NJALA UNIVERSITY

SCHOOL OF BASIC SCIENCES

PHYSICS DEPARTMENT

MATHEMATICAL METHODS – TMAT 211

PAST QUESTIONS AND SOLUTIONS

FEBRUARY 2023

1. a) If $r_1 = 2i - j + k$, $r_2 = i + 3j - 2k$, $r_3 = 2i + j - 3k$ and $r_4 = 3i + 2j + 5k$. Find scalars a , b and c such that $r_4 = ar_1 + br_2 + cr_3$.

   b) Find the work done in moving an object along a vector $r = 3i - 2j - 5k$ if the applied force is $f = 2i - j - k$.

   c) If $A = 3i - j + 2k$, $B = 2i + j - k$ and $C = i - 2j + 2k$, Find

   i) $(A \times B) \times C$

   ii) $A \times (B \times C)$

Solution

1. a) Given that

$r_1 = 2i - j + k$,

$r_2 = i + 3j - 2k$,

$r_3 = 2i + j - 3k$ and

$r_4 = 3i + 2j + 5k$

Then

$ar_1 = a(2i - j + k)$

$br_2 = b(i + 3j - 2k)$

$cr_3 = c(2i + j - 3k)$

$\therefore r_4 = ar_1 + br_2 + cr_3 \Rightarrow 3i + 2j + 5k = a(2i - j + k) + b(i + 3j - 2k) + c(2i + j - 3k)$

$3i + 2j + 5k = 2ai - aj + ak + bi + 3bj - 2bk + 2ci + cj - 3ck$

$\Rightarrow 3i + 2j + 5k = (2a + b + 2c)i + (3b + c - a)j + (a - 2b - 3c)k$

Equating coefficient, we have

$2a + b + 2c = 3..........(1)$

$3b + c - a = 2..........(2)$

$a - 2b - 3c = 5..........(3)$

Solving (i) and (2) simultaneously

$2a + b + 2c = 3..........(1)$

$\underline{6b + 2c - 2a = 4..........(4)}$

$7b + 4c = 7..........(5)$

Solving (1) and (3)

$$2a+b+2c=3..........(1)$$

$$\underline{-2a+4b+6c=-10..........(6)}$$

$$5b+8c=-7..........(7)$$

Solving (5) and (7), we get

$$-14b-8c=-14..........(8)$$

$$\underline{5b+8c=-7..........(7)}$$

$$-9b=-21$$

$$\therefore b=\frac{7}{3}$$

Hence, (5) becomes

$$7\left(\frac{7}{3}\right)+4c=7$$

$$49+12c=21$$

$$\therefore c=-\frac{7}{3}$$

Putting the values of b and c in to (1), we get

$$2a+\frac{7}{3}+2\left(-\frac{7}{3}\right)=3$$

$$6a=9+7$$

$$\therefore a=\frac{16}{6}=\frac{8}{3}$$

Hence the values of the scalars are $a=\frac{8}{3}$, $b=\frac{7}{3}$ and $c=-\frac{7}{3}$

b) Given

Displacement vector $r=3i-2j-5k$

Applied force $f=2i-j-k$

$$Work\ done = force \bullet Displacement = r \bullet f = (3i-2j-5k) \bullet = (2i-j-k)$$

$$Work\ done = force \bullet Displacement = 3(2)-2(-1)-5(-1)$$

$$Work\ done = 6+2+5=13J$$

c) Given

$A = 3i - j + 2k$,

$B = 2i + j - k$ and

$C = i - 2j + 2k$

i) $(A \times B) \times C$

$$A \times B = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} = i\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - j\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} + k\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix}$$

$$A \times B = i\left[(-1 \times -1) - (2 \times 1)\right] - j\left[(3 \times -1) - (2 \times 2)\right] + k\left[(3 \times 1) - (2 \times -1)\right]$$

