#2. a)

$$y_i = x_{1i}'\beta_1 + x_{2i}'\beta_2 + u_i$$

$$= (x_{1i}' \; x_{2i}') \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + u_i$$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} x_{11}' & x_{21}' \\ \vdots & \vdots \\ x_{1N}' & x_{2N}' \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} \quad \text{i.e.,} \quad \mathbb{Y} = (\mathbb{X}_1 \; \mathbb{X}_2) \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \mathbb{U}.$$

sol) WTS: $\hat{\beta}_1 = (\mathbb{X}_1' M_{\mathbb{X}_2} \mathbb{X}_1)^{-1} \mathbb{X}_1' M_{\mathbb{X}_2} \mathbb{Y}$.

$$\mathbb{Y} = \mathbb{X}_1\hat{\beta}_1 + \mathbb{X}_2\hat{\beta}_2 + \hat{\mathbb{U}}$$

↓ multiply both LHS and RHS by $\mathbb{X}_1' M_{\mathbb{X}_2}$

$$\mathbb{X}_1' M_{\mathbb{X}_2} \mathbb{Y} = \mathbb{X}_1' M_{\mathbb{X}_2} \mathbb{X}_1 \hat{\beta}_1 + \underbrace{\mathbb{X}_1' M_{\mathbb{X}_2} \mathbb{X}_2 \hat{\beta}_2}_{①} + \underbrace{\mathbb{X}_1' M_{\mathbb{X}_2} \hat{\mathbb{U}}}_{②}$$

$$= \mathbb{X}_1' M_2 \mathbb{X}_1 \hat{\beta}_1$$

Since ① $M_{\mathbb{X}_2}\mathbb{X}_2 = (I_N - \mathbb{X}_2(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2')\mathbb{X}_2$

$$= \mathbb{X}_2 - \mathbb{X}_2(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{X}_2 = \mathbb{X}_2 - \mathbb{X}_2 = 0.$$

② $\mathbb{X}_1' M_{\mathbb{X}_2} \mathbb{U} = \mathbb{X}_1' [I_N - \mathbb{X}_2(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'] \hat{\mathbb{U}}$

$$= \underbrace{\mathbb{X}_1 \hat{\mathbb{U}}}_{=0} - \mathbb{X}_1'\mathbb{X}_2(\mathbb{X}_2'\mathbb{X}_2)^{-1}\underbrace{\mathbb{X}_2'\hat{\mathbb{U}}}_{=0}$$

Therefore, $\hat{\beta}_1 = (\mathbb{X}_1' M_{\mathbb{X}_2} \mathbb{X}_1)^{-1} \mathbb{X}_1' M_{\mathbb{X}_2} \mathbb{Y}$. //

#2. (b) $x_{1i} = x_{2i}'\pi_2 + v_i$

$$x_{1i} = x_{2i}'\hat{\pi}_2 + \hat{v}_i$$

$$\mathbb{X}_1 = \mathbb{X}_2\hat{\pi}_2 + \hat{\mathbb{V}} \qquad \hat{\pi}_2 = (\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2\hat{\mathbb{V}}.$$

↓ Multiply $\mathbb{X}_1'M_{\mathbb{X}_2}$ to the auxiliary regression.

$$\mathbb{X}_1'M_{\mathbb{X}_2}\mathbb{X}_1 = \mathbb{X}_1'M_{\mathbb{X}_2}\mathbb{X}_2\hat{\pi}_2 + \mathbb{X}_1'M_{\mathbb{X}_2}\hat{\mathbb{V}}.$$

$$= \mathbb{X}_1'[I_N - \mathbb{X}_2(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2']\mathbb{X}_2\hat{\pi}_2$$

$$+ \mathbb{X}_1'[I_N - \mathbb{X}_2(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2']\hat{\mathbb{V}}$$

$$= \mathbb{X}_1'\mathbb{X}_2\hat{\pi}_2 - \cancel{\mathbb{X}_1'\mathbb{X}_2(\mathbb{X}_2'\mathbb{X}_2)^{-1}\mathbb{X}_2'\mathbb{X}_2\hat{\pi}_2}$$

$$+ \mathbb{X}_1'\hat{\mathbb{V}} - \mathbb{X}_1'\mathbb{X}_2(\mathbb{X}_2'\mathbb{X}_2)^{-1}\underbrace{\mathbb{X}_2'\hat{\mathbb{V}}}_{=0}.$$

$$= \mathbb{X}_1'\hat{\mathbb{V}}. \qquad \text{Thus, } \hat{\mathbb{V}} = M_{\mathbb{X}_2}\mathbb{X}_1$$

$$\hat{\beta}_1 = (\hat{\mathbb{V}}'\hat{\mathbb{V}})^{-1}\hat{\mathbb{V}}'\mathbb{Y} = [(M_{\mathbb{X}_2}\mathbb{X}_1)'M_{\mathbb{X}_2}\mathbb{X}_1]^{-1}(M_{\mathbb{X}_2}\mathbb{X}_1)'\mathbb{Y}$$

$$= (\mathbb{X}_1'M_{\mathbb{X}_2}'M_{\mathbb{X}_2}\mathbb{X}_1)^{-1}\mathbb{X}_1'M_{\mathbb{X}_2}'\mathbb{Y}$$

$$= (\mathbb{X}_1'M_{\mathbb{X}_2}\mathbb{X}_1)\mathbb{X}_1'M_{\mathbb{X}_2}\mathbb{Y} \qquad (\text{by } M_{\mathbb{X}_2}'M_{\mathbb{X}_2} = M_{\mathbb{X}_2},\ M_{\mathbb{X}_2}' = M_{\mathbb{X}_2})$$

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