- Generalized Least Squares (continued)

$$\mathbb{Y} = \mathbb{X}\beta + \mathbb{U}, \quad E\{\mathbb{U}\mathbb{U}' | \mathbb{X}\} = \Omega \qquad E\{\mathbb{U} | \mathbb{X}\} = 0$$

$$\hat{\beta}_{GLS} = (\mathbb{X}'\Omega^{-1}\mathbb{X})^{-1}\mathbb{X}'\Omega^{-1}\mathbb{Y} \ : \ \text{BLUE}$$

However, if we don't know Ω, GLS estimator is typically infeasible.

Nevertheless, the idea of GLS can be applied to GMM.

Moment conditions : $\frac{1}{n}\sum_{i=1}^{n} Z_i (y_i - x_i'\beta_0) = 0$ (Assume β_0 is the true cofficient.)

$$Z_i = \begin{pmatrix} Z_{1i} \\ Z_{2i} \\ \vdots \\ Z_{Ji} \end{pmatrix} \qquad J \geqq K$$

(**) $$\left(\begin{array}{c} \frac{1}{n}\sum_{i=1}^{n} Z_{1i} y_i = \frac{1}{n}\sum_{i=1}^{n} Z_{1i} x_i'\beta_0 + V_1 \\ \vdots \\ \underbrace{\frac{1}{n}\sum_{i=1}^{n} Z_{Ji} y_i}_{\boxed{\text{Dependent variable}}} = \underbrace{\frac{1}{n}\sum_{i=1}^{n} Z_{Ji} x_i'\beta_0 + V_J}_{\text{J observations' regression}} \end{array} \right)$$

Since we cannot set the LHS to 0 when $J > K$, $RHS = V = \begin{pmatrix} V_1 \\ \vdots \\ V_J \end{pmatrix}$.

$\Rightarrow$ This can be seen as the linear regression model with J observations and K unknown parameters: β_0.

$\Rightarrow$ Variance - Covariance matrix.

$$\frac{1}{n}\sum_{i=1}^{n} Z_i (\underbrace{y_i - x_i'\beta_0}_{u_i}) = \frac{1}{n}\sum_{i=1}^{n} Z_i u_i \quad \cdots (*)$$

$$\text{Variance - Covariance matrix of } (*) = E\left[\left\{\frac{1}{n}\sum_{i=1}^{n} Z_i u_i\right\}\left\{\frac{1}{n}\sum_{i=1}^{n} Z_i u_i\right\}'\right]$$

$$= \frac{1}{n^2} E\left[\sum_{i=1}^{n}\sum_{k=1}^{n} Z_i Z_k' u_i u_k\right]$$

$$= \frac{1}{n^2} E\left[\sum_{i=1}^{n} Z_i Z_i' u_i^2\right] = \frac{1}{n} E(Z_i Z_i' u_i^2)$$

$\uparrow$ Assuming independence of i and k.

Under Homoskedasticity,

Variance - Covariance matrix of ⊛ $= \frac{1}{n} E(Z_i Z_i' u_i^2) = \frac{1}{n} \sigma_u^2 \underbrace{E(Z_i Z_i')}$

We can estimate this:

$\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i' = \frac{1}{n} \mathbb{Z}'\mathbb{Z}$

Under Heteroskedasticity

$E(Z_i Z_i' u_i^2) \overset{?}{=} \sigma_u^2 E(Z_i Z_i')$ does not hold, but $E(Z_i Z_i' u_i^2)$ can be estimated as we will see later.

Under Homoskedasticity, variance - covariance matrix can be estimated by $\frac{1}{n}\mathbb{Z}'\mathbb{Z}$ except for σ_u^2.

$$\hat{\beta}_{GLS} = (\mathbb{X}'\Omega^{-1}\mathbb{X})^{-1}\mathbb{X}'\Omega^{-1}\mathbb{Y}$$

Ω^{-1} can be replaced by $(\mathbb{Z}'\mathbb{Z})^{-1}$

→ Q: Where is σ_u^2?

