DEF'N: **If a vector $\mathbf{v}$ can be written in terms of a basis set $\mathbf{v}_1, \ldots, \mathbf{v}_n$ as**

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

then the set of coefficients $c_1, c_2 \ldots, c_n$ are called the coordinates of $\mathbf{v}$ with respect to the ordered basis $\mathcal{B} = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$, and the $n \times 1$ column vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is called the coordinate vector of $\mathbf{v}$ with respect to the ordered basis $\mathcal{B} = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$.

EX: **vector $\mathbf{x} = (x_1, x_2)^T$ in R^2 can be written in terms of the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$ as**

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

x_1 and x_2 are the coordinates of $\mathbf{x}$ with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

We can also write $\mathbf{x}$ as a linear combination of any other basis vectors in R^2,

i.e., any two linearly independent vectors $\mathbf{y}$, $\mathbf{z}$:

$$\mathbf{x} = \alpha\,\mathbf{y} + \beta\,\mathbf{z}$$

α, β are then the coordinates of $\mathbf{x}$ with respect to the basis $\mathcal{C} = \{\mathbf{y}, \mathbf{z}\}$. If we *order* the vectors $\mathbf{y}, \mathbf{z}$ ($\mathbf{y}$ being the 1st vector in the basis and $\mathbf{z}$ the 2nd) and denote the ordered basis by:

$$\mathcal{C} = [\mathbf{y}, \mathbf{z}]$$

then $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = (\alpha, \beta)^T$ is the coordinate vector of $\mathbf{x} = [x_1, x_2]^T$ with respect to [$\mathbf{y}, \mathbf{z}$], $[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

So the coordinate vector of x with respect to $[\mathbf{e}_1, \mathbf{e}_2]$ is $[x_1, x_2]^T$, but this will *not* be true for any *other* basis set [y, z].

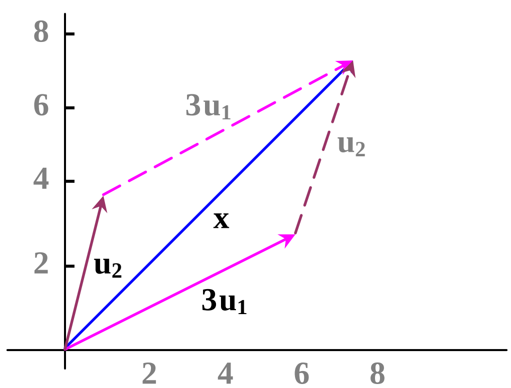
EXAMPLE: In R^2, let: $\mathbf{u}_1 = [2, 1]^T$ and $\mathbf{u}_2 = [1, 4]^T$

(Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent, they form a basis for R^2.) The vector

$$\mathbf{x} = [7, 7]^T = 7\mathbf{e}_1 + 7\mathbf{e}_2$$

can also be written: $\mathbf{x} = 3\mathbf{u}_1 + \mathbf{u}_2$

and so the coordinate vector of x with respect to $[\mathbf{u}_1, \mathbf{u}_2]$ is $[3, 1]^T$:



EXAMPLE: In P_4, let: $p(x) = 2 - 3x + x^2 - x^3/2$

and let $B = [p_0, p_1, p_2, p_3]$ **be the standard basis:**

$$p_0(x) = 1; \quad p_1(x) = x; \quad p_2(x) = x^2; \quad p_3(x) = x^3$$

…Then $$p = 2p_0 - 3p_1 + p_2 - (1/2)p_3$$

…and so the coordinate vector of p **with respect to** $[p_0, p_1, p_2, p_3]$ **is:**

$$[p]_B = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1/2 \end{bmatrix}$$

Example. Consider $\mathcal{B} = \{b_1, b_2\}$ and $\mathcal{C} = \{c_1, c_2\}$ for a vector space V, such that

$b_1 = 4c_1 + c_2$ and $b_2 = -6c_1 + c_2$.

Suppose $[x]_B = [3, 1]^T$. Find $[x]_C$.

Example. Find the coordinate vector of u = (4, 5) relative to $\mathcal{B} = \{(1, 0), (0, 1)\}$ and $\mathcal{B}' = \{(2, 1), (-1, 1)\}$.

Theorem. Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots \mathbf{u}_n\}$ and $\mathcal{B}' = \{\mathbf{u}_1', \mathbf{u}_2', \ldots \mathbf{u}_n'\}$ be bases for a vector space U. If **u** is a vector in U, having coordinate vectors $\mathbf{u}_{\mathcal{B}}$ and $\mathbf{u}_{\mathcal{B}'}$.
Then $\mathbf{u}_{\mathcal{B}}' = P\,\mathbf{u}_{\mathcal{B}}$, where P is the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.

$$P = [\,[u_1]_{\mathcal{B}'} \quad [u_2]_{\mathcal{B}'} \ldots [u_n]_{B'}\,]$$

Example. Consider the bases $\mathcal{B} = \{(1, 2), (3, -1)\}$, $\mathcal{B}' = \{(1, 0), (0,1)\}$.
If **u** is a vector such that $\mathbf{u}_{\mathcal{B}} = (3, 4)^T$. Find $\mathbf{u}_{B'}$

Example . Consider Bases $\mathcal{B} = \{(1, 2), (3, -1)\}$, $\mathcal{B}' = \{(3, 1), (5,2)\}$ on R^2.

Find the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.

If **u** is a vector such that $\mathbf{u}_{\mathcal{B}} = (2, 1)^T$. Find $\mathbf{u}_{B'}$

$\mathbf{u}_{\mathcal{B}'} = P\ \mathbf{u}_{\mathcal{B}}$ and $\mathbf{u}_{\mathcal{B}} = Q\ \mathbf{u}_{\mathcal{B}'}$

$\Longrightarrow$ $\mathbf{u}_{\mathcal{B}'} = P\ \mathbf{u}_{\mathcal{B}} = PQ\,\mathbf{u}_{\mathcal{B}'}$ and $\mathbf{u}_{\mathcal{B}} = Q\ \mathbf{u}_{\mathcal{B}'} = Q\,P\ \mathbf{u}_{\mathcal{B}}$

$\Longrightarrow$ $PQ = I = QP$

$\Longrightarrow$ $Q = P^{-1}$

Theorem. Let $\mathcal{B} = \{\mathbf{u_1}, \mathbf{u_2}, \ldots \mathbf{u_n}\}$ and $\mathcal{B}' = \{\mathbf{u_1'}, \mathbf{u_2'}, \ldots \mathbf{u_n'}\}$ be two bases for R^n .
The transition matrix $\mathcal{B}$ to $\mathcal{B}'$ can be found by using Gauss-Jordan elimination on the n x2n matrix.

$$[\mathcal{B}' : \mathcal{B}] \Longrightarrow [\, I : P \,]$$

Example . Consider Bases Consider Bases $\mathcal{B} = \{(1, 2), (3, -1)\}$,
$\mathcal{B}' = \{(3, 1), (5,2)\}$ on R^2.

Find the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.

If **u** is a vector such that $\mathbf{u}_{\mathcal{B}} = (2, 1)^T$. Find $\mathbf{u}_{B'}$

Example. Let $u_1 = (1, 0, 0)$, $u_2 = (0, 1, 0)$, $u_3 = (0, 0, 1)$ $\mathcal{B} = \{u_1, u_2, u_3\}$
$c_1 = (1,0, 1)$, $c_2 = (0, -1, 2)$ $c_3 = (2, 3, -5)$, and $\mathcal{B}' = \{c_1, c_2, c_3\}$

Find the transition matrix from $\mathcal{B}$ to $\mathcal{B}'$.