Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Senior Section)

Tuesday, 1 June 2010 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D *or* E) *corresponding to the correct answer.*

For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Find the value of $\dfrac{(1\times 2\times 3)+(2\times 4\times 6)+(3\times 6\times 9)+\cdots+(335\times 670\times 1005)}{(1\times 3\times 6)+(2\times 6\times 12)+(3\times 9\times 18)+\cdots+(335\times 1005\times 2010)}$.

(A) $\frac{1}{3}$
(B) $\frac{2}{3}$
(C) $\frac{1}{6}$
(D) $\frac{1}{2}$
(E) $\frac{4}{9}$

2. If a, b, c and d are real numbers such that

$$\frac{b+c+d}{a}=\frac{a+c+d}{b}=\frac{a+b+d}{c}=\frac{a+b+c}{d}=r,$$

find the value of r.

(A) 3
(B) 1
(C) -1
(D) 3 or 1
(E) 3 or -1

3. If $0<x<\frac{\pi}{2}$ and $\sin x-\cos x=\frac{\pi}{4}$ and $\tan x+\frac{1}{\tan x}=\frac{a}{b-\pi^c}$, where a, b and c are positive integers, find the value of $a+b+c$.

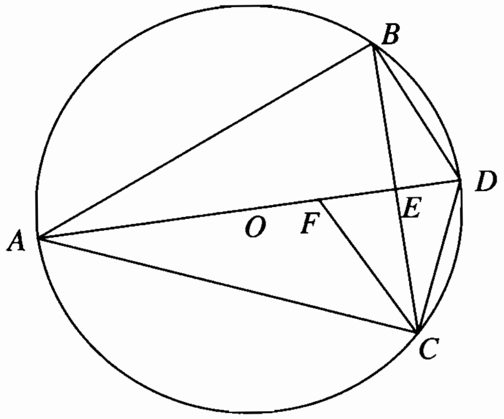
(A) 8
(B) 32
(C) 34
(D) 48
(E) 50

4. Find the value of $\sqrt{14^3+15^3+16^3+\cdots+24^3+25^3}$.

(A) 104
(B) 224
(C) 312
(D) 336
(E) 676

5. In the figure below, ABC is an isosceles triangle inscribed in a circle with centre O and diameter AD, with $AB = AC$. AD intersects BC at E, and F is the midpoint of OE. Given that BD is parallel to FC and $BC = 2\sqrt{5}$ cm, find the length of CD in cm.

(A) $\dfrac{3\sqrt{5}}{2}$
(B) $\sqrt{6}$
(C) $2\sqrt{3}$
(D) $\sqrt{7}$
(E) $2\sqrt{6}$



6. Find the number of ordered pairs (x, y), where x is an integer and y is a perfect square, such that $y = (x-90)^2 - 4907$.

(A) 0
(B) 1
(C) 2
(D) 3
(E) 4

7. Let $S = \{1, 2, 3, \ldots, 9, 10\}$. A non-empty subset of S is considered "Good" if the number of even integers in the subset is more than or equal to the number of odd integers in the same subset. For example, the subsets $\{4, 8\}$, $\{3, 4, 7, 8\}$ and $\{1, 3, 6, 8, 10\}$ are "Good". How many subsets of S are "Good"?

(A) 482
(B) 507
(C) 575
(D) 637
(E) 667

8. If the graph of a quadratic function $f(x)=ax^2+bx+c$ $(a \neq 0)$ passes through two distinct points (r, k) and (s, k), what is $f(r+s)$?

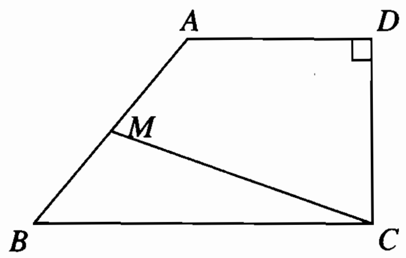
(A) $2k$
(B) c
(C) $k-c$
(D) $2k-c$
(E) None of the above

9. Find the number of positive integers $k < 100$ such that $2(3^{6n})+k(2^{3n+1})-1$ is divisible by 7 for any positive integer n.

