⊙ Basic terms

- Commodities : $\ell = 1, \dots, L$
- Consumers : $i = 1, \dots, I$

consumption set of i : $X_i \subseteq \mathbb{R}^L$

bundle $x_i = (x_{1i}, \dots, x_{Li}) \in X_i$

- endowment of commodities for i : w_i

$w_i = (w_{1i}, \dots, w_{Li}) \in X_i$

$\bar{w}_\ell = \sum_{i=1}^{I} w_{\ell i}$ (total endowment of ℓ)

$\bar{w} = (\bar{w}_1, \bar{w}_2, \dots, \bar{w}_L)$ endowment vector.

- firms : $j = 1, \dots J$

θ_{ij} : share of firm j that i owns

Y_j : production set of firm j, $Y_j \subseteq \mathbb{R}^L$

- allocation $(x, y) = (x_1, \dots, x_I, y_1, \dots, y_J)$

(x, y) is feasible : $\underbrace{\sum_{i=1}^{I} x_{\ell i}}_{\text{consume}} = \underbrace{\bar{w}_\ell}_{\text{endowment}} + \underbrace{\sum_{j=1}^{J} y_{\ell j}}_{\text{production}}$ $\forall \ell$

Pareto optimal (x, y)

if $\nexists$ other feasible allocation (x', y') s.t

(1) $x_i' \succsim x_i$ $\forall i$ (2) $x_i' \succ x_i$, for some i

* Competitive equilibrium (x^*, y^*) with p^*

① profit max : $p^* \cdot y_j^* \geq p^* \cdot y_j$, $\forall y_j \in Y_j$

② Util max s.t. budget constraint.

: $x_i^* \succsim x_i$, $\forall x_i \in B_i^*(p^*)$ subject to

$B_i^* = \{x_i \in X_i \mid p^* \cdot x_i \leq p^* w_i + \sum_{j=1}^{J} \theta_{ij} (p^* y_j^*)\}$

easily, p^*. feasibility for i

③ Market clear $\underbrace{\sum_{i=1}^{I} x_i^*}_{\text{aggregate demand}} = \underbrace{\sum_{i=1}^{I} w_i}_{\text{Endowment}} + \underbrace{\sum_{j=1}^{J} y_j^*}_{\text{Aggregate supply}}$

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max $\bar{w}_m + S(x', q')$ s.t $\sum_i x_i' = \sum_j q_j'$ ⋯(*)

If (x, m, q, z) feasible, & solve (*) ⇒ $v \in bd\, \mathcal{U}(x, q)$

If (x, m, q, z) is C.E, then (x, q) solves (*)

FWT : C.E → P.O

SWT : $v^* \in bd\, \mathcal{U}$ ⇒ (x^*, m^*, q^*, z^*) is CE

& induce v^*

⊙ Partial equilibrium

- Commodities : m_i (money), ℓ
- Consumers : $i = 1, \dots, I$

Consumption set of i : $X_i = \mathbb{R} \times \mathbb{R}_+ \ni (m_i, x_i)$

$u_i(m_i, x_i) = m_i + \phi_i(x_i)$ quasi-linear

→ No wealth effect / perfect substitute

- endowment : $w_i = (w_{m_i}, 0)$ (only money)
- price : $(1, p)$ = (money price, good price)
- firms' production : q_j cost : $C_j(q_j)$

$Y_j := \{(-z_j, q_j) : q_j \geq 0,\ z_j \geq C_j(q_j)\}$

When producing goods, spend z_j

- allocation (x, m, q, z) is feasible.

$\sum_i x_i = \sum_j q_j$ and $\sum_i m_i = \sum_i w_{m_i} - \sum_j z_j$

→ then, $v_i = m_i + \phi_i(x_i)$

⇒ $\sum_i m_i = \sum_i v_i - \sum_i \phi_i(x_i)$

$v \in bd\, \mathcal{U}$ ⇝ $= \bar{w}_m + S(x, q) - \sum_i \phi_i(x_i)$ (proof)

$= \bar{w}_m - \sum_j C_j(q_j) = \bar{w}_m - \sum_j z_j$ //

* Competitive eq. (x^*, m^*, q^*, z^*) with $(1, p^*)$

① Π max : max $p^* q_j - C_j(q_j)$ $p^* \leq C_j'(q_j)$

min z_j s.t $z_j \geq C_j(q_j)$ $z_j^* = C_j^*(q_j)$

(i.e.) max $[-1 \cdot z_j]$

② U max s.t budget constraint.

