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Exercise Set 6

① Determine whether each matrix is pos definite / semi or indefinite. Provide vectors for latter two cases.

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 2 & 4 & 0 \end{bmatrix}$ Leading principle minors: (just a_{11} here)

$a_{11} = 1 > 0$

$a_{22} = 3 > 0$

$a_{33} = 0 \Rightarrow$ cannot be purely positive definite

Next leading principle minor:

$\det\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = (1)(3) - 0 = 3 > 0$

$\det\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = (1)(0) - (2)(2) = -4 \Rightarrow$ indefinite

This matrix is indefinite

Let $V_1 = \langle 1, 0, 0 \rangle$ be a vector yielding a positive value for quadratic form

Then $V_1 A V_1 = 1$

Let $V_2 = \langle -1, 0, 1 \rangle$

Then $V_2 A V_2 = -3$ $\square$

b) $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 2 & 3 & 7 \end{bmatrix}$

Leading Principle Minors:

$a_{11} = 1 > 0$

$\det \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} = 3 > 0$

$\det A = 1(21-9) + 0 + 2(0-6)$
$= 12 - 12 = 0$

$\Rightarrow$ cannot be positive definite

$a_{22}, a_{33} > 0$

$\det \begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} = 7 - 4 > 0$ $\qquad$ $\det \begin{vmatrix} 3 & 3 \\ 3 & 7 \end{vmatrix} = 21 - 9 > 0$

All principle minors are non-negative;

~~det A~~

$$xAx = [x_1 \; x_2 \; x_3] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 3 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 & 3x_2 + 3x_3 & 2x_1 + 3x_2 + 7x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 + 2x_1x_2 + 3x_2^2 + 3x_2x_3 + 2x_1x_3 + 3x_2x_3 + 7x_3^2$$

$$x_1^2 + 2x_1x_2 + 3x_2^2 + 6x_2x_3 + 2x_1x_3 + 7x_3^2 = 0$$

Solving this will give a vector which yields a zero value

(c) $A = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{bmatrix}$

$a_{11} = -1 < 0 \Rightarrow$ indefinite or negative (semi)definite

Can't be negative definite since $\det(A) = 0$

The 2^{nd} order principle minor:

$\det \begin{vmatrix} -1 & -2 \\ -2 & -1 \end{vmatrix} = 1 - 4 = -3 < 0 \Rightarrow$ cannot be negative semidefinite

Therefore, A is indefinite

$$[x_1\ x_2\ x_3] \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 - 2x_3 & 0 & -2x_1 - x_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= -x_1^2 - 2x_1x_3 - 2x_1x_3 - x_3^2$$

$$= -x_1^2 - 4x_1x_3 - x_3^2$$

Let $v = (1, 0, 1) \Rightarrow xAx = -6 < 0$

$v = (-2, 0, 2) \Rightarrow xAx = 8 > 0$

(d) $A = \begin{bmatrix} -3 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & -3 \end{bmatrix}$

$a_{11} = -3 < 0$, $a_{22} = -1$, $a_{33} = -3$

$\det \begin{vmatrix} -3 & 0 \\ 0 & -1 \end{vmatrix} = 3 > 0$ $\quad \det \begin{vmatrix} -3 & 2 \\ 2 & -3 \end{vmatrix}$ $9 - 4 = 5 > 0$ $\quad \det \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} = 3 > 0$

$\det \begin{vmatrix} -3 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & -3 \end{vmatrix} = -3[3] + 0 + 2(2) = -9 + 4 = -5 < 0$

This matrix is negative definite

(2) Use differential function to give argument why the gradient of a function $f: \mathbb{R}^2 \to \mathbb{R}$ at $\bar{x} \in \mathbb{R}^2$ is orthoganol to function's level curve
Then recast argument so that it applies to all $f: \mathbb{R}^n \to \mathbb{R}$

The differential function is defined as (for $\mathbb{R}^2$):

$$\nabla f \cdot \Delta x \in \mathbb{R}^2 = df_{\bar{x}}(\Delta x) = \partial f/\partial x_1 \Delta x_1 + \partial f/\partial x_2 \Delta x_2$$

where ∇f is the gradient and Δx is the the movement or change from some point $\bar{x}$.

If the dot product between two vectors is zero then the vectors are said to be orthoganol.

Choose $\Delta x = 0$ such that we stay on the level curve. Then:

$$\nabla f \cdot \Delta x = \partial f/\partial x_1 (0) + \partial f/\partial x_2 (0) = 0$$

Hence, they are orthoganal

② continued

In a similar way, choose any point on the level curve. $(x_1, x_2, \ldots x_n)$

We know $f(x_1, x_2, \ldots x_n) = C$ where C is some constant value.

