PS #5 (Due 20 Feb, 2019)

1. Let ω be distributed uniformly over $[0,1]$ and define $X_n(\omega) = \omega$ and $X_\infty(\omega) = 1 - \omega$.
   (a) Show that in this case the CDFs of X_n and X_∞ are the same so that $X_n \xrightarrow{d} X_\infty$ holds.
   (b) Show that the CDF of $X_n - X_\infty$ is not concentrated at 0 so that $X_n - X_\infty \xrightarrow{d} 0$ does not hold.
2. In view of the dominated convergence, what is a sufficient condition for the a.s. convergence to imply the 1st mean convergence?
3. Consider a sequence of random variables $\{X_n\}$ defined by
$$X_n(\omega) = \begin{cases} 0 & \text{if } \omega \in [0, 1 - 1/n) \\ n & \text{if } \omega \in [1 - 1/n, 1] \end{cases}$$
   and ω has the uniform distribution over $[0,1]$.
   (a) Is this sequence of random variables asymptotically uniformly integrable?
   (b) Does this sequence of random variables converge in the first mean to zero?
   (c) After showing the result for (b), discuss which of the assumptions on the Lebesgue dominating convergence theorem is violated with this sequence.
4. Suppose $X_n \xrightarrow{d} X_\infty$, where X_∞ is a normal random vector of size k with variance-covariance matrix Ω, which is a $k \times k$ non-singular matrix. Assume that Γ_n converges in probability to Γ, which is the Cholesky factor for Ω, where $\Omega = \Gamma\Gamma'$.
   (a) Consider a mapping from $A \mapsto \det(A)$, where A is a $k \times k$ matrix. Explain why this a continuous function with respect to the k^2 arguments in A.
   (b) Explain why each element of A^{-1} is a continuous function of the k^2 elements of A if the determinant of A is not zero, using Cramer's rule.
   (c) Use the Slutsky's theorem to show that $\Gamma_n^{-1} X_n$ converges in distribution to a vector of normal random vector, where each of the k elements are mutually independent.
   (d) Explain why each of the following equalities hold:
$$\begin{aligned}(\Gamma_n^{-1} X_n)'(\Gamma_n^{-1} X_n) &= X_n{}'(\Gamma_n^{-1})'\Gamma_n^{-1} X_n \\ &= X_n{}'(\Gamma_n')^{-1}\Gamma_n^{-1} X_n \\ &= X_n{}'(\Gamma_n\Gamma_n')^{-1} X_n.\end{aligned}$$
   (e) Use the continuous mapping theorem to show why the first expression on the left-hand side converges in distribution to the chi-square random variable with k degrees of freedom. Above equalities imply that the last expression has the same property.
5. Using the same idea with the proof given for the Chebyshev's inequality, prove Markov's inequality: for any non-negative random variable X with finite mean, for any $\epsilon > 0$,
$$\Pr(X \geq \epsilon) \leq E(X)/\epsilon.$$
6. Suppose for each i, $X_i = v + u_i$ for random variables v and u_i for $i = 1, \ldots, n$. Note that v is common across all i so that every i receives the same random value. Assume that v and u_i for all i are independent and all random variables have mean 0. Assume that random variable v has a finite variance σ_v^2 and u_i has finite variance σ_u^2 for each i.

(a) Show that the condition for the Chebyshev's WLLN does not hold for $n^{-1}\sum_{i=1}^{n} X_i$.

(b) Show that $n^{-1}\sum_{i=1}^{n} X_i$ converges in probability to a random variable v.

7. Suppose $X_{ni} = o_p(1)$ for each $i = 1, \ldots, n$. Does this imply that $\sum_{i=1}^{n} X_{ni}$ converges in probability to 0? If yes, please give a proof. If no, please provide a counter-example.

8. Show that when $\hat{\sigma} \xrightarrow{p} \sigma$ and $\sqrt{n}\left(\hat{\theta} - \theta_0\right) \xrightarrow{d} N(0, \sigma)$, $\sqrt{n}\left(\hat{\theta} - \theta_0\right)/\hat{\sigma} \xrightarrow{d} N(0, 1)$.