Existence of Competitive Equilibrium (continue...)

- Pure Exchange Economy
  - $Y_J = \mathbb{R}^L_-$
  - $X_i = \mathbb{R}^L_+$
  - $\left\{ \begin{array}{l} \succsim_i \text{ is strictly convex, continuous, LNS} \\ w_i \quad \overline{w_l} > 0 \text{ for each } l = 1, \dots, L \end{array} \right\}$ Assumption 1.
  - Define $x_i : \mathbb{R}^L \to \mathbb{R}^L_+ \cup \{\emptyset\}$ such that

$$\begin{cases} x_i(p) \text{ is the } \succsim_i\text{-maximizing bundle on } B_i(p) \text{ if there is one.} \\ x_i(p) = \emptyset, \quad \text{otherwise} \end{cases}$$

  - Define an excess demand function: $Z_i : \mathbb{R}^L \to \mathbb{R}^L \cup \{\emptyset\}$ s.t.

$$Z_i(p) = \begin{cases} x_i(p) - w_i & \text{if } x_i(p) \neq \emptyset \\ \emptyset & \text{if } x_i(p) = \emptyset \end{cases}$$

  - Define aggregate excess demand ; $Z : \mathbb{R}^L \to \mathbb{R}^L \cup \{\emptyset\}$ s.t.

$$Z(p) = \begin{cases} \sum_{i=1}^{I} Z_i(p) & \text{if } Z_i(p) \neq \emptyset \text{ for all } i \\ \emptyset & \text{otherwise} \end{cases}$$

$\boxed{\text{Argnement : there exists a competitive equilibrium if } \exists \text{ some } p^* \geqq 0 \text{ s.t } Z(p^*) = 0}$

$Z : \mathbb{R}^L_{++} \to \mathbb{R}^L \cup \{\emptyset\}$

$\boxed{P1}$ $Z(\cdot)$ is continuous

$\boxed{P2}$ $Z(\cdot)$ is homogeneous degree zero

$\boxed{P3}$ $Z(\cdot)$ satisfies Walras' law.

} from Assumption 1

$\boxed{P4}$ $\forall p \in \mathbb{R}^L_{++}$, $\exists$ some $s > 0$ s.t. $(Z_1(p), \dots, Z_L(p)) > (-s, \dots, -s)$ — Assumption 1

$\boxed{P5}$ If $(p^n : n = 1, 2, \dots)$ has $p^n \in \mathbb{R}^L_{++}$, $p^n \to p^0$ where $p^0 \neq (0, \dots, 0)$, but $p^0_l = 0$, — ⊕ Strong monotonicity.

then $\max\{Z_1(p^n), \dots, Z_L(p^n)\} \to \infty$

Thm Kakutani

Let $A \in \mathbb{R}^n$ be compact & convex. Let $f: A \to 2^A$ be upper hemi continuous

such that for each $a \in A$ $f(a)$ is nonempty & convex.

Then $\exists$ $a^* \in A$ s.t $a^* \in f(a^*)$.

Proposition 2. Consider a pure exchange economy that satisfies Assumption 1.

If $Z(\cdot)$ satisfies $P_1 - P_5$, then $\exists$ $p^* \gg 0$ s.t $Z(p^*) = 0$

proof) Set : $\Delta = \{ p \in \mathbb{R}_+^L : \sum_{\ell=1}^{L} p_\ell = 1 \}$

$\text{int}(\Delta) = \{ p \in \Delta : p_\ell > 0 \text{ for all } \ell = 1, \dots, L \}$

$\text{bd}(\Delta) = \Delta \setminus \text{int}(\Delta)$

Correspondence $f : \Delta \to 2^\Delta$

$$f(p) = \begin{cases} \{ q \in \Delta : Z(p) \cdot q \geq Z(p) \cdot q' \text{ for all } q' \in \Delta \} & \text{If } p \in \text{int}(\Delta) \\ \{ q \in \Delta : p \cdot q = 0 \} & \text{if } p \in \text{bd}(\Delta) \end{cases}$$

Step 1) If $\exists$ a fixed point $p^* \in f(p^*)$, then $p^* \gg 0$ and $Z(p^*) = 0$

To show step 1 : Suppose $p^* \in f(p^*)$

(1a) $p^* \in f(p^*) \Rightarrow p^* \in \text{int}(\Delta)$

$p^* \in \text{bd}(\Delta) \Rightarrow p^* \notin f(p^*) = \{ q \in \Delta : p_\ell^* > 0 \to q_\ell = 0 \}$

cannot have $p^* \in f(p^*)$ since $p_\ell^* > 0$ does not imply $p_\ell^* = 0$.

