# Lecture 6

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## Review from last lecture

Order $\geq_I$ for $I \in \{T, B\}$

$\mu \geq_i \mu'$ if for each $i : \mu(i) \succsim_i \mu'(i)$

$\mu >_i \mu'$ if $\mu \geq_I \mu'$ and there exists an $i \in I$ such that $\mu(i) \succ_i \mu'(i)$

## Theorem (Knuth)

*Suppose preferences are strict. If $\mu$ and $\mu'$ are stable then $\mu >_T \mu'$ if and only if $\mu' >_B \mu$*

**Proof**: Suppos $\mu >_T \mu'$ : to show:

(1) for each $b \in B$, $\mu'(b) \succsim_b \mu(b)$
(2) for some $b \in B$, $\mu'(b) \succ_b \mu(b)$

To show (1): Suppose not, i.e. there is some $b \in B$ such that $\mu(b) \succ_b \mu'(b)$

- In other words will construct a blocking pair to $\mu'$ with $b$, contradicting stability.
- Notice: $\mu(b) \in T$ since $\mu'$ is stable, $\mu'(b) \succsim_b b$ - that is it is individually rational.
- Transitivity: $\mu(b) \succ_b \mu'(b) \succsim_b b$ this implies that $\mu(b) \in T$
- Let $t \in \mu(b)$. Since $\mu(b) \neq \mu'(b) \implies \mu(t) \neq \mu'(t)$ and since preferences are strict and $\mu >_T \mu' \implies \mu(t) \succ_t \mu'(t)$
- Then $(t, b)$ blocks $\mu'$ contradicting that $\mu'$ is stable

To show (2):

We know for all $b \in B, \mu'(b) \succsim_b \mu(b)$ by (1).

Also, we know that $\exists t : \mu(t) \succ_t \mu'(t)$

Now notice that because $\mu(t) \neq \mu'(t)$ one of these matchinigs, maybe both, $t$ has to be matched to some $b$ agent. And that $b$ agent cannot have the same match over $\mu$ and $\mu'$.

So, this implies that $\exists b : \mu(b) \neq \mu'(b)$.

We know that $\mu'(b) \succsim_b \mu(b)$ by (1). Then by strict preferences, we have: $\mu'(b) \succ_b \mu(b)$.

### Idea

Fix two matches $\mu$ and $\mu'$ where:

$$\mu : (T \cup B) \to (T \cup B)$$

$$\mu' : (T \cup B) \to (T \cup B)$$

Define **T-Join** or T-sup.

$$\lambda_T : (T \cup B) \to (T \cup B)$$

$$\lambda_T(t) = \begin{cases} \mu(t) & \mu(t) \succsim_t \mu'(t) \\ \mu'(t) & \mu'(t) \succ_t \mu(t) \end{cases}$$

$$\lambda_T(b) = \begin{cases} \mu(b) & \mu'(b) \succsim_t \mu(b) \\ \mu'(b) & \mu(b) \succ_t \mu'(b) \end{cases}$$

## Proposition

*Suppose preferences are strict and $\mu : (T \cup B) \to (T \cup B)$ and $\mu' : (T \cup B) \to (T \cup B)$ are stable. Then the T-Join of $\mu$ and $\mu'$ is a stable match. Moreover:*

(1) $\lambda_T \geq_T \mu$ and $\lambda_T \geq \mu$
(2) $\mu \geq_B \lambda_T$ and $\mu' \geq_B \lambda_T$

## Lemma 1

Suppose preferences are strict and $\mu$ and $\mu'$ are stable. Then, $\lambda_T(t) = \mu(t)$ if and only if $\lambda_T(\lambda_T(t)) = \mu(\lambda_T(t))$. This is the $b$ agent $(\lambda_T(t))$ point right back at the $t$ who it was assigned to by $\lambda_T$

**Proof**:

Part 1: Suppose that $\lambda_T(t) = \mu(t)$. Now this fact tells us that $\mu(t) \succsim_t \mu'(t)$ by definition of $\lambda_T$. So, one possibility is that they are indifferent. Notice, that if this is the case since preferences are strict, we have to have that $\mu(t) = \mu'(t)$ because preferences are strict. Let's call that match $b$.

Then $\mu'(b) = \mu(b) = \lambda_T(b) = t$.

Then, the case we have to worry about is: $\mu(t) \succ_t \mu'(t)$

Suppose $\lambda_T(t) = \mu(t) = b$. Then: $b \succ_T \mu'(t) \implies \mu'(t) \neq b$ Then, this implies that $\mu(b) = t \neq \mu'(b)$

By strict preferences: Either

(a) $t = \mu(b) \succ_b \mu'(b)$ or
(b) $\mu'(b) \succ_b \mu(b) = t$.

