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ECON 501B

# Midterm 2018

**Instructions:** You have 1 hour and 15 minutes to complete the exam. *All* answers should involve complete proofs to back up your assertions.

**Question 1:** Consider a one-to-one matching environment $\mathcal{E} = (T, B; (\succsim_i : i \in T \cup B))$. Suppose that $\mu, \nu$ are matchings so that $\nu(i) \succsim_i \mu(i)$ for all $i \in T \cup B$. Show the following: If $\mu$ is stable, then $\nu$ is stable.

**Question 2:** Consider a many-to-one matching environment $\mathcal{E} = (T, B; (q_t : t \in T); (\succsim_t : t \in T), (\succsim_b : b \in B))$ with strict preferences. Show the following: If a matching is group stable then it is Pareto efficient.

(Recall: A matching $\mu$ is Pareto efficient if there is no matching $\mu'$ so that (a) for each $i \in T \cup B$, $\mu'(i) \succsim_i \mu(i)$, and (b) for some $i \in T \cup B$, $\mu'(i) \succ_i \mu(i)$. )

**Question 3:** Consider a one-to-one matching environment $\mathcal{E} = (T, B; (\succsim_i^* : i \in T \cup B))$ described as follows: The sets of agents are given by $T = \{t_1, \ldots, t_K\}$ and $B = \{b_1, \ldots, b_K\}$. Preferences are such that:

- For $t_1$: $b_K \succ_{t_1}^* b_{K-1} \succ_{t_1}^* \cdots b_2 \succ_{t_1}^* b_1$.
- For $t_i = t_2 \ldots, t_K$: $b_1 \succ_{t_i}^* b_2 \succ_{t_i}^* \cdots b_{K-1} \succ_{t_i}^* b_K$.
- For $b_i = b_1 \ldots, b_K$: $t_1 \succ_{b_i}^* t_2 \succ_{b_i}^* \cdots t_{K-1} \succ_{b_i}^* t_K$.

*Question 3a:* What are the stable matches? (Your answer should consist of a complete proof that establishes both which matches are stable and which matches are unstable.)

Let $m$ be a matching mechanism for $(T, B)$ satisfying:

- $m(\succsim^*)$ is the T-proposal DA match for the preference profile $\succsim^* = (\succsim_i^* : i \in T \cup B)$, and
- $m(\succsim)$ is the B-proposal DA match for the preference profile $\succsim = (\succsim_i : i \in T \cup B)$ provided $\succsim \neq \succsim^*$.

In what follows, take the true environment to be $\mathcal{E} = (T, B; (\succsim_i^* : i \in T \cup B))$, i.e, so the true preferences are $\succsim^* = (\succsim_i^* : i \in T \cup B)$.

*Question 3b:* Does agent $b_k \in B$ have an incentive to misreport, if all other agents are using a truthful strategy?

*Question 3c:* Does agent $t_k \in T$ have an incentive to misreport, if all other agents are using a truthful strategy?

# Q1 : $\nu$ is IR and no blocking pair.

IR : $\forall i \in T \cup B \quad \nu(i) > \mu(i) \gtrsim i$

Because $\mu$ is stable

Suppose $\exists$ a blocking pair $(t, b)$ for $\nu$

$$b >_t \nu(t) \gtrsim_t \mu(t)$$

$$t >_b \nu(b) \gtrsim_b \mu(b)$$

$(t, b)$ blocks $\mu$ as well. Contradicts $\nu$ being stable

Q2 Proof by contrapositive.

Suppose $\exists \mu'$ Pareto Dominates $\mu$.

$C = \{ i \in T \cup B \mid \mu'(i) \succ_i \mu(i) \}$

C is nonempty by def^n of pareto dominates.

For any $t \in C$, he is matched to group of B agent under $\mu'(t) \subseteq \mu(t) \cup C$

$A \subset B$ : $\forall x \notin B \Rightarrow x \notin A$

For any $b \notin \mu(t) \cup C$ by definition of C:

$\mu'(b) = \mu(b)$
$\neq \mu(t)$

Then $b \notin \mu'(t)$.

