Lecture 5

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Preliminary Review from Last Lecture

Consider environment:

$$\epsilon = (T \cup B, (\succsim)_{i \in T \cup B})$$

And Matching:

$$\mu : (T \cup B) \to (T \cup B)$$

DA Algorithm (T-propsal)

- k proposals $\hat{P}^k : T \to B \cup \{\emptyset\}$
- k tentative accept $\hat{\mu}^k : B \to T \cup \{\emptyset\}$
- $\hat{\mu}^k = \hat{\mu}^{k+1}$

Definition: $\mu^* : (T \cup B) \to (T \cup B)$ is *I-Optimal Stable match* if

1. μ^* is stable
2. $\mu^* \geq_I \mu$ for all stable μ

$\mu^* >_I \mu$ if for all $i \in I$ $\mu^* \geq_I \mu$ and for some $i \in I : \mu^*(i) \succ_i \mu(i)$

Proposition (Thm 2.12 in Roth, Sotomayor?):

Suppose preferences are strict. Then for each $I \in \{T, B\}$ the match $\mu^*_{ID} : (T \cup B) \to (T \cup B)$ is an I-optimal stable match.

Definition: Say b is acheivable for t if there exists some stable match μ with $\mu(b) = t$.

Lemma 1:

In the T-proposal DA algorithm, the following holds for each $k = 1, 2, \ldots$: If $\hat{P}^k(t) = b$ and b is acheivable for t, then $\hat{\mu}^k(b) = t$

Proof:

Suppose the result is not true. Then there exists some k such that:

1. for each $l < k$ and each $t \in T$: if $\hat{P}^l(t) = b$ and b is acheivable for t, then $\hat{\mu}^l(b) = t$
2. There is some $t^* \in T$ such that $\hat{P}^k(t^*) = b^*$, b^* is acheivable for t^* and $\hat{\mu}^k(b^*) \neq t^*$

There is stable matching $\mu : (T \cup B) \to (T \cup B)$ with $\mu(t^*) = b^*$. That is to say, b^* is acheivable for t^*.

Now, $\hat{\mu}^k(b^*) \neq t^* \implies t^{**} = \hat{\mu}^k(b^*)$ and $t^* \neq t^{**}$

Observe $\mu(t^{**}) \neq b^*$

Will show: (t^{**}, b^*) blocks μ, contradicting the fact that μ is stable. Precisely, two points of proof:

First: $t^{**} \succ_{b^*} \mu(b^*)$

To show: $t^{**} \succ_{b^*} \mu(b^*)$ Let:

$$\hat{P}^k(t^*) = \hat{P}^k(t^{**}) = b^*$$

$$\hat{\mu}^k(b^*) = t^{**} \succ_{b^*} t^* = \mu(b^*)$$

Second: $b^* \succ_{t^{**}} \mu(t^{**})$

$b^* \succ_{t^{**}} \mu(t^{**})$

Want to show that for each $l < k$, $\mu(t^{**}) \neq \hat{p}^l(t^{**})$

- if there were an $l \leq k$ such that $\hat{P}^l(t^{**}) = \mu(t^{**}) = b^{**}$, whoever is getting that offer b^{**} is acheivable to t^{**} since μ is stable and $\mu(t^{**}) = b^{**}$.
- By (1) it must be the case that whoever got that offer $\hat{\mu}^l(\hat{\mu}(t^{**}) = b^{**}) = t^{**}$
- Its got to be the case by the DA Algorithm, $\hat{P}^{l+1}(t^{**}) = \mu(t^{**})$
- Then this implies $\hat{P}^k(t^{**}) = \mu(t^{**})$ but this cannot be
- We know on round k, $\hat{P}^k(t^{**}) = b^*$. We know that $\mu(t^*) = b^*$. So, that implies that $\mu(t^{**}) \neq b^*$ and $b^* \neq b^{**}$.

  We know that $\hat{P}^k(t^{**}) \succ_{t^{**}} b$ for $b \neq \{\hat{P}^l(t^{**}) : l = 1, 2, ..., k\}$ by the DA algorithm and strict preferences.