(A) 10
(B) 12
(C) 13
(D) 14
(E) 16

10. Let $ABCD$ be a trapezium with AD parallel to BC and $\angle ADC = 90^\circ$, as shown in the figure below. Given that M is the midpoint of AB with $CM = \frac{13}{2}$ cm and $BC + CD + DA = 17$ cm, find the area of the trapezium $ABCD$ in cm^2.

(A) 26
(B) 28
(C) 30
(D) 33
(E) 35



Short Questions

11. The area of a rectangle remains unchanged when either its length is increased by 6 units and width decreased by 2 units, or its length decreased by 12 units and its width increased by 6 units. If the perimeter of the original rectangle is x units, find the value of x.

12. For $r = 1, 2, 3, \ldots$, let $u_r = 1+2+3+\ldots+r$. Find the value of

$$\frac{1}{\left(\frac{1}{u_1}\right)}+\frac{2}{\left(\frac{1}{u_1}+\frac{1}{u_2}\right)}+\frac{3}{\left(\frac{1}{u_1}+\frac{1}{u_2}+\frac{1}{u_3}\right)}+\cdots+\frac{100}{\left(\frac{1}{u_1}+\frac{1}{u_2}+\cdots+\frac{1}{u_{100}}\right)}.$$

13. If $2010! = M\times 10^k$, where M is an integer not divisible by 10, find the value of k.

14. If $a > b > 1$ and $\frac{1}{\log_a b}+\frac{1}{\log_b a}=\sqrt{1229}$, find the value of $\frac{1}{\log_{ab} b}-\frac{1}{\log_{ab} a}$.

15. For any real number x, let $\lceil x\rceil$ denote the smallest integer that is greater than or equal to x and $\lfloor x\rfloor$ denote the largest integer that is less than or equal to x (for example, $\lceil 1.23\rceil = 2$ and $\lfloor 1.23\rfloor = 1$). Find the value of

$$\sum_{k=1}^{2010}\left\lceil \frac{2010}{k}-\left\lfloor\frac{2010}{k}\right\rfloor\right\rceil.$$

16. Let $f(x)=\frac{x^{2010}}{x^{2010}+(1-x)^{2010}}$. Find the value of

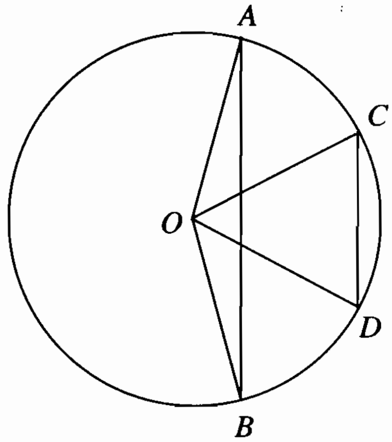
$$f\left(\frac{1}{2011}\right)+f\left(\frac{2}{2011}\right)+f\left(\frac{3}{2011}\right)+\ldots+f\left(\frac{2010}{2011}\right).$$

17. If a, b and c are positive real numbers such that

$$ab + a + b = bc + b + c = ca + c + a = 35,$$

find the value of $(a + 1)(b + 1)(c + 1)$.

18. In the figure below, AB and CD are parallel chords of a circle with centre O and radius r cm. It is given that $AB = 46$ cm, $CD = 18$ cm and $\angle AOB = 3 \times \angle COD$. Find the value of r.



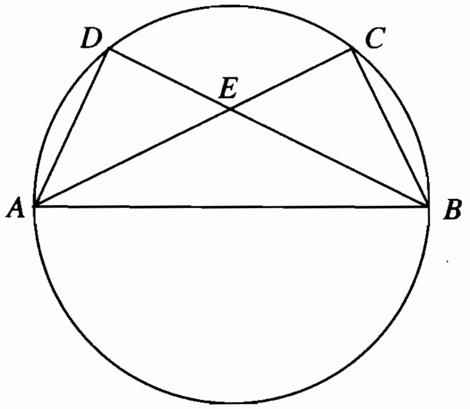
19. Find the number of ways that 2010 can be written as a sum of one or more positive integers in non-decreasing order such that the difference between the last term and the first term is at most 1.

