#1 (a)

$X_n(w) = w \quad X_\infty(w) = 1-w.$ Let $t \in [0,1]$

CDF of $X_n := F_n(t) = Pr(X_n \leqq t) = \int_0^t dw = t.$

CDF of $X_\infty := F_\infty(t) = Pr(X_\infty \leqq t) = \int_0^t d(1-w) = t.$

Therefore, CDF of X_n = CDF of X_∞

By the definition of convergence in distribution,

$F_n(t) \to F_\infty(t), \ \forall t \in [0,1],$ as $n \to \infty$.

Thus, X_n converges in distribution to X_∞. //

(b) $X_n - X_\infty = w - (1-w) = 2w-1 \in [-1,1]$

CDF of $(X_n - X_\infty) = F_{n-\infty}(t) = Pr((X_n - X_\infty) \leqq t)$

$$= \int_{-1}^{t} d(2w-1) = t+1.$$

0 is constant so $X_n - X_\infty \xrightarrow{d} 0 \iff X_n - X_\infty \xrightarrow{P} 0$

$$pr(|X_n - X_\infty| \geqq \varepsilon) \leqq \frac{E(X_n - X_\infty)^2}{\varepsilon^2} = \frac{E(2W-1)^2}{\varepsilon^2} = \frac{4EW^2 - 4EW + 1}{\varepsilon^2} \neq 0$$

Thus, $X_n - X_\infty \not\xrightarrow{P} 0$. i.e., $X_n - X_\infty \not\xrightarrow{d} 0$. //

#2. The sufficient condition is as follows;

$|X_n| < X^*$ and $E\{|X^*|\} < \infty$

Then $X_n \xrightarrow{a.s} X_\infty$ implies $E\{|X_n - X_\infty|\} \to 0$ as $n \to \infty$

#3. (a) asymptotically uniformly integrable means

$$\lim_{n\to\infty} \sup_n E\{\|Y_n\| \cdot 1\{\|Y_n\|>M\}\} = 0$$

That is, we need to show $\|Y_n\| \to 0$ as $n\to\infty$

or $\|Y_n\| < M$ as $n\to\infty$.

$\|Y_n\| \not\to 0$ b/c $X_n(w)=n$ if $w \in [1-\frac{1}{n}, 1]$.

and $\|Y_n\| \not< M$ b/c $X_n(w)=n \to \infty$. if $w\in[1-\frac{1}{n}, 1]$.

Therefore. Not asymptotically uniformly integrable. //

(b) Converge in the first mean to zero.

; $E(\|X_n - 0\|^1) \to 0$ as $n\to\infty$.

But $E(\|X_n-0\|^1) = E\|X_n\| = \begin{cases} 0 & \text{if } w\in[0, 1-\frac{1}{n}) \\ n & \text{if } w\in[1-\frac{1}{n}, 1] \end{cases}$

Thus, $E\|X_n\| \not\to 0$. if $w\in[1-\frac{1}{n}, 1]$. i.e., $E\|X_n\| = 1 \neq 0$.

Therefore. Not converge in the first mean to zero. //

(c) This sequence violate $|X_n| < x^*$ and $E\{|x^*|\} < \infty$.

Because $X_n(w) = n \not< X^*$ as $n\to\infty$. //

4. $X_n \xrightarrow{d} X_\infty$ X_∞ : a random vector with $\Omega = \Gamma\Gamma'$

$T_n \xrightarrow{P} T$

(a) $A \longmapsto \det(A)$.

If $\exists \det(A)$, A has full rank.

Note Let λ_i be eigenvalue for each $i = 1, \dots K$.
Let P_i be eigenvector for 〃 .

