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ECON 501B

ECON 501B: Problem Set 2

Due: Thursday, September 6, 2018

Instructions: Answers should be complete proofs of a claim.

Question 1: Fix an environment $\mathcal{E} = (T, B; (\succsim)_{i \in T \cup B})$. This question will ask you to apply the T-proposer Deferred Acceptance Alogorithm to two environments: First, where T and B each have $M < \infty$ agents. Second, where T and B each have an ~~un~~countable number of agents. The examples are meant to highlight differences/peculiarities that arise, in going from the first setting to the second.

Question 1a. Suppose that, for each $t_i \in T$,

$$b_i \succ_{t_i} b_{i-1} \succ_{t_i} \cdots \succ_{t_i} b_1 \succ_{t_i} t_i,$$

and, for each $j > i$ (if there is some such j), b_j is unacceptable. For each i, $t_{i+1} \succ_{b_1} t_i \succ_{b_1} b_1$. But, for all $b \in B \backslash \{b_1\}$, there are no acceptable T agents.

induction?

1. Consider the market with $M < \infty$ agents on each side of the market. What match results from the T-proposer DA algorithm? How many steps of the algorithm are required to reach this match? $i+2$

infinite

2. Consider the market with a countable number of agents on each side.
   (a) Use the T-proposer DA algorithm. For each k, what matches are tentatively accepted at round k. That is, for each k, what is the k-round match function $\hat{\mu}^k$?
   (b) Does the T-proposer DA algorithm terminate (in the standard sense)? Explain.
   (c) Consider following weaker criterion: Say the T-proposal DA algorithm **weakly terminates** if the sequence of functions $(\hat{\mu}^k : k = 1, 2, 3, \ldots)$ converges pointwise. (See the math appendix for the definition of pointwise convergence.) Does the T-proposal DA algorithm weakly terminate?

Question 1b. Consider an environment where each T agent finds all B agents acceptable. However, they prefer to match with an even B agent over an odd B agent. And, all else equal, they prefer lower numbered agents. Specifically, for each $t \in T$,

$k = 6$

$b_1 < b_{k-1} < b_6 < b_1 < b_3 < b_5$

- for each $j, k = 1, 2, 3, \ldots$, $b_{2j} \succ_t b_{2k-1}$,
- for each $k = 1, 2, 3, \ldots$, $b_{2k} \succ_t b_{2(k+1)}$,
- for each $k = 1, 2, 3, \ldots$, $b_{2k-1} \succ_t b_{2k+1}$, and
- for each $k = 1, 2, 3, \ldots$, $b_k \succ_t t$.

Each B agent finds all T agents acceptable and prefers lower numbered agents. Specifically, for each $b \in B$ agent and each $k = 1, 2, \ldots$, $t_k \succ_b t_{k+1} \succ_b b$.

(1) Assume $b_i \Leftrightarrow t_i$ is [illegible]

1. Consider the market with $M < \infty$ agents on each side of the market. What match results from the T-proposer DA algorithm?

2. Consider the market with a countable number of agents on each side.

   (a) Use the T-proposer DA algorithm. For each k, what matches are tentatively accepted at round k. That is, for each k, what is the k-round match function $\hat{\mu}^k$?

   $\hat{\mu}^a(b_{2i}) = t_i \ \forall i$

   (b) Show that the T-proposal DA algorithm weakly NO terminates, i.e., $(\hat{\mu}^k : k = 1, 2, 3, \ldots)$ converges pointwise.

   $\mu^a(b_{2i-1}) = \phi \ \forall i$

   (c) Write $\mu^\infty : B \to T \cup \{\phi\}$ for the limitting map, i.e., with $\hat{\mu}^\infty(b) = \lim_{k\to\infty} \hat{\mu}^k(b)$ for each $b \in B$. Does this induce a stable match? Either provide a proof or a counterexample.

3. Discuss the qualitative differences between the stable match induced in the finite setting versus the infinite setting.

Question 2: Fix an environment $\mathcal{E} = (T, B; (\succsim)_{i\in T\cup B})$ and an associated matching $\mu : (T \cup B) \to (T \cup B)$. The matching μ is **Pareto Efficient** if there is no matching $\mu' : (T \cup B) \to (T \cup B)$ with (a) for each $i \in T \cup B$, $\mu'(i) \succsim_i \mu(i)$, and (b) for some $i \in T \cup B$, $\mu'(i) \succ_i \mu(i)$.

