SYSTEMS OF LINEAR EQUATIONS – "LINEAR SYSTEMS"

Simplest linear system: $ax = b \Rightarrow x = b/a$

<u>But</u> 3 possibilities: $a = 0,\ b \neq 0 \Rightarrow$ **no solution**

$a = 0,\ b = 0 \Rightarrow x$ **any real no. (infinitely many sol'ns)**

$a \neq 0 \Rightarrow x = b/a$ **(unique sol'n)**

No. of solutions: $0, 1, \infty$

Linear equation in n unknowns: $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

(a_i's: real nos. or scalars)

$$\rightarrow \quad \sum_{i=1}^{n} a_i x_i = b \quad \text{... standard form}$$

System of m linear equations in n unknowns:

$$\begin{array}{lr} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 & Eqn.\ E_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 & Eqn.\ E_2 \\ \vdots \qquad\qquad\qquad\qquad \vdots \quad\ \ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m & Eqn.\ E_m \end{array}$$

$$\rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, \ldots, m \quad \ldots \quad m \times n \text{ linear system}$$

Solution: Set of values $x_1 = c_1$, $x_2 = c_2$ [n-tuple $(x_1, x_2, \dots, x_n) = (c_1, c_2, \dots, c_n)$] satisfying all m equations at once.

EXS:

(a)

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned}$$

(2×2)

(b)

$$\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned}$$

(2×3)

(c)

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 \phantom{{}+x_2} &= 4 \end{aligned}$$

(3×2)

DEFINITIONS: No solution: system is inconsistent

At least one solution: system is consistent

Set of all solutions: solution set

Geometric interpretation/solution:

(a)
$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + x_2 &= 2 \\ -x_1 - x_2 &= -2 \end{aligned}$$

Infinitely many solutions $(x_1, x_2) = (\alpha,\ 2-\alpha)$ (α any real no.)

Parametric representation

EQUIVALENT SYSTEMS:

DEFINITION: Two (or more) linear systems S, S' in the same set of variables are equivalent if (and only if) they have the same solution set.

EX: S S'

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 &= 3 \\ 2x_3 &= 4 \end{aligned} \qquad\qquad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

The following elementary operations on an $m \times n$ system leave the solution set unchanged, i.e., they produce an equivalent $m \times n$ system:

1) **Change the order of any two equations:** $E_i \leftrightarrow E_j$

2) **Multiply an equation by a *nonzero* real no. (scaling):** $cE_i \rightarrow E_i \quad c \neq 0$

3) **Add a (*nonzero*) multiple of one equation to another:** $E_i \pm cE_j \rightarrow E_i$

... So use these operations to obtain an equivalent system that's easier to solve.

DEF'N: An $n \times n$ system is in triangular form if in the kth equation, the coefficients of the first $(k-1)$ variables are zero, *and* the coefficient of x_k is nonzero $(k = 1, \ldots, n)$

Example: 3×3 **system:**

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{33}x_3 &= b_3 \end{aligned}$$

... Solve by back substitution: $x_3 = \dfrac{b_3}{a_{33}}$;

$$x_2 = \frac{b_2 - a_{23}x_3}{a_{22}} ;$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

...and in this case, the solution is unique.

...But this works *only if* a_{33}, a_{22}, a_{11} are all *nonzero*.

If an $n \times n$ system is *not* triangular, use the elementary operations (1)−(3) to generate an equivalent triangular system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

Linear Equations establish connection between Algebra and Matrices

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

- **We have 'm" linear equations and in "n" unknowns**
- **A convenient representation is through matrices.**

Matrix

Definition: An mxn matrix A is an array of 'm" rows and "n" columns of the form

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\qquad a_{ij} \in R \ \textbf{(real number)}
$$

$= [\underline{a_1}, \underline{a_2}, \ldots, \underline{a_n}]$ **"n" columns**

$$
= \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \text{ "m" rows}
$$

DEF'N: Coefficient matrix A of an $m \times n$ system:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

and the augmented matrix of the same system:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

EXAMPLE:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 3x_1 - x_2 - 3x_3 &= -1 \\ 2x_1 + 3x_2 + x_3 &= 4 \end{aligned}$$

Coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix}$

and the augmented matrix of the same system:

$$(A \mid b) = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right]$$

Operations 1, 2, 3 now become elementary row operations, i.e., operations on the *rows* of the coefficient matrix A:

1) **Interchange two rows:** $R_i \leftrightarrow R_j$
2) **Multiply one or more rows by a *nonzero* (real) no.:** $c\,R_i \to R_i$
3) **Replace a row by its sum with a *nonzero* (real) multiple of another row:**

$$R_i \pm cR_j \to R_i$$

Ques: How to do this systematically?

