Exercise Set 1

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1. For each of the following statements, determine whether the statement is true or false, and if false indicate why it's false.

(a) $\{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^3 \times \mathbb{R}^2$

This is false.

Note:

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(a,b,c) : a,b,c \in \mathbb{R}\}$$
$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) : x,y \in \mathbb{R}\}$$

hence:

$$\mathbb{R}^3 \times \mathbb{R}^2 = (\mathbb{R} \times \mathbb{R} \times \mathbb{R}) \times (\mathbb{R} \times \mathbb{R}) = \{((a,b,c),(x,y)) : a,b,c,x,y \in \mathbb{R}\}$$

Consequently, any subset will be of the form $\{((a,b,c),(x,y))\}$ where $a,b,c,x,y \in \mathbb{R}$

(b) $\{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^3 \cup \mathbb{R}^2$

This is true.

(c) $\{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$

This is false. There are no individual three tuples in the subset above that are contained in the set $\mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$ as elements would take the form of ordered lists such as: $((a,b,c),(d,e),(x,y))$ where $a,b,c,d,e,x,y \in \mathbb{R}$

(d) $((2,1,1),(2,1),(1,1)) \in \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$

This is true.

(e) $\{((2,1,1),(2,1),(1,1))\} \subseteq \mathbb{R}^3 \times \mathbb{R}^2 \times \mathbb{R}^2$

This is true.

(f) $\exists n \in \mathbb{N} : \{(2,1,1),(2,1),(1,1)\} \subseteq \mathbb{R}^n$

This is false. By definition, $\mathbb{R}^n = \{x : x = (x_1, x_2, ..., x_n), x_i \in \mathbb{R}\}$

hence, for any given $n \in \mathbb{N}$, there will be only lists with n entries that are contained in $\mathbb{R}^n$, and not $n+m$ entries where $m \in \mathbb{N}$

In other words, any lists put in as elements of a subset of $\mathbb{R}^n$ must have n entries only.

(g) $(1,2,3) \subseteq \mathbb{N}$

This is false because $(1,2,3)$ is not a set.

(h) $(1,2,3) \in \mathbb{N}$

This is false. $(1,2,3) \in \mathbb{N}^3$. It cannot be an element of $\mathbb{N}$ since $\mathbb{N}$ is one dimension.

(i) $(1,2,3) \in \mathbb{R}^3$

This is true.

(j) $\{(2,1,1),(2,1),(1,1)\} = \{(2,1),(2,1,1),(1,1)\}$

This is true. Order does not matter for sets.

(k) $\mathbb{R}^2 \cup \mathbb{R}^3 = \mathbb{R}^3 \cup \mathbb{R}^2$

This is true. For any set X and Y, it is always true that $X \cup Y = Y \cup X$

(l) $\mathbb{R}^2 \times \mathbb{R}^3 = \mathbb{R}^3 \times \mathbb{R}^2$

This is false.

By definition:

$$\mathbb{R}^2 \times \mathbb{R}^3 = \{((a,b),(x,y,z)) : a,b,x,y,z \in \mathbb{R}\}$$

whereas:

$$\mathbb{R}^3 \times \mathbb{R}^2 = \{((x,y,z),(a,b)) : a,b,x,y,z \in \mathbb{R}\}$$

Order matters for the tuples that make up the elements of the cartesian product of two sets, such as $\mathbb{R}^2$ and $\mathbb{R}^3$. Because order matters for n-tuples or *ordered* lists, the cartesian product is not commutative (unless the two sets are equivalent or they are both empty sets).

2. Using only negation, conjuction, and disjuction construct a compound statement for the remaining five truth-value profiles.

Let $\neg$ denote $\sim$.

The five remaining truth profiles (by column) are:

PQ	i.	ii.	iii.	iv.	v.
TT	T	T	F	F	F
TF	T	T	T	F	F
FT	T	F	F	T	F
FF	T	T	F	F	F

(i.) $(P \vee \neg P) \vee (Q \vee \neg Q)$

(ii.) $P \vee \neg Q$

(iii.) $P \wedge \neg Q$

(iv.) $\neg P \wedge Q$

(v.) $(P \wedge Q) \wedge (\neg P \wedge \neg Q)$ or alternatively $(P \wedge \neg P) \wedge (Q \wedge \neg Q)$

3. Let *A* be an *m x n* matrix and let *f* be the function *f(x) = Ax*

(a) What is the domain of *f* and what is the target space of *f*?

