problem 1.2 (a) prove that if $B \subset B'$, then $u^*(B) \leq u^*(B')$

→ proof) suppose $u^*(B) > u(B')$

Then, $\sup_{x \in B} u(x) > \sup_{x \in B'} u(x)$

Let $\Delta := \sup_{x \in B} u(x) - \sup_{x \in B'} u(x) > 0$

By the definition of sup. $\exists x^* \in B$ s.t $u(x^*) > \sup_{x \in B} u(x) - \Delta = \sup_{x \in B'} u(x)$

By assumption, since $B \subset B'$ and $x^* \in B'$

$\exists x^* \in B'$ s.t $u(x^*) > \sup_{x \in B'} u(x)$, contradiction. //

(b) prove that if $y \in B$ but $y \notin C_u(B)$, then $u^*(B) = u^*(B \setminus y)$

proof) Assume $C_u(B) \neq \emptyset$

For any $\underline{x^* \in C_u(B)}$, $u(x^*) = \max_{x \in B} u(x)$

By (a), since $B \setminus y \subseteq B$, $\max_{x \in B \setminus y} u(x) \leq \max_{x \in B} u(x) = u(x^*)$

Since $y \notin C_u(B)$, $y \neq x^*$, $x^* \in B \setminus y$

Thus, $u(x^*) \leq \max_{x \in B \setminus y} u(x)$

$u(x^*) \geq \max_{x \in B \setminus y} u(x) \Rightarrow u(x^*) = \max_{x \in B \setminus y} u(x) \Rightarrow \underline{x^* \in C_u(B \setminus y)}$

$\Rightarrow C_u(B) \subseteq C_u(B \setminus y)$

Trivially, $C_u(B \setminus y) \subseteq C_u(B) \Rightarrow C_u(B \setminus y) = C_u(B) \Rightarrow u^*(B) = u^*(B \setminus y)$. ∎

☆☆

Problem 1.3 $X \supset \{a, b, c\}$.

(a) $C_u : P(X)\setminus\emptyset \to P(X)\setminus\emptyset$

$P(X)\setminus\emptyset = \{\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$.

$P(\{a,b\})\setminus\emptyset = \{\{a\}, \{b\}, \{a,b\}\}$.

$$C_u(\{a,b\}) = \begin{cases} \{a\} & \text{if } u(a) > u(b) \\ \{b\} & \text{if } u(a) < u(b) \\ \{a,b\} & \text{if } u(a) = u(b) \end{cases}$$ //

(b) Suppose $C_u(\{a,b\}) = b. \Rightarrow u(a) < u(b)$

" $C_u(\{b,c\}) = c \Rightarrow u(b) < u(c)$

$C_u(\{a,c\}) = c \Rightarrow u(a) < u(b) < u(c)$ then $u(a) < u(c)$

(c) Suppose $a \in B$, but $a \notin C_u(B)$.

By Q 1.2 (b), $C_u(B \setminus a) = C_u(B)$ //

Prop 1.1 Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function.

$v : X \to \mathbb{R}$ so that $v(x) = f(u(x))$ ($v = f \circ u$) Then, $C_u(B) = C_v(B)$

Problem 1.X <prop. 1.1 proof>

(a) $f: \mathbb{R} \to \mathbb{R}$ is a strictly increasing function

$v: X \to \mathbb{R}$ s.t. $v(x) = f(u(x))$ $\Rightarrow$ $C_u(B) = C_v(B)$

proof)

($\Rightarrow$) Assume $C_u(B) \neq \emptyset$.

WTS: $C_u(B) \subseteq C_v(B)$

$\forall x^* \in C_u(B)$, $x^* \in B$, $u(x^*) = \max_{y \in B} u(y)$ for any $y \in B$, $u(x^*) \geq u(y)$

Since f is strictly increasing, $f(u(x^*)) \geq f(u(y))$

$v(x^*) \geq v(y)$

$\downarrow$ $v(x^*) = \max_{y \in B} v(y)$ 이므로

$\Rightarrow x^* \in C_v(B)$ $\Rightarrow$ $C_u(B) \subseteq C_v(B)$.

($\Leftarrow$) WTS: $C_u(B) \supseteq C_v(B)$

$x^* \in C_v(B)$, $x^* \in B$. $v(x^*) = \max_{y \in B} v(y)$ for any $y \in B$, $v(x^*) \geq v(y)$

f^{-1}를 취해

By f^{-1}, $u(x^*) \geq u(y)$ for any $y \in B$

$\Rightarrow u(x^*) = \max_{y \in B} u(y) \Rightarrow x^* \in C_u(B)$

$\Rightarrow C_u(B) \supseteq C_v(B)$. //

#2 Preference maximization

2.1 Utility function $\Longrightarrow$ preference

$\succsim_u$: preference derived from the utility fcn u.

For all pairs $x, y \in X$, $x \succsim_u y \Leftrightarrow u(x) \geq u(y)$

$\downarrow$ 이걸 이용시.

$$C_u(B) = \{x^* \in B : u(x^*) \geq u(x), \ \forall x \in B\}$$
$$= \{x^* \in B : x^* \succsim_u x, \ \forall x \in B\}$$

$\uparrow$ utility로부터 나온순.

problem 2.1 For every pair $x, y \in X$, indicate $x \succsim y$ or $x \not\succsim y$.

sol) $\succsim$ on X is a subset on $X \times X$

For any $x, y \in \mathbb{R}_+^2$, $x = (x_1, x_2)$ $y = (y_1, y_2)$

(i) $x_1 > y_1 \Rightarrow x \succsim y$

(ii) $x_1 = y_1$, $x_2 \geq y_2 \Rightarrow x \succsim y$

(iii) $x_1 = y_1$, $x_2 < y_2 \Rightarrow x \precsim y$

(iv) $x_1 < y_1 \Rightarrow x \precsim y$

prop 2.1 Let $u: X \to \mathbb{R}$ Let $\succsim_u$ be the preference derived from u

Then (a) $\succsim_u$ is complete : either $x \succsim_u y$ or $y \succsim_u x$

(b) $\succsim_u$ is transitive : $x \succsim_u y$ and $y \succsim_u z \Rightarrow x \succsim_u z$

proof) Let $u: X \to \mathbb{R}$ and $x, y \in U(x)$.

Then either $u(x) \geq u(y) \Rightarrow x \succsim y$

or $u(x) \leq u(y) \Rightarrow x \precsim y$

2.2 preference $\Rightarrow$ utility function.

$$C_{\succsim}(B) = \{x^* \in B : x^* \succsim x, \ \forall x \in B\}$$

☆☆☆

problem 2.2 (a) $x \succsim y \Leftrightarrow x_1 + x_2 \geq y_1 + y_2$

① complete

Let $x, y \in X$ then $(x_1, x_2), (y_1, y_2) \in X$.

If $x_1 + x_2 \geq y_1 + y_2$, $x \succsim y$.

If $x_1 + x_2 \leq y_1 + y_2$, $x \precsim y$. $\therefore \succsim$ is complete.

② transitive.

Let $x, y, z \in X$ and $x_1 + x_2 \geq y_1 + y_2$ and $y_1 + y_2 \geq z_1 + z_2$.

Then $x_1 + x_2 \geq y_1 + y_2 \Leftrightarrow x \succsim y$

$y_1 + y_2 \geq z_1 + z_2 \Leftrightarrow y \succsim z$.

So, $x_1 + x_2 \geq y_1 + y_2 \geq z_1 + z_2$ i.e. $x_1 + x_2 \geq z_1 + z_2 \Leftrightarrow x \succsim z$.

$\therefore \succsim$ is transitive

(b) $x \succsim y \Leftrightarrow x_1 \geq y_1$ and $x_2 \geq y_2$

① complete

Let $x, y \in X$ If $x_1 \geq y_1$ and $x_2 \geq y_2$, $x \succsim y$

If $x_1 \leq y_1$ and $x_2 \leq y_2$, $x \precsim y$

But in the case $x_1 \geq y_1$ and $x_2 < y_2$, we cannot express it as $\succsim$

$\therefore \succsim$ is not complete

(b) ② transitive

Let $x, y, z \in X$ Let $x \succsim y$ and $y \succsim z$.

Then. $\left(\begin{array}{l} x_1 \geqq y_1 \text{ and } x_2 \geqq y_2 \\ y_1 \geqq z_1 \text{ and } y_2 \geqq z_2 \end{array}\right) \Rightarrow x_1 \geqq y_1 \geqq z_1$ and $x_2 \geqq y_2 \geqq z_2$

i.e.. $x_1 \geqq z_1$ and $x_2 \geqq z_2$. $x \succsim z$ $\therefore \succsim$ is transitive.

(c) $x \succsim y \iff x_1 > y_1$ or, $x_1 = y_1$ and $x_2 \geqq y_2$.

① complete.

Let $x, y \in X$. If $\left(\begin{array}{l} x_1 > y_1 \\ \text{or} \\ x_1 = y_1 \text{ and } x_2 \geqq y_2 \end{array}\right)$, $x \succsim y$.

If $\left(\begin{array}{l} x_1 < y_1 \\ \text{or } x_1 = y_1 \text{ and } x_2 \leqq y_2 \end{array}\right)$, $x \precsim y$.

$\therefore \succsim$ is complete.

② transitive

Let $x, y, z \in X$ Let $x \succsim y$ and $y \succsim z$.

Then $x_1 > y_1$ ... (a1) $y_1 > z_1$ ... (b1)

or $x_1 = y_1$ and $x_2 \geqq y_2$ ... (a2) or $y_1 = z_1$ and $y_2 \geqq z_2$. ... (b2)

i) case (a1) and (b1)

$x_1 > y_1 > z_1$ i.e., $x_1 > z_1 \Rightarrow x \succsim z$.

ii) case (a1) and (b2)

$x_1 > y_1$ i.e., $y_1 = z_1$ and $y_2 \geqq z_2$

$\Rightarrow x_1 > z_1 \Rightarrow x \succsim z$.

iii) case (a2) and (b1)

$x_1 = y_1$ and $x_2 \geqq y_2$ and $y_1 > z_1$

$\Rightarrow x_1 > z_1 \Rightarrow x \succsim z$

iv) case (a2) and (b2)

$x_1 = y_1$ and $x_2 \geqq y_2$

$y_1 = z_1$ and $y_2 \geqq z_2$.

$\Rightarrow x_1 = z_1$ and $x_2 \geqq z_2 \Rightarrow x \succsim z$.

$\therefore \succsim$ is transitive

2.2 (d)

$$\begin{cases} \text{Ann}: & 1 \succsim_a 2 \succsim_a 3 \\ \text{Bob}: & 2 \succsim_b 3 \succsim_b 1 \\ \text{Carol}: & 3 \succsim_c 1 \succsim_c 2 \end{cases}$$

$\Rightarrow$ A New pref. $\succsim$: $x \succsim y \Leftrightarrow$ two of the three individuals prefer x to y.

① complete.

