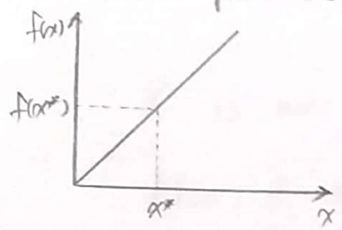
* Fixed Point Theorem

(Def) Given a set X and a function $f: X \to X$,
$x^* \in X$ is a fixed point of f iff $f(x^*) = x^*$

(Note): Existence problems in Economics can often be formulated as a fixed point problem.



We will talk about 3 types of "Fixed point theorem"

- Restrictions on X, every continuous function f on X has a fixed point on X

  Brouwer : $X \subseteq \mathbb{R}^n$ compact and convex
  Kakutani : for correspondence

- Restrictions on f (Contractions)

  → Tarski : f increasing, not continuous. X generalized rectangle.

Brouwer

(Thm) If $X = [a, b] \subseteq \mathbb{R}$ and $f: X \to X$ is continuous,
then f has a fixed point.

proof) $f(a) = a$ or $f(b) = b$, we are done.
Otherwise, $f(a) > a$ and $f(b) < b$
Define $g(x) = f(x) - x$. Then $g(a) > 0$ and $g(b) < 0$
Moreover, g is continuous since f is continuous.
The intermediate value theorem implies that
there is a $x^* \in (a, b)$ such that $g(x^*) = 0$. Hence $f(x^*) = x^*$. //

The assumptions are sufficient but not necessary.

ex) : $X = (0, \frac{1}{3}) \cup (\frac{2}{3}, \infty)$

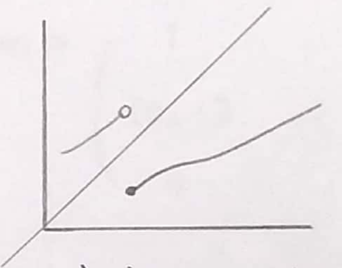
$$f(x) = \begin{cases} \frac{1}{4}, & \text{if } x = \frac{1}{4} \\ 0, & \text{otherwise} \end{cases}$$

X is not closed, not bounded, not connected

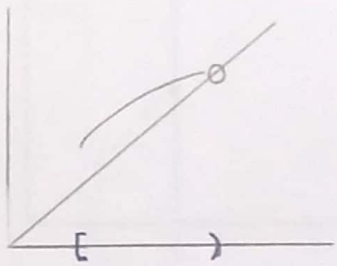
Also, f not continuous

$\Rightarrow$ But, $x^* = \frac{1}{4}$ is a fixed point of f.

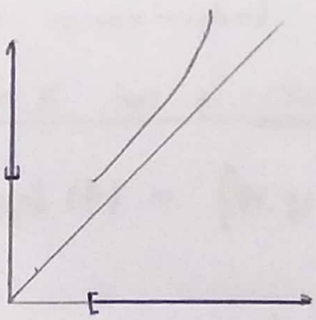
ex 2) For failure of the assumptions



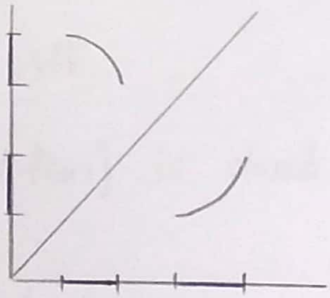
f is not continuous



X is not closed



X unbounded



X not convex / not connected

Thm < Brouwer Fixed point Theorem >

Let $f: X \to X$ be a continuous function mapping a compact and convex set X into itself. Then, f has a fixed point in X.

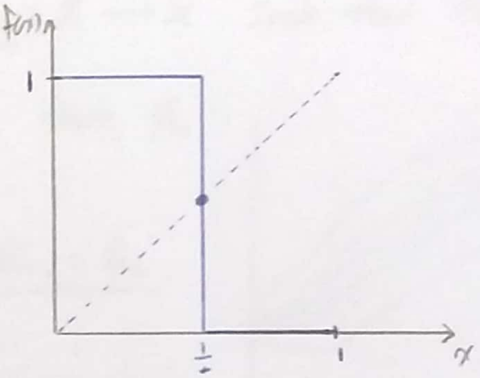
Kakutani

A correspondence on a set X is a function f from X into the set of subsets of X : $f: X \rightrightarrows \mathbb{P}(X)$ ← power set of X

$x^* \in X$ is a fixed point of $f: X \rightrightarrows P(X)$ iff $x^* \in f(x^*)$

ex) $X = [0,1]$

$$f(x) = \begin{cases} 1 & , x < \frac{1}{2} \\ [0,1] & , x = \frac{1}{2} \\ 0 & , x > \frac{1}{2} \end{cases}$$



Def f is convex-valued iff $\forall x \in X$, the set $f(x)$ is convex.

f on X has a closed graph iff

$\text{graph}(f) = \{(x, y) \in X^2 : y \in f(x)\}$ is closed as a subset of X.

Note Any continuous function has a closed graph.

Thm < Kakutani Fixed Point Theorem >

If $X \subseteq \mathbb{R}^n$ is non-empty, compact and convex, then every correspondence $f: X \rightrightarrows P(X)$, that is non-empty valued, convex valued, and has a closed graph has a fixed point. (Sometimes equivalently stated for f uhc and closed-valued.)

proof) For each $t \in \{1, 2, \dots\}$, define $\phi_t : X \rightrightarrows \mathbb{P}(X)$ such that for any $\hat{x} \in X$, $\hat{y} \in \phi_t(\hat{x})$ iff $\hat{y} \in X$ and there is a point (x, y) in the graph of f such that $d((x,y), (\hat{x}, \hat{y})) < \frac{1}{t}$.

