Review

- Matrix Multiplication

$$C = A \cdot B \quad , \quad A_{m\times \underline{\underline{n}}} \cdot B_{\underline{\underline{n}}\cdot p} = C_{m\times p}$$

Example.

Let $A = \begin{bmatrix} 2 & 0 & -5 \\ 1 & 3 & 2 \\ 4 & 1 & -1 \\ 0 & 2 & 7 \end{bmatrix}_{4\times 3}$ and $B = \begin{bmatrix} 5 & 1 \\ -2 & 3 \\ 1 & 0 \end{bmatrix}_{3\times 2}$

$$\Rightarrow \quad AB = \begin{array}{c} \\ \\ r3 \\ \\ \end{array}\begin{bmatrix} 2 & 0 & -5 \\ 1 & 3 & 2 \\ 4 & 1 & -1 \\ 0 & 2 & 7 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 5 & 1 \\ -2 & 3 \\ 1 & 0 \end{bmatrix} \\ \quad\ c2 \end{array} = \begin{bmatrix} 5 & 2 \\ 1 & 0 \\ 17 & \boxed{7} \\ 3 & 6 \end{bmatrix}_{4\times 2} \begin{array}{c} \\ \\ c_{32} \\ \\ \end{array} = C$$

- Diagonal Matrix

$$D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & a_{nn} \end{bmatrix}$$

Eg. $\begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}$, $\begin{bmatrix} 7 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & 1 \end{bmatrix} = I = \{\delta_{ij}\}$ Identity matrix

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Kronecker delta

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

* Key Property of the Identity matrix I,

$$IA = AI = A$$

$$* \quad D = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & a_{nn} \end{bmatrix} \rightarrow D^p = \begin{bmatrix} a_{11}^p & 0 & \cdots & 0 \\ 0 & a_{22}^p & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & a_{nn}^p \end{bmatrix}$$

Example. Given $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$,

$D = \begin{bmatrix} 0 & 5 & 7 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix}$, and $E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$,

Evaluate each of the following.

1) A^{100}

2) B^{100}

3) C^{100}

4) D^{100}

5) E^{100}

6) $(ABC)^3$

7) $A^3 B^3 C^3$

8) $(BC)^4$

9) $B^4 C^4$

* Partitioning Matrix

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 5 & 2 & 7 \\ 1 & 3 & 0 \\ 0 & 4 & 6 \end{bmatrix} = \left[\begin{array}{cc|c} 2 & 0 & -3 \\ 5 & 2 & 7 \\ \hline 1 & 3 & 0 \\ \hline 0 & 4 & 6 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix},$$

where the blocks are

$$A_{11} = \begin{bmatrix} 2 & 0 \\ 5 & 2 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -3 \\ 7 \end{bmatrix}, \quad A_{21} = [1, 3], \dots$$

Example.

If $A = \left[\begin{array}{cc|c} 2 & 4 & 1 \\ -1 & 3 & 0 \\ \hline 5 & 4 & 6 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ and

$$B = \left[\begin{array}{cc|c} 0 & 1 & 3 \\ 2 & -4 & -1 \\ \hline 5 & 8 & 2 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Then $AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$

$$A_{11}B_{11} + A_{12}B_{21} = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [5\ 8]$$

$$= \begin{bmatrix} 8 & -14 \\ 6 & -13 \end{bmatrix} + \begin{bmatrix} 5 & 8 \\ 0 & 0 \end{bmatrix} = \underline{\begin{bmatrix} 13 & -6 \\ 6 & -13 \end{bmatrix}}$$

$$A_{11}B_{12} + A_{12}B_{22} = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [2]$$

$$= \begin{bmatrix} 2 \\ -6 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underline{\begin{bmatrix} 4 \\ -6 \end{bmatrix}}$$

$$A_{21}B_{11} + A_{22}B_{21} = [5\ 4] \begin{bmatrix} 0 & 1 \\ 2 & -4 \end{bmatrix} + [6][5\ 8] = \underline{[38, 37]}$$

$$A_{21}B_{12} + A_{22}B_{22} = [5\ 4] \begin{bmatrix} 3 \\ -1 \end{bmatrix} + [6][2] = \underline{[23]}$$

$$\Rightarrow AB = \left[\begin{array}{cc|c} 13 & -6 & 4 \\ 6 & -13 & -6 \\ \hline 38 & 37 & 23 \end{array}\right] = \begin{bmatrix} 13 & -6 & 4 \\ 6 & -13 & -6 \\ 38 & 37 & 23 \end{bmatrix}$$