If A is an $m \times n$ matrix, then $f : \mathbb{R}^n \to \mathbb{R}^m$.

Hence, the domain is $\mathbb{R}^n$ and the target space is $\mathbb{R}^m$

(b) What conditions on *m, n,* and *A* are both necessary and sufficient for *f* to be one-to-one and onto its target space?

By definition, a function is one-to-one if for every $y \in f(x)$, there exists a unique x in the domain X of f such that $f(x) = y$.

In this case the function $f(x) = Ax$ is one-to-one if there is only one value of x that maps to y for a given y in the target space. In other words, two different values of $x \in X$ cannot map to the same $y \in Y$. It admits of only one solution.

Additionally, a function, by definition, is onto if for every $y \in Y$, or every y in the target, there exists an x in the domain such that $f(x) = y$.

In this case, the function $f(x) = Ax$ is onto if the range of the function is its entire target space. In other words, every $y \in Y$ is covered by some $x \in X$ such that $f(x) = y$ This would admit to at least one solution in x

Consequently, a necessary condition is that there is at least one solution in x. A sufficient condition is that there is only one solution, and this would mean that f is both one-to-one and onto. Also, it must be the case that $m = n$ If a linear tranformation has only one solution, then A is invertible (nonsingular), which requires that it is a square matrix, hence $m = n$. Invertibility also guarantess that $m = n =$ number of pivots.

(c) If *m = 2* and *n = 1* under what conditions (if any) will *f* be one-to-one? Onto?

Simon and Blume note the following facts of matrices A with $m > n$ or more columns than rows:

(1) $Ax = 0$ has one or infinitely many solutions

(2) For any given b, $Ax = b$ has 0, 1, or infinitely many solutions.
(3) If rank A = number of unknowns, $Ax = b$ has 0 or 1 solution for every b

So, it is one-to-one so as long as there is only one solution. This would also mean that there is one pivot.

The function is onto only if the function has at least one solution. However, to be onto will require a pivot in every row. For any matrix of the size $m = 2$ and $n = 1$, I can do elementary row operations which make the bottom element ($a_{2,1}$) 0. So, the rank cannot equal the number of unknowns which excludes this case from being onto. This is to say, there is not a solution for every b or range in the target space $\mathbb{R}^m$

(d) If $m = 2$ and $n = 3$, under what conditions (if any) will f be one-to-one? Onto?

If $m = 2$ and $n = 3$, then there are more unknowns (columns) than equations (rows).

Then, three points follows (Simon and Blume):

(1) $Ax = 0$ has infinitely many solutions.
(2) for any given b, $Ax = b$ has 0 or infinitely many solutions
(3) if rank A = the number of equations (m), $Ax = b$ has infinitely many solutions for every b ($f(x)$)

Hence, it cannot be one-to-one in this case since there is no unique solution for a linear transformation of this form. Every case has infinitely many solutions, or none at all. Furthermore, not every column can have a pivot. This would allow for free variables, which can take give many solutions. If the solutions are not unique, then the equation is not one-to-one.

It can be onto if there is at least one solution (or infinitely many solutions). Additionally, there must be a pivot in every row.

4. You can fool some people all of the time, and you can fool all the people some of the time, but you can't fool all the people all the time.

Let $X = \{a, b, c\}$ and $Y = \{m, w, f\}$. Let the open statement "x can be fooled at time y" be denoted as $\phi(x, y)$. Let the set of pairs (x, y) for which $\phi(x, y) = T$ be represented by the set F.

$$SA1 : \exists x \in X : \forall y \in Y : \phi(x, y) = T$$

In plain English, there exists one person who is fooled everyday.

$$SA2 : \forall y \in Y : \exists x \in X : \phi(x, y) = T$$

Or, for all days, there exists a person who is fooled (not necessarily a unique one person)

$$AS1 : \exists y \in Y : \forall x \in X : \phi(x, y) = T$$

Rather, there exists one day where all people are fooled.

$$AS2 : \forall x \in X : \exists y \in Y : \phi(x, y) = T$$

In other words, for all people, there is one day (not necessarily one unique day) where they are fooled. Everyone is fooled on some day.