So, if you take the derivative of that function, the derivative of a constant is zero.

$$\nabla f \cdot \Delta x = 0$$

And, any dot product of two vectors is perpendicular or orthogonal.

So the gradient is orthogonal to the level curve.

3 The function

$$f(x) = \begin{cases} x^2 & x \le 1 \\ 3x-2 & x \ge 1 \end{cases}$$

is continuous everywhere in $\mathbb{R}$. Show that $f(x)$ is not differentiable at $x=1$

$$\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^-} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{(x^2+2xh+h^2)-x^2}{h} = \lim_{h\to 0} \frac{2xh+h^2}{h} = 2x$$

$$\lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{3(x+h)-2-[3x-2]}{h}$$

$$= \lim_{h\to 0^+} \frac{3x+3h-2-3x+2}{h} = \lim_{h\to 0^+} \frac{3h}{h} = 3$$

$$2x \neq 3 \quad \text{for } x=1 \quad \square$$

4) Determine critical points of the function and whether Hessian is pos/neg (semi) definite or indefinite

a) $x^4 + x^2 - 6xy + 3y^2 = f$

$$\partial f / \partial x = 4x^3 + 2x - 6y = 0$$

$$\partial f / \partial y = 0 + 0 - 6x + 6y = 0$$

$$x = y \Rightarrow \partial f / \partial x = 0 = 4x^3 + 2x - 6x$$

$$4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0$$
$$4x(x+1)(x-1) = 0$$
$$x = 0, 1, -1 \Rightarrow y = 0, -1, 1$$

Solution points:

$(0,0)$ $(1,1)$ $(-1,-1)$

$$F_{xx} = \partial / \partial x \,(4x^3 + 2x - 6y) = 12x^2 + 2$$

$$F_{xy} = \partial / \partial y \,(4x^3 + 2x - 6y) = -6$$

$$F_{yx} = \partial / \partial x \,(-6x + 6y) = -6$$

$$F_{yy} = \partial / \partial y \,(-6x + 6y) = 6$$

$$H = \begin{bmatrix} 12x^2+2 & -6 \\ -6 & 6 \end{bmatrix}$$

4) cont.

$H = \begin{bmatrix} 12x^2+2 & -6 \\ -6 & 6 \end{bmatrix}$ $(0,0) \Rightarrow \begin{bmatrix} 2 & -6 \\ -6 & 6 \end{bmatrix}$ $a_{11} = 2 > 0$

$|H| = 12 - 36 = -24 < 0$

$\Rightarrow$ indefinite.

This is a saddle point at $(0,0)$

$(1,1) \Rightarrow \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$ $a_{11} = 14$

$\det(H) = 84 - 36 = 48 > 0$

$\Rightarrow$ positive definite

$(1,1)$ is a local minimom

$(-1,-1) \Rightarrow H = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$ $a_{11} = 14$,

$|H| = 84 - 36 = 48 > 0$

positive Definite

$(-1,-1)$ is a local minumum

5) $F(x,y) = x^2 - 6xy + 2y^2 + 10x + 2y - 5$

$\partial F/\partial x = 2x - 6y + 10$ $\quad$ $\partial F/\partial y = -6x + 4y + 2$

$2x - 6y + 10 = 0$
$2x = 6y - 10$
$x = 3y - 5$

$-6x + 4y + 2 = 0$
$-6(3y - 5) + 4y + 2 = 0$
$-18y + 30 + 4y + 2 = 0$
$-14y + 32 = 0$
$y = 16/7$

$x = 3(16/7) - 5 = 48/7 - 5 = 48/7 - 35/7$
$= 13/7$

critical pt at $(13/7, 16/7)$

(4) (b) cont.

$F_{xx} = \partial/\partial x\,(2x - 6y + 10) = 2$

$F_{xy} = \partial/\partial y\,(2x - 6y + 10) = -6$

$F_{yx} = \partial/\partial x\,(-6x + 4y + 2) = -6$

$F_{yy} = \partial/\partial y\,(-6x + 4y + 2) = 4$

$H = \begin{bmatrix} 2 & -6 \\ -6 & 4 \end{bmatrix}$ $\quad a_{11} = 2 > 0$

$|H| = 8 - 36 < 0 \Rightarrow$ indefinite

$(13/7, 10/7)$ is a saddle point

(4) (c) $F(x,y,z) = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

$\partial F/\partial x = 2x + 6y - 10$ $\qquad \partial F/\partial z = -3y + 8z - 21$

