- When $E(u_i|x_i) \neq 0$ (continued)

Classical Measurement Error is another case we have a definite idea about the direction of inconsistency.

Classical assumption.

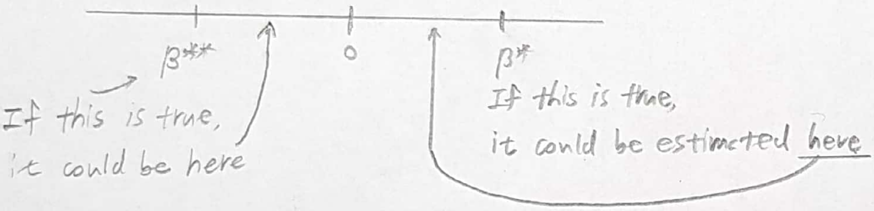
① x_i^* & v_i are uncorrelated : $E(x_i^* v_i)=0$

② v_i & u_i are uncorrelated : $E(v_i u_i)=0$.

In this case, "bias" or "a direction of consistency" is toward zero.

If the coefficient is positive, it is going to be smaller.

If " negative, it is going to be bigger



True model $y_i = \alpha + \beta x_i^* + u_i$

But we cannot see x_i^* so that we should use x_i (observed) where $x_i = \boxed{x_i^*} + v_i$

Then, $y_i = \alpha + \beta(x_i - v_i) + u_i = \alpha + \beta x_i - \beta v_i + u_i$

x_i, v_i : correlated.

$$\hat{\beta} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(\alpha+\beta x_i-\beta v_i+u_i)}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$\sum_{i=1}^{n}(x_i-\bar{x})(-\bar{y}) = -\sum_{i=1}^{n}x_i\bar{y} + \sum_{i=1}^{n}\bar{x}\bar{y} = 0. \quad (\because \sum_{i=1}^{n}x_i = \sum_{i=1}^{n}\bar{x})$$

By $\sum_{i=1}^{n}(x_i-\bar{x})\bar{x} = 0$,

$$\sum_{i=1}^{n}(x_i-\bar{x})x_i = \sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$= \frac{\overbrace{\sum_{i=1}^{n}(x_i-\bar{x})\alpha}^{=0} + \beta\sum_{i=1}^{n}(x_i-\bar{x})x_i - \beta\sum_{i=1}^{n}(x_i-\bar{x})(-\beta v_i+u_i)}{\sum_{i=1}^{n}(x_i-\bar{x})^2} = \beta + \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})(u_i-\beta v_i)}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

(Note) $\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n}x_i^2 - \bar{x}^2 \xrightarrow{p} E(x_i^2) - \{E(x_i)\}^2 = Var(x_i)$

$\frac{1}{n}\sum_{i=1}^{n}x_i^2 \xrightarrow{p} E(x_i^2)$ by LLN

$= Var(x_i^* + v_i)$

$= Var(x_i^*) + Var(v_i)$

Note 2) $\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2(u_i-\beta v_i) = \underbrace{\frac{1}{n}\sum_{i=1}^{n}x_i(u_i-\beta v_i)}_{\xrightarrow{p} E\{x_i(u_i-\beta v_i)\}} - \bar{x}\Big(\underbrace{\frac{1}{n}\sum_{i=1}^{n}(u_i-\beta v_i)}_{\xrightarrow{p} E(u_i-\beta v_i)}\Big)$

$$\xrightarrow{p} E\{x_i(u_i-\beta v_i)\} - E(x_i)\cdot E(u_i-\beta v_i)$$

$$= E\{(x_i^*+v_i)(u_i-\beta v_i)\} - \underbrace{E(x_i^*+v_i)\,\underbrace{E(u_i-\beta v_i)}_{=0}}_{=0}$$

$$= -\beta E(v_i^2) = -\beta\, Var(v_i)$$

If it is large, it is close to 1.
If Not, inconsistency.

$$= \beta - \frac{\beta\, Var(v_i)}{Var(x_i^*)+Var(v_i)} = \beta\left(1-\frac{Var(v_i)}{Var(x_i^*)+Var(v_i)}\right) = \left(\frac{\boxed{Var(x_i^*)}}{Var(x_i^*)+Var(v_i)}\right)\beta.$$

※ How to deal with $E(u_i|x_i)\neq 0$?

- IV (GMM) / Panel data
- MLE
- Non-parametric method / semi-parametric method

• Why OLS fails when $E(u_i|x_i)\neq 0$?

F.O.C $\frac{1}{n}\sum_{i=1}^{n}x_i(\underbrace{y_i-x_i'\hat{\beta}_{OLS}}_{=\hat{u}_i})=0.$

$\rightarrow$ x_i and $\hat{u}_i$ are orthogonal.

