* Second Welfare Theorem (continue..)

· price equilibrium with transfers

$((x^*, y^*), p^*)$: $\exists\ W = (W_1, \dots, W_I)$ s.t.

① $p^* \cdot y_j^* \geq p^* \cdot y_j$ for all $y_j \in Y_J$

② $x_i^* \in B_i^*(p^*, W_i) = \{x_i : p^* x_i \leq W_i\}$

and $x_i^* \succsim_i x_i$ for all $x_i \in B_i^*(p^*, W_i)$

③ $\sum_{i=1}^{I} x_i^* = \bar{w} + \sum_{j=1}^{J} y_j^*$

④ $\sum_{i=1}^{I} W_i = \sum_{i=1}^{I} p^* W_i + \sum_{j=1}^{J} p^* \cdot y_j^*$

· price quasi-equilibrium with transfers

② $x_i \succ x_i^* \Rightarrow p^* \cdot x_i \geq W_i$

· Theorem

Fix an economy where (a) each Y_j is convex and (b) each $\succsim_i$ is convex and LNS

Then, for every pareto optimal allocation (x^*, y^*), $\exists$ a price vector $p^* \neq (0, \dots, 0)$ s.t. $((x^*, y^*), p^*)$ constitute a price quasi-equilibrium with transfers.

prop Fix an economy where (a) each $X_i \subseteq \mathbb{R}^L$ is convex and contains $(0, \dots, 0)$ and (b) each $\succsim_i$ is continuous.

If $((x^*, y^*), p^*)$ is a price quasi-equilibrium with strictly positive transfers for each i, then it is a price equilibrium with transfers.

proof) of thm Fix (x^*, y^*) pareto optimal

$V = \{\sum_i x_i : \text{for each } i,\ x_i \in X_i \text{ and } x_i \succ x_i^*\} \subseteq \mathbb{R}^L$

$Y = \{\sum y_j : \text{for each } j,\ y_j \in Y_j\} \subseteq \mathbb{R}^L$

$\bar{w} + Y = \{z \in \mathbb{R}^L : z = \bar{w} + z' \text{ for some } z' \in Y\}$

Step 1) $\exists\ p^* \neq (0, \dots, 0)$ and $r \in \mathbb{R}$ s.t. (i) $p^* \cdot z \geq r \quad \forall z \in V$

(ii) $r \geq p^* \cdot z \quad \forall z \in \bar{\omega} + Y$

Step 2) Define $W_i = p^* \cdot x_i^*$ and show $\sum W_i = p^* \cdot \bar{\omega} + \sum_j p^* \cdot y_j^*$

④ of quasi-eq

Step 3) If $(x_1, \dots x_I)$ is for each i, $x_i \succsim_i x_i^*$, then $\sum_i p^* \cdot x_i \geq r$

Step 4) $\sum W_i = p^* \cdot w + \sum p^* \cdot y_j^* = r$

Step 5) Show conditions ①+② of a quasi-equilibrium. Get ③ for free

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Step 1) Apply the separating hyperplane theorem.

Need : ① V is convex

② $\bar{\omega} + Y$ is convex

③ $V \cap (\bar{\omega} + Y) = \emptyset$

price vector ↓ total expenditure ↓

If so, the S.H.T. tells us that $\exists\ p^* \neq 0$ and $r \in \mathbb{R}$ satisfying (i), (ii)

(i) V is convex

- $\succsim_i$ is convex $\Rightarrow$ $V_i \equiv \{x_i \in X_i : x_i \succ_i x_i^*\}$ is convex

If $x_i, x_i' \in V_i$, then $x_i > x_i^*$, $x_i' > x_i^*$ $\Rightarrow$ $\succsim_i$ convex gives $\alpha x_i + (1-\alpha) x_i' > x_i^*$ for all $\alpha \in [0,1]$

