Q. IV condition for time-series.

$$Y_t = \beta_1 + \beta_2 Y_{t-1} + \beta_3 X_t + u_t, \quad E(X_t u_t) = 0$$
$$E(u_t) = 0.$$

but $E(Y_{t-1} u_t) \neq 0$ → Endogeneity problem.

⇒ To handle the endogeneity, use IV.

IV candidate : X_{t-1}

① $E(X_{t-1} u_t) = 0$ (This is likely given that $E(u_t X_t) = 0$)

② $E(X_{t-1} Y_{t-1})$ is correlated.

"$\beta_3 \neq 0$" means "Y_{t-1} and X_{t-1} correlated."

i.e., $Y_{t-1} = \beta_1 + \beta_2 Y_{t-2} + \beta_3 X_{t-1} + u_{t-1}$.

---

* Lagged Dependent Variables + Serial correlation. (continued:)

③ Coming up with IV in Data (Natural Experiments).

Examples : Twins, Natural phenomenon, weather, earthquakes e.t.c.

Some form of discontinuity arising from "participation" constraint.

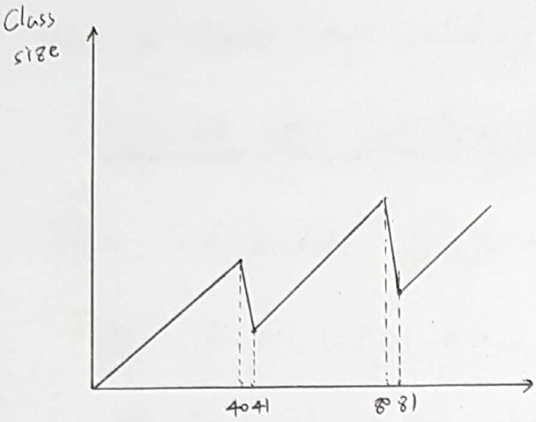
ex) "Class-size effect on test-score", Angrist-Lavy, QJE.

$$Y_{is} = \beta_1 + \beta_2 D_{is} + \beta_3 X_{is} + u_{is}, \quad D_{is} = \begin{cases} 1 & \text{if class size is small} \\ 0 & \text{if } \quad " \quad \text{is not small} \end{cases}$$

↓

Let's think of only one school case.

$$Y_i = \beta_1 + \beta_2 D_i + \beta_3 X_i + u_i, \quad D_i = \begin{cases} 1 & \text{if class size is small} \\ 0 & \text{if } \quad " \quad \text{not small } (\simeq 40) \end{cases}$$



When we see 20 or 30 or even 40, it is just "1" class so that it is small.
But if the total student is 41, there should exist "2" class so that it is not small.

⇓

In this case, if we compare 35-40 vs 41-45
or 38-40 vs 41-43, $\exists$ a bandwidth selection problem.

→ Thus, adjusting its bandwidth, 35-40 vs 20.5 ~ 22.5
38-40 vs 20.5 - 21.5

We can compare them.
(We will talk about it again at Non-parametric part).

* How to test the validity of IV?

Test of over-identifying restrictions.

From IV model, $g_i(\beta) = z_i(y_i - x_i'\beta)$.

Then, ① $E(u_i z_i) = 0$ ② $E(z_i' x_i)$ has full rank.

⇒ This should be checked.

Test of over-identifying restrictions : Check ① $E(u_i z_i) = 0$.

Note < Hansen's J-test >

Under $E(x_i u_i) \neq 0$, (Endogeneity),

$J = J(\hat{\beta}_{gmm}) \xrightarrow{d} \chi^2_{(J-K)}$

$H_0 : E(u_i z_i) = 0 \quad \Rightarrow \quad Pr(J > c \mid H_0) \rightarrow \alpha$ (significant level)

(c: rejection region)

- If we have K IVs, (i.e., J=K), $\frac{1}{N}\sum_{i=1}^{N} z_i(Y - X_i'\hat{b}_{IV}) = 0$, $Y - X_i'\hat{b}_{IV} = \hat{u}_i$

  $\hat{b}_{IV} = (Z'X)^{-1}Z'Y$.

