

# Econ 501A HW 1

David Zynda

August 27, 2018

## Problem 1.1

Suppose  $X = \{a, b, c\}$  and  $u(a) = 1$ ,  $u(b) = 1$ , and  $u(c) = 2$ . For each  $B \subset X$ , compute  $C_u(B)$ .

There are seven such subsets  $B$  of  $X$ :

- (1)  $B = \{a\}$ :  $C_u(B) = a$
- (2)  $B = \{b\}$ :  $C_u(B) = b$
- (3)  $B = \{c\}$ :  $C_u(B) = c$
- (4)  $B = \{a, b\}$ :  $C_u(B) = \{a, b\}$
- (5)  $B = \{a, c\}$ :  $C_u(B) = c$
- (6)  $B = \{b, c\}$ :  $C_u(B) = c$
- (7)  $B = \{a, b, c\}$ :  $C_u(B) = c$

Technically,  $\emptyset$  could also be a subset. But, following the prescription of the notes, we write:

$$C_u : \mathcal{P}(X) \setminus \emptyset \rightarrow \mathcal{P}(X) \setminus \emptyset$$

## Problem 1.2

(a) Prove that  $u^*(B)$  is increasing in  $B$ . That is, prove that if  $B \subseteq B'$ , then  $u^*(B) \leq u^*(B')$ . In words, increasing the set of feasible alternatives never hurts the individual and sometimes helps the individual.

$$B \subset B' \implies \sup_{x \in B} u(x) \leq \sup_{x \in B'} u(x) \implies u^*(B) \leq u^*(B')$$

Let  $B \subset B'$ . Then  $B'$  contains at least one more element than  $B$ . If  $B \subseteq B'$ , then it may contain the same elements. If  $B'$  has more elements than  $B$ , then these elements contained in  $B'$  will either be larger or smaller than the maximum element of  $B$  which is also found, by definition, in  $B'$ . If the elements in  $B'$  not in  $B$  are smaller than the maximum element in  $B$  then  $\sup_{x \in B} u(x) = \sup_{x \in B'} u(x) \implies u^*(B) = u^*(B')$ .

If there is an element in  $B'$  that is not in  $B$  which is greater than all elements in  $B$ , then,  $\sup_{x \in B} u(x) < \sup_{x \in B'} u(x) \implies u^*(B) < u^*(B')$ .

An increasing function is one such that each subsequent value is greater than or equal to the previous value. Hence, in both cases above,  $u(B)$  is an increasing function.

(b) Prove that if  $y \in B$  but  $y \notin C_u(B)$  then  $u^*(B) = u^*(B \setminus y)$ , where  $B \setminus y$  denotes the set consisting of all elements of  $B$  except  $y$ . In words, unchosen alternatives do not affect welfare.

Proof by contraposition:  $u^*(B) \neq u^*(B \setminus y) \implies y \in C_u(B)$

Let  $y \in B$  and assume  $u^*(B) \neq u^*(B \setminus y)$ . Then,  $\sup_{x \in B} u(x) \neq \sup_{x \in B \setminus y} u(x)$ . Hence,  $y$  must be the maximum element in the set of  $B$ . Therefore,  $y \in C_u(B)$ .

## Problem 1.3

Suppose  $X \supset \{a, b, c\}$ . (a) Of course,  $C_u(\{a, n\})$  is a subset of  $\{a, b\}$ . How many subsets of  $\{a, b\}$  are there? How many values might  $C_u(\{a, b\})$  take and what are they?

There are four subsets:

- (1)  $\emptyset$
- (2)  $\{a\}$
- (3)  $\{b\}$
- (4)  $\{a, b\}$

$C_u(\{a, b\})$  could take 3 values. It excludes the empty set, so we exclude case (1) from all further analysis in this part. If  $a > b$  then it will take the value  $a$  unless case (3) where it will only have  $b$ .

If  $b > a$  then it will take the value of  $b$  unless case (2) in which it will take value of  $a$ .

If  $a = b$ , then it will take value  $a$  for case (1), value  $b$  for case (2) and the set  $\{a, b\}$  for case (3).

(b) Suppose that  $C_u(\{a, b\}) = b$ . What can we conclude about  $u(a)$  versus  $u(b)$ ? Suppose further that  $C_u(\{b, c\}) = c$ . What can we conclude about  $C_u(\{a, c\})$ ?

First, we can conclude that  $u(a) < u(b)$ .

Second, we can conclude that  $u(a) < u(c)$  by transitivity.

(c) Suppose  $a \in B$  but  $a \notin C_u(B)$ . What can we conclude about  $C_u(B \setminus a)$  as compared to  $C_u(B)$ ?