$x=1,\ y=3$ Ann: $1 \succsim_a 3$

Bob: $3 \succsim_b 1$ $\therefore 3 \succsim 1$

Carol: $3 \succsim_c 1$

$x=1,\ y=2$ Ann: $1 \succsim_a 2$

Bob: $2 \succsim_b 1$ $\therefore 1 \succsim 2$

Carol: $1 \succsim_c 2$

$x=2,\ y=3$ Ann: $2 \succsim_a 3$

Bob: $2 \succsim_b 3$ $\therefore 2 \succsim 3$

Carol: $3 \succsim_c 2$

$\forall x, y \in X = \{1, 2, 3\}$, $x \precsim y$ or $x \succsim y$. $\therefore \succsim$ is complete.

② transitive.

By ①, we can know $3 \succsim 1$, $1 \succsim 2$, $2 \succsim 3$

If $3 \succsim 1$ and $1 \succsim 2$, $3 \succsim 2$, not $2 \succsim 3$.

$\therefore \succsim$ is not transitive.

(prop 2.2) If $\succsim$ is complete & transitive on a finite set X, then $\succsim$ has a utility representation.

★★★ ⇓

(problem 2.3) (proof).

Define $a \in X$ to be minimal if $x \succsim a$, $\forall x \in X$

(Lemma) In any finite set $A \subseteq X$, there is a minimal element.

proof: If A is singleton, then by completeness, the only element is minimal.

(IH) Assume $|A| = n$ is true. For $|A| = n+1$, let $x \in A$.

Then, for the set of $A \setminus x$, $|A \setminus x| = n$.

So, $\exists$ a minimal element y of $A \setminus x$ (By the IH)

If $x \succsim y$, then y is minimal in A.

If $y \succsim x$, then $z \succsim x$ for all $z \in A \setminus x$. So x is minimal in A. //

Construct a sequence of sets inductively. *proof point

Define X_1, to be the set of elements that are minimal in X.

(Base) By lemma, $X_1 \neq \emptyset$. X_2 is the set of elements that are minimal in $X \setminus X_1$. (So, X_n is the set of the nth smallest element in X)

Assume we have constructed $X_1, \dots, X_k$ by the inductive hypothesis.

(IH) So, X_{k+1} is the set of elements that are minimal in $X \setminus X_1 \setminus \dots \setminus X_k$.

By lemma, we know $X_{k+1} \neq \emptyset$. Since X is finite, we have at most $|X|$ steps. Define utility as $u(x) = k$ if $x \in X_k$. To verify that $u(x)$ represents $\succsim$. Let $a \succsim b$. Then, $b \notin X \setminus X_1 \dots \setminus X_{u(a)}$. So $u(a) \geq u(b)$ ▨

problem 2.3

<Another way>

If $\succsim$ is complete & transitive on a finite set X, then $\succsim$ has a utility representation.

proof) Let $\{\precsim x\}$ be a lower contour set of X.

Define $u(x) := |\{\precsim x\}| = |\{y \mid y \in X,\ y \precsim x\}|$ for some $x \in X$.

X is finite.

($\Rightarrow$) WTS : $x \succsim y \Rightarrow u(x) \geq u(y)$

Assume $z \in \{\precsim y\}$. Then, $z \precsim y$

By transitivity, $z \precsim y \precsim x$ i.e., $z \precsim x$ and so $z \in \{\precsim x\}$.

Thus, $\{\precsim y\} \subset \{\precsim x\} \Rightarrow |\{\precsim y\}| \leq |\{\precsim x\}| \Rightarrow u(y) \leq u(x)$.

($\Leftarrow$) WTS : $u(x) \geq u(y) \Rightarrow x \succsim y$

Assume $u(x) \geq u(y)$. Then $|\{\precsim x\}| \geq |\{\precsim y\}|$.

Suppose $x \prec y$. Let $z \in \{\precsim x\}$ Then, $z \precsim x$.

By transitivity, $z \precsim x \prec y$ i.e., $z \prec y$ Thus $z \in \{\precsim y\}$

$\{\precsim x\} \subsetneq \{\precsim y\} \Rightarrow |\{\precsim x\}| < |\{\precsim y\}|$, contradiction.

Therefore, $x \succsim y$. //

#3. Choice as primitive

3.1 $\succsim$ to choice

Afterwards, this is connected with demand correspondence.

Choice structure $\left[\begin{array}{l} \mathcal{B} \subset P(X) \setminus \emptyset \\ C(\cdot),\ C : \mathcal{B} \to P(X) \quad \text{s.t. } C(B) \subset B,\ \forall B \in \mathcal{B} \end{array}\right.$

(Def 3.1) $\mathcal{B}, C(\cdot)$ satisfies finite nonemptiness if for all finite $B \in \mathcal{B}$, $C(B)$ is nonempty.

(prop 3.1) (a) If $\succsim$ is complete, and B contains just two elements, then $C_{\succsim}(B)$ is nonempty.

Conversly, if $\succsim$ is not complete, then $\exists$ some $B \subset X$ containing just two elements such that $C_{\succsim}(B)$ is empty.

(b) If $\succsim$ is c & t, $\mathcal{B}, C_{\succsim}$ satisfies finite nonemptiness.

(problem 3.1) (a) (proof) Let $B := \{x, y\}$. Since $\succsim$ is complete,

$[x \succsim y \Rightarrow x \in C_{\succsim}(B)]$ or $[y \succsim x \Rightarrow y \in C_{\succsim}(B)]$

Conversly, if $\succsim$ is not complete,

$\exists\ x, y \in X$ s.t $x \not\succsim y$ and $y \not\succsim x$ then $C_{\succsim}(B)$ is empty. ▨

(b) (proof)

We already prove that for any finite set, $\exists$ maximal element in X by induction.

For any finite B, $\exists x^* \in B$ s.t $\forall y \in B : x^* \succsim y \Rightarrow x^* \in C_{\succsim}(B)$.

$C_{\succsim}(B) \neq \emptyset$. ▨

Def 3.2 WAI — If x, y are in B and B', and $x \in C(B)$, $y \in C(B')$ then it must be that $x \in C(B')$

Prop 3.2 — If $\succsim$ is transitive, then $C_{\succsim}$ satisfies WA.

☆ If $\succsim$ is c & t, $C_{\succsim}$ satisfies $\left(\begin{array}{l} \text{finite non-emptiness} \\ \text{WA} \end{array} \right)$.

3.2 Choice to $\succsim$

Def 3.3 — $x \succsim_c y$: x is revealed preferred to y.

if $\exists$ $x \in C(B)$ such that $x, y \in B$.

$x \succ_c y$: x is revealed strictly preferred to y

if $\exists$ $x \in C(B)$, $y \notin C(B)$ such that $x, y \in B$.

Def 3.x — · A c & t preference $\succsim$ rationalizes $C(\cdot)$ over $\mathcal{B}$.

if $C_{\succsim}(B) = C(B)$, $\forall B \in \mathcal{B}$

· A choice rule is rationalizable if $\exists$ a c & t preference that rationalizes it.

☆ Prop 3.3 — Suppose $\mathcal{B}$ contains all duos and trios in X.

If $C(\cdot)$ satisfies WA and $C(B)$ is non-empty for all $B \in \mathcal{B}$, then $C(\cdot)$ is rationalizable and $\succsim_c$ is the unique c & t preference that rationalizes $C(\cdot)$.

proof) ① $\succsim_c$ is complete

Fix $x, y \in X$. $\{x, y\}$ is duo so, $\{x, y\} \in \mathcal{B}$ (assumption)

$C(\{x,y\}) \neq \emptyset$. case 1) $x \in C(\{x,y\}) \Rightarrow x \succsim_c y$

(case 2) $y \in C(\{x,y\}) \Rightarrow y \succsim_c x$

($C(B)$ is nonempty) ($C(\cdot)$ satisfies WA)

$C(\cdot)$ is rationalizable. ② $\succsim_c$ is transitive

Suppose $x \succsim_c y$ and $y \succsim_c z$. WTS: $x \succsim_c z$.

$\{x, y, z\}$ is trio, so $\{x, y, z\} \in \mathcal{B}$.

It suffices to show that $x \in C(x, y, z)$

case 1) $x \in C(x,y,z) \Rightarrow$ Done.

WA on $\{x,y,z\}$ case 2) $y \in C(x,y,z) \Rightarrow$ By WA, $x \in C(x,y,z)$

case 3) $z \in C(x,y,z) \Rightarrow$ By WA, $y \in C(x,y,z) \Rightarrow x \in C(x,y,z)$.

③ $C(B) \subset C_{\succsim_c}(B)$, for all $B \in \mathcal{B}$ WTS: $x \in C(B) \Rightarrow x \in C_{\succsim_c}(B)$

Suppose $x \in C(B)$. Then $x \succsim_c y$, $\forall y \in B$.

Thus $x \in C_{\succsim_c}(B) = \{z \in B : z \succsim_c y, \forall y \in B\}$.

④ $C_{\succsim_c}(B) \subset C(B)$, for all $B \in \mathcal{B}$ WTS: $x \in C_{\succsim_c}(B) \Rightarrow x \in C(B)$

$C(B) = C_{\succsim}(B)$

$C(B)$ is nonempty, so $\exists\, y \in B$: $y \in C(B)$

Suppose $x \in C_{\succsim_c}(B)$. Thus, $x \succsim_c y$.

Then, $\exists$ B' such that $x, y \in B'$, $x \in C(B')$.

Thus, WA implies $x \in C(B)$.

⑤ If $\succsim' \neq \succsim_c$, $C_{\succsim'} \neq C_{\succsim_c}$ because $\mathcal{B}$ contains all duo. ▨

No other preference could rationalize C

Problem 3.2 Suppose that X is finite. Consider a choice rule $C(\cdot)$ defined on all nonempty subsets of X.

$\exists$ c&t $\succsim$ that generates the choice rule $C(\cdot)$

iff $C(\cdot)$ satisfies WAI and $C(B)$ is non-empty for all nonempty $B \subset X$

(prop 2.1b) → If $\succsim$ is c&t, then

($\Rightarrow$) WTS: $C_{\succsim}(B)$ is non-empty. ($\mathcal{B}$, $C_{\succsim}$ satisfies finite non-emptiness)

proof) For any finite set, $\exists$ maximal element in X by induction.

Thus for any finite B, $\exists x^* \in B : \forall y \in B, x^* \succsim y \Rightarrow x^* \in C_{\succsim}(B)$

$\therefore C_{\succsim}(B)$ is non-empty

WTS: $C_{\succsim}(\cdot)$ satisfies WAI. (prop 3.2 If $\succsim$ transitive, $C_{\succsim}(\cdot)$ satisfies WAI holds)

proof) Assume for some $B \in \mathcal{B}$, $x, y \in B$ and $x \in C_{\succsim}(B)$.

By the def of $C_{\succsim}(\cdot)$, $x \succsim y$.

Suppose for some $B' \in \mathcal{B}$, $x, y \in B'$, and $y \in C_{\succsim}(B')$

Then $y \succsim z$, $\forall z \in B'$ By transitivity, $x \succsim z$ for all $z \in B'$.

