Advanced Microeconomics

Consumer theory: optimization and duality

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Introduction

The plan:

1. The utility maximization
2. The expenditure minimization
3. Duality of the consumer problem
4. Some examples

The consumer problem

In general, the consumer problem can be state as:

- choose the best bundle that the consumer can afford
- or: choose a bundle $x \succeq y$ for x and any y's in the budget set B, given prices p and wealth w
- or: if we have a utility function representing $\succeq$, maximize utility subject to the budget constraint (given by p and w).
- the correspondence between prices p, wealth w and the consumer chosen bundle is the demand correspondence.

The utility maximization problem (UMP)

- We will asume that the consumer has a rational, continuous and locally nonsatiated preference relation.
- $u(x)$ is a continuous utility function representing consumer preferences
- the consumption set is $X = \mathbb{R}^L_+$
- The utility maximization problem is defined as:

$$\begin{aligned} \text{Max}_{x\geq 0} \quad & u(x) \\ & \text{subject to } p \cdot x \leq w \end{aligned}$$

- If $u(x)$ is well behaved, then this problem has a solution $x(p, w)$ which is the so-called Walrasian demand correspondence

The UMP (for interior solution)

- The UMP is usually set up as a Kuhn-Tucker sort of problem.
- Let us write down the Lagrange function:

$$\mathcal{L} = u(x) - \lambda(p \cdot x - w)$$

- The first order conditions for an interior solution:

$$\frac{\partial u(x)}{\partial x_1} - \lambda p_1 \leq 0$$

$$\vdots$$

$$\frac{\partial u(x)}{\partial x_L} - \lambda p_L \leq 0$$

$$p \cdot x - w \leq 0$$

which gives (if interior solution) $\frac{\frac{\partial u(x)}{\partial x_l}}{\frac{\partial u(x)}{\partial x_k}} = \frac{p_l}{p_k}$ for all l, k

and hence: $MRS_{lk} = \frac{p_l}{p_k}$

Marginal rate of substitution

- Let's totally differentiate $u = u(x)$ for a zero change in utility:

$$du = 0 = \sum_{l=1}^{L} \frac{\partial u(x)}{\partial x_l}$$

- Lets assume that $dx_l \neq 0$ and $dx_k \neq 0$ and all other $dx_n = 0$:

$$0 = \frac{\partial u(x)}{\partial x_l} dx_l + \frac{\partial u(x)}{\partial x_k} dx_k$$

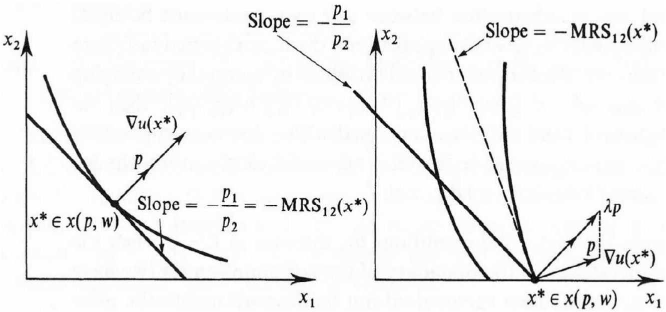
- Rearrange:

$$-\frac{dx_l}{dx_k} = \frac{\frac{\partial \mathbf{u}(x)}{\partial x_\mathbf{l}}}{\frac{\partial \mathbf{u}(x)}{\partial x_\mathbf{k}}} = MRS_{lk}$$

- So at the optimal choice the ratio in which the consumer is willing to give away l for k is equal to the ratio of prices.

The UMP

- The KT procedure says that either $\lambda = 0$ or the budget constraint is binding (which is usually the case).
- Also, we might have $x_l = 0$ and it that case the relevant FOC is satisified with inequality, i.e. $\frac{\partial u(x)}{\partial x_l} < \lambda p_L$ (corner solution)



The UMP with corner solutions

If we suspect there may be corner solutions $x_l = 0$ for some l, then we need to set up the full Kuhn-Tucker problem with:

- The budget constraint $\sum_{l=1}^{l} p_l x_l - w$ with the Lagrange multiplier λ_0
- L inequality constraints $x_l \geq 0$ with the L Lagrange multipilers λ_l

$$\mathcal{L} = u(x) - \lambda_0(\sum_{l=1}^{l} p_l x_l - w) + \sum_{l=1}^{l} \lambda_l(x_l)$$

Then for the inequality constraints it is either ($\lambda_l = 0$ and $x_l > 0$) or ($\lambda_l > 0$ and $x_l = 0$). You have to check all the combinations!