Ans: "Pivoting"

Choose a pivot row and pivot element (in that row):

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right]$$

$Row\,2 - 3\times(Row\,1) \to$ **Row 2**
$Row\,3 - 2\times(Row\,1) \to$ Row 3

$\to$

...then:

$Row\,3 - (Row\,2)/7 \to$ Row 3

$\to$

**If during this process a pivot element turns up zero, interchange rows to obtain a *non*zero pivot element.

... Continue the process until triangular form is obtained (not necessarily unique).

...then solve by back substitution (as before).

BUT... This process breaks down if at any stage *all* possible choices for a pivot element are *zero*.

...What to do in that case?

Ex: 5×5 system: $\left[\begin{array}{rrrrr|r} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrrrr|r} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right]$

All possible pivot elements in column 2 are zero. So, move to column 3 and continue the process:

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrrrr|r} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Now all possible pivot elements in column 4 are zero. So, move to column 5:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

This is called echelon form.

Last two rows represent the equations:

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = -4$$

$$0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = -3$$

system is inconsistent.

change the right-hand side (b_i's) so the system reads

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right] \rightarrow \dots \rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Last two rows now represent: $0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0$

satisfied by any 5-tuple $(x_1, x_2, x_3, x_4, x_5)$.

So, the solution set is all 5-tuples satisfying:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \\ x_3 + x_4 + 2x_5 &= 0 \\ x_5 &= 3 \end{aligned}$$

Note two kinds of variables here:

x_1, x_3, x_5: <u>**lead variables**</u>

x_2, x_4: <u>**free variables**</u>

Move free variables to right-hand side:

$$
\begin{aligned}
x_1 + x_3 + x_5 &= 1 - x_2 - x_4 \\
x_3 + 2x_5 &= -x_4 \\
x_5 &= 3
\end{aligned}
$$

System is now *triangular* in x_1, x_3, x_5.

So for any pair of values (α, β) **assigned to** (x_2, x_4), **system has a unique solution:**

$$
\begin{aligned}
x_5 &= 3 \\
x_3 &= -x_4 - 2x_5 = -\beta - 6 \\
x_1 &= (1 - x_2 - x_4) - (x_3 + x_5) = 4 - \alpha
\end{aligned}
$$

i.e.: $(x_1, x_2, x_3, x_4, x_5) = (4-\alpha,\ \alpha,\ -\beta-6,\ \beta,\ 3)$ α, β <u>**any**</u> **real nos.**

Definition: A matrix is in row echelon form if

1) First nonzero entry in each row is 1;
2) If row k is not all zeros, the no. of leading zeros in row $(k+1)$ is greater than the no. of leading zeros in row k;
3) Any rows with *all zeros* are below the rows having nonzero entries.

DEF'N: Gaussian elimination: The process of using elementary row operations to reduce a linear system to row echelon form (also called row reduction).

NOTE: Because of (1), reduction to row echelon form (as opposed to triangular form) requires the additional step of scaling each row [multiplying by a non-zero real no.: row operation (2)] so that the leading nonzero element is 1.

IMPLICATIONS (for $n \times n$ systems):

1) If the row echelon form of the augmented matrix includes a row that looks like $(0 \;\; 0 \;\; \cdots \;\; 0 \mid 1)$, then the system is inconsistent (no solution).

2) Otherwise, the system is consistent. In this case, there are two possibilities:

   a) If the nonzero rows of the reduced form of the augmented matrix form a triangular system, the solution is unique:

$$\left[\begin{array}{cccc|c} 1 & \times & \times & \times & \times \\ 0 & 1 & \times & \times & \times \\ 0 & 0 & 1 & \times & \times \\ 0 & 0 & 0 & 1 & \times \end{array}\right]$$

b) Otherwise, the system has free variables and there are infinitely many solutions:

...e.g., $\left[\begin{array}{cccc|c} 1 & \times & \times & \times & \times \\ 0 & 1 & \times & \times & \times \\ 0 & 0 & 1 & \times & \times \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ or $\left[\begin{array}{cccc|c} 1 & \times & \times & \times & \times \\ 0 & 0 & 1 & \times & \times \\ 0 & 0 & 0 & 1 & \times \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

BACK TO $m \times n$ SYSTEMS:

DEFINITIONS: **Overdetermined system:**

More equations than unknowns ($m > n$)

Underdetermined system:

Fewer equations than unknowns ($m < n$)

- **Overdetermined systems are usually (but not always) inconsistent;**
- **Underdetermined systems are usually (but not always) consistent with infinitely many solutions (i.e., free variables arise in the eduction/elimination process).**