Math Camp 2018: Exercise Set #1

1. For each of the following statements, determine whether the statement is true or false, and if false indicate *why* it's false.

(a) $\{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^3 \times \mathbb{R}^2$.

(b) $\{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^3 \cup \mathbb{R}^2$.

(c) $\{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$.

(d) $\big((2,1,1),(2,1),(1,1)\big) \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$.

(e) $\big\{\big((2,1,1),(2,1),(1,1)\big)\big\} \subseteq \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$.

(f) $\exists n \in \mathbb{N} : \{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^n$.

(g) $(1,2,3) \subseteq \mathbb{N}$.

(h) $(1,2,3) \in \mathbb{N}$.

(i) $(1,2,3) \in \mathbb{R}^3$.

(j) $\{(2,1,1),(2,1),(1,1)\} = \{(2,1),(2,1,1),(1,1)\}$.

(k) $\mathbb{R}^2 \cup \mathbb{R}^3 = \mathbb{R}^3 \cup \mathbb{R}^2$.

(l) $\mathbb{R}^2 \times \mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^2$.

2. In our first Logic lecture (Lecture #4) we listed compound statements that covered 11 of the 16 possible "profiles" of truth values. Using only negation, conjunction, and disjunction, for each of the remaining five truth-value profiles construct a compound statement that has that truth-value profile.

3. Let A be an $m \times n$ matrix and let f be the function $f(x) = Ax$. Sections 7.3 and 7.4 of Simon & Blume might be helpful for this exercise. Drawing diagrams will also be helpful.

(a) What is the domain of f and what is the target space of f?

(b) What conditions on m, n, and A are both necessary and sufficient for f to be one-to-one and onto its target space?

(c) If $m = 2$ and $n = 1$, under what conditions (if any) will f be one-to-one? Onto?

(d) If $m = 2$ and $n = 3$, under what conditions (if any) will f be one-to-one? Onto?

4. "You can fool some of the people all the time, and you can fool all the people some of the time, but you can't fool all the people all the time." (Attributed by some to Abraham Lincoln.)

Let's say there are three people (Abby, Beth, and Carl) and three times (Monday, Wednesday, and Friday). Let $X = \{a, b, c\}$ and $Y = \{m, w, f\}$. Let's write the (open) statement "x can be fooled at time y" as $\varphi(x, y)$. Note that you can think of φ as a function, $\varphi : X \times Y \to \{T, F\}$, where T or F is the truth value of $\varphi(x, y)$. You can also write $\mathcal{F} = \{(x, y) \mid \varphi(x, y) = T\}$, the set of pairs (x, y) for which x can be fooled at time y.

We have three statements:

SA: You can fool some of the people all the time.
AS: You can fool all the people some of the time.
AA: You can fool all the people all the time.

Note that the first two statements are ambiguous:

SA could mean SA_1: $\exists x \in X$: $\forall y \in Y$: $\varphi(x, y) = T$ or SA_2: $\forall y \in Y$: $\exists x \in X$: $\varphi(x, y) = T$;
AS could mean AS_1: $\exists y \in Y$: $\forall x \in X$: $\varphi(x, y) = T$ or AS_2: $\forall x \in X$: $\exists y \in Y$: $\varphi(x, y) = T$.

For each of the following five statements, draw a diagram of the Cartesian product $X \times Y$, and use the diagram to represent a function φ (or equivalently a set $\mathcal{F}$) for which that statement is True and as many of the other four statements as possible are False. (For each part except (a) there are multiple correct solutions — *i.e.*, multiple correct sets $\mathcal{F} \subseteq X \times Y$. You only need to draw one of them in each part.)

(a) AA
(b) $\mathrm{SA}_1 \wedge \mathrm{AS}_1 \wedge \sim \mathrm{AA}$
(c) $\mathrm{SA}_1 \wedge \mathrm{AS}_2 \wedge \sim \mathrm{AA}$
(d) $\mathrm{SA}_2 \wedge \mathrm{AS}_1 \wedge \sim \mathrm{AA}$
(e) $\mathrm{SA}_2 \wedge \mathrm{AS}_2 \wedge \sim \mathrm{AA}$.

5. Now let X and Y be arbitrary sets and let $\varphi(x, y)$ be an arbitrary open statement (or $\mathcal{F}$ an arbitrary subset of $X \times Y$). Prove that SA_1 implies SA_2 or that AS_1 implies AS_2. (These two implications are clearly equivalent to one another, so you only need to prove one of them.) This proof should require no more than a sentence or two.

Note: Understanding Exercises #4 and #5 will pretty much guarantee that you will have no trouble understanding uniform continuity and uniform convergence. Failing to understand these two exercises will likely mean you'll have a hard time with uniform continuity and uniform convergence.