Singapore Mathematical Society
Singapore Mathematical Olympiad (SMO) 2010
(Open Section, Round 1)

Wednesday, 2 June 2010 **0930-1200 hrs**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

1. Let S be the set of all integers n such that $\dfrac{8n^3 - 96n^2 + 360n - 400}{2n - 7}$ is an integer. Find the value of $\sum_{n \in S} |n|$.

2. Determine the largest value of x for which

$$\left|x^2 - 4x - 39601\right| \geq \left|x^2 + 4x - 39601\right|.$$

3. Given that

$$x = \lfloor 1^{1/3} \rfloor + \lfloor 2^{1/3} \rfloor + \lfloor 3^{1/3} \rfloor + \cdots + \lfloor 7999^{1/3} \rfloor,$$

find the value of $\lfloor \frac{x}{100} \rfloor$, where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y. (For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 30 \rfloor = 30$, $\lfloor -10.5 \rfloor = -11$.)

4. Determine the smallest positive integer C such that $\dfrac{6^n}{n!} \leq C$ for all positive integers n.

5. Let CD be a chord of a circle Γ_1 and AB a diameter of Γ_1 perpendicular to CD at N with $AN > NB$. A circle Γ_2 centred at C with radius CN intersects Γ_1 at points P and Q, and the segments PQ and CD intersect at M. Given that the radii of Γ_1 and Γ_2 are 61 and 60 respectively, find the length of AM.

6. Determine the minimum value of $\sum_{k=1}^{50} x_k$, where the summation is done over all possible positive numbers $x_1, \ldots, x_{50}$ satisfying $\sum_{k=1}^{50} \dfrac{1}{x_k} = 1$.

7. Find the sum of all positive integers p such that the expression $(x-p)(x-13)+4$ can be expressed in the form $(x+q)(x+r)$ for distinct integers q and r.

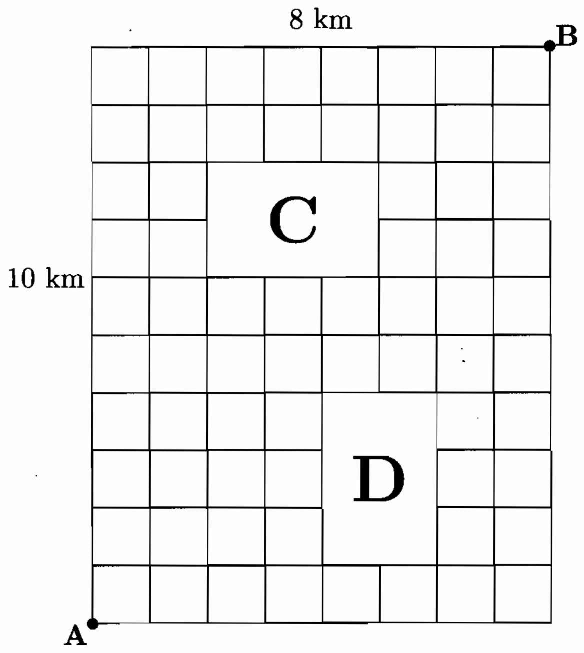
8. Let $p_k = 1 + \dfrac{1}{k} - \dfrac{1}{k^2} - \dfrac{1}{k^3}$, where k is a positive integer. Find the least positive integer n such that the product $p_2 p_3 \cdots p_n$ exceeds 2010.

9. Let B be a point on the circle centred at O with diameter AC and let D and E be the circumcentres of the triangles OAB and OBC respectively. Given that $\sin \angle BOC = \frac{4}{5}$ and $AC = 24$, find the area of the triangle BDE.

10. Let f be a real-valued function with the rule $f(x) = x^3 + 3x^2 + 6x + 14$ defined for all real value of x. It is given that a and b are two real numbers such that $f(a) = 1$ and $f(b) = 19$. Find the value of $(a+b)^2$.

11. If $\cot\alpha + \cot\beta + \cot\gamma = -\dfrac{4}{5}$, $\tan\alpha + \tan\beta + \tan\gamma = \dfrac{17}{6}$ and $\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = -\dfrac{17}{5}$, find the value of $\tan(\alpha+\beta+\gamma)$.

12. The figure below shows a road map connecting two shopping malls A and B in a certain city. Each side of the smallest square in the figure represents a road of distance 1km. Regions C and D represent two large residential estates in the town. Find the number of shortest routes to travel from A to B along the roads shown in the figure.



13. Let $a_1 = 1$, $a_2 = 2$ and for all $n \geq 2$, $a_{n+1} = \dfrac{2n}{n+1}a_n - \dfrac{n-1}{n+1}a_{n-1}$. It is known that $a_n > 2 + \dfrac{2009}{2010}$ for all $n \geq m$, where m is a positive integer. Find the least value of m.

14. It is known that

$$\sqrt{9 - 8\sin 50^\circ} = a + b\sin c^\circ$$

for exactly one set of positive integers (a, b, c), where $0 < c < 90$. Find the value of $\dfrac{b+c}{a}$.

15. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, find the value of $\lfloor \alpha^6 \rfloor$, where $\lfloor x \rfloor$ is the least positive integer not exceeding x.

16. If a positive integer cannot be written as the difference of two square numbers, then the integer is called a "cute" integer. For example, $1, 2$ and 4 are the first three "cute" integers. Find the 2010^{th} "cute" integer.

(Note: A *square number* is the square of a positive integer. As an illustration, $1, 4, 9$ and 16 are the first four square numbers.)

17. Let $\mathrm{f}(x)$ be a polynomial in x of degree 5. When $\mathrm{f}(x)$ is divided by $x-1$, $x-2$, $x-3$, $x-4$ and $x^2 - x - 1$, $\mathrm{f}(x)$ leaves a remainder of $3, 1, 7, 36$ and $x - 1$ respectively. Find the square of the remainder when $\mathrm{f}(x)$ is divided by $x + 1$.

18. Determine the number of ordered pairs of positive integers (a, b) satisfying the equation

$$100(a+b) = ab - 100.$$

(Note: As an illustration, $(1,2)$ and $(2,1)$ are considered as two distinct ordered pairs.)

19. Let $p = a^b + b^a$. If a, b and p are all prime, what is the value of p?

20. Determine the value of the following expression:

$$\left\lfloor \frac{11}{2010} \right\rfloor + \left\lfloor \frac{11 \times 2}{2010} \right\rfloor + \left\lfloor \frac{11 \times 3}{2010} \right\rfloor + \left\lfloor \frac{11 \times 4}{2010} \right\rfloor + \cdots + \left\lfloor \frac{11 \times 2009}{2010} \right\rfloor,$$

where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.

21. Numbers $1, 2, \ldots, 2010$ are placed on the circumference of a circle in some order. The numbers i and j, where $i \neq j$ and $i, j \in \{1, 2, \ \ldots, 2010\}$ form a *friendly* pair if

(i) i and j are not neighbours to each other, and

(ii) on one or both of the arcs connecting i and j along the circle, all numbers in between them are greater than both i and j.

Determine the minimal number of *friendly* pairs.

22. Let S be the set of all non-zero real-valued functions f defined on the set of all real numbers such that

$$\mathrm{f}(x^2 + yf(z)) = x\mathrm{f}(x) + z\mathrm{f}(y)$$

for all real numbers x, y and z. Find the maximum value of f(12345), where f $\in S$.

23. All possible 6-digit numbers, in each of which the digits occur in non-increasing order from left to right (e.g., 966541), are written as a sequence in increasing order (the first three 6-digit numbers in this sequence are 100000, 110000, 111000 and so on). If the 2010^{th} number in this sequence is denoted by p, find the value of $\lfloor \frac{p}{10} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

24. Find the number of permutations $a_1a_2a_3a_4a_5a_6$ of the six integers from 1 to 6 such that for all i from 1 to 5, a_{i+1} does not exceed a_i by 1.

25. Let

$$A = \left(\binom{2010}{0} - \binom{2010}{-1}\right)^2 + \left(\binom{2010}{1} - \binom{2010}{0}\right)^2 + \left(\binom{2010}{2} - \binom{2010}{1}\right)^2 + \cdots + \left(\binom{2010}{1005} - \binom{2010}{1004}\right)^2.$$

Determine the minimum integer s such that

$$sA \geq \binom{4020}{2010}.$$

(Note: For a given positive integer n, $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for $r = 0, 1, 2, 3, \cdots, n$; and for all other values of r, define $\binom{n}{r} = 0$.)

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Open Section, Round 2)

Saturday, 26 June 2010 **0900-1330**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

---

1. Let CD be a chord of a circle Γ_1 and AB a diameter of Γ_1 perpendicular to CD at N with $AN > NB$. A circle Γ_2 centred at C with radius CN intersects Γ_1 at points P and Q. The line PQ intersects CD at M and AC at K; and the extension of NK meets Γ_2 at L. Prove that PQ is perpendicular to AL.

2. Let a_n, b_n, $n = 1, 2, \ldots$ be two sequences of integers defined by $a_1 = 1, b_1 = 0$ and for $n \geq 1$,

$$a_{n+1} = 7a_n + 12b_n + 6$$
$$b_{n+1} = 4a_n + 7b_n + 3.$$

Prove that a_n^2 is the difference of two consecutive cubes.

3. Suppose that $a_1, \ldots, a_{15}$ are prime numbers forming an arithmetic progression with common difference $d > 0$. If $a_1 > 15$, prove that $d > 30,000$.

4. Let n be a positive integer. Find the smallest positive integer k with the property that for any colouring of the squares of a $2n \times k$ chessboard with n colours, there are 2 columns and 2 rows such that the 4 squares in their intersections have the same colour.

5. Let p be a prime number and let x, y, z be positive integers so that $0 < x < y < z < p$. Suppose that x^3, y^3 and z^3 have the same remainder when divided by p, show that $x^2 + y^2 + z^2$ is divisible by $x + y + z$.