A: By Hansen's book, $E(Z_i Z_i' u_i^2) = E(Z_i u_i u_i' Z_i) = E(g_i g_i')$

where $\frac{1}{n}\sum_{i=1}^{n} g_i g_i' = \frac{1}{n}\mathbb{Z}'\mathbb{Z}$

In our class, for convenience, $(\sigma_u^2\Omega)^{-1} = (\mathbb{Z}'\mathbb{Z})^{-1}$

and σ_u^2 will be cancelled.

- From (✱✱), $\frac{1}{n}\sum_{i=1}^{n} Z_{1i} y_i = \frac{1}{n}\sum_{i=1}^{n} Z_{1i} X_i'\beta_0 + V_1$

$\vdots$

$\frac{1}{n}\sum_{i=1}^{n} Z_{Ji} y_i = \frac{1}{n}\sum_{i=1}^{n} Z_{Ji} X_i'\beta_0 + V_J$

$\Rightarrow \mathbb{Z}'\mathbb{Y} = \mathbb{Z}'\mathbb{X}\beta_0 + \mathbb{V}$

$\hat{\beta}_{GLS}$ using $(\mathbb{Z}'\mathbb{Z})^{-1}$ as Ω^{-1} is $(\mathbb{X}'\mathbb{Z}(\mathbb{Z}'\mathbb{Z})^{-1}\mathbb{Z}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Z}(\mathbb{Z}'\mathbb{Z})^{-1}\mathbb{Z}'\mathbb{Y} = \hat{\beta}_{2SLS}$

Thus, when $\Omega^{-1} = (\mathbb{Z}'\mathbb{Z})^{-1}$, $\hat{\beta}_{GLS} = \hat{\beta}_{2SLS}$

- GMM estimator using $(\mathbb{Z}'\mathbb{Z})^{-1}$ as a weighted matrix.

$$\min_b \; [\mathbb{Z}'(\mathbb{Y}-\mathbb{X}b)]'(\mathbb{Z}'\mathbb{Z})^{-1}[\mathbb{Z}'(\mathbb{Y}-\mathbb{X}b)]'$$

$$\Downarrow$$

F.O.C. $2\mathbb{X}'\mathbb{Z}(\mathbb{Z}'\mathbb{Z})^{-1}\mathbb{Z}'(\mathbb{Y}-\mathbb{X}b) = 0$

$$\hat{b}_{GMM} = (\mathbb{X}'\mathbb{Z}(\mathbb{Z}'\mathbb{Z})^{-1}\mathbb{Z}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Z}(\mathbb{Z}'\mathbb{Z})^{-1}\mathbb{Z}'\mathbb{Y}.$$

$$= (\mathbb{X}'P_{\mathbb{Z}}\mathbb{X})^{-1}\mathbb{X}'P_{\mathbb{Z}}\mathbb{Y}$$

$\Rightarrow$ 2SLS is the GMM estimator using $(\mathbb{Z}'\mathbb{Z})^{-1}$ as the weighted matrix.

As we shall see, the 2SLS is efficient among GMM estimators using different weight matrices when Homoskedasticity of u_i given $\mathbb{Z}_i$ holds.

$\Rightarrow$ 2SLS is convenient because it can be estimated without estimating u_i

Therefore, 2SLS estimator is typically used to construct estimator of u_i.

$$\hat{u}_i = y_i - x_i'\hat{\beta}_{2SLS}.$$

point Under Homoskedasticity, ☆

$$\boxed{\text{Using } \underbrace{(\mathbb{Z}'\mathbb{Z})^{-1}}_{\text{good weight matrix.}} \text{ as } \Omega^{-1},\ \hat{\beta}_{GLS} = \hat{\beta}_{2SLS} = \hat{\beta}_{GMM}.}$$

- Properties of GMM estimators.

Remark 1 : IV estimator is a special case of 2SLS.

Remark 2 : 2SLS estimator is a special case of GMM using $(\mathbb{Z}'\mathbb{Z})^{-1}$ as the weight matrix.

