3

The F-statistics is as follows

$$\left[C'\sqrt{n}\left(\hat{\beta}-\beta\right)\right]'\left[\hat{\sigma}^2 C'\left(\frac{1}{N}X'X\right)^{-1}C\right]^{-1}\left[C'\sqrt{n}\left(\hat{\beta}-\beta\right)\right]/r.$$

We have

1. $C'\sqrt{n}\left(\hat{\beta}-\beta\right)\to_d N\left(0, C'\sigma^2 E\left(x_i x_i'\right)^{-1}C\right)$,
2. $\hat{\sigma}^2\to_p \sigma^2$, $\frac{1}{N}X'X\to_p E\left\{x_i x_i'\right\}$, and $\left[\hat{\sigma}^2 C'\left(\frac{1}{N}X'X\right)^{-1}C\right]^{-1}\to_p\left[\sigma^2 C'\left(E\left\{x_i x_i'\right\}\right)^{-1}C\right]^{-1}$ by Slutsky's theorem and Continuous Mapping Theorem.

Let $\Omega=\sigma^2 C'\left(E\left\{x_i x_i'\right\}\right)^{-1}C$, $\Omega=\Gamma\Gamma'$, and $\Gamma_n\to_p\Gamma$ where $\Gamma_n\Gamma_n'=\hat{\sigma}^2 C'\left(\frac{1}{N}X'X\right)^{-1}C$. (The existence of Γ_n is guaranteed as X and C are full rank.) Apply the result we have gotten in Q4 in PS5, the F statistics can be written as

$$\left(\Gamma_n^{-1}C'\sqrt{n}\left(\hat{\beta}-\beta\right)\right)'\left(\Gamma_n^{-1}C'\sqrt{n}\left(\hat{\beta}-\beta\right)\right)/r,$$

where

$$\Gamma_n^{-1}C'\sqrt{n}\left(\hat{\beta}-\beta\right)\to_d N\left(0,I\right).$$

And therefore the F statistics converges in distribution to $\chi^2\left(r\right)/r$ without assuming normality of error term. We also have

$$F\left(r,N-K\right)\to_d\chi^2\left(r\right)/r.$$

Therefore, if N is large, the F statistics approximately follow $F\left(r,N-K\right)$ even without assuming the normality of error term.

4

(a) The OLS estimate is

$$\begin{aligned}\hat{\beta}=\begin{bmatrix}\hat{\beta}_1\\ \hat{\beta}_2\\ \hat{\beta}_3\end{bmatrix}&=\left(\sum_{i=1}^{N}x_i x_i'\right)^{-1}\sum_{i=1}^{N}x_i y_i\\ &=\left(\sum_{i=1}^{N}x_i x_i'\right)^{-1}\sum_{i=1}^{N}x_i\left(x_i'\beta+u_i\right)\\ &=\beta+\left(\sum_{i=1}^{N}x_i x_i'\right)^{-1}\sum_{i=1}^{N}x_i u_i,\end{aligned}$$

where $x_i' = \begin{bmatrix} 1 & x_{2i} & x_{3i} \end{bmatrix}$. Therefore,

$$\begin{aligned}\sqrt{N}\left(\hat{\beta}-\beta\right) &= \left(\frac{1}{N}\sum_{i=1}^{N} x_i x_i'\right)^{-1} \frac{1}{\sqrt{N}}\sum_{i=1}^{N} x_i u_i \\ &\to N\left(0, E\left\{x_i x_i'\right\}^{-1} \Sigma E\left\{x_i x_i'\right\}^{-1}\right),\end{aligned}$$

where Σ is the variance-covariance matrix of $\{u_i\}$. Therefore, the second element of $\hat{\beta}$, $\hat{\beta}_2$ is consistent and asymptotically normal.

(b) If we only regress y_i on constant term and x_{2i}, then

$$\begin{aligned}\tilde{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} &= \left(\sum_{i=1}^{N} \tilde{x}_i \tilde{x}_i'\right)^{-1} \sum_{i=1}^{N} \tilde{x}_i y_i \\ &= \left(\sum_{i=1}^{N} \tilde{x}_i \tilde{x}_i'\right)^{-1} \sum_{i=1}^{N} \tilde{x}_i \left(\tilde{x}_i' \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \beta_3 x_{3i} + u_i\right) \\ &= \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \beta_3 \left(\sum_{i=1}^{N} \tilde{x}_i \tilde{x}_i'\right)^{-1} \sum_{i=1}^{N} \tilde{x}_i \left(\beta_3 x_{3i} + u_i\right) \\ &= \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \beta_3 \left(\sum_{i=1}^{N} \tilde{x}_i \tilde{x}_i'\right)^{-1} \sum_{i=1}^{N} \tilde{x}_i \tilde{u}_i.\end{aligned}$$