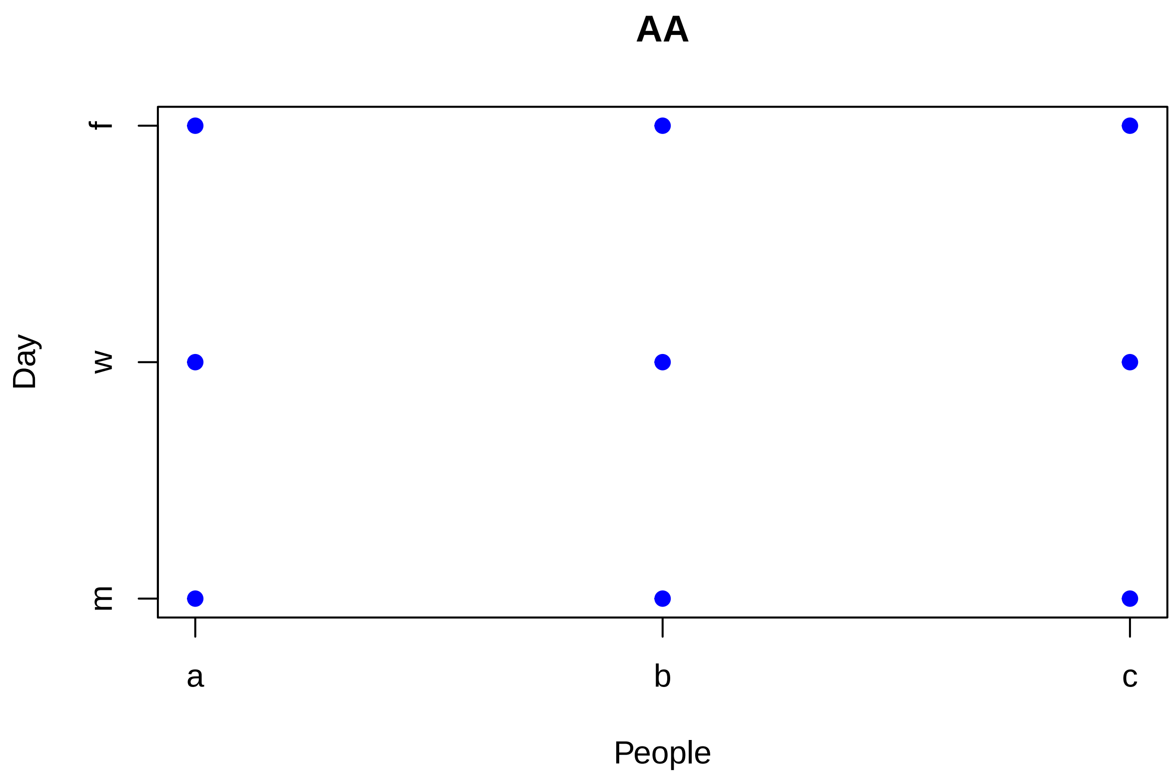
$$AA : \forall x \in X : \forall y \in Y : \phi(x, y) = T$$
$$\sim AA = \sim (\forall x \in X : \forall y \in Y : \phi(x, y)) = \exists x \in X : \exists y \in Y : \phi(x, y) = F$$

Therefore, $\sim AA$ indicates not all of the people are fooled everyday. There exists one person who is not fooled on some day.

(a)

Let the blue dot signify the set of pairs for which x can be fooled at time y.