We are setting the sample correlation of $\hat{u}_i$ & x_i to be zero.

But when $E(u_i|x_i)\neq 0$, then $E(u_ix_i)\neq 0$.

$\Downarrow$

Find Instrumental Variable, Z !

- IV.

What we need to find is a set of K variables $Z_i = \begin{pmatrix} Z_{i1} \\ Z_{i2} \\ \vdots \\ Z_{iK} \end{pmatrix}$ s.t ① $E(Z_i u_i) = 0$.

(because there are K numbers of unknown βs).

Then, we can use $\frac{1}{n}\sum_{i=1}^{n} Z_i(y_i - x_i'\hat{\beta}_{IV}) = 0$, Solve for $\hat{\beta}_{IV}$.

We can call Z_i as instrumental variables.

sol) $\frac{1}{n}\sum_{i=1}^{n} Z_i y_i = \frac{1}{n}\sum_{i=1}^{n} \underset{K\times 1}{Z_i}\,\underset{1\times K}{x_i'}\,\hat{\beta}_{IV}$

$$\begin{pmatrix} Z_{i1} \\ Z_{i2} \\ \vdots \\ Z_{iK} \end{pmatrix}(x_{i1}\ x_{i2} \cdots x_{iK}) = \begin{pmatrix} Z_{i1}x_{i1} & Z_{i1}x_{i2} & \cdots & Z_{i1}x_{iK} \\ Z_{i2}x_{i1} & & & \vdots \\ \vdots & & & \\ Z_{iK}x_{i1} & & \cdots & Z_{iK}x_{iK} \end{pmatrix}$$

$$\frac{1}{n}\sum_{i=1}^{n} Z_i x_i' = \begin{pmatrix} \frac{1}{n}(Z_{11}x_{11} + Z_{21}x_{21} + \cdots + Z_{n1}x_{n1}) & \cdots & \frac{1}{n}(Z_{11}x_{1K} + Z_{21}x_{2K} + \cdots + Z_{n1}x_{nK}) \\ \vdots & & \\ \frac{1}{n}(Z_{1K}x_{11} + Z_{2K}x_{21} + \cdots + Z_{nK}x_{n1}) & \cdots & \frac{1}{n}(Z_{1K}x_{1K} + Z_{2K}x_{2K} + \cdots + Z_{nK}x_{nK}) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{n}\sum_{i=1}^{n} Z_{i1}x_{i1} & \frac{1}{n}\sum_{i=1}^{n} Z_{i1}x_{i2} & \cdots & \frac{1}{n}\sum_{i=1}^{n} Z_{i1}x_{iK} \\ \vdots & & & \vdots \\ \frac{1}{n}\sum_{i=1}^{n} Z_{iK}x_{i1} & \frac{1}{n}\sum_{i=1}^{n} Z_{iK}x_{i2} & \cdots & \frac{1}{n}\sum_{i=1}^{n} Z_{iK}x_{iK} \end{pmatrix}$$

K × K matrix.

Therefore, we need to be able to invert $\frac{1}{n}\sum_{i=1}^{n} Z_i x_i'$, K×K matrix.

$\xrightarrow{p}$ $E(Z_i x_i')$ invertible.

(If Z_i and x_i are independent, then $E(Z_i x_i') = \underset{K\times 1}{E(Z_i)}\,\underset{1\times K}{E(x_i')}$ ⇒ Both of them have rank = 1. Thus, not suitable for IV.)

Therefore, z_i, x_i should be correlated.

※ For IV,

$$\boxed{\begin{array}{l} ① \ E(z_i u_i)=0 \iff cov(z_i, u_i)=0 \\ ② \ cov(z_i, x_i) \neq 0 \end{array}}$$

- If we don't have k IVs, $E(z_{ij} u_i)=0$ $j = 1 \sim J < k$

then
$$\left.\begin{array}{l} E(z_{i1} \cdot (y_i - x_i'\hat{\beta})) = 0 \\ \vdots \\ E(z_{iJ} \cdot (y_i - x_i'\hat{\beta})) = 0 \end{array}\right)$$

$\Downarrow$ Handling $1 \sim J$, $y_i = x_i'\beta + u_i$

$$\begin{pmatrix} E[z_{i1}(u_i - x_i'(\hat{\beta}-\beta))] \\ E[z_{iJ}(u_i - x_i'(\hat{\beta}-\beta))] \end{pmatrix} \Rightarrow \begin{array}{l} E[z_{i1} \cdot x_i'(\hat{\beta}-\beta)] = 0 \\ \vdots \\ E[z_{iJ} \cdot x_i'(\hat{\beta}-\beta)] = 0 \end{array}$$

i.e.,

$$\begin{pmatrix} E(z_{i1} x_i') \\ \vdots \\ E(z_{iJ} x_i') \end{pmatrix} (\hat{\beta}-\beta) = 0$$

$J \times k$ matrix

갯수가 부족하더라도 일단 이건 성립.
그냥 이것만 가지고 추정하기도 하지만, 실제 이것만 가지고 온전한 분석없음...
(Not so much useful...)

