Lecture 8

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Definitions

In many-to-one matching, environment ξ is now defined as $\xi = (T, B; (q_t)_{t \in T}; (\succsim_t), (\succsim_b)_{b \in B})$

- $T = \{t_1, ..., t_{|T|}\}$
- $B = \{b_1, ..., b_{|B|}\}$
- $\mathcal{B} = 2^B \setminus \{\emptyset\} = \{B_* \subseteq B : B_* \neq \emptyset\}$
- $q_t \geq 1$ for t.
- $\succsim_t$ is a preference relation on $\mathcal{B} \cup \{t\}$
- $\succsim_b$ is a preference relation on $T \cup \{b\}$

Responsive

A preference relation $\succsim_t$ is responsive if:

(1) For each $b_1, b_2 \in B$ and $\tilde{B} \subseteq B$ such that $\tilde{B} \cap \{b_1, b_2\} = \emptyset$, $\tilde{B} \cup \{b_1\} \succsim_t \tilde{B} \cup \{b_2\} \iff \{b_1\} \succsim_t \{b_2\}$.

(2) For each $b \in B$ and $\tilde{B} \subseteq B$ with $\tilde{B} \cap \{b\} = \emptyset$:

- $\tilde{B} \cup \{b\} \succsim_t \tilde{B}$ if and only if $\{b\} \succsim_t t$.
- $\tilde{B} \succsim_t \tilde{B} \cup \{b\}$ if and only if $t \succsim_t \{b\}$.

Defition: A matching $\mu : (T \cup B) \to (T \cup B \cup \mathcal{B})$ is a function such that:

(1) each $\mu(t) \in \mathcal{B} \cup \{t\}$ and each $\mu(b) \in T \cup \{b\}$.
(2) $b \in \mu(t) \iff \mu(b) = t$.
(3) Each $|\mu(t)| \leq q_t$ if $\mu(t) \in \mathcal{B}$.

*Note that $\mu : (T \cup B) \to (T \cup B \cup \mathcal{B})$ where the range says that either b matches to $t \in T$, or t matches to himself (T), or b to himself $b \in B$ or, lastly, t matches with a $b \in \mathcal{B}$ meaning some particular subset of b's

Remark: *Suppose we had t_1 and t_2 with:*

$$\mu(t_1) \subseteq B$$
$$\mu(t_2) \subseteq B$$
$$\mu(t_1) \cap \mu(t_2) = \emptyset$$

$$\implies \exists b \in \mu(t_1) \cap \mu(t_2)$$
$$\implies \mu(b) = t_1, \mu(b) = t_2$$
$$\implies t_1 = t_2$$

Hence, it must be that $t_1 \neq t_2$ since we assumed $\mu(t_1) \cap \mu(t_2) = \emptyset$.

Remark: Consider the following case

$$(q_t : t \in T) = (1, 1, ..., 1)$$

Then, it is back to 1-to-1 matching case where $\mu(t) = b$ and $\mu(b) = t$.

Definition: *t blocks $\mu : (T \cup B) \to (T \cup B \cup \mathcal{B})$ if either:*

(1) $t \succ_t \mu(t)$ or
(2) There exists a $\tilde{B} \subsetneq \mu(t) \subseteq B : \tilde{B} \succ_t \mu(t)$

Definition: *Say $b \in B$ blocks μ if $b \succ_b \mu(b)$*

Definition: *A matching is individually rational if no agent $i \in T \cup B$ blocks μ*

Example 1

Let $T = \{t\}$ and $B = \{b_1, b_2\}$. For each $i = 1, 2$, $t \succ_t \{b_i\}$.

But, $B \succ_t t$ and $\mu(t) = B$ is individually rational match even though t prefers himself to any individual $b_i \in B$. That is to say, no individual agent b is acceptable to t, but together they are.

Definition: *A pair (t, b) blocks μ if:*

(1) $\exists \tilde{B} \subseteq B$ with cardinality $|\tilde{B} \cup \{b\}| \leq q_t$ such that: (1a) If $\mu(t) \subseteq B$ then $\tilde{B} \subseteq \mu(t)$ if $\mu(t) = t$ then $\tilde{B} = \emptyset$. (1b) $\tilde{B} \cup \{b\} \succ_t \mu(t)$.
(2) $t \succ_b \mu(t)$.

