Econ 501A HW 1

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Problem 1.1

Suppose $X = \{a, b, c\}$ and $u(a) = 1$, $u(b) = 1$, and $u(c) = 2$ For each $B \subset X$, compute $C_u(b)$

There are seven such subsets B of X:

(1) $B = \{a\}$: $C_u(B) = a$
(2) $B = \{b\}$: $C_u(B) = b$
(3) $B = \{c\}$: $C_u(B) = c$
(4) $B = \{a, b\}$: $C_u(B) = \{a, b\}$
(5) $B = \{a, c\}$: $C_u(B) = c$
(6) $B = \{b, c\}$: $C_u(B) = c$
(7) $B = \{a, b, c\}$: $C_u(B) = c$

Technically, $\emptyset$ could also be a subset. But, following the prescription of the notes, we write:

$$C_u : \mathcal{P}(X) \setminus \emptyset \to \mathcal{P}(X) \setminus \emptyset$$

So, I exclude it.

Problem 1.2

(a) Prove that $u^*(B)$ is increasing in B. That is, prove that if $B \in B'$, the $u^*(B) \leq u^*(B')$. In words, increasing the set of feasible alternatives never hurts the individual and sometimes helps the individual.

$$B \subset B' \implies \sup u(x)_{x \in B} \leq \sup u(x)_{x \in B'} \implies u^*(B) \leq u^*(B')$$

Let $B \subset B'$. Then B' contains at least one more element than B. If $B \subseteq B'$, then it may contain the same elements. If B' has more elements than B, then these elements contained in B' will either be larger or smaller than the maximum element of B which is also found, by definition, in B'. If the elements in B' not in B are smaller than the maximum element in B then $\sup u(x)_{x \in B} = \sup u(x)_{x \in B'} \implies u^*(B) = u^*(B')$.

If there is an element in B' that is not in B which is greater than all elements in B, then, $\sup u(x)_{x \in B} < \sup u(x)_{x \in B'} \implies u^*(B) < u^*(B')$.

An increasing function is one such that each subsequent value is greater than or equal to the previous value (Precisely: $\forall x, y \in \mathbb{R} : x < y \implies f(x) < f(y)$. Hence, in both cases above, $u^*(B)$ is an increasing function.

(b) Prove that if $y \in B$ but $y \notin C_u(B)$ then $u^*(B) = u^*(B \setminus y)$, where $B \setminus y$ denotes the set consisting of all elements of B expect y. In words, unchosen alternatives do not affect welfare.

Proof by contraposition: $u^*(B) \neq u^*(B \setminus y) \implies y \in C_u(B)$

Let $y \in B$ and assume $u^*(B) \neq u^*(B \setminus y)$. Then, the $\sup u(x)_{x \in B} \neq \sup u(x)_{x \in B \setminus y}$. Hence, y must be the maximum element in the set of B. Therefore, $y \in C_u(B)$.

Problem 1.3

Suppose $X \supset \{a, b, c\}$. (a) Of course, $C_u(\{a, n\})$ is a subset of $\{a, b\}$. How many subsets of $\{a, b\}$ are there? How many values might $C_u(\{a, b\})$ take and what are they?

There are four subsets:

(1) $\emptyset$
(2) $\{a\}$
(3) $\{b\}$
(4) $\{a, b\}$

$C_u(\{a, b\})$ could take 3 values. It excludes the empty set, so we exlude case (1) from all further analysis in this part. If $a > b$ then it will take the value a unles case (3) where it will only have b.

If $b > a$ then it will take the value of b unless case (2) in which it will take value of a.

If $a = b$, then it will take value a for case (1), value b for case (2) and the set $\{a, b\}$ for case (3).

(b) Suppose that $C_u(\{a, b\}) = b$. What can we conclude about $u(a)$ versus $u(b)$? Suppose further that $C_u(\{b, c\} = c$. What can we conclude about $C_u(\{a, c\})$?

First, we can conclude that $u(a) < u(b)$.

Second, we can conlude that $u(a) < u(c)$ by transitivity.

(c) Suppose $a \in B$ but $a \notin C_u(B)$ What can we conclude about $C_u(B \setminus a)$ as compared to $C_u(B)$

We can conclude that $C_u(B \setminus a) = C_u(B)$.

Problem 1.4

Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Define a function $v : X \to \mathbb{R}$ so that $v(x) = f(u^*(x))$. Then, $C_u(B) = C_v(B)$.

(a) Prove the above proposition

Let $x, y \in B$ and $u^*(B \setminus x) < u^*(x)$ such that $u^*(y) < u^*(x)$ and by definition $u^*(x), u^*(y) \in \mathbb{R}$.

Define $f : \mathbb{R} \to \mathbb{R}$ to be a strictly increasing function such that:

$$\forall a, b \in \mathbb{R} : a < b \implies f(a) < f(b).$$

Hence, $f(u^*(y)) < f(u^*(x))$. Define $C_v(B) = \arg\max f(u^*(z))_{z \in B}$ Now, $C_v(B) = x$.

