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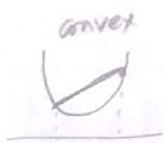
* Concave and convex functions



(Def) Let $C \subseteq \mathbb{R}^n$ be convex

The function $f : C \to \mathbb{R}$ is (strictly) convex

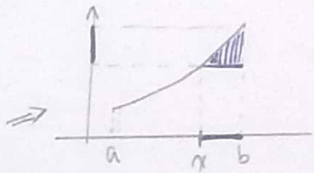
if for all $x', x \in C$ and $\lambda \in [0,1]$, $(1-\lambda) f(x) + f(x') \underset{(>)}{\geqq} f((1-\lambda)x + \lambda x')$

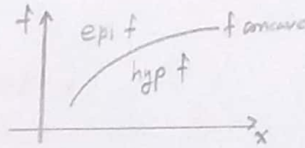
$(\lambda \in (0,1))$

The function $f : C \to \mathbb{R}$ is (strictly) concave if $-f$ is (strictly) convex

The hypograph of the function $f : C \to \mathbb{R}$ is defined as

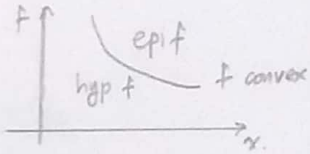
$$\text{hyp } f := \{ (x, y) \in \mathbb{R}^{n+1} \mid x \in C \text{ and } f(x) \geqq y \}$$





The epigraph of the function $f : C \to \mathbb{R}$ is defined as

$$\text{epi } f := \{ (x, y) \in \mathbb{R}^{n+1} \mid x \in C \text{ and } f(x) \leqq y \}$$



(Thm 5.7) Let $C \subseteq \mathbb{R}^n$ be convex and $f : C \to \mathbb{R}$. The f is convex (concave) if and only if its epigraph (hypograph) is convex.

proof) ($\Rightarrow$) Let f is concave. (x, y) and $(x', y') \in \text{hyp } f$, and $\lambda \in [0,1]$.

Then $y^\lambda := (1-\lambda) y + \lambda y' \leqq (1-\lambda) f(x) + \lambda f(x') \leqq f((1-\lambda)x + \lambda x') =: f(x^\lambda)$

This implies that $(x^\lambda, y^\lambda) \in \text{hyp } f$

($\Leftarrow$) Suppose the hypograph is convex.

Given arbitrary $x, x' \in C$ and $\lambda \in [0,1]$,

$(x, f(x))$ and $(x', f(x'))$ lie in the hypograph.

Setting $x^\lambda := (1-\lambda)x + \lambda x'$, we get $(x^\lambda, (1-\lambda) f(x) + \lambda f(x'))$ lies in the hypograph. Therefore, $(1-\lambda) f(x) + \lambda f(x') \leqq f(x^\lambda)$

Since this holds for arbitrary $x, x' \in C$, and $\lambda \in [0,1]$, f is concave. //

Thm 58 Let $C \subseteq \mathbb{R}^n$ be convex and $f: C \to \mathbb{R}$ be concave

If x^* is a local maximizer of f,

then it is a global 〃 that is $f(x) \leq f(x^*)$ for all $x \in C$.

proof) Fix an arbitrary $x \in C$.

Since x^* is a local maximizer, we get that for $\lambda > 0$ small enough

$$f(x^*) \geqq f((1-\lambda)x^* + \lambda x) \geqq (1-\lambda)f(x^*) + \lambda f(x)$$

Rearranging this, we get that $f(x^*) \geqq f(x)$. //

$$(\because f(x^*) \geq (1-\lambda)f(x^*) + \lambda f(x)$$
$$\Downarrow$$
$$\lambda f(x^*) \geq \lambda f(x) \quad \text{i.e.} \quad f(x^*) \geqq f(x))$$

Def Let $X \subseteq \mathbb{R}^n$. $x \in X$ is <u>an extreme point</u> of X

if there are no distinct $x_1, x_2 \in X$ such that $x = \frac{x_1 + x_2}{2}$.

Thm 59 Let $C \subseteq \mathbb{R}^n$ be convex, $X \subseteq C$, and let $f: C \to \mathbb{R}$ be strictly convex.

If <u>x^* maximizes f</u> over X, then it is <u>an extreme point of X</u>.

In particular, if X is compact, then the set of extreme points is non-empty.

proof) Suppose for contradiction that x^* is not an extreme point of X.

Then there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $x^* = \frac{x_1 + x_2}{2}$.

