- Chebyshev's WLLN

$$Y_n := \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_i)$$

$$\underbrace{E\{Y_n^2\}}_{\text{the 2nd moment of } Y_n} \longrightarrow 0 \quad \Rightarrow \quad Y_n \xrightarrow{p} 0$$

$$E\{Y_n^2\} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E\{(X_i-\mu_i)(X_j-\mu_j)\}$$

If $cov(X_i, X_j) = 0$,

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{n} E\{(X_i-\mu_i)(X_j-\mu_j)\} \to \text{fixed}}{n^2 \to \infty} \longrightarrow 0 \quad \text{as } n \to \infty$$

If $\exists$ covariance $= \alpha$

$$= \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} \underbrace{cov(X_i, X_j)}_{=\alpha} = \frac{\alpha n(n-1)}{n^2} \nrightarrow 0$$

$$\left(\text{By } \sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha = \sum_{i=1}^{n} \alpha(n-1) = n\cdot\alpha(n-1)\right)$$

Therefore, for WLLN, cov should be 0.

Thm 5.1 — < kolmogrov's SLLN >

If $\{X_i\}$ is an independent sequence of random variables and $E(X_i) = \mu_i$, $Var(X_i) = \sigma_i^2$ and $\sum_{i=1}^{\infty} \frac{\sigma_i^2}{i^2} < \infty$,

then $\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_i) \xrightarrow{a.s.} 0$ as $n \to \infty$

★★★

※ Central Limit Theorem is used to show convergence in distribution to a normal random variable (vector)

↓

Lindberg's CLT

Thm 5.13 < Lindeberg's Central Limit Theorem >

Let $\{X_{ni}\}$ be independent with means $\{\mu_{ni}\}$ and variances $\{\sigma_{ni}^2\}$

Let $C_n^2 = \sum_{i=1}^{k_n} \sigma_{ni}^2$ and $Y_n = \frac{1}{C_n} \sum_{i=1}^{k_n} (X_{ni} - \mu_{ni})$

If $\lim_{n\to\infty} \frac{1}{C_n^2} \sum_{i=1}^{k_n} E\{(X_{ni}-\mu_{ni})^2 \, 1\{|X_{ni}-\mu_{ni}| > \eta \cdot C_n\}\}$, ... ⊛

then $Y_n \xrightarrow{d} N(0,1)$

Note $Y_n = \frac{1}{C_n} \sum_{i=1}^{k_n} (X_{ni} - \mu_{ni})$

X_{ni} can be $X_{n1}, X_{n2}, \dots, X_{nk_n}$

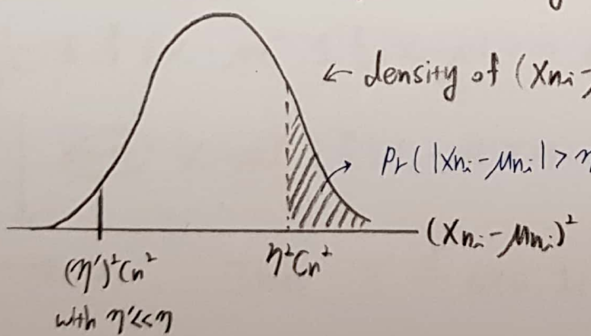
$$\frac{1}{\sqrt{k_n}} \sum_{i=1}^{k_n} (X_{ni}-\mu_{ni}) = \frac{C_n}{\sqrt{k_n}} \cdot \frac{1}{C_n} \sum_{i=1}^{k_n} (X_{ni}-\mu_{ni}) = \frac{\sqrt{C_n^2}}{\sqrt{k_n}} \cdot Y_n$$

$$= \underbrace{\sqrt{\frac{1}{k_n} \sum_{i=1}^{k_n} \sigma_{ni}^2}}_{\text{This is constant}} \cdot Y_n$$

→ If $C_n \to \infty$, $1\{|X_{ni}-\mu_{ni}| > \eta \cdot C_n\}$ will be very small.

i.e., finally it become "0".

→ If we choose some η, there might be a probability like the below graph.



Whatever the $Pr(|X_{ni}-\mu_{ni}| > \eta C_n)$ is,

as $C_n \to \infty$,

$1\{|X_{ni}-\mu_{ni}| > \eta \cdot C_n\} \longrightarrow 0$

→ If $k_n = n$, $\underline{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_{ni} - \mu_{ni})} \to 0$ by chebyshev's WLLN

↓

$$\underset{\substack{\uparrow \\ \text{blow-up factor}}}{\sqrt{n}} \Big(\underbrace{\frac{1}{n} \sum_{i=1}^{n} (X_{ni} - \mu_{ni})} \Big)$$

- This converges to stable object.

