VECTOR SPACES: EXAMPLES & FORMAL DEFINITION:

EUCLIDEAN VECTOR SPACES: $\mathbb{R}^2, \mathbb{R}^3, \ldots, \mathbb{R}^n$

$\mathbb{R}^2$: **Set of all possible ordered pairs of real numbers** $(x_1, x_2)\ldots$
OR all possible 2×1 column vectors (matrices) with real entries, e.g.:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Addition of vectors: $\quad \mathbf{u}+\mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1,\ u_2 + v_2)$

Scalar multiplication of a vector by a scalar (real no.) c**:**

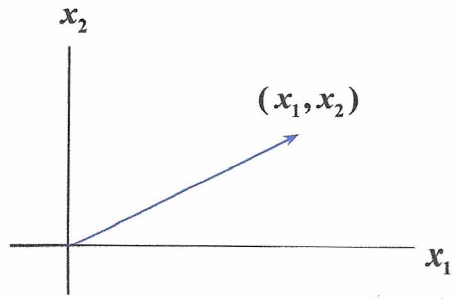
$$c\,\mathbf{x} = c\,(x_1, x_2) = (cx_1, cx_2)$$

NOTE: a) **For any two vectors** $\mathbf{u}, \mathbf{v}$ **in** $\mathbb{R}^2$, $(\mathbf{u}+\mathbf{v})$ **is also in** $\mathbb{R}^2$; **and**
b) **For any vector** $\mathbf{x}$ **in** $\mathbb{R}^2$ **(and any scalar** c**),** $c\,\mathbf{x}$ **is also in** $\mathbb{R}^2$.

...we say that $\mathbb{R}^2$ **is**

a) **closed under vector addition; and**
b) **closed under scalar multiplication**

Geometrically, $\mathbb{R}^2$ **can be represented by the set of all points in a 2-dimensional plane, and a vector** (x_1, x_2) **can be represented as a directed line segment from** $(0,0)$ **to** (x_1, x_2)**:**



$\Rightarrow$ **Vector addition and scalar multiplication can be performed geometrically, and we *define* the length of a vector x as the length of the directed line segment representing x:**

$$l = \sqrt{x_1^2 + x_2^2}$$

For $\mathbb{R}^3$, represent a vector $\mathrm{x} = (x_1, x_2, x_3)$ as a point in 3-dimensional space —or, a directed line segment from the origin $(0,0,0)$ to (x_1, x_2, x_3); and the length of x is:

$$l = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

More generally, $\mathbb{R}^n$ is the space of all ordered n-tuples of real nos. $(x_1, x_2, \ldots, x_n)$... or $n \times 1$ column vectors:

$$\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1, x_2, \ldots, x_n]^T$$

This defies geometric visualization, but we can still *define* the length of x as

$$l = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$$

(we'll call this the *norm* of the vector x, written $\|\mathrm{x}\|$)

With the same definitions of vector addition and scalar multiplication as for $\mathbb{R}^2$ and $\mathbb{R}^3$, $\mathbb{R}^n$ is also closed under these operations.

ANOTHER KIND OF VECTOR SPACE: The set of all $m \times n$ matrices with real elements, $\mathbb{R}^{m \times n}$:

With the usual definitions of matrix addition and scalar multiplication, i.e.,

$$(A+B)_{ij} = a_{ij} + b_{ij}$$

$$\text{and} \qquad (cA)_{ij} = ca_{ij}$$

this set is closed under both operations: if A and B are vectors in $\mathbb{R}^{m \times n}$ ($m \times n$ matrices with real elements), then $(A+B)$ and cA (also $m \times n$ matrices with real elements) are also vectors in $\mathbb{R}^{m \times n}$.

This basic property of closure under vector addition and scalar multiplication is the key ingredient in *defining* a vector space. If it is satisfied, the set of vectors V is said to form a vector space... provided the following conditions (*axioms*) also hold:

