1 Extra material

1.1 Jensens inequality

Finite case: For a real convex function ϕ

$$\phi\left(\sum_{i=1}^{N}\theta_i x_i\right) \leq \sum_{i=1}^{N}\theta_i\phi(x_i)$$

Proof. By induction: n = 2 follows from convexity and for $N+1$:

$$\phi\left(\sum_{i=1}^{N+1}\theta_i x_i\right) = \phi\left(\theta_1 x_1 + (1-\theta_1)\sum_{i=2}^{N+1}\frac{\theta_i}{1-\theta_1}x_i\right)$$

By convexity of the case $n = 2$ the result follows. □

Probability formulation: Let X be some random variable, and let $f(x)$ be a convex function (defined at least on a segment containing the range of X). Then:

$$\mathbb{E}[f(x)] \geq f(\mathbb{E}[X])$$

Proof. Let $c = \mathbb{E}[X]$. Since $f(x)$ is convex there exists a supporting line for $f(x)$ at c:

$$\phi(x) = \alpha(x-c) + f(x)$$

for some α, and $\phi(x) \leq f(x)$. Then

$$\mathbb{E}[f(X)] \geq \mathbb{E}[\phi(\mathbb{X})] = \mathbb{E}[\alpha(X-c) + f(c)] = f(c)$$

□

1.2 Youngs Inequality

Let $a, b, p, q \in \mathbb{R}_{++}$ such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof.

$$ab = \exp\left(\frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q)\right) \leq \frac{1}{p}\exp(\ln(a^p)) + \frac{1}{q}\exp(\ln(b^q)) = \frac{a^p}{p} + \frac{b^q}{q}$$

where the inequality follows from the convexity of the exponential function. □

1.2.1 Standard version

Let $f : [0, c] \longrightarrow R$ be a continuous and strictly increasing function, where $f(0) = 0$.

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(x)dx$$

Proof. By visual inspection. □

1.3 Hölders inequality

1.3.1 Standard version

Let $(S, \sum, \mu)$ be a measure space and let $q \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ then for all measurable real- or complex-valued functions f and g on S

$$\|fg\|_1 \leq \|f\|_p \|g\|_q,$$

in the case $p = 2$ this becomes the Cauchy inequality.

Proof. If $\|f\|_t = 0$ for $t = p$ or $t = q$ the left side is zero almost everywhere, so we can assume that $\|f\|_t > 0$. For $\|f\|_t = \infty$ the right hand side could be infinite large. If $t = \infty$ then $|fg| \leq \|f\|_\infty |g|$ almost everywhere and Hölders inequality follows from the monotonicity of the Lebesque integral (?). Normalizing the equation we can expect

$$\|f\|_p = \|g\|_q = 1.$$

Using Youngs inequality

$$|f(s)g(s)| \leq \frac{|f(s)|^p}{p} + \frac{|g(s)|^q}{q} \ s \in S$$

Integrating both sides:

$$\|fg\|_1 = \int_S |f(s)g(s)|ds \leq \int_s \left(\frac{|f(s)|^p}{p} + \frac{|g(s)|^q}{q} \right) ds = \frac{\|f\|_p}{p} + \frac{\|g\|_q}{q} = 1 = \|f\|_p \|g\|_q$$

□

1.3.2 With sums

Using summation to define the norm :

$$\|x\|_p = \left(\sum_k |x_k|^p \right)^{\frac{1}{p}}$$

Proof. Then (without normalization)

$$\frac{|\sum_k x_k y_k|}{\|x\|_p \|x\|_q} \leq \sum_k \frac{|x_k||y_k|}{\|x\|_p \|x\|_q} \leq \frac{1}{p} \sum \frac{|x_k|^p}{\|x\|_p^p} + \frac{1}{p} \sum \frac{|x_k|^q}{\|x\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1$$

hence

$$|(x,y)| = |\sum_k x_k y_k| \leq \|x\|_p \|x\|_q$$

□

The equality holds (proof) iff $\arg x_j y_j$ and $|x_j|^p |y_j|^p$ are independent of j. For any $x \in \ell^p$ we can choose $y \in \ell^q$ so we get equality (proof?). Hence we get

$$\|x\|_p = \max_{\|y\|_q = 1} \|(x,y)\|_1$$

1.4 Minkowski

For the ℓ^p spaces (same holds for L^p)

$$\|x + p\|_p \leq \|x\|_p + \|y\|_p$$

Proof. (?) Using the second formulation of Hölders inequality and noting that $\|xy\|_1$ could be written as a scalar product and that scalar products are bilinear

$$\|x + y\|_p = \max_{\|u\|_q = 1} |(x + y, u)| \leq \max_{\|u\|_q = 1} |(x, u)| + |(y, u)| \leq \|x\|_p + \|y\|_q.$$

Where we in the last inequality split the max and use Hölder inequality together with fact that $\|u\|_q = 1$. □