- $Var\left(\frac{1}{n}\sum_{i=1}^{n}(X_{ni}-\mu_{ni})\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_{ni}-\mu_{ni})$

(Independency of X_{ni}) $= \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}$

When $k_n = n$,

$$C_n^2 = \sum_{i=1}^{k_n} \sigma_{ni}^2 = \sum_{i=1}^{n} \sigma_{ni}^2 = n \cdot \sigma^2 \quad \text{(By iid)} \cdots ①$$

$$\mu_{ni} = E(X_{ni}) = \mu \quad \text{(By iid)} \cdots ②$$

$$Y_n = \frac{1}{C_n} \sum_{i=1}^{k_n} (X_{ni} - \mu_{ni}) \underset{\substack{\uparrow \\ ①,②}}{=} \frac{1}{\sqrt{n\sigma^2}} \sum_{i=1}^{n} (X_i - \mu) = \frac{1}{\sqrt{n}} \cdot \frac{1}{\sigma} \sum_{i=1}^{n} (X_i - \mu)$$

Check ⊛

$$\lim_{n\to\infty} \frac{1}{C_n^2} \sum_{i=1}^{k_n} E\{(X_{ni} - \mu_{ni})^2 \cdot \mathbb{1}\{|X_{ni} - \mu_{ni}| > \eta \cdot C_n\}\}$$

$$\underset{\substack{\uparrow \\ ①,②}}{=} \frac{1}{n \cdot \sigma^2} \cdot n\, E\{(X_i - \mu)^2 \cdot \mathbb{1}\{|X_i - \mu| > \eta \cdot \sqrt{n\sigma^2}\}\}$$

$$= \frac{1}{\sigma^2} E\{(X_i - \mu)^2\, \mathbb{1}\{|X_i - \mu| > \eta \cdot \sigma\sqrt{n}\}\}$$

Note $(X_i - \mu)^2\, \mathbb{1}\{|X_i - \mu| > \eta \cdot \sigma\sqrt{n}\} \leq \underbrace{(X_i - \mu)^2}_{\text{finite}} < \infty$ as $n \to \infty$

$\longrightarrow 0$. Thus, $Y \xrightarrow{d} N(0, 1)$

* Therefore, Lindberg thm implies $Y_n \xrightarrow{d} N(0,1)$ when $k_n = n$.

Remark Under Homoskedasticity, not necessary to use double script. However, we will use double script result to study such objects as;

ex) kernel density estimator : $\frac{1}{n} \sum_{i=1}^{n} \underbrace{\frac{1}{h_n} K\left(\frac{X_i - x}{h_n}\right)}_{:= Z_{ni}}$

★ **The Lindberg condition guarantees ;**

① there is enough terms with positive variance as $C_n \to \infty$

② there is not a large fraction of sample (i) which has large influence

(most i has small contribution $E\{(X_{ni}-\mu_{ni})^2 \cdot 1\{|X_{ni}-\mu_{ni}| > \eta \cdot C_n\}\} \longrightarrow 0$

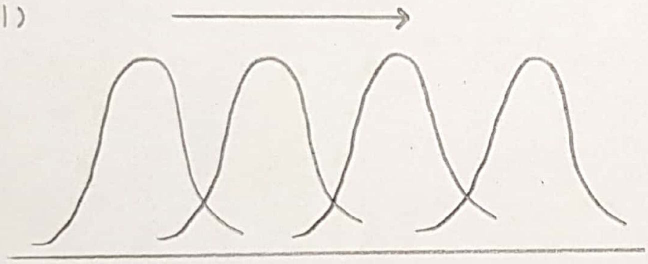
in the sence that, $\forall \eta > 0$,

(Def) Big $O_p(1)$, Little $o_p(1)$ notations and their uses

- $X_n - X_\infty \xrightarrow{P} 0$ we write $X_n = X_\infty + o_p(1)$
- $X_n = O_p(1)$ if $\forall \varepsilon > 0, \exists M_\varepsilon > 0$ such that $\lim_{n\to\infty} Pr\{|X_n| > M_\varepsilon\} < \varepsilon$.

"stochastically bounded"

ex1)



→ This sequence is Not bounded.

→ But stochastically bounded

Choose small $\varepsilon \Rightarrow \exists M_\varepsilon : \lim_{n\to\infty} Pr\{|X_n| > M_\varepsilon\} < \varepsilon$

Thus, we can write $X_n = O_p(1)$: Stochastically bounded

(Called as "tightness", too)

ex2) $X_n \sim N(0,1)$: Not bounded, but stochastically bounded.

$X_n \sim N(\mu_n, 1)$: Not converge but "

$X_n \xrightarrow{d} X_\infty$ then $X_n = O_p(1)$

(Note) If $X_n = O_p(1)$, $Y_n = o_p(1)$. then $\begin{cases} X_n \cdot Y_n = o_p(1) \\ X_n + Y_n = O_p(1) \end{cases}$

If $X_n = O_p(1)$, $Y_n = O_p(1)$, then $\begin{cases} X_n + Y_n = O_p(1) \\ X_n \cdot Y_n = O_p(1) \end{cases}$

※ $X_n \xrightarrow{d} X_\infty \Rightarrow X_n = O_p(1)$

useful & important.