ECON 519 HOMEWORK 1

Inga Deimen, University of Arizona, Department of Economics, Fall 2018 Sol^n

1. Prove the following statement.
   For $x \neq 1$ and $\forall n \in \mathbb{N}$,
   $$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

2. Prove the following statement.
   If A and B are sets, then $A \cap (B - A) = \emptyset$.

3. Prove the following statement.
   Suppose $n \in \mathbb{Z}$. If n^2 is odd, then n is odd.

4. Prove the following statement.
   Let $\mathbb{A} = \{A_i | i \in I\}$ be a family of sets in X, then
   $$(\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} (A_i)^c.$$

5. i) Is $\leq$ an equivalence relation? Prove yes, or explain why no!

   ii) Give an example for a relation that is reflexive but not symmetric!

i) no ~~yes~~: reflexive, symmetric, transitive

$x R x \quad \forall x \in X$

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① Prove the following Statement

For $x \neq 1$ and $\forall n \in \mathbb{N}$

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$$

Proof by Induction:

Let $n = 1$:

$$\sum_{k=0}^{1} x^k = x^0 + x^1 = 1 + x$$

$$\frac{1-x^{1+1}}{1-x} = \frac{1-x^2}{1-x} = \frac{(1-x)(1+x)}{(1-x)} = (1+x)$$

Now, want to show: $\sum_{k=0}^{n} x^k + x^{k+1} = \frac{1-x^{(n+1)+1}}{1-x}$

$$\sum_{k=0}^{n} x^k + x^{k+1} = \frac{1-x^{n+1}}{1-x} + x^{n+1}$$

$$= \frac{1-x^{n+1}}{1-x} + \frac{(1-x)(x^{n+1})}{1-x} = \frac{1-x^{n+1}+x^{n+1}-x^{n+1+1}}{1-x}$$

$$= \frac{1-x^{n+2}}{1-x}.$$

By inductive step, we are done.

② Prove the following Statement:
If A and B are sets:

$$A \cap (B - A) = \emptyset$$

$$\Leftrightarrow A \cap (B \cap A^c)$$

$$\Rightarrow (A \cap B \cap A^c) \quad A \cap A^c = \emptyset$$

$$\Rightarrow A \cap B \cap A^c = \emptyset$$

③ Prove the following: Suppose $n \in \mathbb{Z}$
If n^2 is odd, then n is odd

Contrapositive: If n is even, then n^2 is even

Let $n \in 2k$, where $k \in \mathbb{Z}$

$$n \times n = n^2 = (2k)(2k) = 4k$$

$$= 2 \cdot 2k$$

$$\Rightarrow n^2 \text{ even}$$

④ Prove the following:

Let $A = \{A_i \mid i \in I\}$ be a family of sets in X. Then:

$$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} (A_i)^c$$

$$(\bigcup A_i)^c = (A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_I)^c$$

DeMorgan $= A_1^c \cap A_2^c \cap \ldots \cap A_I^c$

$= \bigcap A_i^c$

⑤ Is $\leq$ an equivalence relation?

Equivalence $\Leftrightarrow$ reflexive, symmetric, transitive.

$x \leq x$ is true ✓

$x \leq y \Leftrightarrow y \leq x$ does not hold.
Assume $x \neq y$. Then if $x \leq y$
it cannot be both,
$x \neq y$ and $y \leq x$.

Clearly transitive. But, not an equivalence relation.

ECON 519 HOMEWORK 2

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1. Prove the following statement.
   Let X be a set and $\{G, *\}$ a group. The set of functions of X into G, endowed with the operation defined by the composition of images, i.e.,

   $$\forall x \in X, (f * g)(x) = f(x) * g(x)$$

   is a group.

2. Let X be a non-empty set. Consider the set of all maps from X to $\mathbb{R}$, denoted by $\mathcal{F} = \{f | f : X \to \mathbb{R}\}$, and the subset of $\mathcal{F}$ consisting of all bounded maps from X to $\mathbb{R}$, denoted by $\mathcal{F}_b = \{f \in \mathcal{F} | \exists k \in \mathbb{R} : |f(x)| < k, \forall x \in X\}$. Show that $\mathcal{F}$ and $\mathcal{F}_b$ are real vector spaces if we define:

   $$(f+g)(x) = f(x) + g(x), \quad \forall f, g \in \mathcal{F},$$
   $$(\lambda f)(x) = \lambda f(x), \quad \forall f \in \mathcal{F}, \lambda \in \mathbb{R}.$$

3. Let $S \subseteq \mathbb{R}$. A real-valued function $f : S \to \mathbb{R}$ is said to be strictly increasing if $z > z' \Rightarrow f(z) > f(z')$ for all $z, z' \in S$. Let U be the set of all real-valued functions on a set X, i.e., U is the set of all functions $u : X \to \mathbb{R}$. [Note that X can be any set.] Define a relation $\sim$ on U by

   $$u \sim u' \Leftrightarrow \exists \text{ a strictly increasing function } f : u(X) \to \mathbb{R} \text{ such that } u' = f \circ u,$$

   where $f \circ u$ is the composition function defined by $\forall x \in X : (f \circ u)(x) = f(u(x))$.
   Prove that $\sim$ is an equivalence relation on U.

4. Let $\{G, *\}$ be a non-empty set with a binay relation $* : G \times G \to G$.

   a) Which of the following statements - if any - imply the other? Give a proof by counterexample.

