Exercise Set 2

David Zynda

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(1) Proof of Injective/Surjective Function

Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$ and define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = ax^n$.

(a) Use your knowledge of proof by induction to verify that if n is even, then f is neither one-to-one nor onto $\mathbb{R}$

Let $n = \{k \in \mathbb{N} | k = 2m \forall m \in \mathbb{N}\}$ for $f(x) = ax^n$

Step 1: Let $n = 2$

$$f(x) = ax^2$$

Is $f(x)$ 1-to-1 or injective? Clearly not. Let $x_1 = 1$ and $x_2 = -1$. Then $f(x_1) = f(x_2)$ where $x_1 \neq x_2$. Hence, $f(x)$ is not one-to-one since each y does not have a unique x.

Secondly, $f(x)$ is not onto or surjective either. As defined above, the target space includes all of $\mathbb{R}$. Yet, it is obvious that the domain does not include all the target space since it excludes all negative numbers. Consequently, the function cannot be onto.

Step 2: Let $S_{n+1} = f(x) = ax^{n+2}$

$$f(x) = ax^{n+2}$$

Notice: raising any ax to the n^{th} even power for all x and a in $\mathbb{R}$ results in a positive number. As such, the function is neither one to one nor onto.

Consequently, by the principle of induction, all even n guarantee the function $f(x) = ax^n$ is neither one-to-one nor onto.

(b) Assuming that the function f is coninuous, determine whether f is one-to-one or onto if n is odd instead of even.

Let $n \in \mathbb{N}$ be an odd number in $f(x) = ax^n$. And, let $m \in \mathbb{R} > 0$ and $a > 0$.

$$x < x + m \implies f(x) < f(x + m)$$

Since $a > 0$. In fact, as x approaches infinity, the limit tends towards infinity. Similarly, as x becomes infinitely negative, the limit approaches negative infinity.

Because of this:

$$\forall y \in \mathbb{R} : \exists M > 0 : x > M \implies f(x) > y$$
$$\forall y \in \mathbb{R} : \exists m > 0 : x > m \implies f(x) < y$$

By the Intermediate Value Theorem, there exists a $c \in [m, M]$ such that $f(c) = y$. Hence, all the target space is in the range, and the function is onto.

Clearly, it is one-to-one as well. Because the function is strictly monotonically increasing given $a > 0$, the function must be one-to-one. Each y has a distinct x.

Now, if $a < 0$, the conditions above still hold, but the function would be monotonically decreasing. If $a = 0$ all bets are off since the function would be neither one-to-one or onto since for all x, $f(x) = 0$.

(2) Proof of the sum of squared natural numbers

Prove that for every $n \in \mathbb{N}$:

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof by Induction:

Let

$$\sum_{i=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6}$$

Step 1 Let $n = 1$:

$$\sum_{i=1}^{1} n^2 = \frac{(1)(1+1)(1+2)}{6} = \frac{6}{6} = 1$$

Hence, the first step is true.

Step 2 Now, show that this statement holds for $n+1$ such that:

$$\sum_{i=k}^{n+1} n^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6} = \frac{(n+1)(n+2)(2n+3)}{6}$$

Let:

$$\sum_{k}^{n+1} n^2 = \sum_{k}^{n+1} k^2 + (n+1)^2$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)(n(2n+1) + 6(n+1))}{6}$$

$$= \frac{(n+1)(n(2n^2 + n + 6n + 6))}{6}$$

$$= \frac{(n+1)(2n^2 + 7n + 6)}{6}$$

$$= \frac{(n+1)(2n+3)(n+2)}{6} = \sum_{i=k}^{n+1} n_k^2$$

Therefore, the statement holds for $n+1$. Thus, by the principle of induction, it is true that:

for every $n \in \mathbb{N}$:

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(3) Determine the lengths of the following vectors:

$$length = norm = ||x|| := \sqrt{x_1^2 + x_2^2 + ... + x_n^2} \forall x \in \mathbb{R}^n$$