max $[m_i + \phi_i(x_i)]$ s.t. $1 \cdot m_i + p^* x_i$ $\leq 1 w_{m_i} + \sum_j \theta_{ij} [\underbrace{p^* q_j^* - C_j(q_j^*)}_{= \pi_j^*}]$

↓

$\phi_i'(x_i^*) \leq p^*$ $m_i + p^* x_i = w_{m_i} + \sum_j \theta_{ij} \cdot \pi_j^*$

③ Market clear

$\underbrace{\sum_i x_i^* = \sum_j q_j^*}_{\text{consume = produce}}$ / $\sum_i m_i^* = \underbrace{\bar{w}_m}_{\text{total money}} - \underbrace{\sum_j z_j^*}_{\text{money for production}}$

$\mathcal{U}(x, q) = \{v \in \mathbb{R}^I : \underbrace{\sum v_i}_{\text{Sum of util. vector}} \leq \underbrace{\bar{w}_m}_{\text{Sum of money}} + \underbrace{S(x, q)}_{\text{surplus}}\}$

$S(x, q) = \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} C_j(q_j)$

⊙ Pure Exchange economy.

- Commodities : $\ell = 1, 2$, $X_i \in \mathbb{R}_+^2$
- consumers : $i = 1, 2$

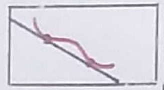
$\succsim_i$ strongly monotone if $x \geq x'$, $x \neq x'$ ⇒ $x \succ_i x'$

strictly convex $x' \succsim x$, $x'' \succsim x$ ⇒ $\alpha x' + (1-\alpha) x'' \succ x$ $(x' \neq x'')$

continuous $x^n \to x$, $z^n \to z$, $x^n \succsim z^n$ ⇒ $x \succsim z$

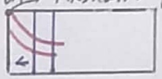
- $w_i = (w_{1i}, w_{2i}) \in \mathbb{R}_+^2$

cf) No convexity ⇒



(cf2) (Evil) (monotone, convex, fora

→ ~~Market clear~~ ⇐



→ w_1 is P.O, NOT C.E.

- allocation $(x_1, x_2) \in \mathbb{R}_+^4$ is feasible

$x_{\ell 1} + x_{\ell 2} \leq \bar{w}_\ell$ ("=" non-wasteful)

- Competitive eq (x_1^*, x_2^*) with $p = (p_1, p_2)$

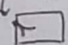
① No firm. ② Util max : $x_i^* = x_i(p^*)$, $\forall i$

③ Market clear : $x_1(p^*) + x_2(p^*) = \bar{w}$

- FWT : C.E. → P.O

(Interior : indiff. curves are tangent at x^*

corner : 〃 not tangent, able to be P.O.



- SWT : P.O ⇒ C.E (eq with transfers)

If $\exists \succsim$ strong monotone, convex, and continuous

(Note) equilibrium with transfer : when $x_i^* \succsim x_i$, $\forall i$,

→ $p_i^* \cdot x_i^* \leq p^* \cdot w_i + T_i$, and $\sum_i T_i = 0$

⊙ C.E (general ver.)

- $\ell = 1, \dots, L$
- $i = 1, \dots, I$

$\succsim$ LNS $\exists$ some $x_i' \in X_i$: $\|x_i - x_i'\| < \varepsilon$ ⇒ $x_i' \succ x_i$

convex

$E = \left(\underbrace{(Y_j)_{j=1}^{J}}_{\text{Firm}} ; \underbrace{(X_i, \succsim_i, \overset{\text{endowment}}{w_i}, \theta_{i1}, \dots, \theta_{iJ})_{i=1}^{I}}_{\text{Consumer}} \right)$

- price equilibrium with transfer " $\exists$ an assignment of W s.t "

① $p^* \cdot y_j^* \geq p^* \cdot y_j$, $\forall y_j \in Y_j$

② $x_i^* \succsim x_i$, $\forall x_i \in B_i^*(p^*, W_i)$

and $x_i^* \in B_i(p^* W_i) = \{x_i \in X_i \mid p^* \cdot x_i \leq W_i\}$

③ $\sum_{i=1}^{I} x_i^* = \bar{w} + \sum_{j=1}^{J} y_j^*$ and $\sum_{i=1}^{I} W_i = p^* \bar{w} + \sum_{j=1}^{J} p^* y_j^*$

* CE ⇒ PET (By $W_i = p^* \cdot w_i + \sum_j \theta_{ij} (p^* y_j^*)$

→ Take Σ : $\sum W_i = p^* \bar{w} + \sum_j p^* y_j^*$

(FWT) PET → P.O $\succsim_i$: LNS $\mathcal{T}$

$\exists x_i' \in X_i$: $\|x_i - x_i'\| < \varepsilon$, $x_i' \succ x_i$

(Lem) Under LNS, $x_i \succsim x_i^*$ ⇒ $p x_i \geq W_i$

(SWT) P.O → P.E.T with Assup. 1, 2

★ prove using silly ver.

