* **Money lotteries & Risk aversion** (MWG 6.C)

- Lotteries over final wealth levels in $[0, \infty)$
- A money lottery is a cumulative distribution function $F: [0, \infty) \to [0, 1]$

x gain wealth ⇓ $F(x)$ probability ⇓

↳ "$F(x)$" is the probability that the lottery yields wealth

- Let δ_x denote the lottery that yields x for sure.
- We consider preferences $\succsim$ over the space of lotteries like F.

and assume $\succsim$ is monotone and continuous
"more is better"

Note: $E(F) = \int_0^\infty x \cdot dF(x)$

$\Delta[0, \infty)$
↳ the space of lottery F.

Def Given a preference $\succsim$ and a lottery F,
the certainty equivalent of F is the value of $c(F) \in [0, \infty)$ such that $F \sim \delta_{c(F)}$

ex) w : Jack's initial wealth level

F : the lottery that gives $\begin{cases} w+6 & w/\ p = \frac{1}{3} \\ w+0 & w/\ p = \frac{2}{3} \end{cases}$ ⇒ Jack had been indifferent between the two options

$\Rightarrow c(F) = w+1$

★ Given that $\succsim$ is continuous & monotone,
Certainty equivalents exist and are unique.

Def A preference $\succsim$ is
- risk neutral if $\delta_{EF} \sim F$ for all F
- risk averse if $\delta_{EF} \succsim F$ for all F ⇒ certainty is more important
- risk loving if $\delta_{EF} \precsim F$ for all F.

"Bernoulli util. fn should be concave" $\overset{\text{iff}}{\Longleftrightarrow}$ risk averse

Def

An Expected Utility representation for such a preference $\succsim$ over $\Delta[0,\infty)$ is a pair of functions

$$U : \Delta[0,\infty) \longrightarrow \mathbb{R} \qquad \text{[VNM utility function]}$$

$$u : [0,\infty) \longrightarrow \mathbb{R} \qquad \text{[Bernoulli utility function]}$$

where $U(F) = \int_0^\infty u(x)\, dF(x)$, $\forall F \in \Delta[0,\infty)$.

and $F \succsim F' \iff U(F) \geq U(F')$

- A preference $\succsim$ over $\Delta[0,\infty)$ has an EU representation iff there exists a function $u : [0,\infty) \longrightarrow \mathbb{R}$ s.t.

$$F \succsim G \iff \underbrace{\int_0^\infty u(x)\, dF(x)}_{=U(F)} \geq \underbrace{\int_0^\infty u(x)\, dG(x)}_{=U(G)}$$

> **Note**
> $\succsim$: risk averse
> $\iff$ Bernoulli util. fcn $u(x)$ concave

Remark

"u is risk-neutral" means $U(EF) = U(F)$

$$u\left(\int_0^\infty x\, dF(x)\right) = \int_0^\infty u(x)\, dF(x) \qquad \text{concave \& convex} \to \text{affine.}$$

risk-averse $\quad u\left(\int_0^\infty x\, dF(x)\right) \geq \int_0^\infty u(x)\, dF(x) \qquad u$: concave

risk-loving $\quad u\left(\int_0^\infty x\, dF(x)\right) \leq \int_0^\infty u(x)\, dF(x) \qquad u$: convex

Jensen's inequality $\quad U(EF) \geq EU(F)$ if u is concave

$\qquad U(EF) \leq EU(F)$ if u is convex

Suppose $r(x) = \alpha u(x)$, $\alpha > 0 \Rightarrow r(x), \alpha u(x)$ are the same

- When u is differentiable,

  risk aversion means $u''(x) \leqq 0$, $\forall x \in (0, \infty)$.

- u is "more risk-averse" than v

  means $\underline{c(F, u) \leqq c(F, v)}$, $\forall F$

  → <u>Certainty equivalent of F will be estimated for u less than for v.</u>

  ex1) Suppose $u''(x) \leqq 0$, $\forall x$ and $v''(x) = 0$, $\forall x$.

  Then, u is more risk averse than v.

  ex2) Suppose $v(x) = \alpha u(x) + \beta$, $\alpha > 0$, $\beta \in \mathbb{R}$.