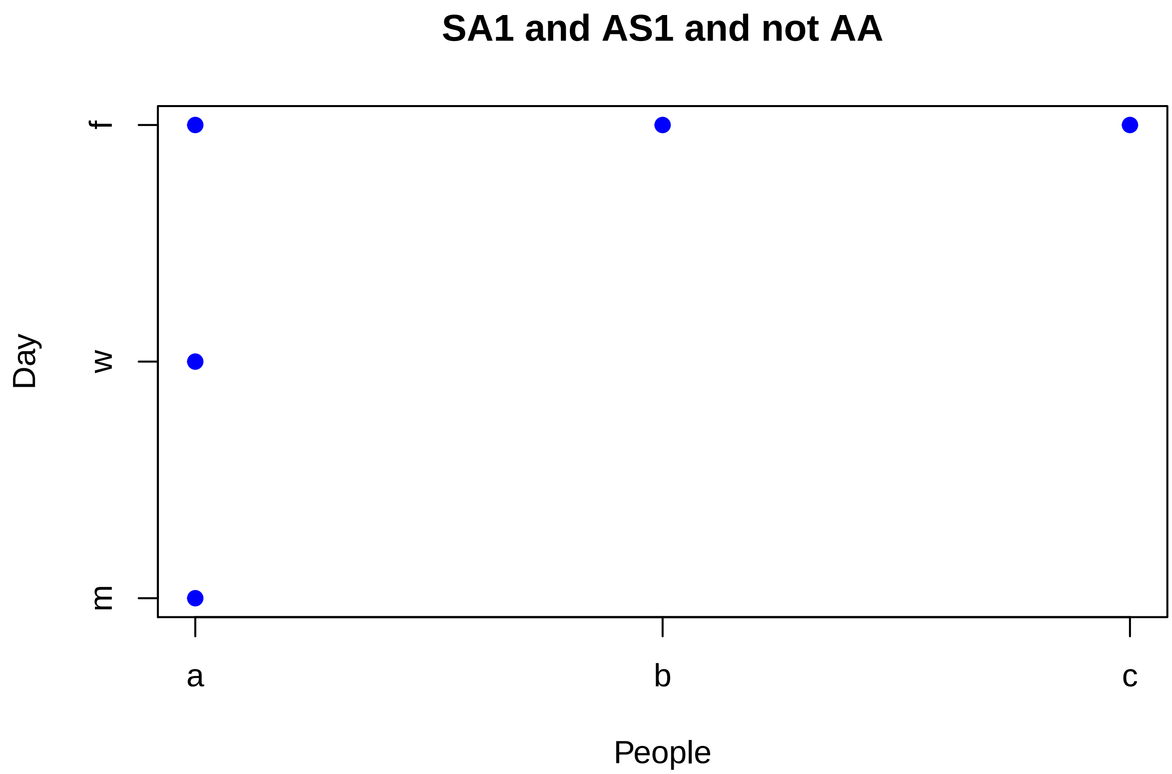
$$AA$$



This is to say, all of the people can be fooled all of the time.

(b)

$$SA_1 \wedge AS_1 \wedge \sim AA$$



One person is fooled everyday (SA1). There is one time for which everyone is fooled (AS1). There is one time where one person is not fooled.

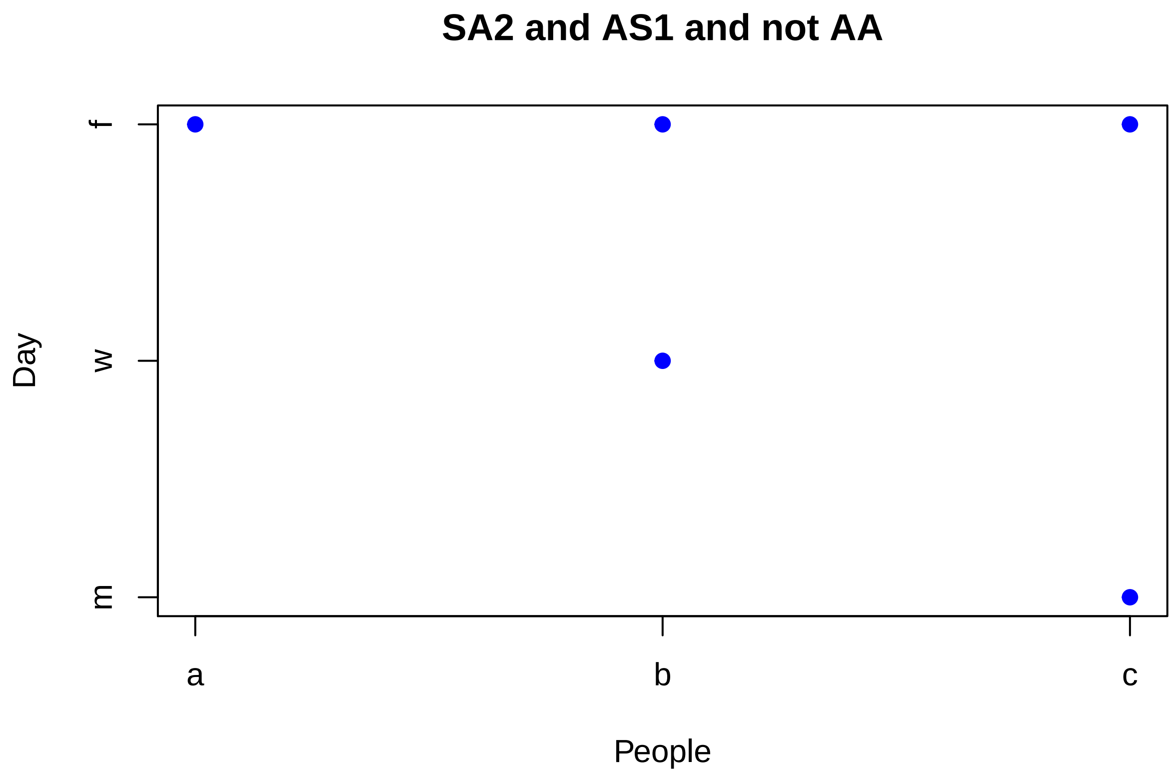
(c)

