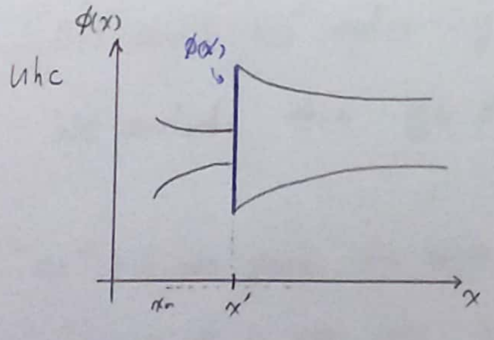
Thm 70 Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. Let $\phi : X \rightrightarrows Y$ be compact-valued.

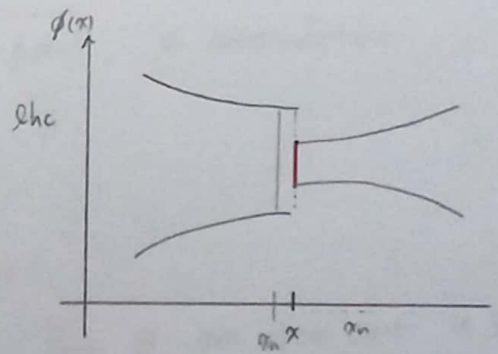
(i) ϕ is uhc at $x \in X$ $\iff$ ϕ is sequentially uhc at $x \in X$.

ϕ is uhc at $x \in X$ if and only if, for every sequence $\{x_n\} \subseteq X$ satisfying $x_n \to x$ and every subsequence $\{y_n\} \subseteq Y$ satisfying $y_n \in \phi(x_n)$, there is a convergent subsequence of $\{y_n\}$ with limit in $\phi(x)$.

(ii) ϕ is lhc at $x \in X$ $\iff$ ϕ is sequentially lhc at $x \in X$.

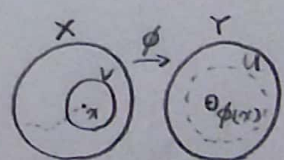
ϕ is lhc at $x \in X$ if and only if $x_n \to x$ and $y \in \phi(x)$ imply that there is a sequence $\{y_n\} \subseteq Y$ satisfying $y_n \in \phi(x_n)$ and $y_n \to Y$.





proof) (i) "$\Rightarrow$"

Suppose ϕ is uhc, $x_n \to x$ and $y_n \in \phi(x_n)$.



Step1) y_n has a bounded subsequence.

Since $\phi(x)$ is compact, $\exists$ a bounded open ball U containing $\phi(x)$.

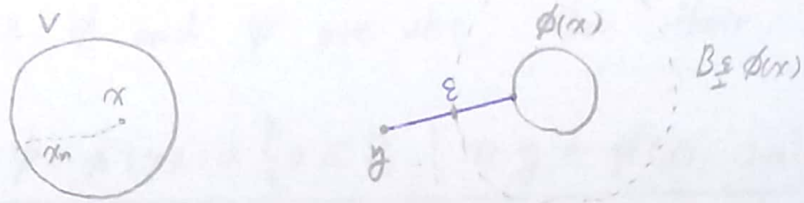
Since ϕ is uhc, $\exists$ $V \subseteq X$ such that $x \in V$ and $\phi(V) \subseteq U$

Since x_n is eventually in V, y_n is eventually in U

Step2) Since $\bar{U}$ is compact, there is a convergent subsequence $\{y_{n_k}\}$ with limit y.

Suppose $y \notin \phi(x)$ (For contradiction)

Let $\varepsilon := \inf_{a \in \phi(x)} d(y, a)$ and note that $\varepsilon > 0$.



the closure of $B_{\frac{\varepsilon}{2}}\phi(x)$

Define $B_{\frac{\varepsilon}{2}}\phi(x) := \bigcup_{a \in \phi(x)} B_{\frac{\varepsilon}{2}}(a)$ and observe that $y \notin \overline{B_{\frac{\varepsilon}{2}}\phi(x)}$

Since ϕ is uhc, $\exists$ an open set $V \subseteq X$ satisfying $x \in V$ and

$\phi(V) \subseteq B_{\frac{\varepsilon}{2}}\phi(x)$ (The definition of uhc)

Therefore, $x_n \in V$ for large n which implies $y_n \in \phi(x_n) \subseteq \phi(V) \subseteq B_{\frac{\varepsilon}{2}}\phi(x)$ for n large enough.

