Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section)

Tuesday, 2 June 2009 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D *or* E) *corresponding to the correct answer.*

For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Suppose that π is a plane and A and B are two points on the plane π. If the distance between A and B is 33 cm, how many lines are there in the plane such that the distance between each line and A is 7 cm and the distance between each line and B is 26 cm respectively?

   (A) 1
   (B) 2
   (C) 3
   (D) 4
   (E) Infinitely many

2. Let $y = (17 - x)(19 - x)(19 + x)(17 + x)$, where x is a real number. Find the smallest possible value of y.

   (A) –1296
   (B) –1295
   (C) –1294
   (D) –1293
   (E) –1292

3. If two real numbers a and b are randomly chosen from the interval (0, 1), find the probability that the equation $x^2 - \sqrt{a}\,x + b = 0$ has real roots.

   (A) $\frac{1}{8}$
   (B) $\frac{5}{16}$
   (C) $\frac{3}{16}$
   (D) $\frac{1}{4}$
   (E) $\frac{1}{3}$

4. If x and y are real numbers for which $|x| + x + 5y = 2$ and $|y| - y + x = 7$, find the value of $x + y$.

   (A) –3
   (B) –1
   (C) 1
   (D) 3
   (E) 5

5. In a triangle ABC, $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$. Find the value of $\cos C$.

(A) $\frac{56}{65}$ or $\frac{16}{65}$

(B) $\frac{56}{65}$

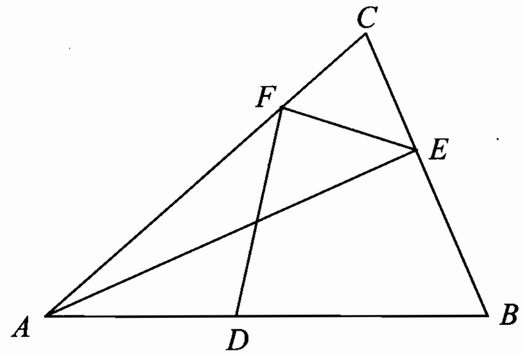
(C) $\frac{16}{65}$

(D) $-\frac{56}{65}$

(E) $\frac{56}{65}$ or $-\frac{16}{65}$

6. The area of a triangle ABC is 40 cm^2. Points D, E and F are on sides AB, BC and CA respectively, as shown in the figure below. If $AD = 3$ cm, $DB = 5$ cm, and the area of triangle ABE is equal to the area of quadrilateral $DBEF$, find the area of triangle AEC in cm^2.

(A) 11
(B) 12
(C) 13
(D) 14
(E) 15



7. Find the value of $\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\cdots+\frac{22}{20!+21!+22!}$.

(A) $1-\frac{1}{24!}$

(B) $\frac{1}{2}-\frac{1}{23!}$

(C) $\frac{1}{2}-\frac{1}{22!}$

(D) $1-\frac{1}{22!}$

(E) $\frac{1}{2}-\frac{1}{24!}$

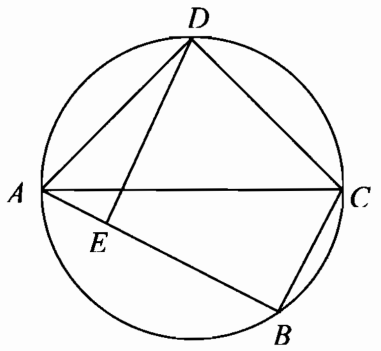
8. There are eight envelopes numbered 1 to 8. Find the number of ways in which 4 identical red buttons and 4 identical blue buttons can be put in the envelopes such that each envelope contains exactly one button, and the sum of the numbers on the envelopes containing the red buttons is more than the sum of the numbers on the envelopes containing the blue buttons.

(A) 35
(B) 34
(C) 32
(D) 31
(E) 62

9. Determine the number of acute-angled triangles (i.e., all angles are less than 90°) in which all angles (in degrees) are positive integers and the largest angle is three times the smallest angle.

(A) 3
(B) 4
(C) 5
(D) 6
(E) 7

10. Let *ABCD* be a quadrilateral inscribed in a circle with diameter *AC*, and let *E* be the foot of perpendicular from *D* onto *AB*, as shown in the figure below. If $AD = DC$ and the area of quadrilateral *ABCD* is 24 cm^2, find the length of *DE* in cm.