$$SA_1 \wedge AS_2 \wedge \sim AA$$

There exists one person who is fooled every day. All people are fooled on some day. There is a time when a person is not fooled.

(d)

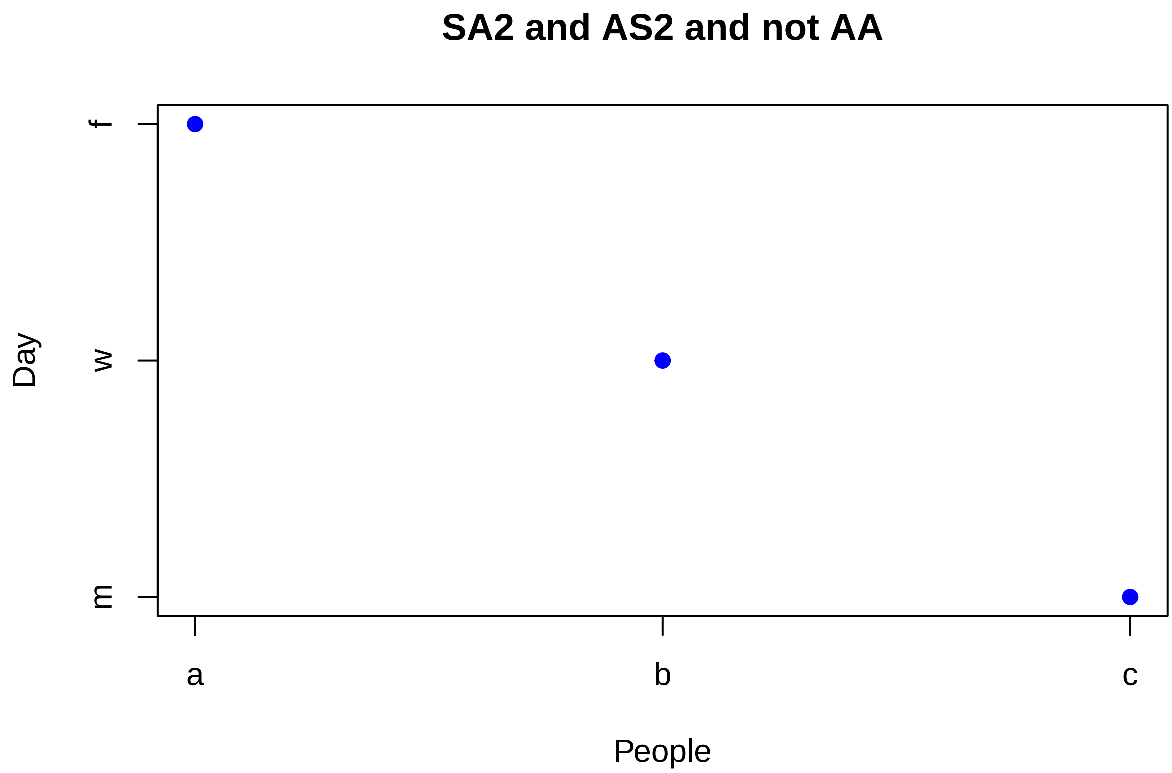
$$SA_2 \wedge AS_1 \wedge \sim AA$$



Everday, there is a person who is fooled. There is one day for which all people are fooled. Not everyone is fooled every day.

(e)

$$SA_2 \wedge AS_2 \wedge \sim AA$$



Everday one person is fooled. Every person is fooled on some day. One day, one person is not fooled.

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5. Now let X and Y be arbitrary sets. Prove that SA1 implies SA2

SA1 implies that every $y \in Y$ has the same $x \in X$ whose pair is included in F. If this is true, it follows that there exists an x for every y. It is not true that because there exists and x for every y implies that the same x guarantees that (x, y) is in the set F for every y.