Matrices

- **Transpose of a matrix**
- **Square matrix**
- **Diagonal matrix**
- **Identity matrix**
- **Triangular matrix**
- **Symmetric matrix**
- **Skew -symmetric matrix**

Operations of Matrices

- **MATRIX ADDITION AND SCALAR MULTIPLICATION:**

IF A = $[a_{ij}]$, B = $[b_{ij}]$ are matrices of size mxn, then

their sum is the mxn matrix given by

A + B = $[a_{ij} + b_{ij}]$ and

k **A** = $[k\, a_{ij}]$

- **Matrix Multiplication**

$$C = A \cdot B = [c_{ij}] = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$(m \times n) \cdot (n \times r) = m \times r$$

If A is $m \times n$ and B is $n \times m$, then both AB and BA are defined... but AB is $m \times m$ while BA is $n \times n$:

EXAMPLE: $A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix}$; $B = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 3\cdot(-2) - 2\cdot 4 & 3\cdot 1 - 2\cdot 1 & 3\cdot 3 - 2\cdot 6 \\ 2\cdot(-2) + 4\cdot 4 & 2\cdot 1 + 4\cdot 1 & 2\cdot 3 + 4\cdot 6 \\ 1\cdot(-2) - 3\cdot 4 & 1\cdot 1 - 3\cdot 1 & 1\cdot 3 - 3\cdot 6 \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 1 & -3 \\ 12 & 6 & 30 \\ -14 & -2 & -15 \end{bmatrix} \quad (3\times 3)$$

...and *BA* is also defined:

$$BA = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 20 & -22 \end{bmatrix} \quad (2\times 2)$$

But NOTE that: $BA \neq AB$ ...i.e., matrix multiplication is *not* commutative.

The $m \times n$ linear system S

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

Can be written as the <u>matrix multiplication</u>

$$A\mathbf{x} = \mathbf{b}$$

Or

As a ***<u>linear combination of the column vectors of A</u>***

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

Ex: 3×3 **system:**

$$\begin{array}{rcl} 3x_1 + 2x_2 + 3x_3 & = & 5 \\ x_1 - 2x_2 + 5x_3 & = & -2 \\ 2x_1 + x_2 - 3x_3 & = & 1 \end{array}$$

THEOREM: **A linear system $A\mathbf{x} = \mathbf{b}$ is consistent (has at least one solution) provided (iff) $\mathbf{b}$ can be expressed as a linear combination of the column vectors of A.**

EXAMPLE:

$$2x_1 + 3x_2 - 2x_3 = 5$$

$$5x_1 - 4x_2 + 2x_3 = 6$$

RULES OF MATRIX ALGEBRA:

For any scalars (real nos.) α, β and any matrices *A, B, C* for which the indicated operations are defined :

1. $A+0=A$
2. $A+(-A)=0$ **(-A is the additive inverse of A)**
3. $A+B = B+A$ **(matrix addition *is* commutative)**
4. $(A+B)+C = A+(B+C)$ **(matrix addition is associative)**
5. $(AB)C = A(BC)$ **(matrix multiplication is associative)**
6. $A(B+C) = AB+AC$ **(matrix multiplication is distributive *wrt* addition)**
7. $(A+B)C = AC+BC$ **(same)**
8. $(\alpha\beta)A = \alpha(\beta A) = \beta(\alpha A)$
9. $\alpha(AB) = (\alpha A)B = A(\alpha B)$
10. $(\alpha+\beta)A = \alpha A+\beta A$
11. $\alpha(A+B) = \alpha A+\alpha B$

Noncommutativity of Matrix Multiplication

1) In general, $BA \neq AB$ (even if both are $n\times n$):

Cancellation is not Valid

2) $AB=0$ does *not* imply $A=0$ or $B=0$ or $BA=0$:

3) If $cA=0$, then c = 0 or A = 0

Powers of a square matrix:

$$A^2 = AA; \quad A^3 = AA^2; \quad \ldots \quad A^n = AA^{n-1} = AA\cdots A$$

...and: $A^0 \equiv I$, ...But if $m \neq n$, A^2 is not defined.

Roots of a square matrix: If $A = B^n$, then $B = \sqrt[n]{A}$

Recall the $n \times n$ identity matrix: $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

For any $n \times n$ matrix A: $AI = IA = A$

NOTATION: The column vectors of I are denoted by $\mathbf{e}_j$,

i.e., $I_n = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$

RULES FOR TRANSPOSES:

1) $(A^T)^T = A$: $\left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T\right]^T =$

2) $(\alpha A)^T = \alpha A^T$: $\left[\alpha \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right]^T =$

3) $(A+B)^T = A^T + B^T$: $\left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right]^T =$

4) $(AB)^T = B^T A^T$: $\left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\right]^T =$

...and $B^T A^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} =$

The other way 'round:

$$(BA)^T = \left[\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right]^T =$$

...and $\quad A^T B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} =$

DEF'N: An $n\times n$ matrix A is <u>nonsingular</u> or invertible if there is an $n\times n$ matrix B such that

$$AB = BA = I$$

...B is called the multiplicative inverse (or inverse) of A.

If B and C are *both* inverses of A, then

$$B = BI = B(AC) = (BA)C = IC = C$$

...i.e.: if A has an inverse, the inverse is unique.

DEF'N: An $n\times n$ matrix is <u>singular</u> or noninvertible if it *does not have* an inverse.

EXAMPLE: $\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$n \times n$ MATRICES:

$n \times n$ matrices can be broadly divided into two classes: nonsingular (invertible) & singular (noninvertible). Each class has a unique set of characteristics that we will develop and exploit throughout the course.

RULES FOR INVERSES:

If A and B are nonsingular (invertible) $n \times n$ matrices,

1. **A^{-1} is nonsingular, and** $(A^{-1})^{-1} = A$
2. **(αA) is nonsingular, and** $(\alpha A)^{-1} = \alpha^{-1} A^{-1}$ **(for $\alpha \neq 0$)**
3. **AB is nonsingular, and** $(AB)^{-1} = B^{-1} A^{-1}$

...by extension of (3), any product of nonsingular $n \times n$ matrices $A_1, A_2, \ldots, A_k$ is also nonsingular, and

$$\begin{aligned}(A_1 A_2 \cdots A_k)^{-1} &= \{A_1 (A_2 \cdots A_k)\}^{-1} \\ &= (A_2 \cdots A_k)^{-1} A_1^{-1} \\ &\vdots \\ &= A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}\end{aligned}$$

...e.g., $(ABC)^{-1} = [(AB)C]^{-1} = C^{-1}(AB)^{-1} = C^{-1}(B^{-1}A^{-1}) = C^{-1}B^{-1}A^{-1}$