Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 1)

Wednesday, 3 June 2009 **0930 – 1200**

Instructions to contestants

1. *Answer ALL 25 questions.*
2. *Write your answers in the answer sheet provided and shade the appropriate bubbles below your answers.*
3. *No steps are needed to justify your answers.*
4. *Each question carries 1 mark.*
5. *No calculators are allowed.*

1. The expression $1000 \sin 10^0 \cos 20^0 \cos 30^0 \cos 40^0$ can be simplified as $a \sin b^0$, where a and b are positive integers with $0 < b < 90$. Find the value of $100a + b$.

2. Let A_1, A_2, A_3, A_4, A_5 and A_6 be six points on a circle in this order such that $\widehat{A_1A_2} = \widehat{A_2A_3}$, $\widehat{A_3A_4} = \widehat{A_4A_5}$ and $\widehat{A_5A_6} = \widehat{A_6A_1}$, where $\widehat{A_1A_2}$ denotes the arc length of the arc A_1A_2 etc. It is also known that $\angle A_1A_3A_5 = 72°$. Find the size of $\angle A_4A_6A_2$ in degrees.

3. Let $P_1, P_2, \ldots, P_{41}$ be 41 distinct points on the segment BC of a triangle ABC, where AB = AC = 7. Evaluate the sum $\sum_{i=1}^{41}(\mathrm{AP}_i^2 + \mathrm{P}_i\mathrm{B} \cdot \mathrm{P}_i\mathrm{C})$.

4. Determine the largest value of x for which

$$|x^2 - 11x + 24| + |2x^2 + 6x - 56| = |x^2 + 17x - 80|.$$

5. Let $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$ be a polynomial in x where a_0, a_1, a_2, a_3, a_4 are constants and $a_5 = 7$. When divided by $x - 2004$, $x - 2005$, $x - 2006$, $x - 2007$ and $x - 2008$, $f(x)$ leaves a remainder of 72, -30, 32, -24 and 24 respectively. Find the value of $f(2009)$.

6. Find the value of $\dfrac{\sin 80^o}{\sin 20^o} - \dfrac{\sqrt{3}}{2\sin 80^o}$.

7. Determine the number of 8-digit positive integers such that after deleting any one digit, the remaining 7-digit number is divisible by 7.

8. It is given that $\sqrt{a}-\sqrt{b}=20$, where a and b are real numbers. Find the maximum possible value of $a - 5b$.

9. Let ABC be a triangle with sides AB = 7, BC = 8 and AC = 9. A unique circle can be drawn touching the side AC and the lines BA produced and BC produced. Let D be the centre of this circle. Find the value of BD^2.

10. If $x=\frac{1}{2}\left(\sqrt[3]{2009}-\frac{1}{\sqrt[3]{2009}}\right)$, find the value of $\left(x+\sqrt{1+x^2}\right)^3$.

11. Let $S = \{1, 2, 3, \ldots\ldots, 30\}$. Determine the number of vectors (x, y, z, w) with $x, y, z, w \in S$ such that $x < w$ and $y < z < w$.

12. Let f(n) be the number of 0's in the decimal representation of the positive integer n. For example, f(10001123) = 3 and f(1234567) = 0. Find the value of

$$f(1) + f(2) + f(3) + \ldots\ldots + f(99999).$$

13. It is given that k is a positive integer not exceeding 99. There are no natural numbers x and y such that $x^2 - ky^2 = 8$. Find the difference between the maximum and minimum possible values of k.

14. Let $S = \{1, 2, 3, 4, \ldots\ldots, 16\}$. In each of the following subsets of S,

$$\{6\},\ \{1, 2, 3\},\ \{5, 7, 9, 10, 11, 12\},\ \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

the sum of all the elements is a multiple of 3. Find the total number of non-empty subsets A of S such that the sum of all elements in A is a multiple of 3.

15. A function $f : \mathbf{R} \rightarrow \mathbf{R}$ satisfies the relation $f(x)f(y) = f(2xy + 3) + 3f(x + y) - 3f(x) + 6x$, where $x, y \in \mathbf{R}$. Find the value of f(2009).

16. Let $\{a_n\}$ be a sequence of positive integers such that $a_1 = 1$, $a_2 = 2009$ and for $n \geq 1$, $a_{n+2}a_n - a_{n+1}^2 - a_{n+1}a_n = 0$. Determine the value of $\frac{a_{993}}{100a_{991}}$.

17. Determine the number of ways of tiling a 4x9 rectangle by tiles of size 1x2.

18. Find the number of 7-digit positive integers such that the digits from left to right are non-increasing. (Examples of 7-digit non-increasing numbers are 9998766 and 5555555; An example of a number that is NOT non-increasing is 7776556)

19. Determine the largest prime number less than 5000 of the form $a^n - 1$, where a and n are positive integers, and n is greater than 1.

20. Determine the least constant M such that

$$\frac{x_1}{x_1+x_2}+\frac{x_2}{x_2+x_3}+\frac{x_3}{x_3+x_4}+\cdots+\frac{x_{2009}}{x_{2009}+x_1} < M,$$

for any positive real numbers $x_1, x_2, x_3, \ldots, x_{2009}$.

