

# Exercise Set 4

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**(1) A set is convex if and only if every convex combination of vectors in S is also in S.**

**(i) If every combination of vectors in S is also in S, then a set is convex.**

This proof is evident from the definition of a convex set, that is:

A set S is convex in vector space V if for any two points  $x, y$  in S and  $\lambda$  in the unit interval  $(0, 1)$  the point  $(1 - \lambda)x + \lambda y$  is in S.

Clearly, if every combination of vectors is convex in S, then there will be two that fulfill that definition of convexity.

**(ii) A set is convex if every convex combination of vectors in S is also in S.**

Let V be a convex vector space such that:

$$(1 - \lambda)y + \lambda x = v \subseteq V$$

For  $\lambda > 0$ . Now, let  $(1 - \lambda) = \lambda_2$  and denote  $\lambda$  as  $\lambda_1$ . Then,

$$\lambda_2 y + \lambda_1 x = v \subseteq V$$

Then, for all  $\{\lambda_n\}$ ,  $\lambda_i \geq 0$  and  $\sum_{i=1}^n \lambda_i = 1$

Now, for the first step in the induction proof, let  $a$  and  $b$  be in the convex set V such that

$$\lambda_1 a + \lambda_2 b = v \subseteq V$$

Let  $\lambda_2 = \lambda_2^* + \lambda_3 = (1 - \lambda_1)$  such that  $(\lambda_1 + \lambda_2^* + \lambda_3 = 1)$ . Multiply  $b$  by a factor of 1, but expressed by:

$$\frac{\lambda_2^* + \lambda_3}{\lambda_2^* + \lambda_3}$$

Such that:

$$\lambda_1 a + (\lambda_2^* + \lambda_3) \left( \frac{\lambda_2^* + \lambda_3}{\lambda_2^* + \lambda_3} \right) b$$

Now, decompose  $b$  into two separate vectors following the scalar fraction:

$$\lambda_1 a + (\lambda_2^* + \lambda_3) \left( \frac{\lambda_2^* b^*}{\lambda_2^* + \lambda_3} + \frac{\lambda_3 c}{\lambda_2^* + \lambda_3} \right)$$

Then, with the help of algebra, it can be shown that any convex combination of 2 vectors in a convex set imply that there are three combinations of complex vectors in that same convex set such that:

$$\lambda_1 a + \lambda_2^* b^* + \lambda_3 c = v \subseteq V$$

Where each  $\lambda$  is nonnegative and the sum of all  $\lambda$  is 1.

Step 2 now requires showing that this is the case for  $(n + 1)$

Similar to above, let  $v_1, \dots, v_n$  be a series of vectors in a convex set. Let  $\lambda_1, \dots, \lambda_n$  be scalars of the vectors which are nonnegative and for which the sum of all scalars is equal to 1. Then, we have:

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = v \subseteq V$$

In a similar manner of the first part of the proof, let  $\lambda_n = (\lambda_n^* + \lambda_{n+1})$ . Now, let:

$$\begin{aligned} \lambda_1 v_1 + \lambda_2 v_2 + \dots + (\lambda_n^* + \lambda_{n+1}) \left( \frac{\lambda_n^* + \lambda_{n+1}}{\lambda_n^* + \lambda_{n+1}} \right) v_n &= v \\ \lambda_1 v_1 + \lambda_2 v_2 + \dots + ((\lambda_n^* + \lambda_{n+1}) \left( \frac{\lambda_n^* v_n^*}{\lambda_n^* + \lambda_{n+1}} + \frac{\lambda_{n+1} v_{n+1}}{\lambda_n^* + \lambda_{n+1}} \right)) &= v \end{aligned}$$

For which we now have:

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n^* v_n^* + \lambda_{n+1} v_{n+1} = v$$

And thus, any convex combination of  $n$  vectors will guarantee that there exists a complex combination of  $n + 1$  vectors.

By the principle of induction, it is true that if  $V$  is a convex set, then every convex combination of vectors in  $S$  is also in  $S$ .