- $z, z' \in V$

$z = \sum x_i$ s.t. $x_i \in V_i$ for each i

$z' = \sum x_i'$ s.t. $x_i' \in V_i$ for each i

$\alpha z + (1-\alpha) z' = \sum_{i=1}^{I} \alpha x_i + \sum_{i=1}^{I} (1-\alpha) x_i' = \sum_{i=1}^{I} \underbrace{(\alpha x_i + (1-\alpha) x_i')}_{\in V_i,\ \text{since } V_i \text{ is convex}}$

$\Rightarrow \alpha z + (1-\alpha) z' \in V$

(2) $\overline{\omega} + Y$ is convex

- given each Y_j is convex $\Rightarrow$ Y is convex (check at home)

$\Rightarrow$ $\overline{\omega}+Y$ is convex (")

($\because$ Sum of convex sets is convex)

(3) $V \cap (\overline{\omega}+Y) = \phi$

Suppose not; $\exists\ z \in V \cap (\overline{\omega}+Y)$

Then, (1) $\exists\ \underbrace{(x_1, \dots, x_I)}_{\text{Call "x"}}$ s.t. $z = \sum_{i=1}^{I} x_i$ and $x_i \succ x_i^*$, $\forall i$ $(z \in V)$

(2) $\exists\ \underbrace{(y_1, \dots, y_J)}_{\text{Call "y"}}$ s.t. $z = \overline{\omega} + \sum_{j=1}^{J} y_j$ $(z \in \overline{\omega}+Y)$

$\Rightarrow$ $\sum_{i=1}^{I} x_i = \overline{\omega} + \sum_{j=1}^{J} y_j$ $\Rightarrow$ (x, y) is a feasible allocation with $x_i \succsim x_i^*$, $\forall i$.

$\Rightarrow$ But, contradicts (x^*, y^*) pareto optimal.

Therefore, $V \cap (\overline{\omega}+Y) = \phi$.

Chosen Step 1

Step 2) Define $W_i = p^* \cdot x_i^*$ and show $\sum_i W_i = p^* \cdot \overline{\omega} + \sum_j p^* \cdot y_j^*$

Take $W_i = p^* \cdot x_i^*$

Note: $x_i^* = (x_{1i}^*, \dots, x_{Li}^*)$, $p^* = (p_1^*, \dots, p_L^*)$

WTS: $\sum_{i=1}^{I} W_i = p^* \cdot \overline{\omega} + \sum_j p^* \cdot y_j^*$

It suffices to show $\sum_{i=1}^{I} p^* \cdot x_i^* = p^* \cdot \overline{\omega} + \sum_{j=1}^{J} p^* \cdot y_j^*$

$$\sum_{i=1}^{I} p^* \cdot x_i^* = \sum_{l=1}^{L} p_l^* \left(\sum_{i=1}^{I} x_{li}^* \right) \underset{\Uparrow}{=} \sum_{l=1}^{L} p_l^* \left(\overline{\omega}_l + \sum_{j=1}^{J} y_{lj}^* \right) = p^* \cdot \overline{\omega} + \sum_{j=1}^{J} p^* \cdot y_j^*$$

Since (x^*, y^*) is feasible

Step 3) If $(x_1, \dots, x_I)$ is s.t. for each i. $x_i \succsim x_i^*$.

then $\sum_i p^* \cdot x_i \geq r$ (Use LNS)

Fix $(x_1, \dots, x_I)$ s.t. $x_i \succsim_i x_i^*$ for each i

WTS: $p^* \cdot \sum_{i=1}^{I} x_i = \sum_{i=1}^{I} p^* \cdot x_i \geq r$

LNS: for each $n \geq 1$, $\exists$ some $(x_1^n, \dots, x_I^n)$ with (a) $x_i^n \succ_i x_i$ for each i
and (b) $\| x_i^n - x_i \| < \frac{1}{n}$ for each i

$\Rightarrow \sum_{i=1}^{I} x_i^n \in V$ for each $n \geq 1$. $x_i^n \succ_i x_i \succsim x_i^*$ ($\because x_i^n \succ_i x_i^* \Rightarrow x_i^n \in V_i$
$\Rightarrow \sum_{i=1}^{I} x_i^n \in V$)