1.5 Cauchy inequality

A scalar product in a linear space X over $\mathbb{C}$ is a real valued function that fulfils

- sesquilinearity: $(ax, y) = a(x,y), \quad (x, ay) = \bar{a}(x,y)$
- Skew symmetry: $(y, x) = \overline{(x,y)}$
- positivity: $(x,x) > 0, \quad x \neq 0$

This also lets us define the norm

$$\|x\| = (x,x)^{\frac{1}{2}}$$

1.5.1 Schwarz Inequality in $\mathbb{C}$

$$|(x, y)| \leq \|x\|\|y\|$$

(This is just a special case of Hölders inequality with $p = 2$)

Proof. For any $t \in \mathbb{R}$ and $y \neq 0$

$$0 \leq \|x + ty\|^2 = \|x\|^2 + 2tRe(x, y) + t^2\|y\|^2 \tag{1}$$

By choosing $t = -Re(x, y)/\|y\|^2$ and multiplying with $\|y\|^2$

$$(Re(x, y))^2 \leq \|x\|^2\|y\|^2 \tag{2}$$

Let $x = ax, a \in \mathbb{C}$ such that $|a| = 1$ and $Im(a(x, y)) = 0$ results in

$$|(x, y)| \leq \|x\|\|y\| \tag{3}$$

This also gives as a corollary that every vector in a scalar product space can be written as

$$\|x\| = \max_{\|y\|=1} \|(x, y)\|_1$$

$\square$

1.6 Parallelogram Identity

Set $t \pm 1$ in (1) and we end up with the parallelogram identity, which is true for all scalar product induced norms.

$$\|x + y\|^2 + \|x - y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \tag{4}$$

Actually every norm that comes from an scalar product must have this equality (von Neumann)

1.7 Polarization Identity

For a real vector space, if the norm fullfills the Parallelogram Identity then it induces a scalar product

$$(x, y) = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right) \tag{5}$$

And for the complex vector space

$$(x, y) = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\right) \tag{6}$$

2 Finite

2.0.1 Lemma x

Let Y be a subset of the metric space X, then $x \in X$ is adherent to Y iff there is a sequence in Y that converges to x.

Proof. ($\Rightarrow$) For each $n \geq 1$ there exists some point $x_n \in B(x; 1/n) \cap Y$ the sequence (x_n) then converges to x.

($\Leftarrow$) If there is a sequence (x_n) converging to x then all balls centered at x contains points from the sequence. □

2.0.2 Every subspace of a normed space of finite dimension is closed

Let $(V, |\cdot|)$ be a normed vector space of finite dimension, and $S \subset V$ a finite dimensional subspace. Then S is closed.

Proof. Let $a \in \overline{S}$ and choose a sequence (a_n), with $a_n \in S$ converging to a **??**. Then (a_n) is a cauchy sequence in V and also a cauchy sequence in S. Since V is finite dimensional, V is a Banach space, S is complete, so (a_n) converges to an element in S. Since limits in a normed space are unique, that limits must be a, so $a \in S$. □

2.0.3 Every finite dimensional subspace of a normed space is closed

Proof. Let $(V, |\cdot|)$ be a normed vector space, and $S \subset V$ a finite dimensional subspace. Then S is closed. Let $x \in V$ and let (a_n) be a sequence in S with converges to x. Because S is of finite dimension, we have a basis $\{x_1 ... x_k$ of$S$. Also $x \in span(x_1 ... x_k, x)$. But as proved in the case when V is finite dimensional, we have that S is closed in $span(x_1 ... x_k, x$ (taken with the normed induced by $(V, |\cdot|)$) with $a_n \to x$, and $x \in S$. □

3 Things to remember

1. *Uniform convex:* For any pair of unit vectors x, y the norm $|(x + y)|/2$ is strictly less then 1, by an amount that depends only on $|x - y|$.
   - L^p spaces are uniformly convex for $1 < p < \infty$
   - The space C is not uniformly convex, not even subadditiv in the maxnorm

3.0.4 Least upper bound property

Is often taken as an axiom. Every nonempty set S of $\mathbb{R}$ must have a least upper bound (supremum).

4 Measure theory

Monotone convergence theory

4.0.5 Reel numbers

Theorem: Let (a_n) be a monotone sequence in $\mathbb{R}$ (i.e. $a_n \leq a_{n+1}$ or $a_n \geq a_{n+1}$ for all $n \geq 1$. Then (a_n) has a finite limit iff it is bounded.

Proof. (Bounded above)

($\Leftarrow$) By the least upper bound property there is a real number $c = \sup_n(a_n)$. For every $\epsilon \geq 0$, there exists a_N such that $a_n > c - \epsilon$, otherwise $c - \epsilon$ would be an lower bound than c. Since (a_n) is monotone $\forall n > N, |c - a_n| = c - a_n \leq c - a_N < \epsilon.$, it is converging to $\sup(a_n)$.