By definition $C_u(B) = \arg\max u^*(z)_{z \in B} = x$.

Hence, $C_u(B) = C_v(B)$.

(b) What if instead f is a weakly increasing function? State and prove an alternative proposition in this case.

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Define a function $v : X \to \mathbb{R}$ so that $v(x) = f(u^*(x))$. Then, $C_u(B) \geq C_v(B)$.

Let $x, y \in B$ and $u^*(B \setminus x) < u^*(x)$ such that $u^*(y) < u^*(x)$ and by definition $u^*(x), u^*(y) \in \mathbb{R}$.

Define $f : \mathbb{R} \to \mathbb{R}$ to be an increasing function such that:

$$\forall a, b \in \mathbb{R} : a < b \implies f(a) \leq f(b).$$

Hence, $f(u^*(y)) \leq f(u^*(x))$. Define $C_v(B) = \arg\max f(u^*(z))_{z \in B}$ Now, $C_v(B) = x' \in B \leq x \in B$. By definition $C_u(B) = \arg\max u^*(z)_{z \in B} = x$.

Hence, $C_u(B) \geq C_v(B)$.

Problem 2.1

Recall the previous problem where $X = \{a, b, c\}$ and $u(a) = 1$, $u(b) = 1$, and $u(c) = 2$. Write down the preference derived from this u. That is for every pair $x, y \in X$, indicate whether $x \succeq y$ or $x \not\succeq y$

(1) $\{a, b\}$: $a \succeq b$ and $b \succeq a$
(2) $\{a, c\}$: $c \succeq a$ and $a \not\succeq c$
(3) $\{b, c\}$: $b \not\succeq c$ and $c \succeq b$

Also note that: $a \succeq a$, $b \succeq b$ and $c \succeq c$.

Problem 2.2

For each of the following four preferences, answer the follwing two questions: Is the preference complete? Is it transitive? Prove your answers

In parts (a), (b), and (c), let $X = \mathbb{R}^2_+$ so an element $x \in X$ is a vector: $x = (x_1, x_2)$ where x_1 and x_2 are nonnegative real numbers.

Note: I will use vector $z = (z_1, z_2)$ as I prove transitivity.

(a) $x \succeq y \iff x_1 + x_2 \geq y_1 + y_2$

The preference is complete.

$$(1)\ x_1 + x_2 < y_1 + y_2$$
$$(2)\ x_1 + x_2 = y_1 + y_2$$
$$(3)\ x_1 + x_2 > y_1 + y_2$$

Which imply:

$$(1)\ y \succ x$$
$$(2)\ x \sim y$$
$$(3)\ x \succ y$$

The preference is transitive. Let $y_1 + y_2 \geq z_1 + z_2$ Then, $x \succeq y \succeq z \iff x_1 + x_2 \geq y_1 + y_2 \geq z_1 + z_2$ and so $x_1 + x_2 \geq z_1 + z_2 \iff x \succeq z$

(b) $x \succeq y \iff x_1 \geq y_1$ **and** $x_2 \geq y_2$

This is not complete. Let $x_1 < y_1$ and $x_2 \geq y_2$. This cannot be represented by any preference relation.

It is transitive. Suppose

$$y_1 \geq z_1$$

$$y_2 \geq z_2$$

Then, we have:

$$x_1 \geq y_1 \geq z_1 \implies x_1 \geq z_1$$

$$x_2 \geq y_2 \geq z_2 \implies x_2 \geq z_2 \implies x \succeq z$$

(c) $x \succeq y \iff$ **either** $x_1 > y_1$ **or both** $x_1 = y_1$ **and** $x_2 \geq y_2$**.**

This is complete.

$$x_1 = y_1 \text{ and } x_2 < y_2 \implies y \succ x$$
$$x_1 > y_1 \implies x \succ y$$
$$x_1 = y_1 \text{ and } x_2 = y_2 \implies x \sim y$$

This is transitive.

$$x_1 > y_1$$
$$y_1 > z_1 \implies x_1 > z_1 \implies x \succeq z$$

Or note the following:

$$x_1 = y_1 = z_1$$
$$x_2 \geq y_2 \text{ and } y_2 \geq z_2 \implies x_2 \geq z_2$$
$$\implies x \succeq z$$

(d) Now let $X = \{1, 2, 3\}$**. There are three individuals, each with their own complete and transitive preferences: For Ann,** $1 \succ_a 2 \succ_a 3$**. For Bob,** $2 \succ_b 3 \succ_b 1$**. For Carol,** $3 \succ_c 1 \succ_c 2$**. Define a new preference** $\succeq$ **such that** $x \succeq y$ **if and only if at least two of the three individuals prefer** y **to** x**. Is this new preference transitive and complete?**

This preference is complete.

$$1 \succeq 2$$
$$3 \succeq 1$$
$$2 \succeq 3$$

So, all bases are covered. Now, this is not transitive because:

$$1 \succeq 2 \succeq 3 \succeq 1$$

Which is circular.