Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Senior Section)

Tuesday, 27 May 2008 0930 – 1200 hrs

Important:

Answer ALL 35 questions.

Enter your answers on the answer sheet provided.

For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D *or* E) *corresponding to the correct answer.*

For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.

No steps are needed to justify your answers.

Each question carries 1 mark.

No calculators are allowed.

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO

Multiple Choice Questions

1. Find the value of $\dfrac{1+3+5+7+...+99}{2+4+6+8+...+100}$.

   (A) $\dfrac{48}{49}$

   (B) $\dfrac{49}{50}$

   (C) $\dfrac{50}{51}$

   (D) $\dfrac{98}{99}$

   (E) $\dfrac{100}{101}$

2. Suppose that x and y are real numbers that satisfy all the following three conditions: $3x-2y=4-p$; $4x-3y=2+p$; $x>y$. What are the possible values of p?

   (A) $p>-1$

   (B) $p<1$

   (C) $p<-1$

   (D) $p>1$

   (E) p can be any real number

3. If $f(x)=x^2+\sqrt{1-x^2}$ where $-1\le x\le 1$, find the range of $f(x)$.

   (A) $\dfrac{1}{2}\le f(x)\le 1$

   (B) $1\le f(x)\le \dfrac{5}{4}$

   (C) $1\le f(x)\le \dfrac{1+2\sqrt{3}}{4}$

   (D) $\dfrac{\sqrt{3}}{2}\le f(x)\le 1$

   (E) $\dfrac{1}{2}\le f(x)\le \dfrac{\sqrt{3}}{2}$

4. If a and b are integers and $\sqrt{7-4\sqrt{3}}$ is one of the roots of the equation $x^2 + ax + b = 0$, find the value of $a + b$.

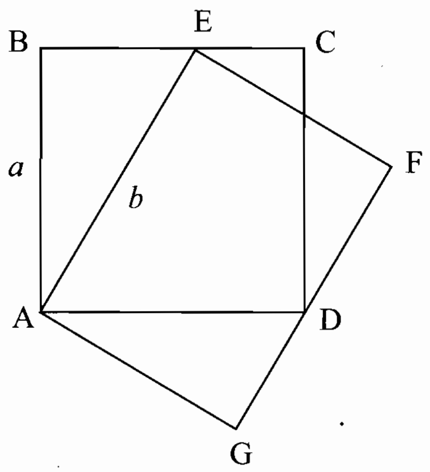
(A) -3
(B) -2
(C) 0
(D) 2
(E) 3

5. A bag contains 30 balls that are numbered 1, 2, …, 30. Two balls are randomly chosen from the bag. Find the probability that the sum of the two numbers is divisible by 3.

(A) $\frac{1}{2}$
(B) $\frac{1}{3}$
(C) $\frac{7}{29}$
(D) $\frac{9}{29}$
(E) $\frac{11}{87}$

6. ABCD is a square with AB = a, and AEFG is a rectangle such that E lies on side BC and D lies on side FG. If AE = b, what is the length of side EF?

(A) $\frac{b}{a}$
(B) $\frac{3a^2}{2b}$
(C) $\frac{4a^2}{3b}$
(D) $\frac{\sqrt{2}\,a^2}{b}$
(E) $\frac{a^2}{b}$



7. Find the value of $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$.

(A) 1

(B) $\frac{3}{2}$

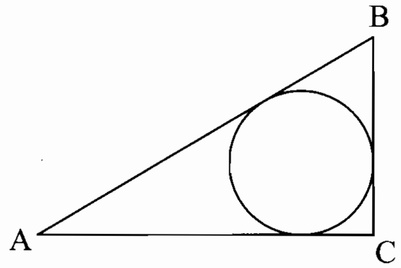
(C) $\frac{7}{4}$

(D) 2

(E) $\frac{5}{2}$

8. A circle with radius x cm is inscribed inside a triangle ABC, where $\angle$ ACB is a right angle. If AB = 9 cm and the area of the triangle ABC is 36 cm^2, find the value of x.