- If we have $J > k$, what to do ?

$\Rightarrow$ One idea is to find the variables of $z_{i1}, \dots, z_{Ji}$, that has best correlation w/ x_i

$$\left.\begin{array}{lll} \text{Regress } x_{1i} \text{ on } z_i \text{ and find the fitted value} & \hat{x}_{1i} = z_i'\hat{\pi}_1 \\ \text{Regress } x_{2i} \text{ on } z_i \text{ and } \quad '' & \hat{x}_{2i} = z_i'\hat{\pi}_2 \\ \vdots & \\ \text{Regress } x_{Ki} \text{ on } z_i \text{ and } \quad '' & \hat{x}_{Ki} = z_i'\hat{\pi}_K \end{array}\right) \text{2 Stage least squares.}$$

- IV and 2SLS estimator. (z_i : $K\times1$ vector, $\mathbb{Z}'\mathbb{X}$: $K\times K$ matrix)

Def < IV >

Under $E(\mathbb{U}|\mathbb{X})\neq 0$, assume $E(z_i u_i)=0 \Leftrightarrow E(\mathbb{Z}'\mathbb{U})=0$.

$\mathbb{Y}=\mathbb{X}\beta+\mathbb{U}$ By $E(z_i u_i)=0$, $\frac{1}{n}\sum_{i=1}^{n} z_i(y_i - x_i'\hat{\beta}_{IV})=0$

$$\parallel$$

$$\frac{1}{n}\mathbb{Z}'(\underbrace{\mathbb{Y}-\mathbb{X}\hat{\beta}_{IV}}_{=\hat{\mathbb{U}}_{IV}})=0 \quad (\because \hat{\mathbb{U}}_{IV} \text{ is orthogonal to } \mathbb{Z})$$

i.e., $\mathbb{Z}'(\mathbb{Y}-\mathbb{X}\hat{\beta}_{IV})=0$

$$\Rightarrow \hat{\beta}_{IV}=(\mathbb{Z}'\mathbb{X})^{-1}\mathbb{Z}'\mathbb{Y}=\left(\sum_{i=1}^{n} z_i x_i'\right)^{-1}\left(\sum_{i=1}^{n} z_i y_i\right)$$

Def < 2SLS >

Regress x_{1i} on z_i : $\hat{\pi}_1=(\mathbb{Z}'\mathbb{Z})^{-1}\mathbb{Z}'\mathbb{X}_1$

the fitted value $\hat{x}_{1i}$: $\hat{\mathbb{X}}_1=\mathbb{Z}\hat{\pi}_1=\mathbb{Z}(\mathbb{Z}'\mathbb{Z})^{-1}\mathbb{Z}'\mathbb{X}_1=P_Z\mathbb{X}_1$

$$\vdots \qquad\qquad \vdots$$

$$\hat{\mathbb{X}}_K=\mathbb{Z}\hat{\pi}_K=\mathbb{Z}(\mathbb{Z}'\mathbb{Z})^{-1}\mathbb{Z}'\mathbb{X}_K=P_Z\mathbb{X}_K$$

Using $\hat{\mathbb{X}}_1 \sim \hat{\mathbb{X}}_K$ as IV,

IV matrix is $[P_Z\mathbb{X}_1 \vdots \cdots \vdots P_Z\mathbb{X}_K]=P_Z[\mathbb{X}_1 \vdots \cdots \vdots \mathbb{X}_K]=P_Z\mathbb{X}$

IV estimating these as IV : $\hat{\beta}_{IV}=(\mathbb{X}'P_Z\mathbb{X})^{-1}\mathbb{X}'P_Z\mathbb{Y}$

$$=(\underline{\mathbb{X}'P_Z}\,\underline{P_Z'\mathbb{X}})^{-1}\underline{\mathbb{X}'P_Z}\mathbb{Y}$$

(In fact, $P_Z\mathbb{X}$ 의 OLS라 생각가능)

※ z_{ij} : when $J<K$, useless

when $J>K$, use them for 2SLS.