Therefore $x \in C_{\succsim}(B')$. $\therefore C_{\succsim}(\cdot)$ satisfies WAI.

WTS: $C(\cdot) = C_{\succsim}(\cdot)$

($\Rightarrow$) Suppose $x \in C(B)$ Then, $x \succsim y$ for all $y \in B$.

Thus $x \in C_{\succsim}(B) = \{s \in B : s \succsim y, \forall s \in B\}$.

($\Leftarrow$) Suppose $x \in C_{\succsim}(B)$. Thus $x \succsim y$. If $x, y \in B'$, then $x \in C(B')$.

By WAI, $x \in C(B)$. $\therefore C(\cdot) = C_{\succsim}(\cdot)$

Therefore, c&t that rationalizes $C(\cdot) \Rightarrow C(\cdot)$ satisfies WAI and $C(\cdot)$ nonempty, $\forall B \subset X$

($\Leftarrow$) prop 3.3 If $C(\cdot)$ satisfies WA and $C(B)$ is nonempty for all $B \in \mathcal{B}$. then $C(\cdot)$ is rationalizable, and $\succsim_c$ is the unique complete and transitive preference that rationalizes $C(\cdot)$

- $\succsim$ is complete : $C(\cdot)$ is defined on all non-empty subsets of X. then $\mathcal{B}$ contains all duos and trios in X.

- $\succsim$ is transitive : Suppose $x \succsim y$, $y \succsim z$. consider $\{x, y, z\} \in \mathcal{B}$.

If $x \in C(\{x,y,z\})$, we are done

If $y \in C(\{x,y,z\})$, $x \succsim y \Rightarrow x \in C(\{x,y,z\})$. by WA.

If $z \in C(\{x,y,z\})$, $y \succsim z \Rightarrow y \in C(\{x,y,z\})$ by WA.

$\Rightarrow \cancel{x} x \in C(\{x,y,z\})$ by WA, $\therefore$ transitive.

- $C(B) = C_{\succsim}(B)$ ($\succsim$ rationalizes $C(\cdot)$)

($\Rightarrow$) Suppose $x \in C(B)$ Then $x \succsim y$, $\forall y, y \in B$. Thus $x \in C_{\succsim}(B)$.

($\Leftarrow$) Suppose $x \in C_{\succsim}(B)$. $x \succsim y$. That is $\exists$ some B', $x, y \in B'$ s.t $x \in C(B')$
By WA, $x \in C(B)$

- $\succsim$ uniqueness. ($\succsim$ that rationalizes $C(\cdot)$ is unique).

If $\succsim \neq \succsim'$, then $C_{\succsim} \neq C_{\succsim'}$ Because $C(\cdot)$ is defined on all non-empty subsets of X, including all duos. //

☆

problem 3.3 $X = \{x_1, \dots x_n\}$. $i > j$ implies $P_i > P_j$

(a) Ann always chose cheapest bottle.

$x_i \succsim x_j$ if $i \leq j$ $\Rightarrow$ $\succsim$ is complete and transitive.

For any B, $C_{\succsim}(B) = \{x_{i^*}\}$ where $i^* = \min \{i \mid x_i \in B\}$

$C_A(B) = \{x_{i^*}\}$ $\therefore C_A(B) = C_{\succsim}(B)$

(b) If $\succsim$ transitive $\rightarrow$ WA (prop 3.2)

Not WA $\Rightarrow$ Not trasitive. (prop 3.2)

Consider $B = \{x_1, x_2, x_3\}$ $B' = \{x_2, x_3, x_4\}$ $x_2, x_3 \in B, B'$

$C_{Bo}(B) = \{x_2\}$ $C_{Bo}(B') = \{x_3\}$

When $x_2, x_3 \in B, B'$, $x_2 \in C(B)$, $x_3 \in C(B')$

but $x_2 \notin C(B')$. $\rightarrow$ This violates WA.

(c) $x_1 \in B$ $\Rightarrow$ Carol chooses nothing

$x_1 \notin B$ $\Rightarrow$ Carol choses chooses the cheapest bottle.

$\Rightarrow$ When $x_1 \in B$, $C(B) = \emptyset$.

However, for rationalizable choice rule, $C(B)$ should be non-empty.

$\Rightarrow$ Carol's choice rule is not rationalizable. //

Problem 3.5

i) The single valued choice rule $C(B) = \{c(B)\}$ Satisfies WA

ii) Given $x, y \in B, B'$, if $C(B) = x$, then $C(B') \neq y$
$(x \neq y)$

iii) Given B, B', if B contains x, y, $C(B) = x$ and $C(B') = y$,
then $x \notin B'$

i) $\Rightarrow$ ii) WA: if x is revealed at least as good as y,
then y cannot be revealed preferred to x.

Given $x, y \in B$, $C(B) = x$ $\Rightarrow$ $C(B) \neq y$. Thus, x is revealed preferred to y.
Therefore, WA guarantees that y cannot be revealed preferred to x.
Since $x, y \in B'$, $C(B') \neq y$. //

ii) $\Rightarrow$ iii) For contradiction, suppose $x, y \in B$, $C(B) = x$, $C(B') = y$ but $x \in B'$.
Since $C(B') = y$, thus $y \in B'$. Then, by our supposition,
$x, y \in B$, $x, y \in B'$ and $C(B) = x$.
Therefore, by ii) we have $C(B') \neq y$. contradiction. //

iii) $\Rightarrow$ i) Since $x, y \in B$, $C(B) = x$ i.e. $C(B) \neq y$, thus x is revealed preferred to y.
Since $C(B') = y$, $x \notin B'$. Therefore, for any set B' such that
$y \in B'$ and y is revealed preferred to other elements in B',
we must have $x \notin B'$. Therefore, y cannot be revealed to x.
Therefore, WARP holds. ▨

Problem 3.6

(a) $x \succsim y \Leftrightarrow x \succsim' y \iff C_{\succsim}(\mathcal{B}) = C_{\succsim'}(\mathcal{B})$

($\Rightarrow$) Let $B \in \mathcal{B}$ and $x \in C_{\succsim}(B)$

Thus, for any $y \in B$, $x \succsim y \Rightarrow x \succsim' y$ for any $y \in B$.

$\Rightarrow x \in C_{\succsim'}(B) \Rightarrow C_{\succsim}(B) \subset C_{\succsim'}(B)$.

Symmetrically, $C_{\succsim'}(B) \subseteq C_{\succsim}(B)$ $\therefore C_{\succsim}(B) = C_{\succsim'}(B)$.

($\Leftarrow$) For any pair $x, y \in X$, let $B = \{x, y\}$.

By assumption, $B \in \mathcal{B}$ and thus $C_{\succsim}(B) = C_{\succsim'}(B)$.

WLOG, let $x \in C_{\succsim}(B)$, thus $x \succsim y$.

let $x \in C_{\succsim'}(B)$, then $x \succsim' y \Rightarrow x \succsim y \Leftrightarrow x \succsim' y$. //

(b) u represents $\succsim \iff C_{\succsim}(B) = C_u(B)$ for all $B \in \mathcal{B}$

($\Rightarrow$) Let $B \in \mathcal{B}$ and let $x \in C_{\succsim}(B)$

Thus, for any $y \in B$, $x \succsim y$.

Since u represents $\succsim$, thus $u(x) \geqq u(y)$ for any $y \in B$.

$\Rightarrow x \in C_u(B) \Rightarrow C_{\succsim}(B) \subset C_u(B)$

Let $x \in C_u(B)$. Thus $u(x) \geqq u(y)$ for any $y \in B$.

$\Rightarrow x \succsim y$ for any $y \in B \Rightarrow x \in C_{\succsim}(B) \Rightarrow C_u(B) \subset C_{\succsim}(B)$. //

($\Leftarrow$) Any pair $x, y \in X$, let $B \equiv \{x, y\}$. By assumption $B \in \mathcal{B}$, $\mathcal{B}$ contains all duos, $C_{\succsim}(B) = C_u(B)$. WLOG, let $x \in C_{\succsim}(B)$. Thus $x \succsim y$ for any $y \in B$

$\Rightarrow x \in C_u(B)$, then $u(x) \geqq u(y)$

Therefore, $x \succsim y \Leftrightarrow u(x) \geqq u(y)$. Thus, u represents $\succsim$ //

(c) Not hold

counter example)

Let $X = \{x, y, z\}$, let $\mathcal{B} = \{\{x, y, z\}\}$, let $B = \{x, y, z\}$

Suppose $x \succ y$, $y \succ z$ and $z \succ x$ and $y \succ' x$, $x \succ' z$, $z \succ' y$

$C_{\succsim}(B) = C_{\succsim'}(B') = \emptyset$ but, $x \succsim y \not\Leftrightarrow x \succsim' y$. //

(d) Not hold

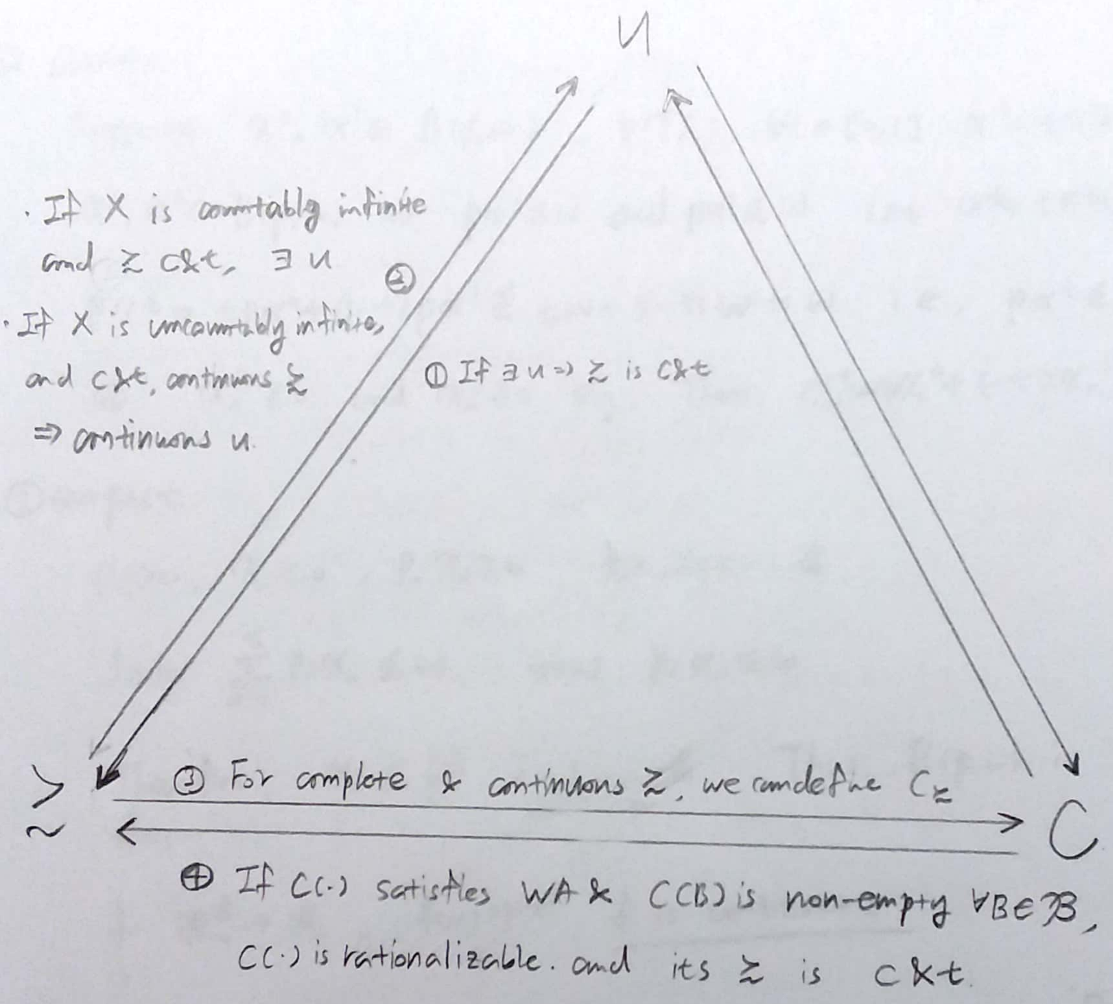
Let $X = \{x, y, z\}$, let $\mathcal{B} = \{\{x, y, z\}\}$, let $B = \{x, y, z\}$

Suppose $x \succ y$, $y \succ z$, $z \succ x$ then $C_{\succsim}(B) = C_u(B) = \emptyset$.