Under Assumption 1 ; P.O ⇒ silly eq.

Under 〃 2 ; silly eq ⇒ P.E.T

⊙ C.E (general ver.)

- $l=1,\dots,L$
- $i=1,\dots,I$

$\succsim$ LNS $\exists$ some $x_i' \in X_i$: $\|x_i - x_i'\| < \varepsilon \Rightarrow x_i' \succ x_i$

convex

$E = \left((Y_j)_{j=1}^{J} : (X_i, \succsim_i, w_i, \theta_{i1}, \dots, \theta_{iJ})_{i=1}^{I}\right)$ endowment

Firm Consumer

• price equilibrium with transfer "$\exists$ an assignment of W s.t."

① $p^*\cdot y_j^* \geq p^*\cdot y_j,\ \forall y_j \in Y_j$

② $x_i^* \succsim x_i,\ \forall x_i \in B_i^*(p^*, W_i)$

and $x_i^* \in B_i(p^* W_i) = \{x_i \in X_i \mid p^*\cdot x_i \leq W_i\}$

③ $\sum_{i=1}^{I} x_i^* = \bar{\omega} + \sum_{j=1}^{J} y_j^*$ and $\sum_{i=1}^{I} W_i = p^*\cdot\bar{\omega} + \sum_{j=1}^{J} p^*\cdot y_j^*$

★ CE ⇒ PET (By $W_i = p^*\cdot\omega_i + \sum_{j=1}^{J}\theta_{ij}(p^*\cdot y_j^*)$ → Take Σ: $\sum W_i = p^*\cdot\bar{\omega} + \sum_j p^* y_j^*$)

FWT PET → P.O $\succsim_i$: LNS $\exists x_i' \in X_i : \|x_i - x_i'\| < \varepsilon$, $x_i' \succ x_i$

Lem1 Under LNS, $x_i \succsim x_i^* \Rightarrow p x_i \geq W_i$

SWT P.O → P.E.T with Assup. 1, 2

★ prove using silly ver.

Under Assumption 1; P.O ⇒ silly eq.

Under " 2; silly eq ⇒ P.E.T

⊙ silly ver. (quasi price equilibrium w/ transfers)

- $l=1,\dots,L$
- $i=1,\dots,I$ · X_i : convex

$\succsim$ LNS and convex

Note $[p^* x_i \leq W_i \Rightarrow x_i^* \succsim x_i \Leftrightarrow [x_i^* \prec_i x_i \Rightarrow p^* x_i > W_i]$

⇓

$x_i \succsim x_i^* \Rightarrow p^*\cdot x_i \geq W_i$

⇓

Contrapositive : $p^*\cdot x_i < W_i \Rightarrow x_i \not\succsim x_i^*$

⇓

$\exists$ some x_i : $p^* x_i = W_i \Rightarrow x_i \prec x_i^*$

prop1 $\succsim_i$ continuous on Y_i. Assume $x_i \succsim x_i^* \Rightarrow p^* x_i \geq W_i$.

If $\exists\ \tilde{x}_i$: $p\cdot\tilde{x}_i < W_i$, $x_i \succsim x_i^* \Rightarrow p\cdot x_i > W_i$

Corollary If $((x^*, y^*), p^*)$ is q-price eq with $W_i > 0, \forall i$, (strongly positive transfer) $((x^*, y^*), p^*)$ is P.E.T.

• quasi-price eq. with transfer

① $p^*\cdot y_j^* \geq p^*\cdot y_j,\ \forall y_j \in Y_j$

② $x_i \succ x_i^* \Rightarrow p^*\cdot x_i \geq W_i$

③ $\sum_i x_i^* = \sum_i \omega_i + \sum_j y_j^*$ and $\sum_{i=1} W_i = p^*\cdot\bar{\omega} + \sum_j p^* y_j^*$

★ proof of SWT Assume (x^*, y^*) is P.O.

