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① Let X be a set and $\{G,*\}$ be a group. The set of functions of X into G, endowed with operation defined by composition of images, i.e.

$$(1) \quad \forall x \in X,\ (f*g)(x) = f(x)*g(x)$$

is a group.

Four conditions of a group:

(i) closure (ii) associativity (iii) identity (iv) inverse

Note: X is a set. G is a set. $\{G,*\}$ is a group

(i) let $f,g \in F = \{f \mid f: X \to G\}$

$$(f*g)(x) = h \overset{\text{by }(1)}{\Rightarrow} h = f(x) * g(x)$$

Since $f \in F$ and $g \in F$, and $F = \{f \mid f: x \to G\}$ this implies $h \in G$

Therefore, F is closed.

(ii) Associativity

Let $f,g,h \in F = \{f \mid f: X \to G\}$

$$[(f*g)*h](x) = (f(x)*g(x))*h(x)$$

$$= \underset{}{\overset{\in G}{f(x)}} * \overset{\in G}{g(x)} * \overset{\in G}{h(x)} = f(x)*(g(x)*h(x))$$

$$= [f*(g*h)](x)$$

Since $\{G,*\}$ is group and associates. there, it associates
f,g,h are in G

(iii) Identity ($\exists e \in S : e * a = a * e = a$)

Let $f, e \in F = \{ f \mid f : X \to G \}$

$\exists e \in G : \forall a \in G : e * a = a * e = a$
since $\{G, *\}$ is a group.

Thus: $(f * e)(x) = f(x) * e(x) = e(x) * f(x)$
$= f(x)$

note: $f(x), e(x) \in G$

Therefore F has identity element

(iv) Inverse:

Let $f, g \in F$

$(f * g)(x) = f(x) * g(x)$

let $f(x) = a$, $g(x) = b$. $f(x), g(x) \in G \Rightarrow$

$a, b \in G$. Let $a * b = e$ where e is identity element

We know $\exists e \in G$ since it is a group.

Therefore; $f(x) * g(x) = a * b = e$
$= (f * g)(x)$

② Let X be a non-empty set. Let:

$\mathcal{F} = \{f \mid f : X \to \mathbb{R}\}$ and the subset $\mathcal{F}$ consisting of all bounded maps from X to $\mathbb{R}$: $\mathcal{F}_b = \{f \in \mathcal{F} \mid \exists k \in \mathbb{R} : |f(x)| < k, \forall x \in X\}$

Show $\mathcal{F}$ and $\mathcal{F}_b$ are real vecter spaces

$$(f+g)(x) = f(x) + g(x) \quad \forall f, g \in \mathcal{F}$$

$$(\lambda f)(x) = (\lambda f)(x) \quad \forall f \in \mathcal{F} \ \lambda \in \mathbb{R}$$

(i) Associative property

let $f, g, h \in \mathcal{F}$

$$\begin{aligned}(f+(g+h))(x) &= f(x) + (g(x) + h(x)) \\ &= (f(x) + g(x)) + h(x) \\ &= ((f+g)+h)(x)\end{aligned}$$

(ii) Commutative

let $f, g \in \mathcal{F}$

$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$$

(iii) Additive Identity $\exists! \underline{0} \in V : x + \underline{0} = \underline{0} + x = x$

$e \in \mathbb{R}$ let $f, g \in \mathcal{F}$ let $g(x) = e$

$$(f+g)(x) = f(x) + g(x) = f(x) + e = f(x)$$

(iv) existence of inverse element $\forall x \in V, \exists ! (-x) : x + (-x) = (-x) + x = 0$

let $f \in \mathcal{F}$ and thus $f(x) \in \mathbb{R}$

~~let $g \in \mathcal{G}$, $g(x) = f(x) \in \mathbb{R}$~~

$0 \in \mathbb{R}$, ~~det~~ if $f(x) \in \mathbb{R}$ then $-f(x) \in \mathbb{R}$

set $g(x) = -f(x)$

$(f+g)(x) = f(x) + g(x) = f(x) - f(x) = 0$

(v) let $\lambda \in \mathbb{R}$, $f, g \in \mathcal{F}$

$(\lambda(f+g)(x)) = \lambda f(x) + \lambda g(x)$

(vi) $\alpha, \beta \in \mathbb{R}$ $f \in \mathcal{F}$

set $\lambda = \alpha + \beta$

$((\alpha+\beta)f)(x) = (\lambda f)(x) = \lambda f(x) = (\alpha+\beta)f(x) = \alpha f(x) + \beta f(x)$