21. Six numbers are randomly selected from the integers 1 to 45 inclusive. Let p be the probability that at least three of the numbers are consecutive. Find the value of $\lfloor 1000p \rfloor$. (Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x).

22. Evaluate $\displaystyle\sum_{k=0}^{\infty} \frac{2}{\pi} \tan^{-1}\left(\frac{2}{(2k+1)^2}\right)$.

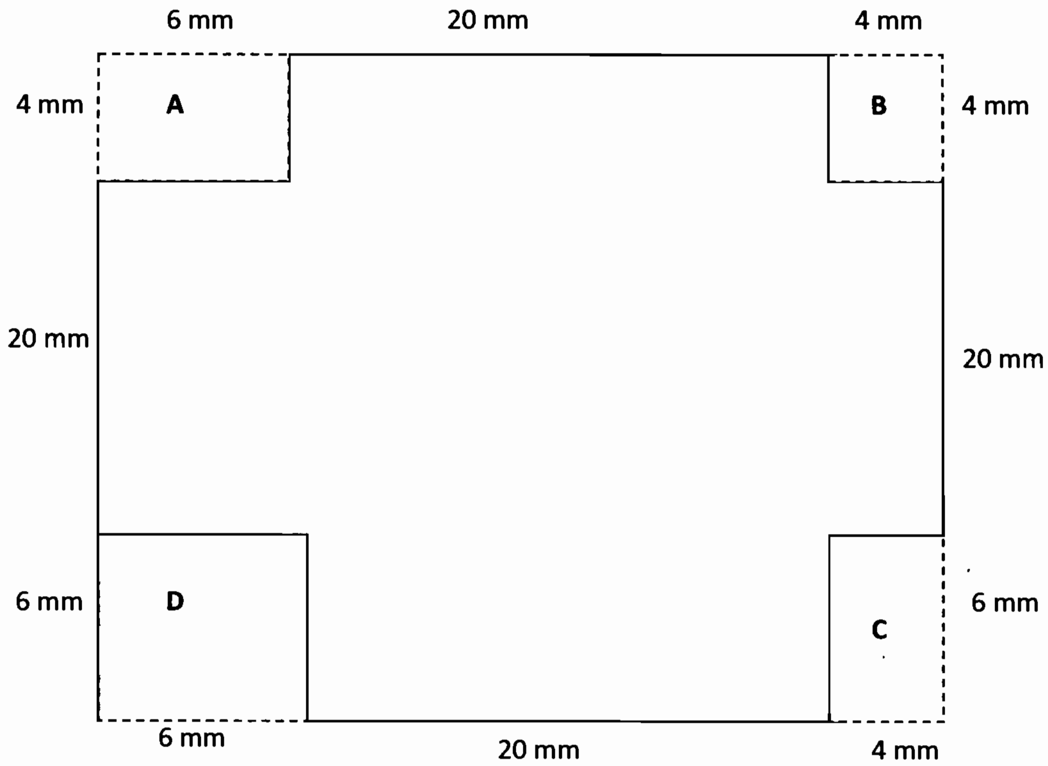
23. Determine the largest prime factor of the sum $\displaystyle\sum_{k=1}^{11} k^5$.

24. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that $x_1 = 3, x_2 = 24$ and

$$x_{n+2} = \frac{1}{4}x_{n+1} + \frac{3}{4}x_n$$

for every positive integers n. Determine the value of $\lim\limits_{n\to\infty} x_n$.

25. A square piece of graph paper of side length 30 mm contains 900 smallest squares each of side length 1mm each. Its four rectangular corners, denoted by A, B, C, D in clockwise order, are cut away from the square piece of graph paper. The resultant graph paper, which has the shape of a cross, is shown in the figure below. Let N denote the total number of rectangles, **excluding** all the squares which are contained in the resultant graph paper. Find the value of $\frac{1}{10}N$.



Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Open Section, Round 2)

Saturday, 4 July 2009 **0900-1330**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let O be the center of the circle inscribed in a rhombus $ABCD$. Points E, F, G, H are chosen on sides AB, BC, CD and DA respectively so that EF and GH are tangent to the inscribed circle. Show that EH and FG are parallel.

2. A palindromic number is a number which is unchanged when the order of its digits is reversed. Prove that the arithmetic progression 18, 37, ... contains infinitely many palindromic numbers.

3. For k a positive integer, define A_n for $n = 1, 2, \ldots,$ by

$$A_{n+1} = \frac{nA_n + 2(n+1)^{2k}}{n+2}, \quad A_1 = 1.$$

Prove that A_n is an integer for all $n \geq 1$, and A_n is odd if and only if $n \equiv 1$ or 2 (mod 4).

4. Find the largest constant C such that

$$\sum_{i=1}^{4}(x_i + \frac{1}{x_i})^3 \geq C$$

for all positive real numbers $x_1, \cdots, x_4$ such that

$$x_1^3 + x_3^3 + 3x_1x_3 = x_2 + x_4 = 1.$$

5. Find all integers x, y and z with $2 \leq x \leq y \leq z$ such that

$$xy \equiv 1 \pmod z, \quad xz \equiv 1 \pmod y, \quad yz \equiv 1 \pmod x.$$