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ECON 501B

ECON 501B: Problem Set 2

Due: Thursday, September 6, 2018

Instructions: Answers should be complete proofs of a claim.

Question 1: Fix an environment $\mathcal{E} = (T, B; (\succsim)_{i \in T \cup B})$. This question will ask you to apply the T-proposer Deferred Acceptance Alogorithm to two environments: First, where T and B each have $M < \infty$ agents. Second, where T and B each have an countable number of agents. The examples are meant to highlight differences/peculiarities that arise, in going from the first setting to the second.

Question 1a. Suppose that, for each $t_i \in T$,

$$b_i \succ_{t_i} b_{i-1} \succ_{t_i} \cdots \succ_{t_i} b_1 \succ_{t_i} t_i,$$

and, for each $j > i$ (if there is some such j), b_j is unacceptable. For each i, $t_{i+1} \succ_{b_1} t_i \succ_{b_1} b_1$. But, for all $b \in B \backslash \{b_1\}$, there are no acceptable T agents.

1. Consider the market with $M < \infty$ agents on each side of the market. What match results from the T-proposer DA algorithm? How many steps of the algorithm are required to reach this match?

2. Consider the market with a countable number of agents on each side.

   (a) Use the T-proposer DA algorithm. For each k, what matches are tentatively accepted at round k. That is, for each k, what is the k-round match function $\hat{\mu}^k$?

   (b) Does the T-proposer DA algorithm terminate (in the standard sense)? Explain.

   (c) Consider following weaker criterion: Say the T-proposal DA algorithm **weakly terminates** if the sequence of functions $(\hat{\mu}^k : k = 1, 2, 3, \ldots)$ converges pointwise. (See the math appendix for the definition of pointwise convergence.) Does the T-proposal DA algorithm weakly terminate?

Question 1b. Consider an environment where each T agent finds all B agents acceptable. However, they prefer to match with an even B agent over an odd B agent. And, all else equal, they prefer lower numbered agents. Specifically, for each $t \in T$,

- for each $j, k = 1, 2, 3, \ldots$, $b_{2j} \succ_t b_{2k-1}$,
- for each $k = 1, 2, 3, \ldots$, $b_{2k} \succ_t b_{2(k+1)}$,
- for each $k = 1, 2, 3, \ldots$, $b_{2k-1} \succ_t b_{2k+1}$, and
- for each $k = 1, 2, 3, \ldots$, $b_k \succ_t t$.

Each B agent finds all T agents acceptable and prefers lower numbered agents. Specifically, for each $b \in B$ agent and each $k = 1, 2, \ldots$, $t_k \succ_b t_{k+1} \succ_b b$.

1. Consider the market with $M < \infty$ agents on each side of the market. What match results from the T-proposer DA algorithm?

2. Consider the market with a countable number of agents on each side.

   (a) Use the T-proposer DA algorithm. For each k, what matches are tentatively accepted at round k. That is, for each k, what is the k-round match function $\hat{\mu}^k$?

   (b) Show that the T-proposal DA algorithm weakly terminates, i.e., $(\hat{\mu}^k : k = 1, 2, 3, \ldots)$ converges pointwise.

   (c) Write $\mu^\infty : B \to T \cup \{\phi\}$ for the limitting map, i.e., with $\hat{\mu}^\infty(b) = \lim_{k\to\infty} \hat{\mu}^k(b)$ for each $b \in B$. Does this induce a stable match? Either provide a proof or a counterexample.

3. Discuss the qualitative differences between the stable match induced in the finite setting versus the infinite setting.

Question 2: Fix an environment $\mathcal{E} = (T, B; (\succsim)_{i\in T\cup B})$ and an associated matching $\mu : (T \cup B) \to (T \cup B)$. The matching μ is **Pareto Efficient** if there is no matching $\mu' : (T \cup B) \to (T \cup B)$ with (a) for each $i \in T \cup B$, $\mu'(i) \succsim_i \mu(i)$, and (b) for some $i \in T \cup B$, $\mu'(i) \succ_i \mu(i)$.

1. Show the following result: If preferences are strict, then any stable match is Pareto Efficient.

2. Does the result also hold if preferences are not strict? Either stregthen the proof you provided above or provide a counter-example, as appropriate.

3. If preferences are strict, is any Pareto Efficient match stable? Either provide a proof or a counter-example, as appropriate.

Question 3: Fix an environment $\mathcal{E} = (T, B; (\succsim)_{i\in T\cup B})$ and recall that we took each $\succsim_i$ to be a complete and transitive preference relation. In class, we defined a binary relation $\geq_T$ on the set of matchings.

For each of the following statements, either provide a proof or a counterexample.

1. The relation $\geq_T$ is complete on the set of all matchings.

2. The relation $\geq_T$ is complete on the set of stable matchings.

3. The relation $\geq_T$ transitive on the set of stable matchings.

Math Appendix

1. For each k, let $f^k : X \to Y$ be a function. The sequence $(f^k : k = 1, 2, \ldots)$ converges pointwise if, for each $x \in X$, the sequence $(f^k(x) : k = 1, 2, \ldots)$ converges.

2. Let R be a binary relation on a set X. Let $X' \subseteq X$.

   - Say R is complete on X' if, for each $x, y \in X'$, xRy.
   - Say R is transitive if, for each $x, y, z \in X$, the following holds: If xRy and yRz, then xRz.