Supplementary notes for Econ 501A

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About these notes

These are supplementary notes for Econ 501A, Fall 2018. Our main reference for this course is the textbook by Mas-Colell, Whinston and Green (1995), *Microeconomic Theory*, which I refer to as MWG. Before reading a section in these notes, I suggest you first read the corresponding sections of MWG — I have tried to indicate that correspondence throughout. For some sections, these notes will add little beyond MWG. For some other sections, these notes contain an alternative presentation or additional commentary related to the material in MWG. Most of the homework problems that I will assign are contained in these notes. At this point, I suggest you read the preface to MWG, that is pages xiii-xvii. In 501A, we will cover approximately the material in MWG Part I.

I have used other textbooks when teaching this course in the past. I do not require or recommend that you consult these other books, but in some places I have indicated the corresponding sections for your information. Kreps (2012), *Microeconomic Foundations I* is an excellent, affordable, recent textbook. However, Kreps is more theoretically-minded and mathematically rigorous than MWG, and I feel that MWG is at a more suitable level for 501A. Kreps writes that the target audience for his book "consists of first-year graduate students who are taking the standard 'theory sequence' and would like to go more deeply into a selection of foundational issues, as well as students who, having taken a first-year graduate course out of one of the standard textbooks, would like a deeper dive." A second interesting – and free – resource is Rubinstein (2012/2015/2016), "Lecture Notes in Microeconomic Theory," which he has generously made available for free download.[1] In 501A, we will approximately cover the material in Rubinstein's book, except for his "Lecture 9. Social Choice," a topic which may be covered in Econ 501B.

I expect that these notes do contain typos and may contain substantive errors. Please let me know if you find such. (I am thankful to previous students who helped me correct errors and gaps in these notes, in particular Wendan Zhang.)

About Economics 501A

Kreps (p1)

"Most people, when they think about microeconomics, think first about the slogan supply equals demand and its picture... with a rising supply function intersecting a falling demand function, determining an equilibrium price and quantity.

But before getting to this picture and the concept of an equilibrium, the picture's constituent pieces, the demand and supply functions, are needed. Those functions arise from choices, choices by firms and by individual consumers. Hence, microeconomic theory begins with choices. Indeed, the theory not only begins with choices; it remains focused on them for a very long time. Most of this volume concerns modeling the choices of consumers, with some attention

[1] It may downloaded at `arielrubinstein.tau.ac.il/books.html` after a brief registration process. The book was last published in 2012 but the pdf I consulted when writing these notes states on its second page, "This is a revised version of the book, last updated January 11th, 2015." As of August 2016, there is a newer version updated January 1, 2016.

paid to the choices of profit-maximizing firms; only toward the end do we seriously worry about equilibrium."

Econ 501A presents the basic economic theory of choice. (Equilibrium is covered in 501B.)

Econ 501A is a course in economic theory. Some of you may do research in economic theory, but perhaps more of you will do research in applied economics. My intent is that 501A is a course in economic theory for applied economists. That said, you will mostly have to wait until your second-year coursework to see applications.

An anecdote about models, and maps: View Google Maps online, or a standard map of the world. Here are three questions:

1. Does Gabon share a border with Nigeria?[2]
2. How tall is the highest peak in Gabon?[3]
3. Which is geographically bigger, Gabon or Iceland?[4]

Similarly, economic models may offer accurate answers to some types of questions, no answers to other types of questions, and misleading answers to other types of questions. As the prominent statistician George Box said, "The most that can be expected from any model is that it can supply a useful approximation to reality: All models are wrong; some models are useful." As others have emphasized, some models are useful for particular purposes.

[2] No, Cameroon is in between the two.

[3] 946m, but of course the map does not indicate that.

[4] Gabon is 2.6 times bigger than Iceland. (See mapfight.appspot.com, or the thetruesize.com. For further explanation, see "Why all world maps are wrong" by Vox, on YouTube.)

Part I
Utility, preference and choice

Our main reference for this part is MWG Chapter 1. (Similar material is covered in Kreps' Chapter 1 and Rubinstein's Lectures 1, 2 and 3.) At this point, I suggest you read MWG 1.A-B.

In this part, we consider the problem of an individual decision maker choosing among sets of alternatives. In the next part, which may be more familiar, we assume that those alternatives are bundles of goods, but in this part we do not make such assumptions.

X is the set of all possible alternatives, sometimes called the *grand set of alternatives*. For example, in the next part we will assume $X = \mathbb{R}^k_+$, where k is the number of goods and the $+$ indicates that one cannot consume a negative quantity of any good.

$B \subset X$ is the subset from which the individual is choosing. MWG call this a *budget set*. I often say *feasible set*. For example in the next part we will consider budget sets $B(p, w) = \{x \in \mathbb{R}^k_+ : px \leq w\}$, where $p \in \mathbb{R}^k_{++}$ is a price vector, $w \in \mathbb{R}_+$ is an amount of wealth, and $B(p, w)$ is the set of product bundles that are affordable given prices p and wealth w.

In this part we study three approaches to modeling the problem of choosing one or more elements from feasible sets B.

1 Utility maximization

The main topic of this section is utility functions. MWG discuss that briefly in 1.B after introducing preferences. I present utility first, in this section, then preference, in the next section. Please read MWG 1.B, "Preference Relations" before reading this section.

Suppose that there exists some function $u : X \to \mathbb{R}$ and the individual chooses from a set B so as to maximize u. We call u a *utility function*. (More generally, we could allow the utility function to take extended-real values, $u : X \to \mathbb{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$.) The individual's choice problem is

$$\max_{x \in B} u(x).$$

One example is where the individual is a profit-maximizing firm, X is a set of alternative business decisions, and $u(x)$ is the profit associated with x.

A second example, where the individual is a consumer rather than a firm, involves a neurobiological story which is not exactly true, but perhaps instructive. Dopamine is a neurotransmitter that is associated with pleasure. It is a molecule and the quantity of it active in the brain could in principle be objectively measured. Suppose that with each alternative there is an associated quantity of dopamine. Suppose that an individual's makes choices so as to maximize the resulting quantity of dopamine. If I understand correctly, Bentham argued similarly that people make choices so as to maximize the resulting quantity of "hedons." Such viewpoints have fallen out of favor.

This is a constrained optimization (a/k/a nonlinear programming) problem. At this point we have put little structure on the objective u and the feasible set B, but later we will, and we will then apply the tools of constrained optimization from Econ 519.

There are two elements to the solution. The first is the maximized value. Let $\mathcal{P}(X)$ denotes the powerset of X, that is the set of subsets of X, so $B \subset X \Leftrightarrow B \in \mathcal{P}(X)$. The maximized value is given by the following function, $u^* : \mathcal{P}(X) \backslash \emptyset \to \mathbb{R}$,

$$u^*(B) = \sup_{x \in B} u(x).$$

I have written sup instead of max, because the latter may not exist. (For example, if $B = [0, 1)$ and $u(x) = x$). Recall that when the max does exist, it equals the sup. Later we will make assumptions on u and B that assure the max exists. A simple, sufficient assumption on just B is that B is finite.

The second part of the solution is the set of maximizers, given by $C_u : \mathcal{P}(X) \backslash \emptyset \to \mathcal{P}(X) \backslash \emptyset$,

$$\begin{aligned} C_u(B) &= \arg\max_{x \in B} u(x) \\ &= \{x^* \in B : u(x^*) \geq u(x)\ \forall x \in B\} \\ &= \{x^* \in B : u(x^*) = u^*(B)\} \end{aligned}$$

Our primary object of interest is the set of maximizers, $C_u(B)$, which we may refer to as the *demand set*. These are the elements that the individual would be willing to choose from the feasible set B.

A particularly convenient case is when $C_u(B)$ contains one and only one element. As we discussed regarding u^*, the max may not exist. In that case, case $C_u(B)$ would be empty. Separately, $C_u(B)$ could contain more than one element. We are assured that $C_u(B)$ contains at most one element if "there are no ties," that is if $x \neq y$ then $u(x) \neq u(y)$. That is to say that u is an injective function (a/k/a a one-to-one function). In that case, we can replace the $\geq$ with $>$ as follows, $C_u(B) = \{x^* \in B : u(x^*) > u(x)\ \forall x \in B\}$. This case of no ties is a convenient special case throughout this part.

Problem 1.1. Suppose $X = \{a, b, c\}$, and $u(a) = 1$, $u(b) = 1$, $u(c) = 2$. For each $B \subset X$, compute $C_u(B)$. [This problem is intended to make sure you understand the previous couple of paragraphs. If so, it should only take a couple of minutes. A first question: How many such sets B are there?]

Importantly, while here we are assuming that there exists some function $u : X \to \mathbb{R}$, we are not making any assumptions on what that function is. For example, we are not assuming that u(ice cream) is greater than u(nuclear apocalypse). In that sense, we are assuming little, and one might wonder what can we can conclude from such an abstract model? The following problems establish some conclusions.

Problem 1.2. (a) Prove that $u^*(B)$ is increasing in B. That is, prove that if $B \subset B'$, then $u^*(B) \leq u^*(B')$. In words, increasing the set of feasible alternatives never hurts the individual and sometimes helps the individual.

(b) Prove that if $y \in B$ but $y \notin C_u(B)$, then $u^*(B) = u^*(B \backslash y)$, where $B \backslash y$ denotes the set consisting of all of the elements of B except for y. In words, unchosen alternatives do not affect welfare.

Problem 1.3. Suppose $X \supset \{a, b, c\}$.

(a) Of course, $C_u(\{a, b\})$ is a subset of $\{a, b\}$. How many subsets of $\{a, b\}$ are there? How many values might $C_u(\{a, b\})$ take, and what are they?

(b) Suppose that $C_u(\{a, b\}) = b$. What can we conclude about $u(a)$ versus $u(b)$? Suppose further that $C_u(\{b, c\}) = c$. What can we conclude about $C_u(\{a, c\})$?

(c) Suppose $a \in B$, but $a \notin C_u(B)$. What can we conclude about $C_u(B \backslash a)$ as compared to $C_u(B)$?

In the case of the profit maximizing firm, it is easy to see that we switched from measuring profit in dollars to measuring profit in cents, the firm's choices would not change. That is if $v(x) = 100u(x)$, $\forall x \in X$, then $C_u = C_v$. More generally any strictly increasing transformation of u leaves C_u unchanged:

Proposition 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. Define a function $v : X \to \mathbb{R}$ so that $v(x) = f(u(x))$. (Sometimes written $v = f \circ u$.*

Then $C_u(B) = C_v(B)$.

Of course $u^*(B)$ generally will not equal $v^*(B)$, but instead $v^*(B) = f(u^*(B))$. The function v is called a strictly increasing transformation of u.

Problem 1.4. (a) Prove the previous proposition. [If you are uncertain of how to proceed, I suggest you begin by writing down five definitions, that of $C_u(B)$, $C_v(B)$, the definition of equality as regards the two previous sets, v, and lastly the definition of strictly increasing as regards the function f. In general, when asked to prove something, a reasonable way to start is by writing down the definitions of all the assumptions, and the definition of the desired conclusion.]

(b) What if instead f is a weakly increasing function? State and prove an alternative proposition in that case.

2 Preference maximization

Please read MWG 1.B, "Preference Relations" before reading this section.

Typesetting note: Kreps uses the symbol $\succeq$ to denote (weak) preference. MWG use the symbol $\succsim$. I have used the same symbol as Kreps in these notes. (The former symbol results from the Latex command \succeq, the latter from \succsim.) A third related symbol is $\succcurlyeq$, produced by the command \succcurlyeq. As far as I am aware, as generally used by economists, these three symbols are intended to mean the same thing.

2.1 From a utility function to a preference

Notice that to compute C_u it suffices to know whether or not $u(x) \geq u(y)$ for each pair x and y in X. One does not have to know the magnitude of the difference $u(x) - u(y)$. (This idea is formalized in the previous proposition regarding strictly increasing transformations of a utility function.) This idea suggests that instead of working with a utility function u, we could work with the binary relation $\succeq_u$ defined on X as follows. For all pairs $x, y \in X$,

$$x \succeq_u y \Leftrightarrow u(x) \geq u(y).$$

Say that $\succeq_u$ is the preference derived from the utility function u. In fact, if one knows the derived preference $\succeq_u$ but not the utility function u, one can still determine the demand set:

$C_u(B) = \{x^* \in B : x^* \succeq_u x \ \forall x \in B\}$.

It follows that if $\succeq_u$ is the same as $\succeq_v$, then $C_u(\cdot) = C_v(\cdot)$.

Problem 2.1. Recall the previous problem where $X = \{a, b, c\}$, and $u(a) = 1$, $u(b) = 1$, $u(c) = 2$. Write down the preference derived from this u. That is for every pair $x, y \in X$, indicate whether $x \succeq y$ or $x \not\succeq y$. [This problem is intended to make sure you understand the previous paragraph. If so, it should only take a couple of minutes. A first question: How many such pairs are there?]

A preference derived from a utility function must satisfy two properties:

Definition 2.1. Let $\succeq$ be a binary relation on X.

(a) $\succeq$ is *complete* if either $x \succeq y$ or $y \succeq x$ (or both), for all $x, y \in X$.

(b) $\succeq$ is *transitive* if $x \succeq y$ and $y \succeq z$ implies $x \succeq z$, for all $x, y, z \in X$.

Proposition 2.1. *For any utility function $u : X \to \mathbb{R}$, the derived preference $\succeq_u$ is complete and transitive.*

MWG speak not about deriving a preference from a utility function, but about whether a preference has a utility representation. A particular utility function u represents a particular preference $\succeq$ if and only if the preference $\succeq_u$ derived from the utility function u is the same as $\succeq$. A preference has a utility representation if there is some utility function that represents it. So the proposition above may be alternatively expressed: if the preference $\succeq$ has a utility representation, then $\succeq$ is complete and transitive. See the proof in MWG.

2.2 From a preference to a utility function

We may consider a preference relation on its own. In fact, the conventional approach is not to assume the existence of a utility function, but instead to assume that the individual has some preference $\succeq$, and from a budget set B the individual chooses those alternatives that are preferred to all other feasible alternatives:

$$C_{\succeq}(B) = \{x^* \in B : x^* \succeq x \ \forall x \in B\}.$$

What assumptions should we make about $\succeq$? Should we consider any binary relation $\succeq$ on X? We saw before that any preference derived from a utility function is complete and transitive. Thus if we assume that an individual chooses according to some preference that is complete and transitive, we are not assuming more than if we instead assumed that she chooses in order to maximize some utility function. MWG, unfortunately, call a preference that is complete and transitive "rational."

Problem 2.2. For each of the following four preferences, answer the following two questions. Is the preference complete? Is it transitive? Prove your answers.

In parts (a), (b) and (c), let $X = \mathbb{R}^2_+$, so an element $x \in X$ is a vector: $x = (x_1, x_2)$, where x_1 and x_2 are both nonnegative reals numbers.

(a) $x \succeq y \Leftrightarrow x_1 + x_2 \geq y_1 + y_2$

(b) $x \succeq y \Leftrightarrow x_1 \geq y_1$ and $x_2 \geq y_2$

(c) $x \succeq y \Leftrightarrow$ either $x_1 > y_1$, or both $x_1 = y_1$ and $x_2 \geq y_2$ [This is called the lexicographic preference.]

(d) Now let $X = \{1, 2, 3\}$. There are three individuals, each with their own complete and transitive preference: For Ann, $1 \succ_a 2 \succ_a 3$. For Bob, $2 \succ_b 3 \succ_b 1$. For Carol, $3 \succ_c 1 \succ_c 2$. Define a new preference $\succeq$ such that $x \succeq y$ if and only if at least two of the three individuals prefer x to y. Is this new preference $\succeq$ complete? Is it transitive? [I believe such an example was introduced by Condorcet.]

It is often convenient for us to work with a utility function instead of a preference. So a central question is whether or not the assumption that an individual chooses according to some complete and transitive preference $\succeq$ implies that there exists some utility function u such that her choice behavior is the same as if she was choosing to maximize u. An equivalent expression of that question: If a preference is complete and transitive, does it have a utility representation? Here are three answers: (1) If X is finite, then yes. (2) If X is countably infinite, then yes. (3) If X is uncountably infinite, as in the next part, then the answer is sometimes but not always. The lexicographic preference mentioned above is an example where there is no utility representation. Here we state, and I ask you to prove (1). The case of (2) is also true, but we skip proving it, which is more difficult. (Kreps and Rubinstein cover it.) We return to case (3) in the next part, where we see that if the preference is not only complete and transitive, but also satisfies a property called continuity, then yes, it must have a utility representation.

Proposition 2.2. *If $\succeq$ is a complete and transitive preference on a finite set X, then $\succeq$ has a utility representation.*

Problem 2.3. Prove the previous proposition. [Here you are asked to prove the existence of a utility representation. When proving existence of something, one approach is to actually construct the desired object, or to prove how it may be constructed. Such a constructive approach is the only way I know to prove this proposition. That is, define a particular utility function $u_{\succeq}$ related to $\succeq$ *and* prove that $u_{\succeq}$ represents $\succeq$ as desired. If you are having trouble getting started, a simpler case to first consider is where there are no ties, i.e., there is no pair of distinct elements $x \neq y$ such that $x \succeq y$ and $y \succeq x$.]

MWG discuss the strict preference relation $\succ$ and the indifference relation $\sim$ that may be derived from a preference relation $\succeq$. (Define $x \succ y$ if $x \succeq y$ and it is not the case that $y \succeq x$. Notice that if $\succeq$ is complete, then $x \succ y$ if and only if it is not the case that $y \succeq x$. Define $x \sim y$ if $x \succeq y$ and $y \succeq x$.) They also state certain properties that the strict preference and indifference relations must have if the corresponding preference relation is complete and transitive. A related question is what properties of the strict preference and indifference relations imply that the corresponding preference relation is complete and transitive. Neither MWG nor we in this course take up that question, but Kreps does.

Proposition 2.3. *Suppose $u : X \to \mathbb{R}$ represents $\succeq$. Then $v : X \to \mathbb{R}$ also represents $\succeq$ if and only if there exists a strictly increasing function $h : u(X) \to \mathbb{R}$ such that $v = h \circ u$.*

3 Choice as primitive

Please read MWG 1.C, "Choice Rules" and 1.D, "The Relationship between Preference Relations and Choice Rules" before reading this section.

3.1 From preference to choice

A *choice structure* has two components. First, a set $\mathscr{B}$ of budget sets, that is nonempty subsets of X, $\mathscr{B} \subset \mathcal{P}(X) \backslash \emptyset$. Second a *choice rule* $C(\cdot)$, that is a set-valued function $C : \mathscr{B} \to \mathcal{P}(X)$ such that $C(B) \subset B$ for all $B \in \mathscr{B}$. (Instead of speaking of a set-valued function we could say that it is a correspondence, and write $C : \mathscr{B} \rightrightarrows X$, but you may not yet be familiar with correspondences, as they have not yet been covered in Econ 519.) We have already seen the choice rule induced by a utility function, C_u, and the choice rule induced by a preference, $C_{\succeq}$. In this section, we consider the properties of $C_{\succeq}$. (We know that C_u has those same properties — why?)

Definition 3.1. The choice structure $\mathscr{B}, C(\cdot)$ satisfies *finite nonemptiness* if for all finite $B \in \mathscr{B}$, $C(B)$ is nonempty.

Proposition 3.1. *(a) If $\succeq$ is complete, and B contains just two elements, then $C_{\succeq}(B)$ is nonempty. Conversely, if $\succeq$ is not complete, then there exists some $B \subset X$ containing just two elements such that $C_{\succeq}(B)$ is empty.*

(b) If $\succeq$ is complete and transitive, $\mathscr{B}, C_{\succeq}$ satisfies finite nonemptiness.

(It is not true that if $\succeq$ is complete but not transitive, then there must exist some set B such that $C_{\succeq}(B)$ is empty. What is true is that if $\succ$ is not transitive, then there exists some B consisting of three elements, such that $C_{\succeq}(B)$ is empty.) It follows that if X is finite, $C_{\succeq}(B)$ is nonempty for all nonempty $B \subset X$.

