* Transferable utility

- Think of the case that utility is generated by matching.

- Change to the environment

  - $\mathcal{E}^T = (T, B; u, v)$

  - $\begin{cases} u : T \times (B \cup \{\emptyset\}) \longrightarrow \mathbb{R} & \text{[utility of all T agents]} \\ v : (T \cup \{\emptyset\} \times B) \longrightarrow \mathbb{R} & \text{[utility of all B agents]} \end{cases}$

  - $(t_i, b_j) \rightarrow \begin{cases} u(t_i, b_j) & \text{[utility of } t_i \text{]} \\ v(t_i, b_j) & \text{[utility of } b_j \text{]} \end{cases}$

    * Unmatched case utility $\begin{cases} (t_i, \emptyset) \rightarrow u(t_i, \emptyset) : & \text{[utility of } t_i \text{ when unmatched]} \\ (\emptyset, b_j) \rightarrow v(\emptyset, b_j) : & \text{[utility of } b_j \text{ when unmatched]} \end{cases}$

  - $W : T \times B \longrightarrow \mathbb{R}$ [Welfare when matching]

    $W(t,b) = u(t,b) + v(t,b)$: Welfare of the match (t,b)

    → Consider utility transfer between t and b

    $\overline{w}(t,b)$ represent t's share of $W(t,b)$

    $\underline{w}(t,b)$ represent b's share of $W(t,b)$

    Thus, $W(t,b) = \overline{w}(t,b) + \underline{w}(t,b)$

    ⇓

    $\begin{cases} u - \overline{w}(t,b) : & \text{represent } t\text{'s transfer} \\ v - \underline{w}(t,b) : & \text{represent } b\text{'s transfer.} \end{cases}$

Example)

$T = \{t_1, t_2\}$ $B = \{b_1, b_2\}$

$u(t_i, \emptyset) = V(\emptyset, b_j) = 0$ for $i=1,2$ and $j=1,2$ (Unmatched ⇒ $u, v = 0$)

$u(t_i, b_j) > 0$, $\forall (t_i, b_j) \in T \times B$

$V(t_i, b_j) > 0$, $\forall (t_i, b_j) \in T \times B$

Q: Can we have a matching μ that is stable with $\mu(t_1) = b_2$ and $\mu(t_2) = b_1$?

→ In this case, we should make a matching Not to be blocked by (t_1, b_1), (t_2, b_2)

→ If stable,

$$\underbrace{\overline{W}(t_1, b_2)}_{\substack{t_1 \text{ is getting} \\ \text{in his current} \\ \text{match}}} + \underbrace{\underline{W}(t_2, b_1)}_{\substack{b_1 \text{ is getting} \\ \text{in his current} \\ \text{match}}} \geq \underbrace{W(t_1, b_1)}_{\text{Welfare produced by } (t_1, b_1)} \quad \cdots \circledast$$

Note: If NOT $\circledast$; $W(t_1, b_1) > \overline{W}(t_1, b_2) + \underline{W}(t_2, b_1)$,

then we could construct transfers

$\left.\begin{matrix} \overline{W}(t_1, b_1) \\ \underline{W}(t_1, b_1) \end{matrix}\right)$ with $\begin{matrix} \overline{W}(t_1, b_1) > \overline{W}(t_1, b_2) \\ \overline{W}(t_1, b_1) > \overline{W}(t_2, b_1) \end{matrix}$

and so, they would block μ.

→ Simarly, if stable ; $\overline{W}(t_2, b_1) + \underline{W}(t_1, b_2) \geq W(t_2, b_2) \quad \cdots \circledast\circledast$

From $\circledast$, $\overline{W}(t_1, b_2) + \underline{W}(t_2, b_1) \geq W(t_1, b_1)$

$$\downarrow \qquad \downarrow$$

$$W(t_1, b_2) - \underline{W}(t_1, b_2) + W(t_2, b_1) - \overline{W}(t_2, b_1) \geq W(t_1, b_1) \quad \cdots \circledast'$$

Adding up $\circledast'$ and $\circledast\circledast$; $\underbrace{W(t_1, b_2) + W(t_2, b_1)}_{\text{welfare of match } \mu} \geq \underbrace{W(t_1, b_1) + W(t_2, b_2)}_{\substack{\text{Welfare that would be produced} \\ \text{if matched } t_1 \to b_1,\ t_2 \to b_2}}$

Conclusion

If μ is stable, then μ is going to maximize the sum of Welfare. (relative to any other matches)

(Def) Given an environment $\mathcal{E}^{TU} = (T, B, u, v)$,

define transfer function $(\overline{w}, \underline{w})$ with $\left(\begin{array}{l}\overline{w} : T \times (B \cup \{\emptyset\}) \longrightarrow \mathbb{R} \\ \underline{w} : T \cup \{\emptyset\} \times B \longrightarrow \mathbb{R}\end{array}\right)$ such that

(1) for each $(t, b) \in T \times B$, $\overline{w}(t,b) + \underline{w}(t,b) = u(t,b) + v(t,b)$

(2) for each $t \in T$, $\underline{\overline{w}(t,\emptyset) = u(t,\emptyset)}$

(3) for each $b \in B$, $\underline{\underline{w}(\emptyset,b) = v(\emptyset,b)}$