$$A \times B = i[1-2] - j[-3-4] + k[3+2]$$

$$A \times B = -i + 7j + 5k$$

$$(A \times B) \times C = \begin{vmatrix} i & j & k \\ -1 & 7 & 5 \\ 1 & -2 & 2 \end{vmatrix} = i\begin{vmatrix} 7 & 5 \\ -2 & 2 \end{vmatrix} - j\begin{vmatrix} -1 & 5 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} -1 & 7 \\ 1 & -2 \end{vmatrix}$$

$$(A \times B) \times C = i\left[(7 \times 2) - (-2 \times 5)\right] - j\left[(2 \times -1) - (5 \times 1)\right] + k\left[(-2 \times -1) - (7 \times 1)\right]$$

$$(A \times B) \times C = i[14+10] - j[-2-5] + k[2-7]$$

$$(A \times B) \times C = 24i + 7j - 5k$$

ii) $$B \times C = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix} = i\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} - j\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix}$$

$$B \times C = i\left[(2 \times 1) - (-2 \times -1)\right] - j\left[(2 \times 2) - (1 \times -1)\right] + k\left[(-2 \times 2) - (1 \times 1)\right]$$

$$B \times C = i[2-2] - j[4+1] + k[-4-1]$$

$$B \times C = -5j - 5k$$

$$A \times (B \times C) = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 0 & -5 & -5 \end{vmatrix} = i\begin{vmatrix} -1 & 2 \\ -5 & -5 \end{vmatrix} - j\begin{vmatrix} 3 & 2 \\ 0 & -5 \end{vmatrix} + k\begin{vmatrix} 3 & -1 \\ 0 & -5 \end{vmatrix}$$

$$A \times (B \times C) = i\left[(-5 \times -1) - (2 \times -5)\right] - j\left[(-5 \times 3) - (0 \times 2)\right] + k\left[(-5 \times 3) - (0 \times -1)\right]$$

$$A \times (B \times C) = i[5+10] - j[-15-0] + k[-15-0]$$

$$A \times (B \times C) = 15i + 15j - 15k$$

2. a) Define Imaginary and Complex numbers and give one example each.

b) If $Z = \dfrac{Z^{'} - 6i}{Z^{'} + 8}$ and $Z^{'} = x + iy$ where x and y are real. Find the real and imaginary parts of Z. Prove that if Z is purely imaginary, the locus of Z on the Argand diagram is a circle; and if Z is purely real then the locus is a straight line.

c) State Demoivre's theorem and use the theorem to show that

i) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

ii) $\sin 2\theta = 2\sin\theta\cos\theta$

Hence, verify that $\tan 2\theta = \dfrac{2\tan\theta}{\left(1-\tan^2\theta\right)}$

d) Given the complex number $Z = (1+i)x + 2(1-2i)y - 3 = 0$, find the values of x and y

Solution

2. a) **Imaginary Numbers:** An imaginary number is a number of the form bi, where b is real and $i = \sqrt{-1}$

Examples of imaginary numbers: $\sqrt{-2} = 2i$

Complex Numbers: Acomplex number is a number of the form $Z = a + bi$, where a and b are real numbers and $i = \sqrt{-1}$. The (real) numbers a and b are called the Real and Imaginary parts of Z.

Examples of imaginary numbers: $Z = 2 + 3i$

b) Given $Z = \dfrac{Z^{'} - 6i}{Z^{'} + 8}$ and $Z^{'} = x + iy$

$$\Rightarrow Z = \frac{(x+yi) - 6i}{(x+yi) + 8} = \frac{x + (y-6)i}{(x+8) + yi}$$

Rationalizing, we get

$$Z = \frac{x + (y-6)i}{(x+8) + yi}\left(\frac{(x+8) - yi}{(x+8) - yi}\right)$$

$$Z = \frac{x(x+8) - xyi + (x+8)(y-6)i - y(y-6)i^2}{(x+8)^2 + y^2}$$

$$Z = \frac{\left[x(x+8) + y(y-6)\right] + \left[(x+8)(y-6) - xy\right]i}{(x+8)^2 + y^2}$$

