<HW9> 522A, Jeong Yeol kim.

#1. $\min \hat{g}(b)' W \hat{g}(b)$ where $\hat{g}(b) = \frac{1}{N}\sum_{i=1}^{N} z_i (y_i - x_i' b)$

F.O.C : $0 = \frac{1}{N}\sum_{i=1}^{N} z_i x_i' \cdot W \cdot \frac{1}{N}\sum_{i=1}^{N} z_i (y_i - x_i' b)$

$$\underset{K\times J}{\hat{G}' = \frac{1}{N}\sum_{i=1}^{N} x_i z_i'} \xrightarrow{p} \underset{K\times J}{E(x_i z_i')} = G' = E\begin{pmatrix} x_{1i} \\ \vdots \\ x_{Ki}\end{pmatrix}(z_{1i} \cdots z_{Ji})$$

$$= E\begin{pmatrix} x_{1i}z_{1i} & \cdots & x_{1i}z_{Ji} \\ \vdots & & \\ x_{Ki}z_{1i} & \cdots & x_{Ki}z_{Ji}\end{pmatrix}$$

Consider $\frac{1}{N}\sum_{i=1}^{N} \underbrace{z_i}_{J\times 1} (\underset{1\times 1}{y_i} - \underset{1\times K}{x_i'} \underset{K\times 1}{b})$ $\quad z_i := \begin{pmatrix} z_{1i} \\ z_{2i} \\ \vdots \\ z_{Ji}\end{pmatrix}$.

$$0 = \hat{G}' W \hat{g}(b) = \hat{G}' W \frac{1}{N}\sum_{i=1}^{N} z_i (y_i - x_i' b)$$

$$= \hat{G}' W \begin{pmatrix} \frac{1}{N}\sum_{i=1}^{N} z_{1i}(y_i - x_i' b) \\ \vdots \\ \frac{1}{N}\sum_{i=1}^{N} z_{Ji}(y_i - x_i' b)\end{pmatrix}.$$

$$\xrightarrow{p} \underset{K\times J}{G'} \underset{J\times J}{W} \underset{J\times 1}{\begin{pmatrix} E[z_{1i}(y_i - x_i')\hat{b}] \\ \vdots \\ E[z_{Ji}(y_i - x_i')\hat{b}]\end{pmatrix}}.$$

$$= \begin{pmatrix} E(x_{1i}z_{1i}) & \cdots & E(x_{1i}z_{Ji}) \\ \vdots & & \vdots \\ E(x_{Ki}z_{1i}) & \cdots & E(x_{Ki}z_{Ji})\end{pmatrix} \begin{pmatrix} W_{11} & \cdots & W_{1J} \\ \vdots & & \\ W_{J1} & \cdots & W_{JJ}\end{pmatrix} \begin{pmatrix} E[z_{1i}(y_i - x_i')\hat{b}] \\ \vdots \\ E[z_{Ji}(y_i - x_i')\hat{b}]\end{pmatrix}$$