MATRIX MULTIPLICATION IS PECULIAR **(counterintuitive):**

1) In general, $BA \neq AB$ **(even if both are** $n \times n$**) :**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix} \quad \text{...but:} \quad \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 7 & 10 \end{bmatrix}$$

2) $AB = 0$ **does** ***not*** **imply** $A = 0$ **or** $B = 0$ **or** $BA = 0$**:**

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{...but:} \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

3) $AC = AD$ **does** ***not*** **imply** $C = D$ **(even if** $A \neq 0$**):**

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

$$\text{...but:} \quad \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \neq \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

For any $n \times n$ **matrix** A **:** $\qquad AI = IA = A$

$$\text{e.g.:} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

NOTATION: **The column vectors of** I **are denoted by** $\mathbf{e}_j$**, i.e.,**

$$I_n = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$$

RULES FOR TRANSPOSES:

1) $(A^T)^T = A$:
$$\left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T\right]^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

2) $(\alpha A)^T = \alpha A^T$:
$$\left[\alpha\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right]^T = \begin{bmatrix} \alpha & 2\alpha \\ 3\alpha & 4\alpha \end{bmatrix}^T = \begin{bmatrix} \alpha & 3\alpha \\ 2\alpha & 4\alpha \end{bmatrix}$$
$$= \alpha\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \alpha\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T$$

3) $(A+B)^T = A^T + B^T$:
$$\left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right]^T = \begin{bmatrix} 1+a & 2+b \\ 3+c & 4+d \end{bmatrix}^T$$
$$= \begin{bmatrix} 1+a & 3+c \\ 2+b & 4+d \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T + \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T$$

4) $(AB)^T = B^T A^T$:
$$\left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\right]^T = \begin{bmatrix} a+2c+3e & b+2d+3f \\ 4a+5c+6e & 4b+5d+6f \end{bmatrix}^T = \begin{bmatrix} a+2c+3e & 4a+5c+6e \\ b+2d+3f & 4b+5d+6f \end{bmatrix}$$

and $\quad B^T A^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} a+2c+3e & 4a+5c+6e \\ b+2d+3f & 4b+5d+6f \end{bmatrix}$ (checks)

The other way 'round:

$$(BA)^T = \left[\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\right]^T = \begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \\ e+4f & 2e+5f & 3e+6f \end{bmatrix}^T$$

$$= \begin{bmatrix} a+4b & c+4d & e+4f \\ 2a+5b & 2c+5d & 2e+5f \\ 3a+6b & 3c+6d & 3e+6f \end{bmatrix}$$

and $\quad A^T B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} = \begin{bmatrix} a+4b & c+4d & e+4f \\ 2a+5b & 2c+5d & 2e+5f \\ 3a+6b & 3c+6d & 3e+6f \end{bmatrix}$ (checks)

DEF'N: An $n \times n$ matrix A is nonsingular or invertible if there is an $n \times n$ matrix B such that

$$AB = BA = I$$

B is called the multiplicative inverse (or inverse) of A.

If B and C are *both* inverses of A, then

$$B = BI = B(AC) = (BA)C = IC = C$$

i.e.: if A has an inverse, the inverse is unique.

DEF'N: An $n \times n$ matrix is singular or noninvertible if it *does not have* an inverse.

EXAMPLE: $\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ ... If B is *any* 2×2 matrix:

$$BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{11} \\ b_{21} & b_{21} \end{bmatrix} \neq I \text{ (for } any\ b_{11}, b_{21} \text{)}$$

$\Rightarrow$ A is singular.

$n \times n$ MATRICES:

$n \times n$ matrices can be broadly divided into two classes: <u>nonsingular</u> (invertible) & <u>singular</u> (noninvertible). Each class has a unique set of characteristics that we will develop and exploit throughout the course.