$\Rightarrow$ If we show the asymptotic properties of GMM estimator, we know the asymptotic properties of 2SLS and IV.

- GMM estimator.

$$\min_{b} \; [\mathbb{Z}'(\mathbb{Y}-\mathbb{X}b)]' W [\mathbb{Z}'(\mathbb{Y}-\mathbb{X}b)]$$

$\downarrow$

F.O.C. w.r.t b : $2\mathbb{X}'\mathbb{Z}(W)[\mathbb{Z}'(\mathbb{Y}-\mathbb{X}b)]=0.$

$$\hat{b}_W = (\mathbb{X}'\mathbb{Z}W\mathbb{Z}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Z}W\mathbb{Z}'\mathbb{Y}.$$

When $\mathbb{Y} = \mathbb{X}\beta_0 + \mathbb{U}$,

$$\hat{b}_W = (\mathbb{X}'\mathbb{Z}W\mathbb{Z}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Z}W\mathbb{Z}'(\mathbb{X}\beta_0 + \mathbb{U})$$
$$= \beta_0 + (\mathbb{X}'\mathbb{Z}W\mathbb{Z}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Z}W\mathbb{Z}'\mathbb{U}$$

① GMM estimator is $\boxed{\text{biased.}}$: $E(\hat{b}_W | \mathbb{X},\mathbb{Z}) \neq \beta_0.$

$$E(\hat{b}_W | \mathbb{X},\mathbb{Z}) = \beta_0 + E\left((\mathbb{X}'\mathbb{Z}W\mathbb{Z}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Z}W\mathbb{Z}'\mathbb{U} \mid \mathbb{X},\mathbb{Z}\right)$$
$$= \beta_0 + (\mathbb{X}'\mathbb{Z}W\mathbb{Z}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Z}W\mathbb{Z}' \underbrace{E(\mathbb{U} \mid \mathbb{X},\mathbb{Z})}_{\neq 0}$$

Why? $\mathbb{X}$, $\mathbb{U}$ correlated.

$\Rightarrow$ Generally, $\hat{\beta}_W$ is not unbiased.

(2) GMM estimator is **consistent** : $\hat{\beta}_W \xrightarrow{p} \beta_0$

$$\hat{\beta}_W = \beta_0 + (\mathbb{X}'\mathbb{Z}W\mathbb{Z}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Z}W\mathbb{Z}'\mathbb{U}$$

Multiplying $\frac{1}{N}$ to each $\mathbb{X}'\mathbb{Z}$, $\mathbb{Z}\mathbb{X}$, $\mathbb{Z}'\mathbb{U}$.

$$= \beta_0 + \left(\frac{1}{N}\mathbb{X}'\mathbb{Z}W\frac{1}{N}\mathbb{Z}'\mathbb{X}\right)^{-1}\frac{1}{N}\mathbb{X}'\mathbb{Z}W\frac{1}{N}\mathbb{Z}'\mathbb{U}$$

$$\frac{1}{N}\mathbb{X}'\mathbb{Z} = \frac{1}{N}\sum_{i=1}^{N} x_i z_i' \xrightarrow{p} E(x_i z_i') \quad \text{by LLN.}$$

$$\frac{1}{N}\mathbb{X}'\mathbb{U} = \frac{1}{N}\sum_{i=1}^{N} z_i u_i' \xrightarrow{p} E(z_i u_i) = 0$$

positive definite

$$\frac{1}{N}\mathbb{X}'\mathbb{Z}W\frac{1}{N}\mathbb{Z}'\mathbb{X} = \left(\frac{1}{N}\sum_{i=1}^{N} x_i z_i'\right)\cdot W\cdot\left(\frac{1}{N}\sum_{i=1}^{N}(z_i x_i')\right) \xrightarrow{p} \underline{E(x_i z_i')}\cdot W\cdot\underline{[E(x_i z_i')]'}$$

Invertible

By continuous mapping theorem,

$$\xrightarrow{p} \beta_0$$

Remark: Suppose W is replaced by $\hat{W}$ s.t $\hat{W} \xrightarrow{p} W$ and it is symmetric. The same argument applies with $\hat{W} \xrightarrow{p} W$ assumption.

