1

(a) Because ω is distributed uniformly over $[0,1]$, the density is $f(\omega) = 1 \quad \forall \omega \in [0,1]$.

$$\begin{aligned} F_{X_n}(t) &= \Pr(X_n \le t) \\ &= \int_0^1 \mathbf{1}\{X_n(\omega) \le t\} f(\omega)\, d\omega \\ &= \int_0^1 \mathbf{1}\{\omega \le t\}\, d\omega. \end{aligned}$$

$$\begin{aligned} F_{X_\infty}(t) &= \Pr(X_\infty \le t) \\ &= \int_0^1 \mathbf{1}\{X_\infty(\omega) \le t\} f(\omega)\, d\omega \\ &= \int_0^1 \mathbf{1}\left\{\underbrace{1-\omega}_{z} \le t\right\} d\omega \\ &= \int_1^0 \mathbf{1}\{z \le t\}\, d(1-z) \\ &= \int_0^1 \mathbf{1}\{z \le t\}\, dz \\ &= F_{X_n}(t). \end{aligned}$$

(b)

$$\begin{aligned} F_{X_n - X_\infty}(t) &= \Pr(X_n - X_\infty \le t) \\ &= \int_0^1 \mathbf{1}\{X_\infty(\omega) - X_\infty(\omega) \le t\} f(\omega)\, d\omega \\ &= \int_0^1 \mathbf{1}\{2\omega - 1 \le t\} f(\omega)\, d\omega \end{aligned}$$

$F_{X_n - X_\infty}(0) = 0.5 \neq F_0(0) = 1.$

2

The sufficient condition is that there exists X^* such that $|X_n| < X^*$ for any n and $E\{|X^*|\} < \infty$.

3

(a) It is easy to see that $X_n \to_p X_\infty = 0$. $E[X_\infty] = 0$. However,

$$\begin{aligned} E[X_n] &= \int_0^1 X_n(\omega)\, d\omega \\ &= \int_0^{1-\frac{1}{n}} 0 d\omega + \int_{1-\frac{1}{n}}^1 n d\omega \\ &= 1. \end{aligned}$$

Therefore, $\{X_n\}$ is not asymptotically uniformly integrable as $E[X_n] \not\to E[X_\infty]$.

(b)

$$\begin{aligned} E[|X_n - 0|] &= E[|X_n|] \\ &= E[X_n] \\ &= 1 \not\to 0. \end{aligned}$$

The second equality is implied by that fact that X_n is nonnegative. Therefore, this sequence of random variables do not converge in the first mean to 0.

(c) To apply Lebesgue dominating convergence theorem, we need to find a random variable X^* which does not depend on n and bounds $|X_n|$. To bound $|X_n|$, we need X^* to be at least as large as $\bar{X}$ where

$$\bar{X}(\omega) = \frac{1}{\omega}.$$

Therefore, $E[|X^*|] \geq E\left[\left|\bar{X}\right|\right] = \int_0^1 \frac{1}{\omega} d\omega = \log 1 - \log 0 = \infty$ which violates the condition that $E[|X^*|] < \infty$. (You can prove that any X that $X(\omega) < \bar{X}(\omega)$ for some ω will not satisfy $|X_n| < X$.)

4

(a) The determinant calculation only involves adding, substracting, and multiplication, which are all continuous operation, and it implies that determinant calculation is also continuous function.

(b) Cramer's rule suggests that calculating the inverse of a matrix is equivalent to calculating the ratios of determinants of matrices. As we know that dividing and determinant calculation are both continuous function, each element of A^{-1} is also a continuous function.

(c) We know that $\Gamma_n \to_p \Gamma$ and $X_n \to_d X_\infty = N(\mu, \Omega)$. By continuous mapping theorem, $\Gamma_n^{-1} \to_p \Gamma^{-1}$. By Slutsky's theorem, we have

$$\begin{aligned}\Gamma_n^{-1} X_n &\to_d \Gamma^{-1} X_\infty \\ &= \Gamma^{-1} N(\mu, \Omega) \\ &= N\left(\mu, \Gamma^{-1}\Omega\left(\Gamma^{-1}\right)'\right) \\ &= N\left(\mu, \Gamma^{-1}\left(\Gamma\Gamma'\right)\left(\Gamma^{-1}\right)'\right).\end{aligned}$$

Notice that Γ is invertible and $\left(\Gamma'\right)^{-1} = \left(\Gamma^{-1}\right)'$ as we have proved in Q5 in PS3. Therefore,

$$\begin{aligned}\Gamma^{-1}\left(\Gamma\Gamma'\right)\left(\Gamma^{-1}\right)' &= \Gamma^{-1}\Gamma\Gamma'\left(\Gamma^{-1}\right)' \\ &= I\end{aligned}$$

and

$$\Gamma_n^{-1} X_n \to_d N(\mu, I)$$

which means each of the k elements in $\Gamma_n^{-1} X_n$ are mutually independent.

