Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Junior Section)

Tuesday, 1 June 2010 0930-1200

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer on the answer sheet and shade the appropriate bubbles below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1. Among the five real numbers below, which one is the smallest?

   (A) $\sqrt[2009]{2010}$; (B) $\sqrt[2010]{2009}$; (C) 2010; (D) $\dfrac{2010}{2009}$; (E) $\dfrac{2009}{2010}$.

2. Among the five integers below, which one is the largest?

   (A) 2009^{2010}; (B) 20092010^2; (C) 2010^{2009}; (D) $3^{(3^{(3^3)})}$; (E) $2^{10} + 4^{10} + \cdots + 2010^{10}$.

3. Among the four statements on real numbers below, how many of them are correct?

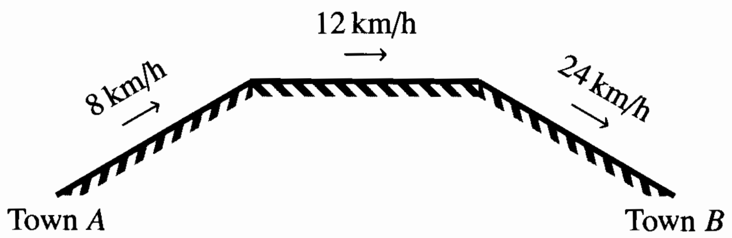
   "If $a < b$ and $a, b \neq 0$ then $\frac{1}{b} < \frac{1}{a}$"; "If $a < b$ then $ac < bc$";
   "If $a < b$ then $a + c < b + c$"; "If $a^2 < b^2$ then $a < b$".

   (A) 0; (B) 1; (C) 2; (D) 3; (E) 4.

4. What is the largest integer less than or equal to $\sqrt[3]{(2010)^3 + 3 \times (2010)^2 + 4 \times 2010 + 1}$?

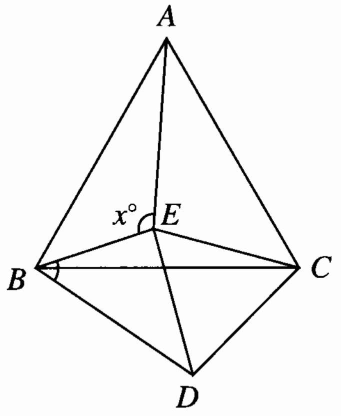
   (A) 2009; (B) 2010; (C) 2011; (D) 2012; (E) None of the above.

5. The conditions of the road between Town A and Town B can be classified as up slope, horizontal or down slope and total length of each type of road is the same. A cyclist travels from Town A to Town B with uniform speeds 8 km/h, 12 km/h and 24 km/h on the up slope, horizontal and down slope road respectively. What is the average speed of his journey?



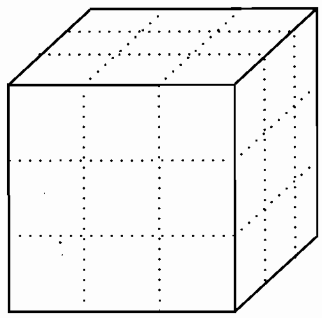
   (A) 12 km/h; (B) $\frac{44}{3}$ km/h; (C) 16 km/h; (D) 17 km/h; (E) 18 km/h.

6. In the diagram, ΔABC and ΔCDE are equilateral triangles. Given that $\angle EBD = 62°$ and $\angle AEB = x°$, what is the value of x?



(A) 100; (B) 118; (C) 120; (D) 122; (E) 135.

7. A carpenter wishes to cut a wooden $3 \times 3 \times 3$ cube into twenty seven $1 \times 1 \times 1$ cubes. He can do this easily by making 6 cuts through the cube, keeping the pieces together in the cube shape as shown:



What is the minimum number of cuts needed if he is allowed to rearrange the pieces after each cut?

(A) 2; (B) 3; (C) 4; (D) 5; (E) 6.

8. What is the last digit of $7^{(7^7)}$?

(A) 1; (B) 3; (C) 5; (D) 7; (E) 9.

9. Given that n is an odd integer less than 1000 and the product of all its digits is 252. How many such integers are there ?

(A) 3; (B) 4; (C) 5; (D) 6; (E) 7.

10. What is the value of

$$(\sqrt{11}+\sqrt{5})^8+(\sqrt{11}-\sqrt{5})^8?$$

(A) 451856; (B) 691962; (C) 903712; (D) 1276392; (E) 1576392.

