- Partial equilibrium (Until today)

* commodities : l and numeraire

* Consumers

- consumption set $\mathbb{R} \times \mathbb{R}_+$: $(m_i, x_i) \in \mathbb{R} \times \mathbb{R}_+$ — endowment : $w_i = (w_{m_i}, 0)$
- $u(m_i, x_i) = m_i + \phi_i(x_i)$, $\phi_i' > 0$, $\phi_i'' < 0$, $\phi(0) = 0$, bounded above

(m_i: numeraire (money))

* firms j

- use numeraire to produce q_j of l
- $C : \mathbb{R}_+ \to \mathbb{R}$, $C' > 0$, $C'' > 0$
- $Y_j = \{(-z_j, q_j) : q_j \geq 0$ and $z_j \geq C_j(q_j)\}$ (z_j: used numeraire.)

* Price : $(1, p)$ (1: numeraire price.)

⇒ Competitive equilibrium (x^*, m^*, q^*, z^*) and $(1, p^*)$ s.t.

① Firms : $\max_{q_j \geq 0} [p^* q_j - C_j(q_j)]$

<profit max>

$\Rightarrow p^* \leq C_j'(q_j)$ with equality if $q_j^* > 0$ (1_j)

$\max_{z_j \geq C_j(q_j^*)} [-1 z_j] \Rightarrow z_j^* = C_j(q_j^*)$ (2_j)

② Consumer <util. max.>

$$\max [m_i + \phi_i(x_i)] \quad \text{s.t.} \quad 1 \cdot m_i + p^* x_i \leq \underbrace{w_{m_i}}_{\text{endowment of money}} + \sum_j \theta_{ij} \underbrace{[p^* q_j^* - C_j(q_j^*)]}_{\text{profit of } j \text{ by the ratio of owning firms}}$$

$$\Rightarrow \begin{cases} \phi_i'(x_i^*) \leq p^* \text{ with equality if } x_i^* > 0 & (3_i) \\ m_i^* + p_i^* x_i^* = w_{m_i} + \sum_j \theta_{ij} [p^* q_j^* - C_j(q_j^*)] & (4_i) \end{cases}$$

③ Market clearing

$$\underbrace{\sum_i x_i^*}_{\text{consumption}} = \underbrace{\sum_j q_j^*}_{\text{production}} \quad \text{and} \quad \sum_i m_i^* = \underbrace{\bar{w}_m}_{\text{sum of entire econ. money}} - \underbrace{\sum_j z_j^*}_{\text{money for production}} \qquad (5)$$

$$\mathcal{U}(x,q) = \{ v \in \mathbb{R}^I : \underbrace{\textstyle\sum v_i}_{\text{Sum of util. vector}} \le \underbrace{\bar{w}_m}_{\text{Sum of money}} + S(x,q) \}$$

$\rightarrow \underset{\text{(boundary)}}{bd}(\mathcal{U}(x,q)) = \{ v \in \mathbb{R}^I : \sum V_i = \bar{w}_m + S(x,q) \}$

$$S(x,q) = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$$: Marshallian surplus

$\Rightarrow$ Feasible allocation (x, m, q, z)

$$\sum_i x_i = \sum_j q_j \quad \text{and} \quad \sum_i m_i = \sum_i w_{m_i} - \sum_j z_j$$

If at competitive eq,

$\bar{w}_m = \sum_i w_{m_i} = \sum_i m_i^* + \sum_j z_j^*$ (comp. eq)

$= \sum_i m_i + \sum_j z_j$ (Just feasible)

$\Rightarrow$ Say allocation (x, m, q, z) induces $v \in \mathbb{R}^I$ if, each i, $V_i = m_i + \phi_i(x_i)$

Lemma1

(1) If (x, m, q, z) is feasible, then it induces $v \in \mathcal{U}(x,q)$

(2) If $v \in bd(\mathcal{U}(x,q))$ and $\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} q_j$, then $\exists\ (m,z)$ such that (x, m, q, z) is feasible and induces v.

proof) (1) Done last class.

(2) WTS : (x, m, q, z) is feasible and induces V.

We have $\sum_i x_i = \sum_j q_j$. Let $z_j = c_j(q_j)$.

$V_i = m_i + \phi_i(x_i)$ Then take $m_i = -\phi_i(x_i)$

by assumption : (x, m, q, z) that induces v

To show feasibility : $\sum_{i=1}^{I} m_i = \sum_{i=1}^{I} V_i - \sum_{i=1}^{I} \phi_i(x_i)$

Since $v \in bd(\mathcal{U}(x,q))$

$= \bar{w}_m + S(x,q) - \sum_{i=1}^{I} \phi_i(x_i)$

$= \bar{w}_m - \sum_{j=1}^{J} c_j(q_j)$ (by def. of $S(x,q)$)

$= \bar{w}_m - \sum_{j=1}^{J} z_j$ (by choice of z_j). //

Remark $\exists$ a close connection between the boundary of $\mathcal{U}(x,q)$, Pareto optimality, and competitive equilibrium problem (*) :

choose (x, q) to solve

$$\max_{(x', q')} [\bar{w}_m + S(x', q')] \text{ s.t. } \sum_{i=1}^{I} x_i' - \sum_{j=1}^{J} q_j' = 0$$

Notice : If (x, m, q, z) is feasible and (x, q) solves problem(*), then the allocation induces $V \in bd(\mathcal{U}(x, q))$

Lemma 2 If (x^*, m^*, q^*, z^*) is a competitive equilibrium allocation, then (x^*, q^*) solves problem (*).

proof) $\exists$ $(1, p^*)$ satisfying conditions (1) - (5)

- (1j) + (3i) says for each i, j

$$\underbrace{\underbrace{\phi_i'(x_i^*) \leq p^*}_{(3i)} \leq c_j'(q_j^*)}_{(1j)} \Rightarrow \sum_{i=1}^{I} \phi_i'(x_i^*) \leq \sum_{j=1}^{J} c_j'(q_j^*)$$

- By (5), $\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$ $\longrightarrow$ In this case, from problem (*) We can know to get its solution by this.

