Econ 519 Homework 12

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1. Prove the following result:
Let $X \subseteq \mathbb{R}^n$. The function $f : X \to \mathbb{R}$ is concave if and only if for all $x, x' \in X$ the function $\phi : [0,1] \to \mathbb{R}$ defined by $\phi(t) = f[x + t(x' - x)]$ is concave.

Solution:

- $\Rightarrow$ If f is concave, then for $\forall x, x' \in X$ and $\forall t, t' \in [0,1], \lambda \in [0,1]$:
  $\lambda\phi(t) + (1-\lambda)\phi(t') = \lambda f[x + t(x' - x)] + (1-\lambda) f[x + t'(x' - x)] \leq$
  $f[\lambda(x + t(x' - x)) + (1-\lambda)(x + t'(x' - x))] =$
  $f[x + (\lambda t + (1-\lambda)t')(x' - x)] = \phi[\lambda t + (1-\lambda)t']$
- $\Leftarrow$ If $\phi(t)$ is concave, then $\forall x, x' \in X; \lambda \in [0,1]$:
  $\lambda f(x) + (1-\lambda) f(x') = \lambda\phi(0) + (1-\lambda)\phi(1) \leq \phi(1-\lambda) =$
  $f[x + (1-\lambda)(x' - x)] = f[\lambda x + (1-\lambda)x']$

2. **Definition:** Let $X \subseteq \mathbb{R}^n$ be open and let $f : X \to \mathbb{R}$ be continuously differentiable. f is pseudo-concave on X if for all $x, x' \in X$

$$f(x) > f(x') \text{ implies } Df(x')(x - x') > 0.$$

Prove the following result:
Let $X \subset \mathbb{R}^n$ be open, let $f : X \to \mathbb{R}$ be continuously differentiable and pseudo-concave, and let $C \subseteq X$ be convex. If $x^* \in C$ and for all $x \in C$, $Df(x^*) \cdot (x - x^*) \leq 0$, then x^* solves the problem

$$\max_{x \in C} f(x).$$

Solution: By contradiction: if $\exists \hat{x} \in C$ s.t. $f(\hat{x}) > f(x^*)$, then this implies $Df(x^*)(\hat{x} - x^*) > 0$, this contradicts to the fact that: for all $x \in C$, $Df(x^*) \cdot (x - x^*) \leq 0$. So x^* solves the problem: $\max_{x \in C} f(x)$.

3. Show that $\Gamma : \mathbb{R}_+ \rightrightarrows \mathbb{R}$ defined by

$$\Gamma(x) = \begin{cases} \left[0, \frac{1}{x}\right] & \text{if } x > 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

is uhc and lhc at every $x > 0$, and that it is lhc but not uhc at $x = 0$.
Solution: For $\forall x_0 > 0$: $\Gamma(x_0) = [0, \frac{1}{x_0}]$.
For uhc: $\forall E \in \mathbb{R}$ and open, and $\Gamma(x_0) = [0, \frac{1}{x_0}] \subseteq E$, $\exists \varepsilon > 0 : Ball_\varepsilon(\frac{1}{x_0}) \subseteq E$. Since $\frac{1}{x}$ is continuous in $\mathbb{R}_{++}$, then the pre image of $Ball_\varepsilon(\frac{1}{x_0})$, denote as F is also open, and $x_0 \in F$, and $\forall x \in F : \frac{1}{x} \in Ball_\varepsilon(\frac{1}{x_0}) \subseteq E$, and $\left\{[0, \frac{1}{x_0}] \cup Ball_\varepsilon(\frac{1}{x_0})\right\} \subseteq E$ i.e. $[0, \frac{1}{x}] \subseteq E$. So $\forall x \in F : \Gamma(x) \subseteq E$. $\Gamma(x)$ is uhc for $x > 0$.

For lhc: $\forall E \in \mathbb{R}$ and open, and $\Gamma(x_0) \cap E \neq \emptyset$, for $\forall y \in \Gamma(x_0) \cap E$: $y \leq \frac{1}{x_0}$. So $\exists \delta > 0$: $y - \delta \in E$, $\frac{1}{x_0} \in (y - \delta, \infty)$, from the continuity of $\frac{1}{x}$ on $\mathbb{R}_{++}$, the pre image of $(y - \delta, \infty)$ is also open, denote as F, and $x_0 \in F$, and $\forall x \in F : \frac{1}{x} > y - \delta$. So $[0, \frac{1}{x}] \cap E \neq \emptyset \, (i.e. : y - \delta \in \left\{[0, \frac{1}{x}] \cap E\right\}), \forall x \in F$. So $\Gamma(x)$ is lhc when $x > 0$.

To prove that at $x = 0$, $\Gamma(x)$ is lhc but not uhc:

At $x = 0 : \Gamma(x) = 0$, $\forall E$ open and $0 \cap E \neq \emptyset$: $\exists F = [0, 1)$ to be open and $0 \in F$, and $\forall x \in F$: $0 \in \Gamma(x)$, so $\Gamma(x) \cap E \neq \emptyset$. So $\Gamma(x)$ is lhc at $x = 0$.

For uhc: $\exists E$ open and $0 \in E$: denote $E = (a, b)$, then $\forall F$ open in $\mathbb{R}_+$ and contains 0: $\exists x \to 0$, i.e. $0 < x < \frac{1}{b}$ then $\frac{1}{x} > b$; $\Gamma(x) = [0, \frac{1}{x}] \nsubseteq E$, so $\Gamma(x)$ is not uhc at $x = 0$.

