Exact and Soft Successive Refinement of the Information Bottleneck

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Motivation

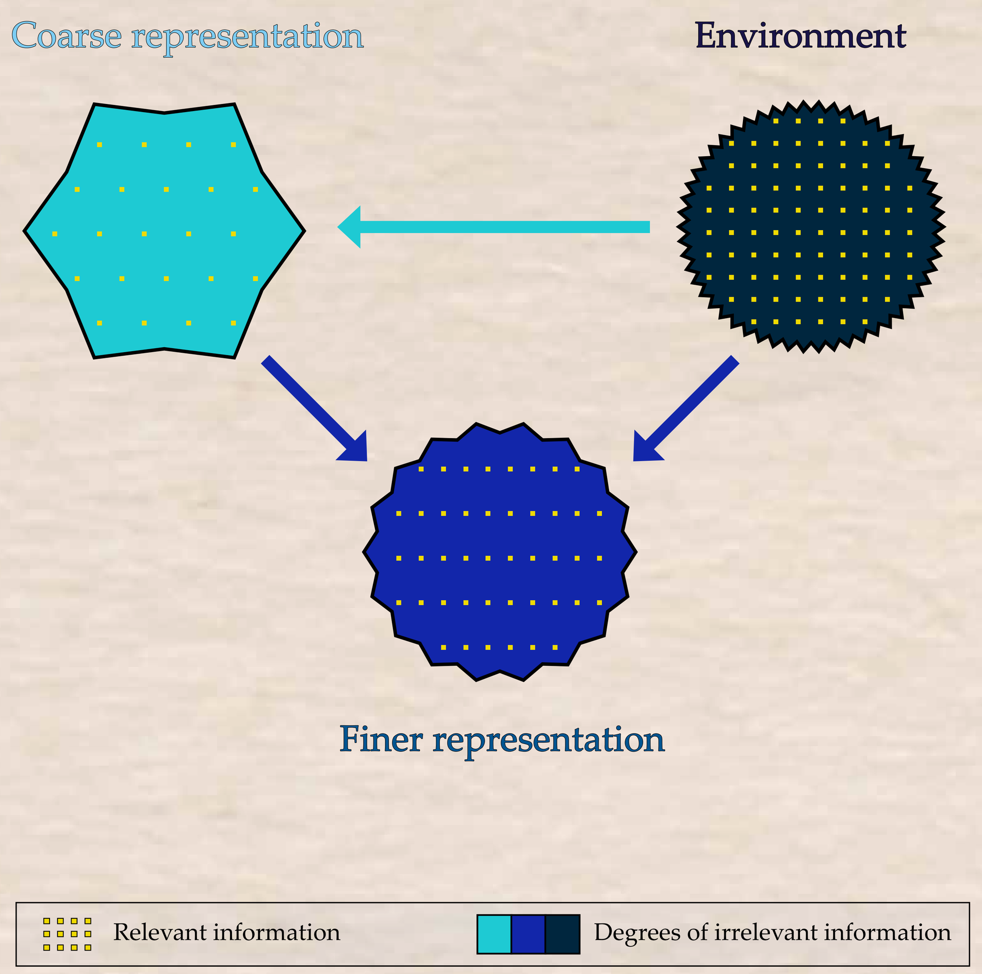
Embodied agents must **balance** their representations' **behavioral relevancy** with the **cognitive cost** of their production and use.

But the right **trade-off varies along time** (e.g. along development), and updating representations comes with a cognitive cost as well.

→ What are the **informationally optimal limits** on the **refinement** and **coarsening** of already existing representations?

The fundamental trade-off

- Trade-off for embodied agents: **Behavioral relevancy** of representations vs. **cognitive cost** of their use.
- Postulate: Evolution poises representations close to **optimality**.
- Thus crucial to understand the optimal limit on feasible trade-offs.
- These are described in information-theoretic terms by the **Information Bottleneck** (IB) method [1].
- However: The method *a priori* disregards the **relationship between representations**, which might crucially affect the informationally optimal limits.