$$= \begin{pmatrix} E(x_{1i} z_{1i}) \cdot W_{11} + \cdots + E(x_{1i} z_{Ji}) \cdot W_{J1} & \cdots & E(x_{1i} z_{1i}) \cdot W_{1J} + \cdots + E(x_{1i} z_{Ji}) \cdot W_{JJ} \\ \vdots & & \\ E(x_{Ki} z_{1i}) W_{11} + \cdots + E(x_{Ki} z_{Ji}) W_{J1} & \cdots & E(x_{Ki} z_{1i}) W_{1J} + \cdots + E(x_{Ki} z_{Ji}) W_{JJ} \end{pmatrix} \begin{pmatrix} E[z_{1i}(y_i - x_i' \hat{\delta})] \\ \vdots \\ E[z_{Ji}(y_i - x_i' \hat{\delta})] \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{J} E(x_{1i} z_{ji}) W_{j1} & \cdots & \sum_{j=1}^{J} E(x_{1i} z_{ji}) W_{jJ} \\ \vdots & & \\ \sum_{j=1}^{J} E(x_{Ki} z_{ji}) W_{j1} & \cdots & \sum_{j=1}^{J} E(x_{Ki} z_{ji}) W_{jJ} \end{pmatrix} \begin{pmatrix} E(z_{1i} [y_i - x_i' \hat{\delta}]) \\ \vdots \\ E(z_{Ji} [y_i - x_i' \hat{\delta}]) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{j=1}^{J} E(x_{1i} z_{ji}) W_{j1} \cdot E[z_{1i}(y_i - x_i' \hat{\delta})] + \cdots + \sum_{j=1}^{J} E(x_{1i} z_{ji}) W_{jJ} \cdot E(z_{Ji} [y_i - x_i' \hat{\delta}]) \\ \vdots \\ \sum_{j=1}^{J} E(x_{Ki} z_{ji}) W_{j1} \cdot E[z_{1i}(y_i - x_i' \hat{\delta})] + \cdots + \sum_{j=1}^{J} E(x_{Ki} z_{ji}) W_{jJ} \cdot E(z_{Ji} [y_i - x_i' \hat{\delta}]) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \sum_{j=1}^{J} \sum_{m=1}^{J} E(x_{1i} z_{ji}) W_{jm} \cdot E[z_{mi} [y_i - x_i' \hat{\delta}]] \\ \vdots \\ \sum_{j=1}^{J} \sum_{m=1}^{J} E(x_{Ki} z_{ji}) W_{jm} \cdot E[z_{mi} [y_i - x_i' \hat{\delta}]] \end{pmatrix}}_{K \times 1} = \underset{K \times 1}{①}$$

//

#2. (a) Asymptotic variance-covariance matrix of GMM.

$$\sqrt{N}(\hat{\beta}_{GMM}-\beta)=\sqrt{N}(X'ZWZ'X)^{-1}X'ZWZ'u$$

$$=\Big(\underbrace{\tfrac{1}{N}X'Z}_{②}W\underbrace{\tfrac{1}{N}Z'X}_{②'}\Big)^{-1}\underbrace{\Big(\tfrac{1}{N}X'Z\Big)}_{②}W\underbrace{\tfrac{1}{\sqrt{N}}Z'u}_{①}$$

① $\frac{1}{\sqrt{N}}Z'u \xrightarrow{d} N(0, E(u_i^2 z_i z_i'))$ by CLT.

② $\frac{1}{N}X'Z \xrightarrow{p} E(x_i z_i')$

$\frac{1}{N}Z'X \xrightarrow{p} E(z_i x_i')$

$$\xrightarrow{d}\big(E(x_i z_i')WE(z_i x_i')\big)^{-1}E(x_i z_i')W\cdot N(0, E(u_i^2 z_i z_i'))$$

let $G := E(z_i x_i')$

$$=(G'WG)^{-1}G'W\cdot N(0, E(u_i^2 z_i z_i'))$$

$$\Omega = E[g_i(\beta)g_i'(\beta)] = E[z_i(y_i-x_i'\beta)\cdot(y_i-x_i'\beta)'z_i']$$

$$=E[z_i u_i u_i' z_i'] = E(u_i^2 z_i z_i')$$

$$=N\big(0, (G'WG)^{-1}G'W\Omega WG(G'WG)^{-1'}\big)$$

$W=W'$, $(G'WG)^{-1'}=(G'WG)^{-1}$

$$=N\big(0, (G'WG)^{-1}G'W\Omega WG(G'WG)^{-1}\big).$$ //

(b) When $W = \Omega^{-1}$,

$$(G'WG)^{-1} G'W\Omega WG (G'WG)^{-1}$$
$$= (G'\Omega^{-1}G)^{-1} G'\Omega^{-1}\Omega\Omega^{-1}G (G'\Omega^{-1}G)^{-1}$$
$$= (G'\Omega^{-1}G)^{-1} G'\Omega^{-1}G (G'\Omega^{-1}G)^{-1} = (G'\Omega^{-1}G)^{-1}. \ //$$