(a) (12, 5)

$$||x|| := \sqrt{12^2 + 5^2} = \sqrt{169} = 13$$

(b) (1,1)

$$||x|| := \sqrt{1^2 + 1^2} = \sqrt{2}$$

(c) (3,3)

$$||x|| := \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$

(d) (3, -3)

$$||x|| := \sqrt{3^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$$

(e) (-1, 1, -1)

$$||x|| := \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3}$$

(f) (1,1,1,1)

$$||x|| := \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$$

(g) (12, 0, 0, 5)

$$||x|| := \sqrt{12^2 + 0^2 + 0^2 + 5^2} = \sqrt{169} = 13$$

(h) (12, -1, 1, 5)

$$||x|| := \sqrt{12^2 + (-1)^2 + 1^2 + 5^2} = \sqrt{171} = 3\sqrt{19}$$

(4) For each vector in #3, determine a vector of length 1 that points in the opposite direction.

(a) (12, 5)

Direction: $(-12k, -5k)$

$$12^2k^2 + 5^2k^2 = 1$$
$$\implies k = \frac{1}{13}$$

So, a vector in the opposite direction of length 1 is: $(-\frac{12}{13}, -\frac{5}{13})$

(b) (1,1)

Direction: $(-1k, -1k)$

$$1^2k^2 + 1^2k^2 = 1$$
$$\implies k = \sqrt{2}$$

So, a vector in the opposite direction of length 1 is:

$$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

(c) (3,3)

Direction: $(-3k, -3k)$

$$3^2k^2 + 3^2k^2 = 1$$
$$\implies k = \frac{1}{\sqrt{18}}$$

So, a vector in the opposite direction of length 1 is: $(-\frac{3}{\sqrt{18}}, -\frac{3}{\sqrt{18}})$

(d) (3, -3)

Direction: $(-3k, 3k)$

$$3^2k^2 + (-3)^2k^2 = 1$$
$$\implies k = \frac{1}{\sqrt{18}}$$

So, a vector in the opposite direction of length 1 is: $(-\frac{3}{\sqrt{18}}, \frac{3}{\sqrt{18}})$

(e) (-1, 1, -1)

Opposite Direction: $(1k, -1k, 1k)$ Set length to 1: $(1k)^2 + (-1k)^2 + (1k)^2 = 1$ This implies $k = \frac{1}{/}\sqrt{3}$ Hence a vector in the opposite direction of length 1 is: $(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

(f) (1,1,1,1)

Opposite Direction: $(-1k, -1k, -1k, -1k)$ Set length to 1: $(-1k)^2 + (-1k)^2 + (-1k)^2 + (-1k^2) = 1$ This implies $k = \frac{1}{\sqrt{4}} = \frac{1}{2}$ Hence a vector in the opposite direction of length 1 is: $(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4})$

(g) (12, 0, 0, 5)

Opposite Direction: $(-12k, 0k, 0k, -5k)$ Set length to 1: $(-12k)^2 + (0k)^2 + (0k)^2 + (-5k^2) = 1$ This implies $k = \frac{1}{\sqrt{169}} = \frac{1}{13}$ Hence a vector in the opposite direction of length 1 is: $(-\frac{12}{13}, 0, 0, -\frac{5}{13})$

(h) (12, -1, 1, 5)

Opposite Direction: $(-12k, 1k, -1k, -5k)$ Set length to 1: $(-12k)^2 + (-1k)^2 + (1k)^2 + (-5k^2) = 1$ This implies $k = \frac{1}{\sqrt{171}}$ Hence a vector in the opposite direction of length 1 is: $(-\frac{12}{\sqrt{171}}, \frac{1}{\sqrt{171}}, -\frac{1}{\sqrt{171}}, -\frac{5}{\sqrt{171}})$

(5) For each of the following vectors, find two vectors u and w that are orthogonal to v and have the same length as v:

(a) $v = (1, 0)$

let $u = (0, 1)$ and $w = (0, -1)$

$$u \cdot v = w \cdot v = 0$$
$$length : l(v) = l(w) = l(u) = 1$$

(b) $v = (1, 1)$

let $u = (1, -1)$ and $w = (-1, 1)$. Then, all vectors share the same length of $sqrt2$ and v is orthogonal to u an w since the dot product is zero (1x1 + (-1)x1 = 0).