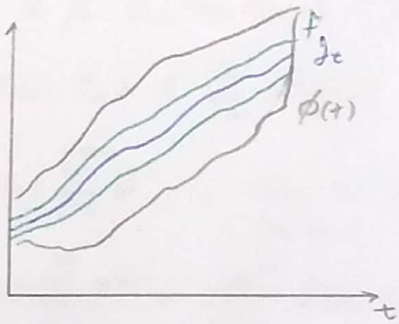
The graph of ϕ_t looks like a tube around the graph of f.

The correspondence ϕ_t is non-empty valued, convex-valued and the set given by the graph of ϕ_t is open relative to $X \times X$.

By the Michael's selection theorem, (e.g. Border (1985))

there exists a continuous function $g_t : X \longrightarrow X$ such that $\forall x \in X$, $g_t(x) \in \phi_t(x)$

g_t is called a continuous selection from ϕ_t.



By Brouwer, $\forall t$, $\exists \hat{x}_t \in X$ s.t $g_t(\hat{x}_t) = \hat{x}_t$.

$\forall t$, by construction of g_t and ϕ_t,

there is a (x_t, y_t) in the graph of f such that $d((x_t, y_t), (\hat{x}_t, \hat{x}_t)) < \frac{1}{t}$

Since $X \times X$ is compact, there is a $x^* \in X$ s.t a subsequence of $(\hat{x}_t, \hat{x}_t)$ converges to (x^*, x^*).

By the triangle inequality, $d((x_t, y_t), (x^*, x^*)) \leq \underbrace{d((x_t, y_t), (\hat{x}_{t_i}, \hat{x}_{t_i}))}_{< \frac{1}{t}} + \underbrace{d((x^*, x^*), (\hat{x}_{t_i}, \hat{x}_{t_i}))}_{\text{converges to } 0} \longrightarrow 0$

along the subsequence (x_t, y_t) must likewise converges to (x^*, x^*)

Since the graph of f is closed, (x^*, x^*) is in the graph of f.

Hence, $x^* \in f(x^*)$. //

Contraction Mapping Theorem

(Def) Let (X, d) be a metric space. A function $f: X \to X$ is a contraction iff there is a number $\beta \in (0,1)$ s.t. $\forall \hat{x}, x \in X$, $d(f(\hat{x}), f(x)) \leq \beta\, d(\hat{x}, x)$

(Thm) < Contraction Mapping Theorem / Banach Fixed point theorem >

Let (X, d) be a metric space and let $f: X \to X$ be a contraction mapping. Then there exists a unique $x^* \in X$ satisfying $f(x^*) = x^*$.
Moreover, for any $x_0 \in X$, the sequence $x_1 = f(x_0)$, $x_n = f(x_{n-1})$ converges to x^*.

proof) Note that $d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq \beta \cdot d(x_n, x_{n-1}) \leq \beta^n d(x_1, x_0)$

For any $m \leq n$, we get $d(x_n, x_m) \leq \sum_{i=m}^{n-1} d(x_{i+1}, x_i) \leq \sum_{i=m}^{n-1} \beta^i d(x_1, x_0)$

$$\leq \beta^m d(x_1, x_0) \sum_{i=0}^{n-m-1} \beta^i \quad (\because d(x_1, x_0) \text{ is indepandent from } i)$$

geometric series

$$\leq \beta^m d(x_1, x_0) \sum_{i=0}^{\infty} \beta^i = \frac{\beta^m}{1-\beta} d(x_1, x_0) \to 0 \text{ as } m \to \infty$$

So, $\frac{\beta^m}{1-\beta} \cdot d(x_1, x_0)$ converges to 0 as $m \to \infty$.

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, it has a limit $x^* \in X$. Since f is a contraction, it is continuous.

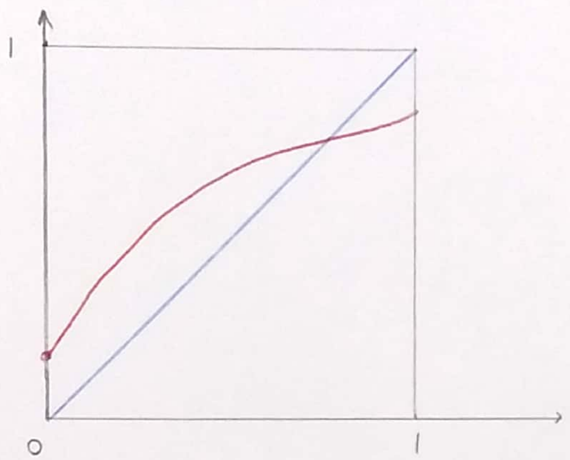
Hence, $f(x^*) = \lim_{n\to\infty} f(x_n) = \lim x_{n+1} = x^*$.

Suppose there is another point $x' \in X$, $x' \neq x^*$ satisfying $f(x') = x'$.
Then, $d(x', x^*) = d(f(x'), f(x^*)) < d(x', x^*)$, which is contradiction.
So, $x' = x^*$. //

Tarski

Thm < Tarski Fixed point theorem > ← (weaker assumption for f / stronger " " X)

Let f be a non-decreasing function mapping the n-dimensional cube $[0,1]^n = [0,1] \times \ldots \times [0,1]$ into itself. Then, f has a fixed point.



- If $f(0)=0$, we are done.
- For $f(0)>0$, since f jumps at most upwards, it cannot cross the 45° line at a jump.

Since $f(1) \leqq 1$, f must cross the 45° line somewhere.

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