TRIANGULAR (LU) FACTORIZATION

If an $n \times n$ matrix A can be reduced to upper triangular form U using *only* row operation III, then the reduction process can be used to generate a factorization of A:

If $A \to \cdots \to U$, **then:** $U = E_k E_{k-1} \cdots E_2 E_1 A$ **(upper triangular)**

and so:

$$A = (E_k E_{k-1} \cdots E_2 E_1)^{-1} U$$

$$= LU$$

where $L = (E_k E_{k-1} \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}$ **is (unit) lower triangular.**

EXAMPLE: $A = \begin{bmatrix} 1 & 4 \\ -2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 \\ 0 & 8 \end{bmatrix} = U$

$$row2 - (-2) \bullet row1 \to row2, \; l_{21} = -2$$

$$E_1 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\Rightarrow LU = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -2 & 0 \end{bmatrix} = A$$

OR: We can also say: $U = E_1 A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} A$ **(adds twice row 1 to row 2)**

and so: $L = E_1^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

EXAMPLE:

$$A = \begin{bmatrix} \mathbf{2} & \mathbf{4} & \mathbf{2} \\ \mathbf{1} & \mathbf{5} & \mathbf{2} \\ \mathbf{4} & \mathbf{-1} & \mathbf{9} \end{bmatrix}$$

TRIANGULAR (LU) FACTORIZATION

If an $n \times n$ matrix A can be reduced to upper triangular (UT) form using *only* row operation III, then it is possible to express the reduction process as a matrix factorization.

EXAMPLE:

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$$

Using only row operation III:

$$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$$

To keep track of the multiples we subtracted: set $l_{21} = 1/2$ and $l_{31} = 2$.

Next eliminate the -9 in the last row:

$$\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \to \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} \equiv U$$

And set $l_{32} = -3$ (multiple of row 2 we subtracted from row 3).

...Call the resulting matrix U, and define

$$L \equiv \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

Then (...verify!):

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} = A$$

i.e.: the matrix A can be factored into a product of a lower-triangular matrix L and an upper-triangular matrix U.

A lower triangular matrix with all 1's along the diagonal is a unit lower triangular matrix.

In terms of elementary matrices, the process in the example can be represented by

$$E_3 E_2 E_1 A = U \tag{1}$$

where $$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}; \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

Since each is nonsingular, multiply (1) from the left by $(E_3 E_2 E_1)^{-1}$**:**

$$A = E_1^{-1} E_2^{-1} E_3^{-1} U$$

Multiplied in this order, the multipliers l_{21}, l_{31}, l_{32} **fill in below the diagonal in the product:**

$$E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = L$$

**** If an $n \times n$ matrix A can be reduced to upper triangular (UT) form using *only* row operation III, then A has an LU factorization, where L is unit lower triangular.**

** Solving system of Linear equations by LU factorization

$AX = b$

→ $LUX = b$

→ $Ly = b$ where $y = Ux$

→ Solve for y

→ $y = Ux$

Example. Solve the following system using 1) LU factorization

2) A^{-1}

$2x + 6y + 2z = 2$

$-3x \quad -8y = 0$

$4x + 9y + 2z = 2$