Then. $P^{-1}AP = D$ where $P = [\underset{K\times 1}{P_1} \cdots \underset{K\times 1}{P_K}]$, $K\times K$ matrix

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_K \end{bmatrix}$$

By definition of eigenvalue, and eigen vector,

$AP_i = \lambda_i P_i$ $AP = [AP_1 \ AP_2 \dots AP_K]$

$$PD = [P_1 \ P_2 \dots P_K] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_K \end{bmatrix} = [\lambda_1 P_1 \ \lambda_2 P_2 \cdots \lambda_n P_n]$$

Thus, $AP_i = \lambda_i P_i$, $i = 1, \dots K$.

Note2 $\det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_K \in \mathbb{R}$.

By the Note and Note 2,

$$A = PDP^{-1} \underset{\text{By } P' = P^{-1}}{=} PDP' = [P_1 \cdots P_K] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_K \end{bmatrix} \begin{bmatrix} P_1' \\ \vdots \\ P_K' \end{bmatrix}$$

$$= \lambda_1 \underset{K\times 1}{P_1} \underset{1\times K}{P_1'} + \lambda_2 \underset{K\times 1}{P_2} \underset{1\times K}{P_2'} + \cdots + \lambda_K \underset{K\times 1}{P_K} \underset{1\times K}{P_K'}$$ maps $\lambda_1 \cdot \lambda_2 \cdots \lambda_K$

Therefore, this is a function w.r.t k^2 argument in A. //

#4 (b) • Cramer's rule.

$$\underset{k\times k}{A}\,\underset{k\times 1}{x} = \underset{k\times 1}{b}, \quad \text{where } x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}$$

Let $A_{(i)}$ be the matrix that ith column is changed as b.

Then, $x_i = \dfrac{\det A_{(i)}}{\det A}$, for $i=1,\dots K$.

?

$A = PDP^{-1}$ by #4 (a).

$A^{-1} = (PDP^{-1})^{-1} = (PDP')^{-1} = P'^{-1}D^{-1}P^{-1} = PD^{-1}P'$

$$D^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\lambda_k} \end{pmatrix}$$

So $A^{-1} = \frac{1}{\lambda_1}P_1P_1' + \frac{1}{\lambda_2}P_2P_2' + \cdots + \frac{1}{\lambda_k}P_kP_k'$

Thus, each element of A^{-1} is a continuous function of the k^2 elements of A.

(c) $X_\infty := [X_1 \cdots X_k]^T$

$$\Omega = cov(X_i, X_j) = \begin{pmatrix} E(X_1-EX_1)(X_1-EX_1) & \cdots & E(X_1-EX_1)(X_k-EX_k) \\ \vdots & & \vdots \\ E(X_k-EX_k)(X_1-EX_1) & \cdots & E(X_k-EX_k)(X_k-EX_k) \end{pmatrix}$$

$T_n \xrightarrow{P} T$ and $X_n \xrightarrow{d} X_\infty$.

Define $g(x) = x^{-1}$. then by continuous mapping theorem,

$T_n^{-1} \xrightarrow{P} \Gamma^{-1}$

By slutsky' thm, $T_n^{-1}X_n \xrightarrow{d} \Gamma^{-1}X_\infty$

X_∞ is a normal random vector. So they are independant each other.

$$\Omega = \begin{pmatrix} EX_1^2 - (EX_1)^2 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & EX_k^2 - (EX_k)^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_k^2 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_k \end{pmatrix} \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_k \end{pmatrix} = \Gamma\Gamma' = \Gamma$$

Thus, $$\Gamma^{-1} X_\infty = \begin{pmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\sigma_k} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix} = \begin{pmatrix} \frac{1}{\sigma_1} X_1 \\ \frac{1}{\sigma_2} X_2 \\ \vdots \\ \frac{1}{\sigma_k} X_k \end{pmatrix}$$

Which is a vector of normal random vector.