1. $\mathrm{u}+\mathrm{v} = \mathrm{v}+\mathrm{u}$ *for any* x *and* y *in* V
2. $(\mathrm{u}+\mathrm{v})+\mathrm{w} = \mathrm{u}+(\mathrm{v}+\mathrm{w})$ *for any* $\mathrm{u}, \mathrm{v}, \mathrm{w}$ *in* V
3. *There is a unique element* 0 *in* V *such that* $\mathrm{v}+0 = \mathrm{v}$ *for any* v *in* V
4. *For any* v *in* V, *there is an element* $-\mathrm{v}$ *in* V *such that* $\mathrm{v}+(-\mathrm{v}) = 0$
5. $c(\mathrm{u}+\mathrm{v}) = c\,\mathrm{x}+c\,\mathrm{y}$ *for any scalar* c *and any* u, v *in* V
6. $(c+d)\mathrm{v} = c\,\mathrm{v}+d\,\mathrm{v}$ *for any scalars* c, d *and any* v *in* V
7. $(cd)\mathrm{v} = c\,(d\,\mathrm{v})$ *for any scalars* c, d *and any* v *in* V
8. $1\cdot\mathrm{v} = \mathrm{v}$ *for all* v *in* V

NOTE: The definition of a vector space says nothing about *multiplication* of vectors. For some vector spaces we may want to define a vector product (as in the case of $\mathbb{R}^{n\times n}$), but this is *not a requirement* for a vector space.

STILL OTHER KINDS OF VECTOR SPACES:

- **$C\,[0,1]$: The set of real-valued, continuous functions of a real variable t defined on the closed interval $[0,1]$: a new kind of 'vector'...**

  Vector addition: If $f(t)$ and $g(t)$ are in $C\,[0,1]$, define the sum $(f+g)$ by

  $$(f+g)(t) = f(t)+g(t)$$

  ...and the scalar multiple (cf) by

  $$(cf)(t) = cf(t)$$

  Then $C\,[0,1]$ is closed under vector addition and scalar multiplication: if $f(t)$ and $g(t)$ are defined and continuous on the interval $[0,1]$, then $f(t)+g(t)$ and $cf(t)$ are also defined and continuous on the same interval. The 8 axioms are also easily verified:

  ...with the zero function defined as: $z(t) = 0$ for all t in $[0,1]$

...and $(-f)$ as: $$(-f)(t) = -f(t) \quad \text{for all } t \text{ in } [0,1]$$

- P_n: The set of all polynomials of degree less than n ...another kind of 'function space':

$$\text{Vector:} \quad p(t) = a_0 + a_1 t + \cdots + a_{n-2} t^{n-2} + a_{n-1} t^{n-1}$$

$$\text{Vector addition } (p+q): \quad (p+q)(t) = p(t) + q(t)$$

$$\text{Scalar multiplication } (cp): \quad (cp)(t) = c\,p(t)$$

$$\text{Zero vector } z(t): \quad z(t) = 0 + 0t + \cdots + 0t^{n-2} + 0t^{n-1}$$

...and the 8 axioms are easily verified.

ASIDE (for future reference): In P_n, a vector p can also be described (in unique fashion) by its set of coefficients $\{a_0, a_1, \ldots, a_{n-1}\}$, and these can be written in the form of an $n \times 1$ column matrix

$$\tilde{p} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

So we can define a one-to-one *mapping* (or *correspondence*) between the vectors in P_n and the $n \times 1$ column vectors in $\mathbb{R}^n$:

$$p(t) = a_0 + a_1 t + \cdots + a_{n-2} t^{n-2} + a_{n-1} t^{n-1} \quad \leftrightarrow \quad \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

...and $$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \leftrightarrow \quad u(t) = u_1 + u_2 t + \cdots + u_{n-1} t^{n-2} + u_n t^{n-1}$$

DEF'N: When this kind of one-to-one mapping is possible between two vector spaces, the spaces are said to be isomorphic.

SUBSPACES:

A subspace of a vector space V is any (nonempty) subset S of V that itself constitutes a vector space; i.e., a subset of V that is closed under vector addition and scalar multiplication.

Since the vectors in S (the elements of S) all belong to V, the other 8 conditions/axioms are automatically satisfied. Only the closure conditions need to be examined and *verified* to determine if S forms a vector space.

Ex: The subset S of $\mathbb{R}^2$ consisting of all vectors $\mathrm{x} = [x_1, x_2]^T$ having $x_2 = 2x_1$

...i.e.: all vectors of the form: $\begin{bmatrix} a \\ 2a \end{bmatrix}$

Multiplying by any scalar c: $$c\begin{bmatrix} a \\ 2a \end{bmatrix} = \begin{bmatrix} ca \\ 2ca \end{bmatrix};$$

