

## Matrices

- Transpose of a matrix
- Square matrix
- Diagonal matrix
- Identity matrix
- Triangular matrix
- Symmetric matrix
- Skew -symmetric matrix

## Operations of Matrices

- **MATRIX ADDITION AND SCALAR MULTIPLICATION:**

If  $A = [a_{ij}]$ ,  $B = [b_{ij}]$  are matrices of size  $m \times n$ , then

their sum is the  $m \times n$  matrix given by

$$A + B = [a_{ij} + b_{ij}] \quad \text{and}$$

$$kA = [k a_{ij}]$$

- **Matrix Multiplication**

$$C = A \cdot B = [c_{ij}] = \sum_{k=1}^n a_{ik} b_{kj}$$

$(m \times n) \cdot (n \times r) = m \times r$

If  $A$  is  $m \times n$  and  $B$  is  $n \times m$ , then both  $AB$  and  $BA$  are defined... but  $AB$  is  $m \times m$  while  $BA$  is  $n \times n$ :

EXAMPLE:

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix}; \quad B = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix}$$

$$\begin{aligned} AB &= \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 3 \cdot (-2) - 2 \cdot 4 & 3 \cdot 1 - 2 \cdot 1 & 3 \cdot 3 - 2 \cdot 6 \\ 2 \cdot (-2) + 4 \cdot 4 & 2 \cdot 1 + 4 \cdot 1 & 2 \cdot 3 + 4 \cdot 6 \\ 1 \cdot (-2) - 3 \cdot 4 & 1 \cdot 1 - 3 \cdot 1 & 1 \cdot 3 - 3 \cdot 6 \end{bmatrix} \\ &= \begin{bmatrix} -14 & 1 & -3 \\ 12 & 6 & 30 \\ -14 & -2 & -15 \end{bmatrix} \quad (3 \times 3) \end{aligned}$$

...and  $BA$  is also defined:

$$BA = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 20 & -22 \end{bmatrix} \quad (2 \times 2)$$

But NOTE that:  $BA \neq AB$  ...i.e., matrix multiplication is *not* commutative.

The  $m \times n$  linear system  $S$

$$\begin{array}{cccc} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

Can be written as the matrix multiplication

$$Ax = b$$

Or

As a linear combination of the column vectors of  $A$

$$Ax = x_1a_1 + x_2a_2 + \cdots + x_na_n$$

Ex:  $3 \times 3$  system:

$$\begin{array}{rcl} 3x_1 + 2x_2 + 3x_3 & = & 5 \\ x_1 - 2x_2 + 5x_3 & = & -2 \\ 2x_1 + x_2 - 3x_3 & = & 1 \end{array}$$

**THEOREM:** A linear system  $Ax = b$  is consistent (has at least one solution) provided (iff)  $b$  can be expressed as a linear combination of the column vectors of  $A$ .

**EXAMPLE:**

$$2x_1 + 3x_2 - 2x_3 = 5$$

$$5x_1 - 4x_2 + 2x_3 = 6$$

### RULES OF MATRIX ALGEBRA:

For any scalars (real nos.)  $\alpha, \beta$  and any matrices  $A, B, C$  for which the indicated operations are defined :

1.  $A + 0 = A$
2.  $A + (-A) = 0$  (-A is the additive inverse of A)
3.  $A + B = B + A$  (matrix addition is commutative)
4.  $(A + B) + C = A + (B + C)$  (matrix addition is associative)
5.  $(AB)C = A(BC)$  (matrix multiplication is associative)
6.  $A(B + C) = AB + AC$  (matrix multiplication is distributive wrt addition)
7.  $(A + B)C = AC + BC$  (same)
8.  $(\alpha\beta)A = \alpha(\beta A) = \beta(\alpha A)$
9.  $\alpha(AB) = (\alpha A)B = A(\alpha B)$
10.  $(\alpha + \beta)A = \alpha A + \beta A$
11.  $\alpha(A + B) = \alpha A + \alpha B$