(c) $v = (1, 0, 0)$

let $u = (0, 1, 0)$ and $w = (0, 0, 1)$

(d) $v = (1, 1, 1)$

of course, length $= \frac{1}{\sqrt{3}}$

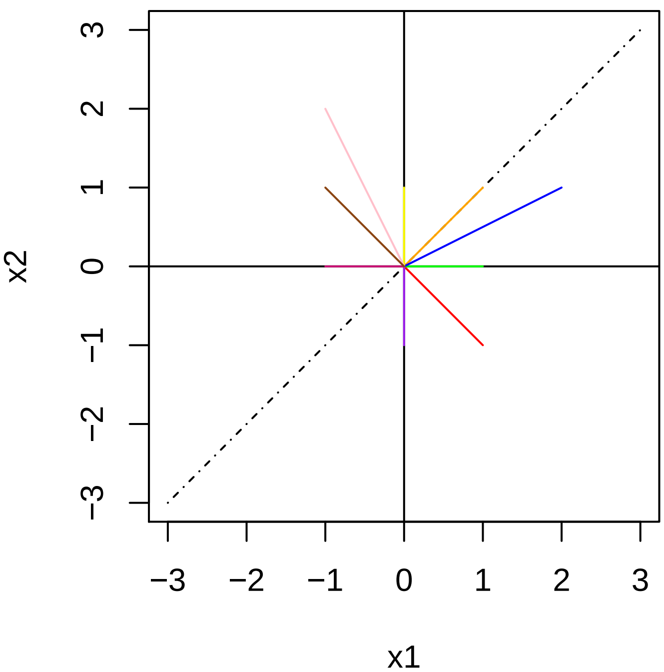
let $u = (0, \sqrt{\frac{\sqrt{3}}{2}}, -\sqrt{\frac{\sqrt{3}}{2}})$. Clearly the length will be $\sqrt{3}$ and the dot product is 0.

let $w = (\sqrt{\frac{\sqrt{3}}{2}}, -\sqrt{\frac{\sqrt{3}}{2}}, 0)$ for a similar and correct result as above with u.

(6) Draw a diagram in $\mathbb{R}^2$ and vector $a = (1, -1)$, the line H, and vectors $(1, 1), (1, 0), (2, 1), (0, -1), (0, 1), (-1, 0), (-1, 2), (-1, 0), (-1, 1)$. Determine $a \cdot v$, which side of the line H each is on both visually and analytically and detemine angle between.

Below is a plot. Notice in the comments to the right of the code the corresponding vectors and colors to aid in matching them on the plot.

```r
par(pty = 's')
plot(c(-3, 3), c(-3, 3), type = 'l', lty=4, ylab = 'x2', xlab='x1') #Line H
abline(h=0)
abline(v=0)
points(c(0,1), c(0,-1), type = 'l', col='red')         #(1,0)  Red
points(c(0,1), c(0,1), type = 'l', col='orange')       #(1,1)  Orange
points(c(0,1), c(0,0), type = 'l', col='green')        #(1,0)  Green
points(c(0,2), c(0,1), type = 'l', col='blue')         #(2,1)  Blue
points(c(0,0), c(0,-1), type = 'l', col='purple')      #(0,-1) Purple
points(c(0,0), c(0,1), type = 'l', col='yellow')       #(0,1)  Yellow
points(c(0,-1), c(0,2), type = 'l', col='pink')        #(-1,2) Pink
points(c(0,-1), c(0,0), type = 'l', col='deeppink3')   #(-1,0) Dark Pink
points(c(0,-1), c(0,1), type = 'l', col='chocolate4') #(-1,1) Brown
```



(a) (1,1)

Dot product: 1x1 + (-1)x1 = 0. This lies on the line H. The angle is 90 degrees visually. Because the dot product is 0, this guarentees the angle is 90 since cos(90 degrees) = 0

(b) (1,0)

Dot product: 1x1 + -1x0 = 1 This lies on the lower-right side of H or below H. The angle appears visually 45 degrees and acute. 1 / (1 * sqrt(2)) = $\frac{1}{\sqrt{2}}$ which is 45 degrees since cos(45 degrees) = $\frac{1}{\sqrt{2}}$ So, the angle is acute.

(c) (2,1)

Dot product: 1x2 + -1x1 = 1 This lies on the southeast side or below H. The angle appears acute. Since the dot product is greater than zero, the angle is acute analytically.

