Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Junior Section)

Tuesday, 27 May 2008 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answers on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.

For the other short questions, write your answer in the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

Multiple Choice Questions

1 How many zeroes does the product of 25^5, 150^4 and 2008^3 end with?

(A) 5
(B) 9
(C) 10
(D) 12
(E) 13

2 Given that $\sqrt{2x+y} + \sqrt{x^2-9} = 0$, find the value(s) of $y - x$.

(A) -9
(B) -6
(C) -9 or 9
(D) -3 or 3
(E) None of the above

3 The number of integers between 208 and 2008 ending with 1 is

(A) 101
(B) 163
(C) 179
(D) 180
(E) 200

4 The remainder when $7^{2008} + 9^{2008}$ is divided by 64 is

(A) 2
(B) 4
(C) 8
(D) 16
(E) 32

5 John has two 20 cent coins and three 50 cent coins in his pocket. He takes two coins out of his pocket, at random, one after the other without replacement. What is the probability that the total value of the two coins taken out is 70 cents?

(A) $\frac{6}{25}$
(B) $\frac{3}{10}$
(C) $\frac{12}{25}$
(D) $\frac{3}{5}$
(E) $\frac{13}{20}$

6 In the following sum, O represent the digit 0. A, B, X and Y each represents distinct digit. How many possible digits can A be?

$$\begin{array}{rl} & A\,O\,O\,B\,A\,O\,O\,B \\ + & B\,O\,O\,A\,B\,O\,O\,A \\ \hline & X\,X\,O\,X\,Y\,Y\,X\,O\,X\,X \end{array}$$

(A) 6
(B) 7
(C) 8
(D) 9
(E) 10

7 The least integer that is greater than $(2+\sqrt{3})^2$ is

(A) 13
(B) 14
(C) 15
(D) 16
(E) 17

8 Let x, y and z be non-negative numbers. Suppose $x + y = 10$ and $y + z = 8$. Let $S = x + z$. What is the sum of the maximum and the minimum value of S?

(A) 16
(B) 18
(C) 20
(D) 24
(E) 26

9 How many integer solutions (x, y, z) are there to the equation $xyz = 2008$?

(A) 30
(B) 60
(C) 90
(D) 120
(E) 150

10 The last two digits of 9^{2008} is

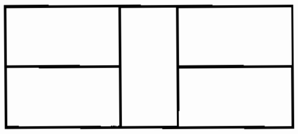
(A) 01
(B) 21
(C) 41
(D) 61
(E) 81

Short Questions

11 Find the remainder when $x^{2008} + 2008x + 2008$ is divided by $x + 1$.

12 Find the maximum value of $\sqrt{x - 144} + \sqrt{722 - x}$.

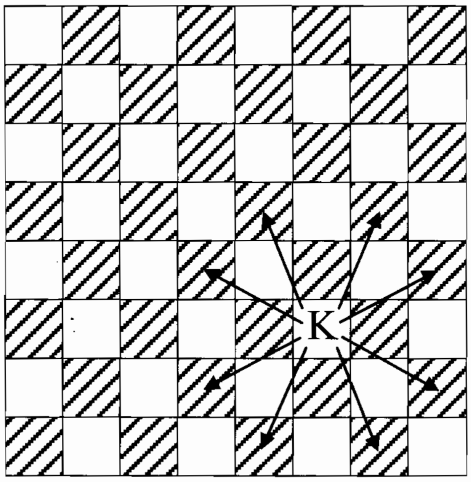
13 Five identical rectangles of area 8 cm^2 are arranged into a large rectangle as shown.



Find the perimeter of this large rectangle.

14 60 students were interviewed. Of these, 33 liked swimming and 36 liked soccer. Find the greatest possible number of students who neither liked swimming nor soccer.

15 As shown in the picture, the knight can move to any of the indicated squares of the 8×8 chessboard in 1 move. If the knight starts from the position shown, find the number of possible landing positions after 20 consecutive moves.



16 Given that $\alpha + \beta = 17$ and $\alpha\beta = 70$, find the value of $|\alpha - \beta|$.

17 Evaluate (in simplest form)

$$\sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{2008 + 2007\sqrt{\cdots}}}}.$$

18 Find the sum of all the positive integers less than 999 that are divisible by 15.

19 A brand of orange juice is available in shop A and shop B at an original price of \$2.00 per bottle. Shop A provides the "buy 4 get 1 free" promotion and shop B provides 15% discount if one buys 4 bottles or more. Find the minimum cost (in cents) if one wants to buy 13 bottles of the orange juice.

20 Anna randomly picked five integers from the following list

$$53,\ 62,\ 66,\ 68,\ 71,\ 82,\ 89$$

and discover that the average value of the five integers she picked is still an integer. If two of the integers she picked were 62 and 89, find the sum of the remaining three integers.