1. Show the following result: If preferences are strict, then any stable match is Pareto Efficient.

2. Does the result also hold if preferences are not strict? Either stregthen the proof you provided above or provide a counter-example, as appropriate.

3. If preferences are strict, is any Pareto Efficient match stable? Either provide a proof or a counter-example, as appropriate.

Question 3: Fix an environment $\mathcal{E} = (T, B; (\succsim)_{i\in T\cup B})$ and recall that we took each $\succsim_i$ to be a complete and transitive preference relation. In class, we defined a binary relation $\geq_T$ on the set of matchings.

For each of the following statements, either provide a proof or a counterexample.

match is preferred by all T agents than other match

1. The relation $\geq_T$ is complete on the set of all matchings. F

2. The relation $\geq_T$ is complete on the set of stable matchings. F

3. The relation $\geq_T$ transitive on the set of stable matchings. T

$$\begin{cases} t_1: b_1 > b_2 > t_1 \\ t_2: b_2 > b_1 > t_2 \\ b_1: t_1 > t_2 > b_1 \\ b_2: t_2 > t_1 > b_2 \end{cases}$$

Math Appendix

1. For each k, let $f^k : X \to Y$ be a function. The sequence $(f^k : k = 1, 2, \ldots)$ converges pointwise if, for each $x \in X$, the sequence $(f^k(x) : k = 1, 2, \ldots)$ converges.

2. Let R be a binary relation on a set X. Let $X' \subseteq X$.

   - Say R is complete on X' if, for each $x, y \in X'$, xRy.
   - Say R is transitive if, for each $x, y, z \in X$, the following holds: If xRy and yRz, then xRz.

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Question 1A $\mathcal{E} = (T, B; (\succsim)_{i \in T \cup B})$

for each $t_i \in T$, $t_i: b_i \succsim_{t_i} b_{i-1} > \ldots > b_1 > t_1$

For each $j > i$, b_j unacceptable for t_i

For each i: $t_{i+1} \succsim_{b_1} t_i \succsim_{b_1} b_1$. But $\forall b \in B \setminus \{b_1\}$ there are no acceptable t agents.

1. Market with $M < \infty$ agents on each side of the market. What are match results from T-proposer algorithm? How many steps are required to reach match?

Match Results: $\hat{M}^k(b_1) = t_M$

and $\forall n \in [2, M]$: $\hat{M}^k(b_n) = b_n$

Will terminate in $k = (i+2)$ steps

Proof by induction:

Step 1: let $M = 2$ and $T = \{t_1, t_2\}$ $B = \{b_1, b_2\}$

T: $t_1: b_1 > t_1 > b_2$
$t_2: b_2 > b_1 > t_2$

B: $b_1: t_2 > t_1 > b_1$
$b_2: b_2 > t_1 \sim t_2$

Next pg →

Round 1

$A^1(t_1) = \{b_1\}$
$A^1(t_2) = B = \{b_1, b_2\}$ $\rightarrow$ $\hat{p}^1(t_1) = b_1$, $\hat{p}^1(t_2) = b_2$ $\rightarrow$ $P(b_1) = \{t_1\}$, $P(b_2) = \{t_2\}$

$\hat{\mu}^1(b_1) = t_1$
$\hat{\mu}^1(b_2) = b_2$

Round 2

$A^2(t_1) = \{b_1\}$
$A^2(t_2) = \{b_1\}$ $\rightarrow$ $\hat{p}^2(t_1) = b_1$, $\hat{p}^2(t_2) = b_1$ $\rightarrow$ $P(b_1) = \{t_1, t_2\}$, $P(b_2) = \{\emptyset\}$ $\rightarrow$ $\hat{\mu}^2(b_1) = t_2$, $\hat{\mu}^2(b_2) = b_2$

Round 3

$A^3(t_1) = \emptyset$
$A^3(t_2) = \{b_1\}$ $\rightarrow$ $\hat{p}^3(t_1) = \{\emptyset$, $\hat{p}^3(t_2) = b_1$ $\rightarrow$ $P(b_1) = \{t_2\}$, $P(b_2) = \{\emptyset\}$ $\rightarrow$ $\hat{\mu}^3(b_1) = t_2$, $\hat{\mu}^3(b_2) = b_2$