  $\hat{b}_{IV}$ solves $Z'(Y - Xb) = 0$

  When J=K, if we use IV residual $\hat{u}_i$ to test $E(u_i z_i) = 0$,

  $\frac{1}{N}\sum_{i=1}^{N} z_i \hat{u}_i = \frac{1}{N}\sum_{i=1}^{N} z_i(Y_i - X_i'\hat{b}_{IV}) = 0$ so that this is always zero.

  (When J=K, $E(u_i z_i) = 0$ always holds).

- However, if we have more than K IVs, (J>K),

  then NOT all equations can be set to zero,

  So that we need to test "extra equalites J-K" : over-identifying restriction test.

④

- Which K equations are set to zero to GMM?

At first, Let's think of GMM estimator when $J > K$

$$\boxed{\min_b \ \hat{g}(b)' W \hat{g}(b) \quad \text{where } \hat{g}(b) = \frac{1}{N} Z'(Y - Xb)}$$

Now, Z is $N \times J$ matrix.

$$\min_b \ \frac{1}{N}(Y - Xb)' Z W \frac{1}{N} Z'(Y - Xb).$$

F.O.C

$$0 = -\frac{1}{N} X' Z W \cdot \frac{1}{N} Z'(Y - Xb) + \left[\frac{1}{N}(Y - Xb)' Z W \cdot \frac{1}{N} Z' X\right]'(-1)$$

$$= -\frac{1}{N} X' Z W \cdot \frac{1}{N} Z'(Y - Xb) - \frac{1}{N} X' Z W Z'(Y - Xb) \cdot \frac{1}{N}$$

$$= -\frac{1}{N} \cdot \frac{1}{N} X' Z W [Z'(Y - Xb)] \quad \text{i.e., } \underset{K\times1}{0} = \underbrace{\underset{K\times N}{X'} \underset{N\times J}{Z} \underset{J\times J}{W}}_{K\times J} \underbrace{[\underset{J\times N}{Z'} \underbrace{(\underset{N\times1}{Y} - \underset{N\times K}{X}\underset{K\times1}{\hat{b}})}_{N\times1}]}_{J\times1}$$

Note $\langle X' Z W [Z'(Y - X\hat{b})] = 0 \rangle$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & & \vdots \\ x_{N1} & \cdots & x_{NK} \end{pmatrix} \begin{pmatrix} \hat{b}_1 \\ \vdots \\ \hat{b}_K \end{pmatrix} + \begin{pmatrix} \hat{u}_1 \\ \vdots \\ \hat{u}_N \end{pmatrix} \rightarrow \begin{array}{l} y_1 = \hat{b}_1 x_{11} + \hat{b}_2 x_{12} + \cdots + \hat{b}_K x_{1K} + \hat{u}_1 \\ \vdots \\ y_N = \hat{b}_1 x_{N1} + \hat{b}_2 x_{N2} + \cdots + \hat{b}_K x_{NK} + \hat{u}_N \end{array}$$

$$X' Z W [Z'(Y - X\hat{b})] = X' Z W [Z' \hat{u}] = 0$$

$$= \underbrace{\begin{pmatrix} x_{11} & \cdots & x_{N1} \\ \vdots & & \vdots \\ x_{1K} & \cdots & x_{NK} \end{pmatrix} \begin{pmatrix} z_{11} & \cdots & z_{1J} \\ \vdots & & \vdots \\ z_{N1} & \cdots & z_{NJ} \end{pmatrix}}_{①} \begin{pmatrix} w_{11} & \cdots & w_{1J} \\ \vdots & & \vdots \\ w_{J1} & \cdots & w_{JJ} \end{pmatrix} \underbrace{\begin{pmatrix} z_{11} & \cdots & z_{N1} \\ \vdots & & \vdots \\ z_{1J} & \cdots & z_{NJ} \end{pmatrix} \begin{pmatrix} y_1 - \hat{b}_1 x_{11} - \hat{b}_2 x_{12} - \cdots - \hat{b}_K x_{1K} \\ \vdots \\ y_N - \hat{b}_1 x_{N1} - \hat{b}_2 x_{N2} - \cdots - \hat{b}_K x_{NK} \end{pmatrix}}_{②}$$

