Lecture 10

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Example

A pairwise stable match may not be group stable

Definition: A matching μ is **group stable** there is no coalition that blocks μ.

Definition: A matching $\mu : (T \cup B) \to (T \cup B \cup \mathcal{B})$ is **pairwise stable** if it is individually rational and there is no blocking pair. That is to say, it is pairwise stable.

With those definitions in mind, take the example where we have pairwise stability but not group stability.

$$T = \{t\}$$
$$B = \{b_1, b_2, b_3\}$$
$$q = 3$$

$$t : B \succ_t \{b_1\} \succ_t \{b_2\} \succ_t \{b_3\} \succ_t \{b_1, b_2\} \succ_t \{b_1, b_3\} \succ_t \{b_2, b_3\} \succ_t t$$
$$\forall i = 1, 2, 3 : t \succ_{b_i} b_i$$

Let $\mu(t) = b_1$.

This match is pairwise stable. Because it makes t worse off, t will not choose to go alone. Additionally, b's cannot block. No pair of agents (t, b_i) would block since no $b_2, b_3 \succ_t b_1$ or any pair.

So, this is pairwise stable, but it is not group stable. Notice the following coalition $C = \{t\} \cup \{b_2, b_3\}$ blocks μ. Then $\mu'(t) = B$ and $\mu(b_i) = t$ for all i. What happened then? What made the difference between pairwise and group stability? The preferences are not responsive:

$\{b_1\} \succ_t \{b_1, b_2\}$ but $\{b_1, b_2, b_3\} \succ_t \{b_1, b_3\}$.

Theorem

Suppose preferences are responsive. A matching μ is group stable if and only if it is pairwise stable.

To prove this theorem, two lemmas are needed.

Lemma 1:

Suppose preferences are responsive. Fix $B_1, B_2, \subset B \setminus \{\emptyset\}$:

(1) B_1 $B_2 \neq \emptyset$ and $|B_2| \geq |B_1|$ In other words, B_1 is not a subset of B_2.
(2) For each $\hat{B} \subseteq B_2$, $B_2 \succsim_t \hat{B}$. This is to say that you can't get rid of stuff and be better off.
(3) $B_1 \succ_t B_2$.

Then, there exists $b_1 \in B_1 \setminus B_2$ *and* $b_2 \in B_2 \setminus B_1$ *such that* $\{b_1\} \succ_t \{b_2\}$*.*

Proof:

We have $B_1 \setminus B_2 = \emptyset$ by assumption. And, we have $|B_2| \geq |B_1| \implies B_2 \setminus B_1 = \emptyset$.

Enermerate:

$$B_1 \setminus B_2 = \{b_1, ..., b_M\}$$
$$B_2 \setminus B_1 = \{b_{M+1}, ..., b_{M+K}\}$$

Suppose contra the hypothesis that for each $m = 1, ..., M$ and each $k = 1, ..., k$:

$$\{b_{k+M}\} \succsim_t \{b_m\}$$

Start with b_1 set. I can write:

$$(B_1 \setminus \{b_1\}) \cup \{b_1\} = B_1$$

By responsive preferences, we have the following:

$$(B_1 \setminus \{b_1\}) \cup \{b_{M+1}\} \succsim_t (B_1 \setminus \{b_1\}) \cup \{b_1\} = B_1$$

Then, keep repeating the processs, throwing out $\{b_2\}, \{b_3\}, ...$ and adding $\{b_{m+2}\}, \{b_{m+3}\}, ...$ That is, we kick out things we like less from the left hand side, and we add back in things that we like better.

So, we have shown that it is true for $k = 1$, and we assume that it is true (inductive hypothesis) for $k \geq 2$ and $k \leq M - 1$.

Then we add the inductive hypothesis: For each $k = 1, ..., M - 1$:

$$\hat{B}_k = (B_1 \setminus \{b_1, ..., b_k\}) \cup \{b_{M+1}, ..., b_{m+k}\} \succsim_t B_1$$

We want to show this is true for the $k + 1$ step. If we can show that $\hat{B}_{k+1} \succsim_t \hat{B}_k$, then by transitivity with the above inductive hypothesis, $\hat{B}_{k+1} \succsim_t B_1$

$$\hat{B}_{k+1} = (B_1 \setminus \{b_1, ..., b_k, b_{k+1}\}) \cup \{b_{M+1}, ..., b_{m+k}\} \cup \{b_{m+k+1}\} \succsim_t (B_1 \setminus \{b_1, ..., b_k, b_{k+1}\}) \cup \{b_{M+1}, ..., b_{m+k}\} \cup \{b_{k+1}\} = \hat{B}_k$$

Therefore, it is true that for any M, $\hat{B}_M \succsim_t \hat{B}_1$. Now, take:

$$\hat{B}_M \succsim_t B_1$$
$$\hat{B}_M = (B_1 \setminus (B_1 \setminus B_2)) \cup \{b_{m+1}, ..., b_{M+M}\}$$

$$\hat{B}_M = (B_1 \cap B_2) \cup \{b_{M+1}, ..., b_{m+m}\} \succsim_t B_1 \succsim_t B_2$$

Now, since $(B_1 \cap B_2) \subseteq B_2$ and $\{b_{M+1}, ..., b_{m+m}\} \subseteq B_2 \setminus B_1 \subseteq B_2$. Therefore, $\hat{B}_M \subseteq B_2$ and so $\hat{B}_M \succsim_t B_2$.

Lemma 2:

Suppose preferences are responsive. Fix $B_1, B_2, \subset B \setminus \{\emptyset\}$*:*

(1) $B_1\ B_2 \neq \emptyset$.
(2) For each $\hat{B} \subseteq B_2$, $B_2 \succsim_t \hat{B}$. This is to say that you can't get rid of stuff and be better off.
(3) $B_1 \succ_t B_2 \succ_t t$.

Then, there exists $b_1 \in B_1 \setminus B_2$ *such that* $\{b_1\} \succ_t t$.

Proof:

Set $B_1 \setminus B_2 = \{b_1, ..., b_M\}$. Then, suppose contra hypothesis $t \succsim_t \{b_m\}$ for each $m = 1, ..., M$.

Responsive preferences indicate the following:

$$B_1 \setminus \{b_1\} \succsim_t (B_1 \setminus \{b_1\}) \cup \{b_1\} = B_1$$

Recall from an earlier lecture that responsive preferences tell us that $\hat{B} \succsim_t \hat{B} \cup \{b\}$ if and only if $t \succsim_t \{b\}$. Here, $\hat{B} = B_1 \setminus \{b_1\}$. Then $\hat{B} = (B_1 \setminus \{b_1\}) \succsim_t \hat{B} \cup \{b_1\} = B_1$.

Then, our inductive hypothesis is that $B_1 \setminus \{b_1, ..., b_m\} \succsim_t B_1$.

$$\hat{B} = B_1 \setminus \{b_1, ..., b_m, b_{m+1}\}$$
$$\hat{B} \succsim \hat{B} \cup \{b_{m+1}\}$$

$$B_1 \setminus \{b_1, ..., b_{m+1}\} \succsim_t B_1 \setminus \{b_1, ..., b_m\}$$

By transitivity:

$$\implies B_1 \setminus \{b_1, ..., b_{m+1}\} \succsim_t B_1$$
$$\iff B_1 \setminus (B_1 \setminus B_2) \succsim_t B_1$$
$$\iff B_1 \cap B_2 \succsim_t B_1$$

$$\implies B_1 \cap B_2 \succ_t B_2$$

Then, a subset of B_2 is greater than B_2. A contradiction of one of our assumptions.