* Linear mappings between normed vector spaces.

Thm 22. Let X and Y be normed vector spaces and $T : X \to Y$ be a linear mapping. If T is continuous at some point $x \in X$, then it is continuous on X.

proof) Suppose T is continuous at $x \in X$ and

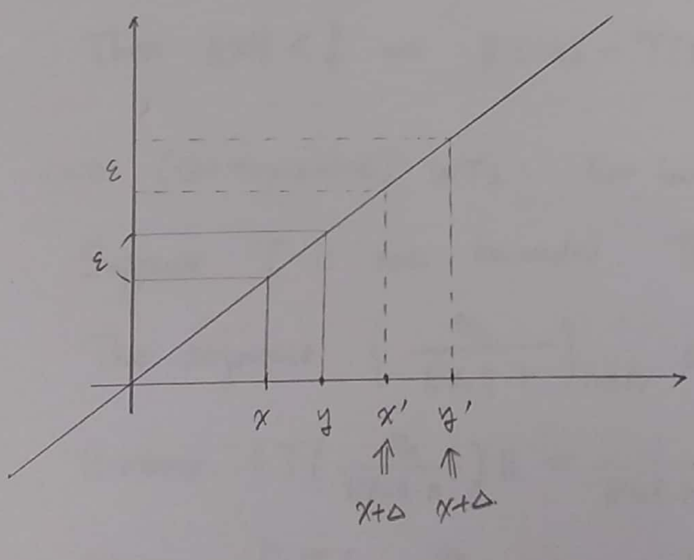
Fix an arbitrary $\varepsilon > 0$. Then, there exists a $\delta > 0$ such that

$\forall y \in B_\delta(x)$, $\| T(y) - T(x) \| < \varepsilon$.

Consider an arbitrary point $x' = x + \Delta \in X$

$\forall y' \in B_\delta(x')$, it holds that $y := y' - \Delta \in B_\delta(x)$.

Hence, $\| T(y') - T(x') \| = \| T(y' - x') \| = \| T(y + \Delta - x - \Delta) \| < \varepsilon$. $\|$



Def Let X and Y be normed vector spaces and $T: X \to Y$ be a linear mapping. T is <u>bounded</u> if there exists $B \in \mathbb{R}$ such that $\forall x \in X$, $\|T(x)\| \leqq B \cdot \|x\|$

(In this case, T maps bounded set into bounded set)

Thm 23 Let X and Y be normed vector spaces.
A linear mapping $T: X \to Y$ is <u>continuous</u> <u>if and only if</u> it is <u>bounded</u>.

proof) ($\Leftarrow$) WTS: T is continuous in $\underline{0}$

Since T is bounded, $\exists B > 0$ such that $\|T(x)\| \leqq B \cdot \|x\|$, $\forall x \in X$.

Fix arbitrary $\varepsilon > 0$ and set $\delta := \frac{\varepsilon}{B}$

Then $\|x\| < \delta \Rightarrow \|T(x) - T(\underline{0})\| \leqq B \cdot \|x\| < \varepsilon$

($\Rightarrow$) (Contrapositive) WTS: Not bounded $\Rightarrow$ Not continuous.

Suppose T is not bounded. Then $\forall n \in \mathbb{N}$, $\exists x_n \in X$ s.t. $T(x_n) > n \|x_n\|$.

The sequence $\left\{ \frac{x_n}{\|x_n\| \cdot n} \right\}_{n \in \mathbb{N}}$ converges to $\underline{0}$ for $n \to \infty$.

However, $\left\| T\left(\frac{x_n}{\|x_n\| \cdot n} \right) \right\| = \frac{1}{\|x_n\| \cdot n} \|T(x_n)\| > 1$

Hence, $\left\{ T\left(\frac{x_n}{\|x_n\| \cdot n} \right) \right\}_{n \in \mathbb{N}}$ does not converge to $\underline{0} = T(\underline{0})$

and T is not sequentially continuous. //

Thm 24 A linear mapping T from a finite-dimensional normed vector space X into a normed vector space Y is continuous

proof) Let $\mathcal{B} := \{v_1, \dots, v_n\}$ be a basis for X.

WTS : For any sequence, $\{x_n\}_{n \in \mathbb{N}} \subseteq X$, with $x_n \to 0$ (as $n \to \infty$), we have $T(x_n) \to \underline{0}$

Each x_n can be written as $x_n = \sum_{i=1}^{m} \alpha_i^n v_i$ for $\alpha_i^n \in \mathbb{R}$.