①
$$\begin{pmatrix} x_{11} z_{11} + x_{12} z_{21} + \cdots + x_{N1} z_{N1} & x_{11} z_{12} + \cdots + x_{N1} z_{N2} & \cdots & x_{11} z_{1J} + \cdots + x_{N1} z_{NJ} \\ \vdots & & & \vdots \\ x_{1K} z_{11} + x_{2K} z_{21} + \cdots + x_{NK} z_{N1} & x_{1K} z_{12} + \cdots + x_{NK} z_{N2} & \cdots & x_{1K} z_{1J} + \cdots + x_{NK} z_{NJ} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^N x_{i1} z_{i1} & \sum_{i=1}^N x_{i1} z_{i2} & \cdots & \sum_{i=1}^N x_{i1} z_{iJ} \\ \vdots & & & \\ \sum_{i=1}^N x_{iK} z_{i1} & \sum_{i=1}^N x_{iK} z_{i2} & \cdots & \sum_{i=1}^N x_{iK} z_{iJ} \end{pmatrix}$$

②
$$\begin{pmatrix} z_{11}(y_1-\hat{b}_1x_{11}-\cdots-\hat{b}_Kx_{1K})+z_{21}(y_2-\hat{b}_1x_{21}-\hat{b}_2x_{22}-\cdots-\hat{b}_Kx_{2K})+\cdots+z_{N1}(y_N-\hat{b}_1x_{N1}-\cdots-\hat{b}_Kx_{NK}) \\ \vdots \\ z_{1J}(y_1-\hat{b}_1x_{11}-\cdots-\hat{b}_Kx_{1K})+z_{2J}(y_2-\hat{b}_1x_{21}-\hat{b}_2x_{22}-\cdots-\hat{b}_Kx_{2K})+\cdots+z_{NJ}(y_N-\hat{b}_1x_{N1}-\cdots-\hat{b}_Kx_{NK}) \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{N} z_{i1}(y_i-\hat{b}_1x_{i1}-\hat{b}_2\cdot x_{i2}-\cdots-\hat{b}_Kx_{iK}) \\ \vdots \\ \sum_{i=1}^{N} z_{iJ}(y_i-\hat{b}_1x_{i1}-\hat{b}_2\cdot x_{i2}-\cdots-\hat{b}_Kx_{iK}) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N} z_{i1}(y_i-\sum_{j=1}^{K}\hat{b}_jx_{ij}) \\ \vdots \\ \sum_{i=1}^{N} z_{iJ}(y_i-\sum_{j=1}^{K}\hat{b}_jx_{ij}) \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{N} z_{i1}\hat{u}_i \\ \vdots \\ \sum_{i=1}^{N} z_{iJ}\hat{u}_i \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \sum_{i=1}^{N} x_{i1}z_{i1} & \sum_{i=1}^{N} x_{i1}z_{i2} & \cdots & \sum_{i=1}^{N} x_{i1}z_{iJ} \\ & \vdots & & \\ \sum_{i=1}^{N} x_{iK}z_{i1} & \sum_{i=1}^{N} x_{iK}z_{i2} & \cdots & \sum_{i=1}^{N} x_{iK}z_{iJ} \end{pmatrix}}_{K\times J} \underbrace{\begin{pmatrix} W_{11} & \cdots & W_{1J} \\ \vdots & & \vdots \\ W_{J1} & \cdots & W_{JJ} \end{pmatrix}}_{J\times J} \begin{pmatrix} \sum_{i=1}^{N} z_{i1}\hat{u}_i \\ \vdots \\ \sum_{i=1}^{N} z_{iJ}\hat{u}_i \end{pmatrix}$$

