501B 11/29.

Existence of Competitive Equilibrium : today's topic

Pure Exchange Economies

- $Y_J = \mathbb{R}^L_-$ (garbage firm)
- $X_i = \mathbb{R}^L_+$

  $\succsim_i$ is strictly convex, continuous, LNS

  ω_i : $\overline{\omega}_\ell > 0$ for each $\ell = 1, \dots, L$

  } Assumption 1

Fix a price vector $p \in \mathbb{R}^L$

- $B_i(p) = \{x_i \in X \mid p \cdot x_i \leq p \cdot \omega_i\}$ ← Not quite the budget set since it doesn't take into account $\theta_{ij} \, p \cdot y_J$
- Under Assumption 1 : there is at most one x_i^* such that x_i^* maximizes i's preferences s.t. $B_i(p)$

  (→ There may be NO x_i^*)

- Define $x_i : \mathbb{R}^L \to \mathbb{R}^L_+ \cup \{\phi\}$ such that

  $\begin{cases} x_i(p) \text{ is the } \succsim_i \text{-maximizing bundle on } B_i(p) \text{ if one exists} \\ x_i(p) = \phi, \text{ otherwise} \end{cases}$

prop 1 Fix a pure exchange economy that satisfies assumption 1.

Then $(x^*, y^*), p^*)$ constitute a Walrasian equilibrium if and only if

(1) $p^* \cdot y_J^* = 0$ and $p^* \geqq (0, \dots, 0)$ $(p^* = p_1^*, \dots, p_L^*)$ [profit max]

(2) $x_i(p^*) = x_i^*$ for each $i = 1, \dots, I$ [util. max]

(3) $y_J^* = \sum_{i=1}^{I} x_i^* - \sum_{i=1}^{I} \omega_i$ [Market clear]

Corollary 1 Fix a pure exchange economy that satisfies Assumption 1.
Then, p is an equilibrium price vector iff

(i) $p \geqq (0, \dots, 0)$

(ii) $x_i(p) \neq \emptyset$ for all $i = 1, \dots, I$

(iii) $\sum_{i=1}^{I} (x_i(p) - w_i) \leqq (0, \dots, 0)$

- Define an excess demand function ; $z_i : \mathbb{R}^L \to \mathbb{R}^L \cup \{\emptyset\}$

$$z_i(p) = \begin{cases} x_i(p) - w_i & \text{if } x_i(p) \neq \emptyset \\ \emptyset & \text{if } x_i(p) = \emptyset \end{cases}$$

- Define aggregate excess demand ; $z : \mathbb{R}^L \to \mathbb{R}^L \cup \{\emptyset\}$ s.t.

$$z(p) = \begin{cases} \sum_{i=1}^{I} z_i(p) & \text{if } z_i(p) \neq \emptyset \quad \forall i = 1, \dots, I \\ \emptyset & \text{otherwise} \end{cases}$$

⇓ Check the properties of aggregate excess demand.

Given $z : \mathbb{R}^L \to \mathbb{R}^L \cup \{\emptyset\}$,

define $\bar{z} : \mathbb{R}^L_{++} \to \mathbb{R}^L \cup \{\emptyset\}$ is z restricted to the domain $\mathbb{R}^L_{++}$.

$p \in \mathbb{R}^L_{++}$: $\bar{z}(p) = z(p)$

properties of aggregate excess demand.

$\boxed{P1}$ $\bar{z}(\cdot)$ is continuous

$\boxed{P2}$ $\bar{z}(\cdot)$ is homogeneous degree zero. ($z(\alpha p) = z(p)$, $\forall p \in \mathbb{R}^L_{++}$ and $\alpha > 0$)

$\boxed{P3}$ $\bar{z}(\cdot)$ satisfies Walras' law. ($p \cdot z(p) = 0$. $\forall p \in \mathbb{R}^L_{++}$)
↳ $\sum_{l=1}^{L} p_l z_l(p_l) = 0$

$\boxed{P4}$ $\forall p \in \mathbb{R}^L_{++}$, $\exists$ some $s > 0$ s.t. $(z_1(p), \dots, z_L(p)) \gg (-s, \dots, -s)$
↑ l^{th} component of $z(p)$.

$\boxed{P5}$ Let $(p^n, n = 1, 2, \dots)$ be such that $p^n \in \mathbb{R}^L_{++}$ and $p^n \to p^0$ where $p^0 \neq \underline{0}$ but $p^0_l = 0$.
Then, $\max \{z_1(p^n), \dots, z_L(p^n)\} \to \infty$ (If $\exists\, p^0_l = 0$, max seq. of $z_l(p^n) \to \infty$)

Suppose we find a $p^* \in \mathbb{R}^L_{++}$ s.t. $z(p^*) = (0, \dots, 0)$ (← corollary is holds)

- Since $p^* \in \mathbb{R}^L_{++}$, $x_i(p^*) \neq \emptyset$
- $\sum_{i=1}^{I} [x_i(p^*) - w_i] = z(p^*) = (0, \dots, 0)$

⇒ Corollary : there exists some (x^*, y^*) s.t. $((x^*, y^*), p^*)$ is a competitive equilibrium.

Prop 2 Consider a pure exchange economy that satisfies Assumption 1.
If $z(\cdot)$ satisfies $P1 \sim P5$, then there exists p^* s.t. $z(p^*) = (0, \dots, 0)$
really only going to need P_1, P_3, P_4, P_5

- Will see : Assumptions + 1 extra condition gives us $P_1 \sim P_5$.

Theorem (Kakutani)

$f: A \rightrightarrows A$

Let $A \subseteq \mathbb{R}^n$ be compact and convex. Let $f: A \to 2^A$ be a upperhemicontinuous correspondence s.t., for each $a \in A$, $f(a)$ is non-empty & convex.
Then there exists some $a^* \in A$ s.t. $a^* \in f(a^*)$

-※ $f(a)$ into some singleton $\{b\} \subseteq A$; $a^* \in f(a^*) \Rightarrow f(a^*) = \{a^*\}$ (Fixed point thm)

proof of proposition 2.)

set : $\Delta = \{p \in \mathbb{R}^L_{++} : \sum_{\ell=1}^{L} p_\ell = 1\}$ ← look for equilibrium price vectors only in this set.
If we find a equilibrium price vector in this set, then done.
If NOT then P_2 tells you that you can't find an equilibrium price vector.

$Int(\Delta) = \{p \in \Delta : p \gg (0, \dots, 0)\}$

$bd(\Delta) = \Delta \setminus Int(\Delta)$

Correspondence : $f: \Delta \to 2^\Delta$ such that

$$f(p) = \begin{cases} \{q \in \Delta : \underbrace{z(p)}_{\text{consume - wealth}} q \geq z(p) \cdot q', \forall q' \in \Delta\} & \text{if } p \in Int(\Delta) \\ \{q \in \Delta : p \cdot q = 0\} & \text{if } p \in bd(\Delta) \end{cases}$$

Step 1 : If p^* is a fixed point of f, then $z(p^*) = 0$.
Step 2 : there exists a fixed point of f.] next class?

//.