The nullspace of an $m \times n$ matrix A, or null A, consists of all solutions to the $m \times n$ homogeneous system $A\mathbf{x} = \mathbf{0}$, and is a subspace of $\mathbb{R}^n$.

The column space of A, or col A, is the subspace of $\mathbb{R}^m$ spanned by the n column vectors of A.

The row space of A, or row A, is the subspace of $\mathbb{R}^{1 \times n}$ spanned by the m row vectors of A.

The rank of A equals the dimension of the row space (or column space):

$$\operatorname{rank} A = \dim(\operatorname{row} A) = \dim(\operatorname{col} A)$$

and also equals the maximum number of independent row vectors (or column vectors) in A.

CONSISTENCY THEOREM FOR LINEAR SYSTEMS:

An $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$ is consistent provided $\mathbf{b}$ belongs to the column space of A. ($\mathbf{b} \in \operatorname{col} A$)

An $m \times n$ homogeneous system $A\mathbf{x} = \mathbf{0}$ will have *only* the trivial solution

$$(\mathbf{x} = \mathbf{0})$$

provided the column vectors of A are linearly independent.

— For an $n \times n$ system, this means the coefficient matrix is *nonsingular*.

— For an $m \times n$ system with *fewer equations than unknowns* ($m < n$), it means the system will have infinitely many solutions (since n column vectors with $m < n$ components cannot be linearly independent).

The linear system $A\mathbf{x} = \mathbf{b}$ is consistent (i.e., has at least one solution) for *every* $\mathbf{b}$ in $\mathbb{R}^m$ provided the columns of A span $\mathbb{R}^m$.

The system has at most one solution for *every* $\mathbf{b}$ in $\mathbb{R}^m$ provided the columns of A are linearly independent.

RANK-NULLITY THEOREM: $\operatorname{rank} A + \dim(\operatorname{null} A) = n$

BASIS FOR $\operatorname{row} A$: To find a basis for $\operatorname{row} A$, reduce A to echelon (or reduced echelon) form U (or A^*) The nonzero rows of U (or A^*) are a are a basis for $\operatorname{row} A$.

BASIS FOR $\operatorname{col} A$: A basis for $\operatorname{col} A$ consists of the columns of A corresponding to the columns associated with the leading 1's in U (or A^*).

ALTERNATE BASIS FOR $\operatorname{col} A$: Reduce A^T to echelon (or reduced echelon) form $(A^T)^*$. The nonzero rows of $(A^T)^*$, turned on end to form $m\times 1$ columns, are a basis for $\operatorname{col} A$.

EXAMPLE: $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \ldots \rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A^*$

$\operatorname{rank} A =$

Basis for $\operatorname{row} A$:

Basis for $\operatorname{col} A$:

$\operatorname{null} A$: Solve $A^*\mathbf{x} = \mathbf{0}$:

Rank-nullity theorem:

Alternate basis for $\operatorname{col} A$: row-reduce A^T:

Alternate basis for $\operatorname{col} A$: row-reduce A^T :

$$A^T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ -1 & -3 & 1 \\ 1 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Basis for $\operatorname{col} A$: $\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

The two bases for $\operatorname{col} A$ are both valid, and are linear combinations of one another:

$$-1 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 1 \cdot \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

and $$3 \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}$$

The two bases for col A are both valid, and are linear combinations of one another:

$$-1\cdot\begin{bmatrix}1\\2\\1\end{bmatrix} - 1\cdot\begin{bmatrix}-1\\-3\\1\end{bmatrix} = \begin{bmatrix}0\\1\\-2\end{bmatrix}$$

and $$3\cdot\begin{bmatrix}1\\2\\1\end{bmatrix} + 2\cdot\begin{bmatrix}-1\\-3\\1\end{bmatrix} = \begin{bmatrix}1\\0\\5\end{bmatrix}$$

NULLSPACE & THE SOLUTION OF NONHOMOGENEOUS SYSTEMS:

Consider the 3×3 system:

$$\begin{aligned} x_1 + x_2 + 3x_3 &= 2\\ x_1 + 2x_2 + 4x_3 &= 3\\ x_1 + 3x_2 + 5x_3 &= 4 \end{aligned} \tag{1}$$

The coefficient matrix A is:

$$A = \begin{bmatrix}1&1&3\\1&2&4\\1&3&5\end{bmatrix}$$

Row-reduce $[A|\,\mathrm{b}] = \left[\begin{array}{ccc|c}1&1&3&2\\1&2&4&3\\1&3&5&4\end{array}\right] \to \left[\begin{array}{ccc|c}1&1&3&2\\0&1&1&1\\0&2&2&2\end{array}\right] \to \left[\begin{array}{ccc|c}1&1&3&2\\0&1&1&1\\0&0&0&0\end{array}\right]$

Here A is singular

$\Rightarrow$ **no solution (inconsistent)** ***or*** **infinitely many solutions.**

...in this case, infinitely many.