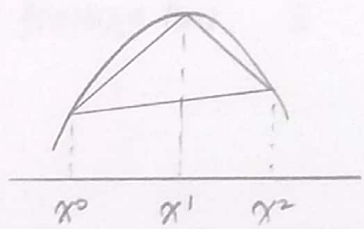
Thus, $f(x^*) < \frac{1}{2}[f(x_1) + f(x_2)]$ because f is strictly convex.

This contradicts optimality of x^*. If X is compact, the Extreme Value thm implies that every continuous function f achieves maximum over X at some $x^* \in X$. If f is in addition strictly convex, (as for example $f(x) = \|x\|^2$) then the first part implies that x^* is an extreme point of X. //

Lemma 11 (Three-chord-lemma)

Let $[a,b] \subseteq \mathbb{R}$, $x^0, x^1, x^2 \in (a,b)$ satisfy $x^0 < x^1 < x^2$ and let $f: [a,b] \to \mathbb{R}$ be concave. Given $x, y \in [a,b]$ satisfying $x \neq y$ define $S(x,y) = \dfrac{f(y)-f(x)}{y-x}$

Then, $S(x^0, x^1) \geqq S(x^0, x^2) \geqq S(x^1, x^2)$



proof) $x^1 = \dfrac{x^2 - x^1}{x^2 - x^0} x^0 + \dfrac{x^1 - x^0}{x^2 - x^0} x^2$

Then, $f(x^1) \geqq \dfrac{x^2 - x^1}{x^2 - x^0} f(x^0) + \dfrac{x^1 - x^0}{x^2 - x^0} f(x^2)$ by concavity.

which implies $f(x^1) - f(x^0) \geqq \dfrac{x^1 - x^0}{x^2 - x^0} [f(x^2) - f(x^0)]$

Hence we get the first enequility. The second follows similarly. //

Thm 60 Let $[a,b] \subseteq \mathbb{R}$ and $f: [a,b] \to \mathbb{R}$ be concave

For any $x, y \in \mathbb{R}$ satisfying $a < x < y < b$, the one-sided dellvative,

$$f'_+(x) := \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'_-(x) := \lim_{h \uparrow 0} \frac{f(x+h) - f(x)}{h}$$ exist and satisfy

$$f'_-(x) \geqq f'_+(x) \geqq \frac{f(y) - f(x)}{y-x} \geqq f'_-(y) \geqq f'_+(y)$$

proof) HW. (HINT: Lemma 11)

Thm 6.1) Let $[a,b] \subseteq \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ be concave.

Then f is continuous on $[a,b]$

proof) Fix $x \in (a,b)$. Then $\lim_{h \downarrow 0} f(x+h) - f(x) = \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \downarrow 0} h = 0$

The analogous argument for $h \uparrow 0$ yields $\lim_{h \to 0} f(x+h) = f(x)$ //

Lemma 13

Let $(a,b) \subseteq \mathbb{R}$ and $f : (a,b) \to \mathbb{R}$ be twice continuously differentiable

Then f is concave if and only if $f''(x) \leqq 0$ for all $x \in (a,b)$

proof) : ($\Rightarrow$) follows from the three - chord lemma //.

($\Leftarrow$) Since f is twice continuously differentiable and $f''(x) \leqq 0$,

this implies (Problem set 8) $f'(x)$ is monotone decreasing.

Given arbitrary $x, x' \in (a,b)$ satisfying $x < x'$ and $\lambda \in (0,1)$,

we set $x^\lambda = (1-\lambda)x + \lambda x'$

Then, $$\frac{f(x^\lambda) - f(x)}{x^\lambda - x} \underset{\substack{\uparrow \\ \text{(mean value thm)}}}{\geqq} \inf_{t \in (x, x^\lambda)} f'(t) \underset{\substack{\uparrow \\ \text{Lemma 11}}}{\geqq} \sup_{t \in (x^\lambda, x')} f'(t) \underset{\substack{\uparrow \\ \text{(Mean value thm)}}}{\geqq} \frac{f(x') - f(x^\lambda)}{x' - x^\lambda}$$

here the first and last inequality follow from the mean value thm.

Note that $x^\lambda - x = \lambda(x' - x)$ and $x' - x^\lambda = (1-\lambda)(x' - x)$

This yields.

$$(1-\lambda)\left(f(x^\lambda) - f(x)\right) \geqq \lambda\left(f(x') - f(x^\lambda)\right)$$

or

$$\underbrace{f(x^\lambda) \geqq (1-\lambda) f(x) + \lambda f(x')}_{(f \text{ is concave})} \quad //$$

Def An $n \times n$ matrix is negative (semi) definite

if for all $x \in \mathbb{R}^n$, $x^T A x < 0$
$(\leq)$

An $n \times n$ matrix is positive (semi) definite

if for all $x \in \mathbb{R}^n$, $x^T A x > 0$
$(\geq)$

twice continuously differentiable

Let $X \subseteq \mathbb{R}^n$ be open if $f : X \to \mathbb{R}$ is of class C^2

Then its Hessian Matrix $D^2 f(x)$ is defined by

$$D^2 f(x) := D \nabla f(x) = \left[\frac{\partial^2 f}{\partial x_i x_j} \right]_{i,j}$$

Thm 6.2 Let $X \subseteq \mathbb{R}^n$ be open and convex and let $f : X \to \mathbb{R}$ be in C^2.