Thus

Real part is $\frac{\left[x(x+8)+y(y-6)\right]}{(x+8)^2+y^2}$

Imaginary part is $\frac{\left[(x+8)(y-6)-xy\right]}{(x+8)^2+y^2}$

When Z is purely imaginary,

$$\frac{\left[x(x+8)+y(y-6)\right]}{(x+8)^2+y^2}=0$$

$$x(x+8)+y(y-6)=0$$

$$x^2+8x+y^2-6y=0$$

$x^2+y^2+8x-6y=0$ Equation of circle

When Z is purely real

$$\frac{\left[(x+8)(y-6)-xy\right]}{(x+8)^2+y^2}=0$$

$$(x+8)(y-6)-xy=0$$

$$xy-6x+8y-48-xy=0$$

$8y-6x-48=0$ Equation of a straight line

c) **Demoivre's Theorem:** Demoivre's theorem states that $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$, for all integral values of n

c) From the theorem

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$$

$$\Rightarrow \cos 2\theta+i\sin 2\theta=(\cos\theta+i\sin\theta)^2$$

$$=(\cos\theta+i\sin\theta)^2=(\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta)$$

$$=\cos^2\theta-\sin^2\theta+2i\sin\theta\cos\theta$$

$$\therefore \cos 2\theta+i\sin 2\theta=\cos^2\theta-\sin^2\theta+2i\sin\theta\cos\theta$$

Equating real and imaginary parts

i) $\cos 2\theta=\cos^2\theta-\sin^2\theta$

ii) $\sin 2\theta=2\sin\theta\cos\theta$

$$\tan 2\theta=\frac{\sin 2\theta}{\cos 2\theta}$$

$$=\frac{2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}=\frac{2\sin\theta\cos\theta/\cos^2\theta}{\left(\cos^2\theta/\cos^2\theta\right)-\left(\sin^2\theta/\cos^2\theta\right)}$$

$$\frac{2\sin\theta/\cos\theta}{1-\left(\sin\theta/\cos\theta\right)^2}$$

$$\therefore \tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$$

d) Given $Z=(1+i)x+2(1-2i)y-3=0$

$\Rightarrow x+xi+2y-4yi-3=0$

$\therefore (x+2y-3)+(x-4y)i=0$

Since $Z=0$

$\Rightarrow x+2y-3=0 \ldots\ldots\ldots..(1)$

$x-4y=0 \ldots\ldots\ldots..(2)$

From (2) $x=4y$, hence (1) becomes

$4y+2y-3=0$

$6y=3$

$\therefore y=\frac{1}{2}$

$\therefore x=4y=4\left(\frac{1}{2}\right)=2$

3. ai) Given $x^2+\cos y+z^3=1$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

ii) The ideal gas equation $PV=nRT$ determines each of the three variables P, V, T as a function of the other two. Show that $\left(\frac{\partial P}{\partial V}\right)\left(\frac{\partial V}{\partial T}\right)\left(\frac{\partial T}{\partial P}\right)=-1$

b) Evaluate the following integrals

i) $\int_0^1\int_0^{x^2} xydydx$

ii) $\int_{-1}^{2}\int_{-1}^{2}\left(2xy^2-3x^2y\right)dydx$

iii) $\int_{-2}^{1}\int_{2}^{4} x^2y^3dydx$

Solution

3. a) Given $x^2 + \cos y + z^3 = 1$,

i) $$2x + 0 + 3z^2 \frac{\partial z}{\partial x} = 0$$

Making $\frac{\partial z}{\partial x}$ the subject, gives

$$\therefore \frac{\partial z}{\partial x} = -\frac{2x}{3z^2}$$

Similarly, $0 - \sin y + 3z^2 \frac{\partial z}{\partial y} = 0$

Making $\frac{\partial z}{\partial y}$ the subject, gives

$$\frac{\partial z}{\partial y} = \frac{\sin y}{3z^2}$$

ii) Given $PV = nRT$

$$\Rightarrow P = nRTV^{-1}$$

$$\therefore \frac{\partial P}{\partial V} = -nRTV^{-2} = \frac{-nRT}{V^2} \,.................(1)$$