Problem 3.1. Prove the previous proposition.

Definition 3.2. The choice structure $\mathscr{B}, C(\cdot)$ fails the *weak axiom of revealed preference* (referred to as WA) if there exist a pair of alternatives x, y, and a pair of budget sets B, B' both of which contain x and y, such that $x \in C(B)$, $y \in C(B')$ and $x \notin C(B')$.

It satisfies WA if it does not fail WA.

Equivalently, WA is satisfied if the following holds. If x, y are in B and B', and $x \in C(B)$, $y \in C(B')$, then it must be that $x \in C(B')$.

Proposition 3.2. *If $\succeq$ is transitive, then $C_{\succeq}$ satisfies WA.*

See the proof in MWG. They assume that $\succeq$ is complete as well, but I believe that is unnecessary for this proposition?

To sum up, we have established that if $\succeq$ is complete and transitive, then $C_{\succeq}$ satisfies finite nonemptiness and WA.

3.2 From choice to preference

We have seen the choice rule induced by a preference, and seen that it satisfies two properties: WA and the property that the demand set is nonempty for all finite budget sets. In this section we begin with a choice rule $C(\cdot)$ and investigate whether there exists some complete and transitive preference $\succeq$ such that $C(\cdot) = C_{\succeq}(\cdot)$, in which case MWG say that the preference *rationalizes* the choice rule. (I prefer to say that the complete and transitive preference *generates* the choice rule.) We see that the answer is yes if C satisfies the two aforementioned properties.

Definition 3.3. The *revealed preference relation*: $x \succeq_C y \Leftrightarrow x \in C(B)$ for some set $B \in \mathscr{B}$ containing x and y.

The *revealed strict preference relation*: $x \succ_C y \Leftrightarrow x \in C(B)$ and $y \notin C(B)$ for some set $B \in \mathscr{B}$ containing x and y.

The choice structure $\mathscr{B}, C(\cdot)$ satisfies the *weak axiom of revealed preference* if it is never the case that $x \succeq_C y$ and $y \succ_C x$.

Definition 3.4. A complete and transitive preference $\succeq$ *rationalizes* a choice rule $C(\cdot)$ over $\mathscr{B}$, if $C_{\succeq}(B) = C(B)$ for all $B \in \mathscr{B}$.

A choice rule is *rationalizable* if there exists a complete and transitive preference that rationalizes it.

Kim Border writes regarding this terminology, and I strongly agree, that

> This terminology is unfortunate because the term "rational" is loaded with connotations, and choice functions that are rational in the technical sense may be nearly universally regarded as irrational by reasonable people. There is also the common belief that if people make choices that are influenced by emotions, then they cannot be rational, but emotions are irrelevant to our definition.

In the previous section we established that if a choice rule is rationalizable then it satisfies finite nonemptiness and WA. Here we establish a partial converse.

Proposition 3.3. *Suppose $\mathscr{B}$ contains all duos and trios in X. If $C(\cdot)$ satisfies WA and $C(B)$ is nonempty for all $B \in \mathscr{B}$, then $C(\cdot)$ is rationalizable, and $\succeq_C$ is the unique complete and transitive preference that rationalizes $C(\cdot)$.*

Proof. Claim 1: $\succeq_C$ is complete. Easy because $\mathscr{B}$ contains all pairs by assumption (iii).

Claim 2: $\succeq_C$ is transitive. Suppose $x \succeq_C y$ and $y \succeq_C z$. We want to show that $x \succeq_C z$. Consider the budget set $\{x, y, z\}$, which is contained in $\mathscr{B}$ by assumption (iii). It suffices to show that $x \in C(x, y, z)$. We know the demand set is nonempty, so there are three cases to consider:

Case 1: $x \in C(x, y, z)$, then we are done.

Case 2: $y \in C(x, y, z)$. WA then implies that $x \in C(x, y, z)$ as desired.

Case 3: $z \in C(x, y, z)$. WA then implies that $y \in C(x, y, z)$, which takes us to previous case that we have already dealt with.

Claim 3: $C(B) \subset C_{\succeq_C}(B)$, for all $B \in \mathscr{B}$.

Suppose $x \in C(B)$. Then $x \succeq_C y$ for all $y \in B$. Thus $x \in C_{\succeq_C}(B) = \{z \in B : z \succeq_C y \text{ for all } y \in B\}$.

Claim 4: $C(B) \supset C_{\succeq_C}(B)$, for all $B \in \mathscr{B}$. (That is, if $x \in C_{\succeq_C}(B)$, then $x \in C(B)$.)

We have assumed that $C(B)$ is nonempty, so there exists some $y \in B$ such that $y \in C(B)$. (This is the only step at which we use the assumption of nonemptiness.)

Suppose $x \in C_{\succeq_C}(B)$. Thus $x \succeq_C y$. That is there exists some budget set B' containing x and y such that $x \in C(B')$. WA then implies that $x \in C(B)$ as desired.

The previous two claims imply $C(B) = C_{\succeq}(B)$.

Claim 5: If $\succeq' \neq \succeq_C$, then $C_{\succeq'} \neq C_{\succeq_C}$. Easy because $\mathscr{B}$ contains all duos.

Claims 1 and 2 establish that $\succeq_C$ is complete and transitive. Claims 3 and 4 then establish that $\succeq_C$ rationalizes C. Finally, Claim 5 shows that no other preference could rationalize C. $\square$

Regarding rationalizability, it would be nice to have an if and only if result. We do have the following such result when X is finite. (Kreps establishes a more complex if and only if result allowing that X is infinite.)

Corollary 3.1. *Suppose that X is finite. Consider a choice rule $C(\cdot)$ defined on all nonempty subsets of X. (I.e., a choice structure where $\mathscr{B} = \mathcal{P}(X) \backslash \emptyset$.) There exists a complete and transitive preference that generates the choice rule $C(\cdot)$ if and only if $C(\cdot)$ satisfies the weak axiom of revealed preference and $C(B)$ is nonempty for all nonempty $B \subset X$.*

Problem 3.2. Prove the immediately previous corollary. (Show that it is implied by previous results. In the language of MWG "there exists a a complete and transitive preference that generates the choice rule" would be replaced with "the choice rule is rationalizable".)

Problem 3.3. $X = \{x_1, x_2, ..., x_n\}$ is a finite set of bottles of wine with distinct prices. It is ordered such that $i > j$ implies the price associated with x_i is greater than the price associated with x_j.

(a) From any nonempty set $B \subset X$, Ann chooses the cheapest bottle. (Her choice rule is single-valued.) Is Ann's choice rule rationalizable? If it is rationalizable, prove that by specifying a preference, showing it is complete and transitive, and showing that it generates Ann's choice rule. If it is not rationalizable, present a proof of that which is based on the previous corollary (i.e., show that Carol's choice rule does not satisfy one of the two conditions which must be satisfied for it to be rationalizable).

(b) From any nonempty set $B \subset X$ containing at least two alternative bottles, Bob chooses the second cheapest bottle. (His choice rule is single-valued.) If B contains just one bottle, Bob chooses it. Is Bob's choice rule rationalizable? If it is rationalizable, prove that by specifying a preference, showing it is complete and transitive, and showing that it generates Bob's choice rule. If it is not rationalizable, present a proof of that which is based on the previous corollary.

(c) From any nonempty set $B \subset X$, Carol chooses the same thing as Ann, unless $x_1 \in B$ in which case Carol chooses nothing. Is Carol's choice rule rationalizable? If it is rationalizable, prove that by specifying a preference, showing it is complete and transitive, and showing that it generates Carol's choice rule. If it is not rationalizable, present a proof of that which is based on the previous corollary.

Problem 3.4. MWG 1.D.4, in which you are asked to show that if a choice structure is rationalizable, then it satisfies the "path-invariance" property.

Problem 3.5. Consider a single-valued choice rule, that is a choice rule where $C(B)$ consists of a single element $c(B)$ for each budget set B. Show that the following three conditions are equivalent, i.e., each of these three conditions implies the other two. [Notice that it would be sufficient to show that (i) implies (ii), (ii) implies (iii), and (iii) implies (i).]

(i) The single-valued choice rule $c(\cdot)$ satisfies the weak axiom of revealed preference.

(ii) Given two budget sets B and B' which each contain the pair of distinct elements x and y, if $c(B) = x$ then $c(B') \neq y$.

(iii) Given two budget sets B and B', if B contains x and y, $c(B) = x$, and $c(B') = y$, then $x \notin B'$.

In the following problem you are asked to show that if $\mathscr{B}$ contains all duos, then (a) $\succeq = \succeq' \Leftrightarrow C_{\succeq} = C_{\succeq'}$, and (b) u represents $\succeq \Leftrightarrow C_u = C_{\succeq}$.

Problem 3.6. (a) Suppose that $\mathscr{B}$ contains all duos. Consider two preferences $\succeq$ and $\succeq'$, and the resulting choice rules $C_{\succeq}$ and $C_{\succeq'}$. Consider the following two conditions: (i) for each pair of alternatives $x, y \in X$, $x \succeq y \Leftrightarrow x \succeq' y$, and (ii) for each budget set $B \in \mathscr{B}$, $C_{\succeq}(B) = C_{\succeq'}(B)$. Prove that (i) $\Leftrightarrow$ (ii).

(b) Again suppose that $\mathscr{B}$ contains all duos. Consider a preference $\succeq$, a utility function u, and the resulting choice rules $C_{\succeq}$ and C_u. Prove that u represents $\succeq$ if and only if $C_{\succeq}(B) = C_u(B)$ for all $B \in \mathscr{B}$.

(c) Now relax the assumption that $\mathscr{B}$ contains all duos. Does the claim in part (a) continue to hold? If so, prove it. If not, provide a counter example, and explain whatever parts of the claim continue to hold.

(d) Continue to relax the assumption that $\mathscr{B}$ contains all duos. Does the claim in part (b) continue to hold? If so, prove it. If not, provide a counter example, and explain whatever parts of the claim continue to hold.

[Note to Asaf for 2019: This problem should get broken up, and some parts of it included earlier.]

Conclusions

In the case where the grand set of alternatives X is finite, we have the following nice equivalence results regarding the three models developed in this Part: Utility, preference and choice. A preference $\succeq$ has a utility representation if and only if $\succeq$ is complete and transitive. Letting $\mathscr{B}$ denote the set of all nonempty subsets of X, $\mathcal{P}(X)\backslash\emptyset$, a choice structure $(\mathcal{P}(X)\backslash\emptyset, C(\cdot))$ is rationalizable if and only if C satisfies WA and $C(B)$ is nonempty for all B. Thus, with finite X, the utility maximization model of section 1 is in a sense equivalent to the complete and transitive preference maximization model of section 2, which is in a sense equivalent to the choice model of section 3 satisfying nonemptiness and WA.

Problem 3.7. Recall the previous paragraph, which describes the relationships between the utility, preference and choice models in the case where X is finite. In this problem you are asked to similarly describe the relationships between those three models in the case where X is infinite. Here you need not prove anything, but rather recapitulate previously proven results.

Part II

Consumer choice

In the previous part, we considered preference, utility and choice over abstract sets of alternatives. In this part, we specialize to the problem of a consumer choosing the best bundle(s) from the set of bundles that she can afford. In the language of the previous part, the grand set of alternatives is the positive orthant, $X = \mathbb{R}^k_+$, and the feasible sets considered are budget sets $B(p, w)$ where p is a price vector, w is a wealth level and $B(p, w)$ is the set of nonnegative bundles that are affordable given prices p and wealth w.

Throughout this part let $P = \mathbb{R}^k_{++}$ and $W = \mathbb{R}_+$.

Much of what we will do regards the mathematical fundamentals of the consumer's problem, but we will also investigate the following questions which may be of more applied interest.

1. The "law of demand" asserts that as the price of a good increases, demand for the good will decrease. When does that law hold?

2. As prices rise, consumers are harmed. How can we estimate the magnitude of that harm?

3. If we know a consumer's preference, how do we compute their demand?

4. If we know a consumer's demand, can we determine their preference?

4 The consumer's budget sets (MWG 2.B/C/D)

Please read the following four sections in MWG before reading this section: 2.A "Introduction," 2.B "Commodities," 2.C "The Consumption Set," 2.D "Competitive Budgets."

4.1 A budget set

Alternatives are bundles of k goods, $x \in \mathbb{R}^k$.

The *nonnegativity constraint* is $x \geq 0$, that is $x_i \geq 0$ for each good $i \in \{1, 2, ..., k\}$.

The grand set of alternatives is $X = \mathbb{R}^k_+$.

The *budget constraint* is $px \leq w$, for some price vector $p \in P = \mathbb{R}^k_{++}$ and wealth (a/k/a income) level $w \in W = \mathbb{R}_+$. Regarding the product "px", note that p and x are both $k-$dimensional vectors. Their product is the dot product $px = p \cdot x = \sum_{i=1}^k p_i x_i$. Writing px to denote the dot product (also know as the inner product) is correct if we take p to be a row vector and x a column vector.

Say that a bundle x is *affordable* if it satisfies the budget constraint, and a bundle is *just affordable* if the budget constraint is binding for that bundle.

The *budget set* is the set of affordable bundles, that is the set of points in the positive orthant that satisfy the budget constraint: $B(p, w) = \{x \in \mathbb{R}^k_+ : px \leq w\}$. The *consumer's problem* is the problem of the individual making choices from such budget sets.

The *budget frontier* is the subset of points in the budget set where the budget constraint is binding, that is the set of bundles that are just affordable, $\bar{B}(p, w) = \{x \in \mathbb{R}^k_+ : px = w\}$. The budget frontier is related to the budget hyperplane: $\{x \in \mathbb{R}^k : px = w\}$. The budget frontier is the intersection of the budget hyperplane and the positive orthant.

Proposition 4.1. *For all $p \in \mathbb{R}^k_{++}$ and $w \in \mathbb{R}_+$, the budget set $B(p, w) = \{x \in \mathbb{R}^k_+ : px \leq w\}$ has the following properties. It is (1) nonempty, (2) convex, (3) compact, (4) it is the intersection of $k+1$ closed half spaces*[5].

These are convenient properties for a constrained optimization (a/k/a nonlinear programming) problem, as we will see in a later section.

[5] In $\mathbb{R}^1$, a closed half space is a half-line, either $[x, +\infty)$ or $(-\infty, x]$ for some $x \in \mathbb{R}$. In $\mathbb{R}^2$ it is a line and all of the points on one side of that line. In $\mathbb{R}^3$ it is a plane and all of the points on one side of the plane. Generally a closed half space in $\mathbb{R}^k$ is the set $\{x \in \mathbb{R}^k : px \leq w\}$ for some $p \in \mathbb{R}^k \backslash \{0\}$ and $w \in \mathbb{R}$ (notice we are not restricting the signs of p and w here).

Proof of convexity. It is true that a half-space is convex, and the intersection of convex sets is convex. So (4) implies (2), but here we prove (2) directly.

Suppose $x^0, x^1 \in B(p, w)$. (Here we use the superscripts ^{0} and ^{1} to denote two different bundles, not exponentiation.) We must show that for all $t \in [0, 1]$, $x^t = tx^1 + (1-t)x^0 \in B(p, w)$.

We have $px^0 \leq w$ and $px^1 \leq w$. Thus $px^t \leq tpx^1 + (1-t)px^0 \leq tw + (1-t)w \leq w$. So x^t is affordable.

We have $x_i^0 \geq 0$ and $x_i^1 \geq 0$ for each good i. Thus $x_i^t = tx_i^1 + (1-t)x_i^0 \geq 0$. So x^t is nonnegative. □

Problem 4.1. (a) Prove that the budget set $B(p, w)$ has the properties listed in the previous proposition. (You need not prove convexity which is proved above.)

(b) The "$_{++}$" in the expression $p \in \mathbb{R}^k_{++}$ indicates that the price of each of the k goods is strictly greater than zero. What if instead we assume only that the price of each good is greater than or equal to zero, that is $p \in \mathbb{R}^k_{+}$. Which of the four previously established properties of $B(p, w)$ must continue to hold? Prove your answer.

(c) Suppose we dispose with the nonnegativity constraint, so instead we define $B(p, w) = \{x \in \mathbb{R}^k : px \leq w\}$ for some $p \in \mathbb{R}^k_{++}$ and $w \in \mathbb{R}_+$. Which of the four previously established properties of $B(p, w)$ must continue to hold? Prove your answer.

Problem 4.2. MWG Exercise 2.D.3, where you are asked to investigate convexity of the budget set under alternative assumptions regarding X.

4.2 The budget correspondence

The *budget correspondence,* $B : \mathbb{R}^k_{++} \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^k_+$, is the function which takes as its input a price vector p and a wealth level w and returns as its output the set of affordable bundles. Note that B is a set-valued function, also known as a correspondence. Instead of "$\rightrightarrows \mathbb{R}^k_+$" we could write "$\to \mathcal{P}(\mathbb{R}^k_+)$", where $\mathcal{P}(\mathbb{R}^k_+)$ denotes the set of all subsets of $\mathbb{R}^k_+$, also known as the power set.

Proposition 4.2. *The budget correspondence is:*

(a) homogeneous of degree zero (that is $B(p, w) = B(tp, tw)$ for all $t \in \mathbb{R}_{++}$),
(b) increasing in w,
(c) decreasing in p,
(d) continuous.

Parts (a), (b) and (c) are simple. Part (d) is not, because we do not yet know what it means for a correspondence to be continuous. It is true that if (p, w) changes a little, then, intuitively, $B(p, w)$ changes only a little. We will later formalize that intuition.

Commentary

Here we take $X = \mathbb{R}^k$. That seems general to me. However our choice of $\mathscr{B}$ is extremely restrictive. I look out in the world and often see nonlinear prices. The standard argument that the possibility of resale prevents prices from being nonlinear seems absurd. (I note that wholesale and retail prices are not equal. Perhaps the argument about resale does suggest that wholesale prices should be linear, at least for buyers that are not too large?) Let me offer four arguments to motivate our choice of $\mathscr{B}$.

(1) The problem we study is truly the problem that consumers generally face. As I suggest above, I think that is false. Perhaps it is approximately the problem that consumers sometimes truly face. As I suggest below, I strongly disagree with the assertion that because this model is not completely true, we ought not to study it.

(2) The problem we study is a standard problem studied by almost all previous economists, so you should study it too. I think that is true, but if that were the primary reason to study it, that would be sad for economics.

(3) The problem we study is mathematically convenient, as captured in the previous proposition establishing properties of $B(p, w)$. This is true and I think it is important to be honest and forthright about this.

(4) We are interested in how consumer's choices change as prices and wealth change, and the problem that we study is perhaps the simplest problem where prices and wealth change. I think this is true and an excellent reason to study this problem. In general in exploring

some phenomenon I think it is perfectly valid to begin with a simple model that captures that phenomenon rather than attempting to develop a true model (as if there was such a thing as a true model). We will see that a comprehensive study of this simplified problem is still complex.

5 The consumer's choice rule (MWG 2.E)

Please read MWG 2.E before reading this section. [[Note from current Asaf to future Asaf: Next year maybe cover this section and the next (WARP/LCD) after the folowing three sections (the consumer's problem, existence of u, and properties of the demand set.). That order would be more standard, though different from MWG.]]

As in part I, we could start with a utility function, preference relation or a choice rule. MWG start with a choice rule, which they call the Walrasian demand correspondence. [[That is not quite right. The Walrasian demand correspondence is equivalent to a choice rule if and only if it is HD0.]] (By comparison, Kreps starts with a preference relation, which quickly leads to a utility function, and from there a choice rule. Rubinstein starts with a preference relation and continues to work in terms of the preference relation, rather than a utility function.) My notes for this section are rough and add little beyond MWG 2.E.

In the language of part I, $X = \mathbb{R}^k$ and $\mathscr{B} = \{A \subset X : A = B(p, w) \text{ for some } p \in \mathbb{R}^k_{++} \text{ and } w \in \mathbb{R}_+\}$.