Definition: *A matching $\mu : (T \cup B) \to (T \cup B \cup \mathcal{B})$ is stable if it is individually rational and there is no blocking pair. That is to say, it is pairwise stable*

Example 2

$$T = \{t_1, t_2\}$$
$$B = \{b_1, b_2, b_3\}$$
$$q_1 = q_2 = 3$$

Preferences:

$$t_1 : \{b_1, b_3\} \succ_t \{b_1, b_2\} \succ_t \{b_2, b_3\} \succ_t \{b_1\} \succ_t \{b_2\} \succ_t t_1 \succ_t B \succ_t \{b_3\}$$
$$t_2 : \{b_1, b_3\} \succ_t \{b_2, b_3\} \succ_t \{b_1, b_2\} \succ_t \{b_3\} \succ_t \{b_1\} \succ_t \{b_2\} \succ_t t_2 \succ_t B$$

And for $i = 1, 2$:

$$b_i : t_2 \succ_{b_i} t_1 \succ_{b_i} b_i$$
$$b_3 : t_1 \succ_{b_3} t_2 \succ_{b_3} b_3$$

Claim: *There is no pairwise stable match.*

Suppose that μ is pairwise stable.

(a) Individually rationality implies that $\mu(t_i) \neq B$ for each $i = 1, 2$.
(b) $\mu(t_1) \neq \{b_i\}$ for any $i = 1, 2, 3$.

And IR $\implies \mu(t_1) \neq \{b_3\}$.

Suppose there were $i = 1, 2$ with $\mu(t_1) = \{b_i\}$. The set $\{b_i, b_3\} \succ_{t_1} \{b_i\}$. Note $|\{b_i, b_3\}| = 2 < q_1$. I added b_3 and now t_1 likes that potential match better. And, b_3 likes it too since $t_1 \succ_{b_3} \mu(b_3)$.

This implies, then, that (t_1, b_3) form a block.

(c) $\mu(t_2) \neq b_i$ for any $i = 1, 2, 3$. If $\mu(t_2) = \{b_i\}$ choose $b_l \in \{b_1, b_2\}$, and $b_l \neq b_i$. Then, $\{b_i, b_l\} \succ_{t_2} \{b_i\} = \mu(t_2)$. Note that $t_2 \succ_{b_l} \mu(b_l)$. This implies that t_2, b_l form a block.

It follows from $(a), (b)$ and (c) that there is some i such that $\mu(t_i) = t_i$. Who would that i be?

- It must be $\mu(t_1) = t_1$ since if $\mu(t_2) = t_2$, then there is some b with $\mu(b) = b$. But, (t_2, b) would form a block. So, this cannot be.
- It must be that $\mu(t_2) = t_2$ by the same argument.
- These two cases imply that $|\mu(t_2)| = 2$.

Two Possibilities:

(1) Match the following:

$$\mu(t_2) = \{b_1, b_2\}$$

$$\mu(t_1) = t_1$$

$$\mu(b_3) = b_3$$

Then, $\tilde{B} = \{b_1\} \subseteq \mu(t_2)$ and (t_2, b_3) form a block:

- $t_2 \succ_{b_3} b_3 = \mu(b_3)$
- $\{b_1, b_3\} \succ_{t_2} \{b_1, b_2\} = \mu(t_2)$

(2) Other possibility is that $\mu(t_2) = \{b_i, b_3\}$ for some $i = 1, 2$ and $\mu(t_1) = 1$.

Here, (t_1, b_l) block for $b_l \in \{b_1, b_2\}$ such that $b_l \neq b_i$. Then, $\{b_l\} \succ_{t_1} t_1$ for $l = 1, 2$. But, we cannot have this. There is no pairwise stable match.

Remark: Nothing would change if we took $q_1 = q_2 = 2$. But, if $q_1 = q_2 = 1$ then there would be a stable match.

Remark: Notice that this example violates reponsive preferences.

$$\{b_i, b_3\} \succ_{t_1} \{b_i\} : i = 1, 2$$

But:

$$t_1 \succ_{t_1} \{b_3\}$$