(c)

$$(G'WG)^{-1} G'W\Omega WG(G'WG)^{-1} - (G'\Omega^{-1}G)^{-1}$$

$\downarrow$ Let $(G'WG) := A$

$$= A^{-1} G'W\Omega WG A^{-1} - A^{-1}A (G'\Omega^{-1}G)^{-1} A A^{-1}$$
$$= A^{-1} \left(G'W\Omega WG - A (G'\Omega^{-1}G)^{-1} A \right) A^{-1}$$
$$= A^{-1} \left(\underset{\sim}{G'W}\Omega \underline{\underline{WG}} - \underset{\sim}{G'W}G (G'\Omega^{-1}G)^{-1} G'\underline{\underline{WG}} \right) A^{-1}$$
$$= A^{-1} G'W \left(\Omega - G(G'\Omega^{-1}G)^{-1}G' \right) WG A^{-1}$$
$$= (G'WG)^{-1}G'W \left(\Omega - G(G'\Omega^{-1}G)^{-1}G' \right) WG (G'WG)^{-1}. \ //$$

#2. d) Let $\Omega := T'T$. Then, $\Omega^{-1} = T^{-1}(T^{-1})'$

Let $C = (T^{-1})'G$.

$$(G'WG)^{-1}G'W\left(\Omega - G(G'\Omega^{-1}G)^{-1}G'\right)WG(G'WG)^{-1}$$

$$= A^{-1}G'W\left(T'T - G(G'T^{-1}(T^{-1})'G)^{-1}G'\right)WGA^{-1}$$

$$= A^{-1}G'WT'\left(I_J - (T')^{-1}G(G'T^{-1}(T^{-1})'G)^{-1}G'T^{-1}\right)TWGA^{-1}$$

By $C = (T^{-1})'G$

$$= A^{-1}G'WT'\left(I_J - C(C'C)^{-1}C'\right)TWGA^{-1}$$

$$= (G'WG)^{-1}G'WT'\left(I_J - C(C'C)^{-1}C'\right)TWG(G'WG)^{-1}. \quad // \quad \cdots ⊛$$

(e) $G = E(z_i x_i')$: $J\times K$ matrix. W : $J\times J$ matrix.

$(G'WG)^{-1}G'WT'$: $K\times J$ matrix

$\underbrace{K\times J \; J\times J \; J\times K}_{K\times K} \; \underbrace{K\times J}_{K\times J} \; J\times J \; J\times J$

$TWG(G'WG)^{-1} = [(G'WG)^{-1}G'WT']'$: $J\times K$ matrix.

Let $(G'WG)^{-1}G'WT' := B$.

Then, ⊛ $= B\left(I_J - C(C'C)^{-1}C'\right)B'$

By the definition of "positive semi-definite",

check $\text{rank}\left(I_J - C(C'C)^{-1}C'\right)$

$$\text{rank}\left(I_J - C(C'C)^{-1}C'\right) = \text{rank}\left(\text{trace}\left[(I_J) - C(C'C)^{-1}C'\right]\right)$$

$$= J - \text{trace}\left(C(C'C)^{-1}C'\right) = J - \text{trace}\left(\underbrace{(C'C)^{-1}C'C}_{=I_K}\right)$$

$$= J - K \geqq 0.$$

Therefore, ⊛ is positive semi-definite. //

#3. (a) $y_i^S = y_i^D$

$$\beta_1^S + \beta_2^S p_i + \beta_3^S x_i + u_i^S = \beta_1^D + \beta_2^D p_i + \beta_3^D z_i + u_i^D$$

$$(\beta_2^S - \beta_2^D) p_i = \beta_1^D - \beta_1^S + \beta_3^D z_i - \beta_3^S x_i + u_i^D - u_i^S$$