Note that $x_n \longrightarrow \underline{0}$ if and only if $\alpha_i^n \longrightarrow 0$ for $i = 1, \dots, n$

(Homework)

For each $n \in \mathbb{N}$, $0 \leqq \|T(x_n)\| = \|\sum_{i=1}^{m} \alpha_i^n T(v_i)\| \leq \sum_{i=1}^{m} \alpha_i^n \|T(v_i)\|$

Since $\alpha_i^n \to 0$, $\forall i$, $\|T(x_n)\| \longrightarrow 0$ and hence $T(x_n) \to \underline{0}$ .//

Def Let X and Y be normed vector spaces.

and let $T : X \to Y$ be a bounded linear mapping.

The sup norm or (linear operator norm) of T is defined by

$$\|T\|_L = \sup \left\{ \frac{\|T(x)\|}{\|x\|}, x \in X \setminus \{\underline{0}\} \right\}$$

We denote $B(X, Y)$ the set of bounded linear mappings from X to Y.

Thm 25 Let X and Y be normed vector spaces. Then $B(X, Y)$ endowed with linear operater norm is a normed vector space

proof) (Homework)

Example : Let $\mathcal{C}^1[0,1]$ be the vector space of continuously differentiable functions $f : [0,1] \longrightarrow \mathbb{R}$, endowed with the sup norm.

Let $\mathcal{C}[0,1]$ be the vector space of the continuous functions endowed with the sup norm.

The mapping $T : \mathcal{C}^1[0,1] \longrightarrow \mathcal{C}[0,1]$ satisfying

$f \longmapsto \underbrace{f'}_{\text{derivative}}$ is linear : $T(\alpha f + \beta g) = \alpha f' + \beta g' = \alpha T(f) + \beta T(g)$

The sequence $\{f_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C}^1[0,1]$ defined by $f_n(x) = \frac{1}{n}\sin(nx)$

satisfies $\|f_n\| \leqq \frac{1}{n}$, hence, $f_n \longrightarrow \underline{0}$.

However, $T(f_n)(x) = \cos(nx)$ satisfies $\|T(f_n(x))\| = 1, \ \forall n \in \mathbb{N}$.

Hence $\{T(f_n)\}$ does not converge to $\underline{0}$

Hence T is not continuous

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* Linear mappings and matrices

◎ Basic facts

- An $n \times m$ matrix is a rectangles array of real #.

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

- The product of a $n \times m$ matrix A with a vector $x \in \mathbb{R}^m$ is

$$Ax = \begin{pmatrix} \sum_{i=1}^{m} a_{1i} x_i \\ \vdots \\ \sum_{i=1}^{m} a_{ni} x_i \end{pmatrix}$$

- The product of $n\times m$ matrix A with an $m\times r$ matrix B is

$$A\cdot B = \begin{pmatrix} \sum_{i=1}^{m} a_{1i}b_{i1} & \cdots & \sum_{i=1}^{m} a_{1i}b_{ir} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{m} a_{ni}b_{i1} & \cdots & \sum_{i=1}^{m} a_{ni}b_{ir} \end{pmatrix}$$

- The set of $n\times m$ matrices endowed with pointwise addition and scalar multiplication is a vector space.

- The canonical basis of $\mathbb{R}^m$ is the set of vectors $\{e_1, \dots e_m\}$ where $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th}$

- An $n\times m$-matrix gives rise to a linear mapping $T_A : \mathbb{R}^m \to \mathbb{R}^n$ defined by $T_A(x) = Ax$.

⌜Note that the linearity of T_A follows from distributivity of matrix multiplication

$$A(\alpha x + \alpha' x') = \alpha A x + \alpha' A x'$$⌟

- Conversely, for any linear mapping $T: \mathbb{R}^m \to \mathbb{R}^n$, we can find an $n\times m$ matrix M_T such that $\forall x \in \mathbb{R}^m$, $T(x) = M_T x$. Let $x = \sum_{i=1}^{m} x_i e_i$ and let $T(e_i) = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$

Then, $T(x) = \sum_{i=1}^{m} x_i T(e_i) = \sum_{i=1}^{m} x_i \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix}$

$$= \begin{pmatrix} a_{11}x_1 + \cdots + a_{1m}x_m \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nm}x_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} x = M_T x$$

