PS7

Q1. 1. Strict & assortative $\succsim$ : $\mu^{NTU}$ is PAM

2. $w(i,j)$ has decreasing difference

$w(\cdot,\cdot) \longrightarrow$ is twice differentiable.

then $w$ has decreasing difference $_{(D.D)}$ iff $w_{xy} \leq 0$

$w$ has increasing difference $_{(I.D)}$ iff $w_{xy} \geq 0$.

D.D) $\forall x_1 > x_0,\ \forall y_1 > y_0$.

$$W(x_1, y_1) - W(x_1, y_0) \geq W(x_0, y_1) - W(x_0, y_0)$$

$$\downarrow$$

$$\frac{W(x_1, y_1) - W(x_1, y_0)}{y_1 - y_0} \geq \frac{W(x_0, y_1) - W(x_0, y_0)}{y_1 - y_0}$$

$$\Downarrow$$

$$W_y(x_1) \geq W_y(x_0)$$

$$\Downarrow$$

$$\lim_{x_1 \to x_0} \frac{W_y(x_1) - W_y(x_0)}{x_1 - x_0} \geq 0$$

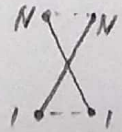
i.e. $W_{yx}(x_0) \geq 0$.

$W_{ij} = 2\alpha(\alpha-1)(i+j)^{\alpha-2} < 0$. Therefore, $\mu^{TU}$ is NAM

3. $W(\mu^{NTU}) = 2\sum_{i=1}^{n}(2i)^{\alpha}$ $\quad W(\mu^{TU}) = 2n(n+1)^{\alpha}$

4. $\dfrac{W(\mu^{NTU})}{W(\mu^{TU})} \in [2^{\alpha-1}, 1]$ TU : non-changed.

But, NTU can be changed




$\Rightarrow \dfrac{(2N)^{\alpha} + 2^{\alpha}}{2\cdot(N+1)^{\alpha}} = \dfrac{2^{\alpha} + \frac{2^{\alpha}}{N^{\alpha}}}{2(1+\frac{1}{N})^{\alpha}} \to 2^{\alpha-1} \quad (N \to \infty)$

Q2.

Lemma 1) If $(x^*, g^*)$ solves $(*)$, then $\exists$ a Pareto optimal allocation $(x^*, m^*, g^*, z^*)$

proof) Choose $z^*$, $m^*$ such that $\forall j$, $z_j^* = c_j(g_j^*)$

and $m^*$ satisfies $\sum_{i=1}^{I} m_i^* = \bar{w}_m - \sum_{j=1}^{J} c_j(g_j^*)$

So defined, the allocation is feasible

Check. $\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} g_j^*$

Since $(x^*, g^*)$ solves $(*)$, $\sum_{i=1}^{I} m_i^* = \bar{w}_m - \sum_{j=1}^{J} c_j(g_j^*)$ by construction.

For any feasible $(x, m, g, z)$, if it pareto dominates $(x^*, m^*, g^*, z^*)$,

then $\sum_{i=1}^{I}(m_i + \phi_i(x_i)) > \sum_{i=1}^{I}(m_i^* + \phi_i(x_i^*))$

Thus,
$$\underline{\bar{w}_m + S(x, g)} = \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(g_j) \quad (\forall j,\ z_j \geq c_j(g_j))$$
$$\geq \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} z_j$$
$$= \sum_{i=1}^{I} m_i + \sum_{i=1}^{I} \phi_i(x_i)$$
$$\underline{>} \sum_{i=1}^{I}(m_i^* + \phi_i(x_i^*)) = \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i^*) - \sum_{j=1}^{J} z_j^*$$
$$= \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i^*) - \sum_{j=1}^{J} c_j(g_j^*)$$
$$= \underline{\bar{w}_m + S(x^*, g^*)}$$

Contradicts that $(x^*, g^*)$ solves $(*)$. //

Q2.

Lemma 2) If $(x^*, m^*, q^*, z^*)$ is pareto optimal.

then it induces a $v \in \mathbb{R}^I$ with $v \in bd(\mathcal{U}(x^*, q^*))$

proof) Suppose $(x^*, m^*, q^*, z^*)$ is pareto optimal and induces $v^* \in \mathbb{R}^I$.

First, claim $\forall j$, $c_j(q_j^*) = z_j^*$. If $\exists j$ s.t $c_j(q_j^*) < z_j^*$.

keep everything else fixed, giving $(z_j^* - c_j(q_j^*))$ to some consumer is a pareto improvement.

Second. $$\sum_{i=1}^{I} v_i^* = \sum_{i=1}^{I} m_i^* + \sum_{i=1}^{I} \phi_i(x_i^*) = \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i^*) - \sum_{j=1}^{J} z_j^*$$

$$= \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i^*) - \sum_{j=1}^{J} c_j(q_j^*) = \bar{w}_m + S(x^*, q^*).$$

For any other feasible allocation $(x, m, q, z)$

$$\sum_{i=1}^{I} v_i^* = \bar{w}_m + S(x^*, q^*) \geq \bar{w}_m + S(x, q) = \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} c_j(q_j)$$

$$\geq \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} z_j$$

$$= \sum_{i=1}^{I} (m_i + \phi_i(x_i)) = \sum_{i=1}^{I} u_i(m_i, x_i) \quad //$$

Q2.