But, transitivity does not hold.

This means we cannot make $\succsim$ that has a utility representation.//

problem 2.7



①, ③ automatically hold
↙ →

☆☆

problem 4.1 (a) Prove $B(p,w)$ properties ① non-empty ② convex ③ compact ④ intersection of $k+1$ closed half spaces.

① non-empty : $\vec{0} \in \mathbb{R}^k_+$, $p \cdot \vec{0} = 0 \leq w \Rightarrow \vec{0} \in B(p,w)$. //

② Convex

Suppose $x^0, x^1 \in B(p,w)$. WTS: $\forall t \in [0,1]$, $x^t = tx^0 + (1-t)x^1 \in B(p,w)$

$x^0, x^1 \in B(p,w) \Rightarrow px^0 \leq w$ and $px^1 \leq w$. Let $x^t = tx^0 + (1-t)x^1$, $\forall t \in [0,1]$.

$px^t = tpx^0 + (1-t)px^1 \leq tw + (1-t)w = w$. i.e., $px^t \leq w \Rightarrow x^t \in B(p,w)$.

⊕ $x_i^0 \geq 0$ and $x_i^1 \geq 0$ $\forall i$. Then, $x_i^t = tx_i^0 + (1-t)x_i^1 \geq 0$. //

③ compact

$p_i > 0$, $x_i \geq 0$, $p_i x_i \geq 0$ for, $i = 1, \dots, k$.

Since $\sum_{i=1}^{k} p_i x_i \leq w$, thus $p_i x_i \leq w$

Therefore, $x_i \leq \frac{w}{p_i}$, $i = 1, \dots, k$. Thus, $B(p,w)$ is bounded.

$f: \mathbb{R}^k_+ \to \mathbb{R}_+$, $f(x) = px$ f is continuous.

Suppose $\{x_n\}$ is a convergent sequence in $B(p,w)$

and $\lim_{n\to\infty} x_n = x^*$. Thus, $\lim_{n\to\infty} f(x_n) = f(x^*)$ i.e. $\lim_{n\to\infty} px_n = px^*$

Since $px_n \leq w$ for every n, $\lim_{n\to\infty} px_n = px^* \leq w$ i.e., $x^* \in B(p,w)$

Thus, $B(p,w)$ is closed.

⑤ intersection of $k+1$ closed half spaces.

$B(p,w) = \{x \in \mathbb{R}^k_+ : px \leq w\}$

$= \{x \in \mathbb{R}^k : px \leq w\} \cap \{x \in \mathbb{R}^k : -e_1 x \leq 0\} \cap \cdots \cap \{x \in \mathbb{R}^k : -e_k x \leq 0\}$

$e_1, \dots, e_k$ unit vector in $\mathbb{R}^k$

※ defn of half spaces: $\{x \in \mathbb{R}^k : px \leq w\}$. for some $p \in \mathbb{R}^k \setminus \{0\}$, $w \in \mathbb{R}$

☆☆
4.1 (b) ($p \in \mathbb{R}^k_+$ case.) $B(p,w)$

① nonempty. $\bar{0} \in B(p,w)$.

② convex : Suppose $x_0, x_1 \in B(p,w)$.

WTS : $t \in [0,1]$, $x^t := t x_0 + (1-t) x_1 \in B(p,w)$.

By $x_0, x_1 \in B(p,w)$, $px_0 \leq w$ and $px_1 \leq w$.

Thus, $px^t = tpx^1 + (1-t)px^0 \leq tw + (1-t)w = w$ i.e., $px^t \leq w$

So, x^t is affordable.

We have $x_i^0 \geq 0$ and $x_i^1 \geq 0$ for each good i.

Thus, $x_i^t = t x_i^1 + (1-t) x_i^0$. Hence, x^t is non-negative. ||

③ NOT COMPACT (x_i is not bounded 않는 이유)

Suppose $p_i = 0$ for some i. Thus, let $x_j = 0$ for $j \neq i$

and thus $px = 0 \leq w$, where is unbounded from above.
(Not bounded)

④ It could not be the intersection of $k+1$ closed half spaces.

Let $X_i := \{x_i \in \mathbb{R}_+ : x_i \geq 0, \forall i, i = \{1, \dots, k\}\}$

Let $H := \{x \in \mathbb{R}^k : p \cdot x \leq w, p = 0 \in \mathbb{R}^k, w \in \mathbb{R}_+\}$

In this case, $\bigcap_{i=1}^{k} X_i \cap H = \{x \in \mathbb{R}^k_+\}$

Which can make closed half spaces less than $k+1$. ||

4.1 (c) $x \in \mathbb{R}^{\ell}$

① non-empty : $\vec{0} \in \mathbb{R}_+^{\ell}$, $p \cdot \vec{0} = 0 \leq w \Rightarrow \vec{0} \in B(p,w)$

② convex :

Suppose $x_0, x_1 \in B(p,w)$, then $x_0, x_1 \in \mathbb{R}^{\ell}$ and $px_0 \leq w$, $px_1 \leq w$.

Thus, for any $t \in [0,1]$, $x^t = tx_0 + (1-t)x_1 \in \mathbb{R}^k$

$px^t = tpx_0 + (1-t)px_1 \leq tw + (1-t)w = w$, $x^t \in B(p,w)$

③ NOT COMPACT

Consider x with $x_i < 0$, $x_j = 0$ for $i \neq j$ Then $px < 0 \leq w$ and $x \in B(p,w)$,

not bounded.

④ No the intersection of $k+1$ closed half-spaces

$B(p,w) = \{x \in \mathbb{R}^{\ell} \mid px \leq w\}$ ← 1개로도 만들 수 있음.

★★★★

#6.1 If consumer's demand is single-valued, then it satisfies WA I.

$\Leftrightarrow$ then it satisfies WA I.

↑ u continuous $\Rightarrow$ Q is non-empty

& strictly convex $\Rightarrow$ Q contains at most one element.

proof) WARP II : For (p^0,w^0), (p^1,w^1) and $x^0 = x(p^0,w^0)$, $x^1 = x(p^1,w^1)$,

$\begin{cases} \text{If } p^1x^0 \leq w^1, \text{ then } p^0x^1 > w^0. \\ \text{If } p^0x^1 \leq w^0, \text{ then } p^1x^0 > w^1 \end{cases}$

WARP I : For given B, B' and $x^0, x^1 \in B$

If $C(B) = x^0$ and $C(B') = x^1$ then $x^0 \notin B'$.

Let $B = B(p^0,w^0)$ and $B' = B(p^1,w^1)$.

Assume $x^0 = x(p^0,w^0) = C(B(p^0,w^0)) = C(B)$

and $x^1 = x(p^1,w^1) = C(B(p^1,w^1)) = C(B')$.

($\Rightarrow$) We have $x^0 \in B$ Suppose $x^1 \in B(p^0,w^0)$. Then $p^0x^1 \leq w^0$.

By WARP II, $p^1x^0 > w^1$. Then, $x^0 \notin B(p^1,w^1)$. So, $x^0 \notin B'$.

Therefore, WA I holds.

($\Leftarrow$) Suppose $p^0x^1 \leq w^0$ Then $x^1 \in B(p^0,w^0)$. i.e., $x^1 \in B$.

By WA I, $x^0 \notin B'$ by $x^1 = C(B')$. So, $p^1x^0 > w^1$.

Therefore, WA II holds.

Problem 8.1

(a) If u is continuous, then $\exists$ some $t \in [0,1]$ s.t $u(x^t) = u(y)$

proof) Let $x^t := tx' + (1-t)x^0$ be a continuous function of t.

and $u(x)$ is a continuous function. WLOG, Let $u(x^0) \leq u(x')$.

$g(t) := u(x^t) = u(tx' + (1-t)x^0)$ is also a continuous function of t.

$g(0) = u(x^0)$, $g(1) = u(x')$ $u(x^0) \leq u(y) \leq u(x')$.

Therefore, by IVT, $\exists t^* \in [0,1]$ s.t. $g(t^*) = u(y)$. ||

(b) If $\succsim$ is continuous, $\exists$ some $t \in [0,1]$ such that $x^t \sim y$.

proof) By Debreu's theorem, since $\succsim$ is c&t and continuous,

it has a continuous utility representation.

$x^t = tx' + (1-t)x^0$ is a continuous function of t, and $u(x)$ is continuous of x

Thus, $u(x^t)$ is a continuous function of t.

Let $T := \{t \in [0,1] \mid x^t \succsim y\}$, $t^* \equiv \inf T$. Then $x^{t^*} \sim y$.

Suppose $x^{t^*} \succ y$. Thus $u(x^{t^*}) > u(y)$ because g is continuous of t.

By continuity, for small enough $\delta > 0$, we have $u(x^{t^*-\delta}) > u(y)$

$\Rightarrow x^{t^*-\delta} \succ y$, $t^* - \delta \in T$, contradicts the assumption that $t^* = \inf T$.

Suppose $x^{t^*} \prec y$. Then $u(x^{t^*}) < u(y)$. By continuity, for small $\delta > 0$,

we have $u(x^{t^*+\delta}) < u(y) \Rightarrow x^{t^*+\delta} \prec y$. Then $t^* + \delta \notin T$.