(1b) $p \in \text{int}(\Delta) \Rightarrow f(p) = \{ q \in \Delta : q_\ell = 0 \text{ if } Z_\ell(p) < \max\{Z_1(p), \dots, Z_L(p)\} \}$

Fix $p \in \text{int}(\Delta)$ and suppose that $Z_\ell(p) < \max\{Z_1(p), \dots, Z_L(p)\}$.

Let $q \in (q_1, \dots, q_L) \in \Delta$ s.t $q_\ell > 0$

Constant $q' \in \Delta$ s.t. $q'_k = q_k + q_\ell$ for some $k \neq \ell$ s.t $z_k(p) > z_\ell(p)$

$\uparrow$ price

$q'_\ell = 0$

$\uparrow$ from $\max\{z_1(p), \dots, z_L(p)\}$

$q'_h = q_h$ for all $h \neq k, \ell$

$z(p) \cdot q' > z(p) \cdot q \Rightarrow q \notin f(p)$

(1c) Fix $p^* \in f(p^*)$

$p^*_\ell > 0$ for all ℓ (1a)

$p^* \in f(p^*) \Rightarrow z_1(p^*) = z_2(p^*) = \dots = z_L(p^*) = c$

If $c \neq 0$, $p^* \cdot z(p^*) = p^* \cdot c = \sum_{\ell=1}^{L} p^*_\ell \cdot c \neq 0$ ($\because p^*_\ell > 0$ for all ℓ)

but, contradiction $\boxed{P3}$ Therefore, $c = 0 \Rightarrow z(p^*) = 0$.

Step 1 Apply kakutani

(2a) Δ is compact and convex ($\Delta = \{p \in \mathbb{R}^L_+ : \sum_{\ell=1}^{L} p_\ell = 1\}$)

(2b) $f : \Delta \longrightarrow 2^\Delta$

$\text{Graph}(f) = \{(p, q) : q \in f(p)\}$

$\boxed{\text{Closed thm : If Graph}(f) \text{ is closed, } f \text{ is uhc.}}$

WTS : Graph(f) is closed Fix a sequence (p^n, q^n) s.t

(1) for each n, $q^n \in f(p^n) \leftarrow (p^n, q^n) \in \text{Graph}(f)$

(2) $\lim_{n \to \infty} (p^n, q^n) = (p, q)$

to show : $(p, q) \in \text{Graph}(f)$ is equivalently $q \in f(p)$

Case A) : $p \in \text{int}(\Delta)$

$\Rightarrow \exists N$ s.t. $\forall n \geq N$, $p^n \in \text{int}(\Delta)$

④

$\Rightarrow Z(p^n) \cdot g^n \geqq Z(p^n) \cdot g'$ for all $g' \in \Delta$ $\leftarrow$ by $g^n \in f(p^n)$

for all $n \geqq N$

$\Rightarrow Z(p^n) \cdot (g^n - g') \geqq 0$ for all $g' \in \Delta$

$\lim_{n\to\infty} Z(p^n) \cdot (g^n - g') = Z(p) \cdot (g - g') \geqq 0$ for all g' $\Rightarrow g \in f(p)$

$\uparrow$

Z is continuous by $\boxed{P1}$

$Z : \mathbb{R}^L_{++} \to \mathbb{R}^L \cup \{\emptyset\}$.

case B) $p \in bd(\Delta)$

$f(p) = \{ g \in \Delta : p \cdot g = 0 \}$ if $p \in bd(\Delta)$

$= \{ g \in \Delta : p_\ell > 0 \Rightarrow g_\ell = 0 \}$.

to show : $p_\ell > 0 \Rightarrow g_\ell = 0$.

