ECON 519 Homework 10

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Question from last homework: 5. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x,y) = (\frac{x^2-y^2}{x^2+y^2}, \frac{xy}{x^2+y^2})$. Is the transformation F invertible near the point $(x,y) = (0,1)$?

Solution:

(i) Since $F(0,1) = (-1,0) = F(0,1+\delta)$ for $\delta \in \mathbb{R}$, so F is not locally invertible at $(0,1)$. Since at any neighbourhood near $(0,1)$, the function F is not locally one-to-one.

(ii) *Or, to use the theorem, and calculate the Jacob Matrix:*

$$DF = \begin{bmatrix} \dfrac{4xy^2}{(x^2+y^2)^2} & \dfrac{4yx^2}{(x^2+y^2)^2} \\ \dfrac{y(y^2-x^2)}{(x^2+y^2)^2} & \dfrac{x(x^2-y^2)}{(x^2+y^2)^2} \end{bmatrix}; \ \det(DF) = \frac{8x^2y^2(x^2-y^2)}{(x^2+y^2)^4}$$

Since det(DF)=0 at $(0,1)$, so F is not locally invertible at $(0,1)$

——*A little bit problematic. Since the theorem gives us the sufficient condition to be locally invertible.*

A counter example: $f(x) = x^3$ *at* $x = 1$, *with* $DF = 0$ *but invertible.*

1. Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = x^2 y + e^x + z$.

   (i) Show that there exists a differentiable function g in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$ such that $g(1, -1) = 0$ and $f(g(y, z), y, z) = 0$.

   (ii) Compute $Dg(1, -1)$.

   Solution:

   (i) From the *Implicit Function Theorem*, for $(y, z) = (1, -1)$, $f(x, y, z) = 0$ has a solution $x = 0$, and since the Jacob Matrix:

   $$D_x f(x, y, z) = \begin{bmatrix} 2xy + e^x \end{bmatrix}; \; |D_x f(0, 1, -1)| = 1 \neq 0$$

   So function g exists in some neighborhood of $(1, -1)$ in $\mathbb{R}^2$ such that $g(1, -1) = 0$ and $f(g(y, z), y, z) = 0$.

   (ii) From *Implicit Function Theorem*:

   $$Dg(1, -1) = -[D_x f(x, y, z)]^{-1} D_{y,z} f(x, y, z) = -1 \cdot \begin{bmatrix} x^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

2. Prove the following result:

   Let V be a vector space and let $X, Y \subseteq V$ be convex. Then the sets

   $$\alpha X := \{z \in V | \exists x \in X : \alpha x = z\}$$

   and

   $$X + Y := \{z \in V | \exists x \in X, y \in Y : x + y = z\}$$

   are convex.

   Note: $X + Y$ is sometimes called the *Minkowski sum* of X and Y.

   Solution:

   $\forall z_1, z_2 \in \alpha X$, $\exists x_1, x_2 \in X$ such that: $z_1 = \alpha x_1, z_2 = \alpha x_2$. Then $\forall \lambda \in [0, 1]$, since $\lambda x_1 + (1 - \lambda) x_2 \in X$, so $\lambda z_1 + (1 - \lambda) z_2 = \alpha[\lambda x_1 + (1 - \lambda) x_2] \in \alpha X$. αX is convex.

   $\forall w_1, w_2 \in X + Y$, $\exists x_1, x_2 \in X; y_1, y_2 \in Y$ such that: $z_1 = x_1 + y_1, z_2 = x_2 + y_2$. Then $\forall \lambda \in [0, 1]$, since $\lambda x_1 + (1 - \lambda) x_2 \in X, \lambda y_1 + (1 - \lambda) y_2 \in Y$, so $\lambda z_1 + (1 - \lambda) z_2 = (\lambda x_1 + (1 - \lambda) x_2) + (\lambda y_1 + (1 - \lambda) y_2) \in X + Y$. $X + Y$ is convex.

3. Show that an arbitrary intersection of convex sets is convex.

   Proof. Denote $D = \bigcap_{i \in \mathcal{I}} D_i$ where D_i is convex for $\forall i \in \mathcal{I}$. Then $\forall d_1, d_2 \in D$, $\lambda \in [0, 1]$: $d_1, d_2 \in D_i$, $\forall i \in \mathcal{I}$, so $\lambda d_1 + (1 - \lambda) d_2 \in D_i$, $\forall i \in \mathcal{I}$. So $\lambda d_1 + (1 - \lambda) d_2 \in D$, D is convex.

4. Prove the following theorem:

   Let $C, D \subseteq \mathbb{R}^n$ be non-empty, closed, disjoint, and convex sets, and suppose that C is bounded. Then there exists a hyperplane (h, β) strictly separating C and D; that is,

$$\exists h \in \mathbb{R}^n, \beta \in \mathbb{R}: \ \forall x \in C, \ \forall y \in D : x \cdot h < \beta < y \cdot h.$$

   Solution:

   (i) To prove $D+(-C)$ is convex and closed. For convexity, from the previous problem: $D, (-C)$ are both convex, so $D+(-C)$ is also convex. For closed: since C is closed and bounded in $\mathbb{R}^n$, so it is compact and sequentially compact: so for any sequence $\{c_n\}$ in C, $\exists c_{nk}$ such that c_{nk} converges in C. Then for $\{e_n\} \in (D-C)$, we have $e_n = d_n - c_n$ where $c_n \in C, d_n \in D$. If e_n converges to e, we want to show $e \in D - C$. Since $\exists c_{nk} \to c \in C$, and $d_{nk} - c_{nk} \to e$ so $d_{nk} \to c + e$, and since D is closed, so $c + e \in D$, so $(c+e) - c = e \in D - C$. So $D - C$ is closed.

   (ii) Since C, D are disjoint, $0 \notin D - C$. Since $D - C$ is closed and convex. So $\exists H(\mathbf{h}, \gamma)$ such that $\mathbf{h}0 < \gamma < \mathbf{h}e, \forall e \in D - C$. So $\forall x \in C, y \in D$: $0 < \gamma < \mathbf{h}(y - x)$, so $\mathbf{h}x < \gamma + \mathbf{h}x < \mathbf{h}y$. Since C is compact, from the Extreme Value Theorem:

$$\sup_x(\gamma + \mathbf{h}x) = \max_x(\gamma + \mathbf{h}x) = \gamma + \mathbf{h}x_0, x_0 \in X$$

   So $\mathbf{h}x < \gamma + \mathbf{h}x \leq \gamma + \mathbf{h}x_0 < \mathbf{h}y$, where $\beta = \gamma + \mathbf{h}x_0$.

5. Let $\{f_j\}_{j \in J}$ be an arbitrary family of convex functions $f_j : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ is convex. Show that $f(x) := \sup\{f_j(x) | j \in J\}$ is convex.

   Solution: $\forall x, y \in X$, and $\lambda \in [0, 1]$:

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= \sup\{f_j(\lambda x + (1-\lambda)y) | j \in J\} \\ &\leq \sup\{\lambda f_j(x) + (1-\lambda) f_j(y) | j \in J\} \\ &\leq \sup\{\lambda f_j(x) | j \in J\} + \sup\{(1-\lambda) f_j(y) | j \in J\} \\ &= \lambda f(x) + (1-\lambda) f(y) \end{aligned}$$