ON THE MATHEMATICAL PROPERTIES OF (p, q) SCALES

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1. Introduction

Throughout the world, music can be a way to show and tell a story. Mathematics can be a way to prove the story. Music is composed by a set of scales; in Western music, scales only contain twelve notes. Throughout this research, if notes are represented as numbers so that a scale becomes a set of numbers, then what are some of the properties that would be desirable for a scale to have from either a musical or mathematical perspective? Which scales have those properties? Can there be other phenomena that branch out from scales and notes? The paper, "Why Twelve Tones", written by Emily Clader [1], discusses the Pythagorean scales, where the scales are formed by the prime numbers 3 and 2 which gives a mathematical reason why twelve is a desirable number of notes. In this paper, we study what happens if we generalize the definition of a "scale" by replacing 3 and 2 with other numbers.

2. Background on Scales

Before investigating, we must first define the terms: n-note scale, k-step property, and the (p, q)-scale.

Definitions with Examples

Definition 2.1 (n-note scale)**.** The n-note scale is the set

$$\{ \frac{3^a}{2^b} \mid a, b \in \mathbb{Z}; a \in [0, n-1]; \frac{3^a}{2^b} \in [1, 2] \}.$$

In other words, the n-note scale is the set of numbers

$$\frac{3^0}{2^0}, \frac{3^1}{2^1}, \frac{3^2}{2^3}, \frac{3^3}{2^4}, \frac{3^4}{2^6}, \ldots, \frac{3^{n-1}}{2^b}, \frac{3^0}{2^{-1}}$$

where n and b are integers. Additionally, the power of 2 in the denominator is chosen so that the resulting fraction is between 1 and 2.

To better understand how scales work, here is an example of a 6-note scale.

Example 2.2. The 6-note scale is the following set:

$$\frac{3^0}{2^0}, \frac{3^1}{2^1}, \frac{3^2}{2^3}, \frac{3^3}{2^4}, \frac{3^4}{2^6}, \frac{3^5}{2^7}, \frac{3^0}{2^{-1}}.$$

In other words

$$1, 1.5, 1.125, 1.6875, 1.265625, 1.8984375, 2.$$

Take note, there are seven fractions, yet it is called a 6-note scale.
Rearranging these elements into ascending order will result in the following:

$$\frac{3^0}{2^0}, \frac{3^2}{2^3}, \frac{3^4}{2^6}, \frac{3^1}{2^1}, \frac{3^3}{2^4}, \frac{3^5}{2^7}, \frac{3^0}{2^{-1}},$$

which results in

$$1, 1.125, 1.265625, 1.5, 1.6875, 1.8984375, 2.$$

The purpose of rearranging these elements into ascending order is to determine the set's k-step property, which will be known as **Definition 2.4** throughout this research.

Remark 2.3. Each note is made of distinct numerators and denominators except for $\frac{3^0}{2^b}$ which has two possible denominators, $b = 0$ and $b = -1$, to satisfy the boundaries $[1, 2]$. A b-formula is provided to find the values of b which can be denoted as

$$b = \left\lceil a \cdot \frac{\log(3 \cdot 2^m)}{\log(2)} \right\rceil - 1$$

such that $a, m \in \mathbb{Z}$ which will be proved in section 5. We now define what it means for a scale to have a step property.

Definition 2.4 (k-step property)**.** A scale has the k-step property if, when its elements are arranged in ascending order, there are exactly k different ratios between consecutive elements.

Example 2.5. Following Example 2.2's set in ascending order, we get the following:

$$\begin{array}{ccccccccccccc} \frac{3^0}{2^0} & & \frac{3^2}{2^3} & & \frac{3^4}{2^6} & & \frac{3^1}{2^1} & & \frac{3^3}{2^4} & & \frac{3^5}{2^7} & & \frac{3^0}{2^{-1}}. \\ & \frac{3^2}{2^3} & & \frac{3^2}{2^3} & & \frac{3^{-3}}{2^{-5}} & & \frac{3^2}{2^3} & & \frac{3^2}{2^3} & & \frac{3^{-5}}{2^{-8}} & \end{array}$$

This 6-note scale has the 3-step property because there are three different ratios: $\frac{3^2}{2^3}, \frac{3^{-3}}{2^{-5}}, \frac{3^{-5}}{2^{-8}}$

Example 2.6. Following Example 2.2, take the 5-note scale.

$$\frac{3^0}{2^0}, \frac{3^1}{2^1}, \frac{3^2}{2^3}, \frac{3^3}{2^4}, \frac{3^4}{2^6}, \frac{3^0}{2^{-1}}.$$

Now in ascending order

$$\begin{array}{ccccccccccc} \frac{3^0}{2^0} & & \frac{3^2}{2^3} & & \frac{3^4}{2^6} & & \frac{3^1}{2^1} & & \frac{3^3}{2^4} & & \frac{3^0}{2^{-1}}. \\ & \frac{3^2}{2^3} & & \frac{3^2}{2^3} & & \frac{3^{-3}}{2^{-5}} & & \frac{3^2}{2^3} & & \frac{3^{-3}}{2^{-5}} & \end{array}$$

This 5-note scale has the 2-step property because there are two different ratios: $\frac{3^2}{2^3}, \frac{3^{-3}}{2^{-5}}$.