If (a) holds the we also have:

- $t \succ_b \mu'(b)$
- $b \succ_t \mu'(t)$ and then both of these would form a blocking pair, contradicting that $\mu'$ is stable.

If (b) holds, then:

- $\mu'(b) \succ_b \mu(b) \implies \lambda_T(b) = \mu(b)$

Part 2: We want that $\lambda_T(b) = \mu(b) \implies \lambda_T(\lambda_T(b)) = \mu(\lambda_T(b))$

Notice that is just same argument before (above) so we are done.

## Proof of Proposition

Want to show:

(1) $\lambda_T$ is a match
(2) $\lambda_T$ is stable.

**Part 1**: want to show:

$$\lambda_T(t) = \mu(t) = b \iff \lambda_T(b) = \mu(b) = t$$

$$\lambda_T(t) = \mu'(t) = b \iff \lambda_T(b) = \mu'(b) = t$$

And this is just Lemma 1.

**Part 2**: want to show that $\lambda_T$ is individually rational since both $\mu$ and $\mu'$ are individually rational. In other words, show that there is no blocking.

- Suppose there is a blocking pair $(t^*, b^*)$ that blocks $\lambda_T$.
- Since $\mu$ is stable, it cannot be that $\lambda_T(t^*) = \mu(t^*)$ and $\lambda_T(b^*) = \mu(b^*)$
- and since $\mu'$ is stable, it cannot be that $\lambda_T(t^*) = \mu'(t^*)$ and $\lambda_T(b^*) = \mu'(b^*)$
- Hence, either (a) $\lambda(t^*) = \mu(t^*)$ and $\lambda_T(b^*) = \mu'(b^*)$ or (b) $\lambda(t^*) = \mu'(t^*)$ and $\lambda_T(b^*) = \mu(b^*)$

If (a):

- $b^* \succ_{t^*} \lambda_T(t^*) = \mu(t^*) \succsim_{t^*} \mu'(t^*)$
- then $t^* \succ_{b^*} \lambda_T(b^*) = \mu'(b^*)$
- This implies that $(t^*, b^*)$ blcoks $\mu'$ and this cannot be.

If (b):

- $b^* \succ_{t^*} \lambda_T(t^*) = \mu'(t^*) \succsim_{t^*} \mu(t^*)$
- $t^* \succ_{b^*} \lambda_T(b^*) = \mu(b^*)$
- This implies that $(t^*, b^*)$ blocks $\mu$ and this cannot be.

## Theorem (Rural Hospital Theorem)

*Suppose preferences are strict. If $\mu$ and $\mu'$ are stable, then the set of agents:*

$$\{i \in T \cup B : \mu(i) = i\} = \{i \in T \cup B : \mu'(i) = i\}$$

### Proof

Define: $I[\mu] = \{i \in I : \mu(i) \neq i$ as matched under $\mu$

$$|T[\mu]| = |B[\mu]|$$

Fix $\mu$ as stable. It suffices to show that:

$$(a) : T[\mu] \subseteq T[\mu_{TD}]$$
$$(b) : B[\mu_{TD}] \subseteq B[\mu]$$

Where || implies cardinality.

If the above two statements (a) and (b) are true:

$$|T[\mu]| \leq |T[\mu_{TD}]| = |B[\mu_{TD}]| \leq |B[\mu]| = |T[\mu]| \implies |T[\mu]| = |B[\mu]| = |B[\mu_{TD}]|$$

Since $T[\mu] \subseteq T[\mu_{TD}]$ and $|T[\mu]| = |T[\mu_{TD}]|$ we can conclude that $|B[\mu]| = |B[\mu_{TD}]|$

To show (a): $T[\mu] \subseteq T[\mu TD]$

- Fix $t \in T[\mu] \implies \mu(t) \neq t$
- $\mu_{TD}(t) \succsim_t \mu(t) \succsim_t t$ by IR or $\mu$. And since preferences are strict, we must have: $\mu(t) \succ_t t$ as $\mu(t) \neq t$ and strict preferences.
- Hence $\mu_{TD} \succ_t t \implies \mu_{TD}(t) \neq t$ and $t \in T[\mu_{TD}]$

To show (b): $B[\mu_{TD}] \subseteq B[\mu]$

- Fix $b \in B[\mu_{TD}]$
- $\mu_{TD}(b) \neq b$
- $\mu(b) \succsim_b \mu_{TD}(b) \succsim_b b$ Now, it must be the case that $\mu_{TD}(b) \succ_b b$ since $\mu_{TD}(b) \neq b$ and strict preferences
- Above implies that $\mu(b) \succ_b b \implies \mu(b) \neq b \implies b \in B[\mu]$

Done