   (i) $\forall g \in G : \exists h \in G : g * h = h * g = g$
   (ii) $\exists \tilde{h} \in G : \forall \tilde{g} \in G : \tilde{g} * \tilde{h} = \tilde{h} * \tilde{g} = \tilde{g}$

   b) Assume that $\{G, *\}$ is a group. Which of the two statements above hold true (if any, or maybe both)?

5. Let A and B be nonempty sets of real numbers, both of them bounded above, and let C be the set

$$C = \{c = a + b \mid a \in A, b \in B\}.$$

Show that C has the supremum that is given by

$$\sup C = \sup A + \sup B.$$

Problem 5.3 Let " $*$ " be a law of internal composition on X that satisfies the associative property and is endowed with the identity element. Prove it that x and y have symmetric elements x^s and y^s, the symmetric element of $x * y$ is $y^s * x^s$

Symmetric $\Leftrightarrow$ inverse
Law Int. Comp $\Leftrightarrow$ closed under $*$

$$x * x^s = e = y * y^s = x * x^s * y * y^s$$
$$= x * y * y^s * x^s$$
$$\neq (x*y) * (y^s * x^s)$$
$$= e$$

No Commutivity

Book Sol^{n}

$$(x * y) * (y^s * x^s) = x * (y * y^s) * x^s$$
$$= x * e * x^s$$
$$= x * x^s$$
$$= e$$

(1) (Problem 5,4) Let X be an arbitrary set and $\{G, *\}$ a group. Show that the set of functions of X into G endowed with the operation defined by composition of images, that is:

$$\forall x \in X \quad (f * g)(x) = f(x) * g(x)$$

is a group

Group:

(1) Closed under $*$

(2) $*$ is an associative law

(3) Endowed w/ identity

(4) Symmetric.

(1) $f(x)$ and $g(x)$ are elements of $\{G, *\}$ Therefore, their composition will be in $\{G, *\}$ by definition, so closed ✓

(2) By same reasoning, if $f(x) * g(x) = (f * g)(x)$ is in the group, so will $g(x) * f(x)$, that is the set inherits the associativity property from the group.

(3) $e(x) = x \quad \forall x \in G, \quad (f \cdot e)(x) = f(x) * e(x)$

$$= f(x) * e$$
$$= f(x)$$

(4) Symmetric. Since $f \in G \Rightarrow f^{-1} \in G$

$$(f * f^{-1})(x) = f(x) * f^{-1}(x) = e$$

Problem 5.9 Prove the following (Thm 5.8)
Let V be the vector space over a field F and let S be a nonempty subset of V. Then S is a vector space if and only if:

$$\forall \alpha, \beta \in F, \forall x, y \in S: \alpha x + \beta y \in S$$

V vector space $\Rightarrow$ S is vector subspace if above is true, Defined on field F, and will inherit those properties (vector addition and scalar mult.)

Notice:

$\alpha = \beta = 1 \Rightarrow S$ is closed
$\alpha = 1, \beta = 0$, we have product of any element in S w/ a scalar in S is in S
$\alpha = 0, \beta = 0 \Rightarrow \exists\, 0 \in S$
$\alpha = -1, \beta = 0 \Rightarrow \exists -x \in S$ (inverse)

(2) Let X be a nonempty ~~subset~~ subset. Consider set of all maps from X to $\mathbb{R}$ denoted by $\mathcal{F} = \{f \mid f: X \to \mathbb{R}\}$, and subset of all bounded maps from X to $\mathbb{R}$ denoted by $\mathcal{F}_b = \{f \in \mathcal{F} \mid \exists k \in \mathbb{R}: |f(x)| < k \;\forall x \in X\}$. Show $\mathcal{F}$ and $\mathcal{F}_b$ are real vector spaces if we define

$$(f+g)(x) = f(x) + g(x) \quad \forall f, g \in F$$
$$(\lambda f)(x) = \lambda f(x) \quad \forall f \in \mathcal{F}, \lambda \in \mathbb{R}$$

① Closed under addition:

$f, g \in \mathcal{F}$, then certainly

$(f+g)(x) = f(x) + g(x)$ is in $\mathcal{F}$

② Closed under scalar mult.

$\lambda f(x) \in \mathcal{F}$ since $\lambda \in \mathbb{R}$
$\| \quad$ and $\mathcal{F}$ is set
$(\lambda f)(x)$ of all $f: X \to \mathbb{R}$

③ Associativity

$(f + (g+h))(x) = f(x) + (g(x) + h(x))$
$= (f(x) + g(x)) + h(x)$
$= ((f+g)+h)(x)$

$\alpha(\beta f(x)) = \alpha\beta(f(x))$

④ Commutativity

$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x)$

⑤ ~~Identity~~ Symmetry

$-f \in \mathcal{F}$ and $(-f + f)(x) = -f(x) + f(x) = \underline{0}$

⑥ Additive Identity.

$0 \in \mathbb{R}$ and hence $\exists\, g : X \to 0 \in \mathbb{R}$ $g \in \mathcal{F}$

$(f+g)(x) = f(x) + g(x) = f(x)$

⑦ Exists multiplicative identity

$$\lambda = 1 \Rightarrow \lambda f(x) = f(\lambda x) = f(x)$$

⑧ Double Dist. $\alpha, \beta \in \mathbb{R}$

$$(\alpha f + \beta g)(x) = (\alpha f)(x) + (\beta g)(x)$$
$$= \alpha (f)(x) + \beta g(x)$$

and f^{-1} exists.

