Econometrics 522A
Spring, 2019
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PS #3 (Due 6 Feb, 2019)

1. Study Krueger and Whitmore (2001) in the syllabus and answer the following questions.
   (a) Explain the differences in the regression model used in column (1), (2), and (3) in Table 2, p.6.
   (b) Explain what's being tested in the P-value statement in the last row in column (1), (2), and (3) in Table 2. How does the test relate to the appropriateness of the random assignment?
   (c) What does β_{1g} estimate in equation (1) p.9? Can you give a causal interpretation?
   (d) Does OLS yield an unbiased estimator of β_{1g}? State assumptions under which it does and discuss if the assumptions are likely to hold in this case.

2. Consider the linear regression model

$$y_i = x_i'\beta + u_i = x_{1i}'\beta_1 + x_{2i}'\beta_2 + u_i,$$

   where $x_i = (x_{1i}', x_{2i}')'$, $\beta_1 \in R^{K_1}$, $\beta_2 \in R^{K_2}$, $\beta = (\beta_1', \beta_2')'$, and $K_1 + K_2 = K$. Also let

$$\mathbf{Y} = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} x_1' \\ \vdots \\ x_N' \end{pmatrix} = (\mathbf{X}_1\mathbf{X}_2) = \begin{pmatrix} x_{11}' & x_{21}' \\ \vdots & \vdots \\ x_{1N}' & x_{2N}' \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}.$$

   Note that

$$\mathbf{Y} = \mathbf{X}_1\beta_1 + \mathbf{X}_\mathbf{2}\beta_\mathbf{2} + \mathbf{U}.$$

   (a) Show that OLS estimator of β_1 can be written as

$$\hat{\beta}_1 = (\mathbf{X}_1' M_{\mathbf{X}_2}\mathbf{X}_1)^{-1}\mathbf{X}_1' M_{\mathbf{X}_2}\mathbf{Y},$$

   where $M_{\mathbf{X}_2} = I_N - \mathbf{X}_2(\mathbf{X}_2'\mathbf{X}_2)^{-1}\mathbf{X}_2'$.
   (b) Relate result (a) to the auxiliary regression result we derived in class.
   (c) Show that if $\mathbf{X}_1'\mathbf{X}_2 = 0$, then $\hat{\beta}_1 = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{Y}$. Interpret the condition and the conclusion.

3. Using R or Stata and Python, and using the linear regression model from PS #2, 14,
   (a) write a simulation program showing the t-test of the coefficient on x_{1i} being 0 performs as it is derived.
   (b) Write a simulation program showing the F-test of the coefficients on x_{1i} and x_{2i} being simultaneously 0 performs as it is derived.

4. Suppose Ω is a $K \times K$ symmetric positive definite matrix. Show that there is a lower triangular matrix Γ such that $\Omega = \Gamma\Gamma'$ by solving the sequence of problems below. This is called Cholesky decomposition. Matrix Γ corresponds to the square root of Ω.
   (a) Show that the result holds when $K = 1$.
   (b) Assume that the result holds for $K = k$ so that for a $k \times k$ positive definite matrix Ω_k, there is a Cholesky decomposition Γ_k such that $\Omega_k = \Gamma_k\Gamma_k'$. Show that the result also holds for $K = k + 1$ by choosing a $k \times 1$ vector γ and a scalar γ_{k+1} appropriately to satisfy:

$$\Omega_{k+1} = \begin{pmatrix} \Omega_k & \omega \\ \omega' & \omega_{k+1} \end{pmatrix} = \begin{pmatrix} \Gamma_k & 0 \\ \gamma' & \gamma_{k+1} \end{pmatrix}\begin{pmatrix} \Gamma_k' & \gamma \\ 0' & \gamma_{k+1} \end{pmatrix}.$$

   In the above notation, ω is a $k \times 1$ given vector and ω_{k+1} is a given scalar. Assume in the derivation that

$$\omega_{k+1} - \omega'\Omega_k^{-1}\omega > 0.$$

(c) Show $\omega_{k+1} - \omega'\Omega_k^{-1}\omega > 0$ by computing

$$\begin{pmatrix} \omega'\Omega_k^{-1}, & -1 \end{pmatrix} \begin{pmatrix} \Omega_k & \omega \\ \omega' & \omega_{k+1} \end{pmatrix} \begin{pmatrix} \Omega_k^{-1}\omega \\ -1 \end{pmatrix}$$

and observing that Ω_{k+1} is a positive definite matrix.

5. Show the following

(a) If B is invertible, then B' is invertible. (Hint: use full rank condition.)

(b) Use (a) to show that when B is invertible, $(B')^{-1} = (B^{-1})'$. (Hint: start from $B'(B')^{-1} = I$ using (a).)