(A) $3\sqrt{2}$
(B) $2\sqrt{6}$
(C) $2\sqrt{7}$
(D) $4\sqrt{2}$
(E) 6

Short Questions

11. Find the number of positive divisors of $(2008^3 + (3 \times 2008 \times 2009) + 1)^2$.

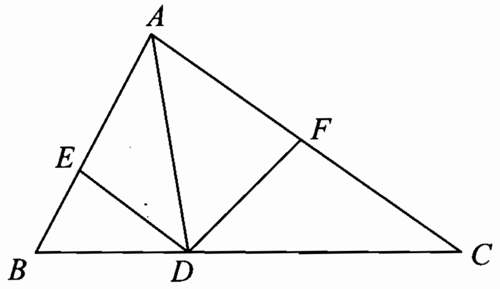
12. Suppose that a, b and c are real numbers greater than 1. Find the value of
$$\frac{1}{1+\log_{a^2b}\left(\frac{c}{a}\right)}+\frac{1}{1+\log_{b^2c}\left(\frac{a}{b}\right)}+\frac{1}{1+\log_{c^2a}\left(\frac{b}{c}\right)}.$$

13. Find the remainder when $(1! \times 1) + (2! \times 2) + (3! \times 3) + \cdots + (286! \times 286)$ is divided by 2009.

14. Find the value of $\left(25+10\sqrt{5}\right)^{1/3}+\left(25-10\sqrt{5}\right)^{1/3}$.

15. Let $a=\dfrac{1+\sqrt{2009}}{2}$. Find the value of $(a^3-503a-500)^{10}$.

16. In the figure below, ABC is a triangle and D is a point on side BC. Point E is on side AB such that DE is the angle bisector of $\angle ADB$, and point F is on side AC such that DF is the angle bisector of $\angle ADC$. Find the value of $\dfrac{AE}{EB}\cdot\dfrac{BD}{DC}\cdot\dfrac{CF}{FA}$.

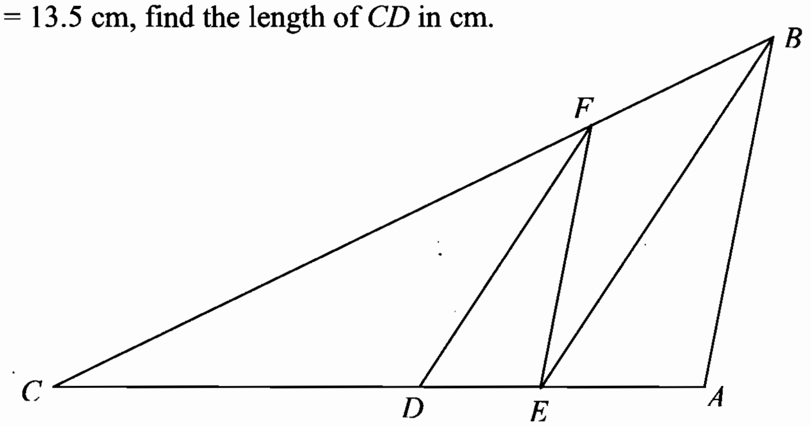


17. Find the value of
$$(\cot 25^\circ - 1)(\cot 24^\circ - 1)(\cot 23^\circ - 1)(\cot 22^\circ - 1)(\cot 21^\circ - 1)(\cot 20^\circ - 1).$$

18. Find the number of 2-element subsets $\{a, b\}$ of $\{1, 2, 3, \ldots, 99, 100\}$ such that $ab + a + b$ is a multiple of 7.

19. Let x be a real number such that $x^2 - 15x + 1 = 0$. Find the value of $x^4 + \frac{1}{x^4}$.

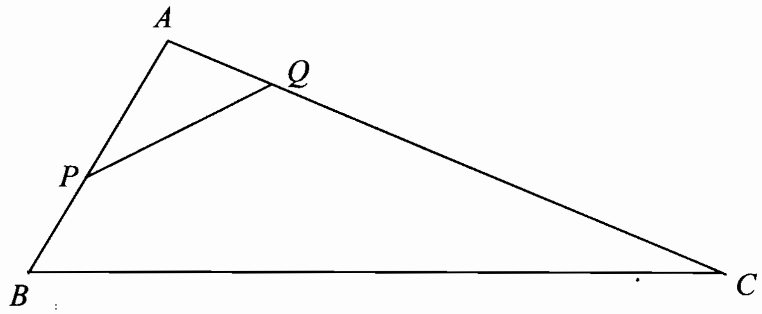
20. In the figure below, ABC is a triangle with $AB = 10$ cm and $BC = 40$ cm. Points D and E lie on side AC and point F on side BC such that EF is parallel to AB and DF is parallel to EB. Given that BE is an angle bisector of $\angle ABC$ and that $AD = 13.5$ cm, find the length of CD in cm.



21. Let $S = \{1, 2, 3, \ldots, 64, 65\}$. Determine the number of ordered triples (x, y, z) such that $x, y, z \in S$, $x < z$ and $y < z$.

22. Given that $a_{n+1} = \frac{a_{n-1}}{1 + n a_{n-1} a_n}$, where $n = 1, 2, 3, \ldots$, and $a_0 = a_1 = 1$, find the value of $\frac{1}{a_{199}\, a_{200}}$.