(b) Now for F_b, F_b is a vector subspace of F. Therefore, we only need to prove:

$$\forall \alpha, \beta \in F : \forall x, y \in S \ \alpha x + \beta y \in S$$

Let x, y be bounded such that

$$|x| \leq M_1 \ , \ |y| \leq M_2$$

$$\alpha |x| \leq \alpha M_1 \ , \ \beta |y| \leq \beta M_2$$

$$\alpha |x| + \beta |y| \leq \alpha M_1 + \beta M_2 = M$$

Therefore, the sum of scalar products is still bounded and in $\mathcal{F}_b$

③ Let $S \subseteq \mathbb{R}$. Real valued function $f: S \to \mathbb{R}$ is said to be str. inc if $z > z' \Rightarrow f(z) > f(z')$ for all $z, z' \in S$. Let U be set of all real valued functions on X, U is set of all functions $u: X \to \mathbb{R}$. Define $\sim$ relation on U

$$u \sim u' \Leftrightarrow \exists \text{ str inc } f: u(X) \to \mathbb{R}$$
$$\text{s.t. } u' = f \circ u$$

$f \circ u$ is composition function $\forall x \in X: (f \circ u)(x) = f(u(x))$
Prove $\sim$ is equivalence relation

① Reflexivity. $u \sim u \Leftrightarrow \exists$ str inc $f: u(X) \to \mathbb{R}$ s.t. ~~and~~ $u' = f \circ u$ or $(f \circ u)(x) = f(u(x))$

Let $f(x) = x$ $\quad f(u(x)) = u(x)$

② Symmetry
$u \sim u' \Leftrightarrow u' \sim u$

The inverse of a str inc func is strictly inc.

$u \sim u' \Leftrightarrow \exists f: u(X) \to \mathbb{R}: u' = f(u(x))$
Let $u' \sim u \Rightarrow \exists f': u(x) \to \mathbb{R} \quad u = f'(u'(x))$
Let $f' = f^{-1}(u'(x)) = u(x)$
Thus relation is symmetric

③ Transitivity - $u_1 \sim u_2 \quad u_2 \sim u_3$
$f_1((u_1(x)) = u_2$
$f_2(u_2(x)) = u_3$
$f_2(f_1(u_1(x)) = u_3$ hence $u_1 \sim u_3$
(And composition of str inc func is inc)

(4) Let $\{G, *\}$ be a nonempty set with a binary relation $*: G \times G \to G$

(a) Which of the following statements if any imply the other?

(i) $\forall g \in G : \exists h \in G : g * h = h * g = g$
(ii) $\exists \tilde{h} \in G : \forall \tilde{g} \in G : \tilde{g} * \tilde{h} = \tilde{h} * \tilde{g} = \tilde{g}$

(ii) $\Rightarrow$ (i), (i) $\not\Rightarrow$ (ii)

Counterexample (i) $\not\Rightarrow$ (ii)

$\{0, 1\}$
$\{1, 0\}$
$\{0, 0\}$
$\{1, 1\}$

if two $0 \to 1$
$1 \to 1$
one 0 one $1 \to 0$

$\{0, 1\} \times \{0, 1\} = \{1, 1\}$
$\{1, 0\} \times \{0, 1\} = \{0, 0\}$
$\{1, 1\} \times \{1, 1\} = \{1, 1\}$
$\{0, 0\} \times \{0, 0\} = \{1, 1\}$
$\{0, 0\} \times \underbrace{\{1, 1\}}_{\text{ident. for } \{0, 0\}} = \{0, 0\}$

$\{1, 0\} \times \underbrace{\{1, 0\}}_{\text{identity for } \{0, 1\}} = \{1, 1\}$

Every "g" has an "h", just not same one.

If $\{G, *\}$ is a group, still holds

Need associativity + (identity)
has h. So, still holds

$(ii) \Rightarrow (i)$, $(i) \not\Rightarrow (ii)$

(5) Let A and B be nonempty sets of real numbers, both bounded above and let C be the set:

$$C = \{c = a + b \mid a \in A, b \in B\}$$

Show C has a supremum $\sup C = \sup A + \sup B$

① $\sup C \leq \sup A + \sup B$ ② $\sup C \geq \sup A + \sup B$

① $\sup(A + B) \leq \sup A + \sup B$

Let $a^* = \sup A$ and $b^* = \sup B$

$c = a + b \quad a \leq a^* \quad b \leq b^*$

$c = a + b \leq a^* + b^*$

C is bounded above, then has $\sup C^*$

$c^* \leq a^* + b^*$

② $\varepsilon > 0$. No number smaller than sup is upper bound

$a > a^* - \varepsilon \qquad b > b^* - \varepsilon$

$$a^* + b^* - 2\varepsilon < a + b = c \leq c^*$$

Then $a^* + b^* \leq c^*$

Therefore $\sup a + \sup b = \sup c$

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ECON 519 HOMEWORK 3

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1. Prove the following theorem from our class.
   Let $(V, \|\cdot\|)$ be a normed vector space. Let $d : V \times V$ be defined by $d(v, w) = \|v - w\|_E$.
   Then $(V, \|\cdot\|)$ is a metric space.

2. Prove the following statement.
   Every convergent sequence in a metric space is bounded.

3. Prove the following theorem from our class.
   Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then

$$\lim_{n\to\infty} x_n = \gamma \in \mathbb{R} \cup \{-\infty, \infty\} \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \sup x_n = \gamma.$$

4. Use the formal definition of the limit to show that

$$\lim_{n\to\infty} \frac{n^2 + 12}{8n^2 + n} = \frac{1}{8},$$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1.$$

5. Prove the following statement.
   Let $\{x_n\}$ and $\{y_n\}$ be convergent sequences in $\mathbb{R}$, with $\{x_n\} \to x$ and $\{y_n\} \to y$.
   Then $\{x_n + y_n\} \to x + y$.