Example 2.7. We observe that the only scale with the 1-step property is the 1-note scale. Following Example 2.2, the 1-note scale is

$$\begin{array}{ccc} \frac{3^0}{2^0} & & \frac{3^0}{2^{-1}}. \\ & \frac{3^0}{2^{-1}} & \end{array}$$

This property is not very interesting to investigate since it only contains the first and last endpoints. Additionally, it is only evenly spaced by the ratio $\frac{3^0}{2^{-1}}$.

Now that we know more about scales and their step properties, we can now discuss more about the kind of scales we are interested in deciphering.

Definition 2.8 ((p, q)-scale)**.** The n-note (p, q)-scale is the set

$$\{ \frac{p^a}{q^b} | \ a, b \in \mathbb{Z}; a \in [0, n-1]; \frac{p^a}{q^b} \in [1, q] \ \}$$

.

Remark 2.9. The n-note $(3, 2)$-scale is the set given by Definition 2.1.

For more information about the $(3, 2)$-scale that will be used throughout this research, Emily Clader, a professor at San Francisco State University, talks about the $(3, 2)$-scale more in-depth. `http://www-personal.umich.edu/~eclader/Tuning.pdf`

Note: Examples 2.2, 2.5, 2.6, and 2.7 are all examples of the $(3, 2)$-scale, but p and q can be different integers. The $(3, 2)$-scale will be the scale primarily used to understand Conjecture 2.10, but what is Conjecture 2.10 we are trying to understand?
Conjecture 2.10 (that will be mentioned momentarily) has already been proven for the $(3, 2)$-scale. The following conjecture is the primary task of this research.

Conjecture 2.10. *If b_n is the denominator of a continued fraction approximation (CFA) of $\frac{\log(p)}{\log(q)}$, then the b_n-note (p,q)-scale has the 1-step property or the 2-step property.*

In other words, Conjecture 2.10 says that approximating $\frac{\log(p)}{\log(q)}$ to a finite fraction, it's resulting denominator, b_n will denote the size of the (p,q)-scale, which will result in the 1-step property or the 2-step property.
To understand Conjecture 2.10, we will need to know what a continued fraction (CF) and continued fraction approximation are, which will be discussed in the next section.

3. Background on Continued Fractions

Continued fractions (CF) have diverse uses and applications in the field of Mathematics, but we will focus on their ability to approximate real numbers. We first introduce definitions to distinguish between two types of CF; finite continued fractions and infinite continued fractions.

Definition 3.1. A finite continued fraction is an expression of the form

$$h_0 + \cfrac{1}{h_1 + \cfrac{1}{h_2 + \cfrac{1}{\ldots + \cfrac{1}{h_n}}}},$$

where $h_0 \in \mathbb{Z}$ and $h_1, h_2, \ldots, h_n \in \mathbb{Z}^+$. It is denoted by $[h_0; h_1, h_2, \ldots, h_n]$.

Definition 3.2. An infinite continued fraction is an expression of the form

$$h_0 + \cfrac{1}{h_1 + \cfrac{1}{h_2 + \cfrac{1}{\ldots}}},$$

where $h_0 \in \mathbb{Z}$ and $h_i \in \mathbb{Z}^+$, $\forall i \in \mathbb{N}$. More precisely, it's the limit of the sequence

$$h_0,\ h_0 + \frac{1}{h_1},\ h_0 + \cfrac{1}{h_1 + \cfrac{1}{h_2}},\ \ldots,$$

and is denoted by $[h_0; h_1, h_2, \ldots]$.

We state a Theorem that provides a key fact about continued fractions.

Theorem 3.3. *Any real number x is equal to either a finite or an infinite CF. Furthermore, it's finite if and only if $x \in \mathbb{Q}$.[2]*

We leave the proof of the theorem up to the curious reader.