③

③
$$\begin{pmatrix} W_{11}\sum_{i=1}^{N}x_{i1}z_{i1}+W_{21}\cdot\sum_{i=1}^{N}x_{i1}z_{i2}\cdots+W_{J1}\cdot\sum_{i=1}^{N}x_{i1}z_{iJ} & \cdots & W_{1J}\sum_{i=1}^{N}x_{i1}z_{i1}+\cdots+W_{JJ}\sum_{i=1}^{N}x_{i1}z_{iJ} \\ \vdots & & \\ W_{11}\sum_{i=1}^{N}x_{iK}z_{i1}+W_{21}\cdot\sum_{i=1}^{N}x_{iK}z_{i2}+\cdots+W_{J1}\cdot\sum_{i=1}^{N}x_{iK}z_{iJ} & \cdots & W_{1J}\sum_{i=1}^{N}x_{iK}z_{i1}+\cdots+W_{JJ}\sum_{i=1}^{N}x_{iK}z_{iJ} \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{N}\sum_{j=1}^{J}x_{i1}z_{ij}W_{j1} & \sum_{i=1}^{N}\sum_{j=1}^{J}x_{i1}z_{ij}W_{j2} & \cdots & \sum_{i=1}^{N}\sum_{j=1}^{J}x_{i1}z_{ij}W_{jJ} \\ & \vdots & & \vdots \\ \sum_{i=1}^{N}\sum_{j=1}^{J}x_{iK}z_{ij}W_{j1} & \sum_{i=1}^{N}\sum_{j=1}^{J}x_{iK}z_{ij}W_{j2} & \cdots & \sum_{i=1}^{N}\sum_{j=1}^{J}x_{iK}z_{ij}W_{jJ} \end{pmatrix} \begin{pmatrix} \sum_{l=1}^{N} z_{l1}\hat{u}_l \\ \vdots \\ \sum_{l=1}^{N} z_{lJ}\hat{u}_l \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{N}\sum_{j=1}^{J}x_{i1}z_{ij}W_{j1}\cdot\sum_{l=1}^{N}z_{l1}\hat{u}_l+\cdots+\sum_{i=1}^{N}\sum_{j=1}^{J}x_{i1}z_{ij}W_{jJ}\sum_{l=1}^{N}z_{lJ}\hat{u}_l \\ \vdots \\ \sum_{i=1}^{N}\sum_{j=1}^{J}x_{iK}z_{ij}W_{j1}\sum_{l=1}^{N}z_{l1}\hat{u}_l+\cdots+\sum_{i=1}^{N}\sum_{j=1}^{J}x_{iK}z_{ij}W_{jJ}\sum_{l=1}^{N}z_{lJ}\hat{u}_l \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{i=1}^{N}\sum_{j=1}^{J}\sum_{l=1}^{N}\sum_{m=1}^{J} x_{i1} z_{ij} W_{jm} z_{lm} \hat{u}_l \\ \vdots \\ \sum_{i=1}^{N}\sum_{j=1}^{J}\sum_{l=1}^{N}\sum_{m=1}^{J} x_{ik} z_{ij} W_{jm} z_{lm} \hat{u}_l \end{pmatrix} = \underset{K\times 1}{\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}}$$

From here,

$$\begin{pmatrix} \sum_{i=1}^{N}\sum_{j=1}^{J}\sum_{l=1}^{N}\sum_{m=1}^{J} x_{i1} z_{ij} W_{jm} z_{lm} \left(y_l - \sum_{n=1}^{K} \hat{b}_n x_{ln}\right) \\ \vdots \\ \sum_{i=1}^{N}\sum_{j=1}^{J}\sum_{l=1}^{N}\sum_{m=1}^{J} x_{ik} z_{ij} W_{jm} z_{lm} \left(y_l - \sum_{n=1}^{K} \hat{b}_n x_{ln}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$\Rightarrow$ Thus, we can know $\exists$ K equations that is set to GMM. //

Coming back to Main story,

We test how close $\hat{g}'(\hat{b}) W \hat{g}(\hat{b})$ is to zero,

using $W = \hat{\Omega}^{-1}$ (optimal weight)

(※ Hansen's J-test $\Rightarrow$ Use efficient GMM so that $W = \hat{\Omega}^{-1}$).