Similarly, $V = nRTP^{-1}$

$$\therefore \frac{\partial V}{\partial T} = nRP^{-1} = \frac{nR}{P} \,.................(2)$$

Also $T = \frac{PV}{nR}$

$$\therefore \frac{\partial T}{\partial P} = \frac{V}{nR} \,.................(3)$$

Hence,

$$\left(\frac{\partial P}{\partial V}\right)\left(\frac{\partial V}{\partial T}\right)\left(\frac{\partial T}{\partial P}\right) = \left(\frac{-nRT}{V^2}\right)\left(\frac{nR}{P}\right)\left(\frac{V}{nR}\right) = -\left(\frac{nRT}{PV}\right)$$

But $T = \frac{PV}{nR}$

$$\Rightarrow \left(\frac{\partial P}{\partial V}\right)\left(\frac{\partial V}{\partial T}\right)\left(\frac{\partial T}{\partial P}\right) = -\left(\frac{nR}{PV}\right)\left(\frac{PV}{nR}\right) = -1$$

$$\therefore \left(\frac{\partial P}{\partial V}\right)\left(\frac{\partial V}{\partial T}\right)\left(\frac{\partial T}{\partial P}\right) = -1$$

i) $$\int_0^1\int_0^{x^2} xydydx = \int_0^1 \left|\frac{xy^2}{2}\right|_0^{x^2} dx$$

$$\therefore \int_0^1\int_0^{x^2} xydydx = \int_0^1 \frac{x^5}{2} dx = \left|\frac{x^6}{12}\right|_0^1 = \frac{1}{12}$$

ii) $$\int_{-1}^2\int_{-1}^2 \left(2xy^2 - 3x^2y\right)dydx = \int_{-1}^2 \left|\frac{2xy^3}{3} - \frac{3x^2y^2}{2}\right|_{-1}^2 dx = \int_{-1}^2 \left|\left(\frac{16x}{3} - \frac{12x^2}{2}\right) - \left(\frac{-2x}{3} - \frac{3x^2}{2}\right)\right|dx$$

$$\int_{-1}^2\int_{-1}^2 \left(2xy^2 - 3x^2y\right)dydx = \int_{-1}^2 \left|\left(\frac{16x}{3} + \frac{2x}{3} + \frac{3x^2}{2} - \frac{12x^2}{2}\right)\right|dx = \int_{-1}^2 \left|\left(6x - \frac{9x^2}{2}\right)\right|dx$$

$$= \int_{-1}^2 \left|\left(6x - \frac{9x^2}{2}\right)\right|dx = \left|\left(\frac{6x^2}{2} - \frac{9x^3}{6}\right)\right|_{-1}^2$$

$$\int_{-1}^2\int_{-1}^2 \left(2xy^2 - 3x^2y\right)dydx = \left|\left(\frac{6x^2}{2} - \frac{9x^3}{6}\right)\right|_{-1}^2 = \left(\frac{24}{2} - \frac{72}{6}\right) - \left(\frac{6}{2} + \frac{9}{6}\right)$$

$$\therefore \int_{-1}^2\int_{-1}^2 \left(2xy^2 - 3x^2y\right)dydx = \left(12 - 3 - 12 - \frac{3}{2}\right) = \left(-3 - \frac{3}{2}\right) = -\frac{9}{2}$$

iii) $$\int_{-2}^1\int_2^4 x^2y^3dydx = \int_{-2}^1 \left|\frac{x^2y^4}{4}\right|_2^4 dx = \int_{-2}^1 \left|\frac{256x^2}{4} - \frac{16x^2}{4}\right|dx = \int_{-2}^1 \left(64x^2 - 4x^2\right)dx = \int_{-2}^1 60x^2dx$$

$$\int_{-2}^1\int_2^4 x^2y^3dydx = \int_{-2}^1 60x^2dx = \left|\frac{60x^3}{3}\right|_{-2}^1 = \frac{60}{3} + \frac{480}{3} = 180$$