$\partial F/\partial y = 6x + 2y - 3z - 5$

$2x + 6y - 10 = 0$
$2x = 10 - 6y$
$x = 5 - 3y$

$\Rightarrow \partial F/\partial y = 6(5 - 3y) + 2y - 3z - 5$
$= 30 - 18y + 2y - 3z - 5$
$= 25 - 16y - 3z$

$-3z - 16y + 25 = 0$
$-3z = 16y - 25$
$z = -16/3\,y + 25/3$

$\partial F/\partial z = -3y + 8(-16/3\,y + 25/3) - 21$
$\Rightarrow y = 1$

4 (c) continued

$y = 1 \Rightarrow z = -16/3(1) + 25/3 = 9/3 = 3$

$y = 1 \Rightarrow x = 5 - 3(1) = 2$

Critical point at $(2, 1, 3)$

$F_{xx} = 2 \quad F_{xy} = 6 \quad F_{xz} = 0$

$F_{yx} = 6 \quad F_{yy} = 2 \quad F_{yz} = -3$

$F_{zx} = 0 \quad F_{zy} = -3 \quad F_{zz} = 8$

$H = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{bmatrix}$ $\quad a_{11} = 2$

$\det \begin{vmatrix} 2 & 6 \\ 6 & 2 \end{vmatrix} = 4 - 36 = -32 < 0$

$\det(H) = 2[16 - 9] - 6[(6)(8) - 0] + 0$

$= (2)(7) - 6(48)$

$= 14 - 288 = -274 < 0$

$\Rightarrow H$ is indefinite at $(2, 1, 3)$

$(2, 1, 3)$ is a saddle point

5) A C^2 function $f: X \to \mathbb{R}$ defined on an open convex set $X \subseteq \mathbb{R}^n$ is concave iif its 2nd degree Taylor polynomial is concave at every $x \in X$, which is the case iif the Hessian matrix of second partial derivatives $D^2 f(x)$ or $H(x)$ is negative semidefinite at every $x \in X$. Sufficient condition for strict concave is if Hessian matrix is negative definite at every $x \in X$

(a) provide a counterexample to show a negative definite Hessian matrix is not necessary for a C^2 function to be strictly concave

$$f(x, y) = -x^4 - y^4$$

$$\partial f / \partial x = -4x^3 \qquad \partial f / \partial y = -4y^3$$

(0,0) critical point

$F_{xx} = -12x^2 \qquad F_{yx} = 0$

$F_{xy} = 0 \qquad F_{yy} = -12y^2$

$$H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{negative semidefinite (and positive too)}$$

Concave condition:

$$t F(x_1, y_1) + (1-t) F(x_2, y_2) \le F(t(x_1, y_1) + (1-t)(x_2, y_2))$$

So, H is not negative definite but function $f(x, y)$ is still concave. (strictly)

5) b For parameters a_1, a_2, a_3

define function $\mathbb{R}^2_{++} \to \mathbb{R}$ by

$f(x_1, x_2) = a_1 \log x_1 + a_2 e^{x_2} + a_3 x_1 x_2$

Determine values that make it concave/convex/neither

A C^2 function is concave only if $f''(x) \leq 0 \quad \forall x \in X$

A C^2 function f on an open convex subset U of $\mathbb{R}^n$ is concave if and only if H is negative semidefinite for all x in U, and convex if positive semidefinite

$\partial f / \partial x_1 = \frac{a_1}{x_1} + a_3 x_2$ $\quad$ $\partial f / \partial x_2 = a_2 e^{x_2} + a_3 x_1$

$F_{x_1 x_1} = \frac{-a_1}{x_1^2}$ $\quad$ $F_{x_1 x_2} = a_3$

$F_{x_2 x_1} = a_3$ $\quad$ $F_{x_2 x_2} = a_2 e^{x_2}$

$$H = \begin{vmatrix} -a_1/x_1^2 & a_3 \\ a_3 & a_2 e^{x_2} \end{vmatrix}$$

$$\det(H) = \left(-a_1/x_1^2\right)\left(a_2 e^{x_2}\right) - a_3^2$$

Positive semidefinite:

Let $a_1 \leq 0$ and $a_2 \geq 0$

Let $a_3^2 \leq \left(-a_1/x_1^2\right)\left(a_2 e^{x_2}\right)$

Negative Semidefinite

Let $a_1 > 0$

Let $a_2 < 0$

and $a_3^2 \leq \left(-a_1/x_1^2\right)\left(a_2 e^{x_2}\right)$

Neither:

let $a_1 < 0$ $a_2 > 0$

and $a_3^2 > \left(-a_1/x_1^2\right)\left(a_2 e^{x_2}\right)$ $\square$