F.O.C : $\mathbb{X}'\mathbb{Z} W [\mathbb{Z}'(\mathbb{Y} - \mathbb{X}\hat{b})] = 0 \Rightarrow \hat{b}_{GMM} = (\mathbb{X}'\mathbb{Z}\hat{\Omega}^{-1}\mathbb{Z}'\mathbb{X})^{-1}\mathbb{X}'\mathbb{Z}\hat{\Omega}^{-1}\mathbb{Z}'\mathbb{Y}$

$$\hat{g}'(\hat{b}) = \frac{1}{N}\mathbb{Z}'(\mathbb{Y} - \mathbb{X}\hat{b}) = \frac{1}{N}\mathbb{Z}'(\mathbb{X}\beta + \mathbb{U} - \mathbb{X}\hat{b}) = \frac{1}{N}\mathbb{Z}'(\mathbb{U} - \mathbb{X}(\hat{b} - \beta))$$

$$\hat{b} = \beta + \underbrace{(\mathbb{X}'\mathbb{Z}\hat{\Omega}^{-1}\mathbb{Z}'\mathbb{X})}_{:=A}{}^{-1}\mathbb{X}'\mathbb{Z}\hat{\Omega}^{-1}\mathbb{Z}'\mathbb{U}$$

$$= \frac{1}{N}\mathbb{Z}'(\mathbb{U} - \mathbb{X}(A^{-1})\mathbb{X}'\mathbb{Z}\hat{\Omega}^{-1}\mathbb{Z}'\mathbb{U})$$

$$= \frac{1}{N}(I - \mathbb{Z}\mathbb{X}A^{-1}\mathbb{X}'\mathbb{Z}\hat{\Omega}^{-1})\mathbb{Z}'\mathbb{U}$$

Thus,

$$\hat{g}'(\hat{b})W\hat{g}(\hat{b}) = \frac{1}{N}\mathbf{u}'\mathbf{Z}\left(I-(\hat{\Omega}^{-1})'\mathbf{Z}'\mathbf{X}(A^{-1})'\mathbf{X}'\mathbf{Z}\right)\hat{\Omega}^{-1}\frac{1}{N}\left(I-\mathbf{Z}'\mathbf{X}A^{-1}\mathbf{X}'\mathbf{Z}\hat{\Omega}^{-1}\right)\mathbf{Z}'\mathbf{u}.$$

Note Hansen's J-test : $J = N\cdot\hat{g}'(\hat{b})W\hat{g}(\hat{b})$ where $W=\hat{\Omega}^{-1}$, $\hat{b}=\hat{b}_{gmm}$

So, multiply N and adjust them as $\sqrt{N}$

By $A^{-1}=(A^{-1})'$

$$N\cdot\hat{g}'(\hat{b})W\hat{g}(\hat{b}) = \underbrace{\frac{\mathbf{u}'\mathbf{Z}}{\sqrt{N}}}_{①}\underbrace{\left(I-(\hat{\Omega}^{-1})'\mathbf{Z}'\mathbf{X}(A^{-1})'\mathbf{X}'\mathbf{Z}\right)}_{③}\underbrace{\hat{\Omega}^{-1}}_{②}\underbrace{\left(I-\mathbf{Z}'\mathbf{X}A^{-1}\mathbf{X}'\mathbf{Z}\hat{\Omega}^{-1}\right)}_{③}\underbrace{\frac{\mathbf{Z}'\mathbf{u}}{\sqrt{N}}}_{①}$$

① $\frac{1}{\sqrt{N}}\mathbf{Z}'\mathbf{u} \xrightarrow{d} N(0, E(u_i^2 z_i z_i'))$ by CLT

② $\hat{\Omega}^{-1} \xrightarrow{p} \Omega^{-1}$ and consider $\Omega = E(u_i^2 z_i z_i')$

By cholesky decomposition, $\Omega^{-1} = \Omega^{-\frac{1}{2}}\Omega^{-\frac{1}{2}'}$ is possible.