Choice rule $C : \mathscr{B} \rightrightarrows \mathbb{R}^k_+$, where $C(B(p, w)) \subset B(p, w)$ for all p, w. Rather than writing $C(B(p, w))$, I will write $Q(p, w)$. (MWG write $x(p, w)$, but I do not like that notation. $X^*(p, w)$ would be okay with me. The * to indicate optimality and the capital letter to indicate that it may be a set.) I instead write $Q(p, w)$. The letter q for quantity. Uppercase because it may be a set. In the case where the consumer chooses a single bundle, I write $q(p, w)$.

$Q : \mathbb{R}^k_{++} \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^k_+$. $Q(p, w) = C(B(p, w))$.

Two main questions: For fixed p, w, what does the set $Q(p, w)$ look like? As p and w vary, how does $Q(p, w)$ vary?

If the Walrasian demand correspondence is a choice rule, then it is HD0. If we do not want to assume at this point that it is a choice rule, we may directly assume that it is HD0.

Proposition 5.1. *The Walrasian demand correspondence corresponds to a choice rule if and only if it is homogeneous of degree zero in (p, w).*

Proof. In saying that Q corresponds to a choice rule, I mean that there exists a choice rule $C : \mathscr{B} \rightrightarrows \mathbb{R}^k_+$ such that $C(B(p, w)) = Q(p, w)$ for all p, w.

Suppose Q does correspond to a choice rule C. Because the budget correspondence is HD0, Q must also be HD0.

Suppose Q is HD0, let $C(B(p, 1)) = Q(p, 1)$ for each p. Notice every budget set can be written with $w = 1$. □

We have $Q(p, w) \subset B(p, w)$. Recall $\bar{B}(p, w)$ is the budget frontier.

Definition 5.1. Q satisfies *Walras' law* if $Q(p, w) \subset \bar{B}(p, w)$ for all p, w.

That is, Walras' law says the consumer only demands bundles that are just affordable. In this section and the next we treat Walras' law as an assumption, in a later section it will instead be a conclusion.

5.1 Comparative statics

Suppose demand is single-valued, so I will write $q(p, w)$ rather than $Q(p, w)$.

Wealth effect. A good is *normal* if demand is increasing in wealth.

Own-price effect. A *Giffen good* is one where demand for a good is strictly increasing in its own price.

Cross-price effect

Elasticities: Here the cross price effect

$$\varepsilon_{ij}(p, w) = \frac{\partial q_i(p, w)}{\partial p_j} \frac{p_j}{q_i(p, w)}$$

Unit free.

Implication of HD0: Euler formula.

Implications of Walras' law: Differentiate $pq(p, w) = w$ with respect to p, and with respect to w.

Problem 5.1. MWG exercise 2.E.8, where you are asked to show that the elasticity of demand for good i with respect to price j is equal to the derivative of log of demand for good i with respect to log of price j.

6 The Weak Axiom of Revealed Preference and the Law of Compensated Demand (MWG 2.F)

Please read MWG 2.F before reading this section. Throughout this section, MWG assume that the consumer's demand function $q : P \times W \to X$, is single-valued, HD0, and satisfies Walras' law. By P I mean $\mathbb{R}^k_{++}$ and by W I mean $\mathbb{R}_+$. [[Note to Asaf: Either use notation P and W throughout, or do not use it here.]]

MWG cover material that I do not cover here: MWG discuss the case where demand is differentiable, beginning in the third paragraph of p33. I would like you to study that part of the chapter, but I do not cover it here nor in lecture.

I cover material that MWG do not cover: MWG assume that demand is single-valued throughout. The law of compensated demand can be established without that assumption as I present here.

Let us begin with the Law of (Uncompensated) Demand. It asserts that the own price effect is nonpositive; no good is Giffen. I.e., in the differentiable case, $\partial q_i/\partial p_i \leq 0$. In the nondifferentiable case we can express that as follows. Fix some good i. Consider two price vectors p^0, p^1 where $p^1_j = p^0_j$ for all $j \neq i$, and $p^1_i > p^0_i$. I refer to p^0 as the old price vector and p^1 as the new price vector. Notice price of good i has increased, but all other prices are unchanged. Supposing demand is single-valued, the Law asserts that demand for good i has decreased, that is $q_i(p^1, w) \leq q_i(p^0, w)$. The Law is in fact not a law; it may fail to hold. In this section, beginning with WA we establish a related law, the Law of Compensated Demand (LCD) that does hold. One expression of the LCD is as follows. Suppose that p^1 and p^0 are as before, and $w^0 = w$. Let $w^1 = p^1 q(p^0, w^0)$. (This change in wealth is called *Slutsky compensation*.) Then $q_i(p^1, w^1) \leq q_i(p^0, w^0)$.

It would be natural to show that the LCD holds by beginning with a complete and transitive preference and deducing the LCD holds for the derived choice rule. We already know that any complete and transitive preference yields a choice rule that satisfies WA. The LCD may be deduced from WA, without reference to an underlying preference. This is the route MWG take.

Definition 6.1. Suppose that the consumer's demand is single-valued, $Q(p, w) = \{q(p, w)\}$. It satisfies the *weak axiom of revealed preference II* if the following holds for all prices wealth pairs (p^0, w^0) and (p^1, w^1). Let $q^0 = q(p^0, w^0)$ and $q^1 = q(p^1, w^1)$. If $p^1 q^0 \leq w^1$ and $q^0 \neq q^1$, then $p^0 q^1 > w^0$.

Somewhat unfortunately, MWG refer to this equivalent property also as the weak axiom of revealed preference (abbreviated WA or WARP). When I want to distinguish the two, I will refer to the weak axiom from part I as WA-I and the weak axiom here in part II as WA-II. We will see that the two are related.

In words, WA-II says if at the new budget set the old demanded bundle is affordable, and new demanded bundle is distinct, then the new demanded bundle is not affordable at the old budget set.

Proposition 6.1. *If the consumer's demand is single-valued then it satisfies the weak axiom of revealed preference II defined above if and only if it satisfies the weak axiom of revelealed preference from part I.*

Problem 6.1. Prove the immediately previous proposition. (Hint: Recall problem 3.5.)

Definition 6.2. The Law of Compensated Demand (LCD): Let w^0 be a wealth level and p^0, p^1 two price vectors. Suppose $w^1 = p^1 q(p^0, w^0)$. Let q^0 denote the bundled demanded at p^0, w^0 and q^1 at p^1, w^1 .

$$(p^1 - p^0)(q^1 - q^0) \leq 0$$

with strict inequality if $q^1 \neq q^0$.

Consider the LCD where only one price has changed as before: Fix some good i. Suppose $p_j^1 = p_j^0$ for all $j \neq i$, and $p_i^1 > p_i^0$. Let q^0 denote the bundled demanded at p^0, w^0. Let $w^1 = p^1 q^0$. Let q^1 denote the bundle demanded at p^1, w^1. LCD implies $q_i^1 \leq q_i^0$ with strict inequality if $q^1 \neq q^0$.

Proposition 6.2. *Suppose demand is single-valued and satisfies Walras' law. WA-II is equivalent to LCD.*

(MWG include the assumption that demand is HD0, but that seems superfluous. I note that WA-II implies HD0.)

Proof that WA-II implies LCD. If $q^1 = q^0$, holds trivially, so suppose $q^1 \neq q^0$. Note $p^1 q^0 \leq w^1$ so $p^0 q^1 > w^0$ by WA. Also $p^0 q^0 = w^0$ by Walras' law. And $p^1 q^1 = w^1 = p^1 q^0$ by Walras' law and the fact that this is a Slutsky-compensated price change.

$$(p^1 - p^0)(q^1 - q^0) = \underbrace{p^1 q^1}_{=w^1} - \underbrace{p^1 q^0}_{=w^1} + \underbrace{p^0 q^0}_{w^0} - \underbrace{p^0 q^1}_{>w^0} < 0$$

□

(Suppose demand satisfies Walras' law, but need not be single-valued. Let $q^0 \in Q(p^0, w^0)$ and $q^1 \in Q(p^1, w^1)$. If $p^1 q^0 < w^1$, then $p^0 q^1 > w^0$. If not $p^0 q^1 > w^0$, then not $p^1 q^0 < w^1$. I.e., if $p^0 q^1 \leq w^0$, then $p^1 q^0 \geq w^0$.)

Proof that LCD implies WA-II for compensated price changes. Consider two price wealth pairs (p^0, w^0) and (p^1, w^1). Further suppose that this is a compensated price change so $w^1 = p^1 q^0$. The LCD asserts that if $q^1 \neq q^0$, then

$$(p^1 - p^0)(q^1 - q^0) = \underbrace{p^1 q^1}_{=w^1} - \underbrace{p^1 q^0}_{=w^1} + \underbrace{p^0 q^0}_{w^0} - p^0 q^1 < 0$$

That implies $p^0 q^1 > w^0$ as desired. □

It is true, but cumbersome to prove, that WA-II for compensated price changes implies WA-II for all price changes. See the argument in MWG.

The LCD implies that Giffen goods are inferior. Specifically, if there is some price increase of good i that would lead to an increase in demand for good i, then there is some wealth increase that would lead to a decrease in demand for good i. Here we prove that. Suppose there exists w^0, p^0 and p^1, such that $p_j^1 = p_j^0$ for all $j \neq i$, $p_i^1 > p_i^0$, and $q_i(p^1, w^0) > q_i(p^0, w^0)$. Let $w^1 = p^1 q(p^0, w^0)$. Notice $w^1 > w^0$. Then

$$\underbrace{q_i(p^1, w^0) - q_i(p^0, w^0)}_{>0} = \underbrace{q_i(p^1, w^0) - q_i(p^1, w^1)}_{\Rightarrow >0} + \underbrace{q_i(p^1, w^1) - q_i(p^0, w^0)}_{\leq 0 \text{ by LCD}}.$$

(The first inequality is by the Giffen assumption. The third is by the LCD. The second inequality is then concluded.)

6.1 Multivalued demand

The assumption that consumer demand is single-valued seems difficult to justify on the basis of preference maximization. (In a future section we will see that demand is single-valued if the consumer's preference is strictly convex, but the assumption of strict convexity is uncomfortable.) Here we relax the assumption that demand is single-valued. (A reference is Kreps chapter 4, specifically section 4.3.)

Proposition 6.3. *If the demand correspondence $Q(\cdot, \cdot)$ satisfies Walras' law and the weak axiom of revealed preference from part I, then the following property holds.*

Fix price wealth pairs (p, w) and (p', w'). Let q denote a bundle demanded at the former (that is $q \in Q(p, w)$), and q' a bundle demanded at the latter.

If $p'q \leq w'$, then $pq' \geq w$.

We may call that property WA-II'.

Problem 6.2. Prove the previous proposition; WA-I plus Walras' law implies WA-II'.

Extra credit: I am not sure about the converse. That is I have not established whether in this context WA-I is equivalent to WA-II'. For extra credit, prove whether or not they are equivalent. (This should be easy.)

Take the previous LCD, allow multivalued demand, and subtract the statement about strict inequality:

Definition 6.3. The Law of Compensated Demand' (LCD') with multivalued demand: Let w be a wealth level and p, p' two price vectors. Let q be a bundle demanded at p, w, that is $q \in Q(p, w)$. Let $w' = p'q$. Let $q' \in Q(p', w')$. Then,

$$(p' - p)(q' - q) \leq 0.$$

Proposition 6.4. *If the demand correspondence $Q(\cdot, \cdot)$ satisfies Walras' law and the weak axiom of revealed preference from part I, then the LCD' holds.*

Problem 6.3. Prove the previous proposition. I do this by showing that given Walras' law, WA-II' implies LCD'.

Extra credit: Again I am not sure about the converse. That is I have not established whether in this context WA-II' is equivalent to LCD'. For extra credit, prove whether or not they are equivalent. (This may not be easy.)

Notice what the LCD' implies in the case of a price increase for a single good i: Let $p'_i > p_i$ and $p'_j = p_j$ for all $j \neq i$. Let $q \in Q(p, w)$. Let $w' = p'q$. Let $q' \in Q(p', w')$. Then $q'_i \leq q_i$. Notice that we began with a strict inequality, $p'_i > p_i$, and ended with a weak inequality, $q'_i \leq q_i$. That is the best we can do allowing multivalued demand. (We cannot conclude that $q'_i < q_i$ — why? Also, if $p' = p$, it could be that $q'_i > q_i$ — why?)

6.2 LCD with nonlinear prices?

A natural question is whether some analog to the LCD holds with nonlinear prices. Here I discuss this in a two-good case.

For simplicity, suppose the price of good 2 is linear: x_2 units of good 2 cost $p_2 x_2$ for some price $p_2 \in \mathbb{R}_{++}$.

The price of good 1 is nonlinear: x_1 units cost $r_1(x_1)$ where $r_1 : \mathbb{R}_+ \to \mathbb{R}_+$, and $r_1(0) = 0$.

The consumer's budget set is $B(r_1, p_2, w) = \{x \in \mathbb{R}^2_+ : r_1(x_1) + p_2 x_2 \leq w\}$.

Let q be a bundle demanded at r_1, p_2, w.

Consider an increase in the nonlinear price of good 1 to $\tilde{r}_1 : \mathbb{R}_+ \to \mathbb{R}_+$ where $\tilde{r}_1(x_1) \geq r_1(x_1)$ for all $x_1 \in \mathbb{R}_+$.

Consider a Slutsky-compensating increase in wealth: $\tilde{w} = p_1 q_1 + \tilde{r}_2(q_2)$.

Let $\tilde{q}$ be a bundle demanded from $B(\tilde{r}_1, p_2, \tilde{w})$.

One version of LCD would assert that $\tilde{q}_1 \leq q_1$. I conjecture that is false. I conjecture it may be true that if instead of assuming $\tilde{r}_1(x_1) > r_1(x_1)$ for all $x_1 > 0$ we had made the stronger assumption that $\tilde{r}'_1(x_1) \geq r'_1(x_1)$ for all x_1, then we could conclude $\tilde{q}_1 \leq q_1$. (There the $'$ denotes the derivative of r_1.)

Problem 6.4. Extra credit: Prove whether or not the two previous conjectures are true. Perhaps it is useful to first establish whether or not some analog to WA-II' holds here.

7 The consumer's problem

This brief section does not have a direct analog in MWG. Broadly, the material beginning here corresponds to MWG's chapter 3, and the material in this section relates to material in MWG 3.D.

In the previous two sections we considered the consumer's choice rule without directly considering preferences. Here we assume that the consumer has a complete and transitive preferences and chooses from budget sets $B(p, w)$ so as to maximize that preference.

In terms of a preference $\succeq$, the *demand set* is

$$Q(p, w) = C_{\succeq}(B(p, w)) = \{x \in B(p, w) : x \succeq x' \text{ for all } x' \in B(p, w)\}.$$

In terms of a utility function, the demand set is

$$Q(p, w) = C_u(B(p, w)) = \arg \max_{x \in B(p,w)} u(x).$$

Our first question, which we address in the next section, is: When can we work in terms of a utility function even if we believe that the true primitive is the preference?

Given a utility function, we can consider both the argmax and the max:

$$v(p, w) = \max_{x \in B(p,w)} u(x),$$

$v : \mathbb{R}^k_{++} \times \mathbb{R}_+$ is called the *indirect utility function.*

8 The existence of a utility representation [MWG 3.C]

Please read MWG 3.C before reading this section.

Suppose we believe that the consumer is actually maximizing a preference, but we would like to model her as maximizing a utility function. In this section we investigate when it is valid to do that.

We know that if X is countable, then a preference $\succeq$ has a utility representation if and only if the preference is complete and transitive. We know that the "only if" part holds more generally: Regardless of X, if a preference has a utility representation, then the preference is complete and transitive. The converse is not true. There are complete and transitive preferences on $\mathbb{R}^k_+$ that do not have utility representations. The standard example is lexicographic preferences, which you may read about in MWG.

A natural question to ask is how can we fill in the blank to make the following true: For a general set X, a preference has a utility representation if and only if the preference is complete, transitive, and ________. It turns out that the property that fills the blank is not simple:

Proposition (Kreps Proposition 1.12)**.** *A preference $\succeq$ on a set X has a utility representation if and only if the following conditions hold:*

(a) The preference is complete and transitive, and

(b) There exists some countable subset X^* of X such that if $x, y \in X$ and $x \succ y$, then $x \succeq x^* \succ y$ for some $x^* \in X^*$.

If X is countable, then (b) is trivially satisified. (We can simply set $X^* = X$ and $x^* = x$.) If X is uncountable as in the consumer's problem, it is not clear how to establish the existence of such an X^*. We take a different route.

Debreu showed that a preference on $\mathbb{R}^k_+$ has a utility representation if the preference is complete, transitive, continuous — continuity of a preference will be defined below. Furthermore, such a preference has a continuous utility representation. (Thus completeness, transitivity and continuity are sufficient but not necessary for the existence of a utility representation. There exist preferences which do have a utility representation but do not have a continuous utility representation.) Before defining continuity of a preference, let us consider continuity of a function.

There are various ways to express continuity of a function $u : \mathbb{R}^k \to \mathbb{R}$. I am somewhat embarassed to admit my fondness for the "epsilon-delta" formulation of continuity, which applies to functions on metric spaces: The function u is continuous if for all $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, x') < \delta$ imples $|u(x) - u(x')| < \epsilon$. Here $d(x, x')$ denotes the distance between the two points. (One could define that distance in various ways which would equally serve our purpose here. A standard notion of distance is the Euclidean distance, a/k/a the L^2 metric. An alternative definition is $d(x, x') = \max_i |x_i - x'_i|$, which is called the Chebyschev distance or the L^∞ metric. In any case, we assume that d is derived from some norm $\|.\|$, so $d(x, y) = \|x - y\|$.) Informally, continuity of u means that if x and x' are sufficiently close, then their utilities are close.

The following is an implication of continuity of u. Suppose $u(x) > u(y)$, then (a) if x' is sufficiently close to x, then $u(x') > u(y)$. Similarly, (b) if y' is sufficiently close to y, then $u(x) > u(y')$. Formally, (a) means for all $x, y \in X$ such that $u(x) > u(y)$ there exists $\delta > 0$ such that if $d(x', x) < \delta$ then $u(x') > u(y)$. (That is true because we can take $\epsilon = u(x) - u(y)$ and apply the definition of continuity from the previous paragraph.) Economically speaking, if the bundle x is strictly better than the bundle y, then all bundles sufficiently near to x are also strictly better than y. This property may be stated in terms of a preference:

Definition 8.1. A complete preference relation $\succeq$ on $X \subset \mathbb{R}^k$ is *continuous* if for all points x, y in X such that $x \succ y$, there exists $\delta > 0$ such that if $d(x', x) < \delta$, then $x' \succ y$, and if $d(y, y') < \delta$, then $x \succ y'$.

Continuity says that the strict upper and lower contour sets, $\{\succ x\}$ and $\{\prec x\}$, are open for all $x \in X$. Equivalently, the (not strict) upper and lower contour sets, $\{\succeq x\}$ and $\{\preceq x\}$, are closed. MWG's definition of continuity is closely related to the statement that these later contour sets are closed. Their definition at first appears to require more than that, but as they note, it is equivalent.

For our purposes below in this section, the essential feature of continuity is part (b) in the following proposition.

Proposition 8.1. *Let X be a convex subset of $\mathbb{R}^k$.*

(a) Let u be a utility function on X. Consider three bundles x^0, x^1 and y such that $u(x^0) \leq u(y) \leq u(x^1)$. For $t \in [0,1]$, let $x^t = tx^1 + (1-t)x_0$.

If u is continuous, then there exists some $t \in [0,1]$ such that $u(x^t) = u(y)$.

(b) Let $\succeq$ by a complete and transitive preference on X. Consider three bundles x^0, x^1 and y such that $x^0 \preceq y \preceq x^1$. For $t \in [0,1]$, let $x^t = tx^1 + (1-t)x_0$.

If $\succeq$ is continuous, then there exists some $t \in [0,1]$ such that $x^t \sim y$.

Problem 8.1. Prove the preceding proposition. Part (a) follows from the Intermediate Value Theorem which follows. Part (b) may be proven using an argument analogous to that in the proof of the intermediate value theorem.

