1

(a) They are different in terms of the controls. In addition to three explanatory variables (race, sex, and free-lunch status) controlled in (1), (2) also controlled for school fixed effects, and entry-grade fiexed effects, and (3) controlled for school-by-entry-wave fixed effect.

(b) It is testing whether the three explanatory variables are jointly significant. It is using F-test as we have seen in the lecture, and the restricted model here is that the coefficients on all three explanatory variables are zeros.

(c) β_{1g} estimate how much the test score percentile rank for a student will be increased if the student was assigned to a small class rather than a regular or regular/aide class. This estimate is causal because of random assignments.

(d) OLS yield an unbiased esimator if $E\left[\epsilon_{isg} \mid SMALL_{is}\right] = 0$. Because of random assignment, this assumption is very likely to be held.

2

(a) Let $M_X = I_N - X\left(X'X\right)^{-1} X$, the OLS regression implies

$$Y = X_1\hat{\beta}_1 + X_2\hat{\beta}_2 + MY,$$

where $\hat{\beta}_1$ and $\hat{\beta}_2$ are OLS estimates, and MY is the residual. Multiply M_{X_2} to both sides of the equation, we have

$$M_{X_2}Y = M_{X_2}X_1\hat{\beta}_1 + M_{X_2}X_2\hat{\beta}_2 + M_{X_2}MY.$$

Notice that $M_{X_2}M = M$ and $M_{X_2}X_2 = 0$, we have

$$M_{X_2}Y = M_{X_2}X_1\hat{\beta}_1 + MY.$$

Multiply X_1' to both sides of the equation,

$$X_1'M_{X_2}Y = X_1'M_{X_2}X_1\hat{\beta}_1 + X_1'MY.$$

Because $MX_1 = 0$, we have

$$\begin{aligned} X_1'M_{X_2}Y &= X_1'M_{X_2}X_1\hat{\beta}_1, \\ \Rightarrow \hat{\beta}_1 &= \left(X_1'M_{X_2}X_1\right)^{-1} X_1'M_{X_2}Y. \end{aligned}$$

(b) Let $\hat{\nu}_{X_1}$ be the residuals of regressing X_1 on X_2, then $\hat{\nu}_{X_1} = M_{X_2}X_1$. Note that $M_{X_2}M_{X_2} = M_{X_2}$ because M_{X_2} is idempotent matrix, the result (a) can be rewritten as

$$\begin{aligned} \hat{\beta}_1 &= \left(X_1'M_{X_2}X_1\right)^{-1} X_1'M_{X_2}Y \\ &= \left(X_1'M_{X_2}'M_{X_2}X_1\right)^{-1} X_1'M_{X_2}Y \\ &= \left(\hat{\nu}_{X_1}'\hat{\nu}_{X_1}\right)^{-1} \hat{\nu}_{X_1}'Y, \end{aligned}$$

which is the same as the auxiliary regression result we derived in class.

(c) If $X_1'X_2 = 0$, then $M_{X_2}X_1 = I_N X_1 - X_2\left(X_2'X_2\right)X_2'X_1 = X_1$. Therefore,

$$\begin{aligned}\hat{\beta}_1 &= \left(X_1'M_{X_2}X_1\right)^{-1}X_1'M_{X_2}Y \\ &= \left(X_1'X_1\right)^{-1}X_1'Y.\end{aligned}$$

This is because when $X_1'X_2 = 0$, they are not correlated. The residuals of regressing X_1 on X_2 is X_1 itself. In this case, we can simply regress Y on X_1 to recover the $\hat{\beta}_1$ without concerning about the omitted variable bias.