Example: For the utility function $u(x) = \sum_{l=1}^{L} a_l x_l$, $l = 2$, find the demand function.

The UMP with corner solutions

$$\mathcal{L} = a_1 x_1 + a_2 x_2 - \lambda_0 (\sum_{l=1}^{l} p_l x_l - w) + \sum_{l=1}^{l} \lambda_l (x_l)$$

FOC's are:

$$\begin{array}{ll} [x_1] & a_1 - \lambda_0 p_1 + \lambda_1 = 0 \\ [x_2] & a_2 - \lambda_0 p_2 + \lambda_2 = 0 \\ [\lambda_0] & p_1 x_1 + p_2 x_2 - w \leq 0, \quad \lambda \geq 0, \lambda(w - p_1 x_1 - p_2 x_2) = 0, \\ [\lambda_1] & x_1 \geq 0, \lambda_1 \geq 0, \lambda_1 x_1 = 0 \\ [\lambda_2] & x_2 \geq 0, \lambda_2 \geq 0, \lambda_2 x_2 = 0 \end{array}$$

Case 1: $\lambda_0 > 0$, and all $\lambda_l = 0$, therefore all $x_l > 0$ (interior solution)

from first 2 FOCs we have: $\frac{a_1}{p_1} = \lambda_0$ and $\frac{a_2}{p_2} = \lambda_0 \rightarrow a_1/p_1 = a_2/p_2$ or $\frac{p_1}{p_2} = \frac{a_1}{a_2}$ or better $\frac{a_1}{p_1} = \frac{a_2}{p_2}$ (the expenditures on one unit of MU are equal).

Only at that price ratio demand is a correspondence: $p_1 x_1 + p_2 x_2 = w$. All other cases are corner solutions.

The UMP with corner solutions

Case 2:

$\lambda_0 > 0$, and $\lambda_1 > 0, \lambda_2 = 0$ therefore $x_1 = 0$ and $x_2 > 0$ (corner solution)

The FOC's become:

$p_2 x_2 = w$ and therefore $x_2 = \frac{w}{p_2}$.

$a_2 - \lambda_0 p_2 = 0 \text{ and } a_1 - \lambda_0 p_1 + \lambda_1 = 0 \Rightarrow \frac{a_1}{p_1} = \lambda_0 - \frac{\lambda_1}{p_1} < \lambda_0 = \frac{a_2}{p_2}, \text{ so}$
$\frac{a_1}{p_1} < \frac{a_2}{p_2}$

Case 3:

$\lambda_0 > 0$, and $\lambda_1 = 0, \lambda_2 > 0$ therefore $x_1 > 0$ and $x_2 = 0$ (corner solution)

The FOC's become:

$p_1 x_1 = w$ and therefore $x_1 = \frac{w}{p_1}$.

$a_1 - \lambda_0 p_1 = 0 \text{ and } a_2 - \lambda_0 p_2 + \lambda_2 = 0 \Rightarrow \frac{a_2}{p_2} = \lambda_0 - \frac{\lambda_2}{p_2} < \lambda_0 = \frac{a_1}{p_1}, \text{ so}$
$\frac{a_2}{p_2} < \frac{a_1}{p_1}$

The demand correspondence

In our problem the final demand is:

$$x(p) = \begin{cases} x_1 = \frac{w}{p_1},\ x_2 = 0. & \text{if } \frac{a_1}{p_1} > \frac{a_2}{p_2} \\ x_1, x_2 :\ p_1 x_1 + p_2 x_2 = w & \text{if } \frac{p_1}{p_2} = \frac{a_1}{a_2} \\ x_2 = \frac{w}{p_2},\ x_1 = 0 & \text{if } \frac{a_1}{p_1} < \frac{a_2}{p_2} \end{cases}$$