(3) GMM estimator is **asymptotically Normal** : $\sqrt{N}(\hat{\beta}_W - \beta_0) \sim N(0, V)$

$$\sqrt{N}(\hat{\beta}_W - \beta_0) = \sqrt{N}(\mathbb{X}'\mathbb{Z}\hat{W}\mathbb{Z}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Z}\hat{W}\mathbb{Z}'\mathbb{U}$$

$$= \underbrace{\left(\frac{1}{N}\mathbb{X}'\mathbb{Z}\hat{W}\frac{1}{N}\mathbb{Z}'\mathbb{X}\right)^{-1}\frac{1}{N}\mathbb{X}'\mathbb{Z}\hat{W}}\ \underbrace{\frac{1}{\sqrt{N}}\mathbb{Z}'\mathbb{U}}$$

$$\left(\frac{1}{N}\mathbb{X}'\mathbb{Z}\hat{W}\frac{1}{N}\mathbb{Z}'\mathbb{X}\right)^{-1}\frac{1}{N}\mathbb{X}'\mathbb{Z}\hat{W} \xrightarrow{p} \left(E(x_i z_i')W[E(x_i z_i')]'\right)^{-1}E(x_i z_i')W$$

$$\frac{1}{\sqrt{N}}\mathbb{Z}'\mathbb{U} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} z_i u_i \xrightarrow[\text{By CLT,}]{d} N(0, E(z_i z_i' u_i^2))$$

By continuous mapping theorem,

$$\xrightarrow{p} \underbrace{\left[E(x_i z_i')W(E(x_i z_i'))'\right]^{-1}}_{:=H_W} E(x_i z_i')W\cdot N(0, E(z_i z_i' u_i^2))$$

$$\sqrt{N}(\hat{\beta}_W - \beta_0) \xrightarrow{d} N\left(0, [E(x_i z_i')W(E(x_i z_i'))']^{-1} E(x_i z_i')W \cdot E(z_i z_i' u_i^2) W E(z_i x_i')[E(x_i z_i')W(E(x_i z_i'))']^{-1}\right)$$

$$= N(0, \underbrace{H_W^{-1} E(x_i z_i') W \cdot E(z_i z_i' u_i^2) W E(z_i x_i') H_W^{-1}}_{⊛})$$

If Homoskedasticity holds, then $E(z_i z_i' u_i^2) = \sigma_u^2 E(z_i z_i')$. (But the formula cannot be simplified.)

If $W = E(z_i z_i' u_i^2)^{-1}$, (weight matrix $W = (Z'Z)^{-1}$)

then the variance-covariance matrix is as follows:

$$⊛ = [E(x_i z_i')W(E(x_i z_i'))']^{-1} E(x_i z_i')W \cdot E(z_i z_i' u_i^2) W E(z_i x_i')[E(x_i z_i')W(E(x_i z_i'))']^{-1}$$

$$= [E(x_i z_i')E(z_i z_i' u_i^2)^{-1}E(z_i x_i')]^{-1} E(x_i z_i') E(z_i z_i' u_i^2)^{-1} \cancel{E(z_i z_i' u_i^2)} \cancel{E(z_i z_i' u_i^2)^{-1}} E(z_i z_i' u_i^2)^{-1} E(z_i x_i')[E(x_i z_i')E(z_i z_i' u_i^2)^{-1}E(z_i x_i')]^{-1}$$

$$= \underbrace{[E(x_i z_i')E(z_i z_i' u_i^2)^{-1}E(z_i x_i')]^{-1}}_{=H^{-1}_{E(z_i z_i' u_i^2)^{-1}}} E(x_i z_i') E(z_i z_i' u_i^2)^{-1} E(z_i x_i') [E(x_i z_i')E(z_i z_i' u_i^2)^{-1}E(z_i x_i')]^{-1}$$

$$= \cancel{H^{-1}_{E(z_i z_i' u_i^2)^{-1}}} \cancel{H_{E(z_i z_i' u_i^2)^{-1}}} H^{-1}_{E(z_i z_i' u_i^2)^{-1}} = H^{-1}_{E(z_i z_i' u_i^2)^{-1}} = [E(x_i z_i')E(z_i z_i' u_i^2)^{-1}E(z_i x_i')]^{-1}$$