Contradiction. Therefore, $\exists$ some t s.t $x^{t^*} \sim y$. ||

#8.2

(a) If $\succsim$ cont, continuous and strictly increasing, it has utility representation

proof) Since $\succsim$ is strictly increasing.

for any $x = (x_1, x_2, \dots, x_k) \in \mathbb{R}_+^k$,

we have $\max_{1 \le i \le k} \{x_i\} \cdot \vec{e} \succsim x \succsim \min_{1 \le i \le k} \{x_i\} \cdot \vec{e}$, where $e = (1, 1, \dots, 1)$

Since $\succsim$ is continuous, by problem 8.1

$\exists\, t \in [0,1]$ s.t $x \sim [t \cdot \max_{1 \le i \le k} \{x_i\} + (1-t) \min_{1 \le i \le k} \{x_i\}] \cdot \vec{e}$

where $s^x \in [\min_{1 \le i \le k} \{x_i\}, \max_{1 \le i \le k} \{x_i\}]$, i.e., $x \sim s^x \vec{e}$

Let $u(x) := s^x$

- The defined u represent $\succsim$ (Check $x \succsim y \iff u(x) \ge u(y)$).

($\Rightarrow$) Since $x \succsim y$, then $s^x \vec{e} \sim x \succsim y \sim s^y \vec{e}$

$\iff s^x \vec{e} \succsim s^y \vec{e} \iff s^x \ge s^y \iff u(x) \ge u(y)$.

($\Leftarrow$) Suppose $u(x) \ge u(y)$. $\Rightarrow s^x \ge s^y \Rightarrow s^x \vec{e} \succsim s^y \vec{e}$ by strictly increasing $\succsim$.

$\Rightarrow x \succsim y$

- The defined u is continuous.

WTS : $\forall \varepsilon > 0 : \exists\, \delta > 0 : d(x,y) < \delta \Rightarrow |u(x) - u(y)| < \varepsilon$

i.e., $u(x) - \varepsilon < u(y) < u(x) + \varepsilon$

Consider $u(x) - \varepsilon < u(y)$, which automatically holds when $u(x) \le \varepsilon$

Assume $u(x) > \varepsilon$.

Then $u(x) - \varepsilon < u(y)$ iff $(u(x)-\varepsilon)\cdot\vec{e} \prec y$

Notice by strictly increasing $\succsim$, $x \succ (u(x)-\varepsilon)\cdot\vec{e}$

Therefore by continuity of $\succsim$,

$\exists\, \delta_1 > 0 : d(x,y) < \delta_1 \Rightarrow y \succ (u(x)-\varepsilon)\vec{e}$.

By similar argument.

$\exists\, \delta_2 > 0 : d(x,y) < \delta_2 \Rightarrow (u(x)+\varepsilon)\vec{e} \succ y$.

Let $\delta := \min\{\delta_1, \delta_2\} \Rightarrow d(x,y) < \delta \Rightarrow (u(x)+\varepsilon)\vec{e} \succ y \succ (u(x)-\varepsilon)\vec{e}$.

i.e., $|u(x) - u(y)| < \varepsilon$. //

(b) If a utility function is continuous and strictly increasing, then the preference derived from that utility function is continuous and strictly increasing.

proof) The existence of utility function $\Rightarrow \succsim$ is complete & transitive.

WTS: $\succsim$ is strictly increasing.

Suppose $x > y$ Then $u(x) > u(y)$.

Since u is strictly increasing, $x \succ y$.

WTS: $\succsim$ is continuous.

Suppose $u(x) > u(y)$. then $x \succ y$.

By continuity of u,

$\forall \varepsilon > 0 : \exists \delta > 0$: for any x' with $d(x,x') < \delta$, $u(x') > u(y)$, then $x' \succ y$

Thus, $\succsim$ is continuous. //

☆☆

Problem 9.2 (a) If each u_i is concave, then u is concave,

so the preference derived from u is convex.

proof) Since u_i is concave for each i, $u_i''(x_i) \leq 0$ for all $x_i \in \mathbb{R}_+$

Consider the Hessian Matrix of u.

$$H = \begin{pmatrix} u_1''(x_1) & 0 & 0 & \cdots & 0 \\ 0 & u_2''(x_2) & & & \vdots \\ \vdots & & \ddots & & 0 \\ 0 & 0 & \cdots & 0 & u_k''(x_k) \end{pmatrix}$$

H is a diagonal matrix and every element on the diagonal is non-positive.

$\Rightarrow H$ is negative semidefinite.

$\Rightarrow$ Thus, u is concave

Let $x^0 \succsim x$, $x' \succsim x$. then $u(x^0) \geq u(x)$, $u(x') \geq u(x)$.

For any $t \in [0,1]$, $u(x^t) = u(tx' + (1-t)x^0) \geq t\,u(x^0) + (1-t)u(x')$

$\geq t\,u(x) + (1-t)u(x) = u(x)$

Thus, $u(x^t) \geq u(x) \Rightarrow x^t \succsim x \Rightarrow \succsim$ is convex. ||

(b) If each u_i is strictly concave, then u is strictly concave,

so the preference derived from u is strictly convex.

proof) Since u_i is strictly concave for each i, $u_i''(x_i) < 0$, $\forall x_i \in \mathbb{R}_+$

Consider Hessian matrix of u.

$$H = \begin{pmatrix} u_1''(x_1) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & 0 \\ 0 & \cdots & 0 & u_k''(x_k) \end{pmatrix}$$

H is a diagonal matrix and each element is negative.

$\Rightarrow H$ is negative definite

$\Rightarrow u$ is strictly concave.

Let $x^0 \succsim x$, $x' \succsim x$. then $u(x^0) \geq u(x)$, $u(x') \geq u(x)$.

For any $t \in (0,1)$, $u(x^t) = u(tx^0 + (1-t)x') > t\,u(x^0) + (1-t)x'$

$\geq t\,u(x) + (1-t)u(x) \geq u(x)$

Thus, $u(x^t) > u(x) \Rightarrow x^t \succ x \Rightarrow \succsim$ is strictly convex !!

☆☆☆

\#11.2 Suppose $\succsim$ c&t, continuous, convex and strictly increasing

(a) Fix $x \in \mathbb{R}^k_{++}$ prove $\exists$ $p \in \mathbb{R}^k_{++}$ and $w \in \mathbb{R}_{++}$ s.t $x \in Q(p,w)$

proof) $\succsim$ is strictly increasing $\Rightarrow$ u is LNS. $\Rightarrow$ Walras' law holds

By prop 11.1, $x^* \in \mathbb{R}^k_{++}$ is optimal $\Leftrightarrow$ $py \geq px^* = w$, $\forall y \succsim x^*$.

$\succsim$ c&t, LNS, conti. $\Rightarrow$

Let $\Upsilon := \{y \in \mathbb{R}^k_{++} : y \succsim x^*\}$ $\succsim$ convex $\Rightarrow$ Υ is convex

$\mathbb{C}$($\underline{\{\succsim x\}}$, upper contour set).

(Walras' law)

$\succsim$ convex $\Rightarrow$ By thm 10.1. If Υ is a convex set and y is a point on the boundary of
(The supporting hyperplane theorem)

Υ. $\exists$ a hyperplane that supports Υ at y.

i.e. $\exists$ $p \neq 0$, w, s.t $px^* = w$ and for any $y \in \Upsilon$. $py \geq w$, $y \succsim x^*$.
(Corollary 11.1)

WTS : $p \in \mathbb{R}^k_{++}$

WLOG, $\exists p_i \leq 0$. $x^* = (x_1^*, \ldots, x_k^*)$ $\Rightarrow$ by strict preference.
$x' = (x_1^*, \ldots, 2x_i^*, \ldots x_k^*)$ $x' \succ x^*$

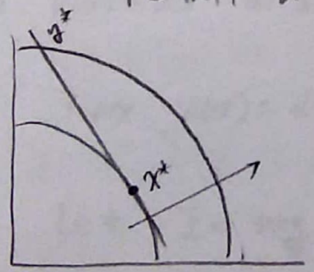
$x' \in \Upsilon$ and also x' is not on the boundary $\Rightarrow$ $px' > w$

But $px' \leq px^* = w$, contradictions.

$p \in \mathbb{R}^k_{++}$ and $x \in \mathbb{R}^k_{++}$ given, then $w \in \mathbb{R}_{++}$. //.

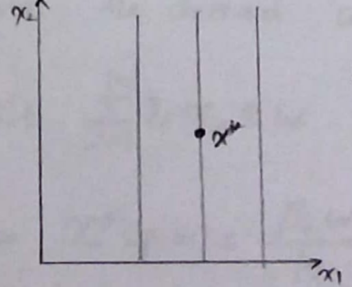
(b)

Counter example

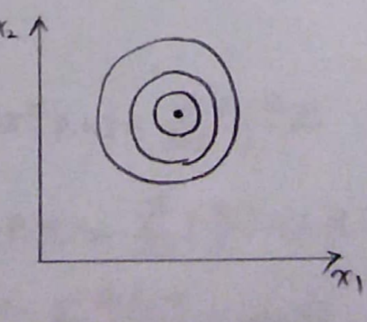


$\succsim$ is not convex.

(c)



$u(x_1, x_2) = x_1$



#11.3 (a) $u(x_1, x_2) = x_1 + x_2$ Compute the demand correspondence

max x_1+x_2 s.t. $p_1x_1+p_2x_2 \leq w$.

$$\mathcal{L} = x_1+x_2 + \lambda(w - p_1x_1 - p_2x_2)$$

KKT: $\frac{\partial \mathcal{L}}{\partial x_1} = 1-\lambda p_1 \leq 0$ $\quad x_1 \cdot \frac{\partial \mathcal{L}}{\partial x_1} = 0$

$\frac{\partial \mathcal{L}}{\partial x_2} = 1-\lambda p_2 \leq 0$ $\quad x_2 \cdot \frac{\partial \mathcal{L}}{\partial x_2} = 0$

$\frac{\partial \mathcal{L}}{\partial \lambda} = w - p_1x_1 - p_2x_2 \geq 0$ $\quad \lambda \cdot \frac{\partial \mathcal{L}}{\partial \lambda} = 0$

$$\therefore x^*(p,w) = \begin{cases} (\frac{w}{p_1}, 0), & p_1<p_2 \\ (\frac{\alpha w}{p_1}, \frac{(1-\alpha)w}{p_2}), & p_1=p_2,\ \alpha\in[0,1] \\ (0, \frac{w}{p_2}), & p_1>p_2 \end{cases}$$

· demand correspondence ⇒ check! ($u(x_1,x_2)$: cxt, continuous, convex, and st. inc)

① $p\cdot x^*(p,w) = w$

② For any $(x_1,x_2) \succsim (x_1^*, x_2^*)$,

$px \geq px^* = w$

(prop 11.1 : $\boxed{x^* \text{ is optimal}}$)

$\succsim$ complete, LNS, continuous ⇒ $py \geq px^*$, $\forall y$, $y \succsim x^*$.