(d) (0,-1)

Dot Product: 1x0 + -1*-1 = 1 This lies on the northwest side of the plot or above H. The angle appears acute. The angle is acute since the dot product is greater than zero.

(e) (0,1)

Dot Product: 1x0 + -1x1 = -1 This lies on the Northwest side of the plot or above H. The angle appear obtuse. The angle is obtuse since the dot product is less than one.

(f) (-1,2)

Dot Product: 1x-1 + -1x2 = -3 This lies on the northwest side, or above H. The angle appears obtuse. The angle is obtuse since the dot product is less than zero.

(-1,0)

Dot product: 1x-1 + -1x0 = -1 This lies on the northwest side or above H. The angle appears obtuse. The angle is obtuse since the dot product is less than zero.

(-1, 1)

Dot Product: -1x1 + -1x1 = -2 This lies on the northwest side or above H. The angle appears obtuse. The angle is obtuse since the dot product is less than zero.

(7) Vectors in $\mathbb{R}^3$

(a) $v = (1, 0, 0)$

$a \cdot v = (1)(1) + (1)(0) + (1)(0) = 1 \mid a \cdot v > 0 \implies$ the angle is acute. Lies on the same side of H

(b) $v = (0, 0, 1)$

$a \cdot v = (1)(0) + (1)(0) + (1)(1) = 1 \mid a \cdot v > 0 \implies$ the angle is acute. Lies on the same side of H

(c) $v = (0, -1, 0)$

$a \cdot v = (1)(0) + (1)(-1) + (1)(0) = -1 \mid a \cdot v < 0 \implies$ the angle is obtuse. Lies on the opposite side of H

(d) $v = (0, 1, 1)$

$a \cdot v = (1)(0) + (1)(1) + (1)(1) = 2 \mid a \cdot v > 0 \implies$ the angle is acute Lies on the opposite side of H

(e) $v = (1, -1, 0)$

$a \cdot v = (1)(0) + (1)(-1) + (1)(1) = 0 \mid a \cdot v = 0 \implies$ the angle is a right angle and the vectors are orthogonal. Lies on the plane of H

(8) Prove that if $n > 1$ then the x_1 axis in $\mathbb{R}^n$ is a closed set.

Please grade me on the first answer below. I have included a second proof too that I thought up. Can you check the second one and tell me if it works, but only grade the first one?

Proof 1:

The compliment of the x_1 axis is defined as $\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | x_2, ..., x_n \neq 0\}$

It must be the case that the compliment is open. For any point in the compliment as a part of $\mathbb{R}^n$, there must be an $\varepsilon > 0$ such that a ball can be constructed around any point which is the subset of that space. The compliment has no boundary points. Therefore, every point in it can have a ball constructed which is a subset of the compliment. The only boundary is the point a which $x_2, ...x_n = 0$ which is contained in x_1. Consequently the compliment is open.

If the compliment is open, then the original set is closed. Therefore, the x_1 axis is a closed set.

Proof 2:

Using contradiction, assume that the x_1 axis in $\mathbb{R}^n$ is an open set. Simply let x_1 denote the entire x_1 axis. Then, by definition:

$$\forall x \in x_1 : \exists \varepsilon > 0 : B(x, \varepsilon) \subseteq x_1$$

Let $a \in \mathbb{R}$ be on the line x_1. Then, there exists a neighborhood around a which is a subset of x_1 for all values of $\varepsilon > 0$. But, x_1 itself is a line. Therefore, any ball around a will not be on the line x_1. By definition, this would mean that x_1 is a closed set. But, we assumed it was open, creating a contradiction. Therefore, x_1 axis in $\mathbb{R}^n$ is closed.

(9) Prove that every finite subset of $\mathbb{R}^n$ is a closed set.

Let $S_1, S_2, ...S_n$ each be a singleton subset of $\mathbb{R}^n$. Because each subset S_n is a singleton, it is closed - that is there is no $\varepsilon > 0$ for which a ball can be a subset of any S_n.

Take the union of each S_n to make the subset S such that:

$$S_1 \cup S_2 \cup ... \cup S_n = S$$

The union of closes sets itself is closed. Therefore, S is closed. Therefore, any finite subset of $\mathbb{R}^n$ is closed.