4. Recall Homework 11, No. 3 and consider the problem

$$\max_{(x,y)\in\mathbb{R}^2} f(x,y) = -\alpha x^2 - \beta y^2$$

subject to $g^1(x,y) \geq 0$ and $g^2(x,y) \geq 0$

where $g^1(x,y) = x + 2y - 2$ and $g^2(x,y) = 2x + y - 2$.

(i) Suppose: $\beta \geq 2\alpha$. Show: In the optimum, the first constraint is binding. Calculate the optimum as well as the supporting Lagrange multipliers.

(ii) Suppose: $\alpha \geq 2\beta$. Show: In the optimum, the second constraint is binding. Calculate the optimum as well as the supporting Lagrange multipliers explicitly.

(iii) What is the solution of the problem and what are the supporting Lagrange multipliers for the remaining parameter configurations?

Solution: Form the Lagrange Objective Function, for $\lambda_1, \lambda_2 \geq 0$:

$$\underset{(x,y)\in\mathbb{R}^2}{L} = -\alpha x^2 - \beta y^2 + \lambda_1(x + 2y - 2) + \lambda_2(2x + y - 2)$$

The F.O.C (*Page 292 on the text book*):

$$\begin{cases} \dfrac{\partial L}{\partial x} = 0 \\ \dfrac{\partial L}{\partial y} = 0 \\ \dfrac{\partial L}{\partial \lambda_1} \geq 0 \text{ and } x + 2y - 2 = 0 \text{ if } \lambda_1 > 0 \\ \dfrac{\partial L}{\partial \lambda_2} \geq 0 \text{ and } 2x + y - 2 = 0 \text{ if } \lambda_2 > 0 \\ \lambda_1 \geq 0 \text{ and } \lambda_1 = 0 \text{ if } x + 2y - 2 > 0 \\ \lambda_2 \geq 0 \text{ and } \lambda_2 = 0 \text{ if } 2x + y - 2 > 0 \end{cases}$$

*Comments:*1. In some questions there are implicitly constraints regarding to $x, y \geq 0$, then remember to add these constraints. 2. Some times need to check the 2nd order conditions, to guarantee the sufficiency and necessity. See more details on Prof. Walker's notes (*Nonlinear Programming and the Kuhn-Tucker Conditions*).

Simplify it:

$$\begin{cases} -2\alpha x + \lambda_1 + 2\lambda_2 = 0 \\ -2\beta y + 2\lambda_1 + \lambda_2 = 0 \\ \lambda_1(x + 2y - 2) = 0 \\ \lambda_2(2x + y - 2) = 0 \\ x + 2y - 2 \geq 0 \\ 2x + y - 2 \geq 0 \\ \lambda_1 \geq 0, \lambda_2 \geq 0 \end{cases}$$

If $\lambda_1 > 0, \lambda_2 = 0$, the first constraint is binding:

$$\begin{cases} 2\alpha x = \lambda_1 \\ \beta y = \lambda_1 \\ x + 2y - 2 = 0 \\ 2x + y \geq 2 \end{cases} \quad \Rightarrow \begin{cases} x = \frac{2\beta}{4\alpha+\beta} \\ y = \frac{4\alpha}{4\alpha+\beta} \\ \lambda_1 = \frac{4\alpha\beta}{4\alpha+\beta} \\ \beta \geq 2\alpha \\ f_{max} = -\frac{4\alpha\beta}{4\alpha+\beta} \end{cases}$$

If $\lambda_1 = 0, \lambda_2 > 0$, the second constraint is binding:

$$\begin{cases} \alpha x = \lambda_2 \\ 2\beta y = \lambda_2 \\ x + 2y - 2 \geq 0 \\ 2x + y = 2 \end{cases} \Rightarrow \begin{cases} x = \frac{4\beta}{4\beta+\alpha} \\ y = \frac{2\alpha}{4\beta+\alpha} \\ \lambda_2 = \frac{4\alpha\beta}{4\beta+\alpha} \\ \alpha \geq 2\beta \\ f_{max} = -\frac{4\alpha\beta}{4\beta+\alpha} \end{cases}$$

If $\lambda_1 = 0, \lambda_2 = 0$:

$$\begin{cases} x = y = 0 \\ -2 \geq 0 \end{cases} \Rightarrow Contradiction.$$

If $\lambda_1 > 0, \lambda_2 > 0$:

$$\begin{cases} x = y = \dfrac{2}{3}, f_{max} = -\frac{4(\alpha+\beta)}{9} \\ \lambda_1 = \frac{8\alpha-4\beta}{9} > 0 \Rightarrow 2\alpha > \beta \\ \lambda_2 = \frac{8\beta-4\alpha}{9} > 0 \Rightarrow 2\beta > \alpha \end{cases}$$

5. Let $[a, b] \subset \mathbb{R}$ and let $f : [a, b] \to [a, b]$ be continuous. Show that there exists $c \in [a, b]$ such that $c = f(c)$. (Such a point is called a fixed point of f.)
Hint: Do you remember the intermediate value theorem?

Solution: Construct the function: $\phi(x) = f(x) - x$, it is continuous. If $\phi(a) = 0$ or $\phi(b) = 0$ then we are done. If not, then $\phi(a) > 0$, and $\phi(b) < 0$, from the intermediate value theorem: for $0 \in (\phi(b), \phi(a))$: $\exists c \in (a, b)$ such that $\phi(c) = 0$, i.e. $f(c) = c$.


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謝謝
ngiyabonga
teşekkür ederim
спасибо
dank je
thank you
gracias
tapadh leat
bedankt
hvala
mauruuru
dziękuję
mochchakkeram
go raibh maith agat
sagolun
sukriya
kop khun krap
obrigado
grazie
arigatō
takk
dakujem
мерси
terima kasih
감사합니다
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