③

$$\hat{b}_{gmm}-\beta = (\mathbf{X}'\mathbf{Z}\hat{\Omega}^{-1}\mathbf{Z}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}\hat{\Omega}^{-1}\mathbf{Z}'\mathbf{u} = \left(\frac{1}{N}\mathbf{X}'\mathbf{Z}\hat{\Omega}^{-1}\frac{1}{N}\mathbf{Z}'\mathbf{X}\right)^{-1}\frac{1}{N}\mathbf{X}'\mathbf{Z}\hat{\Omega}^{-1}\frac{1}{N}\mathbf{Z}'\mathbf{u}.$$

$\Rightarrow$ We can use this skill to part ③.

$$\mathbf{Z}'\mathbf{X}(A^{-1})\mathbf{X}'\mathbf{Z} = \mathbf{Z}'\mathbf{X}(\mathbf{X}'\mathbf{Z}\hat{\Omega}^{-1}\mathbf{Z}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z} = \frac{\mathbf{Z}'\mathbf{X}}{N}\left(\frac{\mathbf{X}'\mathbf{Z}}{N}\hat{\Omega}^{-1}\frac{\mathbf{Z}'\mathbf{X}}{N}\right)^{-1}\frac{\mathbf{X}'\mathbf{Z}}{N}$$

$$\xrightarrow{p} E(z_i x_i')\left[E(x_i z_i')\Omega^{-1}E(z_i x_i')\right]^{-1}E(x_i z_i')$$

Thus, ③ is

$$\left(I-\mathbf{Z}'\mathbf{X}(A^{-1})\mathbf{X}'\mathbf{Z}\hat{\Omega}^{-1}\right) \xrightarrow{p} I-\underbrace{E(z_i x_i')\left[E(x_i z_i')\Omega^{-1}E(z_i x_i')\right]^{-1}E(x_i z_i')\Omega^{-1}}_{=\,▨}$$

Using $\Omega^{-\frac{1}{2}}$ from ② and combining ③ to it,

$$\Omega^{-\frac{1}{2}}\left(I-▨\,\Omega^{-1}\right) = \Omega^{-\frac{1}{2}}\left(\Omega^{\frac{1}{2}}-▨\,\Omega^{-\frac{1}{2}}\right)\Omega^{-\frac{1}{2}} = \left(I-\Omega^{-\frac{1}{2}}▨\,\Omega^{-\frac{1}{2}}\right)\Omega^{-\frac{1}{2}}$$

Considering $\frac{\mathbf{Z}'\mathbf{u}}{\sqrt{N}}$, $\left(I-\Omega^{-\frac{1}{2}}▨\,\Omega^{-\frac{1}{2}}\right)\Omega^{-\frac{1}{2}}\cdot N(0, E(u_i^2 z_i z_i'))$

$$= \left(I-\Omega^{-\frac{1}{2}}▨\,\Omega^{-\frac{1}{2}}\right)\cdot N(0, I_J)$$

Therefore,

$$N \cdot \hat{g}(\hat{b})' W \hat{g}(\hat{b}) = \frac{u'Z}{\sqrt{N}} \left(I - (\hat{\Omega}^{-1})' Z X (A^{-1}) X' Z \right) \hat{\Omega}^{-1} \left(I - Z X A^{-1} X' Z \hat{\Omega}^{-1} \right) \frac{Z'u}{\sqrt{N}}$$

$$\frac{u'Z}{\sqrt{N}} \xrightarrow{d} N(0, E(u_i^2 Z_i Z_i')) = N(0, \Omega) \quad \text{by CLT}$$

$$\hat{\Omega}^{-1} \xrightarrow{p} \Omega^{-1}$$

$$I - Z X (A^{-1}) X' Z \hat{\Omega}^{-1} \xrightarrow{p} I - \underbrace{E(Z_i X_i') \left[E(X_i Z_i') \Omega^{-1} E(Z_i X_i') \right]^{-1} E(X_i Z_i') \Omega^{-1}}_{= \square}$$