Let $V := \{\sum_i x_i \mid x_i \succ x_i^*, \forall i, x_i \in X_i\} \subseteq \mathbb{R}^L$

$Y := \{\sum_j y_j \mid y_j \in Y_j, \forall j\} \subseteq \mathbb{R}^L$

Step1) $\exists\ p^* \neq 0, r \in \mathbb{R}$: $p^*\cdot z \geq r,\ \forall z \in V$; $r \geq p^*\cdot z,\ \forall z \in \bar{\omega} + Y$

Step2) Take $W_i = p^*\cdot x_i^*$ and show $\sum_i W_i = p^*\cdot\bar{\omega} + \sum_j p^* y_j^*$

Step3) If $\forall i$, $x_i \succsim x_i^* \Rightarrow \sum_i p^*\cdot x_i \geq r$

Step4) $\sum_i W_i = p^*\bar{\omega} + \sum_j p^*\cdot y_j = r$

Step5) Use step 1, 3, 4 ⇒ show ①, ②

(x^*, y^*) P.O ⇒ show ③ → Step5)

⊙ Price equilibrium with transfers (Utilitarian efficiency)

- $l=1,\dots,L$, Y_j is convex
- $i=1,\dots,I$, $X_i = \mathbb{R}_+^L$

u_i strictly increasing $x_i' \geq x_i$, $x_i' \neq x_i \Rightarrow u_i(x_i') > u_i(x_i)$

concave

- $\max \sum_{i=1}^{I} \alpha_i \cdot u_i(x_i)$ $(\alpha = (\alpha_1, \dots, \alpha_I) \in \mathbb{R}_+^I)$

Part1

- $\max \sum_{i=1}^{I} \alpha_i u_i(x_i)$ s.t $\sum x_{li} - \sum y_{lj} = \bar{\omega}_l$: problem α'

$\leq$: " α''

solution for $\alpha' \Leftrightarrow$ P.E.T

($\alpha' \Leftrightarrow \alpha''$ (with $\succsim_i$ st. monotone))

Part2

- $\max \underbrace{\sum_{i=1}^{I}\alpha_i u_i}_{\text{concave}}$ s.t $\underbrace{\sum x_{li} - \sum y_{lj} \leq \bar{\omega}_l}_{\text{convex}}$: problem α''

KT : concave form

$\Rightarrow \mathcal{L} = \sum_i \alpha_i u_i - \sum_l \lambda_l [x_{li} - \sum_j y_{lj}]$

$u_i - \sum_i \lambda_i p^*\cdot x_i \Leftrightarrow \alpha_i u_i - p^*\cdot x_i$

Step1) ① V is convex : $z, z' \in V \Rightarrow \alpha z + (1-\alpha)z' \in V$

② $\bar{\omega} + Y$ is convex

③ $V \cap (\bar{\omega} + Y) = \emptyset$ ((x^*, y^*) P.O, but

Suppose $\exists\ z \in V \cap (\bar{\omega}+Y)$ for contradiction)

$\Rightarrow$ ($\exists\ x$ s.t. $z = \sum x_i$: $x_i \succ x_i^*$: P.O contradiction

$\exists\ y$ s.t. $z = \bar{\omega} + \sum_{j=1}^{J} y_j$: feasible)

Step2) Let $W_i = p^*\cdot x_i^*$ → WTS : $\sum_i W_i = p^*\cdot\bar{\omega} + \sum_j p^*\cdot y_j^*$

$\sum_i W_i = \sum_i p^*\cdot x_i^* = \sum_l p_l^*\left(\sum_{i=1}^{I} x_{li}^*\right) = \sum_l p_l^*\left(\bar{\omega} + \sum_j y_{lj}^*\right)$

By feasibility $= p^*\cdot\bar{\omega} + \sum_j p^*\cdot y_j^*$

$\sum_i x_i = \bar{\omega} + \sum_j y_j$

Step3) If $x_i \succsim x_i^*, \forall i \Rightarrow \sum p^* x_i \geq r$

By LNS : $\exists\ x = (x_1^n, \dots, x_I^n)$: $x_i^n \succ x_i$ and $\|x_i^n - x_i\| < \frac{1}{n}$

$\Rightarrow [\sum_i x_i^n \in V \Rightarrow x_i^n \succ x_i \succsim x_i^*] \Rightarrow [p^* \sum_i x_i^n \geq r \Rightarrow p^* \sum_i x_i \geq r]$

Step4) $\sum W_i = p^*\cdot\bar{\omega} + \sum_j p^*\cdot y_j = r$

By feasibility, $\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$ ① By step 3, $\sum_i p^* x_i^* \geq r$ ② By step 1, $\sum_i p^* x_i \leq r$] $\Rightarrow \sum_i p^* x_i = r$.

Step5) By step 1, 4, $p^*\cdot y_j \leq p^*\cdot y_j^*$

If $x_i \succ x_i^*$, $p^*\cdot x_i + p^*\cdot\sum_{k\neq i} x_k \geq r = p^*\cdot\sum_{k=1}^{I} x_k^* \Rightarrow p^*\cdot x_i \geq p^*\cdot x_i^* = W_i$. //