Round 4

$A^4(t_1) = \emptyset$
$A^4(t_1) = \{b_1\}$ $\rightarrow$ $\hat{p}^4(t_1) = \emptyset$, $\hat{p}^4(t_2) = b_1$ $\rightarrow$ $P^4(b_1) = \{t_2\}$, $P^4(b_2) = \emptyset$ $\rightarrow$ $\hat{\mu}^4(b_1) = t_2$, $\hat{\mu}^4(b_2) = b_2$

Round 4 is repeat of Round 3
$k = 4$ $K = 3$ and thus $\exists K = 3 < \infty : \forall k = 4 \geq K = 3 : \hat{P}^k = \hat{P}^K$ and $\hat{\mu}^k = \hat{\mu}^K$

$i = 2$ and 4 rounds imply
total rounds $= (i+2) = K$

Proof by Induction Step 2 :

Show $(i+1)$ agents require $k = (1+i)+2 = (i+3)$ steps

$T = \{t_1, \dots t_i, t_{i+1}\}$ $\quad B = \{b_1, \dots, b_i, b_{i+1}\}$

$t_1 : b_1 > t_1 > b_2 > \dots > b_{i+1}$

$t_2 : b_2 > b_1 > t_2 > b_3 > \dots > b_{i+1}$

$t_3 : b_3 > b_2 > b_1 > t_3 > b_4 > \dots > b_{i+1}$

$\vdots$

$t_{i-1} : b_{i-1} > b_{i-2} > \dots > b_1 > t_{i-1} > b_i > b_{i+1}$

$t_i : b_i > b_{i-1} > \dots > b_1 > t_i > \dots > b_{i+1}$

$t_{i+1} : b_{i+1} > b_i > b_{i-1} > \dots > b_1 > t_{i+1}$

$b_1 : t_{i+1} > t_i > t_{i-1} > \dots > t_1 > b_1$

$b_2 : b_2 > t_i > \dots$

$b_i : b_i > t \dots$

$b_{i+1} : b_{i+1} > t \dots$

Round 1

$A'(t_1) = \{b_1\}$

$A'(t_2) = \{b_1, b_2\}$

$A'(t_3) = \{b_1, b_2, b_3\}$

$\vdots$

$A'(t_{i-1}) = \{b_1, b_2, b_3, \dots, b_{i-1}\}$

$A'(t_i) = \{b_1, b_2, b_3, \dots b_{i-1}, b_i\}$

$A'(t_i) = \{b_1, b_2, b_3, \dots b_{i-1}, b_i, b_{i+1}\} = B$

$\longrightarrow$

$\hat{p}'(t_1) = b_1$

$\hat{p}'(t_2) = b_2$

$\vdots$

$\hat{p}'(t_{i-1}) = b_{i-1}$

$\hat{p}'(t_i) = b_i$

$\hat{p}'(t_{i+1}) = b_{i+1}$

$P'(b_1) = \{t_1\}$

$P'(b_2) = \{t_2\}$

$P'(b_3) = \{t_3\}$

$\vdots$

$P'(b_i) = \{b_i\}$

$P'(b_{i+1}) = \{b_{i+1}\}$

$\longrightarrow$

$\hat{\mu}'(b_1) = t_1$

$\hat{\mu}'(b_2) = b_2$

$\vdots$

$\hat{\mu}'(b_{i+1}) = b_{i+1}$

Round 2

$A^2(t_1) = \{b_1\}$
$A^2(t_2) = \{b_1\}$
$\vdots$
$A^2(t_{i-1}) = \{b_1, b_2, \dots, b_{i-2}\}$
$A^2(t_i) = \{b_1, b_2, \dots, b_{i-1}\}$
$A^2(t_{i+1}) = \{b_1, b_2, \dots b_i\}$

$\longrightarrow$

$\hat{P}^2(t_1) = b_1$
$\hat{P}^2(t_2) = b_1$
$\vdots$
$\hat{P}^2(t_{i-1}) = b_{i-2}$
$\hat{P}^2(t_i) = b_{i-1}$
$\hat{P}^2(t_{i+1}) = b_i$

$P^2(b_1) = \{t_1, t_2\}$
$P^2(b_2) = \{t_3\}$
$P^2(b_{i-1}) = \{t_i\}$
$P^2(b_i) = \{t_{i+1}\}$
$P^2(b_{i+1}) = \emptyset$

