Homework 7 Solutions **Spencer Cooper**

ECON 501B

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1. (a) Strict and assortative preferences. Thus μ^{NTU} is PAM.

   (b) $\omega(i,j)$ has decreasing difference.
   For ω twice differentiale, then ω has DD iff $\omega_{xy} \leq 0$; and ω has ID iff $\omega_{xy} \geq 0$
   $\omega_{ij} = 2\alpha(\alpha-1)(i+j)^{\alpha-2} < 0$
   μ^{TU} is NAM.

   (c) $W(\mu^{NTU}) = 2\sum_{i=1}^{n}(2i)^{\alpha}$
   $W(\mu^{TU}) = 2n(n+1)^{\alpha}$

   (d) Unsolvable. :(
   $\frac{W(\mu^{NTU})}{W(\mu^{TU})} \in [2^{\alpha}, 1]$

2. Lemma 1: If (x^*, q^*) solves problem (*), then there is a Pareto optimal allocation (x^*, m^*, q^*, z^*)
   Proof:
   Choose z^*, m^* such that $\forall j, z_j^* = c_j(q_j^*)$
   and m^* satisfies $\sum_{i=1}^{I} m_i^* = \bar{\omega}_m - \sum_{j=1}^{J} c_j(q_j^*)$
   So defined, the allocation is feasible.
   Check, $\sum x_i^* = \sum q_j^*$ since (x^*, q^*) solves problem (*), and $\sum m_i^* = \bar{\omega_m} - \sum z_j^*$ by construction.
   For any feasible (x, m, q, z), if it Pareto dominates (x^*, m^*, q^*, z^*) then:
   $\sum(m_i + \phi_i(x_i)) > \sum(m_i^* + \phi_i(x_i^*))$
   So $\bar{\omega} + S(x, q) = \bar{\omega}_m + \sum \phi_i(x_i) - \sum c_j(q_j)$ for all j such that $z_j \geq c_j(q_j)$
   $\geq \bar{\omega}_m + \sum \phi_i(x_i) - \sum z_j$
   $= \sum m_i + \sum \phi_i(x_i)$
   $> \sum(m_i^* + \phi_i(x_i^*))$
   $= \bar{\omega}_m + S(x^*, q^*)$
   Contradicts that (x^*, q^*) solves problem (*).

   Lemma 2: If (x^*, m^*, q^*, z^*) is Pareto Optimal, then it induces a $v^* \in \mathcal{R}^I$ with $v^* \in Bd(\mathcal{U}(x^*, q^*))$
   Proof:
   Suppose (x^*, m^*, q^*, z^*) is Pareto Optimal and induces $v^* \in \mathcal{R}^I$
   First, claim $\forall j : c_j(q_j^*) = z_j^*$
   If $\exists j : c_j(q_j^*) < z_j^*$, keep everything fixed, giving $(z_j^* - c_j(q_j^*))$ to some consumer is a Pareto improvement.

Second, $\sum v_i^* = \sum m_i^* + \sum \phi_i(x_i)$
$= \bar{\omega}_m + \sum \phi_i(x_i^*) - \sum z_j^*$
$= \bar{\omega}_m + \sum \phi_i(x_i^*) - \sum c_j(q_j^*)$
$= \bar{\omega}_m + S(x^*, q^*)$
For any other feasible allocation (x, m, q, z), $\sum v_i^* = \bar{\omega}_m + S(x^*, q^*) \geq \bar{\omega}_m + S(x, q)$ for all other x, q
$\Rightarrow \sum v_i^* \geq \sum u_i(m_i, x_i)$

Lemma 3: If (x^*, m^*, q^*, z^*) is Pareto optimal, then (x^*, q^*) solves problem (*)
Proof:
By contradiction:
Suppose (x^*, m^*, q^*, z^*) is Pareto Optimal but it does not solve problem (*):

Step 1 : $\exists (x, q)$ so that $\bar{\omega}_m + S(x, q) > \bar{\omega}_m + S(x^*, q^*)$ and $\sum x_i = \sum q_j$
Construct $m_i = \frac{1}{I}(\bar{\omega}_m + \sum z_j)$ and $z_j = c_j(q_j)$
So defined, $\sum m_i + \sum \phi_i(x_i) = \bar{\omega}_m + \sum \phi_i(x_i) - \sum c_j(q_j)$
$= \bar{\omega}_m + \sum \phi_i(x_i) - \sum z_j$
$= \bar{\omega}_m + S(x, q)$
$> \bar{\omega}_m + S(x^*, q^*)$ from assumption
$\sum m_i^* + \sum \phi_i(x_i)$ from Lemma 2
In conclusion, $\sum u_i(m_i, x_i) > \sum u_i(m_i^*, x_i^*)$
This means someone gets better off under (x, m, q, z)

Step 2: $(x^*, m^*, q^*, z^*) \rightarrow (x, m, q, z)$
Order consumers: $u(x_1, m_1) - u_1(x_1^*, m_1^*) \geq u(x_2, m_2) - u_1(x_2^*, m_2^*) \geq ... \geq u(x_I, m_I) - u_I(x_I^*, m_I^*)$
It must be that $u(x_1, m_1) - u_1(x_1^*, m_1^*) > 0$
There is k such that:
for $i \leq k : u(x_i, m_i) - u_1(x_i^*, m_i^*) \geq 0$
for $i \leq k : u(x_i, m_i) - u_1(x_i^*, m_i^*) < 0$
The total surplus is higher than the total deficit:
$\sum_{i=1}^k u_i(m_i, x_i) + \sum_{i=k+1}^I u_i(m_i, x_i) > \sum_{i=1}^k u_i(m_i^*, x_i^*) + \sum_{i=k+1}^I u_i(m_i^*, x_i^*)$
$\sum_{i=1}^k [u_i(m_i, x_i) - u_i(m_i^*, x_i^*)] > \sum_{i=k+1}^I [u_i(m_i^*, x_i^*) - u_i(m_i, x_i)]$
$\exists \alpha \in (0, 1)$ such that
$\alpha \sum_{i=1}^k [u_i(m_i, x_i) - u_i(m_i^*, x_i^*)] > \sum_{i=1}^k [u_i(m_i^*, x_i^*) - u_i(m_i, x_i)]$
Construct m'
For $i \leq k, m_i' = m_i - \alpha[u_i(m_i, x_i) - u_i(m_i^*, x_i^*)$
For $i > k, m_i' = m_i + [u_i(m_i^*, x_i^*) - u_i(m_i, x_i)]$

Show that (x, m', q, z) Pareto dominates (x^*, m^*, q^*, z^*) :
1. (x, m', q, z) feasible
2. Everybody is as good as under (x^*, m', q^*, z^*)
3. Consumer 1 is strictly better off.

3. Suppose (x^*, m^*, q^*, z^*) and p^* forms a competitive equilibrium.
   Case A: $\sigma < p^*, q^* \to \infty$ no solution to firms maximization.

   Case B: $\sigma = p^*, q^* = x^*, m^* = \bar{\omega}_m - p^* x$
   Consumer Problem:
   $max_x(\bar{\omega}_m - p^* x + \alpha + \beta lnx$

   FOC: $\beta / x^* \leq p^*$ with equality if $x^* > 0$
   $x^* = \beta / \sigma$

   Case C: $\sigma > p^*, q^* = 0, x^* = 0$
   Utility is undefined at $x^* = 0$