$$\therefore p_i = \frac{\beta_1^D - \beta_1^S}{\beta_2^S - \beta_2^D} + \frac{u_i^D - u_i^S}{\beta_2^S - \beta_2^D} + \frac{\beta_3^D z_i - \beta_3^S x_i}{\beta_2^S - \beta_2^D}$$

$$p_i = \frac{y_i^S - \beta_1^S - \beta_3^S x_i - u_i^S}{\beta_2^S} = \frac{y_i^D - \beta_1^D - \beta_3^D z_i - u_i^D}{\beta_2^D}$$

$$\beta_2^D y_i^S - \beta_2^D \beta_1^S - \beta_2^D \beta_3^S x_i - \beta_2^D u_i^S = \beta_2^S y_i^D - \beta_2^S \beta_1^D - \beta_2^S \beta_3^D z_i - \beta_2^S u_i^D$$

$$(\beta_2^D - \beta_2^S) y_i = (\beta_2^D \beta_1^S - \beta_2^S \beta_1^D) + \beta_2^D \beta_3^S x_i - \beta_2^S \beta_3^D z_i + \beta_2^D u_i^S - \beta_2^S u_i^D$$

$$\therefore y_i = \frac{\beta_2^D \beta_1^S - \beta_2^S \beta_1^D}{\beta_2^D - \beta_2^S} + \frac{\beta_2^D \beta_3^S}{\beta_2^D - \beta_2^S} x_i - \frac{\beta_2^S \beta_3^D}{\beta_2^D - \beta_2^S} z_i + \frac{\beta_2^D u_i^S - \beta_2^S u_i^D}{\beta_2^D - \beta_2^S}$$

(b) supply function.

order condition : $\begin{cases} E(x_i u_i^S) = 0 \\ E(z_i u_i^S) = 0 \end{cases}$ and $\underset{1, p_i, z_i}{J} \geq \underset{1, z_i, x_i}{K}$

· $E(x_i u_i^S) = E(E(x_i u_i^S \mid x_i)) = E(x_i E(u_i^S \mid x_i)) = 0.$

· $3 \geq 3$

→ order condition holds.

#3. (b) Supply curve

Explanary variable : $\begin{pmatrix} 1 \\ P_i \\ x_i \end{pmatrix}$ Instrumental variable : $\begin{pmatrix} 1 \\ z_i \\ x_i \end{pmatrix}$

$\therefore$ Order condition : $E\begin{pmatrix} 1 \\ x_i \\ z_i \end{pmatrix} u_i^S = 0$ and $J \geqq K$:

$\Downarrow$

$E(x_i u_i^S) = 0$ and $E(z_i u_i^S) = 0$ "Hold.^"

(c) Demand curve

Explanary variable : $\begin{pmatrix} 1 \\ P_i \\ z_i \end{pmatrix}$ Instrumental variable : $\begin{pmatrix} 1 \\ x_i \\ z_i \end{pmatrix}$

$$\text{Rank}\left(E \begin{pmatrix} 1 \\ x \\ z_i \end{pmatrix} (1 \ \ P_i \ \ z_i) \right) = \text{Rank}\, E \begin{pmatrix} 1 & P_i & z_i \\ x_i & x_i P_i & x_i z_i \\ z_i & z_i P_i & z_i^2 \end{pmatrix} = \text{Rank} \begin{pmatrix} 1 & EP_i & Ez_i \\ Ex_i & Ex_i P_i & Ex_i z_i \\ Ez_i & Ez_i P_i & Ez_i^2 \end{pmatrix}$$

$$P_i = \underbrace{\frac{\beta_1^D - \beta_1^S}{\beta_2^S - \beta_2^D}}_{a} + \frac{u_i^D - u_i^S}{\beta_2^S - \beta_2^D} + \underbrace{\frac{\beta_3^D}{\beta_2^S - \beta_2^D}}_{b} z_i + \underbrace{\frac{\beta_3^S}{\beta_2^D - \beta_2^S}}_{c} x_i$$