**(2) Write out a careful proof of Theorem 21.12. The proof is straightforward but should help one's understanding of quasiconcave functions and reinforce one's ability to prove theorems.**

Theorem 21.12:

Let  $f$  be a function defined on a convex set  $U$  in  $\mathbb{R}^n$ . Then, the following statements are equivalent to each other:

- (a)  $f$  is a quasiconcave function on  $U$ .
- (b) For all  $x, y \in U$  and all  $t \in [0, 1]$ ,  $f(x) \geq f(y) \implies f(tx + (1-t)y) \geq f(y)$
- (c) For all  $x, y \in U$  and all  $t \in [0, 1]$ ,  $f(tx + (1-t)y) \geq \min\{f(x), f(y)\}$

(i)  $a \implies b$

Let  $f(y)$  be defined as  $C_a^+$  such that  $f(y) > a$ . Also, let  $f(x) > f(y)$ . Then,  $f(x) > a$  too. and Consequently,  $f(tx + (1-t)y) \geq f(y) > a$ .

(ii)  $b \implies c$

Let  $f(x) > f(y)$ . Then, we know that  $f(tx + (1-t)y) \geq f(y)$ . Let  $f(x) < f(y)$ . Then, we know that  $f(tx + (1-t)y) \geq f(x)$ . Let  $f(x) = f(y)$ . Then, we know that  $f(tx + (1-t)y) = f(x) = f(y)$

These results can be summarized as follows:

$$f(tx + (1-t)y) \geq \min\{f(x), f(y)\}$$

which concludes the proof.

(iii)  $c \implies a$

Let  $f(x)$  and  $f(y)$  both be greater than  $a$ . Then,  $\min\{f(x), f(y)\} > a \implies \min\{f(x), f(y)\} \in C$ . Then, this means that  $f(tx + (1-t)y) > a$ . So,  $f(tx + (1-t)y) \geq f(x) \in C$

**(3) Here is a useful theorem: If  $f_1$  and  $f_2$  are strictly concave functions defined on a convex subset of a vector space  $V$ , then the sum  $f_1 + f_2$  is also strictly concave.**

**(a) Provide a proof of the theorem above.**

Define concave functions  $f(x)$  and  $g(x)$  on a convex subset  $X$  where:

$$\forall x, y \in X, \lambda \in [0, 1] : f((1 - \lambda)x + \lambda y) \geq (1 - \lambda)f(x) + \lambda f(y)$$

$$\forall a, b \in X, \gamma \in [0, 1] : g((1 - \gamma)a + \gamma b) \geq (1 - \gamma)g(a) + \gamma g(b)$$

Now, add to inequality regarding  $f(x)$  the function  $g(x)$  to both sides such that:

$$f((1 - \lambda)x + \lambda y) + g((1 - \gamma)a + \gamma b) \geq (1 - \lambda)f(x) + \lambda f(y) + g((1 - \gamma)a + \gamma b)$$

Since  $g((1 - \gamma)a + \gamma b) \geq (1 - \gamma)g(a) + \gamma g(b)$  it follows that:

$$\begin{aligned} f((1 - \lambda)x + \lambda y) + g((1 - \gamma)a + \gamma b) &\geq (1 - \lambda)f(x) + \lambda f(y) + g((1 - \gamma)a + \gamma b) \geq (1 - \lambda)f(x) + \lambda f(y) + (1 - \gamma)g(a) + \gamma g(b) \\ \implies f((1 - \lambda)x + \lambda y) + g((1 - \gamma)a + \gamma b) &\geq (1 - \lambda)f(x) + \lambda f(y) + (1 - \gamma)g(a) + \gamma g(b) \end{aligned}$$

Hence, it follows that the sum of the two concave functions defined on a convex set are also concave.