Short Questions

11. Let x and y be real numbers satisfying

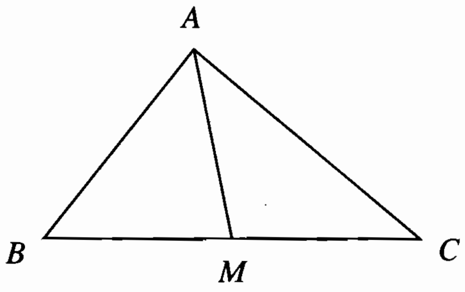
$$y=\sqrt{\frac{2008x+2009}{2010x-2011}}+\sqrt{\frac{2008x+2009}{2011-2010x}}+2010.$$

Find the value of y.

12. For integers $a_1,\ldots,a_n\in\{1,2,3,\ldots,9\}$, we use the notation $\overline{a_1a_2\ldots a_n}$ to denote the number $10^{n-1}a_1+10^{n-2}a_2+\cdots+10a_{n-1}+a_n$. For example, when $a=2$ and $b=0$, $\overline{ab}$ denotes the number 20. Given that $\overline{ab}=b^2$ and $\overline{acbc}=(\overline{ba})^2$. Find the value of $\overline{abc}$.

13. Given that $(m-2)$ is a positive integer and it is also a factor of $3m^2-2m+10$. Find the sum of all such values of m.

14. In triangle ABC, $AB=32$ cm, $AC=36$ cm and $BC=44$ cm. If M is the midpoint of BC, find the length of AM in cm.



15. Evaluate

$$\frac{678 + 690 + 702 + 714 + \cdots + 1998 + 2010}{3 + 9 + 15 + 21 + \cdots + 327 + 333}.$$

16. Esther and Frida are supposed to fill a rectangular array of 16 columns and 10 rows, with the numbers 1 to 160. Esther chose to do it row-wise so that the first row is numbered $1, 2, \ldots, 16$ and the second row is $17, 18, \ldots, 32$ and so on. Frida chose to do it column-wise, so that her first column has $1, 2, \ldots, 10$, and the second column has $11, 12, \ldots, 20$ and so on. Comparing Esther's array with Frida's array, we notice that some numbers occupy the same position. Find the sum of the numbers in these positions.

1	2	3	$\cdots$	$\cdots$	16
17	18	19	$\cdots$	$\cdots$	32
$\cdots$	3 ·	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
145	146	147	$\cdots$	$\cdots$	160

Esther

1	11	21	$\cdots$	$\cdots$	151
2	12	22	$\cdots$	$\cdots$	152
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
10	20	30	$\cdots$	$\cdots$	160

Frida

17. The sum of two integers A and B is 2010. If the lowest common multiple of A and B is 14807, write down the larger of the two integers A or B.

18. A sequence of polynomials $a_n(x)$ are defined recursively by

$$\begin{aligned} a_0(x) &= 1, \\ a_1(x) &= x^2 + x + 1, \\ a_n(x) &= (x^n + 1)a_{n-1}(x) - a_{n-2}(x), \text{ for all } n \geq 2. \end{aligned}$$

For example,

$$\begin{aligned} a_2(x) &= (x^2 + 1)(x^2 + x + 1) - 1 = x^4 + x^3 + 2x^2 + x, \\ a_3(x) &= (x^3 + 1)(x^4 + x^3 + 2x^2 + x) - (x^2 + x + 1) \\ &= x^7 + x^6 + 2x^5 + 2x^4 + x^3 + x^2 - 1. \end{aligned}$$

Evaluate $a_{2010}(1)$.

19. A triangle ABC is inscribed in a semicircle of radius 5. If $AB = 10$, find the maximum value of s^2 where $s = AC + BC$.

20. Find the last two digits of $2011^{(2010^{2009})}$.

21. Your national football coach brought a squad of 18 players to the 2010 World Cup, consisting of 3 goalkeepers, 5 defenders, 5 midfielders and 5 strikers. Midfielders are versatile enough to play as both defenders and midfielders, while the other players can only play in their designated positions. How many possible teams of 1 goalkeeper, 4 defenders, 4 midfielders and 2 strikers can the coach field?

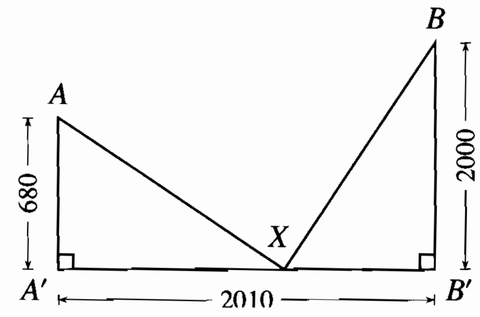
22. Given that $169(157-77x)^2+100(201-100x)^2=26(77x-157)(1000x-2010)$, find the value of x.

23. Evaluate
$$\frac{(2020^2-20100)(20100^2-100^2)(2000^2+20100)}{2010^6-10^6}.$$

24. When 15 is added to a number x, it becomes a square number. When 74 is subtracted from x, the result is again a square number. Find the number x.

