

Econ 501A HW 1

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Problem 1.1

Suppose $X = \{a, b, c\}$ and $u(a) = 1$, $u(b) = 1$, and $u(c) = 2$. For each $B \subset X$, compute $C_u(B)$.

There are seven such subsets B of X :

- (1) $B = \{a\}$: $C_u(B) = a$
- (2) $B = \{b\}$: $C_u(B) = b$
- (3) $B = \{c\}$: $C_u(B) = c$
- (4) $B = \{a, b\}$: $C_u(B) = \{a, b\}$
- (5) $B = \{a, c\}$: $C_u(B) = c$
- (6) $B = \{b, c\}$: $C_u(B) = c$
- (7) $B = \{a, b, c\}$: $C_u(B) = c$

Technically, \emptyset could also be a subset. But, following the prescription of the notes, we write:

$$C_u : \mathcal{P}(X) \setminus \emptyset \rightarrow \mathcal{P}(X) \setminus \emptyset$$

So, I exclude it.

Problem 1.2

(a) Prove that $u^*(B)$ is increasing in B . That is, prove that if $B \subseteq B'$, then $u^*(B) \leq u^*(B')$. In words, increasing the set of feasible alternatives never hurts the individual and sometimes helps the individual.

$$B \subset B' \implies \sup_{x \in B} u(x) \leq \sup_{x \in B'} u(x) \implies u^*(B) \leq u^*(B')$$

Let $B \subset B'$. Then B' contains at least one more element than B . If $B \subseteq B'$, then it may contain the same elements. If B' has more elements than B , then these elements contained in B' will either be larger or smaller than the maximum element of B which is also found, by definition, in B' . If the elements in B' not in B are smaller than the maximum element in B then $\sup_{x \in B} u(x) = \sup_{x \in B'} u(x) \implies u^*(B) = u^*(B')$.

If there is an element in B' that is not in B which is greater than all elements in B , then, $\sup_{x \in B} u(x) < \sup_{x \in B'} u(x) \implies u^*(B) < u^*(B')$.

An increasing function is one such that each subsequent value is greater than or equal to the previous value. Hence, in both cases above, $u(B)$ is an increasing function.

(b) Prove that if $y \in B$ but $y \notin C_u(B)$ then $u^*(B) = u^*(B \setminus y)$, where $B \setminus y$ denotes the set consisting of all elements of B except y . In words, unchosen alternatives do not affect welfare.

Proof by contraposition: $u^*(B) \neq u^*(B \setminus y) \implies y \in C_u(B)$

Let $y \in B$ and assume $u^*(B) \neq u^*(B \setminus y)$. Then, $\sup_{x \in B} u(x) \neq \sup_{x \in B \setminus y} u(x)$. Hence, y must be the maximum element in the set of B . Therefore, $y \in C_u(B)$.

Problem 1.3

Suppose $X \supset \{a, b, c\}$. (a) Of course, $C_u(\{a, n\})$ is a subset of $\{a, b\}$. How many subsets of $\{a, b\}$ are there? How many values might $C_u(\{a, b\})$ take and what are they?

There are four subsets:

- (1) \emptyset
- (2) $\{a\}$
- (3) $\{b\}$
- (4) $\{a, b\}$

$C_u(\{a, b\})$ could take 3 values. It excludes the empty set, so we exclude case (1) from all further analysis in this part. If $a > b$ then it will take the value a unless case (3) where it will only have b .

If $b > a$ then it will take the value of b unless case (2) in which it will take value of a .

If $a = b$, then it will take value a for case (1), value b for case (2) and the set $\{a, b\}$ for case (3).

(b) Suppose that $C_u(\{a, b\}) = b$. What can we conclude about $u(a)$ versus $u(b)$? Suppose further that $C_u(\{b, c\}) = c$. What can we conclude about $C_u(\{a, c\})$?

First, we can conclude that $u(a) < u(b)$.

Second, we can conclude that $u(a) < u(c)$ by transitivity.

(c) Suppose $a \in B$ but $a \notin C_u(B)$. What can we conclude about $C_u(B \setminus a)$ as compared to $C_u(B)$?

We can conclude that $C_u(B \setminus a) = C_u(B)$.

Problem 1.4

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly increasing function. Define a function $v : X \rightarrow \mathbb{R}$ so that $v(x) = f(u^*(x))$. Then, $C_u(B) = C_v(B)$.

(a) Prove the above proposition

Let $x, y \in B$ and $u(B \setminus x) < u^*(x)$ such that $u^*(y) < u^*(x)$ and by definition $u^*(x), u^*(y) \in \mathbb{R}$.