For any $b \in C$: $\mu'(b) \succ_b \mu(b)$

$b \in \mu'(t')$ but $b \notin \mu(t')$

So, $\mu'(t') \neq \mu(t')$. Because of strict preference and Pareto Dominance $\mu'(t) \succ_{t'} \mu(t)$

So, C forms a block to $\mu$.

$\exists t' \in C$

Q3a $M^*$ $M^*(t_1) = b_k$; $M^*(t_i) = b_{i-1}$
for $i = 2, \ldots k$

$M^*$ is unique stable. It suffices to show that it's the outcome of both the T and B-proposal DAA.

T-proposal:

R1 - $t_1$ matched to $b_k$ and $b_1$ matched to $t_2$. They won't change later.

Rk: $b_k$ matched to $t_{k+1}$, they won't change later

B-proposal

R1 - $t_1$ matched to $b_k$, they won't change.

Rk - $t_k$ matched to $b_{k-1}$
They won't change.

$t_1$ and $b_K$ have no incentive to misreport.

b.) If $b_k$ $(k < K)$ misreport through B-proposal DAA, at round $i$, $i < k$ $b_i$ will be matched to $t_{i+1}$ and they won't change later. So, $b_k$ cannot be better off.

c.) If $t_k$ misreport $(k > 1)$ through B-proposal DAA, at round $k-1$ all unmatched $b$ agents will propose to $t_k$, which does not include $(b_1, \dots, b_{k-2})$ who have been matched and won't change. So, $t_k$ cannot get better off.

quilibr soln for b/c

Under truthful reporting, t-proposal deferred acceptance algorithm yields the same match as that from b-proposal. So, its equivalent to treat $m(\cdot)$ as the B-proposal deferred acceptance algorithm. Apply Thm 3 of the class (from October $4^{th}$). And truthful reporting is dominant for any agent.

Q4) ~~Briefly Antisymmetric~~ Want to show first that all agents are matched under $M$ and $M_{TB}$.

For any $M$

Question 4 Contrapositive - $\forall t, \mu_{TB} \prec_t \mu(t)$

Want to show ~~when~~ all agents are matched under $\mu$ and $\mu_{TB}$

For $\mu$, $\forall t, \mu(t) \succ_t \mu_{TB}(t) \succsim_t t$

~~C~~ All $t$ are matched because $|T| = |B|$ all $b$ matched.

For $\mu_{TB}$ to be stable for all $b$

$\mu_{TB}(b) \succsim_b \mu(b) \succ_b b$

Suppose $\exists b$ $\mu_{TB}(b) \prec_b \mu(b)$ Let $t' = \mu(b)$.

$\mu_{TB}(t') \prec_{t'} \mu(t') = b$ $(t', b)$ blocks $\mu_{TB}$.

$|T| = |B|$, all $t$ matched.

Because all matched under $\mu_{TB}$, de there is a $b$ agent newly matched in the last round. Suppose $b^*$ is this guy.

Let $\mu_{TB}(b^*) \succ_{b^*} \mu(b^*)$. Remark: all $(t, \mu_{TB}(t) \neq \mu(t))$

So, for all $b$, $\mu_{TB}(b) \neq \mu(b)$

Q4 Cont.

and $t^* = \mu(b^*)$

We know $\mu_{TB}(t^*) <_{t^*} \mu(t^*)$

At last round of T-Prop DAA, $t^*$ must be matched $t^*$ must be matched to $\mu_{TB}(t^*)$ and $t^*$ must have proposed to $b^*$ before. We know $t^*$ must be acceptable to $b^*$, which implies $b^*$ must be matched to someone at the second to the last round. But $b^*$ rejects this $t$ agent at the last round and switch to $\mu_{TB}(b^*)$

At the last round, $t'$ is unmatched contradicting the last round.