Def] A matrix is in row echelon form ('ref') if:

1. **First nonzero entry in each row is 1;**
2. **If row k is not all zeros, the no. of leading zeros in row $(k+1)$ is greater than the no. of leading zeros in row k;**
3. **Any rows with *all zeros* are below the rows with nonzero entries.**

$$\left[\begin{array}{cccc|c} 1 & \times & \times & \times & \times \\ 0 & 1 & \times & \times & \times \\ 0 & 0 & 1 & \times & \times \\ 0 & 0 & 0 & 1 & \times \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & \times & \times & \times & \times \\ 0 & 1 & \times & \times & \times \\ 0 & 0 & 1 & \times & \times \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{cccc|c} 1 & \times & \times & \times & \times \\ 0 & 0 & 1 & \times & \times \\ 0 & 0 & 0 & 1 & \times \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

DEF] An $m \times n$ coefficient matrix is in reduced row echelon form ('rref') if:

1. **The matrix is in row echelon form; *and***
2. **The first nonzero entry in each row is the *only nonzero entry in its column* (all entries above and below are zero).**

***corresponding elimination process is called Gauss-Jordan reduction.**

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

DEF'N: Homogeneous systems are linear systems in which all the constants $\{b_i\}$ are zero; i.e.:

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & 0 \end{array}\right]$$

RANK OF A MATRIX:

If A is an $n \times n$ matrix, the rank of A, denoted $\operatorname{rank} A$ or r, is the number of nonzero rows in the reduced row echelon form – or *any* echelon form – of A.

Eg.

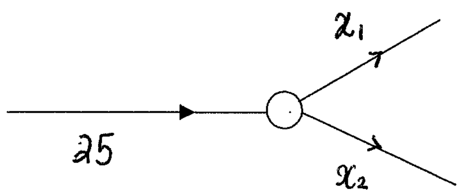
$$[A|b] = \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right] \to \underbrace{\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]}_{\text{REF}} \to \underbrace{\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]}_{\text{RREF}}$$

Applications of System of Linear Equations

1. Polynomial Curve Fitting
   Example: Determine the equation of polynomial of degree two whose graph passes through the points (1, 6), (2, 3), (3, 2).

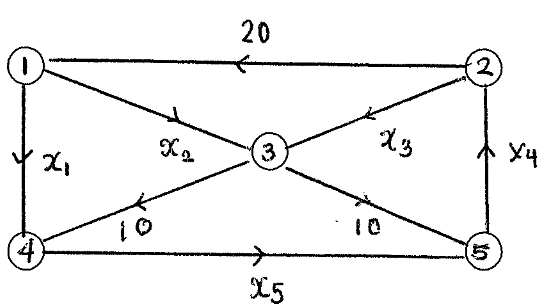
   Let the polynomial be $y = a_0 + a_1x + a_2x^2$

2. Network Analysis: Total flow into a junction is equal to the total flow out of the junction.



$x_1 + x_2 = 25$

Example: Set up a system of linear equations to represent the network shown in the below.



3. Electrical Network

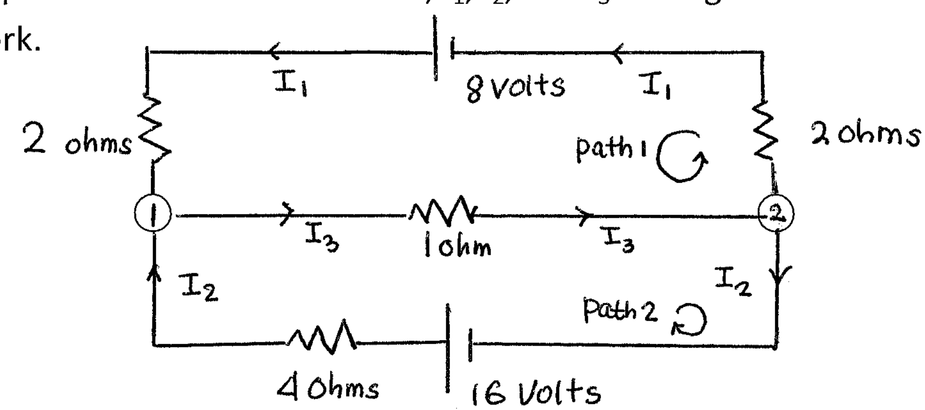
Kirchhoff's Laws:

- All the current flowing into a junction must flow out of it.
- The sum of the product IR (I: current, R: resistance) around a closed path is equal to the total voltage in the path.

Ohm's Law:

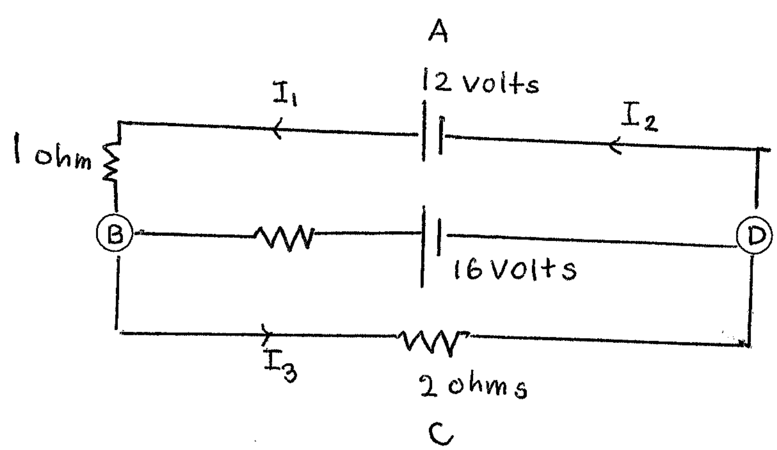
- The voltage drop cross a register is the product of the current passing through it and its resistance; i.e., E = IR

Example. Determine the currents, I_1, I_2, and I_3 through each branch of the network.