where $\tilde{x}_i' = \begin{bmatrix} 1 & x_{2i} \end{bmatrix}$ and $\tilde{u}_i = \beta_3 x_{3i} + u_i$. Because x_{3i} and u_i are independent with x_{2i}, $\sum_{i=1}^{N} \tilde{x}_i \tilde{u}_i \to_p 0$. Futhermore,

$$\begin{aligned}\sqrt{N}\left(\tilde{\beta} - \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}\right) &= \left(\frac{1}{N}\sum_{i=1}^{N} \tilde{x}_i \tilde{x}_i'\right)^{-1} \frac{1}{\sqrt{N}}\sum_{i=1}^{N} \tilde{x}_i \tilde{u}_i \\ &\to N\left(0, E\left\{\tilde{x}_i \tilde{x}_i'\right\}^{-1} \tilde{\Sigma} E\left\{\tilde{x}_i \tilde{x}_i'\right\}^{-1}\right),\end{aligned}$$

where $\tilde{\Sigma}$ is the variance-covariance matrix of $\{\tilde{u}_i\}$. Therefore, the second element of $\tilde{\beta}$, $\tilde{\beta}_2$ is consistent and asymptotically normal.

(c) Because x_{2i} and x_{3i} are independent, the corresponding parts of $E\left\{x_i x_i'\right\}^{-1}$ and $E\left\{\tilde{x}_i \tilde{x}_i'\right\}^{-1}$ which enters in the asymptotica variance of $\hat{\beta}_2$ and $\tilde{\beta}_2$ should be the same. The only difference comes from the difference between Σ and $\tilde{\Sigma}$. Consider the simple homoskedasity case, $\Sigma = \sigma_u^2 I$ and $\tilde{\Sigma} = \left(\sigma_u^2 + \beta_3^2 \sigma_{x_{3i}}^2 + \beta_3 \rho \sigma_u \sigma_{x_{3i}}\right) I$. $\tilde{\beta}$ is asymptotically more efficient if and only if $\beta_3^2 \sigma_{x_{3i}} + \beta_3 \rho \sigma_u < 0$, because when this holds, $\tilde{\Sigma} - \Sigma$ is definite-negative matrix, which implies the asymptotic variance of $\tilde{\beta}_2$ is smaller.

6

(a) In the equilibrium, $y_i = y_i^S = y_i^D$, which implies

$$\beta_1^S + \beta_2^S p_i + u_i^S = \beta_1^D + \beta_2^D p_i + u_i^D$$

$$\Rightarrow p_i = \frac{\beta_1^D - \beta_1^S + u_i^D - u_i^S}{\beta_2^S - \beta_2^D}.$$

As price is nonnegative, the equilibrium exists if

$$\frac{\beta_1^D - \beta_1^S + u_i^D - u_i^S}{\beta_2^S - \beta_2^D} > 0.$$

We believe $\beta_2^S - \beta_2^D > 0$ by supply and demand thoerem. Therefore, the condition is equivalent to

$$u_i^D - u_i^S > \beta_1^S - \beta_1^D.$$

(b) We only observe equilibrium in the data, which means

$$p_i = \frac{\beta_1^D - \beta_1^S + u_i^D - u_i^S}{\beta_2^S - \beta_2^D}$$

and

$$\begin{aligned} y_i &= \beta_1^S + \beta_2^S p_i + u_i^S \\ &= \beta_1^S + \beta_2^S \frac{\beta_1^D - \beta_1^S + u_i^D - u_i^S}{\beta_2^S - \beta_2^D} + u_i^S \\ &= \beta_1^S + \frac{\beta_2^S \left(\beta_1^D - \beta_1^S\right)}{\beta_2^S - \beta_2^D} + \frac{\beta_2^S u_i^D - \beta_2^D u_i^S}{\beta_2^S - \beta_2^D}. \end{aligned}$$

(Substitute p_i into the demand function should give the same results.) If we regress y_i on p_i, the probability limit of the OLS estimator should be

$$\begin{aligned} \frac{\sum_{i=1}^n (p_i - \bar{p})(y_i - \bar{y})}{\sum_{i=1}^n (p_i - \bar{p})^2} \to_p \frac{Cov(y_i, p_i)}{Var(p_i)} &= \frac{\left(\beta_2^S \sigma_D^2 + \beta_2^D \sigma_S^2\right) / \left(\beta_2^S - \beta_2^D\right)^2}{\left(\sigma_D^2 + \sigma_S^2\right) / \left(\beta_2^S - \beta_2^D\right)^2} \\ &= \frac{\left(\beta_2^S \sigma_D^2 + \beta_2^D \sigma_S^2\right)}{\left(\sigma_D^2 + \sigma_S^2\right)}. \end{aligned}$$

(c) The probability limit equals β_2^S or β_2^D if and only if $\sigma_S^2 = 0$ or $\sigma_D^2 = 0$ respectively. The intuition is that we can recover the demand function if and only if there is no variation in supply function, and vice versa.