$((x^*, y^*), p^*)$ with transfers $(W_1, \dots W_I)$

① firms max profit : $p^* \cdot y_j^* \geq p^* \cdot y_j$ for all $y_j \in Y_j$

② $x_i^* \in B(p^*, W_i)$ and $x_i^* \succsim_i x_i$ for all $x_i \in B_i(p^*, W_i)$

③ $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$

④ $\sum_i W_i = p^* \cdot \bar{\omega} + \sum p^* \cdot y_j^*$

<price eq. with transfers>

② $x_i \succ_i x_i^* \Rightarrow p^* x_i \geq W_i$

<Quasi-price Equilibrium with Transfers>

(Thm) Y_j convex, $\succsim_i$ convex & LNS. (x^*, y^*) Pareto Optimal

$\Rightarrow \exists\, p^* \neq 0$ s.t. $((x^*, y^*), p^*)$ is a Quasi-Price eq with transfers.

(prop) Let $\succsim_i$ be a continuous preference relation on a convex $X_i \subseteq \mathbb{R}_+^L$

(*) Let x_i^*, $p \in \mathbb{R}^L$ and W_i be such that $x_i \succ_i x_i^* \Rightarrow p \cdot x_i \geq W_i$.

If $\exists\, \tilde{x}_i \in X_i$ s.t. $p \cdot \tilde{x}_i < W_i$, then $x_i \succ_i x_i^* \Rightarrow p \cdot x_i > W_i$

(proof) Applied to evil example. $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2) \gg 0$, (x_1^*, x_2^*), $x_1^* = (0, \bar{\omega}_2)$

$x_2^* = (\bar{\omega}_1, 0)$

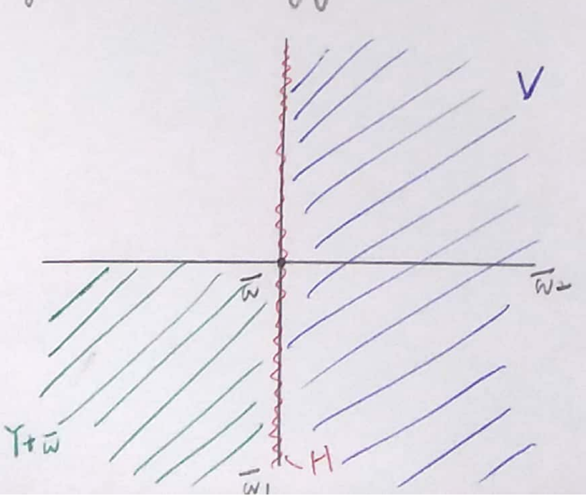
$y^* = (0, 0)$

- $V := \{x_1 + x_2 : x_1 \succ_1 x_1^* \text{ and } x_2 \succ_2 x_2^*\}$

consider the initial endowment.

agent 2 has strongly monotone $\succsim$: $x_2 \geq (0, \bar{\omega}_2)$ with $x_1 \neq (0, \bar{\omega}_2)$

x_2 s.t. $x_{12} > \bar{\omega}_1$



- $Y + \bar{\omega}$

- $p^* \neq (0,0)$ s.t. (1) $p^* \cdot z \geq r$ for all $z \in V$

(2) $p^* \cdot z \leq r$ for all $z \in Y + \bar{\omega}$

$p_2^* = 0$ $\quad W_1 = p_1^* 0 + p_2^* \bar{\omega}_2 = 0$

$p_1^* > 0$ $\quad W_2 = p_1^* \bar{\omega}_1 > 0$

$W_1 = 0$

There is no $\tilde{x}_1$ s.t. $p^*\cdot\tilde{x}_1 < 0$

Note Under the hypothesis of the prop $\Rightarrow$ $p\cdot x_i = W_i$

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proof of prop2

(*) Fix some $x_i \succsim_i x_i^*$ and Suppose, contra hypothesis, that $p\cdot x_i \leq W_i$

For each $n \geq 1$, define $x_i^n = \frac{1}{n}\tilde{x}_i + \frac{n-1}{n}x_i$

* $\tilde{x}_i, x_i \in X_i$ and X_i is convex $\Rightarrow$ $x_i^n \in X_i$ ($\because X_i$ convex)

By assumption : $p\cdot\tilde{x}_i < W_i \Rightarrow \underline{p\cdot x_i^n < W_i}$ ($\because x_i^n \to x_i$ as $n\to\infty$)

By $\succsim_i$ continuous, $\exists\ \bar{N}$ s.t for all $n \geq \bar{N}$, $\underline{x_i^n \succsim_i x_i^*}$

$\Rightarrow$ For all $n \geq \bar{N}$, $x_i^n \succsim_i x_i^*$ and $p\cdot x_i^n < W_i$, contradicting the premise of proposition. ||

Alternate Approach to Welfare

- till now : Pareto optimality
- alternate : utilitarian effciency (Weighted ver. of utilitarian efficiency)
- Under price equilibrium with transfers,

  is there a gap between the two concepts?