21 Suppose the equation $\left|\,|x-a|-b\,\right| = 2008$ has 3 distinct real roots and $a \neq 0$. Find the value of b.

22 Find the value of the integer n for the following pair of simultaneous equations to have no solution.

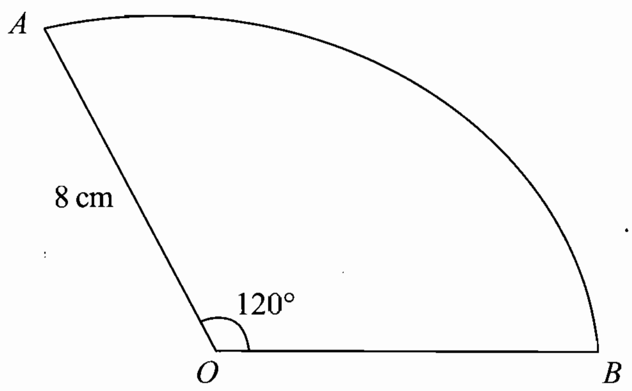
$$\begin{aligned} 2x &= 1 + ny, \\ nx &= 1 + 2y. \end{aligned}$$

23 There are 88 numbers $a_1, a_2, a_3, \ldots, a_{88}$ and each of them is either equals to -3 or -1. Given that $a_1^2 + a_2^2 + \ldots + a_{88}^2 = 280$, find the value of $a_1^4 + a_2^4 + \ldots + a_{88}^4$.

24 Find the value of $\dfrac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{3}-\frac{1}{4}} \times \dfrac{\frac{1}{4}-\frac{1}{5}}{\frac{1}{5}-\frac{1}{6}} \times \dfrac{\frac{1}{6}-\frac{1}{7}}{\frac{1}{7}-\frac{1}{8}} \times \ldots \times \dfrac{\frac{1}{2004}-\frac{1}{2005}}{\frac{1}{2005}-\frac{1}{2006}} \times \dfrac{\frac{1}{2006}-\frac{1}{2007}}{\frac{1}{2007}-\frac{1}{2008}}$.

25 An integer is chosen from the set $\{1, 2, 3, \ldots, 499, 500\}$. The probability that this integer is divisible by 7 or 11 is $\dfrac{m}{n}$ in its lowest terms. Find the value of $m + n$.

26 The diagram shows a sector OAB of a circle, centre O and radius 8 cm, in which $\angle AOB = 120°$. Another circle of radius r cm is to be drawn through the points O, A and B. Find the value of r.



27 The difference between the highest common factor and the lowest common multiple of x and 18 is 120. Find the value of x.

28 Let α and β be the roots of $x^2 - 4x + c = 0$, where c is a real number. If $-\alpha$ is a root of $x^2 + 4x - c = 0$, find the value of $\alpha\beta$.

29 Let m, n be integers such that $1 < m \le n$. Define

$$f(m, n) = \left(1 - \frac{1}{m}\right) \times \left(1 - \frac{1}{m+1}\right) \times \left(1 - \frac{1}{m+2}\right) \times \dots \times \left(1 - \frac{1}{n}\right).$$

If $S = f(2, 2008) + f(3, 2008) + f(4, 2008) + \dots + f(2008, 2008)$, find the value of $2S$.

30 Let a and b be the roots of $x^2 + 2000x + 1 = 0$ and let c and d be the roots of $x^2 - 2008x + 1 = 0$. Find the value of $(a + c)(b + c)(a - d)(b - d)$.

31 4 black balls, 4 white balls and 2 red balls are arranged in a row. Find the total number of ways this can be done if all the balls of the same colour do not appear in a consecutive block.

32 Given that n is a ten-digit number in the form $\overline{2007x2008y}$ where x and y can be any of the digits 0, 1, 2, ... , 9. How many such numbers n are there that are divisible by 33?

33 In triangle ABC, $AB = (b^2 - 1)$ cm, $BC = a^2$ cm and $AC = 2a$ cm, where a and b are positive integers greater than 1. Find the value of $a - b$.

34 How many positive integers n, where $10 \le n \le 100$, are there such that $\frac{n^2 - 9}{n^2 - 7}$ is a fraction in its lowest terms?

35 Let n be a positive integer such that $n^2 + 19n + 48$ is a perfect square. Find the value of n.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Junior Section, Round 2)

Saturday, 28 June 2008 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. In $\triangle ABC$, $\angle ACB = 90°$, D is the foot of the altitude from C to AB and E is the point on the side BC such that $CE = BD/2$. Prove that $AD + CE = AE$.

2. Let a, b, c, d be positive real numbers such that $cd = 1$. Prove that there is an integer n such that $ab \leq n^2 \leq (a+c)(b+d)$.

3. In the quadrilateral $PQRS$, A, B, C and D are the midpoints of the sides PQ, QR, RS and SP respectively, and M is the midpoint of CD. Suppose H is the point on the line AM such that $HC = BC$. Prove that $\angle BHM = 90°$.

4. Six distinct positive integers a, b, c, d, e, f are given. Jack and Jill calculated the sums of each pair of these numbers. Jack claims that he has 10 prime numbers while Jill claims that she has 9 prime numbers among the sums. Who has the correct claim?

5. Determine all primes p such that
$$5^p + 4 \cdot p^4$$
is a perfect square, i.e., the square of an integer.