$\longrightarrow$

$\hat{M}^2(b_1) = t_2$
$\hat{M}^2(b_2) = b_2$
$\vdots$
$\hat{M}^2(b_i) = b_1$
$\hat{M}^2(b_{i+1}) = b_{i+1}$

$\vdots$

Round i

$A^i(t_1) = \emptyset$
$A^i(t_2) = \emptyset$
$\vdots$
$A^i(t_i) = \{b_1\}$
$A^i(t_{i+1}) = \{b_1, b_2\}$

$\hat{P}^i(t_1) = \emptyset$
$\hat{P}^i(t_2) = \emptyset$
$\vdots$
$\hat{P}^i(t_i) = b_1$
$\hat{P}^i(t_{i+1}) = b_2$

$P(b_1) = \{t_i\}$
$P(b_2) = \{t_{i+1}\}$
$\vdots$
$P(b_{i+1}) = \emptyset$

$M^i(b_1) = t_i$
$M^i(b_2) = b_2$
$\vdots$
$M^i(b_{i+1}) = b_{i+1}$

Round $i+1$

$$A^{i+1}(t_1) = \emptyset \qquad \hat{p}^{i+1}(t_1) = \emptyset$$

$$\vdots \qquad \vdots$$

$$A^{i+1}(t_i) = \{b_1\} \qquad \hat{p}^{i+1}(t_i) = b_1$$

$$A^{i+1}(t_{i+1}) = \{b_1\} \qquad \hat{p}^{i+1}(t_{i+1}) = b_1$$

$$P^{i+1}(b_1) = \{t_i, t_{i+1}\} \qquad \mu^{i+1}(b_1) = t_{i+1}$$

$$P^{i+1}(b_2) = \emptyset \qquad \vdots$$

$$\vdots$$

$$P^{i+1}(b_{i+1}) = \emptyset \qquad \mu^{i+1}(b_{i+1}) = b_{i+1}$$

Round $i+2$

$$A^{i+2}(t_1) = \emptyset$$

$$\vdots$$

$$A^{i+2}(t_i) = \emptyset \qquad \hat{p}^{i+2}(t_i) = \emptyset$$

$$A^{i+2}(t_{i+1}) = \{b_1\} \qquad \hat{p}^{i+2}(t_{i+1}) = b_1$$

$$P^{i+2}(b_1) = \{t_{i+1}\} \qquad \mu^{i+2}(b_1) = t_{i+1}$$

$$\vdots \qquad \vdots$$

$$P^{i+2}(b_{i+1}) = \emptyset \qquad \mu^{i+2}(b_{i+1}) = b_{i+1}$$

Round $i+3$

$$A^{i+3}(t_i) = \emptyset \qquad \hat{p}^{i+3}(t_i) = \emptyset$$

$$A^{i+3}(t_{i+1}) = \{b_1\} \longrightarrow \hat{p}^{i+3}(t_{i+1}) = b_1$$

Thus $i+1$ agents means $i+3$ steps to termination.

$$P^{i+3}(b_1) = \{t_{i+1}\} \longrightarrow \mu^{i+3}(b_1) = t_{i+1}$$

$$\vdots \qquad \vdots$$

$$P^{i+3}(b_{i+1}) = \emptyset \qquad \mu^{i+3}(b_{i+1}) = \emptyset$$

By induction need $i+2$ steps for algorithm to terminate

Problem 1A pt 2

Consider a market with a countable number of agents on each side.

(a) Use the T-proposer DA algorithm. For each k, what matches are tentatively accepted at round k?

