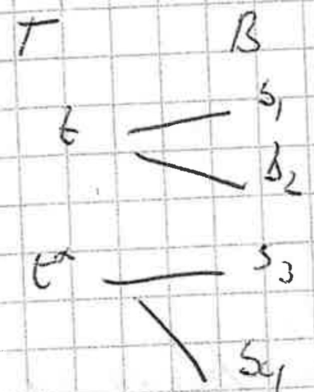
Then, Thm 4 $\Rightarrow$ true prefs $(\succsim_i, \tilde{\succsim}_{-i})$ $C = \{i\}$

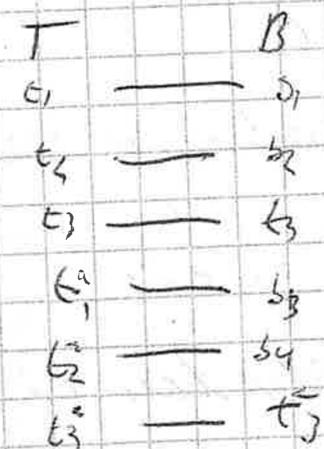
HW 5 Solutions

$T = \{t, t^*\}$ $B = \{b_1\ b_2\ b_3\ b_4\}$ $q_t = q_{t^*}$

μ



$\hat{\mu}$



Q1: Many-to-one $\mathcal{E}_\mu$ One-to-one $\hat{\mathcal{E}}_{\hat{\mu}}$

(One-to-one $\Rightarrow$ many-to-one)

If $\hat{\mu}$ is stable in $\hat{\mathcal{E}}$, then μ is stable in $\mathcal{E}$.

Lemma 1 If $\mathcal{E}$ responsive preference and there is a t such that it he prefers being alone to being matched ($t \succsim_t \tilde{B}$) $\exists\, b \in \tilde{B}$ with $t \succ_t \{b\}$

Proof Let $\tilde{B} = \{b_1, \ldots, b_m\}$. Suppose each $b_m \in \tilde{B}$, $\{b_m\} \succ t$

$$\tilde{B} = \{b_1, \ldots b_m\} = (\{b_1, \ldots, b_{m-1}\} \cup \{b_m\})$$

$$\succsim_t \{b_1, \ldots, b_{m-1}\}$$

$$\succsim_t \{b_1\}$$

$$\succ_t t$$

Lemma 2 If $\succsim$ responsive pref and $B_1 \subseteq B_2$ and $B_1 \succ_t B_2$, then $\exists\, b \in B_2 \setminus B_1$ w/ $t \succ_t \{b\}$

Proof: Suppose $B_2 \setminus B_1 = \{b_1, \dots, b_m\}$ and for each $b_m \in B_2 \setminus B_1$, $\{b_m\} \succsim_t t$

$B_2 = B_1 \cup \{b_1, \dots, b_m\}$

$\succsim_t B_1 \cup \{b_1, \dots, b_{m-1}\}$

$\succsim_t B_1$

Contradict the condition.

$\hat{\mu}$ stable $\Rightarrow$ μ is stable

Proof: (A) No individual blocks μ. (B) No pair blocks μ.

(A) Suppose that t blocks μ. Then,

(i) $t \succ_t \mu(t)$
(ii) $\exists\, \tilde{B} \subseteq \mu(t) \subseteq B$ with $\tilde{B} \succ_t \mu(t)$

Lemma 1 + 2 $\Rightarrow$ in either case, there exists some $b \in \mu(t)$ s.t. $t \succ_t \{b\}$

There exist some duplicate of t, say t_k, with $\hat{\mu}(t_k) = b$, and t_k must preserve same preference as t and $t_k \succ_{t_k} b = \hat{\mu}(t_k)$

$\hat{\mu}$ not IR (Suppose b blocks μ)

$b \succ_b \mu(b) = t$ $\quad$ $\hat{\mu}(b) = \mu(b)$ not IR

(B) Suppose (t^*, b^*) blocks μ.

(1) By definition, $\exists\, \tilde{B} \subseteq \mu(t^*)$ s.t. $|\tilde{B} \cup \{b^*\}| \leq q_{t^*}$ and $\tilde{B} \cup \{b^*\} \succ_{t^*} \mu(t^*)$

(2) $t^\alpha \succ_{b^\alpha} \mu(b^\alpha)$

Let t_k^α be the duplicate of t^α and t_m dup of $\mu(b^\alpha)$

Let t_k^α be the dup of t^α and t_m dup of $\mu(b^\alpha)$

We have $t_k^\alpha \hat{\succ}_{b^\alpha} t_m$.

Discuss (1) $|\mu(t^\alpha)| = q_{t^\alpha}$

Let $\tilde{B} \cup \{b^\alpha\}$ ~~and~~ $= B_1$ and $B_2 = \mu(t^\alpha)$

$B_1 \setminus B_2 = b^\alpha \neq \phi$

$|B_2| \geq |B_1|$

$B_1 \succ_{t^\alpha} B_2$

From these results in lecture notes, $\exists b \in B_2 \setminus B_1$ such that $\{b^\alpha\}$ is preferred to b by t^α

$\{b^\alpha\} \succ_{t^\alpha} \{b\}$

Since $b \in \mu(t^\alpha)$ $\exists t^\alpha$ so that $\hat{\mu}(t_k^\alpha) = b$. and he must prefer b^α to b.