Theorem (Intermediate Value Theorem)**.** *Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Let $v \in \mathbb{R}$. If $f(0) \leq v \leq f(1)$, then there exists some value $t^* \in [0,1]$ such that $f(t^*) = v$.*

Proof. If $f(0) = v$ or $f(1) = v$, we are done, so suppose $f(0) < v < f(1)$.

Let $T^- = \{t \in [0,1] : f(t) \leq v\}$ and let $t^* = \sup T^-$. We will show that $f(t^*) = v$ as desired.

First note that $t^* \in [0,1]$ exists: Every nonempty set of real numbers having an upper bound has a least upper bound, that is the supremum, denoted sup. The set T^- is nonempty, it contains 0, and it has an upper bound, 1.

We will show that given continuity of f, if $f(t^*) \neq v$, then t^* cannot be the least upper bound of T^-.

Suppose $f(t^*) < v$. Thus $t^* < 1$, because $v < f(1)$. Continuity of f then implies that there exists $\delta \in (0, 1-t^*)$, such that $f(t) < v$ for all t within a distance δ from t^*. Consequently $f(t^* + \delta/2) < v$, so t^* is not an upper bound for T^-.

Suppose $f(t^*) > v$. Thus $t^* > 0$, because $f(0) < v$. Continuity of f then implies that there exists $\delta \in (0, t^*)$ such that $f(t) > v$ for all t within a distance δ from t^*. So the interval $(t^* - \delta, t^*]$ is not contained in T^-. So if t^* is an upper bound for T^-, then $t^* - \delta$ is a lesser upper bound for T^-, so t^* cannot be the least upper bound. □

Theorem 8.1 (Debreu)**.** *Let X be a convex subset of $\mathbb{R}^k$. A preference $\succeq$ on X has a continuous utility representation if and only if the preference is complete, transitive and continuous.*

We are going to prove a related result under an additional assumption on $\succeq$.

A function $u : \mathbb{R}^k \to \mathbb{R}$ is strictly increasing if $x > y$ implies $u(x) > u(y)$. (Here $x > y$ means $x_i \geq y_i$ for all i and $x_j > y_j$ for some j. A stronger condition is $x \gg y$, which means $x_i > y_i$ for all i. A weaker condition is $x \geq y$, which means $x_i \geq y_i$ for all i. Notice $x > y \Leftrightarrow (x \geq y$ and $y \ngeq x)$.)

Definition 8.2. A preference on a subset X of $\mathbb{R}^k$ is strictly increasing if $x > y$ implies $x \succ y$.

(Recall $x \succ y$ means $x \succeq y$ and $y \not\succeq x$.)

Proof that if $\succeq$ is complete, transitive, continuous and strictly increasing, and $X = \mathbb{R}^2_+$, then there exists a utility representation:

Proposition 8.2. *A preference on $\mathbb{R}^k_+$ has a continuous and strictly increasing utility representation if and only if the preference is complete, transitive, continuous and strictly increasing.*

Here I prove parts of the proposition and ask you to prove the remaining parts.

Proof that such a preference has some utility representation in the two-good case. Let $D = \{x \in \mathbb{R}^2_+ : x_1 = x_2\}$ denote the diagonal of $\mathbb{R}^2_+$.

Notice that $(\min\{x_1, x_2\}, \min\{x_1, x_2\}) \preceq x \preceq (\max\{x_1, x_2\}, \max\{x_1, x_2\})$ because $\succeq$ is increasing. The preceding proposition then implies that there exists some symmetric bundle $d \in D$ such that $x \sim d^x$.

Notice that if d and d' are two distinct points on the diagonal, then one is strictly preferred to the other, because $\succeq$ is strictly increasing. Thus there cannot be two distinct bundles d and d' on the diagonal such that $x \sim d$ and $x \sim d'$.

We have established that for each bundle x, there exists a unique bundle on the diagonal d^x such that $x \sim d^x$. Define $u(x) = d^x_1$.

So far all we have done is to construct a utility function, that is a map which associates each value in $\mathbb{R}^2_+$ with a single value in $\mathbb{R}$. Now we prove that the utility function we have constructed in fact represents the preference, that is $x \succeq y$ if and only if $u(x) \geq u(y)$.

Suppose $x \succeq y$. Notice $d^x \sim x \succeq y \sim d^y$, so $d^x \succeq d^y$ by transitivity. So $d^x_1 \geq d^y_1$, because $\succeq$ is increasing. That is $u(x) \geq u(y)$, by the definition of u.

If $u(x) \geq u(y)$, then $d^x_1 \geq d^y_1$, by definition of u. So $d^x \succeq d^y$, because $\succeq$ is increasing. Notice $x \sim d^x \succeq d^y \sim y$, so $x \succeq y$ by transitivity. □

Proof that the previously established utility representation is continuous. Fix $x \in X$. We want to show that for all $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|u(x) - u(y)| < \epsilon$, that is $u(x) - \epsilon < u(y) < u(x) + \epsilon$.

Let us show that there exists such δ_1 to imply $u(x) - \epsilon < u(y)$. If $u(x) \leq \epsilon$, we are done, so suppose $u(x) > \epsilon$. Notice $u(x) - \epsilon < u(y)$ if and only if $(u(x) - \epsilon, u(x) - \epsilon) \prec y$. Notice $(u(x) - \epsilon, u(x) - \epsilon) \prec x$. Consequently there exists δ such that $d(x, y) < \delta$ implies $(u(x) - \epsilon, u(x) - \epsilon) \prec y$ as desired.

To show that there exists such δ_2 to imply $u(y) < u(x) + \epsilon$ is similar. Letting $\delta = \min\{\delta_1, \delta_2\}$ then yields the desired conclusion. □

Problem 8.2. (a) Above it is proven, in the two-good case, that if a preference is complete, transitive, continuous and strictly increasing, then it has a utility representation. Prove the same thing in the more general case of k goods. [This is only a slight extension of the proof above. A bit of useful notation: let $e = (1, 1, ..., 1) \in \mathbb{R}^k$, so for any number $x_1 \in \mathbb{R}$, $x_1 e = (x_1, x_1, ..., x_1)$.]

(b) The previous proposition claims ... if and only if ... . Above we have proven only statements regarding the "if" part. Prove the "only if" part. That is prove that if a utility function is continuous and strictly increasing, then the preference derived from that utility function is continuous and strictly increasing.

9 Properties of the demand set

Before reading this section, please read MWG 3.B, and 3.D through p52.

The main question of this section is: For fixed values of p and w, what can we say about properties of the set $Q(p, w)$?

Problem 9.1. Fix some price vector $p \in \mathbb{R}^k_{++}$ and wealth level $w \in \mathbb{R}_+$.

(a) Suppose the consumer has some unknown preference $\succeq$, which may not be complete or transitive. $Q(p, w) = C_{\succeq}(B(p, w))$. Consider the family of sets $\mathcal{Q}(p, w) \subset \mathcal{P}(\mathbb{R}^k_+)$ such that

$$A \in \mathcal{Q}(p, w) \Leftrightarrow \text{there exists some } \succeq \text{ such that } Q(p, w) = A.$$

What is the set $\mathcal{Q}(p, w)$? Prove your answer.

(b) Suppose the consumer has some unknown utility function u. $Q(p, w) = C_u(B(p, w))$. Consider the family of sets $\mathcal{Q}(p, w) \subset \mathcal{P}(\mathbb{R}^k_+)$ such that

$$A \in \mathcal{Q}(p, w) \Leftrightarrow \text{there exists some } u \text{ such that } Q(p, w) = A.$$

What is the set $\mathcal{Q}(p, w)$? Prove your answer.

In this section we establish conditions on $\succeq / u$ which imply restrictions on the family of possibly demanded sets $\mathcal{Q}(p, w)$ considered in the previous problem. Specifically, we establish conditions which imply that $Q(p, w)$ satisfies contains one and only one point, which is on the budget frontier. This conclusion may be familiar from the standard picture of indifference curves from intermediate microeconomics. The conditions of this section imply that indifference curves must have the special features of that standard picture.

From this point on, every preference $\succeq$ is assumed to be complete and transitive (abbreviated c&t) unless otherwise noted.

9.1 Strict monotonicity $\Rightarrow$ Walras' Law

Definition 9.1. Let $\succeq$ be a c&t preference on $X \subset \mathbb{R}^k$. Let $x, y \in X$.

(a) The preference is *increasing* (a/k/a nondecreasing) if $x \geq y$ implies $x \succeq y$.

(b) The preference is *strictly increasing for strict increases in the bundle* if $x \gg y$ implies $x \succ y$.

(c) The preference is *strictly increasing* if $x > y$ implies $x \succ y$.

Recall $x \gg y$ means $x_i > y_i$ for all i, and $x > y$ means $x_i \geq y_i$ for all i and $x_j > y_j$ for some j. It is straightforward to extend these definitions to a utility function. (Replace $x \succ y$ with $u(x) > u(y)$.) If a preference has one of the properties above, then any utility representation of that preference has the corresponding property. The converse is also true.

MWG call (c) above "strongly monotone" and (b) "monotone." The name given for (b) above is from Kreps.

Notice (c) implies (b) and (a). One might imagine that (b) implies (a), but that is not true as we have defined (b) above.

Definition 9.2. Let X be a metric space.

(a) A utility function u on X is *locally nonsatiated* (LNS) if for every $x \in X$ and $\epsilon > 0$, there exists $y \in X$ such that $d(x, y) \leq \epsilon$ and $u(y) > u(x)$.

(b) A c&t preference $\succeq$ on X is *locally nonsatiated* (LNS) if for every $x \in X$ and $\epsilon > 0$, there exists $y \in X$ such that $d(x, y) \leq \epsilon$ and $y \succ x$.

If a preference on $\mathbb{R}^k_+$ is strictly increasing, then it is strictly increasing for strict increases in the bundle. If a preference is strictly increasing for strict increases in the bundle, then it is locally nonsatiated on $\mathbb{R}^k_+$.

Proposition 9.1. *If $\succeq$ is strictly increasing (or u is strictly increasing), or more generally if it is locally nonsatiated, then the budget constraint is binding at every point in $Q(p, w)$.*

Recall, this conclusion is called Walras' law.

Proof. If $px < w$, then there exists $\epsilon > 0$ such that $px' < w$ for all x' such that $d(x, x') \leq \epsilon$. LNS implies that some such x' is strictly preferred to x. That precludes any such x from being in the demand set. □

9.2 Continuity $\Rightarrow$ nonemptiness of demand set

Proposition 9.2. *If c&t $\succeq$ is continuous (or u is continuous), then $Q(p, w)$ contains at least one point and is compact.*

[Given a function $f : \mathbb{R}^k \to \mathbb{R}$, say that that the function is *uppersemicontinuous* at a point x if for all $\epsilon > 0$, there exists $\delta > 0$ such that $\|x' - x\| \leq \delta$ implies $f(x') \leq f(x) + \epsilon$.]

Proof. The extreme value theorem says that if continuous u, and B is compact, then $\arg\max_{x \in B} u(x)$ is nonempty. (It is also true that the argmax set is compact, due to continuity of u.[6]) Debreu's theorem says that if $\succeq$ is c&t and continuous, then it has a continuous utility function. So we are done, but below we prove the proposition directly. [In the 2018 version of 501A, you are not responsible for the following proof.]

If u is continuous, then the induced preference $\succeq$ is continuous.

$B(p, w)$ is closed and bounded. The Heine-Borel theorem states that every closed and bounded subset of $\mathbb{R}^k$ is compact in the sense that every open cover has a finite subcover. We will explain that further in a moment.

[6] If a set of bundles all achieve utility level u^*, then any limit point of that set also achieves u^*.

If $z \in B(p,w)$ and $z \notin Q(p,w)$ then there exists $x \in B(p,w)$ such that $x \succ z$. So if $Q(p,w)$ is empty

$$B(p,w) \subset \cup_{x \in B(p,w)} \{\prec x\}$$

The family of sets on the right hand side is a "cover" of $B(p,w)$: it is a family of sets of which the union equals $B(p,w)$. It is an "open cover" because every set $\{\prec x\}$ is open (because $\succeq$ is continuous).

Because $B(p,w)$ is open, the open cover has a finite subcover, that is there exists a finite set of $\{x^1, x^2, ...x^n\} \in B(p,w)$ such that

$$B(p,w) \subset \cup_{x \in \{x^1,x^2,...,x^n\}} \{\prec x\}.$$

Given a c&t preference, every finite set has a maximal element, that is there exists j such $x^j \succeq x^i$ for all $i \in \{1,...,n\}$. Let $z \in B(p,w)$. Then $z \prec x^i \preceq x^j$ for some i. Transitivity implies $x^j \succeq z$.

Thus $x^j \succeq z$ for all $z \in B(p,w)$. So $x^j \in Q(p,w)$ contradicting that Q is empty.

Claim: $Q(p,w)$ is a closed subset of $B(p,w)$.

We have shown that there exists some $x \in Q(p,w)$. Then $Q(p,w) = \{\succeq x\} \cap B(p,w)$. Both of those later sets are closed, and the finite intersection of closed sets is also closed, so $Q(p,w)$ is closed. □

9.3 Strict convexity ⇒ uniqueness of the demanded bundle

Definition 9.3. Let $\succeq$ be a preference on X, a convex subset of $\mathbb{R}^k$. Given two points x^0, x^1 in X and $t \in (0,1)$, let $x^t = tx^1 + (1-t)x^0$.

(a) $\succeq$ is *convex* if for all x^0, x^1 in X, and $t \in (0,1)$, $x^t \succeq x^0$ or $x^t \succeq x^1$.

(b) $\succeq$ is *strictly convex* if for all distinct x^0, x^1 in X, and $t \in (0,1)$, $x^t \succ x^0$ or $x^t \succ x^1$.

(As usual, here the statement either A or B is true allows the possibility that both A and B are true.)

Convexity of a preference is equivalent to quasi-concavity of its utility representations. Similarly, strict convexity corresponds to strict quasi-concavity.

Convexity is equivalent to the condition that for each $x \in X$, the upper contour set $\{\succeq x\}$ is convex. If the preference is continuous and strictly convex, then the upper contour sets are strictly convex. [I'm not sure what fills in the following blank. Strict convexity of the upper contour sets plus ______ implies that the preference is strictly convex.]

Proposition 9.3. *(a) If $\succeq$ is convex (u is quasi-concave), then $Q(p,w)$ is convex.*

(b) If $\succeq$ is strictly convex (u is strictly quasi-concave), then $Q(p,w)$ contains at most one point.

Proof. (a) The empty set is convex. Suppose $Q(p,w)$ is non-empty. Let x be one of its elements. Then $Q(p,w) = \{\succeq x\} \cap B(p,w)$. That is the intersection of two convex sets, which must itself be convex.

(b) Suppose $Q(p,w)$ contains two distinct points x^0 and x^1. Both points are in the budget set, which is convex, so every convex combination of the two points, $x^t = tx^1 + (1-t)x^0$, $t \in (0,1)$, is in the budget set. Because both are demanded it must be that $x^0 \sim x^1$. Strict convexity of $\succeq$ then implies that x^1 is strictly preferred to both points, which contradicts their being demanded. □

Convexity of a preference may be related to dimishing marginal utility of consumption. The following problem explores this idea in the special context of an additively separable utility function.

Problem 9.2 (Additively separable utility)**.** Many years ago economists typically considered additively separable utility functions: $u(x) = \sum_{i=1}^{k} u_i(x_i)$, where each $u_i : \mathbb{R}_+ \to \mathbb{R}_+$. Here there is no complementarity or substitutability between goods in the sense that the amount of good 3, does not affect the consumer's willingness to trade off quantities of goods 1 and 2, for example. It is common to assume that each u_i is continuous, strictly increasing and concave. Here concavity captures declining marginal utility of consumption for each good.

(a) Show that if each u_i is concave, then u is concave, so the preference derived from u is convex.

(b) Show that if each u_i is strictly concave, then u is strictly concave, so the preference given by u is strictly convex.

10 A bit of convex analysis: Supporting hyperplanes

This section relates to the discussion of supporting hyperplanes in MWG Appendix M.G "Convex Sets and Separating Hyperplanes" and section 3.F "Duality: A Mathematical Introduction. I have also referred to Kreps, Appendix Three: Convexity and his section 9.7, but you need not.

Definition 10.1. A set $Y \subset \mathbb{R}^k$ is *convex* if for all y^0 and y^1 in Y, and all $t \in [0,1]$, the convex combination $y^t = ty^1 + (1-t)y^0$ is also contained in Y.

Furthermore, the set is *strictly convex* if y^t is contained in the interior of Y for all $t \in (0,1)$.

The intersection of convex sets is convex.

Definition 10.2. Given $p \in \mathbb{R}^k$, $p \neq 0$, and $w \in \mathbb{R}$, the *hyperplane* generated by p and w is the set $H_{p,w} = \{x \in \mathbb{R}^k : px = w\}$.

The *closed half-space above* that hyperplane is the set $\{x \in \mathbb{R}^k : px \geq w\}$.

Similarly the closed half-space below the hyperplane is defined replacing $\geq$ with $\leq$. And the open half-spaces are defined replacing the weak inequalities with strict inequalities.

Notice that here we are not assuming $p \gg 0$ and $w \geq 0$. That is true throughout this section, but I will sometimes refer to $p \in \mathbb{R}^k$ and $w \in \mathbb{R}$ as prices and wealth, to perhaps give some economic intuition.

The hyperplane that we have previously encountered is the budget hyperplane, that is the set of all bundles that cost w given prices p. We have also encountered a closed half-space: the budget set is the intersection of the positive orthant and the closed half-space below the budget hyperplane.

Definition 10.3. A hyperplane H supports a set $Y \subset \mathbb{R}^k$ at a point y on the boundary of Y if $y \in H$ and Y is contained in either the closed half-space above H, or Y is contained in the closed half-space below H.

The point y is on that boundary if it is contained in both the closure of Y and the closure of the complement of Y. A set is closed if it contains its own boundary, so if Y is closed, it contains the aforementioned point y.

Remark 10.1. Let Y be a set and H a hyperplane. If $y \in Y$, $y \in H$ and Y is contained in the either the closed half-space above H or the one below H, then y is on the boundary of Y and H supports Y at y.[7]

Theorem 10.1 (The supporting hyperplane theorem)**.** *If $Y \subsetneq \mathbb{R}^k$ is a convex set and y is a point on the boundary of Y, then there exists a hyperplane that supports Y at y.*

(Given two sets Y and X, $Y \subsetneq X$ mean $Y \subset X$ and $Y \neq X$, that is Y is a "proper subset" of X.) I also do not prove this theorem. Again it seems obvious in pictures, at least in the simpler case where Y is compact. Note that there may be multiple supporting hyperplane. Also note if Y is not convex, there need not be any supporting hyperplane.

11 Solving for the demand set

In a previous section we established that if the preference $\succeq$ is continuous and LNS, then there is at least one point in the demand set $Q(p,w) = C_{\succeq}(B(p,w))$, and all points in the demand set are on the frontier of the budget set. Further, if the preference is convex, then the demand set is convex, and if the preference is strictly convex then the demand set is a single point. In this section, we consider how to solve explicitly for the demand set under certain conditions on the preference or utility function. The consumer's problem is a constrained optimization problem, and if u is differentiable then we can apply the Karush-Kuhn-Tucker conditions to identify the solution.

[7] That y is on the boundary of Y means that any neighborhood of y contains points outside of Y. Consider a hyperplane $H = \{x : px = w\}$. Suppose Y is contained in the half space above it, that is $\{x : px \geq w\}$, and suppose $y \in H$, that is $py = w$. The open half space below H contains points arbitrarilly near to y, for example the point $y - p\epsilon$ for small $\epsilon > 0$. That point is outside of Y, so y is on the boundary of Y as desired.

This section relates to MWG's discussion of the KKT conditions for the consumer's problem, beginning on their page 53 and ending near the top of page 56. However, I do not duplicate that material here, so please study that part of MWG especially carefully. In this section I describe some underlying ideas which have some relation to MWG 3.F and section M.G beginning on page 946 of MWG's appendix.