Let $T = \{t_1, t_2, \dots, t_m\}$ and let $B = \{b_1, b_2, \dots, b_n\}$

let $n, m \in [1, \infty)$

$t_1 : b_1 > t_1 > b_2 > \dots > b_n$

$t_2 : b_2 > b_1 > t_2 > b_3 > \dots > b_n$

$\vdots$

$t_{m-1} : b_{m-1} > b_{m-2} > \dots > b_2 > b_1 > t_{m-1} > b_n$

$t_m : b_m > b_{m-1} > \dots > b_2 > b_1 > t_m > b_{m+1} > \dots > b_n$

$b_1 : t_m > t_{m-1} > \dots > t_2 > t_1 > b_1$

$b_2 : b_2 > t_m \sim t_{m-1} \sim \dots t_1$

$b_n : b_n > t_m \sim t_{m-1} \sim \dots \sim t_1$

Round 1

$A^1(t_1) = \{b_1\}$

$A^1(t_2) = \{b_1, b_2\}$

$\vdots$

$A^1(t_{m-1}) = \{b_1, b_2, \dots b_{m-1}\}$

$A^1(t_m) = \{b_1, b_2, \dots, b_m\} = B$ if $m = n$

if $n < m$: last possible match is b_n

$\hat{p}^1(t_1) = b_1$

$\hat{p}^1(t_2) = b_2$

$\vdots$

$\hat{p}^1(t_{m-1}) = b_{m-1}$

$\hat{p}^1(t_m) = b_m$

$P^1(b_1) = \{t_1\}$

$P^1(b_2) = \{t_2\}$

$P^1(b_{m-1}) = \{b_{m-1}\}$

$P^1(b_m) = \{b_m\}$

if $n > m$:

$P^1(b_n) = \phi$

$\forall\ n-1, n, n+1, \dots > m$

$\hat{\mu}^1(b_1) = t_1$

$\hat{\mu}^1(b_2) = b_2$

$\hat{\mu}^1(b_m) = b_m$

$\hat{\mu}^1(b_n) = b_n$

Rand 2

$A^2(t_1) = \{b_1\}$

$A^2(t_2) = \{b_1\}$

$A^2(t_3) = \{b_1, b_2\}$

$\vdots$

$A^2(t_{m-1}) = \{b_1, \dots, b_{m-2}\}$

$A^2(t_m) = \{b_1, \dots, b_{m-1}\}$

$\hat{p}^2(t_1) = b_1$

$\hat{p}^2(t_2) = b_1$

$\hat{p}^2(t_{m-1}) = b_{m-1}$

$\hat{p}^2(t_m) = b_m$

$P^2(b_1) = \{t_1, t_2\}$

$P^2(b_2) = \{t_3\}$

$P^2(b_{m-1}) = \{b_m\}$

$P^2(b_m) = \phi$

$\hat{\mu}^1(b_1) = t_2$

$\hat{\mu}(b_n) = b_n$

Round K

$A^K(t_1) = \emptyset$

$A^K(t_{K-1}) = \{b_1\}$

$A^K(t_K) = \{b_1\}$

$A^K(t_{K+1}) = \{b_1, b_2\}$

$\hat{p}^K(t_1) = \emptyset$

$\hat{p}^K(t_{K-1}) = b_1$

$\hat{p}^K(t_K) = b_1$

$\hat{p}^K(t_{K+1}) = b_2$

$p^K(b_1) = \{t_{K-1}, t_K\}$

$p^K(b_2) = \{t_{K+1}\}$

$p^K(b_n) = \emptyset$

$\boxed{\hat{\mu}^K(b_1) = t_K}$

b has n elements and t has m elements

so if $K \le m$: $\hat{\mu}^K(b_1) = t_K$, $\forall n \neq 1$: $\hat{\mu}^K(b_n) = b_n$

if $K > m$: $\hat{\mu}^K(b_1) = t_m$ since t_m will keep proposing

$\forall n \neq 1$ $\hat{\mu}^K(b_n) = b_n$

(b) Does the T-proposer DA algorithm terminate?

Def$^{\underline{n}}$: T-proposing algorithm terminates if:

$$\exists K < \infty : \forall k \geq K : \hat{p}^k = \hat{p}^K \text{ and } \hat{\mu}^k = \hat{\mu}^K$$

Because the sets are countably infinite, and $\mu^k(b_1) = t_k$, there is no $K < \infty$ such that $\forall k \geq K$: $\hat{p}^k = \hat{p}^K$ and $\hat{\mu}^k = \hat{\mu}^K$ $\square$

(c) Does not converge pointwise since

$$\lim_{k \to \infty} \mu^k(b_1) = t_\infty$$

Assume $\mu^k(b_1)$ terminates. Then $\lim \mu^k(b_1) = t_k$. But, then $\lim \mu^{k+1}(b_1) = t_{k+1} \neq t_k = \lim \mu^k(b_1)$. Hence it does not converge pointwise.

