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ECON 501B

ECON 501B: Problem Set 1

Due: Thursday, August 30, 2018

Instructions: For True/False questions, either provide a proof that the statement is true or provide a counterexample showing that it is false.

Question 1: *True or False.* Fix an environment $(T, B; (\succsim)_{t\in T}, (\succsim)_{b\in B})$ so that the following holds: there exists $t_* \in T$ and $b_* \in B$, with

1. $b_* \succ_{t_*} b$, for each $b \in B \backslash \{b_*\}$, and
2. $t_* \succ_{b_*} t$, for each $t \in T \backslash \{t_*\}$.

Then, in any stable match $\mu : (T \cup B) \to (T \cup B)$, $\mu(t_*) = b_*$.

Question 2: *True or False.* Fix an environment $(T, B; (\succsim)_{t\in T}, (\succsim)_{b\in B})$, so that

$$|\{t \in T : A(t) = B\}| = |\{b \in B : A(b) = T\}|.$$

If all agents have strict preferences and $A(t) = B$, then there is some stable match in which t is matched.

Question 3: In class we said that a pair (t, b) **blocks** a matching $\mu : (T \cup B) \to (T \cup B)$ if (i) $b \succ_t \mu(t)$, and (ii) $t \succ_b \mu(b)$. Consider instead the following definition: A pair (t, b) **blocks*** a matching $\mu : (T \cup B) \to (T \cup B)$ if either (i) $b \succsim_t \mu(t)$ and $t \succ_b \mu(b)$, or (ii) $b \succ_t \mu(t)$ and $t \succsim_b \mu(b)$. Say a matching $\mu : (T \cup B) \to (T \cup B)$ is **stable*** if it is individually rational and there is no block* pair (t, b).

1. If a matching $\mu : (T \cup B) \to (T \cup B)$ is stable* is it stable? Either provide a proof that it is or a counter example.
2. If a matching $\mu : (T \cup B) \to (T \cup B)$ is stable is it stable*? Either provide a proof that it is or a counter example.
3. Is the notion of blocks* stronger or weaker than the notion of blocks? Is the notion of stable* stronger or weaker than the notion of stable?

Question 4: There are three agents on each side of the market: $T = \{t_1, t_2, t_3\}$ and $B = \{b_1, b_2, b_3\}$. Matched agents can share a pie; unmatched agents get no pie. The following describes the fraction of the pie that a t_i agent would get when matched with (b_1, b_2, b_3):

- t_1's Fraction: $(\frac{1}{4}, \frac{2}{4}, \frac{3}{4})$;
- t_2's Fraction: $(\frac{3}{4}, \frac{1}{4}, \frac{2}{4})$;
- t_3's Fraction: $(\frac{2}{4}, \frac{3}{4}, \frac{1}{4})$.

(So, if (t_1, b_1) are matched, t_1 gets $\frac{1}{4}$ of the pie and b_1 gets $\frac{3}{4}$ of the pie.) All agents strictly prefer larger fractions of pie to smaller fractions of pie.

Which matches are (resp. are not) stable? (That is, provide a compelte argument for why each matching is or is not stable.)

Econ 501 B HW 1

Question 1:

T/F: Let $\mathcal{E} = (T, B, (\succ)_{t \in T}, (\succ)_{b \in B})$ and let there exist a $t_* \in T$, $b_* \in B$ such that the following is true:

① $b_* \succ_{t_*} b \quad b \in B \setminus \{b^*\}$
② $t^* \succ_{b_*} t \quad t \in T \setminus \{T^*\}$

Then in any stable match μ: $\mu(t_*) = b^*$

False, Let:

$$T: \begin{array}{l} t_* : t_* \succ b_1 \succ b_2 \succ \\ t_1 : b_1 \succ b_2 \succ t_1 \\ t_2 : b_2 \succ b_1 \succ t_2 \end{array}$$

$$B: \begin{array}{l} b_* : b_* \succ t_1 \succ t_2 \\ b_1 : t_1 \succ t_2 \succ b_1 \\ b_2 : t_2 \succ t_1 \succ b_2 \end{array}$$

So, $\mu(t^*) = t^*$ and $\mu(b^*) = b^*$ and all other matches are stable.

