Proposition: Suppose assumption 1 holds.

(1) Then P1 - P4 hold.

(2) If, in addition, prefs are strongly monotone, then P5 holds

Proof

(1) P1-P3 follow from 501a

P4: $X_i = \mathbb{R}^L_+ \Rightarrow z_{\ell i}(p) = x_{\ell i}(p) - \omega_{\ell i} \geq -\omega_{\ell i}$ for all p and i and ℓ

$\Rightarrow z_\ell(p) \geq -\bar{\omega}_\ell$ for all p and all ℓ

Let $s = \max\{\bar{\omega}_1, \dots, \bar{\omega}_L\}$

$\Rightarrow z_\ell(p) \geq -s$ for all p

(2) Fix a sequence $p^n \in \mathbb{R}^L_{++}$ with $p^n \to p^\circ$ and $p^\circ \neq (0,\dots,0)$ and $p^\circ_\ell = 0$ for some ℓ

* for each i, $x_i(p^\circ) = \phi$: since $p^\circ_\ell = 0$ and $X_i = \mathbb{R}^L_+$, for any $x_i \in B_i(p^\circ)$, can always find an $x_i' \in B_i(p^\circ)$ with $x_i' \geq x_i$ and $x_i' \neq x_i$. By strong monotonicity, $x_i' \succ_i x_i$.
* Will show: if it is not the case that "$\lim_{n\to\infty} \max\{z_1(p^n), \dots, z_L(p^n)\} = \infty$" then there exists some i and some $x_i^* \in B_i(p^\circ)$ s.t. $x_i^* \succsim_i x_i$ for all $x_i \in B_i(p^\circ)$. This contradicts the above.

Suppose $\neg\left(\lim_{n\to\infty} \max\{z_1(p^n), \dots, z_L(p^n)\} = \infty\right)$

$\Rightarrow$ there exists some subsequence - denoted $(p^m : m=1,2,\dots)$ - s.t. $\lim_{m\to\infty} \max\{z_1(p^m), \dots, z_L(p^m)\} = M < \infty$

- Note each $z_\ell(\mathbb{R}^L_{++}) = \{x_\ell' - \bar{\omega}_\ell : \underbrace{x_\ell' \in x_\ell(\mathbb{R}^L_{++})}_{\text{bounded by } [0, \bar{\omega}_L]}\}$ $\Rightarrow$ $z_\ell(\mathbb{R}^L_{++})$ is a closed & bounded set
- there is a subsequence $(p^m : m=1,2,\dots)$ s.t. $\max\{z_1(p^m), \dots, z_L(p^m)\}$ converges
  $\Rightarrow$ converge to a finite M

$\Rightarrow$ for each ℓ, m, $z_\ell(p^m) = x_\ell(p^m) - \bar{\omega}_\ell \in [0, M]$

$\Rightarrow$ for each i, ℓ, m, $x_{\ell i}(p^m) \in [0, M + \bar{\omega}_\ell]$

$\Rightarrow$ for each i, there is a subsequence $(p^k : k=1,2,\dots)$ s.t. $x_i(p^k)$ converges to a pt in X_i ← note subsequence depends on i

let $x_i^* = \lim_{k\to\infty} x_i(p^k)$ ← defined for each i

$\Rightarrow$ if $x_i' \succ_i x_i^*$ then $p^\circ \cdot x_i \geq p^\circ \cdot \omega_i$: (*) ← Quasi-Walrasian Equilibrium

→ will show contrapositive: $p^\circ \cdot \omega_i > p^\circ \cdot x_i \Rightarrow x_i^* \succsim_i x_i$

if $p^\circ \cdot \omega_i > p^\circ \cdot x_i$ then, for k large enough, $p^k \cdot \omega_i > p^k \cdot x_i \Rightarrow x_i \in B_i(p^k) \Rightarrow x_i(p^k) \succsim_i x_i$

$\Rightarrow x_i^* \succsim_i x_i$ by continuity of pref

$\Rightarrow$ will show: there exists i s.t. $x_i' \succ_i x_i^* \Rightarrow p^\circ \cdot x_i > p^\circ \cdot \omega_i$ $\Rightarrow$ $x_i(p^\circ) = x_i^*$ for that i

Use Lemma from SWT + (*)

to show: there exists i s.t. (1) $p^\circ \cdot \omega_i > 0$ and (2) there exists $\tilde{x}_i \in X_i$ with $p^\circ \cdot \tilde{x}_i = 0$ ← Cheaper consumption gets Walrasian Equilibrium

① - since each $\bar{\omega}_\ell > 0$, for each ℓ there exists i with $\omega_{\ell i} > 0$ | choose ℓ s.t. $p^\circ_\ell > 0$

- $p^\circ \neq (0,\dots,0) \Rightarrow$ there exists ℓ with $p_\ell > 0$ | choose i s.t. $\omega_{\ell i} > 0 \Rightarrow p^\circ \cdot \omega_i > 0$ ✓

② $\tilde{x}_i = (0,\dots,0) \in X_i$ and $p^\circ \cdot \tilde{x}_i = 0$ ✓

cheaper