1} \underbrace{(X'Z)^{-1}\,X'Z}_{I_K}\,(Z'Z)^{-1}\,Z'Y \\
&= (Z'X)^{-1} \underbrace{(Z'Z)\,(Z'Z)^{-1}}_{I_K} Z'Y \\
&= (Z'X)^{-1}\,Z'Y \\
&= \hat{\beta}_{IV}.
\end{aligned}$$