DETERMINANTS

The determinant of an $n \times n$ (square) matrix A is a unique scalar (real no.) associated with that matrix, derived from the matrix elements $\{a_{ij}\}$.

The value of this single number *determines* if A is singular or nonsingular

2×2 matrix: If $$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

define: $$\det A \equiv a_{11}a_{22} - a_{21}a_{12}.$$

then the 2×2 matrix A is nonsingular provided: $\det A \neq 0$.

ALTERNATE NOTATION: $$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} =$$

EXAMPLE: $A = \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$:

3×3 matrix: If $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

we can perform the same type of row reduction analytically and show that A is row equivalent to I_3 *iff*:

$$a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \neq 0$$

define:

$$\det A \equiv a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22} \qquad (*)$$

Then a 3×3 matrix A is nonsingular *iff*: $\det A \neq 0$.

$n \times n$ matrix

Consider the 2×2 case: define 2 submatrices M_{11} and M_{12} as

$$M_{11} = (a_{22}) \quad \text{and} \quad M_{12} = (a_{21})$$

where M_{11} is formed from A by deleting its 1st row & 1st column, and M_{12} by deleting its 1st row & 2nd column:

$$M_{11}: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad M_{12}: \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then $\det(A)$ can be written:

$$\det A = a_{11}a_{22} - a_{12}a_{21} = a_{11}\det M_{11} - a_{12}\det M_{12}$$

For the 3×3 case, rearrange (*):

$$\det A = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \qquad (**)$$

Now, for $j = 1, 2, 3$ let M_{1j} denote the 2×2 matrix formed from A by deleting its 1st row & jth column. Then (**) becomes

$$\det A = a_{11}\det M_{11} - a_{12}\det M_{12} + a_{13}\det M_{13} \qquad (***)$$

...where $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \to M_{11} = \qquad ;$

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \to M_{12} = \qquad ; \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \to M_{13} = \qquad .$

i.e., $\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} & \end{vmatrix} - a_{12} \begin{vmatrix} & \end{vmatrix} + a_{13} \begin{vmatrix} & \end{vmatrix}$

Now, if $A = \{a_{ij}\}$ is $n \times n$, let M_{ij} be the $(n-1)\times(n-1)$ submatrix formed by deleting the row & column containing the element a_{ij} (row *i* and column *j*). Then:

$\det M_{ij}$ is called the minor of a_{ij},

...and $A_{ij} = (-1)^{i+j} \det M_{ij}$ is called the cofactor of a_{ij}

For the 2×2 case, rewrite as:

$$\det A = a_{11}a_{22} - a_{12}a_{21} = a_{11}A_{11} + a_{12}A_{12} \qquad (n=2)$$

This is called the cofactor expansion of $\det A$ along the 1st row of A.

Can also write: $\det A = a_{21}(-a_{12}) + a_{22}a_{11} = a_{21}A_{21} + a_{22}A_{22} \quad (n=2)$

This is the cofactor expansion of $\det A$ along the 2nd row of A.

We can also expand by cofactors along columns:

$$\det A = a_{11}a_{22} + a_{21}(-a_{12}) = a_{11}A_{11} + a_{21}A_{21} \text{ (1st column)}$$

or $$\det A = a_{12}(-a_{21}) + a_{22}a_{11} = a_{12}A_{12} + a_{22}A_{22} \text{ (2nd column)}$$

Similarly, for the 3×3 case, (**) becomes:

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \quad \text{(expanded along 1st row)}$$

EXAMPLE:

$$A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$$

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$=$$

**As in the 2×2 case, the determinant of a 3×3 matrix can be expanded along *any row or column*.

...e.g., in the above example, expand along the 2nd column:

$$\begin{aligned} \det A &= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} \\ &= (-1)^3 a_{12} \det M_{12} + (-1)^4 a_{22} \det M_{22} + (-1)^5 a_{32} \det M_{32} \end{aligned}$$

GENERAL $n \times n$ CASE:

Define $\det A$ inductively:

$$\det A = a_{11} \quad \text{for } n = 1$$

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} \quad \text{for } n > 1$$

...where $A_{1j} = (-1)^{1+j} \det M_{1j} \qquad j = 1, \ldots, n$

are the cofactors of the entries in the first row of A.