Then, f is concave if and only if the Hessian matrix $D^2 f(x)$ is negative semi definite, $\forall x \in X$.

* Optimization

Consider constraint optimization problem

$$\max_{x} \quad f(x, \alpha) \quad \text{subject to} \quad x \in C(\alpha)$$

x : choice variable

α : a parameter

f : the objective function

$C(\alpha)$: the constraint set

We define the <u>value function</u> and <u>decision values</u> as

$$V(\alpha) := \max_{x \in C(\alpha)} f(x, \alpha)$$

$$S(\alpha) := \{ y \in C(\alpha) \mid f(y, \alpha) = \max_{x \in C(\alpha)} f(x, \alpha) \}$$

→ Lagrangian thm for equality constraints

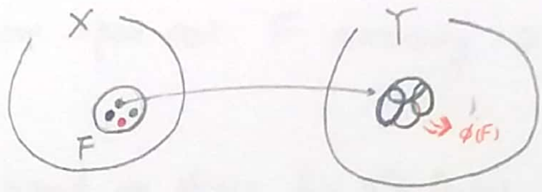
→ Karush - Kuhn - Tucker thm for inequality constraints

* Theorem of the maximum

Def Let X, Y be metric spaces.

A correspondence ϕ from X to Y, $\phi : X \rightrightarrows Y$ associates to each $x \in X$ a subset of Y.

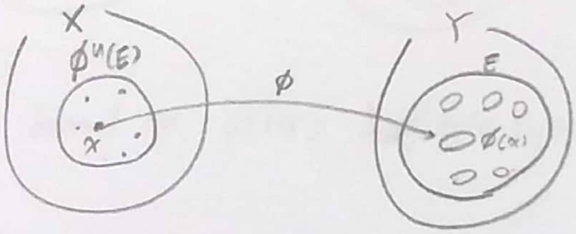
Def: The image of $F \subseteq X$ under ϕ is defined by $\boxed{\phi(F) := \bigcup_{x \in F} \phi(x)}$



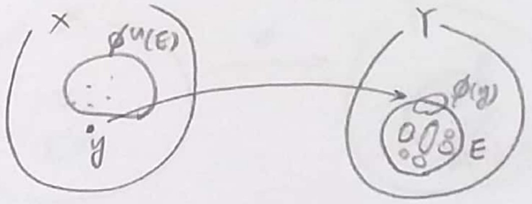
"Totally contained"

- The upper inverse of $E \subseteq Y$ under ϕ is defined by $\phi^u[E] = \{x \in X \mid \phi(x) \subseteq E\}$
- The lower inverse of $E \subseteq Y$ under ϕ is defined by $\phi^l[E] = \{x \in X \mid \phi(x) \cap E \neq \emptyset\}$

"partially contained"

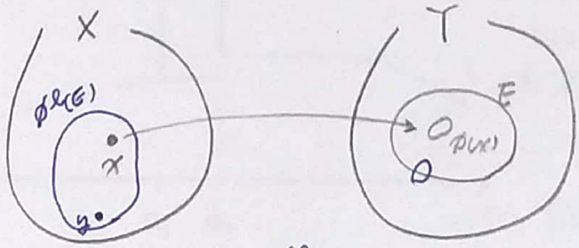


Totally contained



partially contained

Thus, $y \notin \phi^u(E)$



$y \in \phi^l(E)$

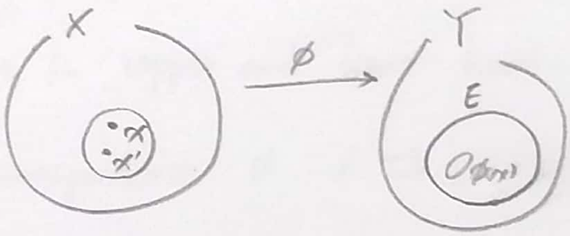
E is upper bound on $\phi(x)$
$\phi(x') \subseteq E$

