- Characterize the effect of an ith observation on the OLS estimate.

$$\boxed{(A-XX')^{-1} = A^{-1} + A^{-1}X(I-X'A^{-1}X)^{-1}X'A^{-1}}$$

$$(A-XX')(A-XX')^{-1} = (A-XX')(A^{-1}+A^{-1}X(I-X'A^{-1}X)^{-1}X'A^{-1})$$

$$= AA^{-1} + AA^{-1}X(I-X'A^{-1}X)^{-1}X'A^{-1} - XX'A^{-1} - XX'A^{-1}X(I-X'AX)^{-1}X'A^{-1}$$

$$= I + X\{I - X'A^{-1}X\}(I-X'A^{-1}X)^{-1}X'A^{-1} - XX'A^{-1}$$

$$= I + XX'A^{-1} - XX'A^{-1} = I.$$

Using the above,

$$\hat{\beta}_{(i)} = (X_{(i)}'X_{(i)})^{-1}X_{(i)}'Y_{(i)}$$

$X_{(i)}$: the matrix that does not have observation "i"

$$X'X = \sum_{j=1}^{N} x_j x_j' \rightarrow X_{(i)}'X_{(i)} = \sum_{j=1}^{N} x_j x_j' - x_i x_i'$$

$$X'Y = \sum_{j=1}^{N} x_j y_j \rightarrow X_{(i)}'Y_{(i)} = \sum_{j=1}^{N} x_j y_j - x_i y_i$$

$$= (X'X - x_i x_i')^{-1}(X'Y - x_i y_i)$$

$$(X'X - x_i x_i')^{-1} = (X'X)^{-1} + (X'X)x_i(1 - x_i'(X'X)^{-1}x_i)^{-1}x_i'(X'X)^{-1}$$

$$= \left[(X'X)^{-1} + (X'X)^{-1}x_i(1 - x_i'(X'X)^{-1}x_i)^{-1}x_i'(X'X)^{-1}\right](X'Y - x_i y_i)$$

$$= \left\{(X'X)^{-1} + \frac{(X'X)^{-1}x_i x_i'(X'X)^{-1}}{1 - x_i'(X'X)^{-1}x_i}\right\}(X'Y - x_i y_i)$$

$$= \overbrace{(X'X)^{-1}X'Y}^{``\hat{\beta}"} - (X'X)^{-1}x_i y_i + \frac{(X'X)^{-1}x_i x_i'\overbrace{(X'X)^{-1}X'Y}^{``\hat{\beta}"}}{1 - x_i'(X'X)^{-1}x_i} - \frac{(X'X)^{-1}x_i \overbrace{\boxed{x_i'(X'X)^{-1}x_i}\, y_i}^{\text{scalar}}}{\boxed{1 - x_i'(X'X)^{-1}x_i}}$$

$$= \hat{\beta} - (X'X)^{-1}x_i y_i\left(1 + \frac{\boxed{x_i'(X'X)^{-1}x_i}}{\boxed{1 - x_i'(X'X)^{-1}x_i}}\right) + \frac{(X'X)^{-1}x_i x_i'\hat{\beta}}{1 - x_i'(X'X)^{-1}x_i}$$

$$= \hat{\beta} - (X'X)^{-1}x_i y_i\left(\frac{1}{1 - x_i'(X'X)^{-1}x_i}\right) + \frac{(X'X)^{-1}x_i x_i'\hat{\beta}}{1 - x_i'(X'X)^{-1}x_i}$$

$$= \hat{\beta} - (X'X)^{-1}x_i\left(\frac{y_i - x_i'\hat{\beta}}{1 - x_i'(X'X)^{-1}x_i}\right) = \hat{\beta} - \frac{(X'X)^{-1}x_i\hat{u}_i}{1 - x_i'(X'X)^{-1}x_i}$$

$$\therefore \hat{\beta} - \hat{\beta}_{(i)} = \frac{(X'X)^{-1}x_i\hat{u}_i}{1 - x_i'(X'X)^{-1}x_i}$$

$$\hat{\beta} - \hat{\beta}_{(i)} = \frac{(X'X)^{-1} x_i \hat{u}_i}{1 - \underline{x_i'(X'X)^{-1}x_i}}$$