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① Prove: Let $(V, \|\cdot\|)$ be a normed vector space. Let $d: V \times V$ be defined by $d(v,w) = \|v-w\|_E$. Then, $(V, \|\cdot\|)$ is a metric space.

3 qualities of metric space:

(i) $d(x,y) > 0$ and $d(x,y) = 0$ if $x=y$
(ii) $d(x,y) = d(y,x)$
(iii) $d(x,z) \le d(x,y) + d(y,z)$

Since $(V, \|\cdot\|)$ contains a set of vectors, only need to prove that $\|\cdot\|_E$ is a metric.

(i) $\forall v \neq a \; \forall w \in V$: $d(v,w) = \sqrt{\sum_{i=1}^{?} (v_i - w_i)^2} = \sqrt{\sum (\alpha)^2}$ where $\alpha > 0 \Rightarrow d(v,w) > 0$

(ii) Clearly. Easy to show: $(v_i - w_i)^2 = (w_i - v_i)^2$

(iii) Use Cauchy Schwartz Inequality

$$\begin{aligned}
\sum (v_i - w_i)^2 &\le \sum \left((v_i - u_i) + (u_i - w_i)\right)^2 \\
&\le \sum (v_i - u_i)^2 + \sum (u_i - w_i)^2 + 2\sum (v_i - u_i)(u_i - w_i) \\
&\le \sum (v_i - u_i)^2 + \sum (u_i - w_i)^2 + 2\sqrt{(v_i - u_i)^2}\sqrt{(u_i - w_i)^2} \\
&= (d_E(v,u))^2 + 2d(v,u)\,d(u,w) + (d_E(u,w))^2 \\
&= (d_E(v,u) + d_E(u,w))^2
\end{aligned}$$

So, taking square root:

$$d(v,w) \le d(v,u) + d(v,w) \qquad \square$$

Problem 2 Every convergent sequence in a metric space is bounded.

A sequence converges to a pt x if $\forall \varepsilon > 0$ $\exists N > \mathbb{N}$: $\forall n > N$. $d(x_n, x) < \varepsilon$

Let $B_\varepsilon(x)$ be the closed ball around x. Therefore $\{x_n\}$ is bounded.

or, a slightly stronger proof: Assume $\{x_n\} \to x$. $\exists N \in \mathbb{N}$: $d(x_n, x) < 1$ $\forall n > N$

set $\varepsilon = 1$. Denote B as: $B = \max\{d(x_1, x), d(x_2, x), \ldots, d(x_n, x), \varepsilon = 1\}$

B is bound for x_n.

$$d(x_i, x_k) \leq d(x_i, x) + d(x, x_k) = 2m.$$

③ Prove the following statement:

Let $\{x_n\}$ be a sequence in $\mathbb{R}$. Then

$$\lim_{n\to\infty} x_n = \gamma \in \mathbb{R} \cup \{-\infty, \infty\} \iff \liminf x_n = \limsup x_n = \gamma$$

(1) "$\Rightarrow$" $\lim_{n\to\infty} x_n = \gamma \Rightarrow \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N \; d(x_n, \gamma) < \varepsilon$

$\Rightarrow \gamma - \varepsilon \leq x_n \leq \gamma$

$\Rightarrow \gamma$ is supremum

and

$\gamma \leq x_n \leq \gamma + \varepsilon$

$\Rightarrow \gamma$ is infimum

Suppose the sequence $\{x_n\}$ converges to a lim. $\gamma \in \mathbb{R}$

Then for every $\varepsilon > 0$, $\varepsilon/2 > 0$ $\exists N \in \mathbb{N}$ such that $\forall n > N: \gamma - \varepsilon/2 < x_n < \gamma + \frac{\varepsilon}{2}$

It follows that $\gamma - \varepsilon < \gamma - \varepsilon/2 \leq \beta_n \leq \alpha_n \leq \gamma + \varepsilon/2 < \gamma + \varepsilon$

$\forall n > N$ Since

$\alpha_n = \sup\{x_k : k \geq n\}$ $\beta_n = \inf\{x_k : k \geq n\}$

If $\alpha_n < \gamma - \varepsilon/2$, then it is not upper bound of $\{x_k : k \geq n\}$

If $\alpha_n > \gamma - \varepsilon/2$, it is not least upper bound of $\{x_k : k \geq n\}$

If $\beta_n < \gamma - \varepsilon/2$, it is not greatest lower bound of $\{x_k : k \geq n\}$

Therefore $\alpha_n, \beta_n \to \gamma$ as $n \to \infty$ so:

$$\limsup x_n = \liminf x_n = \gamma$$

" $\Leftarrow$ " $\gamma = \liminf x_n = \limsup x_n \Rightarrow \lim x_n = \gamma$

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} : \forall n > N \quad d(\alpha_i, \gamma) < \varepsilon \qquad \alpha_i = (\sup x_k : k \geq n)$