**(b) Let  $u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}$  be a utility function defined on  $\mathbb{R}^2$ . The real function  $f(z) = \sqrt{z}$  is strictly concave. Explain why the theorem does not apply in this situation.**

The theorem does not apply to this situation because the utility function is defined on all of  $\mathbb{R}_+^2$ . This is no guarantee that whatever set within  $\mathbb{R}_+^2$  for which the utility function is used is itself a convex set. Notice that the theorem defines the functions on an interval or convex subset. That detail is excluded in this utility function.

Additionally, because the utility function maps to  $\mathbb{R}_+^2$  there is also no guarantee within the bounds of how this function is defined that the resulting set will be concave.

**(c) If  $X_1$  and  $X_2$  are intervals in  $\mathbb{R}$  and if  $f : X_1 \rightarrow \mathbb{R}$  and  $f : X_2 \rightarrow \mathbb{R}$  are strictly convex functions, the the function  $f : X_1 \times X_2 \rightarrow \mathbb{R}$  defined by  $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$  is strictly concave. Provide a proof of this theorem.**

This theorem is really no different than that of part (a). What is different is what the strictly concave functions are defined on. One assumes a convex subset of vector space, and this one is on intervals in  $\mathbb{R}$ . Now, an interval in  $\mathbb{R}$  is going to be a convex subset. For any two points which form an interval on a line, there exists a convex combination that is contained within that interval. This really falls from the definition of a convex set:

$$\forall x, y \in S, \forall \lambda \in [0, 1] : (1 - \lambda)x + \lambda y \subseteq S$$

Clearly any two points  $(x, y)$  on a one dimensional line will be convex under this definition since there can be a convex combination between them provided that  $\lambda \in [0, 1]$ .

Now, if these two intervals are a convex set, and  $f_1$  and  $f_2$  are defined on them, then the theorem begins to look a lot more similar to part (a). But, there is one more detail. Notice that this function maps  $\mathbb{R} \rightarrow \mathbb{R}$ . So it takes functions from convex subsets inbedded in intervals and maps them to convex subsets in the forms of intervals. Because they are strictly concave functions, the addition of the two will be strictly concave.

#### (4) Exercise #10.16 of Simon and Blume

(a) Prove that each of the following norms are in  $\mathbb{R}^2$

$$\begin{aligned} ||(u_1, u_2)|| &= |u_1| + |u_2| \\ ||(u_1, u_2)|| &= \max\{|u_1|, |u_2|\} \end{aligned}$$

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To prove something is in  $\mathbb{R}$  in general, one must show that:

- (i) Contains the zero vector.
- (ii) Closed under scalar multiplication
- (iii) Closed under addition.

For the first norm (the taxi-cab norm):

*Additivity*

Let  $u = (u_1, u_2)$

$$||u + v|| = |u_1 + v_1| + |u_2 + v_2| \leq |u_1| + |u_2| + |v_1| + |v_2| = ||u|| + ||v||$$

*Scalar Multiplication*

$$||\lambda u|| = |\lambda(u_1 + u_2)| = |\lambda||u_1 + u_2| = \lambda||u||$$

*Zero Element*

Let  $u_1, u_2 = 0$ . Then

$$u$$

has the zero element.

**For the second norm:**

Let  $u = (u_1, u_2)$

$$\begin{aligned} ||u + v|| &= \max\{|u_1 + v_1|, |u_2 + v_2|\} \leq \max\{|u_1| + |v_1|, |u_2| + |v_2|\} \leq \max\{|u_1| + |u_2|\} + \max\{|v_1| + |v_2|\} \\ &= \max\{|u_1| + |u_2|\} + \max\{|v_1| + |v_2|\} = ||u|| + ||v|| \end{aligned}$$

$$||\lambda u|| = \max\{|\lambda u_1|, |\lambda u_2|\} = |\lambda| \max\{|u_1|, |u_2|\} = |\lambda| ||u||$$

Lastly, the zero vector exists if  $u_1 = u_2 = 0$ .

(b) What are the other names.

I think the first is just called the max norm, and the last is the taxi-cab or Manhattan Norm.