LOOK AT "NICE" economy

① Each $Y_j \subseteq \mathbb{R}^L$ is convex

② Each $X_i = \mathbb{R}^L_+$

$x_i' \geq x_i$ with $x_i' \neq x_i$ $\Rightarrow u_i(x_i') > u_i(x_i)$ (St. increasing)

③ Each $\succsim_i$ is represented by a utility function u_i that is (strictly increasing & concave.

④ There exists a feasible allocation $(\bar{x}, \bar{y})$ s.t $\sum_{i=1}^{I} \bar{x}_i \gg (0, \ldots, 0)$

- Designer has weights $\alpha = (\alpha_1, \ldots, \alpha_I) \in \mathbb{R}^I_+$
- Choose (x^*, y^*) to solve $\max_{(x,y) \text{ feasible}} \sum_{i=1}^{I} \alpha_i u_i(x_i)$ $\boxed{\text{Problem } \alpha}$
- Utilitarian special case of $\alpha_1 = \cdots = \alpha_I$

Thm Fix a Nice economy.

P.E.T $\Rightarrow$ Problem α

(1) Suppose that $((x^*, y^*), p^*)$ is a price equilibrium with transfers

Then there exists $\alpha = (\alpha_1, \ldots, \alpha_I)$ s.t. (x^*, y^*) solves "Problem α"

(2) Suppose that (x^*, y^*) solves Problem α for some $\alpha = (\alpha_1, \ldots, \alpha_I)$

Then there exists p^* s.t $((x^*, y^*), p^*)$ form a price equilibrium with transfers

Problem $\alpha \Rightarrow$ P.E.T.

part ①

Choose (x, y^*) to solve

$$\max_{\substack{x_i \in X_i,\ i=1,\dots,I \\ y_j \in Y_j,\ j=1,\dots,J}} \sum_{i=1}^{I} \alpha_i u_i(x_i) \quad \text{s.t.} \quad \underbrace{\sum x_{\ell i} - \sum y_{\ell j} = \bar{w}_\ell}_{\text{"feasibility"}} \quad \text{for each } \ell = 1, \dots, L$$

$\boxed{\text{Problem } \alpha'}$

$$\max_{\substack{x_i \in X_i \\ y_j \in Y_j}} \sum_{i=1}^{I} \alpha_i u_i(x_i) \text{ s.t } \sum x_{\ell i} - \sum y_{\ell j} \leqq \bar{w}_\ell \Rightarrow \boxed{\text{Problem } \alpha''}$$

→ problem α = problem α' ($\because$ feasiblity is showed).

→ By $\succsim$ strongly monotone, problem α' = problem α''

- If (x^*, y^*) is a solution to Problem α'' & it satisfies the equality therefore, then it is a solution to problem α'

- $\succsim_i$ strongly monotone with work a solution to problem α' will satisfy two equality constraints

part ②

Fix (x^*, y^*) that solves problem α''.

- Applying Kuhn - Tucker theorem to the space $\prod_{i=1}^{I} X_i \times \prod_{j=1}^{J} Y_j$

- to apply KT : (*) $\prod_{i=1}^{I} X_i \times \prod_{j=1}^{J} Y_j$ is convex

(*) Objective function is concave.

(u_i is concave , so its sum is also concave)

(*) constraints are convex.

$\sum x_{\ell i} - \sum y_{\ell j} = \bar{w}_\ell$: linear constraint ⇒ so, convex.

(*) *slater condition $\Uparrow$ $<$

$\exists\ (\hat{x}, \hat{y})$ s.t. at $(\hat{x}, \hat{y})$ all the constraints are slack

- at $(\tilde{x}, \tilde{y})$ constraints hold with equality

- $\hat{x} = \frac{1}{2}\tilde{x}$, $\hat{y} = \tilde{y}$ $\Rightarrow$ at $(\hat{x}, \hat{y})$ all constraints are slack.

<part① point>

$\Rightarrow$ KT tells you that $\exists\ \lambda_1, \dots, \lambda_L \geq 0$ s.t. can solve problem α"

as the lagrangian with parameter $\lambda_1, \dots, \lambda_L$.

$\bar{w}$ can be cancelled.

$\Downarrow$ b/c constant.

$p^* = (\lambda_1, \dots, \lambda_L)$

Lagrangian : $\boxed{\max_{\substack{x_i \in X_i \\ y_j \in Y_j}} \left[\sum \alpha_i u_i(x_i) - \sum_{l=1}^{L} \lambda_l \left[\sum_i x_{li} - \sum_j y_{lj} \right] \right]}$

p^*_l

① Firms : $p^* y_j^* \geq p^* y_j$ for all y_j

② Consumers : $\max_{x_i}\ u_i(x_i) - \tilde{\lambda}_i \cdot p^* x_i$ for each i. [problem i]

Need to find $\tilde{\lambda}_i = \frac{1}{\alpha_i}$ So we can think this problem

the same as $\max_{x_i} [\alpha_i u_i(x_i) - p^* x_i]$

$\uparrow$

- α_i is tricky

so that it can handle $\tilde{x}_i$ easily.

For each i,

WTS : $\exists\ \tilde{X}_i$ s.t. x_i^* solves [problem i]

Note

Two ways of KT

Find λ $\Rightarrow$ can solve problem"

Solve problem $\Rightarrow$ $\exists\ \lambda$