: Resistance

: Batteries



DEFINITIONS:

Two matrices A and B are equal if:

1) **They have the same dimensions;** ***and***
2) **Their corresponding entries are all equal:**

$$a_{ij} = b_{ij} \quad \text{for all } i, j$$

Scalar multiplication: If A is an m x n matrix and α is any scalar (real no.),

$$\alpha A = [\alpha a_{ij}]$$

i.e., multiply each element of A by α.

Matrix addition: If A and B are matrices having the ***same dimensions*****, their sum is obtained by adding corresponding elements:**

$$A + B = [a_{ij} + b_{ij}]$$

Remember**: Matrix addition is defined**

only for matrices having the same dimensions.

Negative of a matrix: $\quad -A = (-1)\cdot A = [-a_{ij}]$

and matrix subtraction: $\quad A - B = A + (-B) = [a_{ij} - b_{ij}]$

Zero matrix (written 0 or sometimes O): all entries = zero.

$$\Rightarrow \quad A - A = a_{ij} - a_{ij} = [0] = 0$$

($-A$ also called the additive inverse of A)

Transpose of a matrix: Interchange (or *transpose*) the rows & columns of A:

$$A^T = [a_{ji}]$$

If A is $m \times n$, then A^T is $n \times m$.

If $a(i,:)$ is a row vector $(a_1, a_2, ..., a_n)$,

then $(a(i,:))^T$ is the column vector $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

and clearly: $(A^T)^T = A$

Square matrix: an $n \times n$ matrix... $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

Some special square matrices:

Diagonal matrix: $a_{ij} = 0$ for $i \neq j$: $A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

Identity matrix: Diagonal with every $a_{ii} = 1$: $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

The $n \times n$ identity matrix is sometimes denoted by I_n

Upper triangular matrix:

All entries *below* the main diagonal are zero: $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$

Lower triangular matrix: All entries *above* the main diagonal are zero:

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Symmetric matrix: $A^T = A$; **i.e.,** $a_{ji} = a_{ij}$ **for every** i, j

e.g.:

Skew-symmetric (or antisymmetric) matrix: $A^T = -A$; **i.e.,** $a_{ji} = -a_{ij}$

e.g.:

MATRIX MULTIPLICATION:

For an $m \times n$ **linear system**

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \qquad (1)$$

Then the system (1) can be written $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ **and** $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

or

Then the system (1) can be written $A\mathbf{x} = \mathbf{b}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

or

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + & \cdots & + a_{1n}x_n \\ & \vdots & \\ a_{m1}x_1 + & \cdots & + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

EXAMPLE:

1. 3×3 **system:**

$$\begin{aligned} 3x_1 + 2x_2 + 3x_3 &= 5 \\ x_1 - 2x_2 + 5x_3 &= -2 \\ 2x_1 + x_2 - 3x_3 &= 1 \end{aligned}$$

2.

$$\begin{aligned} 2x_1 + 3x_2 - 2x_3 &= 5 \\ 5x_1 - 4x_2 + 2x_3 &= 6 \end{aligned}$$

An $m \times n$ system can also be regarded as a sum of column vectors:

$$\mathbf{b} = A\mathbf{x} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \quad\quad \vdots \quad\quad \ddots \quad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

DEF'N: If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are column vectors in R^m and $c_1, c_2, \ldots, c_n$ are scalars (real nos.), a sum of the form

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \ldots + c_n\mathbf{a}_n = \sum_{i=1}^{n} c_i\mathbf{a}_i$$

is called a linear combination of the vectors $\{\mathbf{a}_i\}$.

$\Rightarrow$ in the linear system $A\mathbf{x} = \mathbf{b}$, the product $A\mathbf{x}$ is a *linear combination of the column vectors of A* with multipliers $c_1 = x_1$, $c_2 = x_2$, ... etc.

MATRIX MULTIPLICATION:

$$C = A \cdot B = [c_{ij}] = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

$$(m \times n) \bullet (n \times r) \quad = \quad m \times r$$

EXAMPLE: 1. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

If A is $m \times n$ and B is $n \times m$,
then both AB and BA are defined,
but AB is $m \times m$ while BA is $n \times n$:

EXAMPLE:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 6 \\ 1 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 0 & 4 \end{bmatrix}$$

AB =

BA =

But NOTE that: $BA \neq AB$ ...i.e., matrix multiplication is *not* commutative.

* THE *ORDER OF THE FACTORS* IN MATRIX MULTIPLICATION MUST *ALWAYS* BE OBSERVED.