Question 1-b

Each T agent finds all B agents acceptable. However, they prefer to match with an even B agent over an odd B agent. And, all else equal, they prefer lower numbered agents. For each $t \in T$:

for each $j, k = 1, 2, 3, \ldots$, $b_{2j} \succ_t b_{2k-1}$
for each $k = 1, 2, 3, \ldots$, $b_{2k} \succ_t b_{2(k+1)}$
for each $k = 1, 2, 3, \ldots$, $b_{2k-1} \succ_t b_{2k+1}$ and
for each $k = 1, 2, 3, \ldots$, $b_k \succ_t t$

Each B agent finds all T agents acceptable and prefers lower numbered agents: $\forall b \in B$, $\forall k = 1, 2, \ldots$ $t_k \succ_b t_{k+1} \succ_b b$

Pt (1) $M < \infty$ agents $T = \{t_1, \ldots t_n\}$ $B = \{b_1, \ldots b_n\}$

Assume m is even, then $m-1$ is odd.

$t_1: b_2 > b_4 > b_6 > \ldots > b_m > b_1 > b_3 > b_5 > \ldots > b_{m-1} > t_1$
$t_2: b_2 > b_4 > b_6 > \ldots > b_m > b_1 > b_3 > b_5 > \ldots > b_{m-1} > t_2$
$\vdots$
$t_n: b_2 > b_4 > b_6 > \ldots > b_m > b_1 > b_3 > b_5 > \ldots > b_{m-1} > t_n$

$b_1: t_1 > t_2 > t_3 > \ldots > t_n$
$\vdots$
$b_n: t_1 > t_2 > t_3 > \ldots > t_n$

Round 1 (10)

$$A^1(t_1) = B \quad A^2(t_1) = B \quad \vdots \quad A^n(t_n) = B \rightarrow \hat{p}^1(t_1) = b_2 \quad \hat{p}^2(t_2) = b_2 \quad \hat{p}^2(t_n) = b_2 \rightarrow p^1(b_1) = \phi \quad p^1(b_2) = T \quad p^1(b_n) = \phi \rightarrow M^1(b_1) = \phi \quad M^1(b_2) = t_1 \quad M^1(b_n) = \phi$$

Round 2

$$A^2(t_1) = \{b_2\} \quad A^2(t_2) = B \setminus \{b_1\} \quad \vdots \quad A^2(t_n) = B \setminus \{b_1\} \rightarrow \hat{p}^2(t_1) = b_2 \quad \hat{p}^2(t_2) = b_4 \quad \hat{p}^n(t_2) = b_4 \rightarrow p^2(b_1) = \phi \quad p^2(b_2) = t_1 \quad p^2(b_4) = T \rightarrow M^2(b_1) = \phi \quad M^2(b_2) = t_1 \quad M^2(b_4) = t_2$$

⋮

Round $n-1$

$$A^{n-1}(t_1) = \{b_2\} \quad A^{n-1}(t_2) = \{b_4\} \quad A^{n-1}(t_{n/2}) = \{b_1\} \quad A^{n-1}(t_{n-1}) = \{b_{m-3}, b_{m-1}\} \quad A^n(t_n) = \{b_{m-3}, b_{m-1}\}$$

$$\rightarrow \hat{p}^{n-1}(t_1) = b_2 \quad \hat{p}^{n-1}(t_2) = b_4 \quad \hat{p}^{n-1}(t_{n-1}) = b_{m-3} \quad \hat{p}^{n-1}(t_{n-1}) = b_{m-3}$$

$$\rightarrow p^{n-1}(b_1) = t_{n/2} \quad p^{n-1}(b_4) = t_2 \quad p^{n-1}(b_{m-3}) = \{t_{n-1}, t_n\} \quad p^{n-1}(b_{m-1}) = \{\phi\}$$

$$\rightarrow M(b_1) = t_{n/2} \quad M(b_4) = t_2 \quad M(b_{m-3}) = t_{n-1} \quad M(b_{m-1}) = \phi$$