⇒ $u(x_1,x_2) \geq u(x_1^*, x_2^*)$

$\downarrow$

$x_1+x_2 \geq x_1^*+x_2^* = \frac{w}{\min\{p_1,p_2\}}$

$\downarrow$

$p_1x_1+p_2x_2 \geq \min\{p_1,p_2\}(x_1+x_2)$

$\geq \min\{p_1,p_2\} \cdot \frac{w}{\min\{p_1,p_2\}}$

$= w.$ //

(b) $u(x) = \alpha x$ for some $\alpha \in \mathbb{R}^n_{++}$ Compute the demand correspondence.

max $u(x) = \alpha x = \sum_{i=1}^{k} \alpha_i x_i$ s.t $\sum_{i=1}^{k} p_i x_i \leq w$

Let $I = \arg\min \{\frac{p_i}{\alpha_i}\}$. Then $x_i^*(p,w) = \frac{\beta_i w}{p_i}$ where $\beta_i = 0$ if $i \notin I$

$\beta_i > 0$ if $i \in I$,

$\sum_{i\in I} \beta_i = 1$

check

① $p\cdot x^*(p,w) = w$

② for any $x \succsim x^*$,

WTS : $px \geq px^* = w$.

$\alpha x \geq \alpha x^*(p,w) = \sum_{i\in I} \frac{\alpha_i \beta_i w}{p_i}$

$px = \sum_i^k p_i x_i = \sum_{i=1}^{k} (\frac{p_i}{\alpha_i})\cdot(\alpha_i x_i) \geq \min(\frac{p_i}{\alpha_i}) \cdot \sum_{i=1}^{k} \alpha_i x_i$

$\geq \min_i \frac{p_i}{\alpha_i} \sum_{i\in I} \frac{\alpha_i \beta_i w}{p_i} = \min \frac{p_i}{\alpha_i} \cdot \frac{1}{\min \frac{p_i}{\alpha_i}} \sum \beta_i w = w.$ //

☆☆☆

#12.1 (a) $\succsim$ C&t, continuous, LNS $\Rightarrow$ LCD holds.

Fix w°, p° and p^1. If $q^\circ \in Q(p^\circ, w^\circ)$, $w^1 = p^1 q^\circ$ and $q^1 \in Q(p^1, w^\circ)$,

then $(p^1 - p^\circ)(q^1 - q^\circ) \leq 0$.

By continuity, $Q(p,w)$ is non-empty.

$$\left(\begin{array}{l} \succsim \text{ C\&t} \Rightarrow \text{WARP I holds} \\ \succsim \text{ LNS} \Rightarrow \text{Walras' law} \end{array} \right) \Rightarrow Q(p,w) \text{ satisfies Walras' law and WARPI} \Rightarrow \text{LCD holds.}$$

proof) WTS : $q^\circ \in Q(p^\circ, w^\circ)$, $w^1 = p^1 q^\circ$, $q^1 \in Q(p^1, w^\circ)$ $\Rightarrow$ $(p^1 - p^\circ)(q^1 - q^\circ) \leq 0$.

By Debreu's theorem, $\succsim$ is C&t, continuous $\Rightarrow$ $\exists$ a continuous utility fcn that represents $\succsim$

Since $B(p,w)$ is compact and u is continuous, $Q(p,w)$ is non-empty. by Exterior value theorem.

Then, Let $q^\circ \in Q(p^\circ, w^\circ)$ and $q^1 \in Q(p^1, w^1)$ and $w^1 = p^1 q^\circ$

Since $\succsim$ is LNS, Walras' law holds Thus $p^\circ q^\circ = w^\circ$, and $p^1 q^1 = w^1$

Since $\succsim$ is C&t, WARP I holds

$q^1 \in Q(p^1, w^1)$ and $w^1 = p^1 q^\circ$ $\Rightarrow$ $q^\circ, q^1 \in B(p^1, w^1)$ & $q^1 \succsim q^\circ$

$q^\circ \in Q(p^\circ, w^\circ)$ $\Rightarrow$ $q^\circ \succsim q''$, $\forall q'' \in B(p^\circ, w^\circ)$

But, $q^1 \notin B(p^\circ, w^\circ)$ by WARPI $\Rightarrow$ $p^\circ q^1 > w^\circ$

Therefore, $(p^1 - p^\circ)(q^1 - q^\circ) = \underbrace{p^1 q^1}_{=w^1} - \underbrace{p^1 q^\circ}_{=w^1} - \underbrace{p^\circ q^1}_{> w^\circ} + \underbrace{p^\circ q^\circ}_{=w^\circ} \leq 0$. //

#12.1(b) Assume $\succsim$ is c&t, continuous, LNS & strictly convex.

WTS: $q^0 = q(p^0, w^0)$, $w' = p' q^0$ and $q' = q(p', w') \Rightarrow (p' - p^0)(q' - q^0) \leq 0$

w/ strict inequality if $q^0 \neq q'$.

proof) By Debreu theorem, $\succsim$ is c&t, continuous $\Rightarrow \exists$ a utility function that represents $\succsim$.

Since $B(p,w)$ is compact and $\succsim$ is continuous, $Q(p,w) = \underset{x \in B(p,w)}{\arg\max}\, u(x)$ is non-empty by Weierstrass theorem.

Lemma Since $\succsim$ is strictly convex, u is strictly quasi-concave $\Rightarrow$ $\boxed{Q \text{ contains at least one element}}$

For contradiction, Suppose $\exists\, x^0, x' \in Q(p,w)$ s.t $x^0 \neq x'$.

Since $B(p,w)$ is convex, $x^t = t x^0 + (1-t) x'$, $t \in (0,1)$, $x^t \in B(p,w)$

By $\succsim$, $x^t \succ x^0 \sim x'$, which is contradiction their being demanded.

Therefore, $Q(p,w) = \{q(p,w)\}$ i.e, $Q(p^0, w^0) = \{q^0\}$ $Q(p', w') = \{q'\}$

Lemma Since $\succsim$ is LNS, u is LNS $\Rightarrow$ $\boxed{\text{Walras' law holds}}$

For contradiction, suppose $\exists\, x \in Q(p,w)$ s.t $px < w$

Then $\exists\, \varepsilon > 0$ s.t. $px' < w$ for all x', $d(x, x') < \varepsilon$.

By LNS, $\exists\, x' \in Q(p,w)$ such that $x' \succ x$, contradiction.

Therefore, Walras' law holds : $px = w$. Then $p^0 q^0 = w^0$, $p' q' = w'$

We know $Q(p,w)$ is single-valued and WARPI holds by $\succsim$ c&t

By our assumption, $q^0 = q(p^0, w^0)$, $q' = q(p', w')$ and $p' q^0 = w'$

#12.1(b) Assume $\succsim$ is c&t, continuous, LNS & strictly convex.

WTS: $q^0 = q(p^0, w^0)$, $q^1 = q(p^1, w^1)$ and $w^1 = p^1 q^0 \Rightarrow (p^1 - p^0)(q^1 - q^0) \leq 0$

with strict inequality if $q^0 \neq q^1$. (Stronger LCD)

proof) By Debreu theorem, $\succsim$ is c&t, continuous $\Rightarrow \exists$ a continuous utility function that represents $\succsim$.

Since $B(p,w)$ is compact and $\succsim$ is continuous, $\underset{x \in B(p,w)}{\text{argmax}}\ u(x)$ is non-empty, by Weierstrauss theorem.

(Lemma1) $\succsim$ is strictly convex $\Rightarrow$ Q contains at most one element.

Lemma proof) For contradiction, Suppose $\exists x^0, x^1 \in Q(p,w)$ s.t $x^0 \neq x^1$.

Since $B(p,w)$ is convex, $x^t = tx^0 + (1-t)x^1$, $t \in (0,1)$, $x^t \in B(p,w)$

Since $\succsim$ is strictly convex, $x^t \succ x^0 \sim x^1$,

which is contradiction their being demanded.

(Lemma2) $\succsim$ is LNS $\Rightarrow$ Walras' law holds.

Lemma proof) For contradiction, Suppose $\exists\, x \in Q(p,w)$ s.t $px < w$.

Then $\exists\, \varepsilon > 0$ s.t. $px' < w$ for all x', $d(x, x') < \varepsilon$

By LNS, $\exists\, x' \in Q(p,w)$ such that $x' \succ x$, contradiction.

By lemma 1, $Q(p,w)$ is single-valued and by $\succsim$ c&t WAI holds.

By lemma 2, Walras' law holds. Then $p^0 q^0 = w^0$ and $p^1 q^1 = w^1$.

Thus, $Q(p^0, w^0) = \{q(p^0, w^0)\}$, $Q(p^1, w^1) = \{q(p^1, w^1)\}$

Assume $q^0 = q(p^0, w^0)$ and $q^1 = q(p^1, w^1)$, and let $p^1 q^0 = w^1$.

Then, $q^0 \in B(p^0, w^0)$, $q^1 \in B(p^1, w^1)$ and $q^0 \in B(p^1, w^1)$ by $p^1 q^0 = w^1$.

By WA I, $q^1 \succsim q^0 \Rightarrow q^1 \notin B(p^0, w^0)$. Then $p^0 q^1 > w^0$.

(Therefore, WARP I holds.) $\cdots$ ⊛

Now, we can show the stronger LCD holds as follows.

WTS: $(p^1 - p^0)(q^1 - q^0) \leq 0$ with strict inequality if $q^0 \neq q^1$.

when $q^1 = q^0$, then $(p^1 - p^0)(q^1 - q^0) = 0$ so we are done.

when $q^1 \neq q^0$, by $p^1 q^1 = w^1$, $p^1 q^0 = w^1$ and $q^1 = q(p^1, w^1)$, $q^1 \succ q^0$.

Thus, $q^1 \notin B(p^0, w^0)$ as we showed above $\Rightarrow p^0 q^1 > w^0$.

Therefore, $(p^1 - p^0)(q^1 - q^0) = \underbrace{p^1 q^1}_{=w^1} - \underbrace{p^0 q^1}_{>w^0} - \underbrace{p^1 q^0}_{=w^1} + \underbrace{p^0 q^0}_{=w^0} < 0$. ||

From ⊛ and this results, $\succsim$ c&t, continuous, LNS & strictly convex

WA I holds

LCD holds

Q is single valued

$\downarrow$

WA I holds

$\downarrow$

Stronger LCD holds.

$\Rightarrow$ WA I = Stronger LCD.

☆☆☆
#12.2 Show that the law of uncompensated demand holds

If $p_i^1 > p_i^0$ for some i and $p_j^1 = p_j^0$ for all $j \neq i$,

then $g_i(p^1, w) \leq g_i(p^0, w)$

proof)

Suppose $g_i(p^1, w) > g_i(p^0, w)$.

$g(p, w)$ solves the problem ; $\max u(x) = \sum_{i=1}^{k} u_i(x_i)$ s.t $px \leq w$, $x_i \geq 0$

By KKT, $\exists\ \lambda^0, \lambda^1$ s.t $u_i'(g_i(p^0, w)) - \lambda^0 p_i^0 \leq 0$ w/ equality if $g_i(p^0, w) > 0$

$\mathcal{L} = \sum_{i=1}^{n} u_i(g_i) - \lambda\left(\sum_{i=1}^{n} p_i g_i - w\right)$, $\frac{\partial \mathcal{L}}{\partial g_i} \leq 0$ with equality if $g_i > 0$

$u_i'(g_i(p^1, w)) - \lambda^1 p_i^1 \leq 0$ w/ equality if $g_i(p^1, w) > 0$.