$d(\beta_i, \gamma) < \varepsilon \qquad \beta_i (= \inf x_k : k \geq n)$

$\beta_i \leq x_i \leq \alpha_i$ and squeeze thm $\Rightarrow$

$\liminf x_n = \lim x_n = \limsup x_n = \gamma$

(4) Use formal defn of limit to show that:

$$\lim \frac{n^2+12}{8n^2+n} = \frac{1}{8} \qquad \left| \frac{n^2+12}{8n^2+n} - \frac{1}{8} \right| < \varepsilon$$

$$\left| \frac{8(n^2+12) - 8n^2 - n}{8(8n^2+n)} \right| < \varepsilon \qquad \frac{8n^2 + 96 - 8n^2 - n}{8(8n^2+n)} = \frac{96-n}{8(8n^2+n)} < \varepsilon$$

$$\frac{96-n}{8} < \varepsilon$$

$$96 - n < 8\varepsilon$$

$$-n < \frac{\varepsilon}{12}$$

$$n > \varepsilon/12$$

⑤ Let $\{x_n\}$ and $\{y_n\}$ be a convergent sequences in $\mathbb{R}$ with $\{x_n\} \to x$ $\{y_n\} \to y$ Then $\{x_n + y_n\} \to x+y$

$$d(x, x_n) < \varepsilon/2 \Rightarrow x_n - \varepsilon/2 \le x \le x_n + \varepsilon/2$$

$$d(y, y_n) < \varepsilon/2 \Rightarrow y_n - \varepsilon/2 \le y \le y_n + \varepsilon/2$$

sum!

$$d(x - x_n + y - y_n) \le d(x, x_n) + d(y, y_n) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$\forall \varepsilon > 0 : \quad |x_n - x| < \varepsilon/2, \quad |y_n - y| < \varepsilon/2$$

$$|x_n - x + y_n - y| \le |x_n - x| + |y_n - y| \le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$$|x_n - x + y_n - y| = |x_n + y_n - (x+y)| < \varepsilon$$

$$\Rightarrow \lim x_n + y_n = x + y$$

David Zynda

ECON 519 — HOMEWORK 4

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1. Define a relation $\sim$ between norms on a given vector space V as follows: $\|x\|_a \sim \|x\|_b$ if there exist positive numbers m and M such that

$$\forall x \in X : m\|x\|_a \leq \|x\|_b \leq M\|x\|_a .$$

   - Prove that on $\mathbb{R}^2$, $\|x\|_1 \sim \|x\|_\infty$. $m = \frac{\|x\|_b}{\|x\|_a}$
   - Prove that $\sim$ is an equivalence relation.

2. Let (X, d) be a metric space and $x, y \in X$ with $x \neq y$. Prove that there exist open sets $U_x, U_y \subset X$ such that $x \in U_x, y \in U_y$ and $U_x \cap U_y = \emptyset$.

3. *Let*(X, d) be a metric space and let $(x_n)_{n \in N}$ be a sequence in (X, d) that converges to x. Show that the set $\{y \in X | \exists n \in N : y = x_n\} \cup \{x\}$ is closed in (X, d). thm 8

4. Let (X, d) be a metric space and $A \subseteq X$. Prove that $\partial A = \overline{A} \setminus \mathit{int}\, A$.

5. - Show that in any metric space, any set $\{x\}$ containing only one element is closed.
   - Prove that in a metric space (X, d) the sets $\overline{B}_\varepsilon(x) = \{y \in X | d(x, y) \leq \varepsilon\}$ are closed for all $x \in X$, $\varepsilon > 0$.

$\{x_n\}$ order matters repeats allowed

$\{y \in X \mid n \in \mathbb{N}\ x_n = y\}$ no repeats order does not matter

A open $\Leftrightarrow$ A^c closed

A not open $\Leftrightarrow$ A^c not closed

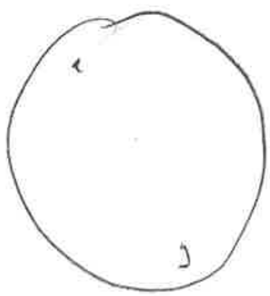
$\frac{1}{2}(\frac{1}{2} + \frac{1}{4}) \leq \frac{1}{2} \leq \frac{1}{2} + \frac{1}{8}$

$\bar{A} \cap \overline{(X \setminus A)}$

$\bar{A} \cap (\text{int } A)^c$

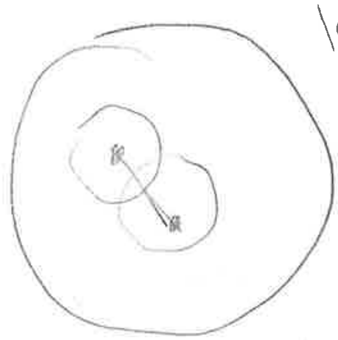
$\bar{A} \setminus \{\text{int } A\}$

δ ball



let

$\delta' = \frac{1}{2} d(x,y)$



Econ 519 HW 4 Solutions

① Define a relation ~ between norms on a given vector space as follows: $\|x\|_a \sim \|x\|_b$ if there exists positive numbers m and M such that:

$$\forall x \in X: m\|x\|_a \leq \|x\|_b \leq M\|x\|_a$$

Prove on $\mathbb{R}^2$ $\|x\|_1 \sim \|x\|_\infty \Rightarrow m\|x\|_1 \leq \|x\|_\infty \leq M\|x\|_1$

$\|x\|_1 = \sum_{i=1}^{2}|x_i| = |x_1|+|x_2|$ $\quad \|x\|_\infty = \max\{|x_1|,|x_2|\}$

Sum of two components will always be greater than or equal to one alone. Therefore

$$\max\{|x_1|,|x_2|\} \leq |x_1|+|x_2|$$

Let $M = 1$.