Successive incorporation of information

Assume the agent

- **First** needs a **coarse** IB optimal representation of its environment,
- Then at a **later** stage, builds on it and on new information from the environment, to design a **finer** IB optimal representation.

To which extent can the finer optimal representation **leverage the information already extracted** in the coarser one?

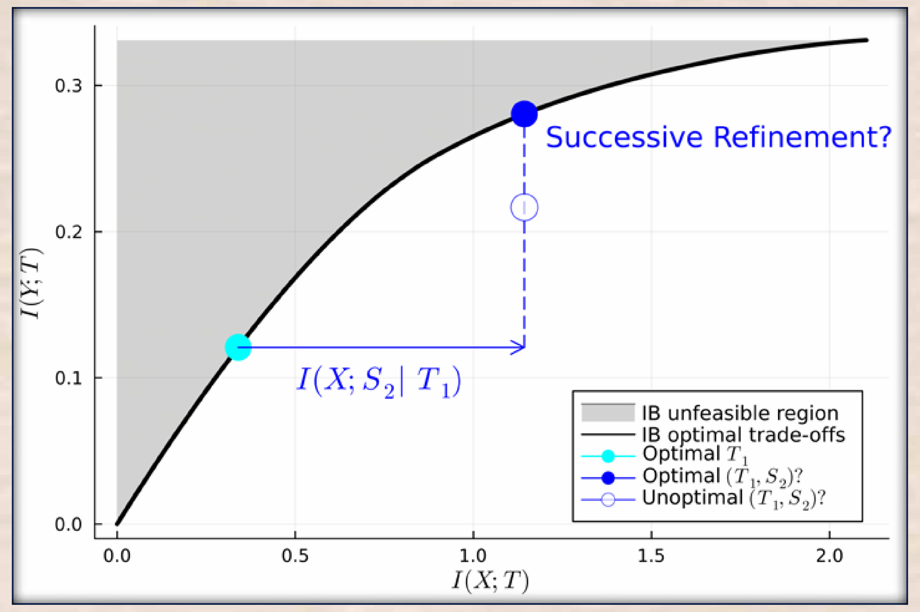
If **all** the information can be leveraged - i.e., if **no cognitive work is wasted** along the refinement process - there is ```**Successive Refinement**'' [2,3].

Formalism

The Information Bottleneck

- ◊ The representation T extracts, within the environment X, information about relevant features Y (e.g., future, perceptions, homeostatic variables, ...)
- ◊ Cognitive cost is $I(X;T)$
- ◊ Behavioral relevancy is $I(Y;T)$
- ◊ "IB" optimal representations solve, for some λ,

$$\underset{\substack{p(T|X):\\ Y-X-T,\ I(X;T)\leq\lambda}}{\arg\max}\ I(Y;T)$$



Exact Successive Refinement (SR)

- ◊ Create IB optimal coarse representation T_1
- ◊ Create finer representation (T_1,S_2) where supplement of information S_2 solves, for some λ_2,

$$\underset{\substack{p(S_2|X,T_1):\\ Y-X-(T_1,S_2),\ I(X;S_2|T_1)\leq\lambda_2}}{\arg\max}\ I(Y;S_2\,|\,T_1)$$