4. a) Distinguish between the following as related to matrices:

i) Column and Row vectors

ii) Lower and Upper triangular matrices

iii) Symmetric and Skew-symmetric matrices

iv) Singular and Non-singular matrices

v) Diagonal and Square matrices

b) Find the adjoint matrix A^+ and the inverse of the matrix if $A = \begin{pmatrix} 1 & 3 & 0 \\ -2 & 3 & 3 \\ 1 & 1 & 4 \end{pmatrix}$

4.a)

i) **Column Matrix or Column Vector:** A matrix that has $m-rows$ and only one column $(m\times 1)$ is called a column Matrix or a Column vector

Row Matrix or Row Vector: A matrix that has only one row and $n-columns$ $(1\times n)$ is called a Row Matrix or a Row vector. Example $\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$

ii) **Lower Triangular Matrix:** A square matrix is a lower triangular matrix if all the entries/elements above the major diagonal are all zero.

Upper Triangular Matrix: A square matrix is an upper triangular matrix if all the entries/elements below the major diagonal are all zero.

iii) **Symmetric Matrix:** A symmetric matrix is a matrix that is equal to its transpose. A symmetric matrix must be a square matrix and is symmetric about its main diagonal

i.e $A = A^T$

Skew-Symmetric Matrix: A Skew- symmetric matrix is a matrix that is equal to the negative of its transpose. A skew-symmetric matrix must also be a square matrix.

i.e $A = -A^T$

iv) **Singular Matrices:** A matrix A is said to be singular if $|A| = 0$.

Non-Singular: A matrix A is said to be singular if $|A| \neq 0$.

v) **Diagonal matrix:** A square matrix that is both lower triangular and upper triangular is called a diagonal.

Square matrix: A matrix in which the number of rows equals the number of columns is called a square matrix.

b) Find the inverse of the matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ -2 & 3 & 3 \\ 1 & 1 & 4 \end{pmatrix}$

Given $A = \begin{pmatrix} 1 & 3 & 0 \\ -2 & 3 & 3 \\ 1 & 1 & 4 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 3 & 0 \\ -2 & 3 & 3 \\ 1 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} 3 & 3 \\ 1 & 4 \end{vmatrix} - 3\begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} + 0\begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix}$$

$$\Rightarrow |A| = \left[(3\times 4)-(1\times 3)\right] - 3\left[(-2\times 4)-(1\times 3)\right] + 0\left[(-2\times 1)-(1\times 3)\right]$$

$\therefore |A| = 9 + 33 = 42$ Since $|A| \neq 0$, therefore the inverse of A exists.

The matrix of cofactor of elements A is:

$$C_{11} = \begin{vmatrix} 3 & 3 \\ 1 & 4 \end{vmatrix} = (3\times 4) - (3\times 1) = 9,$$

$$C_{12} = -\begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} = -(-2\times 4) - (3\times 1) = 11$$

$$C_{13} = \begin{vmatrix} -2 & 3 \\ 1 & 1 \end{vmatrix} = (-2\times 1) - (3\times 1) = -5,$$

$$C_{21} = -\begin{vmatrix} 3 & 0 \\ 1 & 4 \end{vmatrix} = -\left[(3\times 4) - (0\times 1)\right] = -12$$

$$C_{22} = \begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix} = \left[(1\times 4) - (0\times 1)\right] = 4,$$

$$C_{23} = -\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = -\left[(1\times 1) - (3\times 1)\right] = 2$$

$$C_{31} = \begin{vmatrix} 3 & 0 \\ 3 & 3 \end{vmatrix} = \left[(3\times 3) - (3\times 0)\right] = 9,$$

$$C_{32} = -\begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix} = -\left[(1\times 3) - (-2\times 0)\right] = -3$$

$$C_{33} = \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} = \left[(1\times 3) - (-2\times 3)\right] = 9$$

