- Asymtotic Analysis (a set of "approximation" results)

The objectives ① When do we have "good" properties of an estimator?

→ General conditions are better.

② Compare Estimators and decide which ones are better than the others.

(ex) Under basic assumptions ⊕ Normality,
OLS is the best unbiased estimator.
Likewise, the asymtotic property makes estimators better.

※ Approximation concepts need to be defined.

Starting point : sequence of numbers converges to a number

$\lim_{n\to\infty} x_n = x_\infty$ if and only if $\forall \varepsilon > 0, \exists N_0 : \forall n \geq N_0 :$
$|x_n - x_\infty| < \varepsilon$

We want to extend the convergence concept to a sequence of random variables or more generally, a sequence of random vectors.

We want to define a concept of random variables, so that a sequence of random variables can be understood easily.

Example) Throwing a dice infinite times

$\{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 6\} \times \{1, 2, \ldots, 6\} \times \cdots$

→ Any probability to a number realization is equal to 0.

→ Think of a case : a set is realized as a number.

$\{1,2,\ldots,6\} \times \{1,2,\ldots,6\} \times \{1,2,\ldots,6\} \times \cdots$

$\downarrow$ X_1 $\quad$ $\downarrow$ X_2 $\quad$ $\downarrow$ X_3

$X_1 = 1$ w/p $\frac{1}{6}$ $\quad$ $X_2 = 1$ w/p $\frac{1}{6}$

$X_1 = 2$ w/p $\frac{1}{6}$ $\quad$ $\vdots$ $\quad \cdots$

$\vdots$

$X_1 = 6$ w/p $\frac{1}{6}$

Considering such a structure as a sequence,

$\Downarrow$

$\{X_1, X_2, X_3, \ldots\}$.

Then, it is defined as a sequence written by $X_1(\omega), X_2(\omega), \ldots$

That is,

$$\boxed{\begin{array}{l} \text{For each } \omega,\ \{X_1(\omega), X_2(\omega), X_3(\omega), \ldots\} \\ \Rightarrow \text{If we fix } \omega, \text{ then the sequence is realized.} \\ \quad\ \text{If we fix the other } \omega' \neq \omega, \text{ then the sequence is changed.} \end{array}}$$

$$\boxed{X_i : \underset{\substack{\Uparrow \\ \text{probability}}}{\Omega} \longrightarrow \underset{\substack{\Uparrow \\ \text{sample space}}}{\mathbb{R}^d}, \quad \text{random variable is a function}}$$

In the above case (dice), the distribution is uniform.

$$\begin{array}{lcccccccccl} \text{Dice} & 1, & 2, & 3, & 5, & 1, & 6, & 5, & 5, & 4 & \cdots \quad \langle\text{sequence}\rangle \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{corresponding to} & 0, & 1, & 2, & 4, & 0, & 5, & 4, & 4, & 3. & \cdots \end{array}$$

prob $\left(\frac{1}{6}\right)\cdot 0 + \left(\frac{1}{6}\right)^2 \cdot 1 + \left(\frac{1}{6}\right)^3 \cdot 2 + \left(\frac{1}{6}\right)^4 \cdot 4 + \cdots \quad \cdots \longrightarrow$ This will correspond to a number in $[0,1]$

→ $w \in \Omega$ can be understood as a infinite data.

$X_1(w)$ tells us what we compute using w, upto 1st observation...

then $X_n(w)$ " " , upto the nth observation

→ Convergence concept of random variables requires convergence concept of functions.

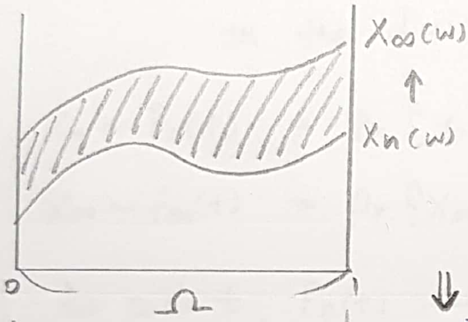
※ ∃ 4 modes of convergence

1 Convergence in rth Mean

$X_i : \Omega \to \mathbb{R}^1$ (d=1 case)

When $E\{|X_n(w) - X_\infty(w)|^2\}^{\frac{1}{2}} \to 0$,

we can say "convergence in 2nd mean"

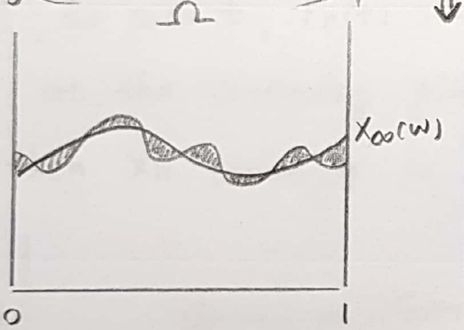


⇓ Extend rth moment!