- ◊ Successive refinement: (T_1,S_2) fulfills IB optimality

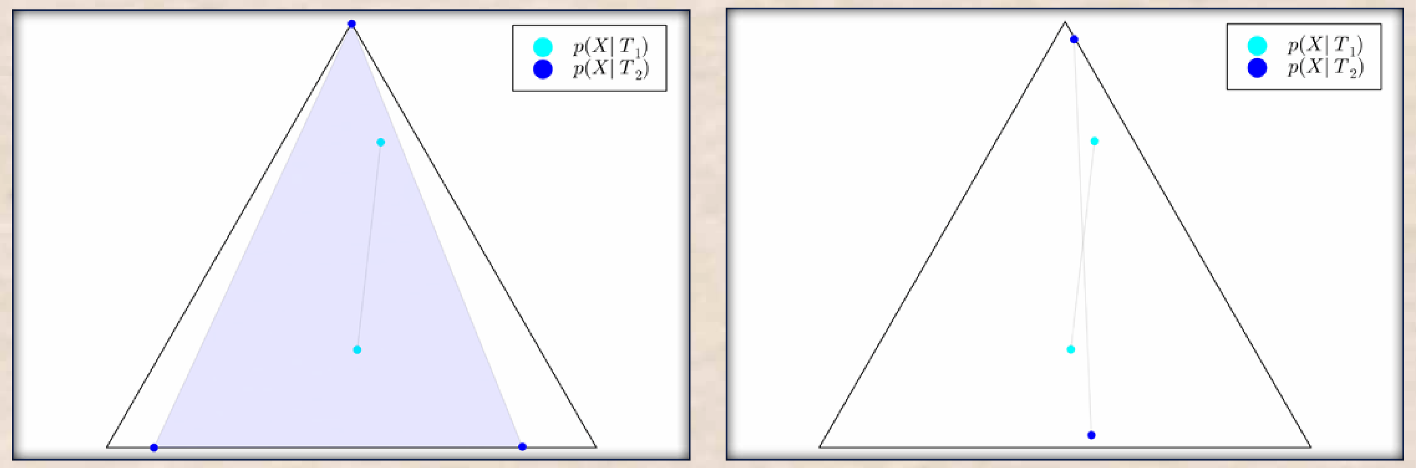
Exact SR: theoretical results

Convex hull characterisation

Successive Refinement: if and only if we can find

- a coarse IB optimal representation T_1,
- a finer representation T_2 which is **also IB optimal**,

such that the coarse decoder's conditional probabilities $p(X\,|\,T_1)$ are in the **convex hull** of the finer one's $p(X\,|\,T_2)$, where X is the environment.



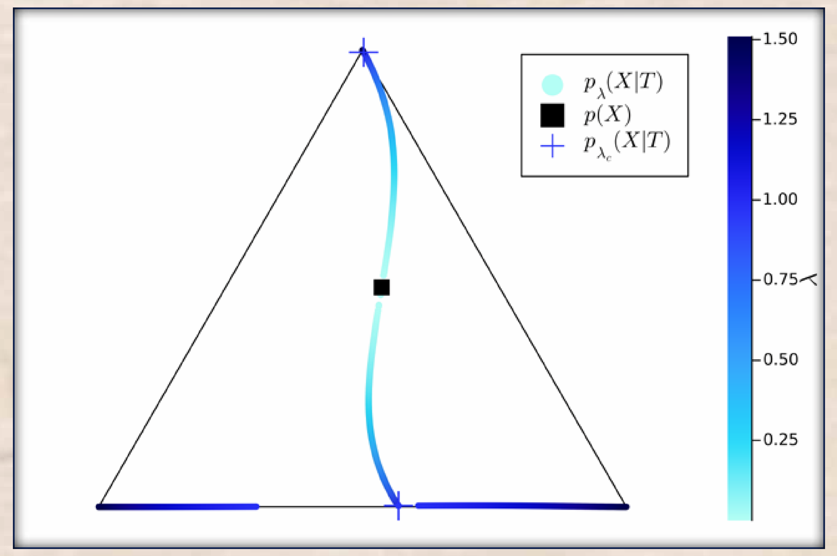
Condition satisfied *Condition not satisfied*

Some IB problems are always successively refinable: if

→ X and Y are jointly Gaussian vectors, or
→ Y is a deterministic function of X, or
→ X and Y are binary,

where X: environment and Y: relevant features.