Lemma 3) If $(x^*, m^*, q^*, z^*)$ is pareto optimal, then $(x^*, q^*)$ solves (*)

proof) Suppose $(x^*, m^*, q^*, z^*)$ is pareto optimal, but it does not solve (*)

Step 1) $\exists\ (x, q)$ so that
1. $\bar{w}_m + S(x,q) > \bar{w}_m + S(x^*, q^*)$
2. $\sum_{i=1}^{I} x_i = \sum_{j=1}^{J} q_j$

Construct $m_i = \frac{1}{I}(\bar{w}_m - \sum_{j=1}^{J} z_j)$ and $z_j = C_j(q_j)$

So defined, $\sum_{i=1}^{I} m_i + \sum_{i=1}^{I} \phi_i(x_i) = \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} C_j(q_j)$

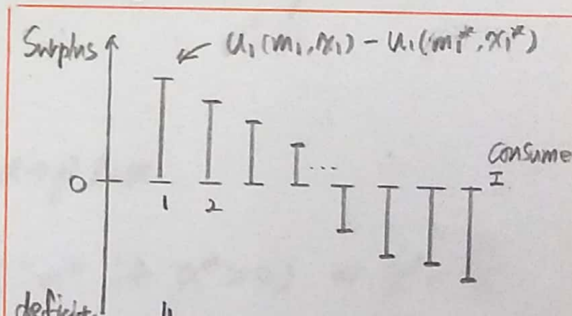
$$= \bar{w}_m + \sum_{i=1}^{I} \phi_i(x_i) - \sum_{j=1}^{J} z_j$$

$$= \bar{w}_m + S(x,q)$$

$$> \bar{w}_m + S(x^*, q^*) = \sum_{i=1}^{I} m_i^* + \sum_{i=1}^{I} \phi_i(x_i^*) \text{ (Lemma 2)}$$

Thus, $\sum_{i=1}^{I} u_i(m_i, x_i) > \sum_{i=1}^{I} u_i(m_i^*, x_i^*)$

It means someone gets better off under $(x, m, q, z)$




$\Downarrow$

$(x^*, m^*, q^*, z^*) \rightarrow (x, m, q, z)$

$\Uparrow$

If this can make up all deficit and $\exists$ a surplus, it will be a new P.O. allocation.

Step 2) Order consumers.

$u_1(m_1,x_1) - u_1(m_1^*,x_1^*) \geq u_2(m_2,x_2) - u_2(m_2^*,x_2^*)$

$\geq \dots \geq u_I(m_I,x_I) - u_I(m_I^*,x_I^*)$

It must be that $u_1(m_1,x_1) - u_1(m_1^*,x_1^*) \geq 0$

There is $k$ s.t.

for $i \leq k$ $\quad u_i(m_i,x_i) - u_i(m_i^*,x_i^*) \geq 0$

for $i < k$ $\quad$ " $\quad$ " $\quad < 0$

The total surplus is higher than the total deficit.

$$\sum_{i=1}^{K} u_i(m_i,x_i) + \sum_{i=K+1}^{I} u_i(m_i,x_i) > \sum_{i=1}^{K} u_i(m_i^*,x_i^*) + \sum_{i=K+1}^{I} u_i(m_i^*,x_i^*)$$

$$\sum_{i=1}^{K} \left(u_i(m_i,x_i) - u_i(m_i^*,x_i^*)\right) > \sum_{i=K+1}^{I} \left(u_i(m_i^*,x_i^*) - u_i(m_i,x_i)\right)$$

$\exists \alpha \in (0,1)$ s.t.

$$\alpha \sum_{i=1}^{K} \left( u_i(m_i, x_i) - u_i(m_i^*, x_i^*) \right) > \sum_{i=K+1}^{I} \left( u_i(m_i^*, x_i^*) - u_i(m_i, x_i) \right)$$

Construct $m'$

For $i \leq k$, $m_i' = m_i - \alpha \left( u_i(m_i, x_i) - u_i(m_i^*, x_i^*) \right)$

For $i < k$, $m_i' = m_i + \left( u_i(m_i^*, x_i^*) - u_i(m_i, x_i) \right)$

Show that $(x, m', q, z)$ pareto dominates $(x^*, m^*, q^*, z^*)$

① $(x, m', q, z)$ is feasible

② Everybody is as good as under $(x^*, m^*, q^*, z^*)$

③ Consumer 1 is strictly better off //

Q3. Suppose $(x^*, m^*, q^*, z^*)$ and $p^*$ forms a competitive equilibrium.

case A) $\sigma < p^*$. $q^* \to \infty$ : NO solution to maximize firm's profit

case B) $\sigma = p^*$, $q^* = x^*$, $m^* = \overline{w}_m - p^* x$

Consumer problem : $\max_{x} \ (\overline{w}_m - p^* x) + \alpha + \beta \ln x$

$\downarrow$ F.O.C. $\frac{\beta}{x^*} \leq p^*$ ("=" if $x^* > 0$) $\Rightarrow x^* = \frac{\beta}{\sigma}$

case C) $\sigma > p^*$, $q^* = 0$, $x^* = 0$ utility is defined at $x^* = 0$. //