As an example, let us take a look at the continued fraction of the real number π. Given that π is not a rational number, we can expect it to be equal to an infinite continued fraction by Theorem 3.3. We know in advance that

$$\pi = [\underset{h_0}{3}; \underset{h_1}{7}, \underset{h_2}{15}, \underset{h_3}{1}, \underset{h_4}{292}, \underset{h_5}{1}, \underset{h_6}{1}, \underset{h_7}{1}, \underset{h_8}{2}, \ldots],$$

which would represent

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}}}}}$$

As stated previously, also by Theorem 3.3, π is the limit of the sequence

$$3,\ 3 + \frac{1}{7},\ 3 + \cfrac{1}{7 + \cfrac{1}{15}},\ \ldots.$$

Notice that each element in the sequence above can be thought of as a finite continued fraction of the form $[h_0; h_1, \ldots, h_i]$, for some $i \in \mathbb{N}$ (technically speaking $i = 0$ can also be thought of as a CF on its own, but it is not relevant to our discussion). In fact, each such element is a rational number that approximates π. We formalize this idea by introducing the definition of a continued fraction approximation (CFA).

Definition 3.4. Let $x \in \mathbb{R}$ and suppose $x = [h_0; h_1, h_2, \ldots]$. Then, we call the rational number $[h_0; h_1, \ldots, h_n]$ the n^{th} CFA of x, for $n \in \mathbb{N}$.

A more loose but intuitive understanding of a CFA can be achieved by first considering the infinite continued fraction of π presented previously. Then, the n^{th} CFA of π is the rational number that results from computing the fraction up to h_n. We illustrate this idea with the following example.

Example 3.5. We compute the 3rd continued fraction approximation (CFA) of π, which is obtained by constructing the infinite continued fraction of π and "truncating" it at h_3; this amounts to ignoring elements past h_3 in the original infinite continued fraction:

$$\pi = [\underset{h_0}{3}; \underset{h_1}{7}, \underset{h_2}{15}, \underset{h_3}{1}, \xcancel{\underset{h_4}{292}, \underset{h_5}{1}, \underset{h_6}{1}, \underset{h_7}{1}, \underset{h_8}{2}, \ldots}],$$

$$\pi = \underset{h_0}{3} + \cfrac{1}{\underset{h_1}{7} + \cfrac{1}{\underset{h_2}{15} + \xcancel{\cfrac{1}{\underset{h_3}{1} + \cfrac{1}{\underset{h_4}{292} + \cfrac{1}{\underset{h_5}{1} + \cfrac{1}{\underset{h_6}{1} + \cfrac{1}{\underset{h_7}{1} + \cfrac{1}{\underset{h_8}{2} + \cdots}}}}}}}}}$$

Then,

$$\pi \approx [\underset{h_0}{3}; \underset{h_1}{7}, \underset{h_2}{15}, \underset{h_3}{1}]$$

$$\pi \approx \underset{h_0}{3} + \cfrac{1}{\underset{h_1}{7} + \cfrac{1}{\underset{h_2}{15} + \cfrac{1}{\underset{h_3}{1}}}}$$

$$\pi \approx \frac{355}{113}$$

And thus, the 3rd CFA of π is $\frac{355}{113}$.

Our main interest in this research in continued fractions is to compute the CFA's of the number $\frac{\log 3}{\log 2}$, and use the denominator of each n^{th} CFA as the size of our scales. However, when computing each CFA with the method described in Example 3.5, this process had two main drawbacks for our purposes:

(1) We want to compute CFA's without having to know in advance the "array" representation of a continued fraction, that is, $[h_0; h_1, h_2, h_3, \ldots]$.
(2) Computing particular CFA's by hand would become increasingly cumbersome.

In order to tackle this problem, we present an algorithm that computes the nth truncation of a real number x, needing the numerical value of said number and nothing more.

Remark 3.6. This pseudocode is designed to produce CFA's of real numbers that have an *infinite* continued fraction. The algorithm would have to be further revised to to also compute CFA's of numbers that have finite continued fractions.

Algorithm 1 Fraction Decomposition

1: **function** FD(x, n) $\triangleright$ $x \in \mathbb{R}$, $n = n$th truncation of CFA $\in \mathbb{Z}^+$
2: $\quad$ $decimalPart \leftarrow x - \lfloor x \rfloor$ // Computing $\{x\}$
3: $\quad$ $wholePart \leftarrow \lfloor x \rfloor$
4: $\quad$ **if** $n = 0$ **then**
5: $\quad\quad$ **return** $wholePart$
6: $\quad$ **end if**
7: $\quad$ **return** $wholePart + \dfrac{1}{\mathrm{FD}(\frac{1}{decimalPart}, n-1)}$
8: **end function**

Algorithm 1 has only one *function* in it (we will also interchangeably refer to it as a *procedure*), called `FD`, which takes in two initial arguments. The first argument, x, is the real number of which we want to compute some CFA. The second argument, n, is an integer that specifies we want to compute the nth CFA. The *function* `FD` makes use of two variables throughout its body:

- On line 2 we define *decimalPart*, which contains the decimal part of the number x, and is obtained by computing, the quantity $x - \lfloor x \rfloor$.
- On line 3 we define *wholePart*, which contians the whole part of the number x, and is obtained by computing the quantity $\lfloor x \rfloor$.