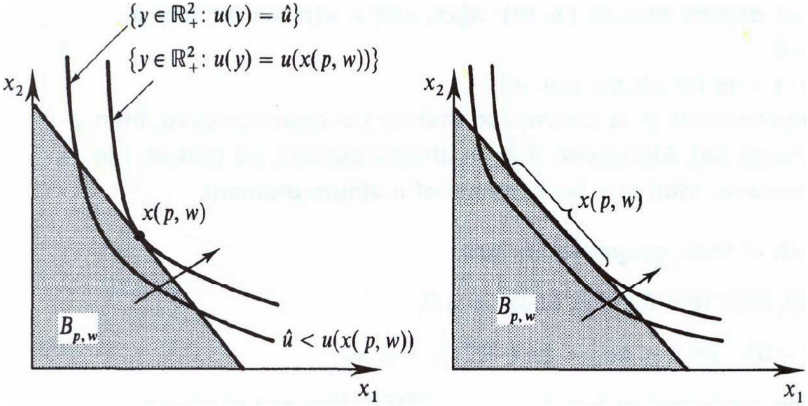
As long as the *bang for the buck* is equal, we have the interior solution, otherwise only corner solutions.

Walrasian demand correspondence

The Walrasian demand correspondence $x(p, w)$ assigns a set of chosen consumption bundles for each price-wealth pair (p, w)

- It can be multi-valued. If single valued we call it a *demand function*
- Under the conditions of continuity and representation of $u(x)$ the Walrasian demand correspondence possesses the following properties:
  1. Homogeneity of degree zero in (p, w)
  2. Walras law: $p \cdot x = w$ (the budget constraint is binding)
  3. Convexity/uniqueness: if $\succeq$ is convex, so that $u(\cdot)$ is quasiconcave, then $x(p, w)$ is a convex set.
     - if $\succeq$ is strictly convex, so that $u(\cdot)$ is strictly quasiconcave, then $x(p, w)$ has just one element.

Walrasian demand correspondence



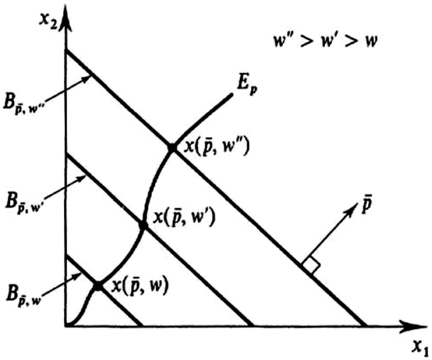
Properties of Walrasian demand

- Wealth effects given the vector of prices p on the demand for good l, partial derivative: $\frac{\partial x_l(p,w)}{\partial w}$.
- In matrix notation:

$$D_w x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \vdots \\ \frac{\partial x_l(p,w)}{\partial w} \\ \vdots \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix}$$

- $\frac{\partial x_l(p,w)}{\partial w} > 0$, good is normal, if all > 0 then demand is normal
- $\frac{\partial x_l(p,w)}{\partial w} < 0$, good is inferior.
- Demand as a function of wealth $x(\bar{p}, w)$, Engel function
- Wealth expansion path: $E_p = \{x(\bar{p}, w) : w > 0\}$
- Income elasticity of demand: $\varepsilon_w = \frac{\partial x(p,w)}{\partial w} \frac{w}{x(p,w)}$, necessity <1, luxury >1

Wealth effects



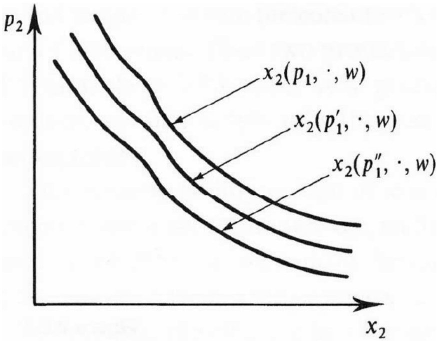
Price effects

- We can measure the effects of prices on the demand for goods.
- The price effect is defined as: $\frac{\partial x_l(p,w)}{\partial p_k}$ and usually > 0. If < 0 then so-called Giffen good.
- In matrix notation

$$D_p x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & & \frac{\partial x_1(p,w)}{\partial p_L} \\ \vdots & \ddots & \\ \frac{\partial x_L(p,w)}{\partial p_1} & \dots & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}$$