$b^\alpha \succ_{t_k^\alpha} b \Rightarrow$ Contradiction

(2) $|\mu(t^\alpha)| < q_{t^\alpha}$

$\tilde{B} \cup \{b^\alpha\} \succ_{t^\alpha} \mu(t^\alpha)$ and by IR

$\mu(t^\alpha) \succsim_{t^\alpha} \tilde{B}$. Thus $\tilde{B} \cup \{b^\alpha\} \succ_{t^\alpha} \tilde{B}$

by monotonicity

By responsive pref. $\{b^*\} \succ_{t^\alpha} t^\alpha$

Because $|\mu(t^\alpha)| < q_{t^\alpha}$

$\exists\, t^\alpha_k$ so that $\hat{\mu}(t^\alpha_k) = t^\alpha_k$ (b^*, t^α_k)

blocks $\hat{\mu}$.

(μ stable $\Rightarrow \hat{\mu}$ stable)

$\hat{\mu}$ has one additional condition to those from class.

If $\mu(t) \subseteq B$, then to each $b \in \mu(t)$ $\exists$ duplicate t_k s.t. $\hat{\mu}(t_k) = b$

Moreover, choose this match such that if $b, b' \in \mu(t)$

and $\{b\} \succ_t \{b'\}$ and $\hat{\mu}(t_j) = b$
$\hat{\mu}(t_k) = b'$

then $j > k$.

To show (A) $\hat{\mu}$ IR. (B) No pair blocks.

(A) $\hat{\mu}$ IR.

If $b \hat{\succ}_b \hat{\mu}(b)$, then $b \succ_b \mu(b)$

If $t_k \hat{\succ}_{t_k} \hat{\mu}(t_k)$, t_k dup t

Then $\succ_t \{\hat{\mu}(t_k)\}$ By responsiveness

$t \succ_t \mu(t) \setminus \{\hat{\mu}(t_k)\}$

(B) No pair blocks. Suppose (t_k^α, b^α) blocks.

means: $b^\alpha \hat{>} \hat{\mu}(t_k^\alpha)$

$t_k^\alpha \hat{>}_{b^\alpha} \hat{\mu}(b^\alpha)$

① $\mu(b^\alpha) = t^\alpha$, ie $\hat{\mu}(b^\alpha) = t^\alpha$ for t_j^α
duplicate t^α

Because $t_k^\alpha >_{b^\alpha} t_j^\alpha \Rightarrow k < j$

$\Rightarrow \{\hat{\mu}(t_k^\alpha)\} \gtrsim_{t^\alpha} \{b^\alpha\}$ Contradiction.

② $\mu(b^\alpha) = t' \neq t^\alpha$

$t^\alpha >_{b^\alpha} \mu(b^\alpha)$

$\Rightarrow \exists \tilde{B} \subseteq \mu(t^\alpha)$ s.t. $\{b^\alpha\} \cup \tilde{B}$

$>_{t^\alpha} \mu(t^\alpha)$ so that

(t^α, b^α) blocks μ

Hence: Since $b^\alpha \hat{>}_{t_k^\alpha} \hat{\mu}(t_k^\alpha)$ $\exists b \in \mu(t^\alpha)$

s.t. $\{b^\alpha\} >_{t^\alpha} \{b\}$ Let $\tilde{B} = \mu(t^\alpha) \setminus \{b\}$

Then $\{b^\alpha\} \cup \tilde{B} >_{t^\alpha} \mu(t^\alpha)$

Q2

Apply RHT on induced $\hat{\mathcal{E}}, \hat{\mu}, \hat{\mu}'$

by inducing environment w/ strict preference

Lemma 4 $\mathcal{E}$ str + responsive pref

Let $\hat{\mu}_{BS}$ be the B-prop algorithm in $\hat{\mathcal{E}}$

Let μ_{BB} induced by $\hat{\mu}_{BB}$

Then μ_{BB} is B-optimal

Lemma 5

Fix $\mathcal{E}$ str + resp pref and μ and μ' stable in $\mathcal{E}$. If $|\mu(t)| < q_t$

~~$|\mu(t)|$~~ $\mu(t) = \mu'(t)$

Proof: For any such $\mu(t) = \mu_{BS}(t)$

TO show $\mu_{B_0}(t) \setminus \mu(t) = \emptyset$

If not there is a $b \in \mu_{BB}(t) \setminus \mu(t)$

B/c μ_{BB} is B optimal, $\mu_{BB}(s)$ must prefer $\mu_{BB}(s)$ to

$t \succ_b \mu(s)$

$|\mu(t)| < q_t$ $\hat{B}$