The midpoint between 2 numbers is always less than or equal to the magnitude of one of the components. So: $m = 1/2$.

$$1/2\|x\|_1 \leq \|x\|_\infty \leq \|x\|_1$$

b) Prove ~ is an equivalence relation

(i) Reflexivity: trivial

(ii) Symmetry: $\|x\|_a \sim \|x\|_b \Rightarrow m\|x\|_a \leq \|x\|_b \leq M\|x\|_a$

$$\frac{m}{mM}\|x\|_a \leq \frac{\|x\|_b}{mM} \leq \frac{M\|x\|_a}{mM} \iff \frac{1}{M}\|x\|_a \leq \frac{\|x\|_b}{mM} \leq \frac{1}{m}\|x\|_a$$

$$\frac{\|x\|_b}{M} \leq \|x\|_a$$

$$\|x\|_a \leq \frac{\|x\|_b}{m}$$

Set $m^* = \frac{1}{M}$ $M^* = \frac{1}{m}$, then

$m^*\|x\|_b \leq \|x\|_a \leq M^*\|x\|_b \Rightarrow$

$\|x\|_b \sim \|x\|_a$.

② Let (X, d) be a metric space and let $x, y \in X$ w/ $x \neq y$. Prove there exists open sets $U_x\ U_y \subset X$ such that

$$x \in U_x \quad y \in U_y \quad U_x \cap U_y = \emptyset$$

Denote $d(x, y)$, $d > 0$ Let $\varepsilon = D/3 > 0$. Construct open balls: $B_\varepsilon(x) = \{u \in X \mid d(u, x) < \varepsilon\}$

$B_\varepsilon(y) = \{u \in X \mid d(u, y) < \varepsilon\}$

And let $U_x = B_\varepsilon(x)$ $U_y = B_\varepsilon(y)$

if $\exists\, u_0 \in U_x \cap U_y$ then $d(u_0, x) < \varepsilon$ $d(u_0, y) < \varepsilon$

So, $d(x, y) < 2\varepsilon < D$ a contradiction.

So, $U_x \cap U_y = \emptyset$

$B_\varepsilon(x)$, $B_\varepsilon(y)$ are open sets $\forall x' \in B_\varepsilon(x)$

$B_{\varepsilon - d(x, x')}(x') \subseteq B_\varepsilon(x)$

$\forall y' \in B_\varepsilon(y)$ $B_{\varepsilon - d(y, y')}(y') \subseteq B_\varepsilon(y)$

③ Let (X, d) be a metric space and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in (X, d) that converges to x.
Show the set $\{y \in X \mid \exists n \in \mathbb{N} : y = x_n\} \cup \{x\}$ is closed in (X, d)

Prove Y^c is open (so Y is closed).

$Y = \{y \in X \mid n \in \mathbb{N} : y = x_n\} \cup \{x\}$ and take some $y \in X \setminus Y = Y^c$ (if $X = Y$, obviously Y is open)

Then, denote $d(x, y) = \varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$

$\exists N \in \mathbb{N} : d(x, x_n) < \varepsilon/2 \ \forall n > N$. Hence;

$d(y, x_n) \geq d(x, y) - d(x, x_n) = \varepsilon/2 \quad \forall n > N$

Let $\delta = \min\{d(y, x_n) \mid n \leq N\} > 0$

$\rho = \min\{\delta, \varepsilon/2\}$

$B_\rho(y) \cap Y = \emptyset \quad B_\rho(y) \subseteq Y^c$

Hence Y^c is open and Y is closed

④ Let (X, d) be a metric space $A \subseteq X$. Let $\partial A = \bar{A} \cap (\overline{X \setminus A})$

Prove $\partial A = \bar{A} \setminus \operatorname{int} A$

$$\cancel{\operatorname{int} A = (A^c \cap \bar{A})^c}$$

$$\cancel{\Rightarrow \partial A = \bar{A} \cap \underbrace{(A^c \cap \bar{A})}_{X \cap (A^c \cap \bar{A}}}$$

$$\partial A = \bar{A} \cap (\overline{X \setminus A})$$
$$= \bar{A} \cap (\operatorname{int} A)^c$$
$$= \bar{A} \setminus \{\operatorname{int} A\}$$

— Risherg said I could be more explicit

⑤ Show that in any metric space, any set $\{x\}$ containing only one element is closed.

Let $A = \{x\} \subseteq X$

Prove $X \setminus A$ is open $\forall y \in X \setminus A$

denote $\varepsilon = d(x, y)$ so $\varepsilon > 0$. Then for open ball $B_\varepsilon(y)$ it doesn't contain x so $B_\varepsilon(y) \subseteq X \setminus A$. So $X \setminus A$ is open. A is closed.

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Prove that in a metric space (X, d) the sets $\bar{B}_\varepsilon(x) = \{y \in X \mid d(x,y) \leq \varepsilon\}$ are closed $\forall x \in X, \varepsilon > 0$

Prove closed balls are closed sets.

$\forall x \ \forall \varepsilon > 0$: $X \setminus \bar{B}_\varepsilon(x) = O_\varepsilon(x)$ is open (prove)

$\forall y \in O_\varepsilon(x)$ Let $\delta = \frac{d(y, x) - \varepsilon}{3}$ then $\delta > 0$ and the open ball $B_\delta(y)$ doesn't contain any points in $\bar{B}_\varepsilon(x)$: $\forall y' \in B_\delta(y)$. If $d(y', x) \leq \varepsilon$ then

$d(x, y) \leq d(y', y) + d(y', x) < \delta + \varepsilon < \delta(y, x) \Rightarrow$ contradiction

So, $d(y', x) > \varepsilon$. So, $B_\delta(y) \subseteq O_\varepsilon(x)$. So, $O_\varepsilon(x)$ is open.