($\Rightarrow$) Let $a_n \to c$ as $n \to \infty$. Then we want to find an K such that $\forall n \in \mathbb{N}: |a_n| \leq K$. Since (a_n) converges is it true that

$$\forall \epsilon > 0 : \exists N : n > N \Longrightarrow |a_n - c| < \epsilon$$

Set $\epsilon = 1$ by the reverse triangle inequality

$$\forall n > N_1 : |a_n| - |c| \leq |a_n - c| < 1,$$

hence $a_n \leq |c| + 1$. $K = \max(|a_1|, |a_2, ...|a_{N_1}|, |c| + 1)$ is finite and does the job. The monotone decreasing sequence can be done analogously.

4.0.6 Lebesque's monotone convergence theorem

Theorem: Let (X, Σ, μ) be a measure space and $(f_1, f_2 ...), f_i \in \Sigma : f_i : X \to [0, \infty]$ be pointwise non-decreasing sequences, i.e

$$0 \leq f_k(x) \leq f_{k+1}(x).$$

Let

$$f(x) := \lim_{k \to \infty} f_k(x)$$

be the pointwise limit for $\forall x \in X$. Then f is $\Sigma-$ measurable and

$$\lim_{k \to \infty} \int f_k d\mu = \int f d\mu \tag{7}$$

This can also take the shape as the sum

$$\lim_{k \to \infty} \sum_k a_{kj} = \sum_k \lim a_{kj} \tag{8}$$

4.0.7 Dominated convergence theorem

Let f_n be a sequence of real-valued measurable functions on a measure space (X, Σ, μ), suppose the sequence converges pointwise to a function f and is dominated by a integrable function g in the sense that

$$|f_n(x)| \leq g(x), \tag{9}$$

$\forall n \in \mathbb{N}$ and $x \in X$ Then f is integrable and

$$\lim_{n\to\infty} \int_S |f_n - f| d\mu = 0 \Longleftrightarrow \lim_{n\to\infty} \int_S f_n d\mu = \int_S f d\mu \tag{10}$$

4.0.8 Fotou's Lemma

Let f_n be a sequence of real-valued non-negative measurable functions on a measure space (X, Σ, μ). Define the function $f : S \to [0, \infty]$ almost everywhere by

$$f(s) = \liminf_{n\to\infty} f_n(s) \quad s \in S \tag{11}$$

Then f is measurable and

$$\int_S f d\mu \leq \liminf_{n\to\infty} \int_S f_n d\mu. \tag{12}$$

5 Things we should know!

5.1 Bolzano - Weierstrass Theorem

Each bounded sequence in $\mathbb{R}^n$ has a convergent subsequence. (or equivalently: A subset of $\mathbb{R}^n$ is sequentially compact iff it is closed and bounded)

5.1.1 Lemma

Every sequence $(a_n) \in \mathbb{R}$ has a monotone subsequence.

Proof. Let us call a positive integer a **Peak** of the sequence if $m > n \to a_m > a_n$. If there are infinitely many peaks we take these as an decreasing subsequence. If we instead has finitely many peaks, suppose N is the last one. Then we can find a sequence that for every value a_{n_j} we have $n_{j+1} > n_j$ and $a_{n_j} \leq a_{n_j+1}$ (otherwise a_{n_j} would be a peak. Hence we have a monotone subsequence $a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \ldots$. □

Continuing the proof:

$n = 1$: Now suppose we have a bounded sequence in $\mathbb{R}$; by the previous lemma there exists a monotone subsequence, necessarily bounded. Then by monotone convergence theorem, thus subsequence converge.

$n \geq 1$: For the bounded sequence $(a_n) \in \mathbb{R}^n$, we use a the following procedure: let (a_n^1) be a subsequence of (a_n) converging in the first coordinate. Let

a_n^2 be a subsequence of (a_n^1) converging in the first two coordinates. Continuing in this manner after n subsequence we are done, and have a subsequence of (a_n) that converges in every coordinate. $\square$

5.2 The Stone-Weirstrass Theorem

5.2.1 The original statement

Suppose f is a continuous complex-valued function defined on the real intervall $[a, b]$. For every $\epsilon > 0$, there exists a polynomial function over $\mathbb{C}$ such that

$$\sup_x \|f(x) - p(x)\| \leq \epsilon \tag{13}$$

5.2.2 Functional analysis

Let S be a compact Hausdorff space, $C(S)$ the set of all real-valued continuous function on S. Let E be a subalgebra of $C(S)$, that is,

- E is a subspace of $C(S)$.
- The product of two functions in E belongs to E.

In addition we impose the following condition on E:

- E Separates points of S, that is, given any pair of points p and q, $p \neq q$, there is a function $f \in E$ such that $f(p) \neq f(q)$
- The product of two functions in E belongs to E.