The cofactor matrix then becomes $A_c = \begin{pmatrix} 9 & 11 & -5 \\ -12 & 4 & 2 \\ 9 & -3 & 9 \end{pmatrix}$

$$\therefore Adj.A = A_c^T = \begin{pmatrix} 9 & 11 & -5 \\ -12 & 4 & 2 \\ 9 & -3 & 9 \end{pmatrix}^T = \begin{pmatrix} 9 & -12 & 9 \\ 11 & 4 & -3 \\ -5 & 2 & 9 \end{pmatrix}$$

Hence, $\therefore A^{-1} = \frac{1}{|A|} Adj.A = \frac{1}{42}\begin{pmatrix} 9 & -12 & 9 \\ 11 & 4 & -3 \\ -5 & 2 & 9 \end{pmatrix}$

5. a) Determine the general solution for each of the following equations

i) $\left(2 + x^2\right)\frac{du}{dx} + 2ux = 0$

ii) $\frac{du^2}{dx^2} + 2\frac{du}{dx} + 5u = 0$

b) Find the particular solution to the differential equation $\left(2 + x^2 y\right)\frac{du}{dx} + xu^2 = 0$ if $u(1) = 2$

Solution

5. a)

iii) Given $\left(2+x^2\right)\frac{du}{dx}+2ux=0$

$\Rightarrow \left(2+x^2\right)du+2uxdx=0$.................(1)

Comparing (1) to the standard exact equation

$N\left(x,u\right)du+M\left(x,u\right)dx=0$

$\Rightarrow N=2+x^2$ and $M=2ux$

$\therefore \frac{\partial N}{\partial x}=2x$ and $\frac{\partial M}{\partial u}=2x$

Hence, $\frac{\partial N}{\partial x}=\frac{\partial M}{\partial u}=2x$

But $N\left(x,u\right)du+M\left(x,u\right)dx=\frac{\partial \phi}{\partial x}dx+\frac{\partial \phi}{\partial u}dx=d\phi=0$

Considering

$\frac{\partial \phi}{\partial x}=2ux$

$\Rightarrow \partial \phi=2ux\partial x$ integrating, we get

$\int \partial \phi=\int 2ux\partial x$

$\phi=ux^2+f\left(u\right)$.................(2)

Differentiating (2) w.r.t u, we have

$\frac{\partial \phi}{\partial u}=x^2+\frac{d}{du}f\left(u\right)$

But $\frac{\partial \phi}{\partial u}=N=2+x^2$

$\therefore 2+x^2=x^2+\frac{d}{du}f\left(u\right)$

$\Rightarrow df\left(u\right)=2du$ integrating

$\int df\left(u\right)=\int 2du$

$f\left(u\right)=2u+c'$.................(3)

Putting (3) into (2), we have

$\phi = ux^2 + 2u + c^{'}$ but $\phi = c$

$\therefore c = ux^2 + 2u + c^{'}$

$c - c^{'} = ux^2 + 2u$ letting $c - c^{'} = A$

$\Rightarrow A = u\left(x^2 + 2\right)$

$\therefore u\left(x\right) = \dfrac{A}{\left(x^2 + 2\right)}$

iv) Given $\dfrac{du^2}{dx^2} + 2\dfrac{du}{dx} + 5u = 0$

$$\frac{du^2}{dx^2} + 2\frac{du}{dx} + 5u = 0.....................(1)$$

Assuming a trial solution $u\left(x\right) = e^{mx}$

$$\Rightarrow \frac{du}{dx} = me^{mx} \text{ and } \frac{d^2u}{dx^2} = m^2e^{mx}$$

Substituting these in to (1), we have

$m^2e^{mx} + me^{mx} + 5e^{mx} = 0$
$\therefore m^2 + m + 5 = 0..............(2)$
Solving (2) using the formula method
$m = \dfrac{-2 \pm \sqrt{4-20}}{2} = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$
$m_1 = -1 + 2i$ and $m_2 = -1 - 2i$
Hence, the two independent solutions are