$\downarrow$

$g_i(p^1, w) > g_i(p^0, w) \geq 0$.

$\Rightarrow$ $\underline{u_i'(g_i(p^1, w)) - \lambda^1 p_i^1} = 0 \geq u_i'(g_i(p^0, w)) - \lambda^0 p_i^0$

Since $u_i'' < 0$, then $u_i'(g_i(p^1, w)) < u_i'(g_i(p^0, w))$

$$-\lambda^1 p_i^1 > -\lambda^0 p_i^0$$

$$\lambda^1 p_i^1 < \lambda^0 p_i^0$$

$$\downarrow\ p_i^1 > p_i^0$$

$$\boxed{\lambda^1 < \lambda^0} \cdots (*)$$

$\sum p_k^1 g_k^1 \overset{*}{=} w = \sum p_k^0 g_k^0$

$g_i^1 > g_i^0$, $p_i^1 > p_i^0 \Rightarrow p_j^1 = p_j^0$

$\Rightarrow \exists\ j \neq i$ s.t $g_j(p^0, w) > g_j(p^1, w)$

$\rightarrow x_i^1 > x_i^0$가 위의 가정이고,
$p_i^1 > p_i^0$ 인 상황이므로
$p_j^1 = p_j^0$이면
어떤 j에 대해 $g_j(p^0, w) > g_j(p^1, w)$
이여야 등호(*) 성립!

$$\mathcal{L} = \sum_{i=1}^{K} u_i(g_i) - \lambda\left(\sum_{i=1}^{n} p_i g_i - w\right)$$

$\Downarrow$ Take a derivative by g_j

$$u_j'(g_j(p^0, w)) - \lambda^0 p_j^0 \leqq 0 \quad \text{w/ equality} \quad \text{if } g_j(p^0, w) > 0.$$

$$u_j'(g_j(p^1, w)) - \lambda^1 p_j^1 \leqq 0 \quad \text{w/} \quad \text{"} \quad \text{if } g_j(p^1, w) > 0$$

$g_j(p^0, w) > g_j(p^1, w) \geqq 0$ 이므로,

$$u_j'(g_j(p^0, w)) - \lambda^0 p_j^0 = 0 \geqq u_j'(g_j(p^1, w)) - \lambda^1 p_j^1$$

$u'' < 0$ 이므로, $u_j'(g_j(p^0, w)) < u_j'(g_j(p^1, w))$

$$-\lambda^0 p_j^0 > -\lambda^1 p_j^1$$

$$\lambda^0 p_j^0 < \lambda^1 p_j^1$$

$$\downarrow \; p_j^0 = p_j^1$$

$$\boxed{\lambda^0 < \lambda^1} \cdots \text{(**)}$$

$\Rightarrow$ (*) and (**) are contradiction.

Therefore, $g_i(p^1, w) \leqq g_i(p^0, w)$ //

(problem 13.1) (a) prove that B satisfies the metric notion of upper and lower semicontinuity described above (page 25 of the lec. note)

proof)

For any $\varepsilon>0$, $\exists\, \delta>0$ s.t. $d((p,w),(p,w'))<\delta \Rightarrow d(B(p,w), B(p,w'))<\varepsilon$

$d(B(p,w), B(p,w')) = \inf\{\varepsilon : B(p,w) \subseteq B(p,w')^{+\varepsilon}\}$ for Lower S.C (LSC)

$\left(\inf\{\varepsilon : B(p,w') \subseteq B(p,w)^{+\varepsilon}\}\right.$ for Upper S.C (USC)

WLOG, suppose $w<w'$. Then, it is trivial $B(p,w) \subseteq B(p,w')^{+\varepsilon}$ for LSC

WTS: $\exists\, \varepsilon$ s.t. the union of all ε-ball of every point in $B(p,w)$ can cover $B(p,w')$

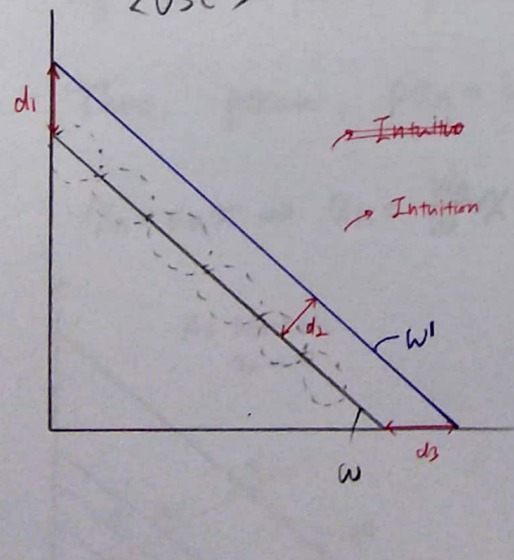
It is sufficient to cover all the points on $B(p,w')$ by using "the longest" bundle.

That is, $\max d(B(p,w), B(p,w')) = \dfrac{|w'-w|}{\min\{p_i\}}$

Take $\delta = \varepsilon \cdot \min\{p_i\}$, where $|w'-w|<\delta$.

Then, $d(B(p,w), B(p,w')) \leq \dfrac{|w'-w|}{\min\{p_i\}} < \dfrac{\delta}{\min\{p_i\}} = \varepsilon$ //

<USC>



~~Intuitive~~

Intuition Choose $\max\{d_1, d_2, d_3\}$ as ε!

(b) prove that B satisfies Sequential upper and lower semicontinuity.

(Def) USC: Let $x_n \in B(p, w_n)$ Let $x_n \to x$, $w_n \to w$ $\Rightarrow x \in B(p,w)$

LSC: Let $x \in B(p,w)$ Let $w_n \to w$ $\Rightarrow \exists x_n$ s.t. $x_n \in B(p, w_n)$ and $x_n \to x$.

- USC

proof) For each n, $x_n \in B(p, w_n) \Rightarrow px_n \leq w_n$

$px_n \to px$, $w_n \to w$ $\Rightarrow$ $px \leq w$ $\Rightarrow x \in B(p,w)$.

- LSC

proof) WLOG, suppose x is on the budget frontier.

Then, we construct the sequence x_n by connecting the line between origin and x, denoted by l.

Pick x_n to be the intersect of l and $B(p,w)$.

Now we prove $x_n \to x$.

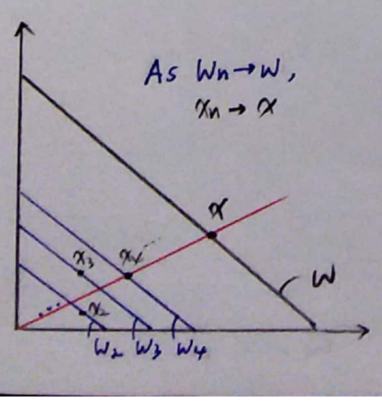
By construction, x_n is on the budget frontier, i.e. $x_n \in B(p,w)$

Since x_n and x are co-linear, then $\exists \alpha_n$: $x_n = \alpha_n \cdot x$

By $x_n \in B(p,w)$, $px = w$, and $w_n \to w$, given.

Thus, $px = w$, $px_n = w_n$ $\Rightarrow$ $p \cdot \alpha_n x = w_n \Rightarrow \alpha_n = \frac{w_n}{px} = \frac{w_n}{w}$

$x_n = \alpha_n x \Rightarrow x_n = \frac{w_n}{w} x \Rightarrow x_n \to x$. ||



☆☆

problem 13.2 If $\succsim$ is continuous and strictly convex, then the demand function is continuous.

proof) $\succsim$ is continuous $\Rightarrow$ u is continuous $\Rightarrow$ Q is non-empty

$\succsim$ is strictly convex $\Rightarrow$ u is quasi-concave $\Rightarrow$ Q contains at most one element

Then, Q is single-valued (correspondence)

u is continuous $\Rightarrow$ V is continuous and Q is uppersemicontinuous (correspondence)

By the result from above,

correspondence Q is usc and single-valued.

$(Q(p,w) = \{\xi(p,w)\}.)$

By the definition of usc,

Q is uppersemicontinuous

$$\forall \varepsilon>0 : \exists \delta>0 : d((p',w'),(p,w))<\delta \Rightarrow Q(p',w') \subseteq Q(p,w)^{+\varepsilon}$$

$$\Leftrightarrow \xi(p',w') \in B_\varepsilon(\xi(p,w))$$

$$\Leftrightarrow \|\xi(p',w') - \xi(p,w)\| < \varepsilon$$

Therefore, $\forall \varepsilon>0 : \exists \delta>0 : d((p',w'),(p,w))<\delta \Rightarrow \|\xi(p',w')-\xi(p,w)\|<\varepsilon$,

the demand function ξ is continuous. //

☆☆
problem 14.1 (i) If u is LNS, then V is strictly increasing in w.

proof) Since u is LNS, $Q(p,w)$ satisfies Walras' law.

Then, $\forall q \in Q(p,w)$, we have $p \cdot q = w$

Let $w' > w \in \mathbb{R}_+$. Then $p \cdot q < w'$

Thus, $\exists \varepsilon > 0$ s.t $B_\varepsilon(q) \subseteq B(p,w')$

Since u is LNS, $\exists \hat{q} \in B_\varepsilon(q)$ s.t. $u(\hat{q}) > u(q) = V(p,w)$.

Since $\hat{q} \in B(p,w')$, $V(p,w') \geqq u(\hat{q}) > u(q) = V(p,w)$.

Therefore, if u is LNS, V is strictly increasing in w. //

(ii) Blank : u is LNS / $q_i > 0$ (interior solution)

If $p' < p$ (i.e., $p_i' \leqq p_i$ for all i and $p_j' < p_j$ for some j) and $\boxed{}$,
then $V(p',w) > V(p,w)$

proof) Since u is LNS, Walras' law holds.

Then, $\forall q \in Q(p,w)$, $p \cdot q = \sum_{i \neq j} p_i q_i + p_j q_j = w$

Assume $\exists p' < p$ such that $p' := (p_1', \dots, p_j', \dots, p_n')$, $p := (p_1, \dots, p_j, \dots, p_n)$
for all $i \neq j$ $p_i' \leqq p_i$ and for some j $p_j' < p_j$.

Then, $p' q < w$. Hence, $\exists \varepsilon > 0$ such that $B_\varepsilon(q) \subseteq B(p',w)$

Since u is LNS, $\exists \tilde{q} \in B_\varepsilon(q)$ s.t $u(\tilde{q}) > u(q) = V(p,w)$

By $\tilde{q} \in B(p',w)$, $V(p',w) \geqq u(\tilde{q}) > u(q) = V(p,w)$.