For instance, if we run the procedure FD(π, 3), the function will return $\frac{355}{113}$.

Remark 3.7. *decimalPart* and *wholePart* are variable names that we have chosen for their descriptive value. Note that they could have been called anything else.

Remark 3.8. For ease of use through the rest of this section, we shall define $\{x\}$ to be the decimal part of the number x. That is, $\{x\} := x - \lfloor x \rfloor$

The following set of instructions loosely describes the procedure followed in the above algorithm. Given a real number x (assumed to have an infinite continued fraction):

(a) Separate x into its whole part and decimal part such that $x = \lfloor x \rfloor + \{x\}$.
(b) If we have reached the desired truncation, we return the value $\lfloor x \rfloor$.
(c) Else, compute the value of $\lfloor x \rfloor$ + (start again at Step (a) and now let $x \leftarrow \frac{1}{\{x\}}$).

The algorithm relies on the natural recursive behavior of continued fractions. Consider the following:

$$\pi = 3 + \cfrac{1}{\boxed{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cdots}}}}}}}}}$$

The above is the same continued fraction expansion of π, but look at the part of the fraction delineated by the black square, think about it in isolation, and realize that this is in itself another continued fraction.

Example 3.9. As a means of illustration, let us compute the 3rd CFA of π once more, but this time using the procedure described by Algorithm 1. The function takes in two arguments, which will be $x = \pi$ and $n = 3$.

Visually, here is what is happening with the algorithm with an initial function call `FD(`π`,3)`:

$$\boxed{\text{Compute CF of } \pi \ \boxed{\text{Compute CF of } \frac{1}{\{\pi\}} \ \boxed{\text{Compute CF of} \frac{1}{\left\{\frac{1}{\{\pi\}}\right\}} \ \boxed{\text{Compute CF of } \frac{1}{\left\{\frac{1}{\left\{\frac{1}{\{\pi\}}\right\}}\right\}}}}}}$$

In more technical terms, what we're doing is:

$$\boxed{\texttt{FD}(\pi, 3) \ \boxed{\texttt{FD}(\frac{1}{\{\pi\}}, 2) \ \boxed{\texttt{FD}(\frac{1}{\left\{\frac{1}{\{\pi\}}\right\}}, 1) \ \boxed{\texttt{FD}(\frac{1}{\left\{\frac{1}{\left\{\frac{1}{\{\pi\}}\right\}}\right\}}, 0)}}}}$$

Which is essentially the same as:

$$\pi \approx \boxed{\underset{h_0}{3} + \frac{1}{\boxed{\underset{h_1}{7} + \frac{1}{\boxed{\underset{h_2}{15} + \frac{1}{\boxed{\underset{h_3}{1}}}}}}}}$$

Notice that the algorithm tries to compute the outermost square, but realizes that before it can finish doing so, it must compute the next biggest square, and so on and so forth. The algorithm stops calling itself recursively once it reaches h_n, since we want this to be our stopping point to construct the nth CFA. As seen in the above diagram, there are no further black boxes we can draw inside the innermost square since we have arrived at the h_n of our desired truncation. At this point in the algorithm, all that's left to do is to compute the CFA from the bottom-up.

The following page provides a breakdown of a sample run of Algorithm 1, when $x = \pi$ and $n = 3$. Once again, the algorithm tries to compute the outermost square, i.e., the outermost function call, `FD(`π`,3)`, but as can be seen in the diagrams above, we are missing the rest of the nested fractions that make up the CFA. It is at this point where we make use of the recursive nature of continued fractions, by now focusing on the continued fraction starting at h_1, i.e., `FD(`$\frac{1}{\{\pi\}}$`, 2)`, and so on and so forth.