Given a complete preference, a bundle $x^* \in \mathbb{R}^k_+$ is optimal in the consumer's problem if and only if x^* is affordable and all strictly better bundles are not affordable. That is $x^* \in C_{\succeq}(B(p,w)) \Leftrightarrow px^* \leq w$ and $py > w$ for all $y \succ x^*$. (That is, x^* is optimal if and only if $\succ \{x^*\}$ is contained in the open half-space above the budget hyperplane.) Given an LNS preference, we know that if x^* is optimal, then $px^* = w$. Here we show if the preference is continuous, then the two strict inequalities in the previous statement, $py > w$ and $y \succ x^*$, may be replaced with weak inequalities. We then relate this to a statement about a supporting hyperplane.

Proposition 11.1. *Suppose the preference is c&t, continuous and LNS.*

A bundle $x^* \in \mathbb{R}^k_+$ is optimal if and only if it is just affordable and all weakly better bundles cost weakly more. That is $x^* \in C_{\succeq}(B(p,w)) \Leftrightarrow py \geq px^* = w$ for all y such that $y \succeq x^*$.

Proof. First, we show that if x^* is optimal, then all weakly better bundles cost weakly more. We do this by showing the contrapositive, that is, if some weakly better bundle costs strictly less, than x^* is not optimal:

Given LNS, we already know that if x^* is optimal, then $px^* = w$. Suppose there exists some $y \succeq x$ such that $py < w$. LNS then implies that there exists some y' close to y such that $py' < w$ and $y' \succ y$. Thus x^* cannot be optimal in this case.

Second, we show that if all weakly better bundles cost weakly more, then x^* is optimal. We do this by showing the contrapositive, that is, if x^* is not optimal, then some weakly better bundle costs strictly less:

Suppose that x^* is not optimal. That is, there exists some $y \succ x^*$ where $py \leq w$. Continuity then implies that for all y' near enough to y, $y' \succ x^*$. Set $y' = (1-\epsilon)y$ for small enough ϵ. Then $y' \succeq x^*$ and $py' < w$. □

Corollary 11.1. *Suppose the preference is c&t, continuous and LNS.*

A bundle x^* is optimal if and only if the budget hyperplane supports the upper contour set $\{\succeq x^*\}$ at x^*, that is $px^* = w$, and $py \geq w$ for all y such that $y \succeq x^*$.

Consider a preference that is c&t, continuous and LNS.

Question 1: At prices p and wealth w, which bundles are demanded? That is, what is $Q(p,w)$.

Question 2: Fixing a bundle x, for which prices p is x demanded given wealth px? Call this $P(x)$, it is an inverse demand function.

The first is a basic economic question. The two questions are clearly related. In particular: $x \in Q(p,w) \Leftrightarrow p \in P(x)$. The previous corollary gives an answer to question 2: $p \in P(x)$ if and only if $H_{p,px}$ supports $\{\succeq x\}$.

Given an arbitrary preference, it may not be easy to determine $P(x)$. It is easy to do so under the conditions of Lagrange/KKT maximization:

...

Given an arbitrary preference, it may not be easy to determine at which points x the budget hyperplane supports the upper contour set. [[I am confident about the following two statements, though I have not rigorously proven them and I do not have a citation.]]

- It is easy to do so for interior points $x \in \mathbb{R}^k_{++}$ given a quasi-concave, (continuously?) differentiable utility function without critical points. Under those conditions, the budget hyperplane given by (p,w) supports a point $x^* \gg 0$ where $px^* = w$ if and only if $\nabla u(x^*) \propto p$. Here $\propto$ means is proportional to; equivalently $\nabla u(x^*) = \lambda p$ for some $\lambda \in \mathbb{R}_{++}$. (Compare this to the Lagrange maximization conditions.)

- We also want to consider points on the boundary of $\mathbb{R}^k_{++}$. Again, given a quasi-concave, differentiable utility function without critical points, it is fairly easy to do so... (the Karush-Kuhn-Tucker conditions).

Problem 11.1. (i) MWG 3.C.6, which introduces and investigates the CES utility function.

(ii) MWG 3.D.5(a), which solves the consumer's problem given CES utility. (Notice, I am not yet asking you to do parts (b), (c), and (d) of 3.D.5.[8])

Problem 11.2. (a) Suppose the consumer's preference is c&t, continuous, convex and strictly increasing. Fix $x \in \mathbb{R}^k_{++}$. Prove that there exists $p \in \mathbb{R}^k_{++}$ and $w \in \mathbb{R}_{++}$ such that $x \in Q(p, w)$.[9] (Hint: Let $x \in \mathbb{R}^k_{++}$. Suppose that $p \in \mathbb{R}^k$ but $p \notin \mathbb{R}^k_{++}$, and suppose $H_{p,px}$ supports the upper contour set $\{\succeq x\}$. Does that contradict $\succeq$ being strictly increasing?)

(b) Relaxing the assumption that the preference is convex, again prove the previous statement or provide a counter example.

(c) Relaxing the assumption that the preference is increasing, again prove the previous statement or provide a counter example.

Problem 11.3. (a) Suppose there are two goods and $u(x_1, x_2) = x_1 + x_2$. Compute the demand correspondence. Further, using the results of this section, prove that what you have computed is indeed the demand correspondence.

(b) More generally, allowing n goods, suppose that u is linear. That is $u(x) = \alpha x$ for some $\alpha \in \mathbb{R}^n_{++}$. Again compute the demand correspondence and prove that what you have computed is indeed the demand correspondence.

[8]I am uncertain about what MWG intend in 3.D.5(c) where they ask you to show that the CES demand function approaches that of linear demand as $\rho \to 1$. Equivalently, let $Q(p, w; \rho)$ denote the demand correspondence for CES utility with paramater ρ. I think they are asking you to show that $Q(p, w; \rho)$ is continuous in ρ at the value $\rho = 1$, but I do not see that is true. In the following discussion suppose $p_1 = p_2$. I think that the following is correct. For $\rho = 1$, $Q(p, w)$ is the entire budget frontier $\overline{B}(p, w)$. For $\rho < 1$, $Q(p, w)$ is a single point in the middle of the budget frontier, that is $q_1 = q_2 = w/(2p_1)$. For $\rho > 1$, $Q(p, w)$ is the two points at the extreme ends of the budget frontier, that is $Q(p, w) = \{(w/p_1, 0), (0, w/p_1)\}$. Thus it seems to me that Q is neither upper or lower semicontinuous at $\rho = 1$ when $p_1 = p_2$. Perhaps it is true that $q(p, w; \rho)$ is single-valued and continuous in ρ whenever $p_1 \neq p_2$?

[9]I think this statement would not be true if instead we replaced the assumption $x \in \mathbb{R}^k_{++}$ with the weaker assumption $x \in \mathbb{R}^k_{+}$ (so we allow that $x_i = 0$ for some goods i). In that case, I wonder what additional conditions on the preference would make the conclusion valid.

12 The Weak Axiom and the Law of Compensated Demand revisited

In the previous three sections we established properties of the demand set $Q(p, w)$ for fixed p, w. In this section and the next we establish properties of the demand correspondence, that is properties regarding how Q varies as p, w vary.

A few sections ago we considered WA in the consumer's problem, and the LCD. In some sense, we did that out of order (although in the same order as MWG). In this section we briefly consider WA and LCD again, after having established conditions where $Q(p, w)$ is single-valued and satisfies Walras' law. (When $Q(p, w)$ is known to be single-valued, I write $q(p, w)$ instead.)

Proposition 12.1. *(a) Suppose the consumer has a c&t, continuous, LNS preference. Then the LCD' holds:*

Fix w^0, p^0, and p^1. If $q^0 \in Q(p^0, w^0)$, $w^1 = p^1 q^0$, and $q^1 \in Q(p^1, w^1)$, then

$$(p^1 - p^0)(q^1 - q^0) \leq 0.$$

(b) Suppose the consumer has a c&t, continuous, LNS, strictly convex preference. Then the stronger LCD holds:

Fix w^0, p^0, and p^1. If $q^0 = q(p^0, w^0)$, $w^1 = p^1 q^0$, and $q^1 = q(p^1, w^1)$, then

$$(p^1 - p^0)(q^1 - q^0) \leq 0.$$

with strict inequality if $q^0 \neq q^1$.

Problem 12.1. Prove the previous proposition.

Problem 12.2 (Additively separable utility). Many years ago economists typically considered additively separable utility functions: $u(x) = \sum_{i=1}^k u_i(x_i)$, where each $u_i : \mathbb{R}_+ \to \mathbb{R}_+$. (Here u_i does not denote $\partial u / \partial x_i$, but rather the amount of utility from good i.)

Suppose that each u_i is twice differentiable with $u_i' > 0$ and $u_i'' < 0$. Show that the law of uncompensated demand holds. That is, if $p_i^1 > p_i^0$ for some i, and $p_j^1 = p_j^0$ for all $j \neq i$, then $q_i(p^1, w) \leq q_i(p^0, w)$.

[I know how to show this using the Karush-Kuhn-Tucker conditions, which is the approach I suggest you take. But it seems to me that the result probably holds more generally. That is, suppose instead that u_i is strictly increasing and concave, but not necessarilly differentiable. I conjecture that the law of uncompensated demand still holds, as above. If someone can prove that, I would give extra credit.]

13 Continuity of the solution to the CP

This section relates to two parts of MWG. First, MWG Proposition 3.D.3(iv), where they state, but do not prove that the indirect utility function is continuous. And second, MWG Chapter 3, Appendix A where they show that the demand correspondence is upper hemicontinuous.

My treatment in this section is somewhat idiosyncratic. I favor a metric notion of continuity of correspondences. I emphasize the connection between continuity of single-valued functions and continuity of correspondences. I refer to upper and lower semicontinuity rather than hemicontinuity. One of my main references on correspondences is Kreps' Appendix Four.

In this section we formalize the following three ideas.

First, regarding the budget correspondence: Fix p, w. Let $x \in B(p, w)$ and $x' \in B(p', w')$. If p', w' is near enough to p, w, then x is near to some bundle in $B(p', w')$, and x' is near to some bundle in $B(p, w)$. To put it differently, if p', w' is near enough to p, w, then neither budget set contains any point that is far outside of the other budget set.

Second, regarding the indirect utility function: Fix p, w, let $q \in Q(p, w)$, $q' \in Q(p', w')$, and suppose the consumer's preference is continuous. If p', w' is near enough to p, w, then q' is neither much better nor much worse than q.

Third, regarding the demand correspondence: Again, fix p, w, let $q \in Q(p, w)$, $q' \in Q(p', w')$, and suppose the consumer's preference is continuous. If p', w' is near enough to p, w, then q' is near to some bundle in $Q(p, w)$. However, q need not be near to some bundle in $Q(p', w')$. To put it differently, if p', w' is near enough to p, w, then $Q(p', w')$ does not contain any points far outside of $Q(p, w)$, however $Q(p, w)$ may contain points far outside of $Q(p', w')$.

The following example illustrates this last idea, that $Q(p, w)$ may contain points far outside of $Q(p', w')$.

Example 13.1. Suppose $n = 2$, $u(x_1, x_2) = x_1 + x_2$, $p_1 = p_2$ and $w > 0$. Then $Q(p, w)$ is the entire budget frontier, that is $Q(p, w) = \{x \in \mathbb{R}^2_+ : px = w\}$, which equals the line segment joining with endpoints $(w/p_1, 0)$ and $(0, w/p_1)$. Now suppose $p'_1 = p_1 + \epsilon$ and $p'_2 = p_2 + 2\epsilon$. Now $Q(p', w')$ contains only a single bundle: $w/(p_1+\epsilon)$. Notice this bundle is near the original demand set, but the original demand set contains points, like $(0, w/p_1)$ that are far outside the new demand set.

Let us first reconsider continuity of a function.

Given a metric space (X, d) and a function $f : X \to \mathbb{R}$, say that that the function is *uppersemicontinuous* (abbreviated usc) at a point x if for all $\epsilon > 0$, there exists $\delta > 0$ such that $d(x', x) \le \delta$ implies $f(x') \le f(x) + \epsilon$. Less formally, f is usc at x if for all $\epsilon > 0$, $f(x') \le f(x) + \epsilon$ for all x' sufficiently near to x. Even less formally, that f is usc at x means: if x' is near enough to x, then $f(x')$ is not much greater than $f(x)$. (Here I have written weak inequalities. We could equivalently define continuity in terms of strict inequalities.)

The function is *lowersemicontinuous* (abbreviated lsc) at x if for all $\epsilon > 0$ there exists $\delta > 0$ such that $d(x', x) \le \delta$ implies $f(x') \ge f(x) - \epsilon$, equivalently $f(x) \le f(x') + \epsilon$. Even less formally, that f is lsc at x means: if x' is near enough to x, then $f(x')$ is not much less than $f(x)$.

The function usc if it is usc at each point x. Similarly, the function is lsc if it is lsc at each point x. The function is continuous at x if it is both upper and lower semicontinuous at x. It is continuous if it is both usc and lsc.

Given a set $S \subset \mathbb{R}^k$. Let $S^{+\epsilon}$ denote the set of all points within a distance ϵ from S. That is $S^{+\epsilon} = \{x \in \mathbb{R}^k : d(x, s) \le \epsilon \text{ for some } x \in S\}$. (Here we could take d to be either the Euclidean distance or the Chebyshev/L_∞ distance.)

Given a non-empty, compact-valued correspondence $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^k$, say the correspondence is *upper semicontinuous* at a point x if for all ϵ, there exists $\delta > 0$ such that $d(x', x) \leq \delta$ implies $F(x') \subseteq F(x)^{+\epsilon}$.

The correspondence is *lower semicontinuous* at x if for all $\epsilon > 0$, there exists $\delta > 0$ such that $d(x', x) \leq \delta$ implies $F(x) \subseteq F(x')^{+\epsilon}$.

Given two sets $S, S' \subset \mathbb{R}^k$, define $d(S, S') = \inf\{\epsilon > 0 : S \subset S'^{+\epsilon} \text{ and } S' \subset S^{+\epsilon}\}$. This is called the Hausdorff distance. The distance d is a metric if we restrict attention to compact sets. (It may not otherwise be a metric: notice that the distance between an open set and its closure is zero.)

The correspondence is continuous at x it is both usc and lsc at x. That is, for all $\epsilon > 0$, there exists δ such that if $d(x, x') \leq \delta$, then $d(F(x), F(x')) \leq \epsilon$.

There are various ways to define continuity of a function. Above we presented the metric, $\epsilon - \delta$ formulation of continuity. Another useful notion of continuity is the sequential notion: f is continuous at x means that if $\lim_{n\to\infty} x_n = x$ then $\lim_{n\to\infty} f(x_n) = f(x)$. Similarly, while above we defined the metric notion of (upper and lower semi) continuity of a correspondence, one often encounters instead sequential definitions:

The aforementioned correspondence F is *sequentially upper semicontinuous* at x means that if $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} f(y_n) \to y$ and $y_n \in F(x_n)\ \forall n$, then $y \in F(x)$. It is *sequentially lower semicontinuous* at x means that if $\lim_{n\to\infty} x_n = x$ and $y \in F(x)$, then there exists a sequence $\{y_n\}$ in the range of F such that $\lim_{n\to\infty} y_n$ and $y_n \in F(x_n)$ for sufficiently large n. (These correspond to definition A4.2 of Kreps.) A function is sequentially continuous at x if it is sequentially upper semicontinuous and sequentially lower continuous at x. As usual, it is continuous if it is continuous at each x in its domain. For correspondences that meet some technical conditions, which the budget correspondence does meet, satisfy sequential upper/lower semicontinuity if and only if they satisfy the previous metric notions of upper/lower semicontinuity.[10] (Economists, such as MWG, often speak of hemicontinuity rather than semicontinuity of correspondences. I, like Kreps, prefer to speak of semicontinuity, noting the analogy between semicontinuity of functions and semicontinuity of correspondences. See Kreps subsection A4.5 if you are interested.)

[10]If you are interested, see Kreps Proposition A4.10.

Problem 13.1. The budget correspondence $B : \mathbb{R}^n_{++} \times \mathbb{R}_+ \rightrightarrows \mathbb{R}^n_+$ is continuous.

(a) Prove that B satisfies the metric notions of upper and lower semicontinuity described above. [Two hints: First, you can prove that if w^1 and w^2 are two wealth levels such that $w^1 < w^2$ then the smallest ϵ such that $B(w^1, p)^{+\epsilon} \supset B(w^2, p)$ is $\epsilon = (w^2 - w^1)/\min_i p_i$. (Notice that $\arg\max_{x\in B(p,w)} d(0, x)$ consists of those bundles where one spends all of one's wealth on the cheapest goods, $\arg\min_i p_i$.) From this first result, you can prove that B is continuous in w. Second, you can prove that if B satisfies the metric notion of continuity in w, then it satisfies the metric notion of continuity in p.]

(b) Prove that B satisfies sequential upper and lower semicontinuity. [Hint regarding the proof of sequential lower semicontinuity: Given a bundle $x \in B(p, w)$, notice $x_i p_i/w$ is the share of wealth spent on good i. Given two distinct budget sets there is a natural isomorphism between based on this share of wealth idea. Specifically, given $x \in B(p, w)$ the corresponding point in $x' \in B(p', w')$ satisfies $x_i p_i/w = x'_i p'_i/w'$ for each good i.]

Proposition 13.1. *If utility u is continuous, then the indirect utility function v is continuous and the demand correspondence Q is uppersemicontinuous.*

Proof. First recall that continuity of u together with compactness of $B(p, w)$ imply that $Q(p, w)$ is nonempty and $v(p, w)$ is well defined.

Fix some initial values p, w.

Claim: v is lsc, that is for any fixed $\epsilon > 0$, there exists $\delta > 0$ such that if p', w' is within a distance δ from p, w, then $v(p', w') \geq v(p, w) - \epsilon$.

Let $q \in Q(p, w)$. So $v(p, w) = u(q)$.

Because u is continuous, there exists $\beta > 0$ such that if $d(q', q) \leq \beta$, then $u(q') \geq u(q) - \epsilon$.

Because B is continuous, there exists $\delta > 0$ such that if p', w' is within δ from p, w, then $B(p', w')$ contains a point within distance β from q.

Combining the two previous lines yields $v(p', w') \geq u(q') \geq v(p, w) - \epsilon$.

Claim: v is usc, that is for any fixed $\epsilon > 0$ there exists $\delta > 0$ such that if p', w' is within a distance δ from p, w, then $v(p', w') \leq v(p, w) + \epsilon$.

Loosely speaking, we will argue that elements near $B(p, w)$ are not much better than elements in $B(p, w)$. Continuity of B implies that if p', w' is near p, w then all elements in $B(p', w')$ are near $B(p, w)$.

Fix large $\gamma > 0$. $B(p, w)^{+\gamma}$ is compact, so u is uniformly continuous on that set. (A continuous function on a compact set is uniformly continuous.)

Fix $\epsilon > 0$. By uniform continuity, there exists $\beta > 0$ such that if x' and x'' are both in $B(p, w)^{+\gamma}$ and $d(x', x'') < \beta$, then $|u(x') - u(x'')| \leq \epsilon$.

Note $\min\{\gamma, \beta\}$ is a positive number. Because B is continuous, there exists $\delta > 0$ such that $B(p', w') \subseteq B(p, w)^{+\min\{\gamma,\beta\}}$ for all p', w' within a distance δ from p, w. For such nearby p', w', let $x' \in B(p', w')$. There exists $x \in B(p, w)$ such that $d(x', x) < \beta$, so $u(x') \leq u(x) + \epsilon \leq v(p, w) + \epsilon$. Because $u(x') \leq v(p, w) + \epsilon$ for every $x' \in B(p', w')$, we have $v(p', w') \leq v(p, w) + \epsilon$ as desired.

Claim: Q is usc, that is for any fixed $\epsilon > 0$, there exists $\delta > 0$ such if p', w' is within δ from p, w, then $Q(p', w') \subset Q(p, w)^{+\epsilon}$.

[[I would like to argue this directly in terms of the metric notion of upper semicontinuity that I have presented, but I do not know how to do so. I give a related sequential argument.]]

Suppose not. There exists some $\epsilon > 0$, such that for each $\delta > 0$, there exists p', w' within a distance δ from p, w such that $Q(p', w')$ contains an element outside of $Q(p, w)^{+\epsilon}$.