23. In the figure below, ABC is a triangle with $AB = 5$ cm, $BC = 13$ cm and $AC = 10$ cm. Points P and Q lie on sides AB and AC respectively such that $\frac{\text{area of } \Delta APQ}{\text{area of } \Delta ABC} = \frac{1}{4}$. Given that the least possible length of PQ is k cm, find the value of k.



24. If x, y and z are real numbers such that $x + y + z = 9$ and $xy + yz + zx = 24$, find the largest possible value of z.

25. Find the number of 0 – 1 binary sequences formed by six 0's and six 1's such that no three 0's are together. For example, 110010100101 is such a sequence but 101011000101 and 110101100001 are not.

26. If $\dfrac{\cos 100^\circ}{1-4\sin 25^\circ \cos 25^\circ \cos 50^\circ} = \tan x^\circ$, find x.

27. Find the number of positive integers x, where $x \neq 9$, such that

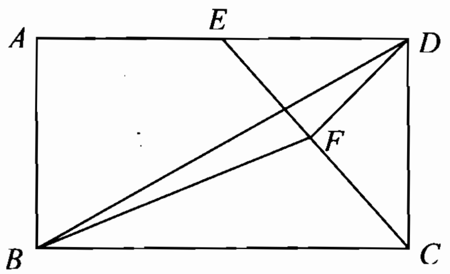
$$\log_{\frac{x}{9}}\left(\frac{x^2}{3}\right) < 6 + \log_3\left(\frac{9}{x}\right).$$

28. Let n be the positive integer such that

$$\frac{1}{9\sqrt{11}+11\sqrt{9}}+\frac{1}{11\sqrt{13}+13\sqrt{11}}+\frac{1}{13\sqrt{15}+15\sqrt{13}}+\cdots+\frac{1}{n\sqrt{n+2}+(n+2)\sqrt{n}}=\frac{1}{9}.$$

Find the value of n.

29. In the figure below, $ABCD$ is a rectangle, E is the midpoint of AD and F is the midpoint of CE. If the area of triangle BDF is 12 cm^2, find the area of rectangle $ABCD$ in cm^2.



30. In each of the following 6-digit positive integers: 555555, 555333, 818811, 300388, every digit in the number appears at least twice. Find the number of such 6-digit positive integers.

31. Let x and y be positive integers such that $27x + 35y \leq 945$. Find the largest possible value of xy.

32. Determine the coefficient of x^{29} in the expansion of $(1+x^5+x^7+x^9)^{16}$.

33. For $n = 1, 2, 3, \dots$, let $a_n = n^2 + 100$, and let d_n denote the greatest common divisor of a_n and a_{n+1}. Find the maximum value of d_n as n ranges over all positive integers.

34. Using the digits 1, 2, 3, 4, 5, 6, 7, 8, we can form 8! (= 40320) 8-digit numbers in which the eight digits are all distinct. For $1 \le k \le 40320$, let a_k denote the kth number if these numbers are arranged in increasing order:

$$12345678,\ 12345687,\ 12345768,\ \dots,\ 87654321;$$

that is, $a_1 = 12345678$, $a_2 = 12345687$, $\dots$, $a_{40320} = 87654321$. Find $a_{2009} - a_{2008}$.

35. Let x be a positive integer, and write $a = \lfloor \log_{10} x \rfloor$ and $b = \left\lfloor \log_{10} \frac{100}{x} \right\rfloor$. Here $\lfloor c \rfloor$ denotes the greatest integer less than or equal to c. Find the largest possible value of $2a^2 - 3b^2$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2009

(Senior Section, Round 2)

Saturday, 27 June 2009 **0930-1230**

INSTRUCTIONS TO CONTESTANTS

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let M and N be points on sides AB and AC of triangle ABC respectively. If

$$\frac{BM}{MA} + \frac{CN}{NA} = 1,$$

show that MN passes through the centroid of ABC.

2. Find all pairs of positive integers n, m that satisfy the equation $3 \cdot 2^m + 1 = n^2$.

3. L et A be an n-element subset of $\{1, 2, \ldots, 2009\}$ with the property that the difference between any two numbers in A is not a prime number. Find the largest possible value of n. Find a set with this number of elements. (Note: 1 is not a prime number.)

4. Let $a, b, c > 0$ such that $a + b + c = 1$. Show that if if $x_1, x_2, \ldots, x_5$ are positive real numbers such that $x_1 x_2 \ldots x_5 = 1$, then

$$(ax_1^2 + bx_1 + c)(ax_2^2 + bx_2 + c) \cdots (ax_5^2 + bx_5 + c) \geq 1.$$

5. In an archery competition, there are 30 contestants. The target is divided in two zones. A hit at zone 1 is awarded 10 points while a hit at zone 2 is awarded 5 points. No point is awarded for a miss. Each contestant shoots 16 arrows. At the end of the competition statistics show that more than 50% of the arrows hit zone 2. The number of arrows that hit zone 1 and miss the target are equal. Prove that there are two contestants with the same score.