Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Open Section, Round 1)

Wednesday, 28 May 2008 **0930-1200**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

1. Determine the number of three-element subsets of the set $\{1, 2, 3, 4, \ldots, 120\}$ for which the sum of the three elements is a multiple of 3.

2. There are 10 students taking part in a mathematics competition. After the competition, they discover that each of them solves exactly 3 problems and any 2 of them solve at least 1 common problem. What is the minimum number of students who solve a common problem which is solved by most students?

3. Evaluate the sum
$$\sum_{n=1}^{6237} \lfloor \log_2(n) \rfloor,$$
where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

4. Determine the number of positive integer divisors of 998^{49999} that are not the divisors of 998^{49998}.

5. Let $p(x)$ be a polynomial with real coefficients such that for all real x,
$$2(1 + p(x)) = p(x-1) + p(x+1)$$
and $p(0) = 8, p(2) = 32$. Determine the value of $p(40)$.

6. In the triangle ABC, $AC = 2BC$, $\angle C = 90°$ and D is the foot of the altitude from C onto AB. A circle with diameter AD intersects the segment AC at E. Find $AE : EC$.

7. In the triangle ABC, $AB = 8, BC = 7$ and $CA = 6$. Let E be the point on BC such that $\angle BAE = 3\angle EAC$. Find $4AE^2$.

8. In the triangle ABC, the bisectors of $\angle A$ and $\angle B$ meet at the incentre I, the extension of AI meets the circumcircle of triangle ABC at D. Let P be the foot of the perpendicular from B onto AD, and Q a point on the extension of AD such that $ID = DQ$. Determine the value of $(BQ \times IB)/(BP \times ID)$.

9. In a convex quadrilateral $ABCD$, $\angle BAC = \angle CAD$, $\angle ABC = \angle ACD$, the extensions of AD and BC meet at E, and the extensions of AB and DC meet at F. Determine the value of

$$\frac{AB \cdot DE}{BC \cdot CE}.$$

10. For any positive integer n, let N_n be the set of integers from 1 to n, i.e., $N_n = \{1, 2, 3, \cdots, n\}$. Now assume that $n \geq 10$. Determine the maximum value of n such that the following inequality

$$\mathrm{i\ n}_{\substack{a,b\in A \\ a\neq b}} |a-b| \leq 10$$

holds for each $A \subseteq N_n$ with $|A| \geq 10$.

11. How many four-digit numbers greater than 5000 can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if only the digit 4 may be repeated?

12. Three girls A, B and C, and nine boys are to be lined up in a row. Let n be the number of ways this can be done if B must lie between A and C, and A, B must be separated by exactly 4 boys. Determine $\lfloor n/7! \rfloor$.

13. Determine the number of 4-element subsets $\{a, b, c, d\}$ of $\{1, 2, 3, 4, \cdots, 20\}$ such that $a + b + c + d$ is divisible by 3.

14. Find how many three digit numbers, lying between 100 and 999 inclusive, have two and only two consecutive digits identical.

15. Find the maximum natural number which are divisible by 30 and have exactly 30 different positive divisors.

16. Determine the number of 0's at the end of the value of the product $1 \times 2 \times 3 \times 4 \times \cdots \times 2008$.

17. Let a_k be the coefficient of x^k in the expansion of $(1 + 2x)^{100}$, where $0 \leq k \leq 100$. Find the number of integers $r : 0 \leq r \leq 99$ such that $a_r < a_{r+1}$.

18. Let a_k be the coefficient of x^k in the expansion of

$$(x+1) + (x+1)^2 + (x+1)^3 + (x+1)^4 + \cdots + (x+1)^{99}.$$

Determine the value of $\lfloor a_4/a_3 \rfloor$.

19. Let a, b, c, d, e be five numbers satisfying the following conditions:

$$a + b + c + d + e = 0, \quad \text{and}$$

$$abc + abd + abe + acd + ace + ade + bcd + bce + bde + cde = 2008.$$

Find the value of $a^3 + b^3 + c^3 + d^3 + e^3$.

20. Let $a_1, a_2, \ldots$ be a sequence of rational numbers such that $a_1 = 2$ and for $n \geq 1$

$$a_{n+1} = \frac{1 + a_n}{1 - a_n}.$$

Determine $30 \times a_{2008}$.

21. Find the number of eight-digit integers comprising the eight digits from 1 to 8 such that $(i+1)$ does not immediately follow i for all i that runs from 1 to 7.

22. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$ where a_0, a_1, a_2, a_3 and a_4 are constants with $a_4 \neq 0$. When divided by $x - 2003$, $x - 2004$, $x - 2005$, $x - 2006$ and $x - 2007$, respectively, $f(x)$ leaves a remainder of $24, -6, 4, -6$ and 24. Find the value of $f(2008)$.

23. Find the number of 10-letter permutations comprising 4 a's, 3 b's, 3 c's such that no two adjacent letters are identical.

24. Let $f(x) = x^3 + 3x + 1$, where x is a real number. Given that the inverse function of f exists and is given by

$$f^{-1}(x) = \left(\frac{x - a + \sqrt{x^2 - bx + c}}{2}\right)^{1/3} + \left(\frac{x - a - \sqrt{x^2 - bx + c}}{2}\right)^{1/3}$$

where a, b and c are positive constants, find the value of $a + 10b + 100c$.

25. Between 1 and 8000 inclusive, find the number of integers which are divisible by neither 14 nor 21 but divisible by either 4 or 6.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Open Section, Round 2)

Saturday, 5 July 2008 **0900-1330**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Find all pairs of positive integers (n, k) so that $(n+1)^k - 1 = n!$.

2. In the acute triangle ABC, M is a point in the interior of the segment AC and N is a point on the extension of the segment AC such that $MN = AC$. Let D and E be the feet of the perpendiculars from M and N onto the lines BC and AB respectively. Prove that the orthocentre of $\triangle ABC$ lies on the cicumcircle of $\triangle BED$.

3. Let n, m be positive integers with $m > n \geq 5$ and with m depending on n. Consider the sequence $a_1, a_2, \ldots, a_m$ where

$$\begin{aligned} a_i &= i && \text{for } i = 1, \ldots, n \\ a_{n+j} &= a_{3j} + a_{3j-1} + a_{3j-2} && \text{for } j = 1, \ldots, m-n \end{aligned}$$

with $m - 3(m-n) = 1$ or 2, i.e., $a_m = a_{m-k} + a_{m-k-1} + a_{m-k-2}$ where $k = 1$ or 2. (Thus if $n = 5$, the sequence is $1, 2, 3, 4, 5, 6, 15$ and if $n = 8$, the sequence is $1, 2, 3, 4, 5, 6, 7, 8, 6, 15, 21$.) Find $S = a_1 + a_2 + \cdots + a_m$ if (i) $n = 2007$, (ii) $n = 2008$.

4. Let $0 < a, b < \pi/2$. Show that

$$\frac{5}{\cos^2 a} + \frac{5}{\sin^2 a \sin^2 b \cos^2 b} \geq 27 \cos a + 36 \sin a.$$

5. Consider a 2008×2008 chess board. Let M be the smallest number of rectangles that can be drawn on the chess board so that the sides of every cell of the board is contained in the sides of one of the rectangles. Find the value of M. (For example, for the 2×3 chess board, the value of M is 3.)