By slutsky theorem,

$$\xrightarrow{d} N(0, \Omega) \cdot \left(I - (\Omega^{-1})' \square' \right) \Omega^{-1} \left(I - \square \Omega^{-1} \right) \cdot N(0, \Omega)$$

$$= N(0, \Omega) \cdot (\Omega^{-\frac{1}{2}})' \left(\Omega^{\frac{1}{2}\prime} - (\Omega^{-\frac{1}{2}})' \square' \right) \Omega^{-\frac{1}{2}} \Omega^{-\frac{1}{2}} \left(\Omega^{\frac{1}{2}} - \square \Omega^{-\frac{1}{2}} \right) \Omega^{-\frac{1}{2}} \cdot N(0, \Omega)$$

$$= N(0, I_J') \cdot \left(I - \Omega^{-\frac{1}{2}\prime} \square' \Omega^{-\frac{1}{2}} \right) \left(I - \underbrace{\Omega^{-\frac{1}{2}} \square \Omega^{-\frac{1}{2}}}_{4} \right) \cdot N(0, I_J')$$

4 $\Omega^{-\frac{1}{2}} \square \Omega^{-\frac{1}{2}} = \Omega^{-\frac{1}{2}} E(Z_i X_i') \left[E(X_i Z_i') \Omega^{-1} E(Z_i X_i') \right]^{-1} E(X_i Z_i') \Omega^{-\frac{1}{2}}$

$$\underbrace{\underset{J\times J}{\Omega^{-\frac{1}{2}}} \underset{J\times K}{E(Z_i X_i')}}_{J\times K} \; \underbrace{\underset{K\times J}{E(X_i Z_i')} \underset{J\times J}{\Omega^{-1}} \underset{J\times K}{E(Z_i X_i')}}_{K\times K} \; \underbrace{\underset{K\times J}{E(X_i Z_i')} \underset{J\times J}{\Omega^{-\frac{1}{2}}}}_{K\times J}$$

Let $\Omega^{-\frac{1}{2}} E(Z_i X_i') := \underset{J\times K}{B}$

$$= B (B'B)^{-1} B', \quad J\times J \text{ matrix}$$

$$= N(0, I_J')(I - B(B'B)^{-1}B')(I - B(B'B)^{-1}B') \cdot N(0, I_J)$$

$\Downarrow$ Idempotent!

$$= N(0, I_J')(I_J - B(B'B)^{-1}B') \cdot N(0, I_J) \sim \chi^2_{\underbrace{\text{rank}(I_J - B(B'B)^{-1}B')}_{5}}$$

5 Using the property of trace,

$$\text{trace}(I_J - B(B'B)^{-1}B') = \text{trace}(I_J) - \text{trace}(B(B'B)^{-1}B')$$

$$= J - \text{trace}((B'B)^{-1}B'B) = J - \text{trace}(I_K) = J - K.$$

$$\sim \chi^2_{(J-K)}$$

Through J-test, we can check $H_0: E(u_i z_i) = 0$

That is, $J = N \cdot \hat{g}(\hat{b})' \hat{\Omega}^{-1} \hat{g}(\hat{b}) \xrightarrow{d} \chi^2_{(J-K)}$.

Then, if $H_0: E(u_i z_i) = 0$ cannot be rejected,

we have ① $E(u_i z_i) = 0$ in even $J > K$ case (over-identifying restrictions).

After that, we need to check ② Full rank condition of $E(z_i x_i')$.

It is easy to check $E(z_i x_i')$ has full rank. b/c we have data.

Overidentification.

point When $J > K$, in order to use IV,

Check ① $E(u_i z_i) = 0$ by J-test $\overset{d}{\sim} \chi^2_{(J-K)}$

② $E(z_i x_i')$ is invertible