Def. • A correspondence $\phi : X \rightrightarrows Y$ is <u>upper hemi-continuous</u> at x if whether x is in the upper inverse of an open set $E \subseteq Y$ there is an open set F containing x that satisfies $F \subseteq \phi^u(E)$

• A correspondence $\phi : X \rightrightarrows Y$ is <u>lower hemi-continuous</u> at x whenever x is in the lower inverse of an open set $E \subseteq Y$, there is an open set F containing x that satisfies $F \subseteq \phi^\ell(E)$

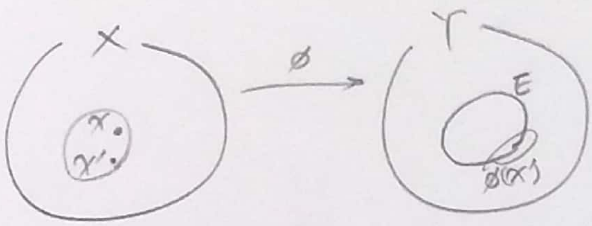
E is ~~upper~~ lower bound on $\phi(x)$.
$\phi(x') \cap E \neq \emptyset$.

$\Rightarrow$ E is upper bound on $\phi(x)$ for $x' \in B_\varepsilon(x)$: $\phi(x) \subseteq E$

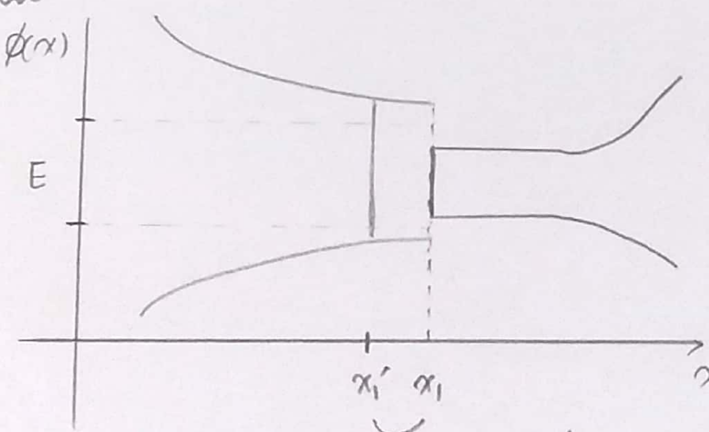


$x \to x'$ close, $E \to \phi(x)$ close.

$\Rightarrow$ E is lower bound on $\phi(x)$ for $x' \in B_\varepsilon(x)$: $\phi(x') \cap E \neq \emptyset$



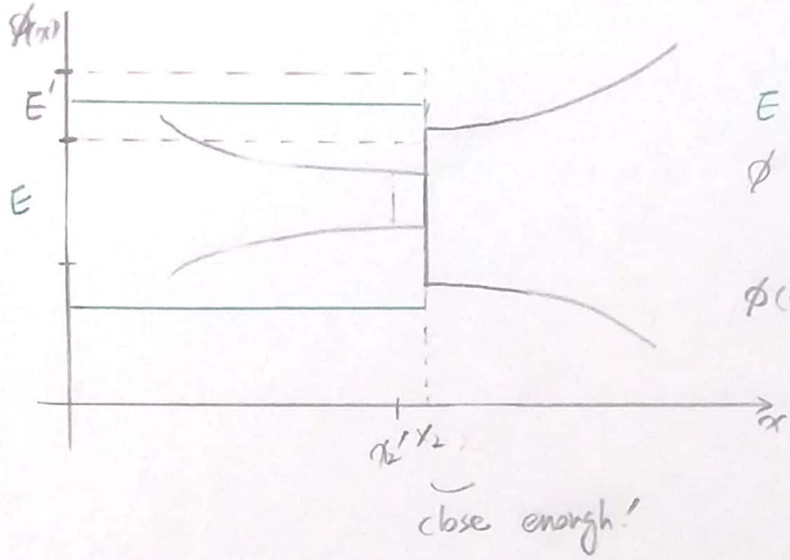
& pictures!



$E \supseteq \phi(x_1)$ but $E \nsupseteq \phi(x_1')$: ϕ is not uhc in x_1

$E \cap \phi(x_1') \neq \emptyset$: ϕ is lhc in x_1

close enough!



$E \supset \phi(x_2')$ also $\phi(x_2)$

ϕ is uhc at x_2

$\phi(x_2') \cap E' = \emptyset$: ϕ is not lhc at x_2

(Def) · A correspondence $\phi : X \rightrightarrows Y$ is continuous at x if it is upper and lower hemi-continuous at x

· A correspondence $\phi : X \rightrightarrows Y$ is compact-valued at x if $\phi(x)$ is compact