Lecture 5

David Zynda

September 4, 2018

Preliminary Review from Last Lecture

Consider environment:

$$\epsilon = (T \cup B, (\succsim)_{i \in T \cup B})$$

And Matching:

$$\mu : (T \cup B) \to (T \cup B)$$

DA Algorithm (T-propsal)

- k proposals $\hat{P}^k : T \to B \cup \{\emptyset\}$
- k tentative accept $\hat{\mu}^k : B \to T \cup \{\emptyset\}$
- $\hat{\mu}^k = \hat{\mu}^{k+1}$

Definition: $\mu^* : (T \cup B) \to (T \cup B)$ is *I-Optimal Stable match* if

1. μ^* is stable
2. $\mu^* \geq_I \mu$ for all stable μ

$\mu^* >_I \mu$ if for all $i \in I$ $\mu^* \geq_I \mu$ and for some $i \in I : \mu^*(i) \succ_i \mu(i)$

Proposition (Thm 2.12 in Roth, Sotomayor?):

Suppose preferences are strict. Then for each $I \in \{T, B\}$ the match $\mu^*_{ID} : (T \cup B) \to (T \cup B)$ is an I-optimal stable match.

Definition: Say b is acheivable for t if there exists some stable match μ with $\mu(b) = t$.

Lemma 1:

In the T-proposal DA algorithm, the following holds for each $k = 1, 2, ...$: If $\hat{P}^k(t) = b$ and b is acheivable for t, then $\hat{\mu}^k(b) = t$

Proof:

Suppose the result is not true. Then there exists some k such that:

1. for each $l < k$ and each $t \in T$: if $\hat{P}^l(t) = b$ and b is acheivable for t, then $\hat{\mu}^l(b) = t$
2. There is some $t^* \in T$ such that $\hat{P}^k(t^*) = b^*$, b^* is acheivable for t^* and $\hat{\mu}^k(b^*) \neq t^*$

There is stable matching $\mu : (T \cup B) \to (T \cup B)$ with $\mu(t^*) = b^*$.

Now, $\hat{\mu}^k(b^*) \neq t^* \implies t^{**} = \hat{\mu}^k(b^*)$ and $t^* \neq t^{**}$

Observe $\mu(t^{**}) \neq b^*$

Will show: (t^{**}, b^*) blocks μ, contradicting the fact that μ is stable.

$$t^{**} \succ_{b^*} \mu(b^*)$$

$$b^* \succsim_{t^{**}} \mu(t^{**})$$

First

$$t^{**} \succ_{b^*} \mu(b^*) \implies \hat{P}^k(t^*) = \hat{P}^k(t^{**}) = b^*$$

$$\hat{\mu}^k(b^*) = t^{**} \succ_{b^*} t^* = \mu(b^*)$$

Second

$b^* \succsim_{t^{**}} \mu(t^{**})$

- observe that for each $l < k$, $\mu(t^{**}) \neq \hat{p}^l(t^{**})$
- if there were an $l \leq k$ such that $\hat{P}^l(t^{**}) = \mu(t^{**}) = b^{**}$, whoever is getting that offer b^{**} is acheivable to t^{**} by the fact that whoever got that offer is preferable.
- By (1) it must be the case that whoever got that offer $\hat{\mu}^l(\hat{\mu}(t^{**}) = b^{**}) = t^{**}$
- Its got to be the case by the DA Algorithm, $\hat{P}^{l+1}(t^{**}) = \mu(t^{**})$
- Then this implies $\hat{P}^k(t^{**}) = \mu(t^{**})$ but this cannot be
- We know on round k, $\hat{P}^k(t^{**}) = b^*$. We know that $\mu(t^*) = b^*$. So, that implies that $\mu(t^{**}) \neq b^*$ and $b^* \neq b^{**}$.

We know that $\hat{P}^k(t^{**}) \succsim_{t^{**}} b$ for $b \neq \{\hat{P}^l(t^{**}) : l = 1, 2, ..., k\}$ by the DA algorithm and strict preferences.

On round k make proposal to b^*: $b^* = \hat{P}^k(t^{**}) \succsim_{t^{**}} \mu(t^{**})$

Proof of Proposition for $I = T$

Fix a stable match $\mu : (T \cup B) \to (T \cup B)$. Required to prove: for each $t \in T$ agent, the $\mu_{TD}(t) \succsim_t \mu(t)$. Fix some $t \in T$ One possibility: if, for each k, $\hat{P}^k(t) \neq \mu(t)$. Does not choose best for whatever reason. In the DA Algorithm (μ_{TD})I make first offer to best, second to second best, so certainly here, $\mu_{TD}(t) \succ_t \mu(t)$

Suppose there is some k such that $\hat{P}^k(t) = \mu(t)$. By Lemma 2, $\hat{\mu}^k(\mu(t)) = t \implies \hat{P}^{k+1}(t) = \hat{P}^k(t)$. Then, $\hat{\mu}^{k+1}(\mu(t)) = t$

Under the DA Algorithm, this implies $\mu_{TD}(t) = \mu(t)$ and certainly $\mu_{TD}(t) \succsim_t \mu(t)$ and we are done. Strict preferences are important for the result.

Indifference

Many properties considered require that all agents have strict preferences. But, what if this isn't so? What if one agent is indifferent? In particular, can a matching be *I-Optimal*?

Example 1

$T = \{t_1, t_2, t_3\}$ and $B = \{b_1, b_2, b_3\}$

$$t_1 : b_2 \sim b_3 \succ b_1 \succ t_1$$
$$t_2 : b_2 \succ b_1 \succ t_2 \succ b_3$$
$$t_3 : b_3 \succ b_1 \succ t_3 \succ b_2$$

$$b_1 : t_1 \succ t_2 \succ t_3 \succ b_1$$
$$b_2 : t_1 \succ t_2 \succ b_2 \succ t_3$$
$$b_3 : t_1 \succ t_3 \succ b_3 \succ t_2$$

If μ is stable and t_2 is matched,

$$\mu(t_2) \in \{b_2, b_1\}$$

If μ is stable and t_3 is matched,

$$\mu(t_3) \in \{b_3, b_1\}$$

Suppose that $\mu(t_2) = b_1$ then $\mu(t_3) = b_3 \implies \mu(t_1) = b_2$. Then, we have stable match 1.

Another possibility: $\mu(t_2) = b_2$. Then $\mu(t_3) = b_3$ and $\mu(t_1) = b_1$. But the (t_1, b_3) form a block, so that is not going to work.

So, if $\mu(t_2) = b_2$, it must be the case that $\mu(t_3) = b_1$ and $\mu(t_1) = b_3$. Then we have stable match 2.

In match 1, t_3 is pretty happy since he gets his best match. In match 2, t_2 gets his best. And, b_2 prefers match 1, but b_3 prefers match 2.

This can only happen because t_1 is indifferent.

So, there are no *I-Optimal* stable matchings when one agent is indifferent.

Theorem (Knuth)

Suppose preferences are strict. If μ and μ' are stable matches, then $\mu >_T \mu'$ if and only if $\mu' >_B \mu$.

Definition Let $\mu' : (T \cup B) \to (T \cup B)$ is an *I-Pessimal* stable match is:

1. μ' is stable
2. And $\mu \geq_I \mu'$ for all other stable matches μ.

Corollary

Suppose preferences are strict. Then μ_{ID} is J-pessimal.