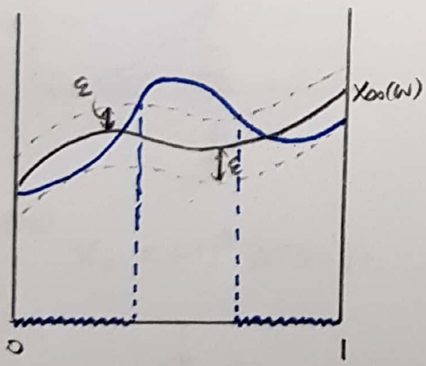
$$\lim_{n\to\infty} E\,\|X_n - X_\infty\|^r \to 0$$

i.e., $\{X_n\}$ converges in rth mean to X_∞

i.e., $X_n \xrightarrow{r} X_\infty \iff X_n - X_\infty \xrightarrow{r} 0$



2 Convergence in Probability



Choose an arbitrary ε.

If $X_n(w)$ is within $X_\infty(w)$, the case that the sum of probability is going to 1 is convergence in Prob.

That is,

$\forall \varepsilon > 0$, $Pr\{|X_n(w) - X_\infty(w)| < \varepsilon\} \to 1$ if $n \to \infty$

i.e., $X_n \xrightarrow{P} X_\infty \iff X_n - X_\infty \xrightarrow{P} 0$

3 Almost sure convergence

$\{X_n\}$ converges almost surely to X_∞

i.e., $Pr\{\lim_{n\to\infty} X_n(w) = X_\infty(w)\} = 1$ where $w \in \Omega$

i.e., $X_n \xrightarrow{a.s} X_\infty \iff X_n - X_\infty \xrightarrow{a.s} 0$

4 Convergence in distribution

; Convergence of the distribution function of X_n to the distribution function of X_∞

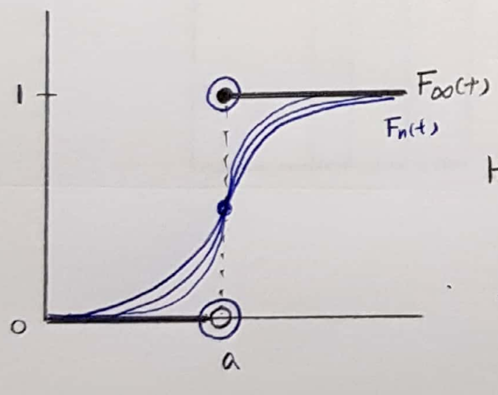
$X_n \sim F_n(t) = Pr\{X_n \leq t\}$

$X_\infty \sim F_\infty(t) = Pr\{X_\infty \leq t\}$

If, for each t, $F_n(t) \longrightarrow F_\infty(t)$ as $n \to \infty$

at the continuity point of $F_\infty(t)$,

then X_n converges in distribution to X_∞



In this case, $F_\infty(t)$ discontinues at a.

However, "At all continuous point of $F_\infty(t)$"

$F_n(t) \longrightarrow F_\infty(t)$

This means $F_n(t)$ converges in distribution of $F_\infty(t)$.

ex) $X_n = (-1)^n \cdot N(0,1)$

$\downarrow$

$X_\infty \sim N(0,1)$

⇐ Their realized value are not the same.
But their distribution can be equal.

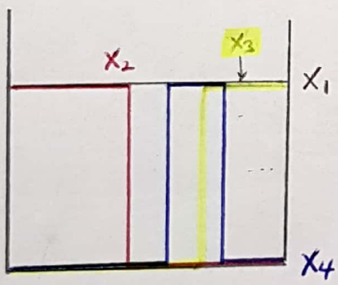
⇒ Relationships of 4 modes of convergence.

$$
\begin{tikzcd}
X_n - X_\infty \xrightarrow{a.s} 0 \arrow[d, Leftrightarrow] & & \\
X_n \xrightarrow{a.s} X_\infty \arrow[dr] & & \\
 & X_n \xrightarrow{p} X_\infty \arrow[r] \arrow[d, Leftrightarrow] & X_n \xrightarrow{d} X_\infty \arrow[d, phantom, "\Uparrow\ \not\Downarrow"] \\
X_n \xrightarrow{r\text{th}} X_\infty \arrow[ur] \arrow[d, Leftrightarrow] & X_n - X_\infty \xrightarrow{p} 0 & X_n - X_\infty \xrightarrow{d} 0 \arrow[d, Leftrightarrow] \\
X_n - X_\infty \xrightarrow{r\text{th}} 0 & & X_n - X_\infty \xrightarrow{p} 0
\end{tikzcd}
$$

Note X_∞ is a constant number.

$X_n \xrightarrow{p} X_\infty \iff X_n \xrightarrow{d} X_\infty$

Example) $X_n \xrightarrow{a.s} X_\infty \;\underset{\not\Longleftarrow}{\overset{\theta}{\Longrightarrow}}\; X_n \xrightarrow{p} X_\infty$



When $n \to \infty$,

$X_n \xrightarrow{a.s} X_\infty$: all X_n will go to X_∞

$X_n \xrightarrow{p} X_\infty$: many of X_n will go to X_∞, but not guranteed for all //