For any W matrix, $⊛ \geq \boxed{[E(x_i z_i') E(z_i z_i' u_i^2)^{-1} E(z_i x_i')]^{-1}}$

→ This is the smallest variance-covariance matrix.

Remark We could use a consistent estimator of $E(z_i z_i' u_i^2)$ to form the optimal weight matrix. $\frac{1}{N}\sum_{i=1}^{N} z_i z_i' \hat{u}_i^2$, where $\hat{u}_i = y_i - x_i'\hat{\beta}_{2SLS}$

* $E(u_i | x_i) \neq 0$ cases

⇓

- Measurement error
- Simultaneity (For today)
- Lagged Dependent Variables & Serial correlation. (For today)
- Sample selection.
- Functional Misspecification.

Solution.

⇒ IV, Panel data analysis. (Measurement error, Simultaneity, Lagged Dependent Variables & Serial correlation)

⇒ MLE, Semi-parametric analysis, Control function Approach. (Sample selection)

⇒ Non/Semi-parametric analysis. (Functional Misspecification)

→ We already checked IV for Measurement error. (3/13 note).

* IV for the Simultaneous Equation Model. [point: use excluded exogenous variables as IVs]

Let y_i & x_{2i} be endogenous variables. ⇒ $E(u_i | x_{2i}) \neq 0$.

Let x_{3i} be a exogenous variable. ⇒ $E(u_i) = 0$ & $E(u_i x_{3i}) = 0$.

<Structural Equation>

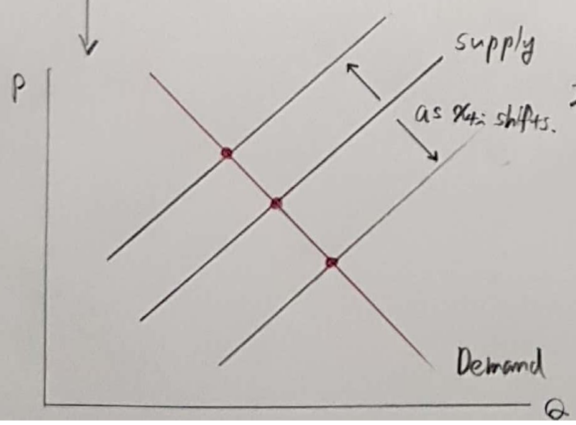
triangular system.

$$y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + u_i \quad \text{(Demand function)}.$$

$$x_{2i} = \Pi_1 + \Pi_2 x_{3i} + \Pi_3 x_{4i} + V_i \Rightarrow$$

Note reduced form: Endogeneous variable expressed by exogeneous variables and error terms.

⊛
- $E(x_{4i} u_i) = 0$
- $E(V_i) = 0$. $E(V_i x_{3i}) = 0$, $E(V_i x_{4i}) = 0$.
- $E(u_i V_i) \neq 0$ (Since $E(x_{2i} u_i) \neq 0$, u_i and V_i are correlated).

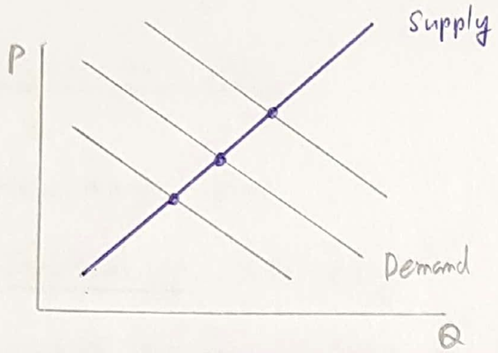


If ∃ a supply fcn such that

Such as $y_i = \beta_1^S + \beta_2^S x_{2i} + \beta_3^S x_{4i} + u_i^S$,

we can get a demand curve by using x_{4i}

(x_{4i}: excluded exogeneous variable from demand fcn).