But now, this implies $y \in \overline{B_{\frac{\varepsilon}{2}}\phi(x)}$, a contradiction.

We conclude that $y \in \phi(x)$. //

"$\Leftarrow$" (We prove the contrapositive)

Suppose ϕ is not uhc at x. Then $\exists$ an open set $U \subseteq Y$ such that $\phi(x) \subseteq U$.

but $\forall n \in \mathbb{N}$, $\exists$ $x_n \in B_{\frac{1}{n}}(x)$ such that $\phi(x_n) \nsubseteq U$.

That is, $\exists$ a $y_n \in \phi(x_n)$ such that $y_n \notin U$.

Since $Y \setminus U$ is closed, all converging sequences in $Y \setminus U$

have a limit in Y / U.

Therefore, no subsequence $\{y_{n_k}\}$ can have a limit in U. //

(No proof for (ii))

(Thm11) Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$, $Z \subseteq \mathbb{R}^k$.

Let $\phi : X \rightrightarrows Y$ and $\psi : Y \rightrightarrows Z$

Suppose ϕ and ψ are uhc, then their composition $\psi \circ \phi$ defined by

$\psi \circ \phi(x) := \{ z \in Z \mid \exists\, y \in \phi(x)$ such that $z \in \psi(y) \}$. is uhc.

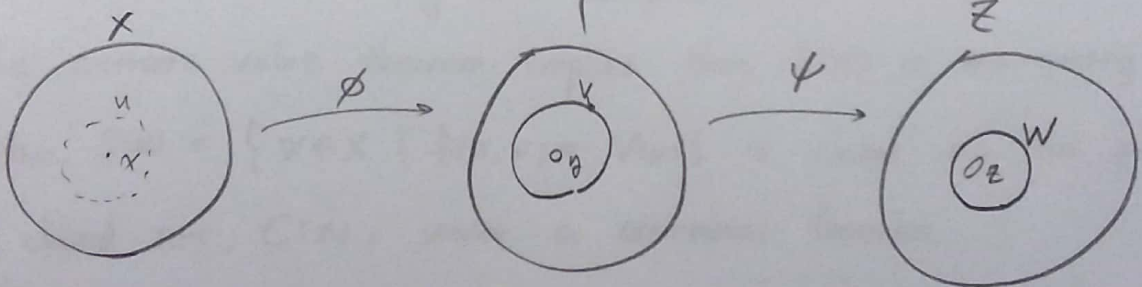
proof) Let $W \subseteq Z$ be open and define $U := \{ x \in X \mid \psi \circ \phi(x) \subseteq W \}$.

We show that U is open.

Let $V := \{ y \in Y \mid \psi(y) \subseteq W \}$ and

Note that $U = \{ x \in X \mid \phi(x) \subseteq W \}$.

Then V is open because ψ is uhc is therefore U is open because ϕ is uhc. //



Thm72 < Theorem of the Maximum (Berge) >

Let $X \subseteq \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^p$.

Suppose $f : X \times \Omega \longrightarrow \mathbb{R}$ is continuous and that $C : \Omega \rightrightarrows X$ is non-empty valued, compact valued, and continuous.

Consider the parameterized optimization problem: $\boxed{\max_{x \in C(\alpha)} f(x, \alpha)}$

Then the value function $\boxed{V(\alpha) = \sup_{x \in C(\alpha)} f(x, \alpha)}$ is continuous and the solution correspondence $S : \Omega \rightrightarrows X$ defined by $S(\alpha) := \arg\max_{x \in C(\alpha)} f(x, \alpha)$ is non-empty valued, compact-valued and uhc.

proof)

Step1) : $S(\alpha)$ is non-empty and compact valued.