Round n

$$A^n(t_1) = \{b_2\} \quad A^n(t_2) = \{b_4\} \quad A^n(t_{n/2}) = \{b_1\} \quad A^n(t_{n-1}) = \{b_{m-3}\} \quad A^n(t_n) = \{b_{m-1}\}$$

$$\rightarrow \hat{p}^n(t_1) = b_2 \quad \hat{p}^n(t_2) = b_4 \quad \hat{p}^n(t_{n/2}) = b_1 \quad \hat{p}(t_{n-1}) = b_{m-3} \quad \hat{p}(t_n) = b_{m-1}$$

$$\rightarrow p^n(b_1) = t_{n/2} \quad p^n(b_4) = t_2 \quad p^n(b_{m-3}) = t_{n-1} \quad p^n(b_{m-1}) = t_n$$

$$\rightarrow M(b_1) = t_{n/2} \quad M(b_4) = t_2 \quad M(b_{m-3}) = t_{n-1} \quad M(b_{m-1}) = t_n$$

Round $n+1$ will repeat Round n + algorithm terminates match results for $M < \infty$:

$M(b_1) = t_{n/2}$ for n even or $t_{(n+1)/2}$ for n odd

$M(b_2) = t_1$

$\vdots$

$M(b_4) = t_2$

$\vdots$

$M(b_m) = t_{(n/2)-1}$ for n even and m even

$\vdots$

$M(b_{m-3}) = t_{n-1}$

$M(b_{m-1}) = t_n$

(2) Consider market w/ countable number of agents

(a) For each k, what matches are tenitavily accepted at round k?

Based on last TA-Algorithm in pt (1), we can generalize:

<u>Round k</u>

$A^k(t_1) = \{b_2\}$
$A^k(t_2) = \{b_4\}$
$\vdots$
$A^k(t_n) = \{b_{2n}\}$
$\rightarrow$
$\hat{p}^k(t_1) = b_2$
$\vdots$
$A^k(t_n) = b_{2n}$
$\rightarrow$
$p^k(b_1) = \emptyset$
$p^k(b_2) = t_1$
$p^k(b_4) = t_2$
$\vdots$
$p^k(b_{2n}) = t_n$
$p^k(b_{2n+1}) = \emptyset$
$p^k(b_{2n+2}) = \{t_{n+1}, t_{n+2}, \dots\}$
$\rightarrow$
$\hat{M}^k(b_1) = b_1$
$\hat{M}^k(b_2) = t_1$
$\vdots$
$\hat{M}^k(b_{2n}) = t_n$
$\vdots$
$\hat{M}^k(b_{2n+2}) = t_{n+1}$

$\Rightarrow \hat{M}^k(b_{2k}) = t_k$

(b) Show that the algorithm weakly terminates pointwise

$$\lim_{k\to\infty} \hat{\mu}^{k}(x) = L$$

b_2 t_1
b_4 t_2
b_6 t_3
b_8 t_4

$$\lim \hat{\mu}^{1}(x) = \begin{cases} t_1 & x = b_2 \\ 0 & \text{o/w} \end{cases}$$

$$\Rightarrow \quad \lim_{k\to\infty} \hat{\mu}^{k}(x) = t_i \quad \text{if } x = b_i \ \forall i$$

$$\lim_{k\to\infty} \hat{\mu}^{k}(x) = b_{2i-1} \quad \text{if } x = b_{2i-1}$$

Subsequence of even/odd numbers converge pointwise. For all b, the algorithm converges pointwise sense every b converges to the same point as $k \to \infty$. This is because of ordering of the preferences of t and b. Once t_1 is matched with b_2, and so forth, there is no blocking pair and they will continue to match with each other. Odd number b's will not have a match because there are an infinite number of even b's who the t's would rather be with.

$$\hat{p}^{k}(t_k) = \hat{p}^{k+1}(t_k) \Rightarrow \lim_{k\to\infty} \hat{\mu}^{k}(x) = \lim_{k\to\infty} \mu^{k+1}(x)$$

(c) $\hat{\mu}^{\infty}(b) = \lim_{k \to \infty} \mu^k(b)$ does this induce a stable match

By definition, a stable match is induced when there is no blocking pair and a match is rational.

Let:

$$\hat{\mu}^{\infty}(b_{2i}) = t_i \text{ for all } i$$

and $\hat{\mu}^{\infty}(b_{2i-1}) = \emptyset$ for all i

Since t_i has the following preferences: $b_2 > b_4 > b_6 > \dots > b_{2k}$ $k = n, n+1, \infty$

and b's match with lowest i, it follows

that each b_{2i} will match with the i^{th} t_i

If b_{2i} were paired with t_{i-1}, b_{2i} would prefer that, but t_i would not have a rational match since it would be with b_{2i+1}.