In the same manner,
we can get a supply curve
by shifting the demand curve.

☆☆☆

⇒ Point $\boxed{\text{Structural Equation coefficients can be recovered if there are excluded exogenous variables accounting for included endogenous variables.}}$

- The condition to recover Structural Equation coefficients.
  - ① order condition
  - ② Rank condition (Full rank of IVs)

① Order condition (Necessary condition)

If there are m included endogenous variables,
there have to exist at least m excluded exogenous variables.

(Intuitively, we need to be able to solve equations by # of endogenous v. = # of exogenous v.)

⇒ i.e., there exist enough moment conditions.

※ Need to check : $E(Z_i u_i)=0$ and $J \geqq K$.

(J: the length of the vector Z_i,
K: 〃 the vector X_{is}).

(→ $J \geqq K$ does not mean that $\text{rank}\, E(Z_i X_i') \geqq K$
Thus, we should check it. ⇓)

② Rank condition. (Full rank of IV)

$$X_i = \begin{pmatrix} 1 \\ x_{3i} \\ x_{2i} \end{pmatrix} \qquad Z_i = \begin{pmatrix} 1 \\ x_{3i} \\ x_{4i} \end{pmatrix}$$

$K=3$. $\qquad J=3$

⇒ These are satisfying the order condition. ($J=3 \geq K=3$).

※ Need to check : $\text{rank } E(Z_i X_i') \geq K$

$$E(Z_i X_i') = E\left[\begin{pmatrix} 1 \\ x_{3i} \\ x_{4i} \end{pmatrix} (1 \quad x_{3i} \quad x_{2i})\right] = E\left[\begin{pmatrix} 1 \\ x_{3i} \\ x_{4i} \end{pmatrix} (1 \quad x_{3i} \quad \pi_1 + \pi_2 x_{3i} + \pi_3 x_{4i} + v_i)\right]$$

$$= E\begin{pmatrix} 1 & x_{3i} & \pi_1 + \pi_2 x_{3i} + \pi_3 x_{4i} + v_i \\ x_{3i} & x_{3i}^2 & \pi_1 x_{3i} + \pi_2 x_{3i}^2 + \pi_2 x_{3i} x_{4i} + v_i x_{3i} \\ x_{4i} & x_{3i} x_{4i} & \pi_1 x_{4i} + \pi_2 x_{3i} x_{4i} + \pi_3 x_{4i}^2 \end{pmatrix}$$

from ⊛ at page ①.

$$= \begin{pmatrix} 1 & E(x_{3i}) & \pi_1 + \pi_2 E(x_{3i}) + \pi_3 E(x_{4i}) + \underbrace{E(v_i)}_{=0} \\ E(x_{3i}) & E(x_{3i}^2) & \pi_1 E(x_{3i}) + \pi_2 E(x_{3i}^2) + \pi_2 E(x_{3i} x_{4i}) + \underbrace{E(v_i x_{3i})}_{=0} \\ E(x_{4i}) & E(x_{3i} x_{4i}) & \pi_1 E(x_{4i}) + \pi_2 E(x_{3i} x_{4i}) + \pi_3 E(x_{4i}^2) \end{pmatrix} := A$$

⇒ Let's check the determinant.

$$\det A = \det \begin{pmatrix} 1 & E(x_{3i}) & \boxed{\pi_1} + \boxed{\pi_2 E(x_{3i})} + \pi_3 E(x_{4i}) \\ E(x_{3i}) & E(x_{3i}^2) & \boxed{\pi_1 E(x_{3i})} + \boxed{\pi_2 E(x_{3i}^2)} + \pi_3 E(x_{3i} x_{4i}) \\ E(x_{4i}) & E(x_{3i} x_{4i}) & \boxed{\pi_1 E(x_{4i})} + \boxed{\pi_2 E(x_{3i} x_{4i})} + \pi_3 E(x_{4i}^2) \end{pmatrix}$$