Since $C(\alpha)$ is non-empty and compact, and f is continuous, the extreme value theorem implies that $S(\alpha)$ is non-empty.

Also, $S(\alpha) = \{ x \in X \mid f(x, \alpha) = V(\alpha) \}$ is closed as the preimage of a closed set, $C(\alpha)$, under a continuous function.

Since $S(\alpha) \subseteq C(\alpha)$, $S(\alpha)$ is bounded and therefore compact. //

Step2) : $S(\alpha)$ is uhc.

We use Thm 70 : Consider an arbitrary sequence α_n satisfying $\alpha_n \rightarrow \alpha$ and a sequence x_n satisfying $x_n \in S(\alpha_n) \subseteq C(\alpha_n)$.

Since C is lhc and compact-valued, $\exists$ a subsequence $\{x_{n_k}\}$ converging to $x \in C(\alpha)$. Given arbitrary $z \in C(\alpha)$, since C is lhc, $\exists$ a subsequence $\{z_{n_k}\} \subseteq X$ with $z_{n_k} \in C(\alpha_{n_k})$ and $z_{n_k} \rightarrow z$.

⑤

Since x_{n_a} solves the maximization ploblem by assumption,

we get that $f(x_{n_a}, \alpha) \geqq f(z_{n_a}, \alpha)$

Since f is continuous, we can take limits to obtain $f(x,\alpha) \geqq f(z,\alpha)$

Since $z \in C(\alpha)$ is arbitrary, we conclude that $x \in S(\alpha)$.

Since S is compact-valued, we conclude by thm 70 that S is uhc. //

Step 3) $V(\alpha)$ is continuous

Note that we can write $V(\alpha) = f(S(\alpha), \alpha)$

Interpreting f as a correspondence, V is the composition of two uhc correspondences and therefore uhc by thm 71.

Since it is single-valued, it is continuous. (Problem set 11. 5(ii)). //

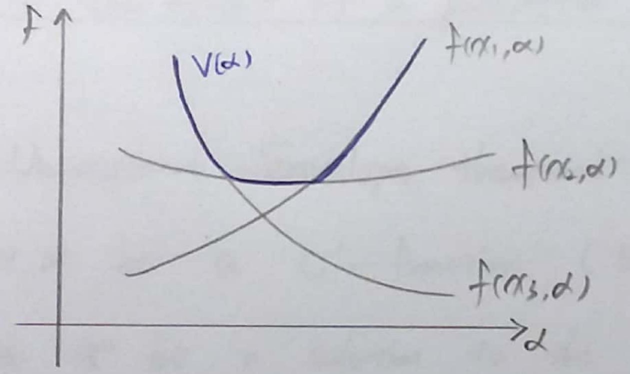
$\max_{x \in C(\alpha)} f(x, \alpha) \Rightarrow x$ is changed according to α. : $S(\alpha)$

< The Envelope theorem >

The value function $V: \Omega \to \mathbb{R}$ for a maximization problem gives the maximization attainable value of the objective function for each value of the parameter

$$V(\alpha) = \max_{x} \{ f(x, \alpha) \mid x \in C(\alpha) \} = f(x^*, \alpha), \ x^* \in S(\alpha).$$

Example: Maximum-value function as "upper envelope" of a family curve.



Thm 7.3 (Concavity of the value function)

Consider the following problem and the associated value function

$$V(\alpha) = \max_{x} \{ f(x, \alpha) \mid g(x, \alpha) \geqq 0 \}$$

Suppose the objective function f is concave in (x, α) ((in x and α)) and that all constraint functions $g^j(\cdot)$, $j = 1, \dots, c$, are quasi-concave.

Then, $V(\cdot)$ is concave.

Thm 7.4 (Convexity of the Value function)

Consider the following problem and the associated value function

$$V(\alpha) = \max_{x} \{ f(x,\alpha) \mid \underline{g(x) \geq 0} \} \quad \text{where } \alpha \in \Omega \text{ a convex set.}$$

$\boxed{\text{Note}}$: $\underline{g \text{ is NOT a function of } \alpha.}$

If the objective function f is convex in the parameters α for any given X, then $V(\cdot)$ is convex.