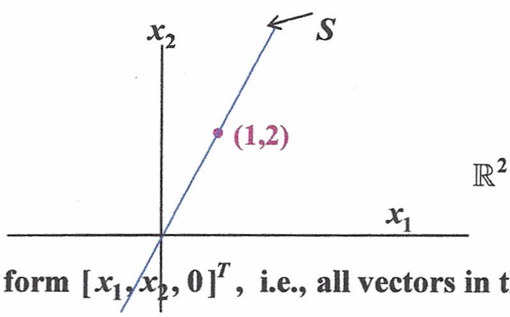
and adding any 2 such vectors:

$$\begin{bmatrix} a \\ 2a \end{bmatrix} + \begin{bmatrix} b \\ 2b \end{bmatrix} = \begin{bmatrix} a+b \\ 2a+2b \end{bmatrix} = \begin{bmatrix} (a+b) \\ 2(a+b) \end{bmatrix}$$

yields a vector of the same form, and so these vectors are also elements of S.

Thus, S is closed under vector addition and scalar multiplication and hence forms a subspace of $\mathbb{R}^2$.

Geometric picture:



Ex: In $\mathbb{R}^3$, all vectors of the form $[x_1, x_2, 0]^T$, i.e., all vectors in the x_1, x_2-plane:

$$\mathrm{u} + \mathrm{v} = [u_1, u_2, 0]^T + [v_1, v_2, 0]^T = [u_1 + v_1,\ u_2 + v_2,\ 0]^T$$

$$c\,\mathrm{u} = c\,[u_1, u_2, 0]^T = [c\,u_1, c\,u_2, 0]^T$$

This set is closed under addition and scalar multiplication, and hence is a subspace of $\mathbb{R}^3$.

[**Comment:** Any subspace other than V itself, or the set $\{0\}$ containing only the zero
vector (also a subspace!), is called a **proper subspace** of V.]

Ex: The subset S of $\mathbb{R}^2$ consisting of all vectors $\mathbf{x} = [x_1, 1]^T$.

S is ***not*** a subspace of $\mathbb{R}^2$, since

Ex: The subset S of $R^{n \times n}$ consisting of all skew-symmetric $n \times n$ matrices:

These are all matrices of the form: $$A = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{bmatrix}$$

true for the sets of all **symmetric**, **diagonal**, and **triangular** $n \times n$ matrices.

not true for the sets of all **singular** or **nonsingular** $n \times n$ matrices:

SPAN AND SPANNING SETS:

DEF'N: Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in V, the set of ***all possible*** linear combinations

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \sum_{i=1}^{n} c_i\mathbf{v}_i \qquad (c_1, c_2, \ldots, c_n \text{ all real nos.})$$

is called the **span** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, or **span**$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

Ex: In $\mathbb{R}^3$, the span of the vectors e_1 and e_2 is the set of all vectors of the form

$$c\,e_1 + d\,e_2 = c\begin{bmatrix}1\\0\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}c\\d\\0\end{bmatrix}$$

Clearly, $\text{span}\{e_1, e_2\}$ is a subspace of $\mathbb{R}^3$ (closure under addition & scalar multiplication).

Geometrically, it consists of all vectors in 3-space that lie in the x_1x_2-plane. and $\text{span}\ \{e_1, e_2, e_3\}$ is all vectors of the form

$$c_1 e_1 + c_2 e_2 + c_3 e_3 = \begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

which comprises *all* vectors in $\mathbb{R}^3$. So:

$$\text{span}\ \{e_1, e_2, e_3\} = \mathbb{R}^3$$

THEOREM: If $v_1, v_2, \ldots, v_n$ are any vectors in V, then $\text{span}\{v_1, v_2, \ldots, v_n\}$ is a subspace of V.

PROOF:

In $\mathbb{R}^3$, if two vectors v and w can be used to define a plane in 3-space, this plane is the geometrical representation of $\text{span}\ \{\, v, w \,\}$.

$\text{span}\{v_1, v_2, \ldots, v_n\}$ is also called the subspace of V spanned by $v_1, v_2, \ldots, v_n$.

If $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = V$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are said to **span** or **generate** V; ...and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be a **spanning set** for V.

DEF'N: The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a **spanning set** for V provided (if and only if) *every vector in V* can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

EXS: Do the following sets span $\mathbb{R}^3$?

1) $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, [1, 2, 3]^T\}$

2) $S = \{[1, 1, 1]^T, [1, 1, 0]^T, [1, 0, 0]^T\}$

3) $S = \{[1, 0, 1]^T, [0, 1, 0]^T\}$

4) $S = \{[1, 2, 4]^T, [2, 1, 3]^T, [4, -1, 1]^T\}$