(A) 2.2
(B) 2.6
(C) 3
(D) 3.4
(E) 3.8



9. How many positive integers n are there such that $7n + 1$ is a perfect square and $3n + 1 < 2008$?

(A) 6
(B) 9
(C) 12
(D) 15
(E) 18

10. Find the minimum value of $(a+b)\left(\frac{1}{a}+\frac{4}{b}\right)$, where a and b range over all positive real numbers.

(A) 3
(B) 6
(C) 9
(D) 12
(E) 15

Short Questions

11. Find the smallest integer n such that $n(\sqrt{101}-10)>1$.

12. Given that x and y are positive real numbers such that $(x+y)^2=2500$ and $xy=500$, find the exact value of x^3+y^3.

13. Find the smallest positive integer N such that $2^n>n^2$ for every integer n in $\{N, N+1, N+2, N+3, N+4\}$.

14. The lengths of the sides of a quadrilateral are 2006 cm, 2007 cm, 2008 cm and x cm. If x is an integer, find the largest possible value of x.

15. Find the number of positive integers x that satisfy the inequality $\left|3+\log_x \frac{1}{3}\right|<\frac{8}{3}$.

16. Two bullets are placed in two consecutive chambers of a 6-chamber pistol. The cylinder is then spun. The pistol is fired but the first shot is a blank. Let p denote the probability that the second shot is also a blank if the cylinder is spun after the first shot and let q denote the probability that the second shot is also a blank if the cylinder is not spun after the first shot. Find the smallest integer N such that $N\geq\frac{100p}{q}$.

17. Find the value of $\left(\log_{\sqrt{2}}(\cos 20^\circ)+\log_{\sqrt{2}}(\cos 40^\circ)+\log_{\sqrt{2}}(\cos 80^\circ)\right)^2$.

18. Find the number of ways for 5 persons to be seated at a rectangular table with 6 seats, 2 each on the longer sides and 1 each on the shorter sides. The seats are not numbered.

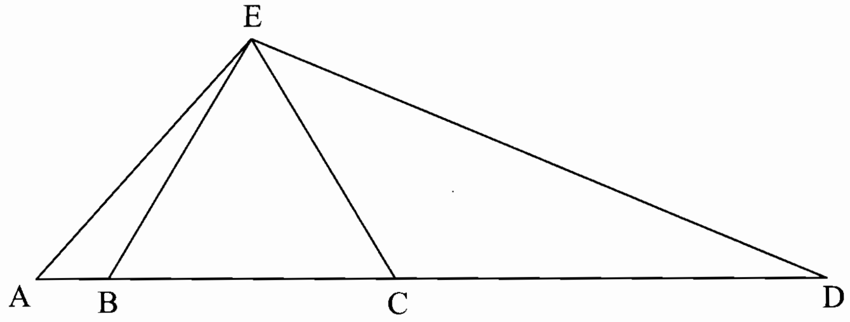
19. Find the remainder when $(x-1)^{100}+(x-2)^{200}$ is divided by x^2-3x+2.

20. Suppose that ABC is a triangle and D is a point on side AB with AD = BD = CD. If $\angle$ ACB = x°, find the value of x.

21. If x, y and z are positive integers such that $27x + 28y + 29z = 363$, find the value of $10(x + y + z)$.

22. Find the value of $\dfrac{\tan 40^\circ \tan 60^\circ \tan 80^\circ}{\tan 40^\circ + \tan 60^\circ + \tan 80^\circ}$.

23. In the figure below, ADE is a triangle with $\angle \mathrm{AED} = 120^\circ$, and B and C are points on side AD such that BCE is an equilateral triangle. If AB = 4 cm, CD = 16 cm and BC = x cm, find the value of x.



24. Suppose that x and y are positive integers such that $x + 2y = 2008$ and xy has the maximum value. Find the value of $x - y$.

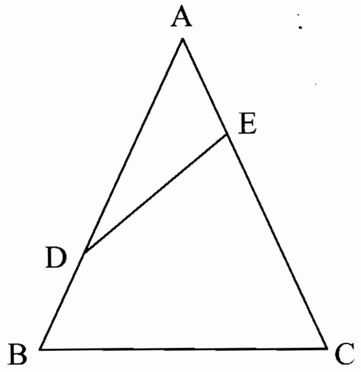
25. If $\cos(2A) = -\dfrac{\sqrt{5}}{3}$, find the value of $6\sin^6 A + 6\cos^6 A$.