→ If this is close to "1", the gap is big
If " " "0", " small.

$$\boxed{(A + XX')^{-1} = A^{-1} - A^{-1}X(I + X'A^{-1}X)^{-1}X'A^{-1}}$$

$$\hat{\beta} = (X'X)^{-1}X'Y = (X_{(i)}'X_{(i)} + x_i x_i')^{-1}(X_{(i)}'Y_{(i)} + x_i y_i)$$

$$= \left[(X_{(i)}'X_{(i)})^{-1} - (X_{(i)}'X_{(i)})^{-1}x_i\{1 + x_i'(X_{(i)}'X_{(i)})^{-1}x_i\}^{-1}x_i'(X_{(i)}'X_{(i)})^{-1}\right](X_{(i)}'Y_{(i)} + x_i y_i)$$

$$= (X_{(i)}'X_{(i)})^{-1}X_{(i)}'Y_{(i)} + (X_{(i)}'X_{(i)})^{-1}x_i y_i - \frac{(X_{(i)}'X_{(i)})^{-1}x_i x_i'\hat{\beta}_{(i)}}{1 + x_i'(X_{(i)}'X_{(i)})^{-1}x_i} - \frac{(X_{(i)}'X_{(i)})^{-1}x_i\{x_i'(X_{(i)}'X_{(i)})^{-1}x_i\}}{1 + x_i'(X_{(i)}'X_{(i)})^{-1}x_i}$$

$$= \hat{\beta}_{(i)} + \frac{(X_{(i)}'X_{(i)})^{-1}x_i \boxed{(y_i - x_i'\hat{\beta}_{(i)})}}{1 + x_i'(X_{(i)}'X_{(i)})^{-1}x_i} \quad = \tilde{u}_i$$

$$\therefore \hat{\beta} - \hat{\beta}_{(i)} = \frac{(X_{(i)}'X_{(i)})^{-1}x_i \tilde{u}_i}{1 + x_i'(X_{(i)}'X_{(i)})^{-1}x_i}$$

$$x_i'\hat{\beta} - x_i'\hat{\beta}_{(i)} = \underbrace{\frac{x_i'(X_{(i)}'X_{(i)})^{-1}x_i}{1 + \underline{x_i'(X_{(i)}'X_{(i)})^{-1}x_i}}}(y_i - x_i'\hat{\beta}_{(i)})$$

⊛

$$0 \leq \frac{⊛}{1 + ⊛} \leq 1$$

If ⊛ is very large, $\frac{⊛}{1+⊛} \to 1$

↓

This means the variance of x_i is larg

Estimate $x_i'\beta$ for a given x_i by OLS $x_i'\hat{\beta}_{(i)}$

$$\sigma^2 x_i'(X_{(i)}'X_{(i)})^{-1}x_i$$

→ If $x_i'\beta$ is difficult to estimate,

we can understand there exists the effect from i like above.

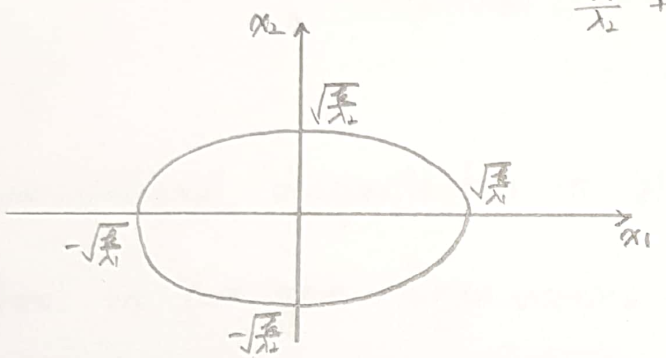
Think of a quadratic form of $x_i'(\mathbb{X}_{(i)}'\mathbb{X}_{(i)})^{-1}x_i$

Assume its dimension is 2.