(3) In the finite setting, we can ascertain a concrete iteration of steps and guarantee matchings, since the algorithm terminates.

(14)

In the infinite case, the algorithm will not always terminate, and not all b's will be matched with f's per se, even if it is their preference to do so.

Question 2

(15)

(1) If preferences are strict, than any stable matching is pareto efficient.

Assume no stable matching is pareto efficient, and Preferences are strict.

Then for a stable match $\mu(t_0) = b_0$, there is an alternative which makes t_0, b_0 no better or worse off such that $\mu(t_0) \succsim_{t_0} \mu'(t_0) = b$

than, for t_0: $b_0 \succsim b$. But, we assumed Preferences were strict. Therefore, we have a contradiction

(2) Counter example

$t = \{t_1, t_2\}$ $\quad B = \{b_1, b_2\}$

$t_1: b_1 \succsim b_2 \succsim t_1$
$t_2: b_2 \succsim b_1 \succsim t_2$

$b_1: t_2 \succsim t_1 \succsim b_1$
$b_2: t_1 \succsim t_2 \succsim b_2$

then: $\mu(b_1) = t_1$ $\quad \mu'(b_1) = t_2$
$\mu(b_2) = t_2$ $\quad \mu'(b_2) = b_1$

for b: $\mu'(b) \succsim \mu(b)$

hence it is not pareto efficient.

5) If preferences are strict, is any Pareto Efficient match stable? 16

Yes.

Assume Pareto efficient match is not stable.

Then $\exists \mu'(i)$ such that $\mu'(i) \succsim \mu(i)$ for $i \in T \cup B$.

But we assumed preferences are strict.

Therefore, this is a contradiction.

Additionally strict preferences imply one unique stable matching for the problem. So, any other matching would form a blocking pair, making some agent worse off. Then, the stable matching is pareto efficient.

Question 3

(17)

(1) The relation $\succsim_T$ is complete on the set of all matchings

False.

Let: $T = \{t_1, t_2\}$ and $B = \{b_1, b_2\}$

$t_1: b_1 \succ b_2 \succ t_1$ $\quad$ $b_1: t_1 \succ t_2 \succ b_1$

$t_2: b_1 \succ b_2 \succ t_2$ $\quad$ $b_2: t_2 \succ t_1 \succ b_2$

$\mu'(b_1) = t_1$ $\quad$ $\mu(b_1) = t_2$

$\mu'(b_2) = t_2$ $\quad$ $\mu(b_2) = t_1$

then b_1 prefers one ordering and b_2 prefers another

so matching is not complete

(2) The relation $\succsim_T$ is complete on set of all stable matchings

$T = \{t_1, t_2, t_3\}$ $\quad$ $B = \{b_1, b_2, b_3\}$

$t_1: b_2 \sim b_3 \succ b_1 \succ t_1$ $\quad$ $b_1: t_1 \succ t_2 \succ t_3 \succ b_1$

$t_2: b_2 \succ b_1 \succ t_2 \succ b_3$ $\quad$ $b_2: t_1 \succ t_2 \succ b_2 \succ t_3$

$t_3: b_3 \succ b_1 \succ t_3 \succ b_2$ $\quad$ $b_3: t_1 \succ t_3 \succ b_3 \succ t_2$

Then 2 stable matchings:

$\mu(t_2) = b_1,\ \mu(t_3) = b_3\ \ \mu(t_1) = b_2$

$\mu'(t_2) = b_2,\ \mu(t_3) = b_1,\ \mu(t_1) = b_3$

t's and b's cannot agree on best match since t_3 and b_2 prefer match 1 and t_2, b_3 like match 2. Therefore relation is not complete.

③ The relation is transitive on set of stable matchings.

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Assume b's like μ at least as well as μ'. In other words, $\mu \geq_s \mu'$. Suppose they like μ' at least as well as μ''. Then $\mu' \geq_s \mu''$. By transitivity:

$$\mu \geq \mu' \geq \mu'' \Rightarrow \mu \geq \mu''$$

If one assumes $\mu \geq \mu'$ and $\mu' \geq \mu''$ while $\mu'' \geq \mu$, then this implies a contradiction since:

$$\mu'' \geq \mu \Rightarrow \mu' \geq \mu'' \geq \mu \Rightarrow \mu' \geq \mu$$