Exercise Set 4

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(1) A set is convex if and only if every convex combination of vectors in S is also in S.

(i) If every combination of vectors in S is also in S, then a set is convex.

This proof is evident from the definition of a convex set, that is:

A set S is convex in vector space V if for any two points x, y in S and λ in the unit interval $(0, 1)$ the point $(1-\lambda)x + \lambda y$ is in S.

Clearly, if every combination of vectors is convex in S, then there will be two that fulfill that definition of convexity.

(ii) A set is convex if every convex combination of vectros in S is also in S.

Let V be a convex vector space such that:

$$(1-\lambda)y + \lambda x = v \subseteq V$$

For $\lambda > 0$. Now, let $(1-\lambda) = \lambda_2$ and denote λ as λ_1. Then,

$$\lambda_2 y + \lambda_1 x = v \subseteq V$$

Then, for all $\{\lambda_n\}$, $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$

Now, for the first step in the induction proof, let a and b be in the convex set V such that

$$\lambda_1 a + \lambda_2 b = v \subseteq V$$

Let $\lambda_2 = \lambda_2^* + \lambda_3 = (1-\lambda_1)$ such that $(\lambda_1 + \lambda_2^* + lambda_3 = 1)$. Multiply b by a factor of 1, but expressed by: $\frac{\lambda_2^*+\lambda_3}{\lambda_2^*+\lambda_3}$

Such that:

$$\lambda_1 a + (\lambda_2^* + \lambda_3)(\frac{\lambda_2^* + \lambda_3}{\lambda_2^* + \lambda_3})b$$

Now, decompose b into two separate vectors following the scalar fraction:

$$\lambda_1 a + (\lambda_2^* + \lambda_3)(\frac{\lambda_2^* b^*}{\lambda_2^* + \lambda_3} + \frac{\lambda_3 c}{\lambda_2^* + \lambda_3})$$

Then, with the help of algebra, it can be shown that any convex combination of 2 vectors in a convext set imply that there are three combinations of complex vectors in that same convex set such that:

$$\lambda_1 a + \lambda_2^* b^* + \lambda_3 c = v \subseteq V$$

Where each λ is nonnegative and the sum of all λ is 1.

Step 2 now requires showing that this is the case for $(n+1)$

Similar to above, let $v_1, ..., v_n$ be a series of vectors in a convex set. Let $\lambda_1, ...\lambda_n$ be scalars of the vectors which are nonnegative and for which the sum of all scalars is equal to 1. Then, we have:

$$\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n v_n = v \subseteq V$$

In a similar manner of the first part of the proof, let $\lambda_n = (\lambda_n^* + \lambda_{n+1})$. Now, let:

$$\lambda_1 v_1 + \lambda_2 v_2 + ... + (\lambda_n^* + \lambda_{n+1})(\frac{\lambda_n^* + \lambda_{n+1}}{\lambda_n^* + \lambda_{n+1}})v_n = v$$

$$\lambda_1 v_1 + \lambda_2 v_2 + ... + ((\lambda_n^* + \lambda_{n+1})(\frac{\lambda_n^* v_n^*}{\lambda_n^* + \lambda_{n+1}} + \frac{\lambda_{n+1} v_{n+1}}{\lambda_n^* + \lambda_{n+1}}) = v$$

For which we now have:

$$\lambda_1 v_1 + \lambda_2 v_2 + ... + \lambda_n^* v_n^* + \lambda_{n+1} v_{n+1} = v$$

And thus, any convex combination of n vectors will guarantee that there exists a complex combination of $n + 1$ vectors.

By the principle of induction, it is true that if V is a convex set, then every convex combination of vectors in S is also in S.

(2) Write out a careful proof of Theorem 21.12. The proof is straightforward but should help one's understaind of quasiconcave functions and reinforce one's ability to prove theorems.

Theorem 21.12:

Let f be a function defined on a convex set U in $\mathbb{R}^n$. Then, the following statements are equivalent to each other:

(a) $f is a quasiconcave function on U.
(b) For all $x, y \in U$ and all $t \in [0, 1]$, $f(x) \geq f(y) \implies f(tx + (1 - t)y \geq f(y)$
(c) For all $x, y \in U$ and all $t \in [0, 1]$, $f(tx + (1 - t)y \geq min\{f(x), f(y)\}$