- Moreover, if $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $S \in L(\mathbb{R}^n, \mathbb{R}^l)$ with associated matrices M_T and M_S, then
  $T: \mathbb{R}^m \to \mathbb{R}^n$ $\quad S: \mathbb{R}^n \to \mathbb{R}^l$

$$\underbrace{M_{S \circ T}}_{l \times m} = \underset{l \times n}{M_S} \cdot \underset{n \times m}{M_T} \quad = l \times m$$

$$(S \circ T)(x) = S[T(x)] = S[M_T x] = M_S [M_T x] = [M_S M_T] x = M_{S \circ T} x$$

↑
"composition"

$$n \downarrow \overset{\xrightarrow{m}}{\underset{n \times m}{(A_{nm})}} \underset{m}{(x)}$$

Thm 26 Let $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ be a linear mapping with standard matrix representation $A = |a_{ik}|$ with $i = 1, \dots, n$, $k = 1, \dots, m$

Define $\underline{\mu = \max\{|a_{ik}|, i = 1, \dots, n, k = 1, \dots, m\}}$

Then $\underline{\mu \leqq \|T\| \leqq \mu \sqrt{n \cdot m}}$

proof) (Upperbound)

Given some $x \in X$, define $y := T(x) = Ax$

Then, the ith component of y is given by $y_i = \sum_{k=1}^{m} a_{ik} x_k$

The Cauchy-Schwartz inequality implies

$$|y_i| = \left|\sum_{k=1}^{m} a_{ik} x_k\right| \leq \sqrt{\sum_{k=1}^{m} a_{ik}^2}\sqrt{\sum_{k=1}^{m} x_k^2} \leq \sqrt{m\cdot\mu^2}\cdot\|x\|$$

Hence. $\|T(x)\| = \|y\| = \sqrt{\sum_{i=1}^{n} y_i^2} \leq \sqrt{nm}\,\mu\,\|x\|$

This implies $\|T\| \leq \sqrt{nm}\cdot\mu$

(Lower bound)

Fix i, k such $a_{ik} = \mu$.

Then $\|T\| = \sup\{\|T(x)\| \mid x \in \mathbb{R}^m \text{ and } \|x\|=1\}$

$\geq \|T(e_k)\| = \mu$. //

Remark

$\|T\|$: operator norm (sup)

$\|x\|$ in X] norm.
$\|T(x)\|$ in Y

☆ Be careful, please.

From ⑧ page.
⊛

Thm 7) Let $R \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $S \in L(\mathbb{R}^n, \mathbb{R}^p)$.
n×m matrix; p×n matrix

Then $T = S\circ R \in L(\mathbb{R}^m, \mathbb{R}^p)$ and $\underbrace{\|T\| \leq \|S\circ R\|}_{1st} \leq \|S\|\cdot\|R\|$ (2nd: $\|S\circ R\| \leq \|S\|\cdot\|R\|$)

proof) 1st ⇒ Homework

Second part : $\forall x \in X$, $\|S\circ R(x)\| = \|S[R(x)]\| \leq \|S\|\cdot\|R(x)\| \leq \|S\|\|R\|\|x\|$
by lemma 7

Def) If $T \in L(X, X)$, we say T is an operator and write $L(X)$ for $L(X,X)$.
$\Omega(\mathbb{R}^n)$ denotes the set of invertible linear operators on $\mathbb{R}^n$.

Lemma 7 If $T: X \to Y$ is a bounded linear mapping on $x \in X$,
then $\|T(x)\| \leq \|T\| \cdot \|x\|$

(*) move in front of Thm 27.

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* useful property

WTS: $\Omega(\mathbb{R}^n) \subset L(\mathbb{R}^n)$ open and inverse operator is continuous.
$\Rightarrow$ (Inverse function theorem)

Thm 28 $T \in L(\mathbb{R}^n)$ is invertible if and only if $\ker T = \{\underline{0}\}$ ☆☆☆

sketch: $n = \dim(\ker T) + \operatorname{rank} T$

Thm 29 Let S and T be invertible operators in $L(\mathbb{R}^n)$.
Then $S \circ T \in L(\mathbb{R}^n)$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

proof) sketch
Suppose $x \in \mathbb{R}^n \setminus \{\underline{0}\}$. Since T is invertible, $T(x) \neq \underline{0}$
Since S is invertible, $S[T(x)] \neq \underline{0}$.
Hence, $\ker S \circ T = \{\underline{0}\}$
Now, Thm 28 implies that $S \circ T$ is invertible.
$(S \circ T) \circ (T^{-1} \circ S^{-1}) = S \circ (T \circ T^{-1}) \circ S^{-1} = S \circ S^{-1} = I$.
$(T^{-1} \circ S^{-1}) \circ (S \circ T) = T^{-1} \circ (S^{-1} \circ S) \circ T = T^{-1} \circ T = I$.