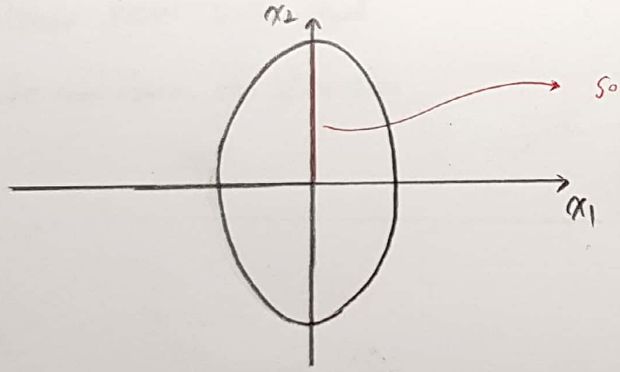
Then, $(x_1\ \ x_2)\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} = \lambda_1 x_1^2 + \lambda_2 x_2^2 = k$

i.e.,

$\frac{x_1^2}{\lambda_2} + \frac{x_2^2}{\lambda_1} = \frac{k}{\lambda_1\lambda_2}$ i.e., $\frac{x_1^2}{\frac{k}{\lambda_1}} + \frac{x_2^2}{\frac{k}{\lambda_2}} = 1$



$\downarrow$ $\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}^{-1} = \begin{pmatrix}\frac{1}{\lambda_1} & 0\\ 0 & \frac{1}{\lambda_2}\end{pmatrix}$ by inverse



So, we can understand

$\exists$ high influence from x_2.

When is $x_i'(\mathbb{X}_{(i)}'\mathbb{X}_{(i)})^{-1}x_i$ "large"?

$= \|x_i\|^2\left[\frac{x_i'}{\|x_i\|}(\mathbb{X}_{(i)}'\mathbb{X}_{(i)})^{-1}\frac{x_i}{\|x_i\|}\right]$: x_i is normalized by $\|x_i\|$.
(Thus, if $\|x_i\|^2$ is big, $x_i'(\mathbb{X}_{(i)}'\mathbb{X}_{(i)})^{-1}x_i$ is big)

In order to know which direction is bigger,

$\max_{x_i} x_i'(\mathbb{X}_{(i)}'\mathbb{X}_{(i)})^{-1}x_i$ s.t $\|x_i\| = 1$

Then, $x_i'(\mathbb{X}_{(i)}'\mathbb{X}_{(i)})^{-1}x_i$ maximized at the eigen-vector corresponding to the maximum eigen-value of $(\mathbb{X}_{(i)}'\mathbb{X}_{(i)})^{-1}$!

$x_i'(X_{(i)}'X_{(i)})^{-1}x_i$ $\qquad$ $\underbrace{X_{(i)}'X_{(i)}}_{:=A}$: symmetric.

$$x_i'Ax_i = (Qy)'A(Qy) = y'Q'AQy = y'Dy \quad \text{where } Q'AQ = D$$

$$= (y_1 \cdots y_N)\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_N \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \lambda_1 y_1^2 + \cdots + \lambda_N y_N^2$$

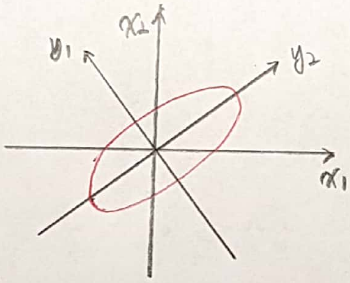
① Take orthogonal diagonalization to $X_{(i)}'X_{(i)}$

Then, we can get eigen vectors and $Q = [V_1 \cdots V_N]$ where $\|V_i\| = 1$.

② Set $|Q| = 1$, then Q - unitvectors will be new axis.

③ Draw new axis and

We can know its direction.



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ex)

$$\widehat{\log wage} = \underset{(0.104)}{0.284} + \underset{(0.007)}{0.092}\, educ + \underset{(0.0017)}{0.0041}\, exper + \underset{(0.03)}{0.022}\, tenure$$

$H_0 : \beta_{exper} = 0$ v. $H_1 : \beta_{exper} > 0$.

$$t_{exper} = \frac{0.0041}{0.0017} \simeq 2.41, \quad \text{reject } H_0.$$

(2.41 > 1.96 95% confi. interval).