$u_1\left(x\right) = C_1e^{(-1+2i)x}$ and $u_2\left(x\right) = C_2e^{(-1-2i)x}$
The general solution becomes
$u\left(x\right) = u_1\left(x\right) + u_2\left(x\right)$
$u\left(x\right) = C_1e^{(-1+2i)x} + C_2e^{(-1-2i)x} = C_1\left(e^{-x}.e^{2xi}\right) + C_2\left(e^{-x}.e^{-2xi}\right) = e^{-x}\left[C_1e^{2xi} + C_2e^{-2xi}\right]$
$u\left(x\right) = e^{-x}\left[C_1e^{2xi} + C_2e^{-2xi}\right] = e^{-x}\left[\left(C_1\cos 2x + iC_1\sin 2x\right) + \left(C_2\cos 2x - iC_2\sin 2x\right)\right]$
$u\left(x\right) = e^{-x}\left[\left(C_1 + C_2\right)\cos 2x + i\left(C_1 - C_2\right)\sin 2x\right]$
Letting $\left(C_1 + C_2\right) = A$ and $i(C_1 - C_2) = B$, we have
$u\left(x\right) = e^{-x}\left[A\cos 2x + B\sin 2x\right]$

b) Given $\left(2 + x^2y\right)\dfrac{du}{dx} + xu^2 = 0$ if $u\left(1\right) = 2$

$$\Rightarrow \left(2 + x^2y\right)du + xu^2dx = 0..............(1)$$

Comparing (1) to the standard exact equation

$$N(x,u)du + M(x,u)dx = 0$$

$$\Rightarrow N = 2 + x^2 u \text{ and } M = xu^2$$

$$\therefore \frac{\partial N}{\partial x} = 2xu \text{ and } \frac{\partial M}{\partial u} = 2xu$$

Hence, $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial u} = 2xu$

But $N(x,u)du + M(x,u)dx = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial u}dx = d\phi = 0$

Considering

$$\frac{\partial \phi}{\partial x} = xu^2$$

$\Rightarrow \partial\phi = xu^2 \partial x$ integrating, we get

$$\int \partial\phi = \int xu^2 \partial x$$

$$\phi = \frac{u^2 x^2}{2} + f(u) \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Differentiating (2) w.r.t u, we have

$$\frac{\partial \phi}{\partial u} = ux^2 + \frac{d}{du} f(u)$$

But $\frac{\partial \phi}{\partial u} = N = 2 + x^2 u$

$$\therefore 2 + x^2 u = x^2 u + \frac{d}{du} f(u)$$

$\Rightarrow df(u) = 2du$ integrating

$$\int df(u) = \int 2du$$

$$f(u) = 2u + c' \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Putting (3) into (2), we have

$\phi = \frac{u^2 x^2}{2} + 2u + c'$ but $\phi = c$

$$\therefore c = \frac{u^2 x^2}{2} + 2u + c'$$

$c - c' = \frac{u^2 x^2}{2} + 2u$ letting $c - c' = B$

$$\Rightarrow B = \frac{u^2x^2}{2} + 2u$$

$$\therefore u^2x^2 + 4u - 2B = 0 \ldots\ldots\ldots\ldots(4)$$

Solving (4) using the formula method

$$u(x) = \frac{-4 \pm \sqrt{16 + 8x^2B}}{2x^2} = \frac{-2 \pm \sqrt{4 + 2x^2B}}{x^2}$$

$$u(x) = \frac{-2}{x^2} \pm \frac{1}{x^2}\sqrt{4 + 2x^2B}$$

Using the condition $u(1) = 2$

$$\Rightarrow 2 = -2 \pm \sqrt{4 + 2B}$$

$$4 = \sqrt{4 + 2B} \qquad \therefore B = 6$$

$$u(x) = \frac{-2}{x^2} \pm \frac{1}{x^2}\sqrt{4 + 12x^2} = \frac{-2}{x^2} \pm \frac{2}{x^2}\sqrt{1 + 3x^2}$$

$u(x) = \dfrac{-2}{x^2} + \dfrac{2}{x^2}\sqrt{1 + 3x^2}$ the plus sign is retained so that $u(1) = 2$