Another Solution: (Pg 672 Fuente)

Denote closed ε-ball as $B_\varepsilon[X]$. We will show that $a \in B_\varepsilon[X]$. By definition of limit point, there exists a sequence $\{y_n\}$ in $B_\varepsilon[X]$ that converges to a. Because $\{y_n\} \in B_\varepsilon[X]$ we have $d(y_n, x) \leq \varepsilon$, $\forall n$. Using triangle inequality:

$$d(a,x) \leq d(a,y_n) + d(y_n,x) \leq d(a,y_n) + \varepsilon$$

$$y_n \to a \Rightarrow d(a,y_n) \to 0 \Rightarrow d(a,y_n) < \varepsilon$$

$$a \in B_\varepsilon[X] \Rightarrow B_\varepsilon[X]$$

contains all its limit points. Therefore it is closed

DAVID ZYNDA

ECON 519 HOMEWORK 5

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1. Prove the following statement. Pg 62 Fuente
   A set A in a metric space (X, d) is closed if and only if it contains all it limit points.

2. Prove the following statement.
   Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$. Then f is continuous if and only if the preimage of every open set in Y is open in X:

$$f^{-1}(A) \text{ is open in } X \; \forall A \subseteq Y \text{ open.}$$

3. Show that in any normed vector space $(X, \|\cdot\|)$ the norm is a continuous function from X to $\mathbb{R}$.

We will discuss here two stronger notions of continuity. Throughout, let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$.

Definition 1

- f *is* uniformly continuous *if, for all* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, for all* $x \in X$, $d(x, x') < \delta$ *implies* $\rho(f(x), f(x')) < \varepsilon$.
- f *is* Lipschitz continuous *if there exists a constant* $K > 0$ *such that, for all* $x, x' \in X$, $\rho(f(x), f(x')) \leq K\, d(x, x')$.

4. (i) Show that f being Lipschitz continuous implies that f is uniformly continuous, which in turn implies that f is continuous.

   (ii) Let $f : (0, 1) \to \mathbb{R}$ be given by $f(x) = 1/x$ and $g : (0, 1) \to \mathbb{R}$ be given by $f(x) = \sqrt{x}$. Which notions of continuity do f and g satisfy? Explain.

5. Show that if $C \subset X$ is compact and $f : C \to Y$ is continuous then f is uniformly continuous.

Econ 519 HW5 Soln

① Prove the following statement:

A set A in a metric space (X,d) is closed iff it contains all its limit points

"⇒" A is a closed in X and x is a limit point of A. We prove $x \in A$.

$\forall \varepsilon > 0 \quad B_\varepsilon(x) \cap (A \setminus \{x\}) \neq \phi$. Let $\varepsilon = 1/n$

Set $x_n \in B_{1/n}(x) \cap (A \setminus \{x\})$

$\lim_{n\to\infty} x_n = x$. Since A is closed, then $\{x_n\}$ has limit $x \in A$

So if A is closed, then contains all limit points

Alternatively, the book says (pg 62)

⇒ Assume A is closed. ⇒ A^c is open. Then, for any $x \in A^c$, there exist some $\varepsilon > 0$ s.t.

$$B_\varepsilon(x) \subseteq A^c \Leftrightarrow B_\varepsilon(x) \cap A = \phi$$

Hence, no point in A^c can be a limit point of A and it follows that all such points must be contained in A

"⇐" A contains all its limit pts ⇒ A^c is open

Contra: If A^c is not open, then it contains some limit pts of x

There exist points in A^c w/ property that no open ball around them lies entirely in A^c. Let x be one such point. For $\varepsilon > 0$, $B_\varepsilon(x)$ contains at least one point in A, necessarily different from x because $x \in A^c$. Hence X is a lim pt of a lying in A^c

Alternatively,

Suppose A contains all its limit points. then if a sequence $\{y_n\}$ converges to its limit $y \in X$, we want to show $y \in A$. If $y = y_n$ for some n, we are done. If not, we prove y is a lim pt of A. Since $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ $\forall n > N$ $d(y, y_n) < \varepsilon$ so:

$B_\varepsilon(y) \cap (A \setminus \{y\}) \neq \emptyset \quad \forall \varepsilon > 0$

y is lim pt of A so $y \in A$ from the assumption. Then by Thm 8, A is closed.

② Prove the following statement:
Let (X, d) and (Y, ρ) be metric spaces and $f: X \to Y$
Then, f is continuous if and only if the preimage of every open set in Y is open in X.

$$f^{-1}(A) \text{ is open in } X \ \forall A \subseteq Y \text{ open}$$

$\Rightarrow$

If A is continuous: and $A \subseteq Y$ is open, then to Prove $f^{-1}(A)$ is open in X:

for $\forall x \in f^{-1}(A)$, $f(x) = y \in A$ since $A \subseteq Y$ is open
$\exists \varepsilon > 0 \ B_\varepsilon(y) \subseteq A$

From definition of continuity: $\exists \delta > 0$ if $d(x, x') < \delta$
then $d(f(x), f(x')) < \varepsilon$. $f(x) \in B_\varepsilon(y)$
Since $B_\varepsilon(y) \subseteq A$, so, $B_\delta(x) \subseteq f^{-1}(A)$. So, $f^{-1}(A)$ is open in X.