26. Let N be the positive integer for which the sum of its two smallest factors is 4 and the sum of its two largest factors is 204. Find the value of N.

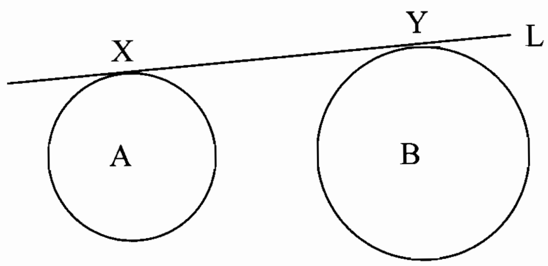
27. If $S = \displaystyle\sum_{k=1}^{99} \frac{(-1)^{k+1}}{\sqrt{k(k+1)}\,(\sqrt{k+1} - \sqrt{k})}$, find the value of $1000S$.

28. A teacher wrote down three positive integers on the whiteboard: 1125, 2925, N, and asked her class to compute the least common multiple of the three numbers. One student misread 1125 as 1725 and computed the least common multiple of 1725, 2925 and N instead. The answer he obtained was the same as the correct answer. Find the least possible value of N.

29. The figure below shows a triangle ABC where AB = AC. D and E are points on sides AB and AC, respectively, such that AB = 4DB and AC = 4AE. If the area of the quadrilateral BCED is 52 cm^2 and the area of the triangle ADE is x cm^2, find x.



30. The figure below shows two circles with centres A and B, and a line L which is a tangent to the circles at X and Y. Suppose that XY = 40 cm, AB = 41 cm and the area of the quadrilateral ABYX is 300 cm^2. If a and b denote the areas of the circles with centre A and centre B respectively, find the value of $\dfrac{b}{a}$.



31. Find the maximum value of $3\sin\left(x+\dfrac{\pi}{9}\right)+5\sin\left(x+\dfrac{4\pi}{9}\right)$, where x ranges over all real numbers.

32. Find the number of 11-digit positive integers such that the digits from left to right are non-decreasing. (For example, 12345678999, 55555555555, 23345557889.)

33. Find the largest positive integer n such that $\sqrt{n-100}+\sqrt{n+100}$ is a rational number.

34. Let $S = \{ 1, 2, 3, \ldots, 20 \}$ be the set of all positive integers from 1 to 20. Suppose that N is the smallest positive integer such that exactly eighteen numbers from S are factors of N, and the only two numbers from S that are not factors of N are consecutive integers. Find the sum of the digits of N.

35. Let $a_1, a_2, a_3, \ldots$ be the sequence of all positive integers that are relatively prime to 75, where $a_1 < a_2 < a_3 < \cdots$. (The first five terms of the sequence are: $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 7$, $a_5 = 8$.) Find the value of a_{2008}.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2008

(Senior Section, Round 2)

Saturday, 28 June 2008 **0930-1230**

Instructions to contestants

1. *Answer ALL 5 questions.*
2. *Show all the steps in your working.*
3. *Each question carries 10 mark.*
4. *No calculators are allowed.*

1. Let $ABCD$ be a trapezium with $AD \parallel BC$. Suppose K and L are, respectively, points on the sides AB and CD such that $\angle BAL = \angle CDK$. Prove that $\angle BLA = \angle CKD$.

2. Determine all primes p such that
$$5^p + 4 \cdot p^4$$
is a perfect square, i.e., the square of an integer.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that
(i) $f(2u) = f(u+v)f(v-u) + f(u-v)f(-u-v)$ for all $u, v \in \mathbb{R}$, and
(ii) $f(u) \geq 0$ for all $u \in \mathbb{R}$.

4. There are 11 committees in a club. Each committee has 5 members and every two committees have a member in common. Show that there is a member who belongs to 4 committees.

5. Let $a, b, c \geq 0$. Prove that
$$\frac{(1+a^2)(1+b^2)(1+c^2)}{(1+a)(1+b)(1+c)} \geq \frac{1}{2}(1+abc).$$