  $v'' = \alpha u''$ → If $\alpha \leq 1$, $v'' \leqq u''$, u is more

  If $\alpha > 1$, $v'' > u''$, v is more

Def The Arrow-Pratt coefficient of <u>Absolute risk aversion</u>

$$\boxed{a(x, u) = -\frac{u''(x)}{u'(x)}}$$

ex) $v(x) = \alpha u(x) + \beta$ → $v' = \alpha u'$, $v'' = \alpha u''$

$$a(x,u) = -\frac{u''}{u'}$$
$$\parallel$$
$$a(x,v) = -\frac{v''}{v'} = -\frac{\alpha u''}{\alpha u'} = -\frac{u''}{u'}$$

prop $\boxed{\text{The following are equivalent}}$

(a) u is more risk averse than v

(b) $a(x, u) \geqq a(x, v)$, $\forall x \in (0, \infty)$

(c) $\exists$ an <u>increasing concave function</u> f such that

$u(x) = f(v(x))$, $\forall x$ i.e., (u is more concave)

④

- More risk-averse $\Rightarrow$ more willing to pay for insurance
- $\begin{pmatrix} \text{riskless asset} \\ \text{risky asset} \end{pmatrix}$: more risk-averse $\rightarrow$ less risky asset. //

#23.3 $\succsim$ over $\mathcal{L}$ has EU representation $\Rightarrow$ $\succsim$ is C & t, continuous and IA.

proof) $\exists\ u_1, u_2, \dots u_N$ s.t $\forall L \in \mathcal{L}$, $U(L) = \sum_{i=1}^{N} P_i u_i$

Lottery shows each probability for each u_i
So, $(P_1, \dots, P_N)$ is the prob. vector.

① $\succsim$ is complete : for $L, L' \in \mathcal{L}$, $U(L)$ and $U(L')$

$\Rightarrow$ either $L \succsim L'$ or $L' \succsim L$, complete.

② $\succsim$ is transitive : for $L, L', L'' \in \mathcal{L}$, let $L \succsim L'$, $L' \succsim L''$

$\Rightarrow$ $u(L) \geq u(L')$, $u(L') \geq u(L'')$ $\Rightarrow$ $L \succsim L''$, transitive

③ $\succsim$ is continuous : for $L, L', L'' \in \mathcal{L}$

(By def from MWG)

$A = \{\alpha \in [0,1] : \alpha L + (1-\alpha) L' \succsim L''\}$ is closed $\Rightarrow$ $\succsim$ is continuous

Suppose $\{\alpha_n\} \subseteq A$, and $\alpha_n \longrightarrow \alpha^*$.

WTS : $\alpha^* \in A$.

for every n, $\alpha_n L + (1-\alpha_n) L' \succsim L''$ $\Rightarrow$ $U(\alpha_n L + (1-\alpha_n) L') \geq U(L'')$

$\downarrow$ By the linearity of u,

$\alpha_n u(L) + (1-\alpha_n) u(L') \geq u(L'')$

So, LHS converges to $\alpha^* u(L) + (1-\alpha^*) u(L') \geq u(L'')$

Then, in the same way, $u(\alpha^* L + (1-\alpha^*) L') \geq u(L'')$ $\Rightarrow$ $\alpha^* \in A$. //

④ $\succsim$ satisfies IA, for $L, L', L'' \in \mathcal{L}$, and $\alpha \in (0,1)$

IA : $L \succsim L'$ iff $\alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$

$L \succsim L'$ $\Leftrightarrow$ $u(L) \geq u(L')$ $\Leftrightarrow$ $\sum_{i=1}^{N} P_i^L u_i \geq \sum_{i=1}^{N} P_i^{L'} u_i$

$\Leftrightarrow$ $\alpha \sum_{i=1}^{N} P_i u_i + (1-\alpha) \sum_{i=1}^{N} P_i'' u_i \geq \alpha \sum_{i=1}^{N} P_i' u_i + (1-\alpha) \sum_{i=1}^{N} P_i'' u_i$

$\Leftrightarrow$ $u(\alpha L + (1-\alpha) L'') \geq u(\alpha L' + (1-\alpha) L'')$ $\Leftrightarrow$ $\alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$

#23.5 If $W_F \succsim W_{F'}$, then $F \succsim_L F'$

- complete and transitive

- Continuity : If $\forall x, y, z$ s.t. $x \succ y \succ z$
(Another version of definition) then $\exists \alpha x + (1-\alpha) z \sim y$

$\Downarrow$

If $F \succ_L F' \succ_L F''$, $\alpha F + (1-\alpha) F'' \sim F'' \nsim F'$. Not continuous.

- IA

$F \succ_L F' \nRightarrow \alpha F + (1-\alpha) F'' \succ_L \alpha F' + (1-\alpha) F''$, Not IA. ||