Thus there is a sequence $p^n, w^n \to p, w$, $q^n \in Q(p^n, w^n)$, $q^n \notin Q(p, w)^{+\epsilon}$. Because B is locally bounded, the sequence q^n eventually falls inside a compact set, so it has a convergent subsequence. $q^{n_m} \to q^\infty \notin Q(p, w)^{+\epsilon/2}$. Because v is continuous, $u(q^\infty) = v(p, w)$, but then $q^\infty \in Q(p, w)$, which is a contradiction. $\square$

A correspondence $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^k$ is single-valued if there exists some corresponding function $f : \mathbb{R}^k \to \mathbb{R}^k$ such that $F(x) = \{f(x)\}$. If a single-valued correspondence $F = \{f\}$ is uppersemicontinuous, then the corresponding function f is continuous.

Problem 13.2. Prove that if preferences are continuous and strictly convex, then the demand function is continuous. [Hint: This only requires you to combine several previous results.]

14 The indirect utility function

MWG discuss the indirect utility function in section 3.D, particularly Proposition 3.D.3. Please read that section in MWG before this one in these notes. Rubinstein discusses the indirect preference.

One of our main questions is: As prices change, how much better or worse off is the consumer? Here we further develop the foundations to answer that question.

Recall the indirect utility function: $v(p, w) = \max_{x \in B(p,w)} u(x)$. Throughout most of this section we assume that v is well-defined, that is, for each $p \in \mathbb{R}^n_{++}$ and $w \in \mathbb{R}_+$, $Q(p, w)$ is non-empty, so $\max_{x \in B(p,w)} u(x)$ is a real number. Recall, we know that v is indeed well-defined if u is continuous.[11]

Proposition 14.1. *Suppose the indirect utility v function is well defined. It is homogeneous of degree zero.*

This is an immediate consequence of the fact that the budget correspondence is HD0.

Proposition 14.2. *Suppose the indirect utility v function is well defined.*

(a) It is increasing in w and decreasing in p.

(b) If in addition u is LNS, then v is strictly increasing in w.

[11] If we want to consider discontinuous u, then we can more generally define $v(p, w) = \sup_{x \in B(p,w)} u(x)$, but to keep things simple I assume that u is continuous.

Proof. We could define more generally $v(B) = \max_{x \in B} u(x)$ for arbitrary budget sets $B \subset \mathbb{R}^k_+$. Of course, v is increasing in B. Furthermore $B(p, w)$ is increasing in w and decreasing in p (that is, increasing in $-p$). Thus v is increasing in w and decreasing in p. $\square$

Problem 14.1. (i) Prove part (b) of the previous proposition.

(ii) Under what conditions is v strictly decreasing in p? That is, state and prove a proposition of the following form: If $p' < p$ (that is $p'_i \leq p_i$ for all i and $p'_j < p_j$ for some j) and _________, then $v(p', w) > v(p, w)$. (Please fill in the blank and prove that the resulting statement is true.)

Proposition 14.3. *Suppose the indirect utility v function is well defined. It is quasiconvex, that is, the set $\{(p, w) : v(p, w) \leq \bar{v}\}$ is convex for every $\bar{v}$.*

Proof. Let $(p^t, w^t) = t(p^1, w^1) + (1 - t)(p^0, w^0)$ for $t \in [0, 1]$. We want to show that v is quasiconvex, that is $v(p^t, w^t) \leq \max\{v(p^0, w^0), v(p^1, w^1)\}$. Equivalently, if $v(p^0, w^0) \leq \bar{v}$ and $v(p^1, w^1) \leq \bar{v}$, then $v(p^t, w^t) \leq \bar{v}$.

We often consider the budget correspondence, which describes the set of affordable bundles for each price, wealth pair. Now consider the correspondence which describes the set of unaffordable bundles: $B^c(p, w) = \{x \in \mathbb{R}^k_+ : px > w\}$. We will show that the graph of B^c is convex, and this implies that v is quasiconvex (without conditions on u apart from v being well-defined).

That the graph of B^c is convex means that if x is a bundle such $p^1x > w^1$ and $p^0x > w^0$, then $p^tx > w^t$. That is true because $p^tx = tp^1x + (1 - t)p^0x > tw^1 + (1 - t)w^0 = w^t$. Conversely, if $p^tx \leq w^t$, then either $p^0x \leq w^0$or $p^1x \leq w^1$, or both.

Let $q \in Q(p^t, w^t)$. The previous step implies either $q \in B(p^0, w^0)$ or $q \in B(p^1, w^1)$. In the former case, $v(p^t, w^t) = u(q) \leq v(p^0, w^0) \leq \bar{v}$. Similarly, in the latter case, $v(p^t, w^t) \leq v(p^1, w^1) \leq \bar{v}$. In both cases, $v(p^t, w^t) \leq \bar{v}$ as desired. $\square$

Up to this point, the assumption that u is continuous is convenient, but I believe unnecessary. The assumption that u is continuous is essential for the following proposition.

Proposition 14.4. *If u is continuous, then v is continuous.*

We proved that in the previous section.

14.1 The indirect preference

Instead of considering the indirect utility function, we could consider the consumer's preference over budget sets. Let $\succeq$ be the individual's standard preference over a set of all possible alternatives X. (We could call $\succeq$ the direct preference, though we have not previously referred to it that way.) Let $\mathscr{B}$ be a family of subsets of X. Given two budget sets B and B' in $\mathscr{B}$, say that $B' \succeq^* B$ if for every x in B there exists x' in B' such that $x' \succeq x$. This new preference $\succeq^*$ is called the *indirect preference.*

Notice that the indirect preference is well defined whether or not $C_\succeq$ satisfies nonemptiness.

Problem 14.2. (a) If the individual's choice function $c_\succeq$ is single-valued for two budget sets $B, B' \in \mathscr{B}$, then $B' \succeq^* B$ if and only if $c_\succeq(B') \succeq c_\succeq(B)$. (That is, the set B' is indirectly prefered to the set B if and only if the thing chosen from B' is directly prefered to the thing chosen from B.) Prove that.

(b) Assume that u represents $\succeq$. Also assume that $C_u(B)$ is nonempty for all $B \in \mathscr{B}$, so $v(B) = \max_{x \in B} u(x)$ is well-defined for all B. Under these two assumptions, it is true that v represents $\succeq^*$. Prove that.

(c) Continue to assume that u represents $\succeq$. Now assume that perhaps $C_u(B)$ is empty for some B, so v may be ill-defined. Consider instead $\tilde{v}(B) = \sup_{x \in B} u(x)$. (That is well defined whether or not C_u satisfies nonemptiness.) Does $\tilde{v}$ still represent $\succeq^*$? Prove that it does or give a counter example.

(d) Previously in this section we established several properties of the indirect utility function (it is HD0, increasing in ..., quasiconvex, and continuous). Without relying on u or v, state and prove analogous properties of the indirect preference $\succeq^*$ over the consumer's budget sets.

15 The expenditure minimization problem - MWG 3.E

Much of this section is close to MWG 3.E — please read that section first. [[Note to Asaf: This section should get reworked. To prove that the solutions correspond we need the results from the next section that H is nonempty and the analog to Walras law: $h(p,v) = v$.]]

So far we have considered the Consumer's Problem (abbreviated CP). (Note MWG instead refer to the Utility Maximization Problem, abbreviated UMP, by which they mean the same thing.) The CP a/k/a UMP:

$$\begin{aligned} Q(p,w) &= \arg \max_{x \in B(p,w)} u(x) \\ v(p,w) &= \max_{x \in B(p,w)} u(x). \end{aligned}$$

We now consider the Expenditure Minimization Problem (abbreviated EMP):

$$\begin{aligned} H(p,\nu) &= \arg \min_{x : u(x) \geq \nu} px \\ e(p,\nu) &= \min_{x : u(x) \geq \nu} px. \end{aligned}$$

H is called the Hicksian demand correspondence and e the expenditure function. I refer to the value ν as the utility target.

Why do we consider the EMP?

Recall that as prices change "real" wealth changes. E.g., if the price of good 1 goes up, the budget set shrinks, so the consumer is in a sense poorer (even if w remains unchanged.) We want to consider the effect of prices changes without real wealth changes. So far we have considered Slutsky compensation: Suppose that q^0 is chosen at p^0, w^0. Consider $Q(p, pq^0)$. That is what the consumer would buy at prices p if they had enough wealth to buy the original bundle q^0 at prices p. Note that Slutsky compensation is in a sense too much. Suppose $p^1 > p^0$, then $B(p^1, p^1 q^0) \supset B(p^0, w^0)$. So $v(p^1, p^1 q^0) \geq v(p^0, w^0)$.

If Slutsky compensation is too much, what is the "right" level of compensation. It is given by e, and referred to as Hicks compensation. A main result of this section is that $H(p,\nu) = Q(p, e(p,\nu))$. That is H is the demand of a Hicks-compensated consumer.

The EMP is "dual" to the CP. As an aside, what does dual mean? I am not completely sure. MWG write

> The term "dual" is meant to be suggestive. It is usually applied to pairs of problems and concepts that are formally similar except that the role of quantities and prices, and/or maximizaiton and minimization, and/or objective function and constraint, have been reversed.
>
> (MWG footnote 13 on page 58)

However, I do not think that is quite right. Their section 3.F, titled "Duality: A Mathematical Introduction" suggests duality is a mathetical concept regarding convex sets. Larry Blume similarly writes

> The word 'duality' is often used to invoke a contrast between two related concepts, as when the informal, peasant, or agricultural sector of an economy is labelled as dual to the formal, or profit-maximizing, sector. In microeconomic analysis, however, 'duality' refers to connections between quantities and prices which arise as a consequence of the hypotheses of optimization and convexity. Connected to this duality are the relationship between utility and expenditure functions (and profit and production functions), primal and dual linear programs, shadow prices, and a variety of other economic concepts. In most textbooks, the duality between, say, utility and expenditure functions arises from a sleight of hand with the first-order conditions for optimization. These dual relationships, however, are not naturally a product of the calculus; they are rooted in convex analysis and, in particular, in different ways of describing a convex set. This article will lay out some basic duality theory from the point of view of convex analysis, as a remedy for the microeconomic theory textbooks the reader may have suffered.

> (Blume 2008, “Duality” entry in the New Palgrave Dictionary of Economics, 2nd Ed)

That seems authoritative. However, if duality is truly about convexity, I am a bit confused because we do not need to assume that the upper contour sets are convex for most of the results in this section. (Budget sets are convex, but perhaps we could establish some analogous results without that.)

We are interested in the EMP primarily because understanding it will help us better understand the CP. We will establish that in a sense the solutions to the two problems coincide. First let us consider Rubinstein’s “dual turtle” (from his Lecture 6):

> Consider the following two statements:
>
> (1) The maximal distance a turtle can travel in 1 day is 1 km.
>
> (2) The minimal time it takes a turtle to travel [at least] 1 km is 1 day.
>
> In conversation, these two statements would seem to be equivalent. In fact this equivalence relies on two assumptions:
>
> For (1) to imply (2), we need to assume that the turtle travels a positive distance in any period of time. Contrast this with the case in which the turtle’s speed is 2 km/day, but after half a day it must rest for half a day. In this case, the maximal distance it can travel in 1 day is 1 km, though it is able to travel this distance in only half a day.
>
> For (2) to imply (1), we need to assume that the turtle cannot “jump” a positive distance in zero time. Contrast this with the case in which the turtle’s speed is 1 km/day, but after a day of traveling it can “jump” 1 km. Thus, it can travel 2 km in 1 day (and if you don’t believe that a turtle can jump, think about a “frequent consumer” scheme in which the number of points “jumps” after the consumer reaches a certain point level).
>
> We will now show that the above assumptions are sufficient for the equivalence of (1) and (2). Formally, let $m(t)$ be the maximal distance the turtle can travel in time t and assume that m is strictly increasing and continuous. We can then show that the statement “the maximal distance a turtle can travel in t^* is x^*” is equivalent to the statement “the minimal time it takes a turtle to travel x^* is t^*”. If the maximal distance that the turtle can travel within t^* is x^* and if it covers the distance x^* in $t < t^*$, then by the strict monotonicity of m the turtle can cover a distance larger than x^* in t^*, a contradiction. If it takes t^* for the turtle to cover the distance x^* and if it travels the distance $x > x^*$ in t^*, then by the continuity of m the turtle will already be beyond the distance x^* at some $t < t^*$, a contradiction.

Analogously, we are interested in equivalence of the following two statements:

(1) The maximum utility the consumer can achieve given wealth w and prices p is v.

(2) The minimum wealth required for the consumer to achieve utility v given prices p is w.

We are also interested in the relationship between Q and H.

First let us note that the EMP is trivial if $\nu \leq u(0)$, where here 0 means the bundle $(0, ..., 0) \in \mathbb{R}^k_+$. In that case $e(p,\nu) = 0$ and $H(p,\nu) = \{0\}$. Further if $\nu > u(x)$ for all $x \in \mathbb{R}^k_+$, then the constraint set in the EMP is empty, so the problem is not well defined. So we typically consider $\nu \in [u(0), \sup_x u(x))$. (If $\nu > \sup_x u(x)$, then we have already mentioned that there is no solution. If $\nu = \sup_x u(x)$, then there might be a solution, but not if u is globally insatiable in the sense that for all x there exists x' so that $u(x') > u(x)$. If u is LNS it is globally insatiable.)

The solutions to the CP and EMP coincide.

Proposition 15.1 (The solutions to CP and EMP coincide, version 1)**.** *Fix a price vector $p \in \mathbb{R}^k_{++}$.*

(a) Suppose u is LNS. Fix $w \in \mathbb{R}_+$. If $q \in Q(p,w)$, then $q \in H(p,u(q))$, and $e(p,u(q)) = w$.

(b) Suppose u is continuous. Fix $\nu \in \mathbb{R}$. If $h \in H(p,\nu)$, then $h \in Q(p,ph)$, and $v(p,ph) = \nu$.

Note in (a) that $u(q) = v(p,w)$. Similarly note in (b) that $ph =$

$e(p,\nu)$. Also note that this proposition carries no bite if Q or H empty. For example, if $Q(p,w)$ is empty, then part (a) says nothing.

A wordier statement closer to that in MWG: Fix p. (a) If a bundle q is optimal in the CP with wealth w, then q is optimal in the EMP with utility target $u(q)$, and the minimum expenditure in that EMP is w. (b) If a bundle h is optimal in the EMP with utility target v, then h is optimal in the CP with wealth equal to ph, and the maximum utility in that CP is v.

Proof. (a) A first case to consider is $w = 0$. Then $q = 0$, and $H = \{0\}$, $e = 0$ as we have previously discussed. We now suppose that $w > 0$. We prove that if q is not optimal in the EMP of part (a), then it is not optimal in the CP:

Suppose $q \notin H(p,u(q))$, then there exists x such that $u(x) \geq u(q)$ and $px < pq$. By LNS, there exists x' near to x so that $px' < pq$ and $u(x') > u(q)$. If $q \in B(p,w)$ then $x' \in B(p,w)$ and $x' \succ q$, so $q \notin Q(p,w)$.

We have established if $q \in Q(p,w)$, then $q \in H(p,u(q))$. LNS implies Walras law: $pq = w$. We know if $h \in H(p,v)$, then $e(p,v) = ph$. So $e(p,u(q)) = pq = w$.

(b) We already discussed the case where $v \leq u(0)$. So suppose $v > u(0)$. We prove that if h is not optimal in the CP of part (b), then it is not optimal in the EMP:

If $h \in H(p,v)$, then $h \neq 0$, so $ph > 0$. If $h \notin Q(p,e(p,v))$, then there exists x such that $u(x) > u(h)$ and $px \leq ph$. By continuity of u, there exists nearby x' such that $px' < px$ and $u(x') > u(h)$. If $u(h) \geq v$, then $u(x') \geq v$, and $px' < ph$, so $h \notin H(p,v)$.

Below we show that if $h \in H(p,v)$, then $u(h) = v$. We know that if $h \in Q(p,w)$, then $v(p,w) = u(h)$. In combination that implies $v(p,e(p,v)) = v$. □

We can also establish at least part of the proposition by means of a previous result: [This part needs to be refined.] Recall we previous established the following.

Proposition 15.2. *Suppose the preference is c&t, continuous and LNS.*

A bundle $x^* \in \mathbb{R}^k_+$ is optimal [in the Consumer's Problem] if and only if it is just affordable and all weakly better bundles cost weakly more. That is $x^* \in C_{\succeq}(B(p,w)) \Leftrightarrow py \geq px^* = w$ for all y such that $y \succeq x^*$.

Suppose $x^* \in Q(p,w)$. Noting that then $v(p,w) = u(x^*)$, the statement on the right hand side of the "$\Leftrightarrow$" in the proposition above implies that $px^* \leq py$ for all y such that $u(y) \geq u(x) = v(p,w)$,which is to say that $x^* \in H(p,v(p,w))$, so $e(p,v(p,w)) = px^* = w$.

Suppose $x^* \notin Q(p,w)$. The statement on the right hand side of the "$\Leftrightarrow$" in the proposition above must not hold. That is, there exists some $y \succeq x^*$ such that $py < px^*$. Thus $x^* \notin H(p,u(x^*))$.

Suppose $x^* \notin Q(p,e(p,v))$. The statement on the right hand side of the "$\Leftrightarrow$" in the proposition above must not hold. That is, there exists some $y \succeq x^*$ such that $py < px^*$. Thus $x^* \notin H(p,u(x^*))$.

Suppose $x^* \in H(p,v)$

So we have established $x^* \in Q(p,w) \Leftrightarrow x^* \in H(p,v(p,w))$

Proposition 15.3 (The solutions to CP and EMP coincide, version 2)**.** *If u is LNS and continuous, then for all $p \in \mathbb{R}^k_{++}$, $w \in \mathbb{R}_+$ and $\nu \in [u(0), \sup_x u(x))$,*

(a) $Q(p,w) = H(p,v(p,w))$,
(b) $H(p,\nu) = Q(p,e(p,\nu))$,
(c) $e(p,v(p,w)) = w$, and
(d) $v(p,e(p,\nu)) = \nu$.

Problem 15.1. Prove the immediately preceding proposition. [Compared to the proposition that preceeds it, this one additionally asserts that $Q(p,w) \supset H(p,v(p,w))$ and $H(p,v) \supset Q(p,e(p,v))$. These inclusions may be proved directly or I believe they may be established by application of the previous propositions.]

Problem 15.2. The Cobb-Douglas utility function is

$$u(x) = \prod_{i=1}^{k} x_i^{\alpha_i}$$

where $\alpha_i \in (0,1)$ and $\sum_{i=1}^k \alpha_i = 1$. For this utility function derive (a) the demand function, (b) the indirect utility function, (c) the expenditure function, (d) the Hicksian demand function.

15.1 Rubinstein's version of the EMP

We have previously considered the problem of choosing the $\succeq$ $-$optimal bundle from $B(p,w)$ — call that the consumer's problem. Fixing a reference bundle z, consider the closely related problem of choosing the $\succeq$-optimal bundle from $B(p,pz)$ — that is the best bundle among those that are not more expensive than z. Let us denote that problem "Prime(p,z)." For a fixed value of p, the prime problem is the same as the standard consumer problem of choosing the best bundle that costs not more than $w = pz$. (As p varies, the prime problem differs from the consumer's problem. The prime problem yields the Slutsky compensated demand function.)

[The solutions to the prime problem are

$$\{x \in X : px \leq pz, \text{ and } x \succeq y \text{ for all } y \text{ such that } py \leq pz\}.$$

The solutions to the dual problem are

$$\{x \in X : x \succeq z, \text{ and } px \leq py \text{ for all } y \text{ such that } y \succeq z\}.$$

]

In this section we consider the "dual problem":

$$\min_{x : x \succeq z} px.$$

That is the problem of choosing the cheapest bundle that is not worse than z. Let us denote that problem "Dual(p,z)." The dual problem is a version of what is elsewhere referred to as the expenditure minimization problem. In this section we study the dual problem and the relationship between the prime and dual problems. Our main reference is Rubinstein, Lecture 6, subsection "A Dual Consumer."