Although not directly relevant to our discussion, it is interesting to note that the value of the second argument of our algorithm, n, always truly denotes the CFA of the number corresponding to the first argument of our algorithm, x. For instance, when `FD(`π`,3)`, $x = \pi$ and $n = 3$, which means that we are computing the 3rd CFA of π. But note that this is still true for subsequent function calls. When we call `FD(`$\frac{1}{\{\pi\}}$`, 2)`, $x = \frac{1}{\{\pi\}}$ and $n = 2$. Because we are, in a way, looking at the continued fraction starting from h_1, we truly are computing the "2nd" CFA of the continued fraction of $\frac{1}{\{\pi\}}$.

```
FD(π, 3) //x = π = 3.1415..., n = 3
 ||nextDecomp ← 0.1415... //x − ⌊x⌋
 ||wholePart ← 3 //⌊x⌋
 ||if n = 0 then //n = 3, so we skip to after "end if"
 ||  return wholePart
 ||end if
 ||return 3 + 1/FD(1/0.1415..., 3 − 1) // x = 1/0.1415... = 7.0625..., n = 2
           ||nextDecomp ← 0.0625... //x − ⌊x⌋
           ||wholePart ← 7 //⌊x⌋
           ||if n = 0 then //n = 2, so we skip to after "end if"
           ||  return wholePart
           ||end if
           ||return 7 + 1/FD(1/0.0625..., 2 − 1) // x = 1/0.0625... = 15.9965..., n = 1
                     ||nextDecomp ← 0.9965... //x − ⌊x⌋
                     ||wholePart ← 15 //⌊x⌋
                     ||if n = 0 then //n = 1, so we skip to after "end if"
                     ||  return wholePart
                     ||end if
                     ||return 15 + 1/FD(1/0.9965..., 1 − 1) //x = 1/0.9965... = 1.0034...,
                     n = 0
                               ||nextDecomp ← 0.0034... //x − ⌊x⌋
                               ||wholePart ← 1 //⌊x⌋
                               ||if n = 0 then //n = 0, enter if block!
                               ||  return 1 // When we reach the end
                               of a "return statement", we go back to
                               where this function call started.
```

4. Illustrations and Evidence

Now that we have defined the terminology in Conjecture 2.10, we are now prepared to illustrate it in some examples.

Example 4.1. Let $p = 3$ and $q = 2$. Then, our conjecture states that the n-note (3,2)-scale should have the 2-step property whenever n is the denominator of a CFA of $\frac{\log 3}{\log 2}$.

The CF for $\frac{\log(3)}{\log(2)}$ is $[1; 1, 1, 2, 2, 3, 1, 5, 2, 23, ...]$

Now that we have the CF, let's calculate the CFAs. For the sake of space, we will only calculate the first four truncations:

$$\frac{\log(3)}{\log(2)} \approx 1 + \frac{1}{1} \implies \frac{\log(3)}{\log(2)} \approx 2$$

$$\frac{\log(3)}{\log(2)} \approx 1 + \frac{1}{1+\frac{1}{1}} \implies \frac{\log(3)}{\log(2)} \approx \frac{3}{2}$$

$$\frac{\log(3)}{\log(2)} \approx 1 + \frac{1}{1+\frac{1}{1+\frac{1}{2}}} \implies \frac{\log(3)}{\log(2)} \approx \frac{8}{5}$$

$$\frac{\log(3)}{\log(2)} \approx 1 + \frac{1}{1+\frac{1}{1+\frac{1}{2+\frac{1}{2}}}} \implies \frac{\log(3)}{\log(2)} \approx \frac{19}{12}$$

The denominators of the CFAs are $1, 2, 5$, and 12, respectively. These denominators are the sizes of the scales that we will be observing. Note that the 1-note scale is not particularly interesting as it only contains the two end-point notes, but because of the nature of this scale, it clearly has the 1-step property since there is only one ratio, $\frac{3^0}{2^{-1}}$, between these two notes as seen in Example 2.7.

The 2-note scale is

$$\begin{array}{ccccc} \frac{3^0}{2^0} & & \frac{3^1}{2^1} & & \frac{3^0}{2^{-1}} \\ & \frac{3^1}{2^1} & & \frac{3^{-1}}{2^{-2}} & \end{array}$$

The 5-note scale is

$$\begin{array}{ccccccccccc} \frac{3^0}{2^0} & & \frac{3^2}{2^3} & & \frac{3^4}{2^6} & & \frac{3^1}{2^1} & & \frac{3^3}{2^4} & & \frac{3^0}{2^{-1}} \\ & \frac{3^2}{2^3} & & \frac{3^2}{2^3} & & \frac{3^{-3}}{2^{-5}} & & \frac{3^2}{2^3} & & \frac{3^{-3}}{2^{-5}} & \end{array}$$

The 12-note scale is

$$\begin{array}{ccccccccccccccccccccccccc} \frac{3^0}{2^0} & & \frac{3^7}{2^{11}} & & \frac{3^2}{2^3} & & \frac{3^9}{2^{14}} & & \frac{3^4}{2^6} & & \frac{3^{11}}{2^{17}} & & \frac{3^6}{2^9} & & \frac{3^1}{2^1} & & \frac{3^8}{2^{12}} & & \frac{3^3}{2^4} & & \frac{3^{10}}{2^{15}} & & \frac{3^5}{2^7} & & \frac{3^0}{2^{-1}} \\ & \frac{3^7}{2^{11}} & & \frac{3^{-5}}{2^{-8}} & & \frac{3^7}{2^{11}} & & \frac{3^{-5}}{2^{-8}} & & \frac{3^7}{2^{11}} & & \frac{3^{-5}}{2^{-8}} & & \frac{3^{-5}}{2^{-8}} & & \frac{3^7}{2^{11}} & & \frac{3^{-5}}{2^{-8}} & & \frac{3^7}{2^{11}} & & \frac{3^{-5}}{2^{-8}} & & \frac{3^{-5}}{2^{-8}} & \end{array}.$$

Each of these scales has the 2-step property because there are exactly two different ratios between consecutive notes, as denoted by the arrows. Although this only proves the conjecture for the first few n-note scales, it suggests that it works for the other n-note scales as well.