(d) The first equality holds as the transpose of a product of two matrices equals to the product of the transposes of these two matrices. The second equality holds as the results we have shown in Q5 in PS3. The third equality holds as the $\left(AA'\right)^{-1} = \left(A'\right)^{-1} A^{-1}$ if A is invertible.

(e) Multiplication of two matrices is a continuous operation because it only involves addition and multiplication of elements in those two matrices. In addition, we have proved $\Gamma_n^{-1} X_n \to_d N(\mu, I)$, therefore, by applying continuous mapping theorem

$$\left(\Gamma_n^{-1} X_n\right)'\left(\Gamma_n^{-1} X_n\right) \to_d \left(N(\mu, I)\right)'\left(N(\mu, I)\right) \sim \chi^2(k).$$

5

$$\begin{aligned}E(X) &= \int_0^\infty t f_X(t)\,dt \\ &\geq \int_\epsilon^\infty t f_X(t)\,dt \\ &\geq \int_\epsilon^\infty \epsilon f_X(t)\,dt \\ &= \epsilon \int_\epsilon^\infty f_X(t)\,dt \\ &= \epsilon \Pr(X \geq \epsilon).\end{aligned}$$

Therefore, we have the Markov's inequality holds as

$$\Pr(X \geq \epsilon) \leq \frac{E(X)}{\epsilon}.$$

6

Notice that $\frac{1}{n}\sum_{i=1}^{n} X_i = v + \frac{1}{n}\sum_{i=1}^{n} u_i$, and $E[X_i] = \mu_i = 0$.

(a)

$$
\begin{aligned}
E\left\{\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_i)^2\right)\right\} &= E\left\{\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-0)^2\right)\right\} \\
&E\left\{\left(v+\frac{1}{n}\sum_{i=1}^{n}u_i\right)^2\right\} \\
&= E\left\{v^2+\frac{1}{n^2}\sum_{i=1}^{n}u_i^2\right\} \\
&= \sigma_v^2+\frac{1}{n}\sigma_u^2 \\
&\to \sigma_v^2 \neq 0,
\end{aligned}
$$

Therefore, the conidtion of Chebyshev's WLLN $E\left\{\left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_i)^2\right)\right\} \to 0$ does not hold.

(b) Rather than $\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_i)$, consider $\frac{1}{n}\sum_{i=1}^{n}(X_i-v) = \frac{1}{n}\sum_{i=1}^{n}u_i$. Because

$$
\begin{aligned}
E\left\{\left(\frac{1}{n}\sum_{i=1}^{n}u_i\right)^2\right\} &= E\left\{\frac{1}{n^2}\sum_{i}u_i^2\right\} \\
&= \frac{1}{n^2}\sigma_u^2 \\
&\to 0
\end{aligned}
$$

holds now, by applying Chebyshev's inequality to $\frac{1}{n}\sum_{i=1}^{n}(X_i-v)$, we have

$$
\begin{aligned}
\Pr\left\{\left|\frac{1}{n}\sum_{i=1}^{n}(X_i-v)\right|>\epsilon\right\} &= \Pr\left\{\left|\frac{1}{n}\sum_{i=1}^{n}(X_i)-v\right|>\epsilon\right\} \\
&\to 0
\end{aligned}
$$

as $n\to\infty$, which implies $\frac{1}{n}\sum_{i=1}^{n}(X_i)$ converges in probability to a random variable v.

7

No, it does not. A counter-example could be

$$
X_{ni}(\omega) = \begin{cases} 1 & \text{if } \omega \in \left[\frac{i-1}{n}, \frac{i}{n}\right] \\ 0 & \text{otherwise} \end{cases}.
$$

$X_{ni} \to_p 0$ obviously. However, $\sum_{i=1}^{n} X_{ni}(\omega) = 1 \quad \forall \omega$, which does not converge in probability to 0.

8

By applying the third equation of Slutsky theorem,

$$\sqrt{n}\left(\hat{\theta} - \theta_0\right)/\hat{\sigma} \to_d N(0, \sigma)/\sigma = N(0, 1).$$