※ <u>What is the effect of a parameter change on the maximum?</u>

Thm 7.5 (Unconstraint Envelope theorem)

Let $f(x,\alpha)$ be a C^2- function. (twice differentiable & continuous)

and let x^0 be a solution to the maximization problem of f for α^0.

Then $V(\alpha) = \max_{x} f(x,\alpha) = f[x(\alpha),\alpha]$ is differentiable at α^0

and $\underline{DV(\alpha^0) = D_\alpha f(x^0,\alpha^0)}$

Idea: $DV(\alpha^0) = \underbrace{D_x f(x^0,\alpha^0) \cdot DX(\alpha^0)}_{\text{indirect effect through change in } x} + \overbrace{D_\alpha f(x^0,\alpha^0)}^{\text{direct effect of } \alpha}$

FOC : $D_x f(x^0,\alpha^0) \cdot DX(\alpha^0) = 0$. So, $DV(\alpha^0) = D_\alpha f(x^0,\alpha^0)$

Recall: Lagrange Problem

Thm (Lagrange)

Let x^* be an optimal solution of $\max_x \{f(x) \mid g(x)=0\}$

where f and g are C^1-functions. and rank $Dg(x^*) = c \leq n$

Then. there exist unique Lagrange multipliers $\lambda_i^* \in \mathbb{R}$ such that.

$$Df(x^*) + \lambda^{*T} Dg(x^*) = \underline{0}$$

Thm 76 < Envelope Theorem for Lagrange >

Let $V(\alpha) = \max_x \{f(x,\alpha) \mid g(x,\alpha)=0\}$ where f and g are C^2-functions.

If x° is a solution of this problem for α°.

(i.e., it satisfies sufficient second-order conditions for a strict local maximum)

Then V is differentiable at α° and $DV(\alpha^\circ) = D_\alpha \mathcal{L}(\lambda^\circ, \alpha^\circ)$

$$= D_\alpha f(x^\circ,\alpha^\circ) + \lambda^T D_\alpha g(x^\circ,\alpha^\circ)$$

proof) Differentiate the Lagrangian function:

(1) $$\mathcal{L}(x^\circ,\alpha^\circ) = f(x^\circ,\alpha^\circ) + \lambda^T g(x^\circ,\alpha^\circ)$$

implies that the first order condition is

(2) $$D_x \mathcal{L}(x,\alpha^\circ) = D_x f(x,\alpha^\circ) + \lambda^T D_x g(x,\alpha^\circ) = \underline{0}$$

(3) $$D_\lambda \mathcal{L}(x,\alpha^\circ) = g(x,\alpha^\circ) = \underline{0}$$

By assumption, the decision rule for problem $x(\alpha)$ is a well-defined and differentiable function.

Substituting x into the objective, we obtain the value function

$$V(\alpha) = f[x(\alpha), \alpha]$$

Differentiating $V(\cdot)$ and using (2), we get

$$DV(\alpha^0) = D_\alpha f(x^0,\alpha^0) + \underbrace{D_x f(x^0,\alpha^0)}_{\text{(by (2))}} \cdot DX(\alpha^0)$$

(4) $$= D_\alpha f(x^0,\alpha^0) - \lambda^T D_x g(x^0,\alpha^0) \cdot DX(\alpha^0)$$

Substituting $X(\alpha)$ into (3) and differentiating with respect to α.

$$D_x g(x^0,\alpha^0)\, DX(\alpha^0) + D_\alpha g(x^0,\alpha^0) = 0 \quad \rightarrow \quad D_x g(x^0,\alpha^0)\, DX(\alpha^0) = - D_\alpha g(x^0,\alpha^0)$$

Using this expression, we can reduce (4) to

(minus sign is cancelled with $-\lambda^T$)

$$DV(\alpha^0) = D_\alpha f(x^0,\alpha^0) + \lambda^T D_\alpha g(x^0,\alpha^0) \quad . \,||$$