  On round k make proposal to b^*: $b^* = \hat{P}^k(t^{**}) \succ_{t^{**}} \mu(t^{**})$

Alternatively *(because David is still confused by what he recorded from class above)* MIT offers the following proof for the above:

We say that b is achievable for t if there is some stable matching μ with $\mu(t) = b$. For a contradiction, suppose a man (t) is rejected by an achievable woman b at some stage of the deferred acceptance algorithm.

Consider the first step of the algorithm in which a man t is rejected by an achievable woman b. Let μ be a stable matching where $\mu(t) = b$. Then b tentatively accepted some other man t' at this step. So:

(i) $t' \succ_b t$ since this is the first time a man is rejected by an achievable woman.
(ii) $b \succ_{t'} \mu(t')$

By (i) and (ii), (t', b) blocks μ, contradicting the stability of μ. [1]

Proof of Proposition for $I = T$

Fix a stable match $\mu : (T \cup B) \to (T \cup B)$. Required to prove: for each $t \in T$ agent, the $\mu_{TD}(t) \succsim_t \mu(t)$. Fix some $t \in T$ One possibility: if, for each k, $\hat{P}^k(t) \neq \mu(t)$. Does not choose best for whatever reason. In the DA Algorithm (μ_{TD})I make first offer to best, second to second best, so certainly here, $\mu_{TD}(t) \succ_t \mu(t)$

Suppose there is some k such that $\hat{P}^k(t) = \mu(t)$. By Lemma 2, $\hat{\mu}^k(\mu(t)) = t \implies \hat{P}^{k+1}(t) = \hat{P}^k(t)$. Then, $\hat{\mu}^{k+1}(\mu(t)) = t$

Under the DA Algorithm, this implies $\mu_{TD}(t) = \mu(t)$ and certainly $\mu_{TD}(t) \succsim_t \mu(t)$ and we are done. Strict preferences are important for the result.

[1] https://ocw.mit.edu/courses/economics/14-16-strategy-and-information-spring-2016/lecture-slides/MIT14_16S16_Matching.pdf

Indifference

Many properties considered require that all agents have strict preferences. But, what if this isn't so? What if one agent is indifferent? In particular, can a matching be *I-Optimal*?

Example 1

$T = \{t_1, t_2, t_3\}$ and $B = \{b_1, b_2, b_3\}$

$$t_1 : b_2 \sim b_3 \succ b_1 \succ t_1$$
$$t_2 : b_2 \succ b_1 \succ t_2 \succ b_3$$
$$t_3 : b_3 \succ b_1 \succ t_3 \succ b_2$$

$$b_1 : t_1 \succ t_2 \succ t_3 \succ b_1$$
$$b_2 : t_1 \succ t_2 \succ b_2 \succ t_3$$
$$b_3 : t_1 \succ t_3 \succ b_3 \succ t_2$$

If μ is stable and t_2 is matched,
$$\mu(t_2) \in \{b_2, b_1\}$$

If μ is stable and t_3 is matched,
$$\mu(t_3) \in \{b_3, b_1\}$$

Suppose that $\mu(t_2) = b_1$ then $\mu(t_3) = b_3 \implies \mu(t_1) = b_2$. Then, we have stable match 1.

Another possibility: $\mu(t_2) = b_2$. Then $\mu(t_3) = b_3$ and $\mu(t_1) = b_1$. But the (t_1, b_3) form a block, so that is not going to work.

So, if $\mu(t_2) = b_2$, it must be the case that $\mu(t_3) = b_1$ and $\mu(t_1) = b_3$. Then we have stable match 2.

In match 1, t_3 is pretty happy since he gets his best match. In match 2, t_2 gets his best. And, b_2 prefers match 1, but b_3 prefers match 2.

This can only happen because t_1 is indifferent.

So, there are no *I-Optimal* stable matchings when one agent is indifferent.

Theorem (Knuth)

Suppose preferences are strict. If μ and μ' are stable matches, then $\mu >_T \mu'$ if and only if $\mu' >_B \mu$.

Definition Let $\mu' : (T \cup B) \to (T \cup B)$ is an *I-Pessimal* stable match is:

1. μ' is stable
2. And $\mu \geq_I \mu'$ for all other stable matches μ.

Corollary

Suppose preferences are strict. Then μ_{ID} is J-pessimal.