Exact SR: minimal numerical experiments



- In general, exact successive refinement does not always hold
- But visually, convex hull condition often ``**close**'' to being satisfied

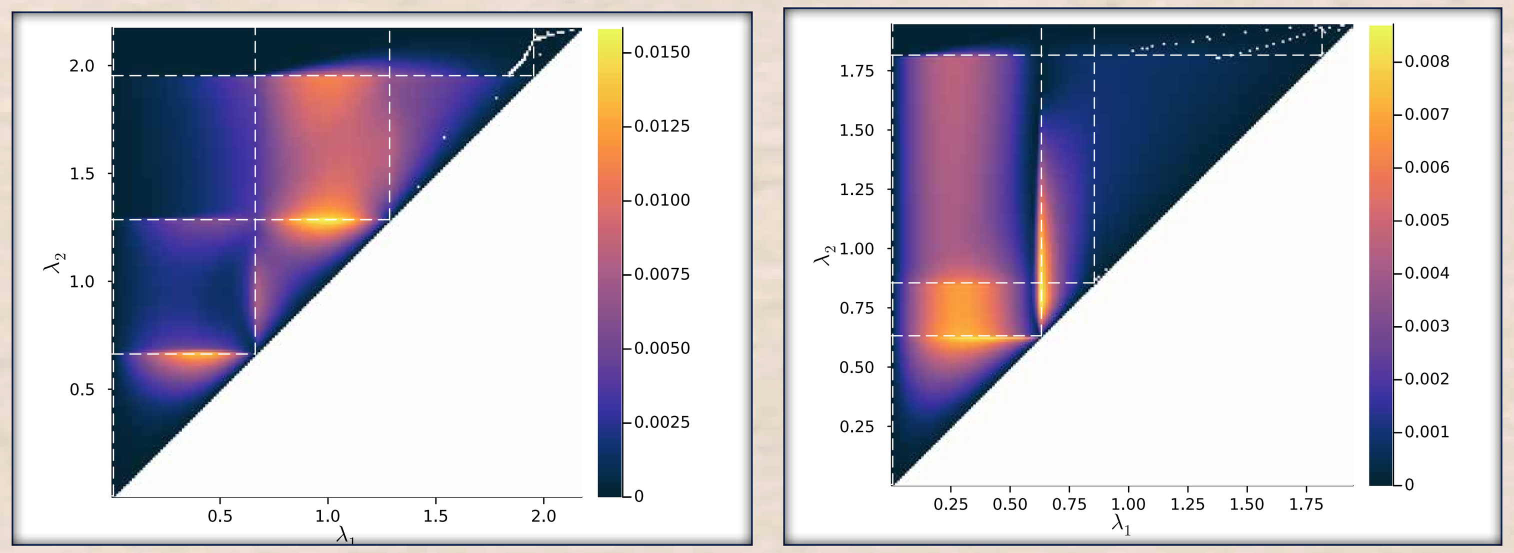
⟶ Almost no information discarded along refinement?

Soft Successive Refinement

Information discarded along representation's refinement can be quantified [4] with the **unique information** [5]:

$$\min\ I_r(X;T_1\,|\,T_2)$$

over all $r(X,T_1,T_2)$ such that $r(X,T_1)=p(X,T_1)$ and $r(X,T_2)=p(X,T_2)$.



Unique information, example 1 *Unique information, example 2*

- Rich structure which is not accessible with the binary notion of exact successive refinement
- Unique information rises then sharply drops close to IB bifurcations
- Sharp transition due to that of bottlenecks' trajectories on the probabilty simplex
- Unique information always significantly low, as compared to system's global information

All our results on representations' refinement apply equally well to their coarsening.

Our numerical results on minimal examples suggest that

◊ Exact and soft successive refinement are shaped by IB bifurcations:
IB optimal cognitive systems should carefully weigh their representations' granularity, so that dynamically adapting it is cognitively cheap.

◊ Successive refinement is often nearly satisfied:
The cognitive cost of (informationally optimal) incremental learning seems typically significantly limited.

These results shed light on the structure of agents' dynamical construction of meaning from the constraints inherent to their embodied interaction with the environment.

References

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