Example 4.2. When p and q are powers of each other and $p > q$, such as letting $p = 16$ and $q = 4$, then the following occurs:

Observe that $\frac{\log(16)}{\log(4)}$ results in a rational number, 2. So the CF for $\frac{\log(16)}{\log(4)}$ is [2].

This means that the only truncation we can see is:

$$\frac{\log(16)}{\log(4)} = 2.$$

As noted previously, this means that we will only be looking at the 1-note scale because the denominator of a whole number is always 1.

The 1-note (16,4) scale is

$$\frac{16^0}{4^0} \underset{\frac{16^0}{4^{-1}}}{\curvearrowright} \frac{16^0}{4^{-1}} .$$

Because the scale only contains two notes, we can see that it has the 1-step property. This example suggests that we can get the 1-step property from any p and q when they are powers of each other and $p > q$. This will be proven later.

Example 4.3. Similarly, when p and q are powers of each other and $p < q$, such as letting $p = 3$ and $q = 27$, then the following occurs:

When evaluated, $\frac{\log(3)}{\log(27)}$ results in a rational number, $\frac{1}{3}$.

This means that the resulting CF is $[0; 3]$.

The only non-zero truncation that we can see is:

$$\frac{\log(3)}{\log(27)} = 0 + \frac{1}{3} \implies \frac{\log(3)}{\log(27)} = \frac{1}{3}$$

Unlike the previous example, we are able to observe the 3-note scale:

$$\frac{3^0}{9^0} \underset{\frac{3^1}{9^0}}{\curvearrowright} \frac{3^1}{9^0} \underset{\frac{3^1}{9^0}}{\curvearrowright} \frac{3^2}{9^0} \underset{\frac{3^{-1}}{9^{-1}}}{\curvearrowright} \frac{3^0}{9^{-1}} .$$

Initially, it may seem that the 3-note scale has the 2-step property. However, when evaluating the two ratios for this scale, it is shown that they are indeed equivalent to each other. So, the 3-note scale has the 1-step property. This example suggests that we can also get the

1-step property from any p and q, given that $p > q$, which will be proven in the next section.

Example 4.4. When p is multiplied by a power of q, i.e. $p = p' \cdot q^k$, then the conjecture for (p, q) is the equivalent to the conjecture for (p', q).

For example, let $p = 12$ and $q = 2$, where $p = 3 \cdot 2^2$ (i.e. $p' = 3$).

When evaluated, the CF of $\frac{\log 12}{\log 2}$ is $[3; 1, 1, 2, 2, 3, 1, 5, 2, 23, ...]$. When taking a closer look at this, one can see that, apart from the whole number component of the CF, it matches with the CF of $\frac{\log 3}{\log 2}$.

The first four truncations of the CF for $\frac{\log 12}{\log 2}$ are

$$\frac{\log(12)}{\log(2)} \approx 3 + \frac{1}{1} \implies \frac{\log(12)}{\log(2)} \approx 4$$

$$\frac{\log(12)}{\log(2)} \approx 3 + \frac{1}{1 + \frac{1}{1}} \implies \frac{\log(12)}{\log(2)} \approx \frac{7}{2}$$

$$\frac{\log(12)}{\log(2)} \approx 3 + \frac{1}{1 + \frac{1}{1+\frac{1}{2}}} \implies \frac{\log(12)}{\log(2)} \approx \frac{18}{5}$$

$$\frac{\log(12)}{\log(2)} \approx 3 + \frac{1}{1 + \frac{1}{1+\frac{1}{2+\frac{1}{2}}}} \implies \frac{\log(12)}{\log(2)} \approx \frac{43}{12}$$

Because the whole number component of the CF does not affect the denominators of the truncations, it can be seen that the denominators of the CFAs for $\frac{\log 12}{\log 2}$ are exactly the same as the denominators of the CFAs for $\frac{\log 3}{\log 2}$. This means that we will get the same n-note scales. To confirm this, let's look at the 5-note (12,2) scale:

$$\frac{12^0}{2^0} \xrightarrow[\frac{12^2}{2^7}]{} \frac{12^2}{2^7} \xrightarrow[\frac{12^2}{2^7}]{} \frac{12^4}{2^{14}} \xrightarrow[\frac{12^{-3}}{2^{-11}}]{} \frac{12^1}{2^3} \xrightarrow[\frac{12^2}{2^7}]{} \frac{12^3}{2^{10}} \xrightarrow[\frac{12^{-3}}{2^{-11}}]{} \frac{12^0}{2^{-1}}$$

When comparing the 5-note (12,2) scale with the 5-note (3,2) scale from Example 4.1, it can be seen that when evaluating the fractions in the scale as well as the ratios, the decimal equivalents are exactly the same. Because the (3,2)-scale demonstrated the 2-step property, the (12,2)-scale will have the 2-step property as well.