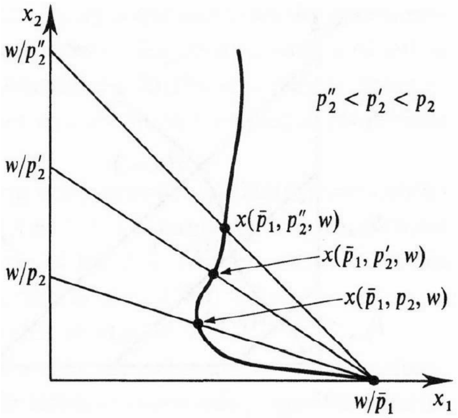
- In that context we can define the own- and cross-price elasticity of demand $\frac{\partial x_l(p,w)}{\partial p_l}\frac{p_l}{x_l(p,w)}$ and $\frac{\partial x_l(p,w)}{\partial p_k}\frac{p_l}{x_k(p,w)}$ where $k \neq l$

Demand for good as a function of own price

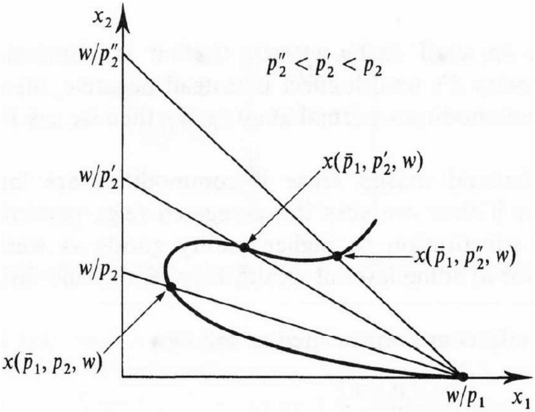


Offer curve

OC - a locus of points demanded in over all possible values of one of the prices (in $\mathbb{R}^2$).



The Giffen good

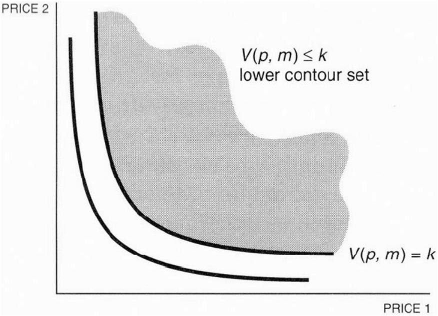


The indirect utility function

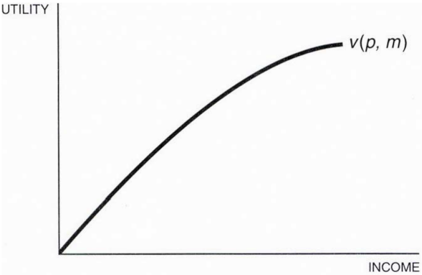
- Once we have the optimal choice, $x(p, w)$ we can plug it back into the utility function.
- $u(x(p, w)) = v(p, w)$ is the indirect utility function
- it says what the level of utility is, given prices and wealth **and** utility maximization

What for?

- for example we can find the levels of prices that generate same utility given wealth



- or the relationship wealth and utility at fixed prices



Expenditure minimization

- We can go back and redefine our problem.
- Instead of UMP, let us think of the consumer that has a desired level of utility.
- He wants to obtain this level of utility at the lowest possible expenditure.
    - the analogy of the production level and the cost minimization is obvious
- The problem is set up as follows:

$$\min_{x \geq 0} px \text{ subject to } u(x) \geq u$$

- The solution is $h(p, u)$, the demand for goods given prices and utility, the so called Hicksian demand (contrast it to $x(p, w)$).