$$= \text{Rank} \begin{pmatrix} 1 & a + bE(z_i) + cE(x_i) & E(z_i) \\ E(x_i) & aE(x_i) + bE(x_i z_i) + cE(x_i^2) & E(x_i z_i) \\ E(z_i) & aE(z_i) + bE(z_i^2) + cE(z_i x_i) & E(z_i^2) \end{pmatrix}$$

$$= c \cdot \text{Rank} \begin{pmatrix} 1 & E(x_i) & E(z_i) \\ E(x_i) & E(x_i^2) & E(x_i z_i) \\ E(z_i) & E(z_i x_i) & E(z_i^2) \end{pmatrix}$$

$$= c \cdot cov(x_i, z_i) = \frac{\beta_3^S}{\beta_2^D - \beta_2^S} cov(x_i, z_i) \geqq K$$

$\therefore$ rank condition : $\dfrac{\beta_3^S}{\beta_2^D - \beta_2^S} cov(x_i, z_i) \geqq K$.

#3. (d) Demand curve.

explamary variable. $\begin{pmatrix} 1 \\ p_i \\ z_i \end{pmatrix}$. Instrumental variable $\begin{pmatrix} 1 \\ x_i \\ z_i \end{pmatrix}$

$$y_i = \frac{\beta_2^D \beta_3^S}{\beta_2^D - \beta_2^S} x_i + \frac{\beta_2^S \beta_3^D}{\beta_2^S - \beta_2^D} z_i + \frac{\beta_2^D \beta_1^S - \beta_2^S \beta_1^D}{\beta_2^D - \beta_2^S} + \frac{\beta_2^D}{\beta_2^D - \beta_2^S} u_i^S + \frac{\beta_2^S}{\beta_2^D - \beta_2^S} u_i^D$$

$$= b_0 + b_1 x_i + b_2 z_i + b_3 u_i^S + b_4 \cdot u_i^D.$$

$$\hat{\beta} = \begin{pmatrix} \hat{b}_0 \\ \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = (\mathbb{X}' \mathbb{Z} W \mathbb{Z}' \mathbb{X})^{-1} \mathbb{X}' \mathbb{Z} W \mathbb{Z}' \mathbb{Y}.$$

$$= \left(\frac{1}{N} \mathbb{X}' \mathbb{Z} W \frac{1}{N} \mathbb{Z}' \mathbb{X}\right)^{-1} \frac{1}{N} \mathbb{X}' \mathbb{Z} W \frac{1}{N} \mathbb{Z}' \mathbb{U} + \beta.$$

$$\xrightarrow{p} \beta + \left(E(x_i z_i') E(u_i^2 z_i z_i')^{-1} E(z_i x_i')\right)^{-1} E(x_i z_i') E(u_i^2 z_i z_i')^{-1} E(z_i u_i)$$

$$\sqrt{N}(\hat{\beta} - \beta) = \sqrt{N} (\mathbb{X}' \mathbb{Z} W \mathbb{Z}' \mathbb{X})^{-1} \mathbb{X}' \mathbb{Z} W \mathbb{Z}' \mathbb{U}$$

$$= \left(\frac{1}{N} \mathbb{X}' \mathbb{Z} W \frac{1}{N} \mathbb{Z}' \mathbb{X}\right)^{-1} \frac{1}{N} \mathbb{X}' \mathbb{Z} W \cdot \frac{1}{\sqrt{N}} \mathbb{Z}' \mathbb{U}$$

$$\xrightarrow{d} \left[E(x_i z_i') E(\hat{u}_i^2 z_i z_i')^{-1} E(z_i x_i')\right]^{-1} E(x_i z_i') E(u_i^2 z_i z_i')^{-1} \cdot N(0, \Omega)$$

$$= N\left(0, \left(E(x_i z_i') E(\hat{u}_i^2 z_i z_i')^{-1} E(z_i x_i')\right)^{-1}\right)$$