4

(a) For one by one positive definite matrix $\Omega_1 = [a]$, we have $a > 0$, and $\sqrt{a}$ exists. Then

$$\begin{aligned}\Omega_1 &= \left[\sqrt{a}\right]\left[\sqrt{a}\right]' \\ &= \Gamma\Gamma'.\end{aligned}$$

(b) If Ω_k is $k \times k$ positive definite matrix and can be decomposed to $\Gamma_k\Gamma_k'$ where Γ_k is lower triangular matrix, then Γ_k has full rank. (Because positive definite matrix has full rank, and $rank\left(\Omega_k\right) \leq rank\left(\Gamma_k\right)$). Choose $\gamma = \Gamma_k^{-1}\omega$, and let $\gamma_{k+1} = \sqrt{\omega_{k+1} - \omega'\Omega_k^{-1}\omega}$. ($\gamma_{k+1}$ exists and by assumption). Then

$$\begin{aligned}\underbrace{\begin{bmatrix}\Gamma_k & 0 \\ \gamma' & \gamma_{k+1}\end{bmatrix}}_{\Gamma_{k+1}}\begin{bmatrix}\Gamma_k' & \gamma \\ 0' & \gamma_{k+1}\end{bmatrix} &= \begin{bmatrix}\Gamma_k\Gamma_k' & \Gamma_k\gamma \\ \gamma'\Gamma_k' & \gamma'\gamma + \gamma_{k+1}^2\end{bmatrix} \\ &= \begin{bmatrix}\Gamma_k\Gamma_k' & \omega \\ \omega' & \gamma'\gamma + \omega_{k+1} - \omega'\Omega_k^{-1}\omega\end{bmatrix} \\ &= \begin{bmatrix}\Gamma_k\Gamma_k' & \omega \\ \omega' & \gamma'\gamma + \omega_{k+1} - \gamma'\Gamma_k'\left(\Gamma_k\Gamma_k'\right)^{-1}\Gamma_k\gamma\end{bmatrix} \\ &= \begin{bmatrix}\Gamma_k\Gamma_k' & \omega \\ \omega' & \gamma'\gamma + \omega_{k+1} - \gamma'\Gamma_k'\left(\Gamma_k'\right)^{-1}\left(\Gamma_k\right)^{-1}\Gamma_k\gamma\end{bmatrix} \\ &= \begin{bmatrix}\Gamma_k\Gamma_k' & \omega \\ \omega' & \gamma'\gamma + \omega_{k+1} - \gamma'\gamma\end{bmatrix} \\ &= \begin{bmatrix}\Gamma_k\Gamma_k' & \omega \\ \omega' & \gamma'\gamma + \omega_{k+1}\end{bmatrix} \\ &= \Omega_{k+1}.\end{aligned}$$

Therefore, by construction, Ω_{k+1} can be decomposed to $\Gamma_{k+1}\Gamma_{k+1}'$.

(c) Given positive definite matrix Ω, for any vector v, we have $v'\Omega v > 0$. Now that Ω_{k+1} is symmetric positive definite matrix, by choosing $v = \begin{pmatrix} \Omega_k^{-1}\omega \\ -1 \end{pmatrix}$, we have

$$\begin{aligned} v'\Omega_{k+1}v &= \begin{pmatrix} \omega'\Omega_k^{-1} & -1 \end{pmatrix} \begin{pmatrix} \Omega_k & \omega \\ \omega' & \omega_{k+1} \end{pmatrix} \begin{pmatrix} \Omega_k^{-1}\omega \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} \omega'\Omega_k^{-1} & -1 \end{pmatrix} \begin{pmatrix} \omega - \omega \\ \omega'\Omega_k^{-1}\omega - \omega_{k+1} \end{pmatrix} \\ &= \omega_{k+1} - \omega'\Omega_k^{-1}\omega > 0 \end{aligned}$$

5

(a) If B is invertible, then B has full rank. Therefore, B' has full rank, and B' is invertible.

(b) If B is invertible, by result (a), B' is invertible. We have

$$B'\left(B'\right)^{-1} = I.$$

Transpose both sides,

$$\left(\left(B'\right)^{-1}\right)' B = I.$$

$$\begin{aligned} &\Rightarrow \left(\left(B'\right)^{-1}\right)' = B^{-1}, \\ &\Rightarrow \left(B'\right)^{-1} = \left(B^{-1}\right)'. \end{aligned}$$