$\Rightarrow$ Solution to Problem (*). //

< First Welfare Theorem >

$\boxed{\text{If } (x^*, m^*, q^*, z^*) \text{ is a competitive equilibrium, then it is Pareto optimal.}}$

Note (x, m, q, z) is Pareto optimal if it is feasible and then is no other feasible allocation (x', m', q', z') s.t.

(1) $u_i(m_i', x_i') \geqq u_i(m_i, x_i)$ for all $i = 1, \ldots, I$

(2) $u_i(m_i', x_i') > u_i(m_i, x_i)$ for some $i = 1, \ldots, I$

proof)

Suppose (x^*, m^*, q^*, z^*) is a Competitive equilibrium

but that is NOT Pareto optimal.

(x^*, m^*, q^*, z^*) is feasible, so "NOT Pareto optimal" implies

that there is some feasible (x, m, q, z) with

① $m_i + \phi_i(x_i) \geq m_i^* + \phi_i(x_i^*)$ for all i

② $m_i + \phi_i(x_i) > m_i^* + \phi_i(x_i^*)$ for some i

↓ SUM UP!

$$\Rightarrow \sum_{i=1}^{I} m_i + \sum_{i=1}^{I} \phi_i(x_i) > \sum_{i=1}^{I} m_i^* + \sum_{i=1}^{I} \phi_i(x_i^*) \qquad \text{(Equation 1)}$$

↖ The result of our supposition

(From the definition of competitive equilibrium)

Since (x^*, m^*, q^*, z^*) is a Competitive equilibrium,

Lemma 1 says that (x^*, q^*) solves problem (*)

from $u(x, q)$

$$\Rightarrow \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i^*) - \sum_{j=1}^{J} c_j(q_j^*) \geq \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$$

(Since (x, m, q, z) satisfies aggregate demand = supply)

$$\underline{\bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i^*) - \sum_{j=1}^{J} z_j^*} = \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i^*) - \sum_{j=1}^{J} c_j(q_j^*) \quad \leftarrow (\text{Comp. eq} \Rightarrow \sum z_j^* = \sum c_j(q_j^*))$$

$$\geq \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$$

) ← production technology $z_j \geq c_j(q_j)$ i.e., $-c_j(q_j) \geq -z_j$

$$\geq \underline{\bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} z_j}$$

From ⊛, $\sum m_i^* + \sum z_j^* - \bar{w}_m = \sum m_i + \sum z_j$.

$$\sum_{i=1}^{I} m_i^* + \cancel{\sum_{j=1}^{J} z_j^*} + \sum_{i=1}^{I} \phi_i(x_i^*) - \cancel{\sum_{j=1}^{J} z_j^*} \geq \sum_{i=1}^{I} m_i + \cancel{\sum_{j=1}^{J} z_j} + \sum_{i=1}^{I} \phi_i(x_i) - \cancel{\sum_{j=1}^{J} z_j}$$

$$\sum_{i=1}^{I} m_i^* + \sum_{i=1}^{I} \phi_i(x_i^*) \geq \sum_{i=1}^{I} m_i + \sum_{i=1}^{I} \phi_i(x_i), \text{ Contradicting Eq 1.}$$

<Second Welfare Theorem>

Fix some (x^*, q^*) s.t. $\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*$

distribution of wealth

$\downarrow$

If $v^* \in bd(\mathcal{U}(x^*, q^*))$, then $\exists$ some (m^*, z^*) and $W_m^* = (W_{m_1}^*, \dots, W_{m_I}^*)$ with $\overline{W_m^*} = \overline{W_m}$ s.t.

(1) (x^*, m^*, q^*, z^*) is a competitive equilibrium &

(2) (x^*, m^*, q^*, z^*) induces v^*

Idea) Since $v^* \in bd(\mathcal{U}(x^*, q^*))$, then (x^*, q^*) solves problem $(*)$

- Lagrange multipliers of the equality constraint : λ
- $p^* = \lambda$ (If we choose $p^* = \lambda$, it makes competitive eq satisfies $(j \sim 3_i)$)
- Satisfies $(1j)-(3_i)$ of competitive eq. for all i, j
- $z_j^* = C_j(q_j^*)$ satisfies $(2j)$
- for competitive equilibrium need ;

(a) $m_i^* + p^* x_i^* = W_{m_i} + \sum \theta_{ij} [p^* q_j^* - C_j(q_j^*)]$ $\leftarrow$ lower i's endowment of numeraire by $1 & lower her consumption of m_i by $1 & retain budget balance

(b) $\sum_{i=1}^{I} m_i^* = \sum_{i=1}^{I} W_{m_i}^* - \sum_{j=1}^{J} z_j^*$ $\uparrow$ From (5)

$\Downarrow$

Since $v^* \in bd(\mathcal{U}(x^*, q^*))$ can ensure that everything adds up to satisfy market clearing.