25. Given that x and y are positive integers such that $56 \le x+y \le 59$ and $0.9 < \frac{x}{y} < 0.91$, find the value of $y^2 - x^2$.

26. Let AA' and BB' be two line segments which are perpendicular to $A'B'$. The lengths of AA', BB' and $A'B'$ are 680, 2000 and 2010 respectively. Find the minimal length of $AX + XB$ where X is a point between A' and B'.

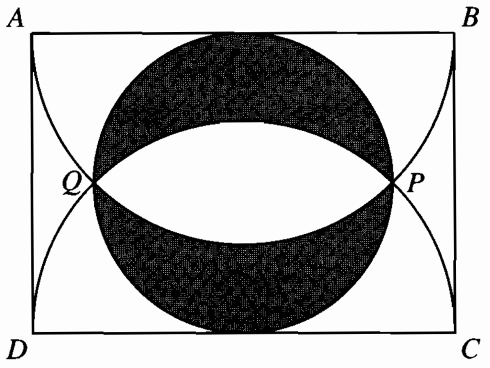


27. The product $1\times 2\times 3\times \cdots \times n$ is denoted by $n!$. For example $4! = 1\times 2\times 3\times 4 = 24$. Let $M = 1!\times 2!\times 3!\times 4!\times 5!\times 6!\times 7!\times 8!\times 9!$. How many factors of M are perfect squares?

28. Starting from any of the L's, the word *LEVEL* can be spelled by moving either up, down, left or right to an adjacent letter. If the same letter may be used twice in each spell, how many different ways are there to spell the word *LEVEL*?

		L		
	L	*E*	*L*	
L	*E*	V	*E*	*L*
	L	*E*	*L*	
		L		

29. Let $ABCD$ be a rectangle with $AB = 10$. Draw circles C_1 and C_2 with diameters AB and CD respectively. Let P, Q be the intersection points of C_1 and C_2. If the circle with diameter PQ is tangent to AB and CD, then what is the area of the shaded region?



30. Find the least prime factor of

$$1\underbrace{0000\cdots00}_{2010\text{-many}}1.$$

31. Consider the identity $1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$. If we set $P_1(x) = \frac{1}{2}x(x+1)$, then it is the unique polynomial such that for all positive integer n, $P_1(n) = 1 + 2 + \cdots + n$. In general, for each positive integer k, there is a unique polynomial $P_k(x)$ such that

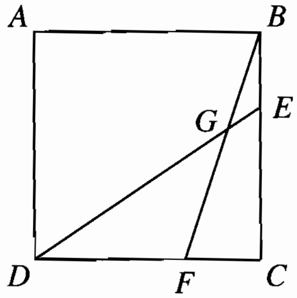
$$P_k(n) = 1^k + 2^k + 3^k + \cdots + n^k \quad \text{for each } n = 1, 2, \ldots.$$

Find the value of $P_{2010}(-\frac{1}{2})$.

32. Given that $ABCD$ is a square. Points E and F lie on the side BC and CD respectively. such that $BE = CF = \frac{1}{3}AB$. G is the intersection of BF and DE. If

$$\frac{\text{Area of } ABGD}{\text{Area of } ABCD} = \frac{m}{n}$$

is in its lowest term, find the value of $m + n$.



33. It is known that there is only one pair of positive integers a and b such that $a \leq b$ and $a^2 + b^2 + 8ab = 2010$. Find the value of $a + b$.

34. The digits of the number 123456789 can be rearranged to form a number that is divisible by 11. For example, 123475869, 459267831 and 987453126. How many such numbers are there?

35. Suppose the three sides of a triangular field are all integers, and its area equals the perimeter (in numbers). What is the largest possible area of the field?

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Junior Section, Round 2)

Saturday, 25 June 2010 **0930-1230**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

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1. Let the diagonals of the square $ABCD$ intersect at S and let P be the midpoint of AB. Let M be the intersection of AC and PD and N the intersection of BD and PC. A circle is incribed in the quadrilateral $PMSN$. Prove that the radius of the circle is $MP - MS$.

2. Find the sum of all the 5-digit integers which are not multiples of 11 and whose digits are 1, 3, 4, 7, 9.

3. Let $a_1, a_2, \ldots, a_n$ be positive integers, not necessarily distinct but with at least five distinct values. Suppose that for any $1 \leq i < j \leq n$, there exist k, ℓ, both different from i and j such that $a_i + a_j = a_k + a_\ell$. What is the smallest possible value of n?

4. A student divides an integer m by a positive integer n, where $n \leq 100$, and claims that

$$\frac{m}{n} = 0 \cdot 167a_1a_2 \cdots .$$

Show the student must be wrong.

5. The numbers $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2010}$ are written on a blackboard. A student chooses any two of the numbers, say x, y, erases them and then writes down $x + y + xy$. He continues to do this until only one number is left on the blackboard. What is this number?