Lemma 8 Let $T \in L(\mathbb{R}^n)$ and denote by I the identity mapping in $\mathbb{R}^n$

(i) If $\|T\| < 1$, then $(I - T)$ is invertible and $\|(I-T)^{-1}\| \leq \dfrac{1}{1 - \|T\|}$

(ii) If $\|I - T\| < 1$, then T is invertible.

Thm 30 Let T and S be linear operators in $\mathbb{R}^n$

If T is invertible and $\|S-T\| < \frac{1}{\|T^{-1}\|}$, then S is invertible

This implies that $\Omega(\mathbb{R}^n)$ is open in $L(\mathbb{R}^n)$

Moreover, $\|S^{-1}\| \leq \dfrac{\|T^{-1}\|}{1-\|T^{-1}\circ(T-S)\|}$

$S = T-T-S$
$= T - I\circ(T-S)$
$= T\circ(I-T^{-1}(T-S))$

proof) Note that $S = T - I\circ(T-S)$
$= T\circ(I-T^{-1}(T-S)) \cdots (*)$

B/c $\|T^{-1}\circ(T-S)\| \leqq \|T^{-1}\|\cdot\|T-S\| < 1$ (By assumption)

Now, lemma 8 implies $I - T^{-1}\circ(T-S)$ is invertible.

Hence S is invertible as composition of two invertible operators,

and $S^{-1} = \underbrace{(I-T^{-1}\circ(T-S))^{-1}\circ T^{-1}}_{\text{from }(*)}$

$\|S^{-1}\| \leqq \|I-T^{-1}\circ(T-S)\|^{-1}\cdot\|T^{-1}\|$ (By lemma 7)

$\leqq \dfrac{1}{1-\|T^{-1}\circ(T-S)\|}\cdot\|T^{-1}\|$ (By lemma 8)

$\|(I-T)^{-1}\| \leq \dfrac{1}{1-\|T\|}$

Thm 31 The function $(\cdot)^{-1} : \Omega(\mathbb{R}^k) \longrightarrow \Omega(\mathbb{R}^n)$ that assigns to each invertible operator its inverse is continuous.

proof) Fix arbitrary $T \in \Omega(\mathbb{R}^n)$ and $\varepsilon > 0$.

WTS: $\exists\, \delta > 0$ such that $\|T-S\| < \delta \Rightarrow \|T^{-1}-S^{-1}\| < \varepsilon$

Note that $S^{-1}-T^{-1} = (I-T^{-1}\circ S)\circ S^{-1} = T^{-1}\circ(T-S)\circ S^{-1}$

Hence $\|S^{-1}-T^{-1}\| \leqq \|T^{-1}\|\cdot\|T-S\|\cdot\|S^{-1}\|$

We can choose a small bound on $\|T-S\|$,

and $\|T\|$ is a constant. So, we need to find a bound a $\|S^{-1}\|$

If $\|T-S\| < \dfrac{1}{2\|T^{-1}\|}$, Thm 30 implies that $\|S^{-1}\| \leqq \dfrac{\|T^{-1}\|}{1-\|T^{-1}\circ(T-S)\|}$

$$\leqq \frac{\|T^{-1}\|}{1-\|T^{-1}\|\cdot\|T-S\|} \leqq 2\|T^{-1}\|$$

If we set $\delta := \min\left\{ \dfrac{1}{2\|T^{-1}\|}, \dfrac{\varepsilon}{2\|T^{-1}\|^2} \right\}$

then $\|T-S\| < \delta$ implies $\|S^{-1}-T^{-1}\| \leqq \|T^{-1}\|\cdot\|T-S\|\cdot\|S^{-1}\|$

$$< \|T^{-1}\|\cdot\delta\cdot 2\|T^{-1}\| = 2\|T^{-1}\|^2\cdot\frac{\varepsilon}{2\|T^{-1}\|^2} = \varepsilon \text{ . } \|$$