20. Find the largest possible value of n such that there exist n consecutive positive integers whose sum is equal to 2010.

21. Determine the number of pairs of positive integers n and m such that

$$1! + 2! + 3! + \cdots + n! = m^2.$$

22. The figure below shows a circle with diameter AB. C and D are points on the circle on the same side of AB such that BD bisects $\angle CBA$. The chords AC and BD intersect at E. It is given that $AE = 169$ cm and $EC = 119$ cm. If $ED = x$ cm, find the value of x.



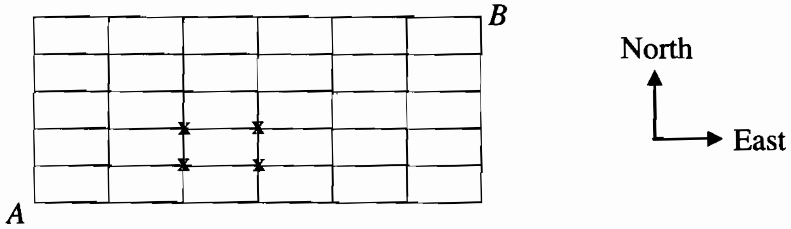
23. Find the number of ordered pairs (m, n) of positive integers m and n such that $m + n = 190$ and m and n are relatively prime.

24. Find the least possible value of $f(x) = \dfrac{9}{1+\cos 2x} + \dfrac{25}{1-\cos 2x}$, where x ranges over all real numbers for which $f(x)$ is defined.

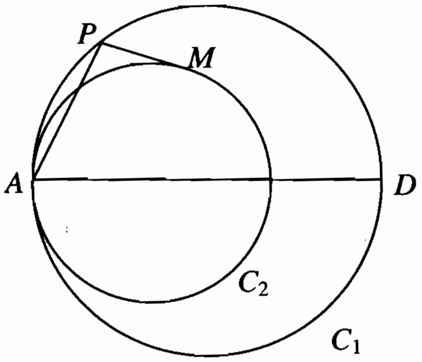
25. Find the number of ways of arranging 13 identical blue balls and 5 identical red balls on a straight line such that between any 2 red balls there is at least 1 blue ball.

26. Let $S = \{1, 2, 3, 4, \ldots, 100000\}$. Find the least possible value of k such that any subset A of S with $|A| = 2010$ contains two distinct numbers a and b with $|a - b| \le k$.

27. Find the number of ways of traveling from A to B, as shown in the figure below, if you are only allowed to walk east or north along the grid, and avoiding all the 4 points marked x.



28. Two circles C_1 and C_2 of radii 10 cm and 8 cm respectively are tangent to each other internally at a point A. AD is the diameter of C_1 and P and M are points on C_1 and C_2 respectively such that PM is tangent to C_2, as shown in the figure below. If $PM = \sqrt{20}$ cm and $\angle PAD = x^\circ$, find the value of x.



29. Let a, b and c be integers with $a > b > c > 0$. If b and c are relatively prime, $b + c$ is a multiple of a, and $a + c$ is a multiple of b, determine the value of abc.

30. Find the number of subsets $\{a, b, c\}$ of $\{1, 2, 3, 4, \ldots, 20\}$ such that $a < b - 1 < c - 3$.

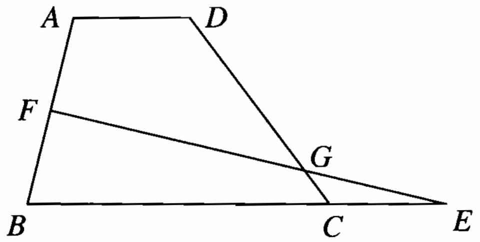
31. Let $f(n)$ denote the number of 0's in the decimal representation of the positive integer n. For example, $f(10001123) = 3$ and $f(1234567) = 0$. Let

$$M = f(1) \times 2^{f(1)} + f(2) \times 2^{f(2)} + f(3) \times 2^{f(3)} + \cdots + f(99999) \times 2^{f(99999)}.$$

Find the value of $M - 100000$.