- Fix $p_\ell > 0$
- If there exists a subsequence of p^n contained in $bd(\Delta)$, then we are done.
  - for n large enough, $p_\ell^n > 0 \Rightarrow g_\ell^n = 0 \Rightarrow \lim_{n\to\infty} g_\ell^n = 0 = g_\ell$
- If No subsequences is contained in $bd(\Delta)$,

  $\exists$ a subsequence contained in $int(\Delta)$: $(p^n : n = 1, 2, \ldots)$

  $\uparrow$

  Take subsequence to be the full sequence

  $\Rightarrow p_\ell^n > 0$ since interior

  $\Rightarrow \exists\, \varepsilon > 0$ and $\bar{N}$ s.t. $\forall n \geqq \bar{N}$, $p_\ell^n > \varepsilon$.

  We focus on subsequence consists of $n \geqq \bar{N}$

  By P5 : $\max \{ Z_1(p^n), \ldots, Z_L(p^n) \} \to \infty$

  By P4 : for each $n \geqq \bar{N}$, there exists S_n : $(-Z_1(p^n), \ldots, -Z_L(p^n) < S_n, \ldots, S_n))$

$$\underline{\varepsilon z_\ell(p^n)} \le \underset{\substack{\uparrow \\ \text{By } p_\ell^n > \varepsilon}}{p_\ell^n z_\ell(p^n)} \overset{\boxed{P3}\ \text{Walras' law } (\because \sum_{\ell=1}^{L} p_\ell^n z_\ell(p^n) = 0)}{=} \underbrace{-\sum_{k\neq\ell} p_k^n z_k(p^n) < S_n \sum_{k\neq\ell} p_k^n}_{\boxed{P4}\ -z_k(p^n) < S_n} \underset{\substack{\uparrow \\ \sum_{k\neq\ell} p_k^n < 1 \\ (\because p_\ell^n > 0)}}{<} \underline{S_n}$$

$\Rightarrow \underline{z_\ell(p^n) < \frac{S_n}{\varepsilon}}$ for all $n \geq \bar{N}$.

($z_\ell(p^n)$ is bounded).

But, $\max\{z_1(p^n), \dots, z_L(p^n)\} \to \infty$

$\Rightarrow \exists \hat{N}$ s.t. $\forall n \geq \hat{N}$, $\underline{\max\{z_1(p^n), \dots, z_L(p^n)\} > z_\ell(p^n)}$

(Look at (16))

$\Rightarrow z_\ell(p^n)$ is bounded. Then $\underbrace{\lim_{n\to\infty} z_\ell(p^n)}_{= z_\ell(p)} < \infty$. $z_\ell(\cdot)$ continuous

But : $\max\{z_1(p), \dots, z_L(p)\} = \lim_{n\to\infty} \max\{z_1(p^n), \dots, z_L(p^n)\} = \infty$

Thus, apply 16 : $q_\ell = 0$.

(2c) f is non-empty

* $p \in \text{int}(\Delta)$: WTS : $f(p) \neq \emptyset$

- $g: \Delta \to \mathbb{R}$ s.t. $g(q) = z(p) \cdot q$ $\Rightarrow$ g is continuous (by P1)
- $f(p) \neq \emptyset$ if $g(\cdot)$ obtains a maximum on Δ.

$\Rightarrow$ g is continuous on compact set

* $p \in \text{bd}(\Delta)$: $\exists \ell$ s.t. $p_\ell = 0$

Take q s.t. $q_\ell = 1$ and $q_k = 0$ for $k \neq \ell$

$\Rightarrow p \cdot q = 0$ $\Rightarrow$ $f(p) \ni q$. //

(2-d) f is convex valued : $f(p)$ is convex

- $p \in int(\Delta)$ $\quad q', q'' \in f(p) \Rightarrow q' \cdot z(p) = q'' \cdot z(p) \geq q \cdot z(p)$ for all $q \in \Delta$

Take $\alpha \in [0,1]$ and

$$\underbrace{(\alpha q' + (1-\alpha) q'')}_{\in f(p)} \cdot z(p) = q' \cdot z(p) \geq q \cdot z(p) \text{ for all } q.$$

- $p \in bd(\Delta)$: $q', q'' \in f(p)$

$\alpha q' + (1-\alpha) q'' \in f(p) \Rightarrow$ if whenever $p_\ell > 0$, $\alpha q_\ell' + (1-\alpha) q_\ell'' = 0$

but when $p_\ell > 0$, $q_\ell' = q_\ell'' = 0$. //

P1 P2 P3 from assumption 1.

&
P4

$\boxed{P_5} \Rightarrow$ St. monotone ⊕ assumption 2.