Therefore, if u is LNS and $p' < p$, $V(p',w) > V(p,w)$. //

☆☆

problem 14.2

(a) If $C_{\succsim}$ is single valued for $B, B' \in \mathcal{B}$,

then $B' \succsim^* B$ iff $C_{\succsim}(B') \succsim C_{\succsim}(B)$

($\Rightarrow$) Let $B' \succsim^* B$. By definition, $C_{\succsim}(B) \in B$.

Then $\exists x' \in B'$ s.t $x' \succsim C_{\succsim}(B)$. By definition, $C_{\succsim}(B') \succsim x'$, $\forall x' \in B'$

Thus, $C_{\succsim}(B') \succsim x' \succsim C_{\succsim}(B)$ i.e., $C_{\succsim}(B') \succsim C_{\succsim}(B)$ by transitivity.

($\Leftarrow$) By definition, $\forall x \in B$, $C_{\succsim}(B) \succsim x$. Since $C_{\succsim}(B') \succsim C_{\succsim}(B)$,

$C_{\succsim}(B') \succsim x$, for all $x \in B$.

Then, $\forall x \in B$: $\exists C_{\succsim}(B') \in B'$ and $C_{\succsim}(B') \succsim x \Rightarrow B' \succsim^* B$. //

(b) Assume u represents $\succsim$. and $Q(B) = \emptyset$ for all $B \in \mathcal{B}$

$V(B) = \max_{x \in B} u(x)$ well defined for all B. Show v represents $\succsim^*$

$V(B') \geqq V(B) \Leftrightarrow B' \succsim^* B$

proof)

($\Rightarrow$) Suppose $V(B) \leqq V(B)' \Rightarrow \max_{x \in B} u(x) \leq \max_{x' \in B'} u(x')$.

Let $x_* \in \arg\max_{x \in B} u(x)$ and $x_*' \in \arg\max_{x' \in B'} u(x)$ Then $u(x_*) \leqq u(x_*')$

$\Rightarrow x_* \precsim x_*'$.

and $x_* \in \arg\max_{x \in B} u(x)$ for any $x \in B$, $u(x) \leqq u(x_*) \Rightarrow x \precsim x_*$.

Then $x \precsim x_* \precsim x_*'$ i.e., $x \precsim x_*'$ for any $x \in B$, by transitivity.

For any $x \in B$, $\exists x_*' \in B'$ such that $x_*' \succsim x \Rightarrow B' \succsim^* B$. //

($\Leftarrow$) Suppose $B' \succsim^* B$. Then for any $x \in B$, $\exists x' \in B'$ s.t. $x' \succsim x$

Since V is well-defined, let $x_* \in \arg\max_{x \in B} u(x)$, $x_*' \in \arg\max_{x' \in B'} u(x)$

Thus, $\exists x' \in B'$ s.t. $x' \succsim x_*$. By definition, we have $x_*' \succsim x'$ for any $x' \in B$.

Therefore, $x_*' \succsim x' \succsim x_* \succsim x$ i.e., $x_*' \succsim x_* \Rightarrow u(x_*') \geqq u(x_*)$

$\Rightarrow V(B') \geqq V(B)$. //

#14.2 (c) Assume u represents $\succsim$, $C_u(B)$ is nonempty for some B.

$\tilde{V}(B) := \sup_{x \in B} u(x)$ Does $\tilde{V}$ still represent $\succsim^*$?

sol) Let $\tilde{V}(B) := \sup_{x \in B} u(x)$

Let $u(x) = x$ and suppose $B = [0, 1)$, $B' = [0, 1]$

Then, $\tilde{V}(B) = \tilde{V}(B') = 1$

However, $1 \in B'$ but $1 \notin B$ by our suppostion

Thus, there is no $x \in B$: $x \succsim 1$, but $1 \in B'$.

Therefore, $\tilde{V}(B)$ does not represent $\succsim^*$. //

☆☆

problem 1.4.2(d)

① V is HD0

$B(p,w)$ is HD0 $\Rightarrow$ $B(p,w) = B(\lambda p, \lambda w)$, $\lambda > 0$

$\Rightarrow B(p,w) \sim^* B(\lambda p, \lambda w)$ Then, $V(p,w) = V(\lambda p, \lambda w)$. //

② V is increasing in W

Let $W' \geqq W \Rightarrow B(p,w') \supseteq B(p,w) \Rightarrow B(p,w') \succsim^* B(p,w)$

$\Rightarrow V(p,w') \geqq V(p,w)$. //

③ V is decreasing in P

Let $P' \geqq P \Rightarrow B(p,w) \supseteq B(p',w) \Rightarrow B(p,w) \succsim^* B(p',w)$

$\Rightarrow V(p,w) \geqq V(p',w)$. //

④ V is strictly increasing in W if u is LNS.

Assume u is LNS. Then Walras' law holds.

Let $w' > w$. then, $w' = p \cdot x' > w = p \cdot x$

$\Rightarrow B(p,w') \supsetneq B(p,w) \Rightarrow B(p,w') \succ^* B(p,w) \Rightarrow V(p,w') > V(p,w)$. //

⑤ V is continuous if u is continuous.

Let $B(P_n,W_n) \succsim^* \hat{B}$ and $B(P_n,W_n) \to B(p,w)$

Then, $\forall x \in \hat{B}$, $\exists x_n' \in B(P_n,W_n)$: $x_n' \succsim x$.

By $B(P_n,W_n) \to B(p,w)$, $\exists x' \in B(p,w)$ s.t. $x_n' \to x'$ and $x' \succsim x$.

Thus, $B(p,w) \succsim^* \hat{B}$.

⑥ V is quasi-convex.

Let $X := \{B \mid B(p,w) \precsim^* \hat{B}\}$

Choose $B', B'' \in X$

Then $\forall x' \in B' : \exists y' \in \hat{B}$ s.t. $x' \precsim y'$

and $\forall x'' \in B'' : \exists y'' \in \hat{B}$ s.t. $x'' \precsim y''$.

Let $t \in (0,1)$ and $y := \max\{y', y''\}$. Then $y \in \hat{B}$

By $x' \precsim y'$ and $x'' \precsim y''$, $tx' \precsim ty'$ and $(1-t)x'' \precsim (1-t)y''$.

Let $x^t := tx' + (1-t)x''$

Then, $x^t \precsim ty' + (1-t)y'' \precsim ty + (1-t)y = y \in \hat{B}$.

$x^t \in tB' + (1-t)B''$, $y \in \hat{B}$, $x \precsim y \Rightarrow tB' + (1-t)B'' \precsim^* \hat{B}$. //

$\succsim$ on X

a continuous utility representation

(WA I)

If $x, y \in B, B'$ and $x \in C(B)$, $y \in C(B')$, then $x \in C(B')$.

$\succsim$ C & t, $\succsim$ continuous

$\succsim$ strictly increasing

$\succsim$ strictly convex

$\oplus$ (X is a convex set)

u conti.

Q is non-empty

V is continuous. (well-defined)

Q is uppersemicon.

Q is usc & ξ is conti

Q is single-valued. $\Rightarrow$ (WA II) holds

$Q(p,w) = \{\xi(p,w)\}$

If $p^1 g^0 \leq w^1$, $p^0 g^1 > w^0$

WA II = LCD (stronger) $\rightarrow$ given $w^1 = p^1 g^0$, $(p^1 - p^0)(g^1 - g^0) \leq 0$. w/ < when $g^1 \neq g^0$

u strictly quasi concave $\rightarrow$ Q contains at most one element.

V is strictly inc. in w

u LNS $\rightarrow$ Walras' law $px = w$

x^* is optimal

$\Leftrightarrow$ $py \geq px^* = w$, $\forall y \succsim x^*$

(budget hyperplane supports $\{\succsim x^*\}$ at x^*, $px^* = w$ and $py \geq w$, $\forall y \succsim x^*$)

$\cdot$ $\succsim \Rightarrow u$

- $\succsim$ C & t on a finite X
- $\succsim$ C & t on a countably infinite X
- or $\succsim$ C & t, conti. on a uncountably infinite X. ($\Rightarrow$ continuous u)

$\cdot$ $C \Rightarrow \succsim$

If $C(\cdot)$ satisfies WA and $C(B)$ is non-empty,

$C(\cdot)$ is rationalizable & unique $\succsim$ c & t.

$(C(\cdot) = C_{\succsim}(\cdot))$

- $u \Rightarrow \succsim$ (automatically)

  If $\exists u$, then $\succsim$ is c&t

- $\succsim \Rightarrow u$

  If $\succsim$ c&t on a finite X,

  If $\succsim$ c&t on a countablely infinite X,

  $\Rightarrow$ $\succsim$ has a util. representation.

  If $\succsim$ c&t on a uncountably infinite X and CONTINUOUS,

  $\Rightarrow$ $\succsim$ has a continuous util. rep.

- $\succsim \Rightarrow C$ (automatically / we can define $C_{\succsim}$).

  If $\succsim$ c&t $\Rightarrow$ "$\mathcal{B}, C_{\succsim}$" finite nonempty & WAI holds.

- $C \Rightarrow \succsim$

  If $C(\cdot)$ satisfies WA and $C(B)$ is non-empty,

  $\underline{C(\cdot) \text{ is rationalizable}}$ & $\underline{\succsim \text{ c\&t}}$

  $C(\cdot) = C_{\succsim}(\cdot)$

$\succsim$ c&t $\Rightarrow$ WAI

$\succsim$ c&t and continuous

$\Downarrow$

u is continuous

$\Uparrow$ ⊕ $\succsim$ on convex X

$\succsim$ on has a continuous utility representation

$\succsim$ c&t, continuous, st. increasing $\Longleftrightarrow$ u continuous, st. increasing.

$\succsim$ st. inc $\Rightarrow$ u st. inc $\Rightarrow$ Walras' law

$\succsim$ continuous $\Rightarrow$ u continuous $\Rightarrow$ Q is non-empty

$\succsim$ st. convex $\Rightarrow$ u st. quasi concave $\Rightarrow$ Q contains atmost one point

Q is singlevalued (WAI = WAII)

$\succsim$ c&t, LNS, continuous $\Rightarrow$ x^* is optimal $\Longleftrightarrow$ $py \geq px^* = w$, $\forall y \succsim x^*$

$x^* \in C_{\succsim}(B\{p,w\})$

(i.e., budget hyper plane supports $\{\succsim x^*\}$ at x^*, $px^* = w$ and $py \geq w$ $\forall y$, $y \succsim x^*$

$\Downarrow$

LCD.

(⊕ st. convex $\Rightarrow$ Stronger LCD)

$\left.\begin{array}{l}\left.\begin{array}{l}\succsim \text{ continuous} \\ \succsim \text{ st. convex}\end{array}\right] Q \text{ singlevalued} \\ \succsim \text{ st. inc} \Rightarrow \text{Walras law}\end{array}\right] : \text{WAII} = \text{LCD}.$

$\succsim$ continuous $\Rightarrow$ u continuous $\Rightarrow$ V continuous

$\Rightarrow$ Q usc $\xrightarrow{\oplus\, u \text{ st. convex}}$ Q continuous.