ELEMENTARY MATRICES:

GOAL: Solve a linear system $A\mathrm{x} = \mathrm{b}$ by using a sequence of simple matrix multiplications instead of row operations.

Equivalent systems:

Multiply both sides of an $m \times n$ linear system $\quad A\mathrm{x} = \mathrm{b} \quad$ **(*)**

by a *nonsingular* $m \times m$ matrix M:

$$MA\mathrm{x} = M\mathrm{b} \qquad (**)$$

Any solution x of (*) will also be a solution of (**).

any solution of (**) is also a solution of (*)

$\Rightarrow$ (*) & (**) are equivalent systems.

So: To obtain an easier-to-solve equivalent system, multiply both sides of $A\mathrm{x} = \mathrm{b}$ by a sequence of *nonsingular* matrices $E_1, E_2, \ldots, E_k$ <u>to obtain a simpler system</u>

$$U\mathrm{x} = \mathrm{c}$$

where $U = E_k E_{k-1} \cdots E_2 E_1 A = MA$

and $\mathrm{c} = E_k E_{k-1} \cdots E_2 E_1 \mathrm{b} = M\mathrm{b}$

$(MA\,\mathrm{x} = M\mathrm{b})$

Since all the E_i are nonsingular, the product $M = E_k E_{k-1} \cdots E_2 E_1$ is also nonsingular...

$\Rightarrow$ the systems $A\mathrm{x} = \mathrm{b}$ and $U\mathrm{x} = \mathrm{c}$ are equivalent.

DEF'N: An elementary matrix is an $n \times n$ matrix obtained from the identity I_n by performing a *single* elementary row operation (Type I, II or III).

⇒ **3 types of elementary matrix:**

TYPE I: **Interchange two rows of I: e.g., for :**

$$E_{\mathrm{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{row } 2 \leftrightarrow \text{row } 3$$

If A is any 3×3 matrix,

$$E_{\mathrm{I}}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

and $$AE_{\mathrm{I}} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} =$$

i.e.: multiplying on the left by E_{I} interchanges the 2nd and 3rd rows of A; multiplying on the right by E_{I} interchanges the 2nd and 3rd columns of A.

TYPE II: **Multiply a row of I by a nonzero scalar:**

e.g., $$E_{\mathrm{II}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} \alpha \cdot \text{row } 3$$

Then $$E_{\mathrm{II}}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

and $$AE_{\mathrm{II}} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix} =$$

i.e.: multiplying on the left by E_{II} multiplies the 3rd row of A by α;
multiplying on the right by E_{II} multiplies the 3rd column of A by α.

TYPE III: Add a nonzero scalar multiple of one row of I to another row: e.g.,

$$E_{\mathrm{III}} = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{array}{l} \alpha \cdot \text{row } 3 + row1 \\ \\ \\ \end{array}$$

Then $$E_{\mathrm{III}}A = \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =$$

and $$AE_{\mathrm{III}} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & \alpha \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

i.e.: multiplying on the left by E_{III} adds α times the 3rd row of A to the 1st row; multiplying on the right by E_{III} adds α times the 1st column to the 3rd column.

THEOREM: If E is an elementary matrix (type I, II or III), then E is nonsingular, and E^{-1} is an elementary matrix of the same type.

Type I: E_{I}: interchange two rows of I

$$E_{\mathrm{I}}^{-1} = E_{\mathrm{I}}$$

Type II if E_{II} multiplying a row of I by a nonzero scalar α

Then $E_{\mathrm{II}}^{-1} = (1/\alpha)E_{\mathrm{II}}$ multiplying the same row by $(1/\alpha)$

Type III: If E_{III} is formed by adding a nonzero scalar multiple α of row i to row j,

$$\text{i.e. } E_{\mathrm{III}} = \begin{bmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots \\ 0 & \cdots & 1 & \cdots & & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & \alpha & \cdots & 1 & \cdots & 0 \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{array}{l} \\ \\ \text{row } i \\ \\ \text{row } j \\ \\ \\ \end{array} \qquad E_{\mathrm{III}}^{-1} = \begin{bmatrix} 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & & & \vdots \\ 0 & \cdots & 1 & \cdots & & \cdots & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & \cdots & -\alpha & \cdots & 1 & \cdots & 0 \\ \vdots & & & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{array}{l} \\ \\ \text{row } i \\ \\ \text{row } j \\ \\ \\ \end{array}$$

* The process of row-reducing an $n \times n$ matrix (Gaussian elimination) is equivalent to a succession of multiplications by a sequence of elementary matrices $\{E_i\}$.

DEF'N: An $n \times n$ matrix B <u>is row equivalent</u> to another $n \times n$ matrix A if B can be obtained from A by a *finite* number of elementary row operations such that $B = E_k E_{k-1} \cdots E_2 E_1$ A,

where $E_1, E_2, \ldots, E_k$ elementary matrices

⇒ Two obvious results:

1) If A is row equivalent to B, then B is row equivalent to A.
2) If A is row equivalent to B, and B is row equivalent to C, then A is row equivalent to C.

THEOREM: If A is an $n \times n$ matrix, the following statements are *equivalent*:

i) A is nonsingular (invertible)
ii) The linear system $A\mathrm{x} = 0$ has only the trivial solution $(\mathrm{x} = 0)$
iii) A is row equivalent to the $n \times n$ identity I

ALSO: The $n \times n$ linear system $A\mathrm{x} = \mathrm{b}$ has a unique solution provided (*iff*) A is nonsingular.

So: If A is an $n \times n$ matrix, the following are equivalent:

- A is nonsingular (invertible)
- The $n \times n$ linear system $A\mathrm{x} = \mathrm{b}$ has a unique solution $(\mathrm{x} = A^{-1}\mathrm{b})$ for any b
- The (homogeneous) linear system $A\mathrm{x} = 0$ has only the trivial solution $(\mathrm{x} = 0)$
- $\operatorname{rank} A = n$
- A is row equivalent to the $n \times n$ identity matrix I (reduced row echelon form of A is I)
- A is a product of elementary matrices