(vii) $(\alpha(\beta f))(x) = \alpha(\beta f(x)) = (\alpha\beta)f(x) = ((\alpha\beta)f(x))$

(viii) let $\lambda = 1$

$(\lambda f)(x) = \lambda f(x) = (1)f(x) = f(x)$

② For $\mathcal{F}_b$

Thm 5.8:

Let V be a vector space over field F, and let S be a nonempty subset of V. Then S is a vector subspace if:

$$\forall \alpha, \beta \in F \text{ and } \forall x, y \in S : \alpha x + \beta y \in S$$

let $\mathcal{F}_b$ be a subset of $\mathcal{F}$. From X, construct a bounded sequence s.t. the map f from $X \to \mathbb{R}$ is bounded.

Then $\mathcal{F}_b$ is nonempty

let $\lambda \in \mathbb{R}$ and $f, g \in \mathcal{F}_b$

$\exists k \in \mathbb{R} : |f(x)| < k \;\; \forall x \in X$

and $\exists l \in \mathbb{R} : |g(x)| < l \;\; \forall x \in X$

$$|f(x)| + |g(x)| < k + l = M$$

Let $h(x) = |f(x)| + |g(x)|$

then $h(x) \in \mathcal{F}_b$. Closure under addition

$$|\lambda f(x)| < \lambda k \Rightarrow |\lambda||f(x)| < \lambda k = M$$

Define $h(x) = \lambda f(x)$

then $\exists M \in \mathbb{R} \; |h(x)| < M \;\; \forall x \in X$

$$\Rightarrow h(x) \in \mathcal{F}_b$$

So, $\mathcal{F}_b$ is a vector subspace of $\mathcal{F}$, inheriting all vector space properties

③ Let $S \subseteq \mathbb{R}$. A real-valued function $f : S \to \mathbb{R}$ is strictly inc. if $\forall z > z' \Rightarrow f(z) > f(z')$

Let U be the set of all real-valued functions on a set X.
$u : X \to \mathbb{R}$

$u \sim u' \Leftrightarrow \exists$ a strictly inc function $f : u(X) \to \mathbb{R} : u' = f(u(x))$

Equivalence requires transitivity, reflexivity, and symmetricity

<u>Reflexive:</u>

$u \sim u \Leftrightarrow \exists$ a strictly increasing function
$f : u(X) \to \mathbb{R} : u = f(u(x))$

let $f(X) = X$, then $f(u(x)) = u(x)$

<u>Symmetric:</u>

$u_1 \sim u_2 \Leftrightarrow \exists$ str. inc $f : u_1(x) \to \mathbb{R} : u_2 = f(u_1(x))$

$u_2 \sim u_1 \Leftrightarrow \exists$ str. inc $f : u_2(x) \to \mathbb{R} : u_1 = f(u_2(x))$

the inverse of a strictly increasing function is a strictly increasing function

$x < y \Leftrightarrow f(x) < f(y) \Rightarrow f^{-1}(x) < f^{-1}(y) \Leftrightarrow f^{-1}(f(x)) < f^{-1}(f(y))$
$\Leftrightarrow x < y$

So, it is true that this relation is symmetric. For any str. increasing function mapping $u \to \mathbb{R}$, its inverse is also strictly increasing

