Lecture 7

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Last Class Clarification: Optimality versus pareto dominance

T-optimal is pareto dominant for the T agents, not for everyone, relative to the set of all stable matches. You can ask, for instance, is it also T-optimal relative to the set of all stable matches? Not necessarily. Is it optimal relative to the set of all IR matches? Yes you can show it is.

Interest in the rural hospital theorem: In any two stable matches, the set of unmatched interests is exactly the same. Hospitals in rural areas - maybe doctors will match to them when they werent before. But, the set of unmatched agents will be the same in any matches. So, nothing you can do.

Now, this is technically many-to-one. Rural hospital needs 20 posititions, but maybe only have 3. So, what can be done? Well, changing the matching will not fill those positions, no matter what, according to this theorem.

Stability

We defined stability so that no individual so that no individual has an incentive to deviate from the match, and no pair of agents has an incentive to deviate from the match. What about a group of agents, though? A group could be 2, or 1, or even 10. So, we will start by thinking about what if a group of agents has an incentive to deviate from a match? So, we turn our attention to a coalition having an incentive to deviate.

Let match $u = \mu$ for sake of convenience.

Definition A nonempty coalition $C \subset T \cup B$ blocks a matching u if there exists a matching u' such that for each $i \in C$ the following hold:

(1) $u'(i) \in C$. Each member of C is matched to another member of the coalition C. If agent t is a part of this coalition, it could be the case that u' matches t to either t or b so as long as they are both in C.
(2) $u'(i) \succ_i u(i)$. each member of C picks his match under u' to his match under u.

When these conditions hold we will say that u' dominates u (for coalition C).

Definition The matching $u : (T \cup B) \to (T \cup B)$ is in the *core* if there is no nonempty blocking coalition C.

The idea of a core started in cooperative game thoery. It is about a group of players blocking something and making a deviation together. Here we apply it to a specific cooperative game that is about matching. When we get to many-to-one markets, we will have to redefine carefully. Why didn't we talk about this until now? Well, let's turn to a theorem.

Theorem 1 (1-to-1 Environment)

A matching μ is in the core if and only if it is stable

This tells us that if we are interested in what the cores are, we only have to look at stable matches.

To show its stable, show that it is IR and no blocking pair.

Proof

For the first part, we want to show a core matching must be stable:

Individually Rational - If u is in the core, then u is stable. To see this, fix a core matching u. First note that for each coalition $C = \{i\}$ (a singleton coaltion) there is no matching u' with $u'(i) = i \succ_i u(i)$. Now, certainly, there is a matching with $u'(i) = i$. The fact there is no matching satisfying $\succ_i$, this implies that $u(i) \succsim_i i$ and so this implies the matching is individually rational. This is just to say that it cannot be the case that $u'(i) = i \succ_i u(i)$.

Stable - Now, we want to show that there is no blocking pair. For each $C = \{t, b\}$ there is no u' with:

(1) $u'(t) = b$
(2) $u'(b) = t$
(3) $u'(t) \succ_t u(t)$
(4) $u'(b) \succ_b u(b)$

There is some u' with (1) and (2) satisfied. But, to be in the core there is no blocking coalition, all four together cannot be satisfied. But, let the (1) and (2) be satisfied and call $u''(t) = b$ and $u''(b) = t$

Either it must be the case that (3) $u(t) \succsim_t u''(t) = b$ or (4) $u(b) \succsim_b u''(b) = t$. (This contrasts with definition of blocking pair where instead of *either*, *or* we have *both* , *and* prefer one another to their match). Since one of these conditions hold, $\{t, b\}$ is not a blocking pair.

For the second part, we say that if u is stable, then u is in the core. Take the contrapositive: we want to show that if u is not in the core, then u is not stable.

So, suppose that u is not in the core. Then we can find a u' and coalition C such that u' dominates u for coalition C if there exists some $i \in C$ such that $u'(i) = i$. Then, $u'(i) = i \succ_i u(i)$. This implies, however, that u is not individually rational.

If there doesn't exist some i such that the above is true, then for all $i \in C$, $u'(i) \neq i$ implies that for all $t \in C$, $u'(t) = b \in B$ which implies also that $u(b) = t \in T$. By the condition in the definition of a non-empty coalition C blocking above, it must hold that: $u'(t) = b \succ_t u(t)$ and $u'(b) = t \succ_b u(b)$.

Since we assumed it is not in the core, we showed that either a person would stay with himself and the matching would then not be IR, or we could find a pair so they would block on their own.