...In fact, we can use *any row* or *column* of A for the cofactor expansion(s):

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \qquad i = 1,\ldots,n$$

<u>or</u>

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \qquad j = 1,\ldots,n$$

...where $$A_{ij} = (-1)^{i+j} \det M_{ij}$$

Thus any nth-order determinant is reduced to a combination of n determinants, each
of order $(n-1)$, i.e., $(n-1)\times(n-1)$.

<u>In practice,</u> expand along the row or column containing the most zeros: e.g., choose the first column of

$$A = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{bmatrix}$$

for the cofactor expansion:

$$\det A = \begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix} =$$

Determinant of a triangular matrix:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{vmatrix} =$$

i.e.: The determinant of an $n\times n$ triangular matrix A is just the product of the diagonal elements.

CRAMER'S RULE:

Method for computing the inverse of a nonsingular matrix A and the solution to the linear system $A\mathbf{x} = \mathbf{b}$, using determinants.

Cramer's Rule: If A is a nonsingular $n \times n$ matrix and $\mathbf{b}$ is any column vector in $\mathbb{R}^n$,
then the (unique) solution $\mathbf{x} = \{x_i\}$ to the linear system $A\mathbf{x} = \mathbf{b}$ is given by:

$$x_i = \frac{\det A_i}{\det A} \qquad i = 1, 2, \ldots, n$$

where A_i is the matrix obtained by replacing the *i* th column of A by $\mathbf{b}$.

...e.g., if $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, then:

$$x_1 = \frac{1}{\det A} \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}; \quad x_2 = \frac{1}{\det A} \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}; \quad x_3 = \frac{1}{\det A} \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} \det M_{11} - a_{12} \det M_{12} + a_{13} \det M_{13}$$

$$= (-1)^2 a_{11} \det M_{11} + (-1)^3 a_{12} \det M_{12} + (-1)^4 a_{13} M_{13}$$

$$= a_{11} A_{11} + a_{13} A_{13} + a_{13} A_{13}$$

$\det M_{ij}$ is called the minor of a_{ij} ,

and $A_{ij} = (-1)^{i+j} \det M_{ij}$ is called the cofactor of a_{ij}

...and the inverse of A may be calculated from:

$$A^{-1} = \frac{1}{\det A} \cdot \operatorname{adj} A ,$$

where $\operatorname{adj} A$ (the adjoint of A) is defined by:

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

...i.e., each element a_{ij} of A is replaced by its corresponding cofactor, A_{ij}, *and* the result-ing matrix is then transposed.

Properties of Determinants

a) $\det A^T = \det A$

b) If A has a row or column containing all zeros, $\det A = 0$

c) If A has two (or more) identical rows or columns, $\det A = 0$

d) An $n \times n$ matrix A is singular (noninvertible) *iff* $\det A = 0$

e) If A and B are $n \times n$ matrices: $\det(AB) = \det A \cdot \det B$

and…

ANOTHER WAY TO CALCULATE $\det A$:

Effect of elementary row operations:

a) Interchange two rows (or columns) of A $\Rightarrow$ $\det A \rightarrow -\det A$

b) Multiply a row (or column) by a nonzero scalar c $\Rightarrow$ $\det A \rightarrow c \det A$

c) Add a multiple of one row to another row: $\Rightarrow$ $\det A \rightarrow \det A$

$\Downarrow$

d) Multiply the $n \times n$ matrix A by a scalar c $\Rightarrow$ $\det A \rightarrow c^n \det A$

e) $\det(A^{-1}) = \dfrac{1}{\det(A)}$

So: Two methods of calculating $\det A$:

a) The basic definition in terms of descending cofactor expansions; or

b) Reduction to triangular form, keeping track of the no. of row interchanges used along the way.

EXAMPLE:

$$\begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} =$$