Since $X_1, \ldots, X_k$ are mutually independent,

$\frac{1}{\sigma_1} X_1, \ldots, \frac{1}{\sigma_k} X_k$ are also mutually indepandent. //

#4. (d) $\Gamma_n^{-1} X_n \xrightarrow{d} \Gamma^{-1} X_\infty$, $\Omega = \begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_k^2 \end{pmatrix}$, $\Gamma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_k \end{pmatrix}$.

equation 1)

$$\Gamma^{-1} X_\infty = \begin{pmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & & 0 \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \frac{1}{\sigma_k} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix} = \begin{pmatrix} \frac{X_1}{\sigma_1} \\ \vdots \\ \frac{X_k}{\sigma_k} \end{pmatrix}.$$

(a) $(\Gamma_n^{-1} X_n)'(\Gamma_n^{-1} X_n) \longrightarrow \left(\frac{X_1}{\sigma_1} \cdots \frac{X_k}{\sigma_k} \right) \begin{pmatrix} \frac{X_1}{\sigma_1} \\ \vdots \\ \frac{X_k}{\sigma_k} \end{pmatrix} = \frac{X_1^2}{\sigma_1^2} + \cdots + \frac{X_k^2}{\sigma_k}$

(b) $X_n'(\Gamma_n^{-1})' \xrightarrow{d} X_\infty'(\Gamma_n^{-1})' = (X_1 \cdots X_k) \begin{pmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & & \frac{1}{\sigma_k} \\ 0 & \cdots & 0 & \frac{1}{\sigma_k} \end{pmatrix} = \left(\frac{X}{\sigma_1} \cdots \frac{X_k}{\sigma_k} \right)$

$$\Gamma_n^{-1} X_n \xrightarrow{d} \Gamma^{-1} X_\infty = \begin{pmatrix} \frac{X_1}{\sigma_1} \\ \vdots \\ \frac{X_k}{\sigma_k} \end{pmatrix}$$

By slutsky, $X_n'(\Gamma_n^{-1})' \Gamma_n^{-1} X_n = \frac{X_1^2}{\sigma_1^2} + \cdots + \frac{X_k^2}{\sigma_k}$

Therefore, $(\Gamma_n^{-1} X_n)'(\Gamma_n^{-1} X_n) = X_n'(\Gamma_n^{-1})' \Gamma_n^{-1} X_n$

equation 2)

ⓒ. $X_n'(\Gamma_n')^{-1} \xrightarrow{d} X_\infty'(\Gamma_n')^{-1} = (X_1 \cdots X_k)\begin{pmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\sigma_k} \end{pmatrix}$

$$= \left(\frac{X_1}{\sigma_1} \cdots \frac{X_k}{\sigma_k}\right)$$

$$\Gamma_n^{-1}X_n \xrightarrow{d} \Gamma_n^{-1}X_\infty = \begin{pmatrix} \frac{X_1}{\sigma_1} \\ \vdots \\ \frac{X_k}{\sigma_k} \end{pmatrix}.$$

$$X_n'(\Gamma_n')^{-1}\Gamma_n^{-1}X_n \xrightarrow{d} \frac{X_1^2}{\sigma_1^2} + \cdots \frac{X_k^2}{\sigma_k^2}$$

Therefore by ⓑ, ⓒ, equation 2 hold.

equation 3)

ⓓ $X_n'(\Gamma_n\Gamma_n')^{-1}X_n \xrightarrow{d} X_\infty'(\Gamma_n\Gamma_n')^{-1}X_\infty$

$$= (X_1 \cdots X_k)\begin{pmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_k^2 \end{pmatrix}^{-1}\begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix}$$

$$= (X_1 \cdots X_k)\begin{pmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\sigma_k^2} \end{pmatrix}\begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix} = \frac{X_1^2}{\sigma_1^2} + \cdots + \frac{X_k^2}{\sigma_k}$$

By ⓒ and ⓓ, equation 3 holds.