$\Rightarrow$ **general solution is:** $\mathrm{x} = \begin{bmatrix}1-2\alpha\\1-\alpha\\\alpha\end{bmatrix}$ **(α any real no.)** **(2)**

...or: $$\mathrm{x} = \begin{bmatrix}1\\1\\0\end{bmatrix} + \alpha\begin{bmatrix}-2\\-1\\1\end{bmatrix} \tag{2a}$$

NULLSPACE OF A MATRIX:

If A is an $m \times n$ matrix, let null A be the set of all solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

$$\text{null}\,A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

Since $A\mathbf{x} = \mathbf{0}$ always has at least the trivial solution $\mathbf{x} = \mathbf{0}$, null A is a *nonempty* set.

If $\mathbf{x} \in \text{null}\,A$ and c is any scalar,

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c \cdot \mathbf{0} = \mathbf{0};$$

$\Rightarrow$ $c\mathbf{x}$ is also in null A.

And, if $\mathbf{x}, \mathbf{y} \in \text{null}\,A$, then:

$$A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

$\Rightarrow$ $(\mathbf{x}+\mathbf{y})$ is also in null A.

Thus, null A is a *subspace* of $\mathbb{R}^n$; i.e.:

> The set of all solutions to the $m \times n$ homogeneous system $A\mathbf{x} = \mathbf{0}$ forms a subspace of $\mathbb{R}^n$, called the nullspace of the matrix A.

EX: Find null A for

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

LIST OF EQUIVALENCIES FOR $n \times n$ MATRICES (UPDATE)

If A is an $n \times n$ matrix, the following statements are equivalent (i.e., any one of them implies all the others)

LIST A	LIST B
A is nonsingular (has an inverse)	A is singular (has *no* inverse)
$\operatorname{rank} A = n$	$\operatorname{rank} A < n$
The column vectors of A are linearly independent & span $\mathbb{R}^n$ (i.e., form a *basis* for $\mathbb{R}^n$)	The column vectors of A are linearly *dependent* & do *not* span $\mathbb{R}^n$ (do *not* from a basis for $\mathbb{R}^n$)
The row vectors of A are linearly independent & span $\mathbb{R}^{1\times n}$ (i.e., form a *basis* for $\mathbb{R}^{1\times n}$)	The row vectors of A are linearly *dependent* & do *not* span $\mathbb{R}^{1\times n}$ (do *not* from a basis for $\mathbb{R}^{1\times n}$)
The linear system $A\mathrm{x} = \mathrm{b}$ has a *unique* solution (for any b): $\mathrm{x} = A^{-1}\mathrm{b}$	The linear system $A\mathrm{x} = \mathrm{b}$ has either *no* solution or *infinitely many* solutions, depending on b
The homogeneous system $A\mathrm{x} = 0$ has only the trivial solution ($\mathrm{x} = 0$)	The homogeneous system $A\mathrm{x} = 0$ has infinitely many solutions
null A contains only the zero vector $[\dim(\operatorname{null} A) = 0]$	$1 \le \dim(\operatorname{null} A) \le n$
A is row equivalent to the $n \times n$ identity matrix	A is *not* row equivalent to the $n \times n$ identity matrix
A can be expressed as a product of elementary matrices $\{E_i\}$	A *cannot* be expressed as a product of elementary matrices $\{E_i\}$

LIST OF PROPERTIES FOR $m \times n$ MATRICES

If A is an $m \times n$ matrix, the following are true:

LIST A ($m < n$)	LIST B ($m > n$)
$0 \leq \text{rank} A \leq m$	$0 \leq \text{rank} A \leq n$
The homogeneous system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions	The nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ may have no solution if $\mathbf{b} \neq \mathbf{0}$
$\text{rank} A + \dim(\text{null} A) = n$	$\text{rank} A + \dim(\text{null} A) = n$
The column vectors of A cannot be linearly independent	The column vectors of A cannot span $\mathbb{R}^m$
The row vectors of A cannot span $\mathbb{R}^{1 \times n}$	The row vectors of A cannot be linearly independent
The row space and column space of A have the same dimension, equal to $\text{rank} A$	The row space and column space of A have the same dimension, equal to $\text{rank} A$
If $\text{rank} A = m$, the row vectors of A are linearly independent	If $\text{rank} A = n$, the column vectors of A are linearly independent
The system $A\mathbf{x} = \mathbf{b}$ is consistent iff* $\mathbf{b}$ is in the column space of A	The system $A\mathbf{x} = \mathbf{b}$ is consistent iff* $\mathbf{b}$ is in the column space of A (same)
The system $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ in $\mathbb{R}^m$ iff the column vectors of A span $\mathbb{R}^m$	The system $A\mathbf{x} = \mathbf{b}$ *cannot be* consistent for all $\mathbf{b}$ in $\mathbb{R}^m$
	The system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$ in $\mathbb{R}^m$ iff the column vectors of A are linearly independent

* iff: "if and only if"