Transitive

$u_1 \sim u_2$ and $u_2 \sim u_3$ then $u_1 \sim u_3$

$u_1 \sim u_2 \Rightarrow \quad u_2 = f_1(u_1(x))$

$u_2 \sim u_3 \Rightarrow u_3 = f_2(u_2(x))$

$\Leftrightarrow u_3 = f_2(f_1(u(x)) \Rightarrow u_1 \sim u_3$

$(f_1 \circ f_2)(x)$ is strictly increasing; that is composition of two increasing functions is strictly increasing

$x > y \Rightarrow f(x) > f(y)$ let g also be strictly increasing

$x > y \Rightarrow f(x) > f(y)$

let $f(x) = a$ $f(y) = b \Rightarrow b > a$

$\Rightarrow g(b) > g(a) \Leftrightarrow g(f(x)) > g(f(y))$

Therefore, the relation $\sim$ is transitive.

So, $u \sim u'$ is an equivalence relation

④ $\{G, *\}$ is a non-empty binary relation

$* : G \times G \to G$

(i) $\forall g \in G : \exists h \in G : g * h = h * g = g$

(ii) $\exists \tilde{h} \in G : \forall \tilde{g} \in G : \tilde{g} * \tilde{h} = \tilde{h} * \tilde{g} = \tilde{g}$

(i) $\not\Rightarrow$ (ii)

(i) says every g has an h (maybe not the same h) such that $g * h = h * g = g$

(ii) says there is (one) $\tilde{h}$ such that every $\tilde{g}$ has: $\tilde{g} * \tilde{h} = \tilde{h} * \tilde{g} = \tilde{g}$

let $G_1 \times G_2 \to G_1 = *$

if $g_1 \in G = (g_1, g_2) = 1$

$G_1 \times G_2 \to (g_2, g_1)^{\in G_1}$ and $(g_1, g_2)^{\in G_1}$ otherwise

if $g_1 = 2$ $G \times G \to (\underset{G_1}{\underset{\cap}{g_2}}, g_1)$ and $(g_1, \underset{G_1}{\underset{\cap}{g_2}})$ ow

let G have 4 elements

$g =$ $(1, 1)$ $h = (1,1), (1,2), (2,2)(2,1) \Rightarrow g * h = g$

$g =$ $(1, 2)$ $h = (1,1), (1,2) \Rightarrow g * h = g$

$g =$ $(2, 2)$ $h = (2,1)(2,2)(1,1)(1,2) \Rightarrow g * h = g$

$g =$ $(2, 1)$ $h = (2,1)(2,2) = g * h = g$

hence $\forall g \in G : \exists h \in G : g * h = h$

but $\neg(\exists h \in G : \forall g \in G : g * h = g)$ since h's are not all the same

(b) If $\{G, *\}$ is a group, then

(i) holds since by definition a group must have an identity element playing the role of g

(ii) also holds, again, let $\tilde{h} = e$, the identity element. Then, since $\{G, *\}$ is a group, it must be the case:

$$\tilde{g} * h = h * \tilde{g} = \tilde{g}$$

and also since (ii) $\Rightarrow$ (i)

if (ii) is true, and it is, (i) is also true

(5) A and B are nonempty sets bounded from above.

$$C = \{c = a + b \mid a \in A, b \in B\}$$

Show C has a supremum $\sup C = \sup A + \sup B$

(i) $\sup C \overset{?}{\geq} \sup A + \sup B$

$a \leq \sup A \qquad b \leq \sup B$

$a = a + b - b \leq \sup(A+B) - b$

$a \leq \sup(A+B) - b \quad \forall a \in A, b \in B$

$\sup A \leq \sup(A+B) - b$

$\Rightarrow b \leq \sup(A+B) - \sup A$

$\Rightarrow \sup B \leq \sup(A+B) - \sup A$

$\Rightarrow \sup A + \sup B \leq \sup(A+B) = \sup C$

$\sup(A+B) = \sup C \leq \sup A + \sup B$

$a \leq \sup A \quad \forall a$
$b \leq \sup B \quad \forall b$

so $\sup(A+B) = \sup C \leq \sup A + \sup B$

$\Rightarrow \sup C = \sup(A+B) = \sup A + \sup B$