#4 (e) $\Gamma_n^{-1} X_n' \xrightarrow{d} \Gamma_n^{-1} X_\infty = \begin{pmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{\sigma_K} \end{pmatrix} \begin{pmatrix} X_1 \\ \vdots \\ X_K \end{pmatrix} = \begin{pmatrix} \frac{X_1}{\sigma_1} \\ \vdots \\ \frac{X_K}{\sigma_K} \end{pmatrix}$

Thus each element of $\Gamma_n^{-1} X_n'$ converges to $N(0, 1)$

Because $X_\infty \sim$ Normal. if its element X_i has variance σ_i, then using $g(x_i) = \frac{x_i}{\sigma_i}$ for each i.

by continuous mapping thm, $X_i \sim N(0, 1)$

so that $\Gamma_n^{-1} X_\infty \sim N(0, I)$

$(\Gamma_n^{-1} X_n)' (\Gamma_n^{-1} X_n) \xrightarrow{d} (\Gamma_n^{-1} X_\infty)' (\Gamma_n^{-1} X_\infty) = (\text{Normal} \times \text{Normal})$

$= \begin{pmatrix} \frac{X_1}{\sigma_1} & \cdots & \frac{X_K}{\sigma_K} \end{pmatrix} \begin{pmatrix} \frac{X_1}{\sigma_1} \\ \vdots \\ \frac{X_1}{\sigma_K} \end{pmatrix} = \frac{X_1^2}{\sigma_1^2} + \cdots + \frac{X_K^2}{\sigma_K^2} = \sum_{i=1}^{K} \left(\frac{X_i}{\sigma_i}\right)^2$

$= \chi_K^2$

Also, $X_n'(\Gamma_n \Gamma_n')^{-1} X_n \xrightarrow{d} X_\infty'(\Gamma_n \Gamma_n')^{-1} X_n = \frac{X_1^2}{\sigma_1^2} + \cdots + \frac{X_K^2}{\sigma_K^2} = \chi_K^2$ //

#5. $X \sim (\mu, Var(X))$.

By chebyshev inequality, $Pr(|X-\mu| \geqq \varepsilon) \leqq \dfrac{E|X-\mu|^2}{\varepsilon^2}$

WTS: $Pr(X \geqq \varepsilon) \leqq E(X)/\varepsilon$.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^{\infty} x \cdot f(x)\,dx = \int_0^{\varepsilon} x \cdot f(x)\,dx + \int_{\varepsilon}^{\infty} x \cdot f(x)\,dx$$

$$\geqq \int_{\varepsilon}^{\infty} x \cdot f(x)\,dx \underset{(\because x \geqq \varepsilon)}{\geqq} \int_{\varepsilon}^{\infty} \varepsilon \cdot f(x)\,dx = \varepsilon \int_{\varepsilon}^{\infty} f(x)\,dx = \varepsilon \cdot Pr(X \geqq \varepsilon). \quad //$$

#6. $X_i = V + u_i$, V, u_i: independent. $V \sim (0, \sigma_V^2)$, $u_i \sim (0, \sigma_u^2)$

(a) Chebyshev's WLLN condition.

$$Y_n := \frac{1}{n}\sum_{i=1}^{n}(X_i - \mu_i) \qquad E(Y_n^2) \to 0 \Rightarrow Y_n \xrightarrow{p} 0.$$

$$E(X_n^2) = E\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] = E\left[\left(\frac{1}{n}\Sigma(V+u_i)\right)^2\right] = E\left[\left(\frac{1}{n}\sum_{i=1}^{n} V + \frac{1}{n}\sum_{i=1}^{n} u_i\right)^2\right]$$

$$= E\left(\frac{1}{n}\sum_{i=1}^{n} V\right)^2 = E(V^2) = \sigma_V^2 \qquad \text{Thus } E(X_n^2) \not\to 0. \quad //$$