$\Leftarrow$ If $f^{-1}(A)$ is open in X $\forall A \subseteq Y$ open Then,
$\forall x \in X \ \forall \varepsilon > 0$ $B_\varepsilon(f(x))$ is open so $f^{-1}(B_\varepsilon(f(x)))$ is open and $x \in f^{-1}(B_\varepsilon f(x)))$. So $\exists \delta > 0 : B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$
So, for $\forall x \in X \ \forall \varepsilon > 0 \ \exists \delta > 0 \ \forall d(x, x') < \delta$:
$d(f(x), f(x')) < \varepsilon$

③ Show that in any normed vector space $(X, \|\cdot\|)$ the norm is a continuous function from X to $\mathbb{R}$

Construct a metric space (X, d) $(\mathbb{R}, \rho)$

$d(x,y) = \|x-y\|$ $\rho(x,y) = |x-y|$

$\forall x \in X$ $\|x\| \in \mathbb{R}$ is well defined and unique, so $\|\cdot\|$ is a function from X to $\mathbb{R}$. From the definition.

$\forall x \in X$ $\forall \varepsilon > 0$ $\exists \delta = \varepsilon$: $\forall d(x, x') < \varepsilon$ $\|x - x'\| < \varepsilon$,

$\rho(\|x\|, \|x'\|) =$

$|\|x\| - \|x'\|| \leq \|x - x'\| < \varepsilon$

So, $\|\cdot\|$ is continuous

(4) Show Lipschitz continuous $\Rightarrow$ uniform continuity $\Rightarrow$ continuity

f is Lipschitz cont. if there exist a constant $K > 0$ such that $\forall x, x' \in X$, $\rho(f(x), f(x')) \leq K \cdot d(x, x')$

$$\forall \varepsilon > 0 \quad \exists \delta = \varepsilon / K \quad \forall x \in X \; d(x, x') < \delta$$
$$\Rightarrow \rho(f(x), f(x')) \leq K d(x, y) \leq \varepsilon$$

f uniformly cont, $\forall x \; \forall \varepsilon > 0 \; \exists \delta > 0$:
$\forall \, d(x, x') < \delta \Rightarrow \rho(f(x), f(x')) < \varepsilon$
$f(x)$ is continuous

(ii) $f : (0,1) \to \mathbb{R}$ $\quad f(x) = 1/x$ $\qquad g : (0,1) \to \mathbb{R}$ $\quad g(x) = \sqrt{x}$

Not Lip
Not Unif
Cont

All but Lip

$f(x) = 1/x$. Prove continuity: $\forall a, b \in \mathbb{R}_+$ open in $\mathbb{R}$:
$f^{-1}(a,b) = (0,1) \cap (1/b, 1/a)$ is open in $(0,1)$
$\therefore$ Cont

Not uniformly cont: $\exists \varepsilon = 1$ $\forall \delta > 0$ $\exists \delta$ x_n, y_n = [illegible] :
$\exists N \in \mathbb{N}$ $\forall n > N$ $d(x_n, y_n) < \delta$ but $\rho(f(x_n), f(y_n)) = 1 \geq 1$

Not unif. cont $\Rightarrow$ not Lip Cont.

(4) $f(x) = \sqrt{x}$

$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in (0,1) : d(x, x') < \delta \Rightarrow \rho(f(x), f(x')) < \varepsilon$

Set $\delta = \varepsilon^2/4$

$$\sqrt{x} - \sqrt{y} < \sqrt{\delta + y} - \sqrt{y} = \frac{\delta}{\sqrt{\delta + y} + \sqrt{y}} < \sqrt{\delta} = \frac{\varepsilon}{2} < \varepsilon$$

Therefore also cont.

Not Lip. Cont $\quad \forall k > 0 \quad \exists x = \frac{1}{4k^2} \quad x' = x/2$

$$\frac{\rho(f(x), f(x'))}{d(x, x')} = \frac{\sqrt{x} - \sqrt{x'}}{x - x'} = \frac{1}{\sqrt{x} + \sqrt{x'}} > \frac{1}{2\sqrt{x}} = \frac{2k}{2} = k$$

(5) If $C \subset X$ is compact and $f : C \to Y$ is cont. then f is uniform cont. Pg 99

Let $\varepsilon > 0$. f is cont. for each point x in C we can find a positive number $\delta(x) : d(x,y) < \delta(x) \Rightarrow \rho(f(x), f(y)) < \varepsilon/2$

For each $x \in C$, let $B(x)$ be set of all points $y \in C$ for which $d(x,y) < \delta(x)/2$. Collection of all such $B(x)$'s is open cover of C. C is compact, there are finite set of points in $C : \{x_1, \dots x_n\}$

$C \subseteq B(x_1) \cup \dots \cup B(x_n)$

Let $\delta = \frac{\min \{\delta(x_1) \dots \delta(x_n)\}}{2}$

$\delta > 0$ then (finite collection of pos #'s)

Let $x, y \in C$. $d(x,y) < \delta$. $\exists x_m$: $x \in B(x_m)$

$\Rightarrow d(x, x_m) < \frac{\delta(x_m)}{2}$

$d(y, x_m) \le d(y, x) + d(x, x_m) < \delta + \frac{\delta(x_m)}{2} \le \delta(x_m)$

x and y are sufficiently close to x_m

Therefore, $\rho[f(y), f(x)] \le \rho[f(y), f(x_m)]$

$+ \rho[f(x_m), f(x)] < \varepsilon$