These findings suggest that when $p = p' \cdot q^k$, the (p, q)-scale is equivalent to the (p', q)-scale and has the 2-step property if and only if the (p', q)-scale has the 2-step property, which is proven in the following section.

5. Proofs

Theorem 5.1. *Conjecture 2.10 holds strictly with the* ***2-step property*** *when* $p = 3 \cdot 2^k$ *and* $q = 2 \; \forall \; k \in \mathbb{Z}_{\geq 0}$. *That is, the* b_n*-note* $(3 \cdot 2^k, 2)$*-scale holds the 2-step property.*

Proof. We first show that the CFA of $\frac{\log 3}{\log 2}$ and $\frac{\log(3 \cdot 2^k)}{\log 2}$ share the same denominators b_n for any nth CFA of both numbers.

Let

$$\frac{\log 3}{\log 2} = [h_0; h_1, h_2, \ldots]$$

be the CF of $\frac{\log 3}{\log 2}$.

Then, consider

$$\begin{aligned} \frac{\log(3 \cdot 2^k)}{\log 2} &= \frac{\log 3}{\log 2} + \frac{\log 2^k}{\log 2} \\ &= \frac{\log 3}{\log 2} + k. \end{aligned}$$

Since k is an integer, we can conclude that the CF of $\frac{\log(3 \cdot 2^k)}{\log 2}$ is equal to the CF of $\frac{\log 3}{\log 2}$ plus k, that is,

$$\frac{\log(3 \cdot 2^k)}{\log 2} = [h_0 + k; h_1, h_2, \ldots].$$

Therefore, their CFA's share the same denominators b_n.

Now, let

$$\begin{aligned} A &= b_n\text{-note } (3, 2)\text{-scale}, \\ B &= b_n\text{-note } (3 \cdot 2^k, 2)\text{-scale}. \end{aligned}$$

We will show that $A = B$.

"$A \subseteq B$": Assume that $x \in A$. Then

$$x = \frac{3^a}{2^b} \in [1, 2].$$

Then, consider

$$x = x \cdot \frac{(2^k)^a}{(2^k)^a},$$

which is still in $[1, 2]$.

Then,

$$\begin{aligned} x &= x \cdot \frac{(2^k)^a}{(2^k)^a} \\ &= \frac{3^a}{2^b} \cdot \frac{(2^k)^a}{(2^k)^a} \\ &= \frac{(3 \cdot 2^k)^a}{2^{b+ka}} \in [1, 2]. \end{aligned}$$

Let $a' = a$ and $b' = b + ma$. Thus, $x \in B$.

"$A \supseteq B$": Assume $x \in B$. Then,

$$x = \frac{(3 \cdot 2^k)^a}{2^b} \in [1, 2]. \tag{1}$$

We distinguish between two cases, one where $a = 0$ and a second one when $a \neq 0$. When $a = 0$, it is evident that our choices for b must be $b = 0$, resulting in

$$\frac{(3 \cdot 2^k)^0}{2^0} = \frac{3^0}{2^0} = 1 \in A,$$

and $b = -1$, resulting in

$$\frac{(3 \cdot 2^k)^0}{2^{-1}} = \frac{3^0}{2^{-1}} = 2 \in A.$$

For our second case, we know the exact value of b for all $a \neq 0$ thanks to the $b-$formula defined in Remark 2.3 within Section 1.

$$\begin{aligned} b &= \left\lceil a \cdot \frac{\log(3 \cdot 2^k)}{\log 2} \right\rceil - 1 \\ &= \left\lceil a \cdot \frac{\log 3}{\log 2} + ak \right\rceil - 1 \\ &= \left\lceil a \cdot \frac{\log 3}{\log 2} \right\rceil + ak - 1. \end{aligned}$$

Let $w = \lceil a \cdot \frac{\log 3}{\log 2} \rceil - 1 \in \mathbb{Z}^+$.
Then,

$$b = ak + w.$$

Now, plugging into (1),

$$\begin{aligned} x &= \frac{(3 \cdot 2^m)^a}{2^b} \\ &= \frac{(3 \cdot 2^k)^a}{2^{ak+w}} \\ &= \frac{3^a}{2^w} \cdot \cancel{\frac{(2^k)^a}{2^{ak}}} \\ &= \frac{3^a}{2^w} \in [1, 2]. \end{aligned}$$

Let $a' = a$ and $b' = w$. Thus, $x \in A$.