(b) $\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} V$

Let $X_n := \frac{1}{n}\sum_{i=1}^{n} X_i$

$$E(|X_n - V| \geqq \varepsilon) \leqq \frac{E(X_n - V)^2}{\varepsilon^2} = \frac{E\left(\frac{1}{n}\sum_{i=1}^{n} X_i - V\right)^2}{\varepsilon^2} = \frac{E\left(\frac{1}{n}\sum_{i=1}^{n} X_i - V)\right)^2}{\varepsilon^2}$$

$$= \frac{E\left(\frac{1}{n}\sum_{i=1}^{n}(V+u_i-V)\right)^2}{\varepsilon^2} = \frac{E\left(\frac{1}{n}\sum_{i=1}^{n} u_i\right)}{\varepsilon^2} \longrightarrow 0 \quad \text{as } n \to \infty. \quad //$$

7. $X_{ni} = o_p(1)$ for $i=1,\dots,n$.

$\forall i$, $X_{ni} = o_p(1) \Longrightarrow \sum_{i=1}^{n} X_{ni} \xrightarrow{p} 0$?

$X_{ni} = o_p(1)$ means $X_{ni} \xrightarrow{p} 0$.

Let $X_{ni} = \frac{i}{n^2}$ Then, $X_{ni} \xrightarrow{p} 0$. (counterexample)

$$E\left(\left|\frac{i}{n^2}\right| \geq \varepsilon\right) \leq \frac{E\left(\frac{i}{n^2}\right)^2}{\varepsilon^2} = \frac{E(i^2)}{n^4\varepsilon^2} \leq \frac{n^2}{n^4\varepsilon^2} = \frac{1}{n^2\varepsilon^2} \to 0 \text{ as } n\to\infty.$$

$$\sum_{i=1}^{n} X_{ni} = \sum_{i=1}^{n} \frac{i}{n^2} = \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{n(n+1)}{2n^2}$$

$$E\left(\left|\sum_{i=1}^{n} X_{ni}\right| \geq \varepsilon\right) \leq \frac{E\left(\sum_{i=1}^{n}\frac{i}{n^2}\right)}{\varepsilon^2} = \frac{\frac{n(n+1)}{2n^2}}{\varepsilon^2} \longrightarrow \frac{1}{2} \neq 0 \text{ as } n\to\infty. \;//$$

#8. $\left[\hat{\sigma} \xrightarrow{p} \sigma \text{ and } \sqrt{n}(\hat{\theta}-\theta_0) \xrightarrow{d} N(0,\sigma)\right] \Rightarrow \sqrt{n}(\hat{\theta}-\theta_0)/\hat{\sigma} \xrightarrow{d} N(0,1$

proof) Assume $\hat{\sigma} \xrightarrow{p} \sigma$ and $\sqrt{n}(\hat{\theta}-\theta_0) \xrightarrow{d} N(0,\sigma)$

Define $g(x) = \frac{x}{\sigma}$

By continuous mapping theorem, $\hat{\sigma} \xrightarrow{p} \sigma \Rightarrow g(\hat{\sigma}) \xrightarrow{p} g(\sigma)$

i.e., $\frac{\hat{\sigma}}{\sigma} \xrightarrow{p} \frac{\sigma}{\sigma} = 1$.

$\frac{\hat{\sigma}}{\sigma} \xrightarrow{p} 1$ implies $\frac{\hat{\sigma}}{\sigma} \xrightarrow{d} 1$. Then, by slutsky theorem,

$\sqrt{n}(\hat{\theta}-\theta_0)\cdot\frac{\sigma}{\hat{\sigma}} \xrightarrow{d} N(0,\sigma)$. using $\frac{\sigma}{\hat{\sigma}} \xrightarrow{d} 1$.

By continuous mapping theorem, using $g(x)$,

$\frac{1}{\sigma}\left(\sqrt{n}(\hat{\theta}-\theta_0)\cdot\frac{\sigma}{\hat{\sigma}}\right) \xrightarrow{d} \frac{1}{\sigma}N(0,\sigma)$ Therefore, $\frac{\sqrt{n}(\hat{\theta}-\theta_0)}{\hat{\sigma}} \xrightarrow{d} N(0,1)$. //