We can conclude that  $C_u(B \setminus a) = C_u(B)$ .

## Problem 1.4

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a strictly increasing function. Define a function  $v : X \rightarrow \mathbb{R}$  so that  $v(x) = f(u^*(x))$ . Then,  $C_u(B) = C_v(B)$ .

(a) Prove the above proposition

Let  $x, y \in B$  and  $u(B \setminus x) < u^*(x)$  such that  $u^*(y) < u^*(x)$  and by definition  $u^*(x), u^*(y) \in \mathbb{R}$ .

Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  to be a strictly increasing function such that:

$$\forall a, b \in \mathbb{R} : a < b \implies f(a) < f(b).$$

Hence,  $f(u^*(y)) < f(u^*(x))$ . Define  $C_v(B) = \arg \max_{z \in B} f(u^*(z))$ . Now,  $C_v(B) = x$ .

By definition  $C_u(B) = \arg \max_{z \in B} u^*(z) = x$ .

Hence,  $C_u(B) = C_v(B)$ .

(b) What if instead  $f$  is a weakly increasing function? State and prove an alternative proposition in this case.

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be an increasing function. Define a function  $v : X \rightarrow \mathbb{R}$  so that  $v(x) = f(u^*(x))$ . Then,  $C_u(B) \geq C_v(B)$ .

Let  $x, y \in B$  and  $u(B \setminus x) < u^*(x)$  such that  $u^*(y) < u^*(x)$  and by definition  $u^*(x), u^*(y) \in \mathbb{R}$ .

Define  $f : \mathbb{R} \rightarrow \mathbb{R}$  to be an increasing function such that:

$$\forall a, b \in \mathbb{R} : a < b \implies f(a) \leq f(b).$$

Hence,  $f(u^*(y)) \leq f(u^*(x))$ . Define  $C_v(B) = \arg \max_{z \in B} f(u^*(z))$ . Now,  $C_v(B) = x' \in B \leq x \in B$ . By definition  $C_u(B) = \arg \max_{z \in B} u^*(z) = x$ .

Hence,  $C_u(B) \geq C_v(B)$ .

## Problem 2.1

Recall the previous problem where  $X = \{a, b, c\}$  and  $u(a) = 1$ ,  $u(b) = 1$ , and  $u(c) = 2$ . Write down the preference derived from this  $u$ . That is for every pair  $x, y \in X$ , indicate whether  $x \succeq y$  or  $x \not\succeq y$

- (1)  $\{a, b\}$ :  $a \succeq b$  and  $b \succeq a$
- (2)  $\{a, c\}$ :  $c \succeq a$  and  $a \not\succeq c$
- (3)  $\{b, c\}$ :  $b \not\succeq c$  and  $c \succeq b$

Also note that:  $a \succeq a$ ,  $b \succeq b$  and  $c \succeq c$ .

## Problem 2.2

For each of the following four preferences, answer the following two questions: Is the preference complete? Is it transitive? Prove your answers

In parts (a), (b), and (c), let  $X = \mathbb{R}_+^2$  so an element  $x \in X$  is a vector:  $x = (x_1, x_2)$  where  $x_1$  and  $x_2$  are nonnegative real numbers.

- (a)  $x \succeq y \iff x_1 + x_2 \geq y_1 + y_2$

The preference is complete.

$$(1) x_1 + x_2 < y_1 + y_2 (2) x_1 + x_2 = y_1 + y_2 (3) x_1 + x_2 > y_1 + y_2$$

Which imply:

$$(1) y \succ x (2) x \sim y (3) x \succ y$$

The preference is transitive. Let  $y_1 + y_2 \geq z_1 + z_2$  Then,  $x \succeq y \succeq z \iff x_1 + x_2 \geq y_1 + y_2 \geq z_1 + z_2$  and so  $x_1 + x_2 \geq z_1 + z_2 \iff x \succeq z$

- (b)  $x \succeq y \iff x_1 \geq y_1$  **and**  $x_2 \geq y_2$

This is not complete. Let  $x_1 < y_1$  and  $x_2 \geq y_2$ . This cannot be represented by any preference relation.

It is transitive. Suppose

$$y_1 \geq z_1, y_2 \geq z_2$$

Then, we have:

$$x_1 \geq y_1 \geq z_1 \implies x_1 \geq z_1, x_2 \geq y_2 \geq z_2 \implies x_2 \geq z_2$$

\_\_\_(c)  $x \succeq y \iff$  either  $x_1 > y_1$  or both  $x_1 = y_1$  and  $x_2 \geq y_2$ .