The set $\{v_1, v_2, \ldots, v_n\}$ is a spanning set for V provided (if and only if) *every vector in V* can be written as a linear combination of $v_1, v_2, \ldots, v_n$.

A minimal spanning set is one with no unnecessary or redundant elements: *all* the vectors in the set are *needed* to span V.
In order to decide if a set of vectors $\{v_1, v_2, \ldots, v_n\}$ constitutes a minimal spanning set, we need to consider if/how the vectors of the set 'depend' on each other. So we introduce the notions of linear dependence and linear independence...

DEF'N: The vectors $v_1, \ldots, v_n$ are said to be linearly independent if

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = \underset{\sim}{0} \qquad (3)$$

implies that all the scalars $\{c_i\}$ must be zero;

OR... in other words... if the vector equation (3) has only the zero (trivial) solution for the coefficients $c_1, c_2 \ldots, c_n$.

DEF'N: The vectors $v_1, \ldots, v_n$ are said to be linearly dependent if there exist scalars $\{c_i\}$ *not all zero* such that:

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n = \underset{\sim}{0}$$

OR... in other words... if the vector equation (3) has (one or more) nonzero solutions.

EXAMPLE:

$$u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

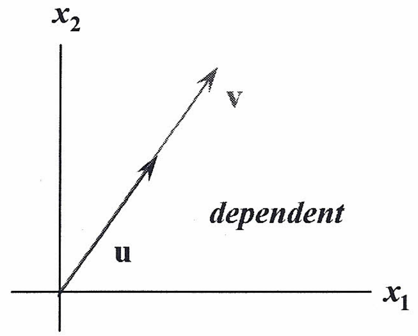
GEOMETRICALLY SPEAKING:

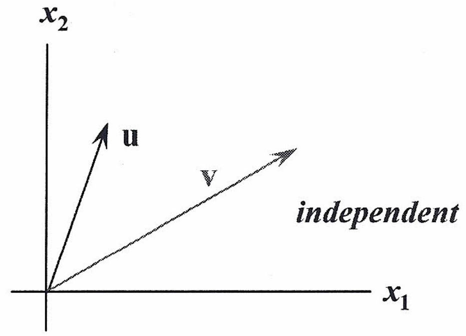
In $\mathbb{R}^2$, if two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent, then:

$$c_1\mathbf{u} + c_2\mathbf{v} = \underset{\sim}{0} \qquad \text{(with } c_1, c_2 \text{ not both zero)}$$

...so: $\mathbf{u} = -(c_2/c_1)\mathbf{v}$ (if $c_1 \neq 0$) **...or:** $\mathbf{v} = -(c_1/c_2)\mathbf{u}$ (if $c_2 \neq 0$)

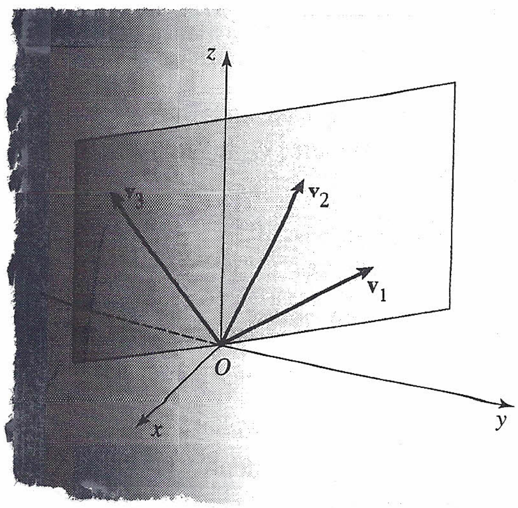
...i.e., one must be a simple scalar multiple of the other $\Rightarrow$ the vectors are *collinear*:

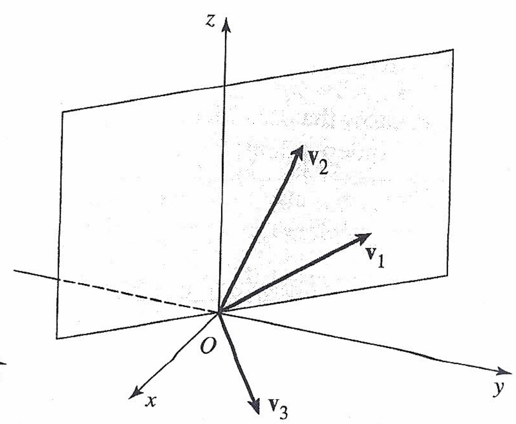




...And the same in $\mathbb{R}^3$: any two vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent if they are not scalar multiples of one another. In that case, $\mathbf{u}$ and $\mathbf{v}$ do not lie on the same line through the origin $(0,0,0)$, and so together they define a *plane* ; and any vector in that plane can be written as a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

$\Rightarrow$ If $\mathbf{w} = (w_1, w_2, w_3)$ lies *in* that plane, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly *dependent*; If $\mathbf{w}$ lies *outside* that plane, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly *independent.*





EXAMPLE:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

EXAMPLE:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}; \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}; \quad \mathbf{w} = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

there will be *non*trivial solutions (c_1, c_2, c_3). Thus u, v, w are linearly *dependent.*

These results can be summarized/generalized as:

THEOREM: If $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ are vectors in $\mathbb{R}^n$, the set $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ is linearly *dependent* provided (*iff*) the $n \times n$ matrix $V = [\mathbf{v}_1\ \mathbf{v}_2 \ldots \mathbf{v}_n]$ is singular (noninvertible).