In a sense to be described, the solutions to the Prime and Dual problems coincide. The solution to the Prime problem is $Q(p,pz)$. The solution to the dual problem is

$$\hat{H}(p,z) = \arg \min_{x \in \mathbb{R}^k_+ : x \succeq z} pz.$$

The main result is that the solution to the prime and dual problems coincide in the following sense. If $\succeq$ is continuous and LNS, then x is a solution to Prime(p,x) if and only if x is a solution to Dual(p,x). That is, fixing a price vector p, x is a best bundle among those not more expensive than x if and only if x is a cheapest bundle among those at least as good as x.

Proposition 15.4 (The solutions to the EMP and CP coincide, Rubinstein's version)**.** *Suppose $\succeq$ is c&t, continuous and LNS. For all $p \in \mathbb{R}^k_{++}$, x is a solution to Prime(p,x) if and only if x is a solution Dual(p,x), that is $x \in Q(p,px) \Leftrightarrow x \in \hat{H}(p,x)$.*

Problem 15.3. Prove the preceding proposition. (Hint: I believe that we have already proved something which implies the statement that you are asked to prove here.)

16 Properties of the solution to EMP

Much of this section is close to MWG 3.E — please read that section first.

Proposition 16.1 (Properties of the Hicksian demand set)**.** *Fix $p \in \mathbb{R}^k_{++}$ and $v \in [u(0), \sup_x u(x))$. The Hicksian demand set $H(p,v)$ has the following properties.*

(a) If u is continuous, then $H(p,v)$ is nonempty.

(b) If u is continuous, then for all $h \in H(p,v)$, $u(h) = v$.

(c) If u is quasiconcave ($\succeq$ is convex), then $H(p,v)$ is convex. If u is continuous and strictly quasiconcave ($\succeq$ is strictly convex), then $H(p,v)$ contains at most one point.

Proof. (b) We show the contrapositive: if $u(x) > v \geq u(0)$, then $x \notin H(p,v)$. [Notice that px is LNS for all $x \neq 0$.] Given such x, by continuity, there exists x' slightly less than x such that $px' < px$ and $u(x') > u(x)$. (Take $x' = (1-\epsilon)x$ for sufficiently small $\epsilon > 0$.) Thus $x \notin H(p,v)$ because x' is feasible and cheaper. □

Problem 16.1. Prove (a) and (c). MWG omit (a). They assign the proof of (c) as Exercise 3.E.4.

Proposition 16.2. *The Hicksian demand correspondence $H : \mathbb{R}^k_{++} \times [u(0), \sup_x u(x)) \rightrightarrows \mathbb{R}^k_+$ satisfies the following properties.*

(a) H is HD0 in p;

(b) If u is LNS and continuous, then H is upper semicontinuous in p and v.

(c) The Law of Hicks-Compensated Demand: Fix a target utility level v and two price vectors p^0 and p^1. Suppose that $h^0 \in H(p^0, v)$ and $h^1 \in H(p^1, v)$. Then

$$(p^1 - p^0)(h^1 - h^0) \leq 0.$$

(a) is trivial. I do not prove (b) and MWG do not state it. [You are not responsible for the proof, but if you want to see it, I suggest Kreps Proposition 10.3.] MWG state (c) in their proposition (3.E.4), but they add the assumptions that u is continuous and LNS, and H is single-valued. That seems superfluous? Perhaps under such conclusions one can establish a strict inequality, but MWG do not do that.

Proof of (c). That $h \in H(p, v)$ implies $ph \leq px$ for all x such that $u(x) \geq u(v)$. We know $u(h') \geq u(v)$. So $ph \leq ph'$. Similarly $p'h' \leq p'h$. Thus $ph' + p'h \geq p'h' + ph$. Rearranging yields the desired inequality. □

Proposition 16.3 (Properties of the expenditure function)**.** *Suppose u is continuous and LNS. The expenditure function $e : \mathbb{R}^k_{++} \times [u(0), \sup_x u(x)) \to \mathbb{R}_+$ is*

(a) HD1 in p;

(b) Strictly increasing in v and nondecreasing in p;

(c) Concave in p; and

(d) Continuous in p and v.

Neither MWG nor I prove (d). [Again, you are not responsible for the proof, but if you want to see it, I suggest Kreps Proposition 10.3.]

Proof. (a) Follows from H being HD0 in p.

(b) Let $v^H > v^L \geq u(0)$ and $h \in H(p, v^H)$. By continuity there exists x slightly smaller than h so that $px < ph$ and $u(x) > v^L$. Thus $e(p, v^L) \leq px < e(p, v^H)$. You are asked to show something more general than the fact that e is nondecreasing in p below.

(c) Fix $v \in [u(0), \sup_x u(x))$. Consider two price vectors p^0 and p^1, and a convex combination p^t. Let $x \in H(p^t, v)$. Of course $u(x) \geq v$, x is feasible in the EMP at p^0, v and p^1, v. So $e(p^0, v) \leq p^0 x$ and $e(p^1, v) \leq p^1 x$. Then

$$\begin{aligned} e(p^t, v) &= p^t x \\ &= tp^1 x + (1-t)p^0 x \\ &\geq te(p^1, v) + (1-t)e(p^0, v). \end{aligned}$$

□

Problem 16.2. Let $f^*(t) = \min_{x \in X} f(x, t)$. Suppose f^* is well-defined for all t, which is true for example if f is continuous in x and X is compact. Show that if f is nondecreasing in t, then f^* is nondecreasing in t. Apply that result to show that e is nondecreasing in p.

Problem 16.3. MWG 3.E.2, where you are asked to verify for the case of Cobb-Douglas utility that e and H indeed have the properties established above.

17 A bit more convex analysis: Separating hyperplanes

This section relates to MWG Appendix M.G “Convex Sets and Separating Hyperplanes” and section 3.F “Duality: A Mathematical Introduction. I have also referred to Kreps, Appendix Three: Convexity and his section 9.7, but you need not.

Recall our discussion of hyperplanes and half-spaces in a previous section. There we discussed supporting hyperplanes. Here we discuss separating hyperplanes.

Definition 17.1. A hyperplane H *strictly separates* a set $Y \subset \mathbb{R}^k$ from a point $x \in \mathbb{R}^k$ if Y is contained in the open half-space on one side of H and x is contained in the open half-space on the other side of H.

In the language of budgets and affordability, the budget hyperplane $H_{p,w}$ strictly separates Y from x if every point in Y is strictly affordable (that is $py < w$), while x is not affordable ($px > w$), or the opposite.

Theorem 17.1. *If $Y \subset \mathbb{R}^k$ is closed and convex, and $x \in \mathbb{R}^k \backslash Y$, then there exists a hyperplane that strictly separates Y and x.*

This may be called the strict separating hyperplane theorem regarding a closed convex set and a point. I do not prove this. It seems obvious in pictures. It is not simple to prove rigourously.

One might more generally consider the separation of two convex sets.

We now get to "duality."

Proposition 17.1. *Every closed, convex $Y \subsetneq \mathbb{R}^k$ consists of the intersection of all of the closed half-spaces containing Y.*

Proof. There exists $x \in \mathbb{R}^k \backslash Y$. So the separating hyperplane theorem tells us that there exists at least one closed half-space containing Y. Of course the intersection of all of the closed half-spaces containing Y contains Y. So what needs to be shown is that this intersection does not contain any point outside of Y. Let x be such a point, that is $x \in \mathbb{R}^k \backslash Y$. The strict separating hyperplane theorem above tells us that there exists a half space which does contain Y but does not contain x. Thus x is not in the intersection of all of the half spaces containing Y. □

Given a set $Y \subset \mathbb{R}^k$ and a "price vector" $p \in \mathbb{R}^k$, one might ask, what is the smallest amount of wealth w such that each $y \in Y$ is affordable, that is $py \leq w$. If there exists at least one most expensive bundle that solves $\max_{y \in Y} py$, then that is the answer. More generally, the answer is given by the corresponding sup:

Definition 17.2. Given a set $Y \subset \mathbb{R}^k$, the support function of Y, $s_Y : \mathbb{R}^k \to [-\infty, \infty]$ is defined as

$$s_Y(p) = \sup_{y \in Y} py.$$

MWY define the support function in terms of inf rather than sup. They define it only for nonempty closed sets Y— I am not sure why. Kreps defines it for closed convex sets. It is true that the support function is nicest for nonempty, convex, compact sets $Y \subset \mathbb{R}^k$.

Corollary 17.1. *(a) If $Y \subset \mathbb{R}^k$ is closed and convex, then*

$$Y = \cap_{p \in \mathbb{R}^k} \{x \in \mathbb{R} : px \leq s_Y(p)\}.$$

(b) If X and Y are both closed, convex subsets of $\mathbb{R}^k$ and $s_X = s_Y$, then $X = Y$.

(By $s_X = s_Y$, I mean $s_X(p) = s_Y(p)$ for all $p \in \mathbb{R}^k$. It may be that $s_Y(p) = +\infty$. The "half" space where $px \leq +\infty$ is the whole space $\mathbb{R}^k$. Similarly the half space where $px \leq -\infty$ is the empty set.)

Problem 17.1. Prove part (b) of the previous corollary directly, by proving the contrapositive: if X and Y are both closed, convex, proper subsets of $\mathbb{R}^k$ and $X \neq Y$, then there exists some p for which $s_X(p) \neq s_Y(p)$. [This follows from a previous proposition in this section.]

We can generalize (b). Recall the convex hull of a set is the smallest convex set that contains the original set.

Proposition 17.2. *Let X and Y be arbitrary subsets of $\mathbb{R}^k$. Then the closure of the convex hull of X is equal to the closure of the convex hull of Y if and only if $s_X = s_Y$.*

We have seen some properties of the expenditure function...

Proposition 17.3. *If Y is nonempty, convex and compact, its support function is HD1, continuous and convex; for all $p \neq 0$ the support function is real-valued and the* sup *is attained, and if in addition Y is strictly convex then the sup is attained at a unique point.*

If we defined the support function in terms of inf rather sup, then it would be concave rather than convex.

Theorem 17.2 (The Duality Theorem)**.** *Let $Y \subset \mathbb{R}^k$ be a nonempty closed set, and let $s(\cdot)$ be its support function. Fix a point $p^0 \in \mathbb{R}^k$. The support function is differentiable at p^0 if and only there exists a unique $y^0 \in Y$ such that $p^0 \cdot y^0 = s(p^0)$. And in that case, $\nabla s(p^0) = y^0$.*

I do not prove that if there exists a unique y^0, then s is differentiable. I do prove that if s is differentiable, then there exists a unique y^0 and $\nabla s(p^0) = y^0$.

Proof. Assume s is differentiable at p^0. Then $s(p^0) < \infty$. In that case, because Y is closed, the sup is attained: there exists a $y^0 \in Y$ such that $s(p^0) = p^0 y^0$. I will show that y^0 is unique and $\nabla s(p^0) = y^0$.

Define $\phi(p) = py^0 - s(p)$. The function ϕ is differentiable is differentiable at p^0, because s is. We know $\phi(p) \leq 0$ for all p and $\phi(p^0) = 0$. So the FOC must hold: $0 = \nabla\phi(p^0) = y^0 - \nabla s(p^0)$, thus $y^0 = \nabla s(p^0)$. (Because $\nabla s(p^0)$ is a single value, y^0 must be unique.) $\square$

18 Further relationships between q, v, h, and e (MWG 3.G)

Please study MWG 3.G itself carefully. In this section I add only a few minor comments.

18.1 $h = \nabla_p e$

If we know the Hicksian demand function, it is trivial to get the expenditure function: $e(p, v) = ph(p, v)$. Here we consider going the other direction, from the expenditure function to the Hicksian demand function.

MWG Proposition 3.G.1 establishes conditions under which $h_i(p, v) = \partial e(p, v)/\partial p_i$. In vector notation, $h(p, v) = \nabla_p e(p, v)$.

Consider an increase in the price of good j: $p_j^1 > p_j^0$, while $p_i^1 = p_i^0$ for all $i \neq j$. Let $\Delta = p_j^1 - p_j^0 > 0$ be the size of the increase.

$$
\begin{aligned}
e(p^1, v) - e(p^0, v) &= p^1 h^1 - p^0 h^0 \\
&= \underbrace{p^1 h^1 - p^1 h^0}_{\approx 0} + \underbrace{p^1 h^0 - p^0 h^0}_{= \Delta h_j^0}
\end{aligned}
$$

where "≈ 0" holds in sense that that term is second order. That is true not because h^1 is close to h^0 but because h^1 is exactly expenditure minimizing at p^1 and h^0 is approximately expenditure minimizing at p^1, so costs little more than h^1.

Differentiating the equation $h = \nabla_p e$ with respect to price we get that the Jacobian of h is equal to the Hessian of e. Hessian matrices are symmetric and negative semidefinite, so we conclude that the Jacobian of h has those properties. (See MWG Proposition 3.G.2)

18.2 The Slutsky equation

We do not directly observe h. If we know q we can recover h via the Slutsky equation:

Proposition 18.1. *Suppose the consumer's preference is continuous, convex and LNS, and has utility representation u. Then for all (p, w),*

$$
\frac{\partial h_i}{\partial p_j} = \frac{\partial q_i}{\partial p_j} + \frac{\partial q_i}{\partial w} q_j
$$

where q is evaluated at (p, w) and h is evaluated at (p, v) where $\nu = v(p, w)$.

Proof.

$$
\begin{aligned}
h_i(p, \nu) &= x_i(p, e(p, \nu)) \\
\frac{\partial h_i}{\partial p_j} &= \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}\frac{\partial e}{\partial p_j} \\
&= \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} h_j(p, \nu) \\
&= \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} q_j(p, w)
\end{aligned}
$$

$\square$

We may rearrange the Slutsky equation as follows:

$$\frac{\partial q_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - \frac{\partial q_i}{\partial w} q_j.$$

The first term on the right is a substitution effect, that is the effect of a change in relative prices while holding real wealth constant. The second term is a wealth effect.

We might also decompose the effect of a discrete price change into a substition effect (SE) and wealth effect (WE): Consider an initial wealth level w_0, new and old prices p^1 and p^0, and $v^0 = v(p^0, w^0)$:

$$\begin{aligned} & q(p^1, w^0) - q(p^0, w^0) \\ & = q(p^1, w) - \overbrace{h(p^1, v^0)}^{=q(p^1, e(p^1, v^0))} + h(p^1, v^0) - \overbrace{q(p^0, w^0)}^{=h(p^0, v^0)} \\ & = \underbrace{\left(h(p^1, u^0) - h(p^0, v^0)\right)}_{\text{SE}} + \underbrace{\left(q(p^1, w^0) - q(p^1, e(p^1, u^0))\right)}_{\text{WE}} \\ & = \underbrace{\left(q(p^1, e(p^1, u^0)) - q(p^0, w^0)\right)}_{\text{SE}} + \underbrace{\left(q(p^1, w^0) - q(p^1, e(p^1, v^0))\right)}_{\text{WE}} \end{aligned}$$

Above we have decomposed the price effect into a substituation effect at the original real wealth level plus a wealth effect at the new prices. We could also decompose it into a wealth effect at the old prices plus a substitution effect at at the new real wealth level:

$$\begin{aligned} & q(p^1, w^0) - q(p^0, w^0) \\ & = q(p^1, w) - h(p^0, v^1) + h(p^0, v^1) - q(p^0, w^0) \\ & = \underbrace{\left(h(p^1, v^1) - h(p^0, v^1)\right)}_{\text{SE}} + \underbrace{\left(q(p^0, e(p^0, v^1)) - q(p^0, w^0)\right)}_{\text{WE}} \end{aligned}$$

18.3 Roy's identity

We have previously considered the issue of deriving q from u. It turns out that it is easier to derive q from v, via Roy's identity: MWG Proposition 3.G.4

Problem 18.1. MWG 3.G.2: "Verify for the case of a Cobb Douglas utility function that all of the propositions in section 3.G hold."

Problem 18.2. MWG 3.G.4 parts (a), (b) and (c) [but not part (d)]. For part (b) they are asking you to show the following. Let I,J be a partitition of $\{1, 2, ..., k\}$ that is $I \cap J = \emptyset$ and $I \cup J = \{1, 2, ..., k\}$. Let $\succeq$ be a preference that has an additively separable utility representation. If $x \succeq y$ for some x and y where $x_j = y_j$ for all $j \in J$, then $x' \succeq y'$ for all x', y' where $x'_i = x_i$ and $y'_i = y_i$ for all $i \in I$ and $x'_j = y'_j$ for all $j \in J$. What has just been written may be hard to digest. In the case where $k = 3$, one implication is that if $(x_1, x_2, x_3) \succeq (y_1, y_2, x_3)$ then $(x_1, x_2, x'_3) \succeq (y_1, y_2, x'_3)$ for all values $x'_3 \in \mathbb{R}_+$.

Problem 18.3. MWG 3.G.7, where you are to consider the "indirect demand function." I think that the first part of this problem, where you are asked to show that $g(x) = \nabla u(x)/(x \cdot \nabla u(x))$, relies on some unstated assumptions regarding the consumer's preference. State those assumptions.

19 Integrability (MWG 3.H)

MWG break this section into two parts: "Recovering Preferences from the Expenditure Function" and "Recovering the Expenditure Function from Demand." Please study the first part. Regarding the second part, it may be useful to know that in principal if you know q you may be able to determine e, but in this class we are not covering how to do so. (It requires solving a system of partial differential equations: $\nabla_p e(p, v) = q(p, e(p, v))$.)

Under what conditions is a demand function rationalizable? That is given a proposed demand function q, does there exist a utility function u that generates q? Given q can we compute that u? MWG 3.H take up these questions.

Suppose the consumer has a continuous, convex preference. Then for each utility level v, the upper contour set $U(v) = \{x : u(x) \geq v\}$ is a closed convex set. Recall that if we know the support function of such a set then we know the set itself. The expenditure function $e(p, v) = \min_{x \in U(v)} pv$ is almost the support function $s_{U(v)}(p) =$

$\sup_{x\in U(v)} px$. In particular, for $p \in \mathbb{R}^k_{--}$, $s_{U(v)}(p) = e(-p, v)$. We know that s is continuous in p, so from e we can determine the support function s not only for $p \in \mathbb{R}^k_{--}$ but for p in the closure of the previous set, that is $\mathbb{R}^k_{-}$. If the preference is increasing, then for p outside of $\mathbb{R}^k_{-}$, we have that $s_{U(v)}(p) = +\infty$. So if the consumer's preference is continuous, convex and increasing, from her expenditure function we can recover the support function of each of her upper contour sets, and thus we can recover those upper contour sets themselves.

20 Welfare (MWG 3.I)

This section might be entitled "consumer welfare" or "consumer surplus." MWG give it the title "welfare evaluation of economic changes." Our main reference is MWG 3.I, please read all of that section. I have also consulted Kreps chapter 12, but you need not.

Consider a move from p^0 to p^1. How good or bad is it for the consumer? Here are three examples of price changes we might be interested in. In October 2016 avocado prices rose to more than double their usual level, how bad is that for avocado consumers? In Tucson a number of big apartment buildings aimed at students have recently opened. As a result, rents around campus have fallen, perhaps by 10 or 20 percent. How good is that? In 2007 Apple introduced the iPhone. We can think of a product introduction as a reduction in the product's price from infinity. How good is that?

We already know something about when price changes are good or bad. Obviously, if $p^1 \geq p^0$, then bad. If $p^1 \leq p^0$, then good. More generally if $p^1q^0 \leq p^0q^0$ where $q^0 \in Q(p^0, w^0)$, then good. But how good or bad?

Bentham might suggest

$$\Delta v = v(p^1, w^0) - v(p^0, w^0).$$

If u is real, that is a measure of how good the change is, measured in utils. It is nonnegative in the case of a price reduction. But we usually don't think that utils are real. Dollars (or your currency of choice) are real.