Define $f : \mathbb{R} \rightarrow \mathbb{R}$ to be a strictly increasing function such that:

$$\forall a, b \in \mathbb{R} : a < b \implies f(a) < f(b).$$

Hence, $f(u^*(y)) < f(u^*(x))$. Define $C_v(B) = \arg \max_{z \in B} f(u^*(z))$. Now, $C_v(B) = x$.

By definition $C_u(B) = \arg \max_{z \in B} u^*(z) = x$.

Hence, $C_u(B) = C_v(B)$.

(b) What if instead f is a weakly increasing function? State and prove an alternative proposition in this case.

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function. Define a function $v : X \rightarrow \mathbb{R}$ so that $v(x) = f(u^*(x))$. Then, $C_u(B) \geq C_v(B)$.

Let $x, y \in B$ and $u^*(B \setminus x) < u^*(x)$ such that $u^*(y) < u^*(x)$ and by definition $u^*(x), u^*(y) \in \mathbb{R}$.

Define $f : \mathbb{R} \rightarrow \mathbb{R}$ to be an increasing function such that:

$$\forall a, b \in \mathbb{R} : a < b \implies f(a) \leq f(b).$$

Hence, $f(u^*(y)) \leq f(u^*(x))$. Define $C_v(B) = \arg \max_{z \in B} f(u^*(z))$. Now, $C_v(B) = x' \in B \leq x \in B$. By definition $C_u(B) = \arg \max_{z \in B} u^*(z) = x$.

Hence, $C_u(B) \geq C_v(B)$.

Problem 2.1

Recall the previous problem where $X = \{a, b, c\}$ and $u(a) = 1$, $u(b) = 1$, and $u(c) = 2$. Write down the preference derived from this u . That is for every pair $x, y \in X$, indicate whether $x \succeq y$ or $x \not\succeq y$

- (1) $\{a, b\}$: $a \succeq b$ and $b \succeq a$
- (2) $\{a, c\}$: $c \succeq a$ and $a \not\succeq c$
- (3) $\{b, c\}$: $b \not\succeq c$ and $c \succeq b$

Also note that: $a \succeq a$, $b \succeq b$ and $c \succeq c$.

Problem 2.2

For each of the following four preferences, answer the following two questions: Is the preference complete? Is it transitive? Prove your answers

In parts (a), (b), and (c), let $X = \mathbb{R}_+^2$ so an element $x \in X$ is a vector: $x = (x_1, x_2)$ where x_1 and x_2 are nonnegative real numbers.

Note: I will use vector $z = (z_1, z_2)$ as I prove transitivity.

(a) $x \succeq y \iff x_1 + x_2 \geq y_1 + y_2$

The preference is complete.

$$(1) x_1 + x_2 < y_1 + y_2 (2) x_1 + x_2 = y_1 + y_2 (3) x_1 + x_2 > y_1 + y_2$$

Which imply:

$$(1) y \succ x (2) x \sim y (3) x \succ y$$

The preference is transitive. Let $y_1 + y_2 \geq z_1 + z_2$ Then, $x \succeq y \succeq z \iff x_1 + x_2 \geq y_1 + y_2 \geq z_1 + z_2$ and so $x_1 + x_2 \geq z_1 + z_2 \iff x \succeq z$

(b) $x \succeq y \iff x_1 \geq y_1$ and $x_2 \geq y_2$

This is not complete. Let $x_1 < y_1$ and $x_2 \geq y_2$. This cannot be represented by any preference relation.

It is transitive. Suppose

$$y_1 \geq z_1, y_2 \geq z_2$$

Then, we have:

$$x_1 \geq y_1 \geq z_1 \implies x_1 \geq z_1, x_2 \geq y_2 \geq z_2 \implies x_2 \geq z_2 \implies x \succeq z$$

(c) $x \succeq y \iff$ either $x_1 > y_1$ or both $x_1 = y_1$ and $x_2 \geq y_2$.

This is complete.

$$x_1 = y_1 \text{ and } x_2 < y_2 \implies y \succ x$$

$$x_1 > y_1 \implies x \succ y$$

$$x_1 = y_1 \text{ and } x_2 = y_2 \implies x \sim y$$

This is transitive.

$$x_1 > y_1 y_1 > z_1 \implies x_1 > z_1 \implies x \succ z$$

Or note the following:

$$x_1 = y_1 = z_1 x_2 \geq y_2 \text{ and } y_2 \geq z_2 \implies x_2 \geq z_2 \implies x \succ z$$