Finally, $A = B$. As stated by Remark 2.9, given that the conjecture has already been proven true for the (3,2)-scale, we know that B has the 2-step-property. Thus, it must be the case that A also has the 2-step property. Thus, proof is completed as required.

□

Theorem 5.2. *Conjecture 2.10 holds for cases of p and q in which p and q are powers of each other. More specifically, the (p,q)-scale possesses the* ***1-step property*** *(2.4) when $p = q^m$ and $q = p^m$, $\forall m \in \mathbb{Z}^+$.*

***Note:** This logarithmic property is of essence to the following proof:

(1) $\log(q^m) = m \cdot \log(q)$

Proof. **Case One:**
We will begin our proof of Theorem 5.2 by first exploring the (p,q)-scale when $p = q^m$. In this case,

$$\begin{aligned}\frac{\log(p)}{\log(q)} &= \frac{\log(q^m)}{\log(q)} \\ &= \frac{m \cdot \log(q)}{\log(q)} \\ &= m \cdot \frac{\log(q)}{\log(q)} \\ &= m.\end{aligned}$$

Our new fraction, $\frac{m}{1}$ expressed as a continued fraction is

$$[a_0; a_1] = [0; m^{-1}] = 0 + \frac{m}{1}.$$

Since our *only* denominator in this continued fraction is 1, **Conjecture 2.10** (2.10) holds given that we have demonstrated above the *1-note scale* possesses the *1-step property.*

Recall from the definition of the 1-step property in **Section 2, Example 2.7** (2.7) of this paper, that the 1-step property only contains the two *boundary ratios*,

$$\frac{p^0}{q^0}, \frac{p^0}{q^{-1}}.$$

Our only truncation in this fraction, $\frac{m}{1}$, can also be expressed as $\frac{1}{m^{-1}}$ which is equal to $\frac{p^0}{q^{-1}}$. Making our only truncation equal to the *right-hand* boundary ratio outlined in the definition of the 1-step property.

For visual reference,

$$\frac{p^0}{q^0} \qquad \frac{p^0}{q^{-1}}$$

$$\frac{p^0}{q^{-1}}$$

Our final ratio is simply $\frac{m}{1}$ for all $m \in \mathbb{Z}^+$, demonstrating the denominator simplifies to 1 in any case of the (q^m, q)-scale. In conclusion, our (p,q)-scale possesses the 1-step property when $p = q^m$. □

Proof. **Case Two:**
Now we will explore why our (p,q)-scale possesses the 1-step property when $q = p^m$.
Let $q = p^m$, where

$$\begin{aligned}\frac{\log(p)}{\log(q)} &= \frac{\log(p)}{\log(p^m)} \\ &= \frac{\log(p)}{m \cdot \log(p)} \\ &= \frac{1}{m} \cdot \frac{\log(p)}{\log(p)} \\ &= \frac{1}{m}.\end{aligned}$$

Our new fraction, $\frac{1}{m}$ expressed as a continued fraction is

$$[a_0; a_1] = [0; m] = 0 + \frac{1}{m}$$

Now that we have shown $\frac{\log(p)}{\log(p^m)} = \frac{1}{m}$, we continue our exploration of the 1-step property of the m-note (p, q)-scale.

Keep in mind that **Theorem 5.2** (5.2) is concerned with the exponential powers of either p or q. That is, we are not so much concerned with the value of m, but the exponential power of p as p^m. More specifically, our main concern is what happens when q is *any power* of p. So, our middle fraction $\frac{p^1}{q^m}$ can be rewritten as a series of infinite powers of m,

$$\frac{p^1}{p^m}, \frac{p^1}{p^{m+1}}, \frac{p^1}{p^{m+2}}, \cdots, \frac{p^1}{p^{m+(n+1)}}$$

Let's break down the step sizes of the m-note (p, q)-scale when $q = p^m$, which is

$$\begin{array}{ccccccccccc} \frac{p^0}{q^0} & & \frac{p^1}{q^0} & & \frac{p^2}{q^0} & & \cdots & & \frac{p^{m-1}}{q^0} & & \frac{p^0}{q^{-1}} \\ & \frac{p^1}{q^0} & & \frac{p^1}{q^0} & & & & & & \frac{p^0}{q^{-1}} & \end{array}$$

At first glance, there are two different step sizes here, $\frac{p^0}{q^{-1}}$ and $\frac{p^1}{q^0}$. But in this case, we know that $q = p^m$ so $\frac{p^1}{q^o}$ is equal to $\frac{q^0}{p^{-1}}$. Therefore there is only *one* step size between each (p, q) ratio. In conclusion, the (p, q)-scale possesses the 1-step property when $q = p^m$.

$\square$

References

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