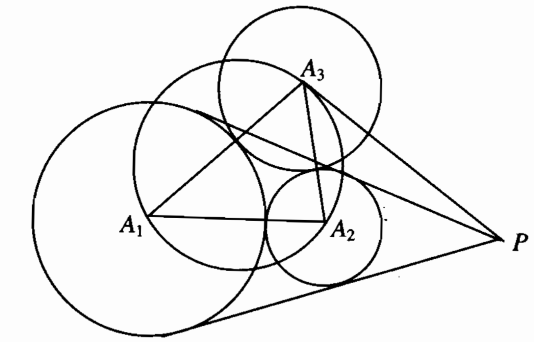
32. Determine the odd prime number p such that the sum of digits of the number $p^4 - 5p^2 + 13$ is the smallest possible.

33. The figure below shows a trapezium $ABCD$ in which $AD \parallel BC$ and $BC = 3AD$. F is the midpoint of AB and E lies on BC extended so that $BC = 3CE$. The line segments EF and CD meet at the point G. It is given that the area of triangle GCE is 15 cm^2 and the area of trapezium $ABCD$ is k cm^2. Find the value of k.



34. Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial in x where the coefficients $a_0, a_1, a_2, \ldots, a_n$ are non-negative integers. If $P(1) = 25$ and $P(27) = 1771769$, find the value of $a_0 + 2a_1 + 3a_2 + \cdots + (n+1)a_n$.

35. Let three circles Γ_1, Γ_2, Γ_3 with centres A_1, A_2, A_3 and radii r_1, r_2, r_3 respectively be mutually tangent to each other externally. Suppose that the tangent to the circumcircle of the triangle $A_1A_2A_3$ at A_3 and the two external common tangents of Γ_1 and Γ_2 meet at a common point P, as shown in the figure below. Given that $r_1 = 18$ cm, $r_2 = 8$ cm and $r_3 = k$ cm, find the value of k.



Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2010

(Senior Section, Round 2)

Saturday, 25 June 2010 **0930-1230**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

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1. In the triangle ABC with $AC > AB$, D is the foot of the perpendicular from A onto BC and E is the foot of the perpendicular from D onto AC. Let F be the point on the line DE such that $EF \cdot DC = BD \cdot DE$. Prove that AF is perpendicular to BF.

2. The numbers $\frac{1}{1}, \frac{1}{2}, \ldots, \frac{1}{2010}$ are written on a blackboard. A student chooses any two of the numbers, say x, y, erases them and then writes down $x + y + xy$. He continues to do this until only one number is left on the blackboard. What is this number?

3. Given $a_1 \geq 1$ and $a_{k+1} \geq a_k + 1$ for all $k = 1, 2, \cdots, n$, show that

$$a_1^3 + a_2^3 + \cdots + a_n^3 \geq (a_1 + a_2 + \cdots + a_n)^2.$$

4. An infinite sequence of integers, $a_0, a_1, a_3 \ldots$, with $a_0 > 0$, has the property that for any $n \geq 0$, $a_{n+1} = a_n - b_n$, where b_n is the number having the same sign as a_n, but having the digits written in the reverse order. For example if $a_0 = 1210$, $a_1 = 1089$ and $a_2 = -8712$, etc. Find the smallest value of a_0 so that $a_n \neq 0$ for all $n \geq 1$.

5. Let p be a prime number and let $a_1, a_2, \ldots, a_k$ be distinct integers chosen from $1, 2, \ldots, p-1$. For $1 \leq i \leq k$, let $r_i^{(n)}$ denote the remainder of the integer na_i upon division by p, so $0 \leq r_i^{(n)} < p$. Define

$$S = \{n : 1 \leq n \leq p-1,\ r_1^{(n)} < \cdots < r_k^{(n)}\}.$$

Show that S has less than $\frac{2p}{k+1}$ elements.