(Def) Given $\mathcal{E}^{TU}$ and $(\overline{w}, \underline{w})$ transfer functions,

this defines an NTU environment $\mathcal{E}^{NTU} = (T ; B ; (\succsim_i)_{i \in T \cup B})$

- $b \succsim_t b'$ iff $\overline{w}(t,b) \geqq \overline{w}(t,b')$
- $b \succsim_t t$ iff $\overline{w}(t,b) \geqq \overline{w}(t,\emptyset)$, etc

(Def) $\underline{\mu \text{ is stable relative to } (\mathcal{E}^{TU} : \overline{w}, \underline{w})}$

if $\underline{\text{it is stable in the NTU environment induced by } (\mathcal{E}^{TU} ; (\overline{w}, \underline{w}))}$.

Welfare of a match μ

$$M[\mu] = \{ i \in T \cup B : \mu(i) \neq i \}$$

$$W[\mu] = \sum_{t \in T \cap M[\mu]} u(t, \mu(t)) + \sum_{b \in B \cap M[\mu]} v(\mu(b), b) + \sum_{t \in T \setminus M[\mu]} u(t,\emptyset) + \sum_{b \in B \setminus M[\mu]} v(\emptyset, b)$$

$\Updownarrow$

$$W[\mu] = \sum_{t \in T \cap M[\mu]} \left[u(t, \mu(t)) + v(t, \mu(t)) \right] + \sum_{t \in T \setminus M[\mu]} u(t,\emptyset) + \sum_{b \in B \setminus M[\mu]} v(\emptyset, b)$$

$\Rightarrow$ Check an example for Welfare of a match μ.

Example 2) $T=\{t_1, t_2\}$ $B=\{b_1, b_2\}$

	b_1	b_2
t_1	20 2	18 2
t_2	18 1	5 1

$\left.\begin{array}{l} u(t,\emptyset)=0 \\ v(\emptyset,b)=0 \end{array}\right\} \forall t,b$

NOTE) Assortative prefs
① $\forall t,b$ has the same $\succsim$
② ~~Ranked~~

- If we were in a NTU world, $\left(\begin{array}{l} t_i: b_1 \succsim_{t_i} b_2 \succsim_{t_i} t_i \\ b_j: t_1 \succsim_{b_j} t_2 \succsim_{b_j} b_j \end{array}\right)$ Assortative prefs.

→ Unique stable match : $\left.\begin{array}{l} \mu(t_1)=b_1 \\ \mu(t_2)=b_2 \end{array}\right\}$ Positive Assortative Match (PAM)

- Will want to understand : (1) What matches are stable with transfers?
  (2) What matches maximize welfare?

- Comparing Welfares of PAM and NAM

① PAM : (t_1,b_1), (t_2,b_2)
$u(t_1,b_1)+u(t_2,b_2)+V(t_1,b_1)+V(t_2,b_2)$
$= 20+5+2+1 = 28$

$<$

② NAM : (t_2,b_1), (t_1,b_2)
$u(t_2,b_1)+u(t_1,b_2)+V(t_2,b_1)+V(t_1,b_2)$
$= 18+18+1+2 = 39$

- Utility transfers cases

① transfers case A	② transfers case B
$(\overline{w}(t_1,b_1), \underline{w}(t_1,b_1)) = (11,11)$	$(\overline{w}(t_1,b_1), \underline{w}(t_1,b_1)) = (11,11)$
$(\overline{w}(t_2,b_2), \underline{w}(t_2,b_2)) = (3,3)$	$(\overline{w}(t_2,b_2), \underline{w}(t_2,b_2)) = (3,3)$
$(\overline{w}(t_1,b_2), \underline{w}(t_1,b_2)) = \underline{(16,4)}$	$(\overline{w}(t_1,b_2), \underline{w}(t_1,b_2)) = \underline{(10,10)}$
$(\overline{w}(t_2,b_1), \underline{w}(t_2,b_1)) = (4,15)$	$(\overline{w}(t_2,b_1), \underline{w}(t_2,b_1)) = (9.5,9.5)$
$b_2 \succsim_{t_1} b_1 \succsim_{t_1} t_1$ $\quad$ $t_2 \succsim_{b_1} t_1 \succsim_{b_1} b_1$	$b_1 \succsim_{t_1} b_2 \succsim_{t_1} t_1$ $\quad$ $b_1 \succsim_{t_2} b_2 \succsim_{t_2} t_2$
$b_1 \succsim_{t_2} b_2 \succsim_{t_2} t_2$ $\quad$ $t_1 \succsim_{b_2} t_2 \succsim_{b_2} b_2$	$t_1 \succsim_{b_1} t_2 \succsim_{b_1} b_1$ $\quad$ $t_1 \succsim_{b_2} t_2 \succsim_{b_2} b_2$ ⇒ Induced NTU environment assortative preference.
⇒ $\mu(t_1)=b_2$, $\mu(t_2)=(b_1)$: PAM	⇒ stable relative to $\overline{w}, \underline{w}$: PAM.

⇒☆ key point : A stable match can be changed depending on <u>how to transfer utility</u>.