If V is *non*singular, the set of its column vectors $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ is linearly *independent.*

To determine whether n vectors form a linearly independent set, construct an $n \times n$ matrix V whose column vectors are the vectors in question (in any order you wish), and evaluate $\operatorname{rank} V$. If $\operatorname{rank} V < n$, the vectors are a dependent set; if $\operatorname{rank} V = n$, the vectors are an independent set.

EXAMPLE.

DETERMINE THE FOLLOWING SETS ARE LINEARLY DEPENDENT OR INDEPENDENT.

1. $S = \{ \mathbf{i}, \mathbf{j}, \mathbf{k}\}$, $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$

2. $S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\mathbf{v}_1 = (2, -1, 0, 3)^T$, $\mathbf{v}_2 = (1, 2, 5, -1)^T$, $\mathbf{v}_3 = (7, -1, 5, 8)^T$

3. $S = \{ \mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$, $P_1 = 1 - x$, $P_2 = 5 + 3x - 2x^2$, $P_3 = 1 + 3x - x^2$ in polynomial of degree 2.

4. $S = \{ E_1, E_2, E_3, E_4\}$, $E_1 = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$, $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

5. $\{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\mathbf{v}_1 = (1, -2, 3)$, $\mathbf{v}_2 = (5, 6, -1)$, $\mathbf{v}_3 = (3, 2, 1)$

Example.

1. Let $\{ v_1, v_2\}$ be linearly independent. Prove that $\{ v_1 + v_2, v_1 - v_2\}$ is also linearly independent.

2. Show that $\{ x + 1, x - 1, - x + 5\}$ is linearly dependent in P_1.

3. Determine whether the set $\{ \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \}$ is linearly dependent.

Basis & Dimension:

Consider $S = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}; \quad \mathbf{u}_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}; \quad \mathbf{u}_3 = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$$

S is a subspace of $\mathbb{R}^3$.
But S can be constructed from just $\mathbf{u}_1$ and $\mathbf{u}_2$, since

$$\mathbf{u}_3 = 3\mathbf{u}_1 + 2\mathbf{u}_2 \qquad \text{(by inspection)} \qquad (1)$$

$\Rightarrow$ $\mathbf{u}_3$ is already *in the span of* $\mathbf{u}_1$ and $\mathbf{u}_2$.

$$\begin{aligned} c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 &= c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3(3\mathbf{u}_1 + 2\mathbf{u}_2) \\ &= (c_1 + 3c_3)\,\mathbf{u}_1 + (c_2 + 2c_3)\,\mathbf{u}_2 \\ &= d_1\mathbf{u}_1 + d_2\mathbf{u}_2 \end{aligned}$$

$$\Rightarrow \qquad S = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\}$$

i.e., $\mathbf{u}_3$ is somehow ‘redundant’ or ‘repetitive.’

Now rewrite the dependence of $\mathbf{u}_3$ on $\mathbf{u}_1$ and $\mathbf{u}_2$ [eqn. (1)] as:

$$3\mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3 = \underset{\sim}{0} \qquad (2)$$

Since the 3 coefficients are nonzero, we can solve for any vector in terms of the other 2.

$$\Rightarrow \quad \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_2\} = \text{Span}\,\{\mathbf{u}_2, \mathbf{u}_3\} = \text{Span}\,\{\mathbf{u}_1, \mathbf{u}_3\}$$

A spanning set for a vector space V is minimal if the vectors in the set are linearly independent; in this case, this ‘basic’ set of vectors provides a set of ‘building blocks’ for the entire vector space.

To determine whether n vectors form a linearly independent set, construct an $n \times n$ matrix X whose column vectors are the vectors in question (in any order you wish), and evaluate $\operatorname{rank} X$.
If $\operatorname{rank} X < n$, the vectors are a dependent set;
if $\operatorname{rank} X = n$, the vectors are an independent set.

Theorem. Let $x_1, \ldots, x_n$ be n vectors in R^n and let $X = (x_1, \ldots, x_n)$.
The vectors $x_1, \ldots, x_n$ will be linearly <u>dependent if and only if X is singular</u>.

Example. Determine whether the following vectors are linearly independent.

1.

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}; \quad u_2 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}; \quad u_3 = \begin{bmatrix} -1 \\ 3 \\ 8 \end{bmatrix}$$

2.

$$u = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \quad v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}; \quad w = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

BASIS & DIMENSION:

DEF'N: The vectors $x_1, \ldots, x_n$ form a basis for the vector space V provided:

1) $v_1, \ldots, v_n$ are linearly independent; *and*
2) $v_1, \ldots, v_n$ span V

EX: Show the following set is a basis for R^3

1. 'Standard' basis $\{e_1, e_2, e_3\}$: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$

a 3. $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};$

4. [(1, 2, 3), (0, 1, 2), (-2, 0, 1)] ...etc.

Ex. Show that the following set is a basis for R^2

1. S = { (1, 1), (1, -1)}

2. S = $\{ [2,1]^T , [1,4]^T \}$

Ex: **Basis for $\mathbb{R}^{2\times 2}$:**

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Ex. Basis for the function space P^4
(set of all polynomial functions of degree < 4)

An arbitrary vector in the space has the form:

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

So if we choose as a basis:

$$p_0(x) = 1$$
$$p_1(x) = x$$
$$p_2(x) = x^2$$
$$p_3(x) = x^3$$

Then the vector p can be written: $p = a_0 p_0 + a_1 p_1 + a_2 p_2 + a_3 p_3$

The vectors $1, x, x^2, x^3$ are considered the standard basis for P^4.