Recall Hicks compensation. Given initial prices and wealth p^0, w^0, and new prices p^1, consider a Hicks compensating wealth level, w^1, which satisfies the following.

$$(p^0, w^0) \sim^* (p^1, w^1)$$

Throughout this section, let us assume that preferences are continuous and LNS, so there is a unique w^1 satisfying the above expression. (Recall $\succeq^*$ is the indirect preference, which corresponds to the indirect utility function.) The difference between the two wealth levels w^0 and w^1 is a measure of the change in welfare for an uncompensated move from (p^0, w^0) to (p^1, w^0). Let $CV = w^0 - w^1$. That is again nonnegative, like Δv, in the case of a price reduction. The *compensating variation* (abbreviated CV) is the net revenue of the imaginary compensator. CV is how much wealth the consumer would be willing to give up to move from p^0 to p^1, given initial wealth w^0:

$$(p^0, w^0) \sim^* (p^1, w^0 - CV)$$

$$CV = \underbrace{e(p^0, \nu^0)}_{=w^0} - \underbrace{e(p^1, \nu^0)}_{=w^1}$$

To make things more explicit, we might write $CV(p^0, p^1, w^0)$ above.

There are other money measures (see MWG's mention of "money metric indirect utility function"). There is one particular other called the *equivalent variation* (abbreviated EV). The EV is the amount that you would have to compensate the consumer to forgo a price change from p^0 to p^1, that is

$$(p^0, w^0 + EV) \sim^* (p^1, w^0)$$

Like CV and Δv, the EV is nonnegative in the case of a price reduction. Like the CV, the EV is a difference in the expenditure function evaluated at p^1 and p^0. However, while the CV is that difference evaluated at ν^0, the EV is that difference evaluated at ν^1.

$$EV = e(p^0, \nu^1) - e(p^1, \nu^1)$$

Suppose that only the price of good i has changed, that is $p^1_j = p^0_j$ for all $j \neq i$. We know from a previous section that $\partial e / \partial p_i = h_i$, when demand is single-valued. (Recall the fundamental theorem of calculus: if $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then $f(b) - f(a) = \int_a^b f'(x)dx$.) So CV and EV may be expressed as areas under Hicksian demand curves:

$$CV = \int_{p_i^1}^{p_i^0} h_i(p, \nu^0) dp_i$$

$$EV = \int_{p_i^1}^{p_i^0} h_i(p, \nu^1) dp_i$$

Notice the difference between the previous two integrals is that ν^0 has been replaced with ν^1.

All three expressions Δv, CV and EV are positive if and only if $(p^1, w^0) \succ^* (p^0, w^0)$.

In undergraduate economics one often considers what MWG refer to as the *area variation* (AV), that is the area under the Walrasian demand curve:

$$AV = \int_{p_i^1}^{p_i^0} q_i(p, v^0) dp_i$$

(Throughout this section, I am assuming that h and q are single-valued. Kreps discusses that issue.)

See the picture in MWG comparing CV, EV and AV. If there are no wealth effects, that is $\partial q_i / \partial w = 0$, over the relevant range, then $h_i(p, v^0) = h_i(p, v^1) = q_i(p, w^0)$, so CV=EV=AV.

Problem 20.1. (a) We have claimed that all four values CV, EV, AV and Δv have the same sign. Prove that.

(b) Suppose that $p^1 \geq p^0$, what is aforementioned sign? What if instead $p^1 \leq p^0$.

(c) Suppose that $p^1 \geq p^0$ and $p^1 \neq p^0$. Under what conditions could we conclude that none of CV, EV and AV are equal to zero?

Problem 20.2. Consider a change in the price of a single good. Show that if the good is normal, then $EV \geq AV \geq CV$. (A main step is to show that if q_i is increasing in w, then h_i is increasing in ν.)

Problem 20.3. MWG 3.I.6

Problem 20.4. Varian 10.1, which is reproduced here in the notation of MWG: Ellsworth's utility function is $U(x_1, x_2) = \min\{x_1, x_2\}$. Ellsworth has wealth $w = \$150$ and initially faces prices $p_1 = p_2 = 1$. Ellsworth's boss is thinking of sending him to another town where the price of good 1 is still 1 but the price of good 2 is 2. The boss offers no raise in pay. Ellsworth, who understands compensating and equivalent variation perfectly, complains bitterly. He says that although he doesn't mind moving for its own sake and the new town is just as pleasant as the old, having to move is as bad as a cut in pay of $\$A$. He also says he wouldn't mind moving if when he moved he got a raise of $\$B$. What are A and B equal to?

21 SARP and GARP

[[We are not covering this section in Fall 2017.]]

Suppose we observe the following data: From $B(p^1, w^1)$ the consumer chooses q^1, from $B(p^2, w^2)$ the consumer chooses q^2,... $B(p^n, w^n)$ the consumer chooses q^n, for a finite number of budget sets n. Under what conditions does there exist a c&t preference that would generate this choice behavior? We know that WA is necessary, but not sufficient.

21.1 SARP (MWG 3.J)

Lemma 21.1. *Suppose a consumer with a c&t, strictly convex preference chooses q from $B(p, w)$.*

Then we know that $q \succ x$ for all x such that $px < w$ and $x \neq q$.

In the following, we have in mind a c&t, strictly convex preference.

Definition 21.1. (a) Take any finite set of demand data:
q^1 is chosen from $B^1 = B(p^1, w^1)$,
q^2 is chosen from $B^2 = B(p^2, w^2)$,
...
q^n is chosen from $B^n = B(p^n, w^n)$,
where $q^i \neq q^j$ for all $i, j \in \{1, ..., n\}$.

If $p^i q^j \leq w^i$, then the data reveal directly that q^i is strictly preferred to q^j, written $q^i \succ {}^d q^j$.

(b) If $q^i \succ^d q^k$ and $q^k \succ^d q^j$, then the data indirectly reveal that q^i is strictly preferred to q^j, written $q^i \succ^r q^j$. (Here i, k and j need not be distinct.)

Similarly, if $q^i \succ^r q^k$ and $q^k \succ^r q^j$, then $q^i \succ^r q^j$.

(c) The data satisfy the Strong Axiom of Revealed Preference (SARP) if no strict revealed preference cycles exist, that is there is no x^i where $x^i \succ^r x^i$.

Theorem 21.1. *If every finite set of demand data satisfies SARP, then there is a c&t preference consistent with the data.*

21.2 GARP

Lemma 21.2. *Suppose a consumer with a c&t, LNS preference chooses q from $B(p, w)$.*

Then we know that $q \succeq x$ for all x such that $px \leq w$.

And we know that $q \succ x$ for all x such that $px < w$.

In the following, we have in mind a c&t, LNS preference.

Definition 21.2. (a) Take any finite set of demand data:

q^1 is chosen from $B^1 = B(p^1, w^1)$,

q^2 is chosen from $B^2 = B(p^2, w^2)$,

...

q^n is chosen from $B^n = B(p^n, w^n)$.

If $p^i q^j \leq w^i$, then the data reveal directly that q^i is weakly preferred to q^j, written $q^i \succeq^d q^j$.

If $p^i q^j < w^i$, then the data reveal directly that q^i is strictly preferred to q^j, written $q^i \succ^d q^j$.

(b) If $q^i \succeq^d q^k$ and $q^k \succeq^d q^j$, then the data indirectly reveal that q^i is weakly preferred to q^j, written $q^i \succeq^r q^j$. (Here i, k and j need not be distinct.)

Furthermore, if in addition either $q^i \succ^d q^k$ or $q^k \succ^d q^j$, then the data indirectly reveal that q^i is strictly preferred to q^j, written $q^i \succ^r q^j$.

Similarly, If $q^i \succeq^r q^k$ and $q^k \succeq^r q^j$, then $q^i \succeq^r q^j$.

Furthermore, if in addition either $q^i \succ^r q^k$ or $q^k \succ^r q^j$, then $q^i \succ^r q^j$.

(c) The data satisfy the Generalized Axiom of Revealed Preference (GARP) if no strict revealed preference cycles exist, that is there is no x^i where $x^i \succ^r x^i$.

Theorem 21.2. *A finite set of demand data is consistent with choice according to a c&t, LNS preference if and only if it satisfies GARP.*

Further if it satisfies GARP, then it is consistent with choice according to a c&t, strictly increasing and convex preference.

Part III
Preference under uncertainty

22 Background

Consider a risky financial situation. One's total wealth is given by a random variable X, of which the values are dollar amounts. Perhaps one has a choice between two situations X and X'. For example,

Example 22.1. Ina's total wealth is \$1 million. There is a 1/100 chance that her house would burn down. In that case her total wealth would be only \$1/4 million. So her current situation is given by the random variable X, where $\Pr[X = 1m] = 0.99$ and $\Pr[X = 0.25m] = 0.01$.

An insurer proposes that for a certain payment of \$10,000 it will pay Ina \$3/4 million if her house does burn down. If Ina purchases this insurance, then her financial situation is described by the new random variable X', where $\Pr[X' = 990,000] = 1$.

We are interested in preferences over objects like X and X'.

Example 22.2 (EMV maximization)**.** Mo's preference over risky financial situations is followings. Mo weakly prefers X to X' if and only if the expected monetary value of X is greater than or equal to that of X', that is $\mathrm{E}[X] \geq \mathrm{E}[X']$.

For example, Mo would reject the insurance mentioned in the previous example, because there $\mathrm{E}[X] = 992,500$ and $\mathrm{E}[X'] = 990,000$.

If I understand correctly, several hundred years ago it was often suggested that one should maximize the expected monetary value.

Example 22.3 (St. Petersburg Paradox)**.** Consider a lottery ticket that yields \$2 with probability 1/2, \$4 with probability 1/4, \$8 with probability 1/8, ... (More generally, for every $n \in \{1, 2, 3, ...\}$, the lottery ticket yields $\$2^n$ with probability $1/2^n$.)

How much would you be willing to pay for this lottery ticket?

Its expected monetary value is $\frac{1}{2}2 + \frac{1}{4}4 + \frac{1}{8}8 + ... = 1 + 1 + 1 + ... = +\infty$.

Example 22.4. Andy owns a lottery ticket that will yield \$20 million with probability 1/2, otherwise \$0. Andy currently has only \$1,000 in total wealth. Someone offers to purchase the ticket from Andy for \$9 million. Should Andy sell it?

Bill is in the same position as Andy, except Bill's current wealth is \$1 billion. Should Bill sell the ticket for \$9 million?

In the early 17 hundreds, Cramer and Bernoulli proposed that one should not maximize expected monetary value, but rather expected utility value. Cramer wrote "the mathematicians estimate money in proportion to its quantity, and men of good sense in proportion to the usage that they may make of it." How much usage would Andy make from winning \$9m? How much from an additional \$11m?

23 Expected Utility (MWG 6.B)

Our main reference for this section is MWG 6.B, please read that first. I do not repeat the contents of that section here, but add a few comments.

I mostly like MWG's presentation of expected utility. They provide useful graphical intuition for the three-outcome case, particularly in figures 6.B.1 and 6.B.5. (It is also common to consider a slightly different graphical depiction in that case, known as the Marschak-Machina triangle, which you may see here.)

Two minor comments on notation: MWG use $L = (p_1, ..., p_N)$ to denote a lottery. I would prefer to write p in place of L. MWG write L, L', L'' for three lotteries. I would prefer to write L, P, Q or similar, rather than using so many primes. Nonetheless, I stick with MWG's notation here.

Problem 23.1. It is important to understand what is meant by a mixture of lotteries. This introductory problem is meant to clarify that. The meaning is different than that of a mixture of consumption bundles in part II.

Suppose that L is a lottery which gives the outcome 2 apples with probability one, and L' is the lottery which gives the outcome 2 bananas with probability one. For example it could be that the space of possible outcomes is C ={2 apples, 2 bananas}. In that case $L = (1, 0)$ and $L' = (0, 1)$.

(a) What is the lottery $L'' = \frac{1}{2}L + \frac{1}{2}L'$? What outcomes does it give, with what probabilities?

(b) Given C as previously defined, does $\mathscr{L}$ contain the lottery that gives 2 apples and 2 bananas with probability one? Why or why not? What about the lottery that gives 1 apple and 1 banana with probability one? Why or why not?

(c) We think of lotteries as probability distributions. We could instead think of lotteries as random variables. Suppose that X is a random variable of which the outcome is 2 apples with probability one and X' is a random variable of which the outcome is 2 bananas with probability one. How might you interpret $\frac{1}{2}X + \frac{1}{2}X'$?

Problem 23.2. MWG 6.B.1, where you are asked to establish some implications of the independence axiom.

Problem 23.3. Show that if a preference over lotteries has an expected utility representation, then the preference is complete, transitive, continuous and satisfies the independence axiom.

The main result in this section is that the converse holds. That is if a preference over the space of lotteries is complete, transitive, continuous and satisfies the independence axiom then it has an expected utility representation. To prove this we construct a specific expected utility representation: $U(L)$ is defined to be the value such that $L \sim U(L) * \overline{L} + (1 - U(L)) * \underline{L}$, where $\overline{L}$ is the best lottery and $\underline{L}$ the worst. The first step is to show that for each L there exists a unique value $U(L) \in [0, 1]$ such that the previous indifference relationship holds. The next step is to show that U in fact represents $\succeq$. The proof for both of those steps is similar to that establishing a utility representation in Part II. (A difference is that while is part II we assumed that the preference is strictly increasing, here we get something similar as an implication of independence.)

The main step is to show that $U(L)$ as previously defined is linear. (We previously proved that a utility function has the expected utility form if and only if it is linear.) That is, for all pairs of lotteries L^0, L^1 and $t \in [0, 1]$, we have $U(L^t) = tU(L^1) + (1 - t)U(L^0)$, where $L^t = tL^1 + (1 - t)L^0$ as usual. We prove that U is linear here.

By definition of U we have

$$L^1 \sim U(L^1)\overline{L} + (1 - U(L^1))\underline{L}$$

and similarly for L^0. Applying the independence axiom twice yields,

$$\begin{aligned} L^t &\sim t[U(L^1)\overline{L} + (1 - U(L^1))\underline{L}] + (1 - t)L^0 \\ &\sim t[U(L^1)\overline{L} + (1 - U(L^1))\underline{L}] + (1 - t)[U(L^0)\overline{L} + (1 - U(L^0))\underline{L}] \end{aligned}$$

Rearranging terms in last lottery we get

$$L^t \sim [tU(L^1) + (1 - t)U(L^0)]\overline{L} + [1 - tU(L^1) - (1 - t)U(L^0)]\underline{L}.$$

By definition of U, we have $U(L^t) = tU(L^1) + (1-t)U(L^0)$ as desired.

Problem 23.4 (Allais Paradox)**.** There are three possible outcomes: C = {win \$5m, win \$1m, win \$0), where "m" stands for million. Consider the following four lotteries, lottery 1A is $(0, 1, 0)$, that is win \$1m for sure, lottery 1B is $(0.1, 0.89, 0.01)$, lottery 2A is $(0, 0.11, 0, 89)$, and lottery 2B is $(0.1, 0, 0.9)$. How would you choose between lottery 1A and lottery 1B? How would you choose between lottery 2A and 2B?

A significant fraction of people would choose 1A over 1B, and would choose 2B over 2A. Show that that pair of choices is inconsistent with the maximization of a utility function having the expected utility form.

Problem 23.5. Consider the "maxmin" preference over lotteries where the individual has a complete and transitive preference over final outcomes and prefers lottery F to F' if the worst outcome that could occur in F is preferred to the worst outcome that could occur in F'. Is this preference over lotteries complete? Transitive? Continuous? Does it satisfy the independence axiom? Prove your answers.

24 Money lotteries and risk aversion (MWG 6.C)

Our main reference for this section is MWG 6.C. Please read that first. These notes add little to that section.

In this section we consider lotteries of which the outcomes are nonnegative quantities of money. Whereas in the previous section, we assumed that there are a finite number of outcomes, we relax that here. (This introduces some technical issues that MWG ignore, and I also ignore here. Kreps' chapters 5 and 6 discusses them, but you need not read that.) A lottery is a cumulative distribution function: $F : [0, \infty) \to [0, 1]$. The value $F(x)$ is the probability that the lottery yields an amount of money less than or equal to x.

We write $\mathrm{E}\, F$ to denote the expected value of F, that is $\mathrm{E}\, F = \int_0^\infty x\, dF(x)$.

We write δ_x to denote the lottery that yields the quantity $x \in [0, \infty)$ for sure.

We consider a preference $\succeq$ over lotteries F.

Definition 24.1. The preference $\succeq$ is *risk neutral* if $\delta_{\mathrm{E}\, F} \sim F$ for all F. It is *risk loving* if $\delta_{\mathrm{E}\, F} \preceq F$ for all F. It is *risk averse* if $\delta_{\mathrm{E}\, F} \succeq F$ for all F. It is *strictly risk averse* if $\delta_{\mathrm{E}\, F} \succ F$ for all F such that $F \neq \delta_{\mathrm{E}\, F}$.

Definition 24.2. Given a preference $\succeq$, the certainty equivalent of F, $c(F)$, is the value such that $F \sim \delta_{c(F)}$.

We have in mind a preference that is strictly increasing, so that $x > y$ implies $\delta_x \succ \delta_y$, and continuous over outcomes, so that for each F there exists a quantity $c(F)$ such that $F \sim \delta_{c(F)}$. (That the preference is strictly increasing implies that there is at most one such value $c(F)$.)

See MWG example 6.C.1, which shows that if insurance is actuarially fair, then a strict risk averter will fully insure.

See MWG example 6.C.2, which considers demand for a risky asset.

Problem 24.1. [[Diversification: Bernoulli writes "Another rule which may prove useful can be derived from our theory. This is the rule that it is advisable to divide goods which are exposed to some danger into several portions rather than to risk them all together. Again I shall explain this more precisely by an example. Sempronius owns goods at home worth a total of 4000 ducats and in addition possesses 8000 ducats worth of commodities in foreign countries from where they can only be transported by sea. However, our daily experience teaches us that of ten ships one perishes. Under these conditions I maintain that if Sempronius trusted all his 8000 ducats of goods to one ship his expectation of the commodities is worth 6751 ducats. That is..."]]

24.1 Comparison across individuals

Definition 24.3. Given a twice-differentiable Bernoulli utility function $u(\cdot)$ for money, the Arrow-Pratt coefficient of absolute risk aversion at x is $a(x) = -u''(x)/u'(x)$.

See MWG example 6.C.4, regarding constant absolute risk aversion (abbreviated CARA) utility: $u(x) = -e^{-ax},\ a > 0$.

Consider two utility functions u_1 and u_2 and the corresponding coefficients of absolute risk aversion $a_1(x) = -u_1''(x)/u_1'(x)$ and similarly for a_2 and u_2.

Definition 24.4. We say that u_2 is more risk averse than u_1 if $a_2(x) \geq a_1(x)$ for all x.

Proposition 24.1. *The following are equivalent*

(i) u_2 is more risk averse than u_1.

(ii) There exists an increasing, concave function $f(\cdot)$ such that $u_2(x) = f(u_1(x))$.

(iii) $c(F, u_2) \leq c(F, u_1)$ for any lottery F.

See the proof in MWG that (i) and (ii) are equivalent.

Problem 24.2. MWG 6.C.6 part (a): prove the equivalence of (ii) and (iii) above. [Hint: Jensen's inequality.]

24.2 Comparison across wealth levels

Definition 24.5. The Bernoulli utility function $u(\cdot)$ exhibits decreasing absolute risk aversion if $a(x)$ is a decreasing function of x.

We could now reapply the results of the previous section, treating the individual at two different wealth levels as two different individuals.

Problem 24.3. MWG Exercise 6.C.8. In MWG Example 6.C.2 continued (on their page 192), they show that a more risk averse individual will invest less in a risky asset. In this exercise you are asked to show that if an individual has decreasing absolute risk aversion they will invest less in the risky asset at lower wealth levels.

Definition 24.6. Given a Bernoulli utility function $u(\cdot)$, the coefficient of relative risk aversion at x is $r(x) = -xu''(x)/u'(x)$.

Example 24.1. Constant relative risk aversion (CRRA) utility $u(c) = \frac{c^{1-\eta}-1}{1-\eta}$ for $\eta \neq 1$, or $u(c) = \ln(c)$ for $\eta = 1$.