Question 2 T/F let $\mathcal{E} = (T, B, (\succsim)_{i \in T \cup B})$

$|\{t \in T : A(t) = B\}| = |\{b \in B : A(b) = T\}|$

If all agents have strict preferences and $A(t) = B$ there is some stable match where t is matched

$A(t) = B \Rightarrow$ all $b \in B$ are acceptable to t

Let $2 = |\{t \in T : A(t) = B\}| = |\{b \in B : A(b) = T\}|$

and so $T = \{t_1, t_2, t_3$
$B = \{b_1, b_2\}$

$t_1 \quad b_1 > b_2 > t_1$
$t_2 \quad b_1 > b_2 > t_2$
$t_3 \quad b_2 > t_3 > b_1$

$b_1 \quad t_1 > t_2 > t_3 > b_1$
$b_2 \quad t_2 > t_1 > t_2 > b_2$

$\mu(t_1) = b_1$

$\mu(t_2) = b_2$

$\mu(t_3) = t_3$

$t_3 \in T$ not matched. All other conditions met. Therefore statement is false

Question 3

(t,b) blocks a match $\mu : (T \cup B) \to (T \cup B)$
(i) $b \succ_t \mu(t)$
(ii) $t \succ_b \mu(b)$

Let blocks* mean the following:
a pair (t,b) blocks* a $\mu : (T \cup B) \to (T \cup B)$
if either
(i) $b \succsim_t \mu(t)$ and $t \succ_b \mu(b)$
OR (ii) $b \succ_t \mu(t)$ and $t \succsim_b \mu(b)$

A matching $\mu : (T \cup B) \to (T \cup B)$ is Stable* if it is IR and there is no block* pair

① If a matching $\mu : (T \cup B) \to (T \cup B)$ is stable* is it stable?

i.e. Stable* $\Rightarrow$ stable

TRUE: Contrapositive Not stable $\Rightarrow$ not stable*

Not stable $\Rightarrow \exists (t,b) :\ b \succ_t \mu(t)$ and $t \succ_b \mu(b)$

then: $b \succsim_t \mu(t)$ and $t \succ_b \mu(b)$ ✓

② If a matching is stable is it stable*?

False

$t_1 :\ b_1 \sim b_2 \succ t_1$ $\quad \mu(t_1) = b_2$
$t_2 :\ b_1 \succ b_2 \succ t_2$ $\quad \mu(t_2) = b_1$

$b_1 :\ t_1 \succ t_2 \succ b_1$ $\quad$ Not Stable*, but stable
$b_2 :\ t_1 \succ t_2 \succ b_2$

③ Is notion of blocks* stronger or weaker than blocks?
Is notion of blocks stronger or weaker than blocks*?

stable* is stronger.

stable* $\Rightarrow$ stable ; Not stable $\Rightarrow$ not stable*

not stable means there is a block

Therefore: blocks $\Rightarrow$ blocks*
and blocks is stronger and blocks* weaker
but not vice versa

Question 4 There are 3 agents on each side of the market

$T = \{t_1 \ t_2 \ t_3\}$
$B = \{b_1 \ b_2 \ b_3\}$

Match agents share a pie. Unmatched do not.
Fractions t_i gets with pairing w/ (b_1, b_2, b_3)

$t_1 \ (1/4, 2/4, 3/4)$ $b_1 \ (3/4 \ 2/4 \ 1/4)$
$t_2 \ (3/4, 1/4, 2/4)$ $b_2 \ (1/4, 3/4 \ 2/4)$
$t_3 \ (2/4, 3/4, 1/4)$ $b_3 \ (2/4, 1/4, 3/4)$

$t_1: b_3 > b_2 > b_1 > t_1$ $b_1: t_1 > t_2 > t_3 > b_1$
$t_2: b_1 > b_3 > b_2 > t_2$ $b_2: t_2 > t_3 > t_1 > b_2$
$t_3: b_2 > b_1 > b_3 > t_3$ $b_3: t_3 > t_1 > t_2 > b_3$

(i) $\mu(t_1) = b_3$
$\mu(t_2) = b_1$ Stable
$\mu(t_3) = b_2$

(ii) $\mu(t_1) = b_3$
$\mu(t_2) = b_2$
$\mu(t_3) = b_1$

(b_1, t_2) block

$t_2 \succ_{b_1} \mu(b_1) = t_3$

$b_1 \succ_{t_2} \mu(t_2) = b_2$

And so on.