### Noncommutativity of Matrix Multiplication

- 1) In general,  $BA \neq AB$  (even if both are  $n \times n$ ):

### Cancellation is not Valid

- 2)  $AB = 0$  does *not* imply  $A = 0$  or  $B = 0$  or  $BA = 0$ :
- 3) If  $cA = 0$ , then  $c = 0$  or  $A = 0$

### Powers of a square matrix:

$$A^2 = AA; \quad A^3 = AA^2; \quad \dots \quad A^n = AA^{n-1} = AA \cdots A$$

...and:  $A^0 \equiv I$ , ... But if  $m \neq n$ ,  $A^2$  is not defined.

**Roots of a square matrix:** If  $A = B^n$ , then  $B = \sqrt[n]{A}$

**Recall the  $n \times n$  identity matrix:** 
$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

**For any  $n \times n$  matrix  $A$ :**  $AI = IA = A$

**NOTATION:** The column vectors of  $I$  are denoted by  $e_j$ ,

i.e.,  $I_n = (e_1, e_2, \dots, e_n)$

#### **RULES FOR TRANSPOSES:**

1)  $(A^T)^T = A$ : 
$$\left[ \left[ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T \right]^T =$$

2)  $(\alpha A)^T = \alpha A^T$ : 
$$\left[ \alpha \left[ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right]^T =$$

3)  $(A+B)^T = A^T + B^T$ : 
$$\left[ \left[ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right]^T =$$

4)  $(AB)^T = B^T A^T$ : 
$$\left[ \left[ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \left[ \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \right]^T \right]^T =$$

...and  $B^T A^T = \left[ \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} \right]^T \left[ \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \right]^T =$

The other way 'round:

$$(BA)^T = \left[ \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \right]^T =$$

...and  $A^T B^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix} =$

**DEF'N:** An  $n \times n$  matrix  $A$  is nonsingular or invertible if there is an  $n \times n$  matrix  $B$  such that

$$AB = BA = I$$

... $B$  is called the multiplicative inverse (or inverse) of  $A$ .

If  $B$  and  $C$  are *both* inverses of  $A$ , then

$$B = BI = B(AC) = (BA)C = IC = C$$

...i.e.: if  $A$  has an inverse, the inverse is unique.

**DEF'N:** An  $n \times n$  matrix is singular or noninvertible if it *does not have* an inverse.

EXAMPLE:  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

### **$n \times n$ MATRICES:**

**$n \times n$  matrices can be broadly divided into two classes: nonsingular (invertible) & singular (noninvertible). Each class has a unique set of characteristics that we will develop and exploit throughout the course.**

### **RULES FOR INVERSES:**

**If  $A$  and  $B$  are nonsingular (invertible)  $n \times n$  matrices,**

- 1.  $A^{-1}$  is nonsingular, and  $(A^{-1})^{-1} = A$**
- 2.  $(\alpha A)$  is nonsingular, and  $(\alpha A)^{-1} = \alpha^{-1} A^{-1}$  (for  $\alpha \neq 0$ )**
- 3.  $AB$  is nonsingular, and  $(AB)^{-1} = B^{-1} A^{-1}$**

**...by extension of (3), any product of nonsingular  $n \times n$  matrices  $A_1, A_2, \dots, A_k$  is also nonsingular, and**

$$\begin{aligned}(A_1 A_2 \cdots A_k)^{-1} &= \{A_1 (A_2 \cdots A_k)\}^{-1} \\ &= (A_2 \cdots A_k)^{-1} A_1^{-1} \\ &\vdots \\ &= A_k^{-1} A_{k-1}^{-1} \cdots A_1^{-1}\end{aligned}$$

**...e.g.,**  $(ABC)^{-1} = [(AB)C]^{-1} = C^{-1}(AB)^{-1} = C^{-1}(B^{-1}A^{-1}) = C^{-1}B^{-1}A^{-1}$