$\Rightarrow$ for each n, $p^* \cdot \sum_{i=1}^{I} x_i^n \geq r \Rightarrow p^* \cdot \sum_{i=1}^{I} x_i \geq r$ (by (b))

(By step 1 and $\sum_{i=1}^{I} x_i^n \in V$)

Step 4) $\sum W_i = p^* \cdot \bar{\omega} + \sum p^* \cdot y_j^* = r$

WTS: $\sum_{i=1}^{I} W_i = r$

By Step 2: suffices to show $\sum_{i=1}^{I} p^* \cdot x_i^* = r$

By Step 3: $\sum_{i=1}^{I} p^* \cdot x_i^* \geq r$

Note Since (x^*, y^*) is feasible,

$\sum_i x_i^* = \bar{\omega} + \sum_j y_j^*$

implies that

$\bar{\omega} + \sum_j y_j^* \in \bar{\omega} + Y$

& $\sum x_i^* \in \bar{\omega} + Y$

$\Downarrow$

$p^* \cdot \sum_{i=1}^{I} x_i^* \leq r$ (from step 1)

$\Rightarrow \sum_{i=1}^{I} p^* \cdot x_i^* \leq r$

Step 5) Show conditions ① + ② of a quasi-equilibrium.

5a) $(y_1^*, \dots, y_J^*)$ maximizes profit given p^* (for each j)

Fix a firm j and $y_j \in Y_j \Rightarrow y_j + \sum_{k \neq j} y_k^* \in Y$

$$p^* \cdot \bar{\omega} + p^* \cdot y_j + p^* \cdot \sum_{k \neq j} y_k^* \underset{\text{Step 1}}{\leq} r \underset{\text{Step 4}}{=} p^* \cdot \bar{\omega} + \sum_{j=1}^{J} p^* \cdot y_j^* \Rightarrow p^* \cdot y_j \leq p^* \cdot y_j^*$$

5b) Suppose $x_i > x_i^*$ for some i

$$p^* \cdot x_i + p^* \cdot \sum_{k \neq i} x_k^* \geq r = p^* \cdot \sum_{k=1}^{I} x_k^*$$

By Step 3   By Step 1 (def of W_i) and Step 4

Then, $p^* \cdot x_i \geq p^* \cdot x_i^* = W_i$ ||

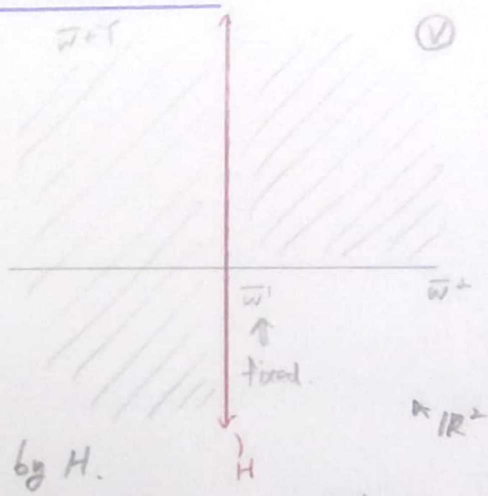
→ Through this process, we can get the condition for price quasi-eq. with st. positive tr.

V

Evil example) $(x_1^*, x_2^*) = (W_1, W_2) = ((0, \bar{w}_2), (\bar{w}_1, 0))$

$V = \{ x_1 + x_2 : x_1 + x_2 \in \mathbb{R}_+^2$ and each $x_i \succsim w_i \}$

$Y = \mathbb{R}_-^2$



By definition,
$\mathbb{R}^+$ can be separated by H.

$(p_1^*, p_2^*) \neq 0$ but $p_1^* > 0$
$p_2^* = 0$

then $W_2 = p_2^* \cdot x_2^* = 0$