Note: strict preferences and $|T| = |B|$ and
$|A(t) = B| = |A(b) = T|$ hence
in a stable match all agents are matched

Homework 1 Solutions

ECON 501A

Prof. Freidenberg

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1. False.
   $T = \{t^*\}$ and $B = \{b^*\}$
   Where $t^* \succ_t b^*$ and $b^* \succ_b t^*$
   Then the only stable match is $\mu(t^*) = t^*$ and $\mu(b^*) = b^*$

2. False.
   $|\{t \in T : A(t) = B\}| = |\{b \in B : A(b) = T\}|$ and preferences are strict.
   $\Rightarrow$ For any t agent s.t. $A(t) = B : \exists$ a stable matching μ where $\mu(t) \in B$.
   Counter example:
   $T = \{t_1, t_2\}$ and $B = \{b_1, b_2, b_3\}$
   $t_1 : b_1 \succ b_2 \succ b_3 \succ t - 1$
   $t_2 : b_3 \succ b_1 \succ b_2 \succ t - 2$
   $b_1 : t_1 \succ t_2 \succ b_1$
   $b_2 : t_1 \succ t_2 \succ b_2$
   $b_3 : t_2 \succ b_3 \succ t_1$
   Then there is a unique stable matching:
   $\mu(t_1) = b_1$, $\mu(t_2) = b_3$, $\mu(b_2) = b_2$
   This is a contradiction since b_2 should have $A(b) = T$.

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3. Stable* $\Rightarrow$ Stable:
   Contrapositive:
   $\not{S}$table $\Rightarrow$ $\not{S}$table*
   If a match μ is not stable:
   $\exists(t, b)$ s.t. $b \succ_t \mu(t)$
   $t \succ_b \mu(b)$ then b $\succsim_t \mu(t), t \succ_b \mu(b) \Rightarrow \mu$ is not stable*.

   Stable $\Rightarrow$ Stable* is not true:
   For $T = \{t^*\}$ and $B = \{b^*\}$
   $t_1 : b_1 \succsim b_2 \succ t_1$
   $t_1 : b_1 \succsim b_2 \succ t_2$
   $b_i : t_1 \succ t_2 \succ b_i$
   t_2 is the only T agent who can be in a blocking pair.
   $\mu(t_1) = b_2$
   $\mu(t_2) = b_1$
   This is stable but not stable* as (b_1, t_1) block*.

   Stable* is stronger. Block is stronger.

4. Because of strict preferences, all agents find all other agents acceptable. Also, $|T| = |B|$, then in any stable matching, all agents are matched. Check the cases:

   (a) $\mu(t_1) = b_1$:
   t_1 gets $\frac{1}{4} \Rightarrow$ for (t_1, b_2). To not be a block $\Rightarrow \mu(b_2) \succ_{b_2} \frac{2}{4}$
   $\Rightarrow \mu(b_2) = t_2$

$\Rightarrow \mu(t_3) = b_3$
This is a stable match. All b agents get their best options.

(b) $\mu(t_1) = b_3$
b_3 gets $\frac{1}{4} \Rightarrow$ for (t_2, b_3). To not be a block $\Rightarrow \mu(t_2) \succ_{t_2} \frac{2}{4}$
$\Rightarrow \mu(t_2) = b_1$
$\Rightarrow \mu(t_3) = b_2$
This is a stable match. All t agents get their best options.

(c) $\mu(t_1) = b_2$:

i. $\mu(t_1) = b_2, \mu(t_2) = b_1, \mu(t_3) = b_3$
t_3 and b_1 get $\frac{1}{4}$ and will block.
$\Rightarrow$ Not a stable match

ii. $\mu(t_1) = b_2, \mu(t_2) = b_3, \mu(t_3) = b_1$
All agents get $\frac{1}{2}$ and will not block.
$\Rightarrow$ A stable match.

#2

$T = \{t_1, t_2, t_3\}$ and $B = \{b_1, b_2\}$:

- $t_1 : b_1 \succ b_2 \succ t_1$,
- $t_2 : b_1 \succ b_2 \succ t_2$,
- $t_3 : b_2 \succ t_3 \succ b_1$,
- $b_1 : t_1 \succ t_2 \succ t_3 \succ b_1$,
- $b_2 : t_3 \succ t_1 \succ t_2 \succ b_2$.

The unique stable matching is that $\mu(t_1) = b_1$, $\mu(t_2) = t_2$, $\mu(t_3) = b_2$. The stable matching is unique because the pairs (t_1, b_1) and (t_3, b_2) find their mates as their first options.

t_1 and t_2 have the acceptable set as B, b_1 and b_2 have the acceptable set as T, so the condition of the question is satisfied. While t_2, whose acceptable set is B, remains unmatched in the unique stable matching.