By a property of matrix,
We can eliminate this when calculating its determinant.

$$= \det \begin{pmatrix} 1 & E(x_{3i}) & \pi_3 E(x_{4i}) \\ E(x_{3i}) & E(x_{3i}^2) & \pi_3 E(x_{3i} x_{4i}) \\ E(x_{4i}) & E(x_{3i} x_{4i}) & \pi_3 E(x_{4i}^2) \end{pmatrix} = \pi_3 \cdot \det \begin{pmatrix} 1 & E(x_{3i}) & E(x_{4i}) \\ E(x_{3i}) & E(x_{3i}^2) & E(x_{3i} x_{4i}) \\ E(x_{4i}) & E(x_{3i} x_{4i}) & E(x_{4i}^2) \end{pmatrix}$$

$$= \Pi_3 \cdot \det\left\{ E\left(\begin{pmatrix} 1 \\ x_{3i} \\ x_{4i} \end{pmatrix} (1 \ \ x_{3i} \ \ x_{4i}) \right) \right\} = \Pi_3 \cdot \det E(z_i z_i')$$

⇒ Therefore, if there is no perfetly collinearity b/w $1, x_{3i}, x_{4i}$, and $\Pi_3 \neq 0$, then rank $E(z_i x_i') \geq K$ holds.

c.f) What if ∃ two endogenous variables ?

$$y_i = \beta_1 + \beta_2 x_{2i} + \beta_3 x_{3i} + u_i$$

$$x_{2i} = \Pi_1^2 + \Pi_2^2 x_{4i} + \Pi_3^2 x_{5i} + V_{2i}$$

$$x_{3i} = \Pi_1^3 + \Pi_2^3 x_{4i} + \Pi_3^3 x_{5i} + V_{3i}$$

① exogenous variables : x_{4i}, x_{5i} $J=2$
endogenous variables : x_{2i}, x_{3i} $K=2$
⇒ Order condition is satisfied.

② • If $\Pi_2^2 = \Pi_2^3$ and $\Pi_3^2 = \Pi_3^3$,

∃ the perfect collinearity between x_{2i} and x_{3i}

(i.e., in fact, ∃ only one exogenous varicble, Not two).

⇒ rank $E(z_i x_i') \geq K$ DOES NOT HOLD.

• If $\Pi_2^2 = \Pi_3^2 = 0$, then ∃ no IV for x_{2i}

or

If $\Pi_2^3 = \Pi_3^3 = 0$, then ∃ no IV for x_{3i}

⇒ Thus, rank $E(z_i x_i') \geq K$ DOES NOT HOLD.

Note Since ∃ this kind of difficulty,
Usually we use only One endogenous variable as the IV for simultaneous equation cases.

* Lagged Dependent Variables + Serial correlation. [point: use lagged exogenous variables as IVs]

$$y_t = \beta_1 + \beta_2 y_{t-1} + \beta_3 X_t + U_t$$

$$y_{t-1} = \beta_1 + \beta_2 y_{t-2} + \beta_3 X_{t-1} + U_{t-1}.$$

⇒ What is IV? Lagged exogenous variables : X_{t-1}.

: $E(U_t X_t) = 0$ and $E(U_t X_{t-1}) = 0$ (This assumption need to be maintained)

⇒ Rank condition is Not usually checked without writing down a full time-series model.

(It is the weakness of this approach.
We don't know how many ts there are, and even though we know it, it is very hard to calculate the rank).

① Coming up with IV within Economic Models
② 〃 〃 in Data. (Natural Experiments)

① Hansen - Singleton Euler Equation Estimation.
Rosenzweig - Schultz health production Estimation.

⇒ Read the presentation file.