The expenditure function

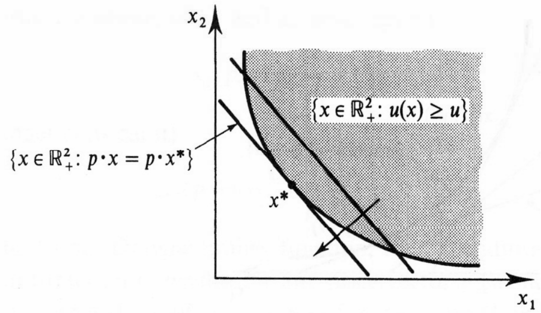
Once we have the solution to the problem, we can calculate the actual expenditure:

$$e(p,u)=\sum_{l=1}^{L} p_l h(p,u)$$

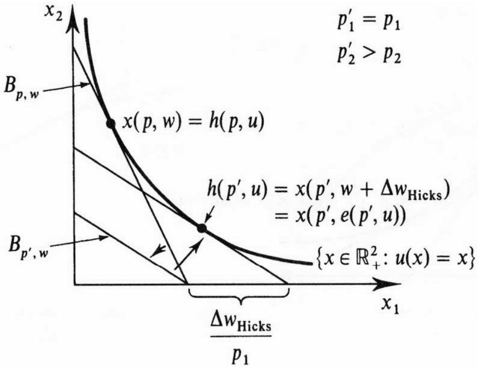
It is the "cost" of generating/obtaining a level of utility u given the set of prices.

- Why is it useful?
  - given the prices it determines a one-to-one relationship between money/expenditure and utility

Expenditure minimization



Hicksian demand and effects of a price change



Duality of the consumer problem

The two problems can be related to each other:

1. $x(p, w) = h(p, v(p, w))$
2. $x(p, e(p, u)) = h(p, u)$
3. $e(p, v(p, w)) = w$
4. $v(p, e(p, u)) = u$

Some more nice properties

- To recover Hicksian demand from expenditure function
  - If $u(\cdot)$ is a continuous utility function. For all p and u the Hicksian demand $h(p,u)$ is the derivative vector of the expenditure function with respect to prices.

$$h(p,u) = \nabla_p e(p,u)$$

  - To recover Walrasian demand from indirect utility function (Roy's identity - check for assumptions):

$$x(p,w) = -\frac{1}{\nabla_w v(p,w)} \nabla_p v(p,w)$$

The Slutsky Equation

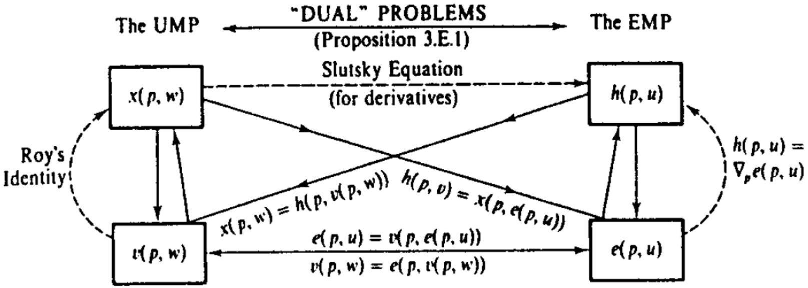
Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$. Then for all (p, w), and $u = v(p, w)$, we have

$$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} x_k(p,w)$$

and we can also rewrite it as:

$$\underbrace{\frac{\partial h_l(p,u)}{\partial p_k}}_{\text{Substitution effect}} - \underbrace{\frac{\partial x_l(p,w)}{\partial w} x_k(p,w)}_{\text{Income effect}} = \frac{\partial x_l(p,w)}{\partial p_k}$$

Recap


"DUAL" PROBLEMS
(Proposition 3.E.1)
The UMP
The EMP
Slutsky Equation
(for derivatives)
x(p, w)
h(p, u)
v(p, w)
e(p, u)
Roy's Identity
h(p, u) = ∇pe(p, u)
x(p, w) = h(p, v(p, w))
h(p, u) = x(p, e(p, u))
e(p, u) = v(p, e(p, u))
v(p, w) = e(p, v(p, w))