Introducing Many-to-One Markets

So, we have been talking about 1-to-1 because it is easier to get ideas and math right. But, most matching markets are many-to-one or many-to-many. With a strong assumption on preferences, however, many of the ideas in one-to-one will transfer over to many-to-one. We won't need to reprove things, per se, because we will be able to covert a many-to-one market to a one-to-one.

We will still have T and B agents. But, we are in a world where T agents can potentially match to many B agents. But B agents can only match to one t. We will think of a t agent as either being matched to some B^* where $B^* \in \mathcal{B} = (2^B \setminus \{\emptyset\})$ and 2^B = set of subsets of B or to himself t.

Now t can match with some b agents, but he might not be able to match with everyone out there. So, he (t) is going to have a quota. A quota for t is some $q_t \geq 1$ and it tells us the maximum number of b agents t can match to. With this, now the environment looks different:

$$\varepsilon = (T, B_i; (q_t)_{t \in T}, ; (\succsim_t)_{t \in T}, (\succsim_{b \in B}))$$

Here now for a t agent the preference relation $\succsim_t$ is a binary relation on $\mathcal{B} \cup \{t\}$. For b, $\succsim_b$ is still as before a binary relation on $T \cup \{b\}$.

There is a little bit of awkwardness in this notation. If you think about what $\mathcal{B} \cup \{t\}$ is saying: $\{B_* : B_* \subseteq B, B_* \neq \emptyset\} \cup \{t\}$, so it could be $B_* \succsim_t t$, or $\{b_1, b_2\} \succsim_t t$ or $\{b_1\} \succsim t$.

And $T \cup \{b\} = \{t_2, ..., t_{|T|}\} \cup \{b\}$.

Before, $\succsim_t$ was a relation on $B \cup \{t\}$. Now it is a relation on $\{B_* : B_* \subseteq B, B_* \neq \emptyset\} \cup \{t\}$. So the two changes in the environment is (i) adding the quotas and (ii) changing what t's preferences are over.

Now, we will assume that these preferences are complete and transitive as they were before in all cases for both t and b.

Remark: As in the 1-to-1 environment, we are implicitly assuming that there are no externalities in the match. This is a much stronger assumption now. Imagine colleges and students. As a student, you might very well care which students that college is matched to - maybe you want to go to a place with strong students. We have ruled that out by saying that your preferences are only over the other side of the market (above). I have to admit now that I do not care what students are going to that college.

Definition: Call $\succsim_t$ *responsive* if the following holds:

(1) for each b_1, b_2 and some subset $\bar{B} \subset B$ with $\bar{B} \cap \{b_1, b_2\} = \emptyset$, then $\{b_1\} \succsim_t \{b_2\}$ if and only if $\{b_1\} \cup \bar{B} \succsim_t \{b_2\} \cup \bar{B}$.

    I am adding b_1, b_2 and saying it is not in $\bar{B}$. So, we are going to say adding them to $\bar{B}$ will not change our ranking.

(2) for each $b \in B$ and $\bar{B} \subseteq B$ with $\bar{B} \cap \{b\} = \emptyset$:

- $\bar{B} \cup \{b\} \succsim_t \bar{B}$ if and only if $\{b\} \succsim_t t$
- $\bar{B} \succsim_t \bar{B} \cup \{b\}$ if and only if $t \succsim_t \{b\}$.

Preferences are *responsive* if, for each $t \in T$, $\succsim_t$ is responsive. *Preferences over groups of agents are determined by preferences over individuals.*

In a sense, pragmatically, it is a strong restriction even though it is mathematically elegant. It rules out certain types of compliments and substitutes. For compliments, think about t being a college and two different types of students you might admit. Students differ in scores and personalities. Think about a case where b_1 is a top student and b_2 is okay. While b_1 is a super top student, he has personality issues and drags everyone in the group down. All else equal, you prefer b_1 over b_2. But now, add b_1 to a group of "okay" students, whereas b_2 may raise the group. b_2 has complimentarities with the group that b_1 does not have. So, alone, I prefer b_1 to b_2, but in context of a group, I prefer b_2 over b_1.

For substitutes, think about t being a hospital and b_1 and b_2 being residents. Let b_1 be a general practitioner (GP) and b_2 a dermatologist. All else equal, I prefer a GP over a dermatologist. But, if I have lots of GP's already, then I may prefer the dermatologist to the GP. In that case, this b_1 and the set of GP's are subsititutes. In that case, I will prefer to get myself a dermatologist since I have lots of GP's already.

But, there are some cases where this assumption overall fits. Maybe colleges only know rankings, and not personalities, of students. Just keep in mind that it is a strong assumptions.

Book defines responsive preferences in terms of a match. Here we do differently.