RULES FOR INVERSES:

If A and B are nonsingular (invertible) $n \times n$ matrices,

1. A^{-1} **is nonsingular, and** $(A^{-1})^{-1} = A$
2. (αA) **is nonsingular, and** $(\alpha A)^{-1} = \alpha^{-1} A^{-1}$ (for $\alpha \neq 0$)
3. AB **is nonsingular, and** $(AB)^{-1} = B^{-1} A^{-1}$

1): If $C = A^{-1}$, then $AC = CA = I$; or, $CA = AC = I$.

Thus, $A = C^{-1} = (A^{-1})^{-1}$

2):

$$(\alpha^{-1} A^{-1})(\alpha A) = (\alpha^{-1}\alpha)(A^{-1}A) = 1 \cdot (I) = I \text{ ; and}$$
$$(\alpha A)(\alpha^{-1} A^{-1}) = (\alpha\alpha^{-1})(AA^{-1}) = 1 \cdot (I) = I$$

3):

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I \text{ ; and}$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

...by extension of (3), any product of nonsingular $n \times n$ matrices $A_1, A_2, \ldots, A_k$ is also nonsingular, and

$$\begin{aligned} (A_1 A_2 \cdots A_k)^{-1} &= \{A_1(A_2 \cdots A_k)\}^{-1} \\ &= (A_2 \cdots A_k)^{-1} A_1^{-1} \\ &\vdots \\ &= A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1} \end{aligned}$$

...e.g., $(ABC)^{-1} = [(AB)C]^{-1} = C^{-1}(AB)^{-1} = C^{-1}(B^{-1}A^{-1}) = C^{-1}B^{-1}A^{-1}$

COMPUTING THE INVERSE:

How do we *calculate* the inverse of an $n \times n$ matrix A?

3×3 case: Suppose $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ **is the inverse of A; then**

$$AB = A[\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3] = [A\mathbf{b}_1, A\mathbf{b}_2, A\mathbf{b}_3] = I = [\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3] \qquad (1)$$

or: $$A\mathbf{b}_1 = \mathbf{e}_1;\quad A\mathbf{b}_2 = \mathbf{e}_2;\quad A\mathbf{b}_3 = \mathbf{e}_3$$

This amounts to solving 3 separate linear systems for the three column vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$:

$$\begin{aligned} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} &= 1 \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} &= 0 \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} &= 0 \end{aligned} \;;\quad \begin{aligned} a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} &= 0 \\ a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} &= 1 \\ a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} &= 0 \end{aligned} \;;\quad \begin{aligned} a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} &= 0 \\ a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} &= 0 \\ a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} &= 1 \end{aligned}$$

These three systems can be solved by constructing and row-reducing the three 3×4 augmented matrices:

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 1 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right] ; \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 1 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right] ; \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 1 \end{array}\right]$$

OR, equivalently, by row-reducing 'in one go' the single $3\times \underline{6}$ augmented matrix

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right] = [A \mid I]$$

The final result will be:

$$[A|I] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & b_{11} & b_{12} & b_{13} \\ 0 & 1 & 0 & b_{21} & b_{22} & b_{23} \\ 0 & 0 & 1 & b_{31} & b_{32} & b_{33} \end{array}\right] = [I|A^{-1}]$$

(2)

This works for *<u>any</u>* $n \times n$ matrix.

Also, it's clear from (2) that the set of equations (1) defining the inverse of A will have a solution when (and only when) A can be row-reduced to the $n \times n$ identity, I. So:

> **An $n \times n$ matrix A is invertible if (and only if) $\text{rank} A = n$. The inverse of A can**
> **then be calculated by row reduction of the $n \times 2n$ augmented matrix $[A|I]$:**
>
> $$[A|I] \rightarrow [I|A^{-1}]$$

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$: $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right]$

$$\rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 1/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right]$$

So: $$A^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, **the above process produces** $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

...and so a 2×2 matrix is invertible (nonsingular) if (and only if) $ad - bc \neq 0$.

DEF'N: The quantity $ad - bc$ is called the <u>determinant of A</u>, or <u>det A</u>.