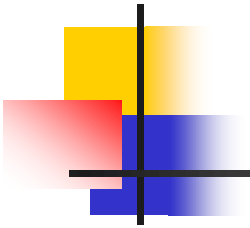
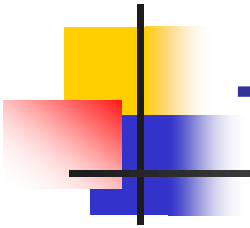
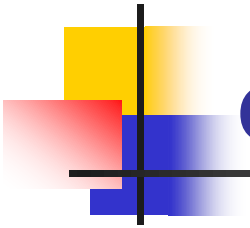


[Chapter 7]
Introduction to Computational Complexity: The Sorting Problem



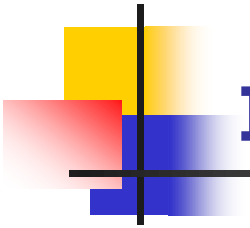
Two Types of Complexity Analysis

- Algorithm Analysis
  - Different approaches to solve the same problem with the hope of finding efficient algorithms.
  - Time and space complexity.
- Problem Analysis
  - In matrix multiplication problem, there is an algorithm in $\Theta(n^3)$
    - But the time complexity of Strassen's is $\Theta(n^{2.81})$ and it is even possible to be in $\Theta(n^{2.38})$.
  - Therefore, we do not want to know the complexity of an algorithm, but the complexity of a problem.
  - It is a computational complexity analysis.
    - The result is stated by a lower bound for a problem.
    - (e.g.) $\Omega(n^2)$ for the matrix multiplication. Meaning?



Computational Complexity

- Objective
  - To develop a $\Theta(f(n))$ algorithm for the problem of $\Omega(f(n))$.
  - It is impossible to develop an algorithm that is lower than a lower bound for the problem.
- Example: Sorting problem
  - Exchange sort: $\Theta(n^2)$
  - Merge sort: $\Theta(n \lg n)$
  - The lower bound for the sorting problem is $\Omega(n \lg n)$
  - Fortunately, we found an algorithm that touches the lower bound of the sorting problem.
- Our new objective
  - How to find such a lower bound



Insertion Sort (1/2)

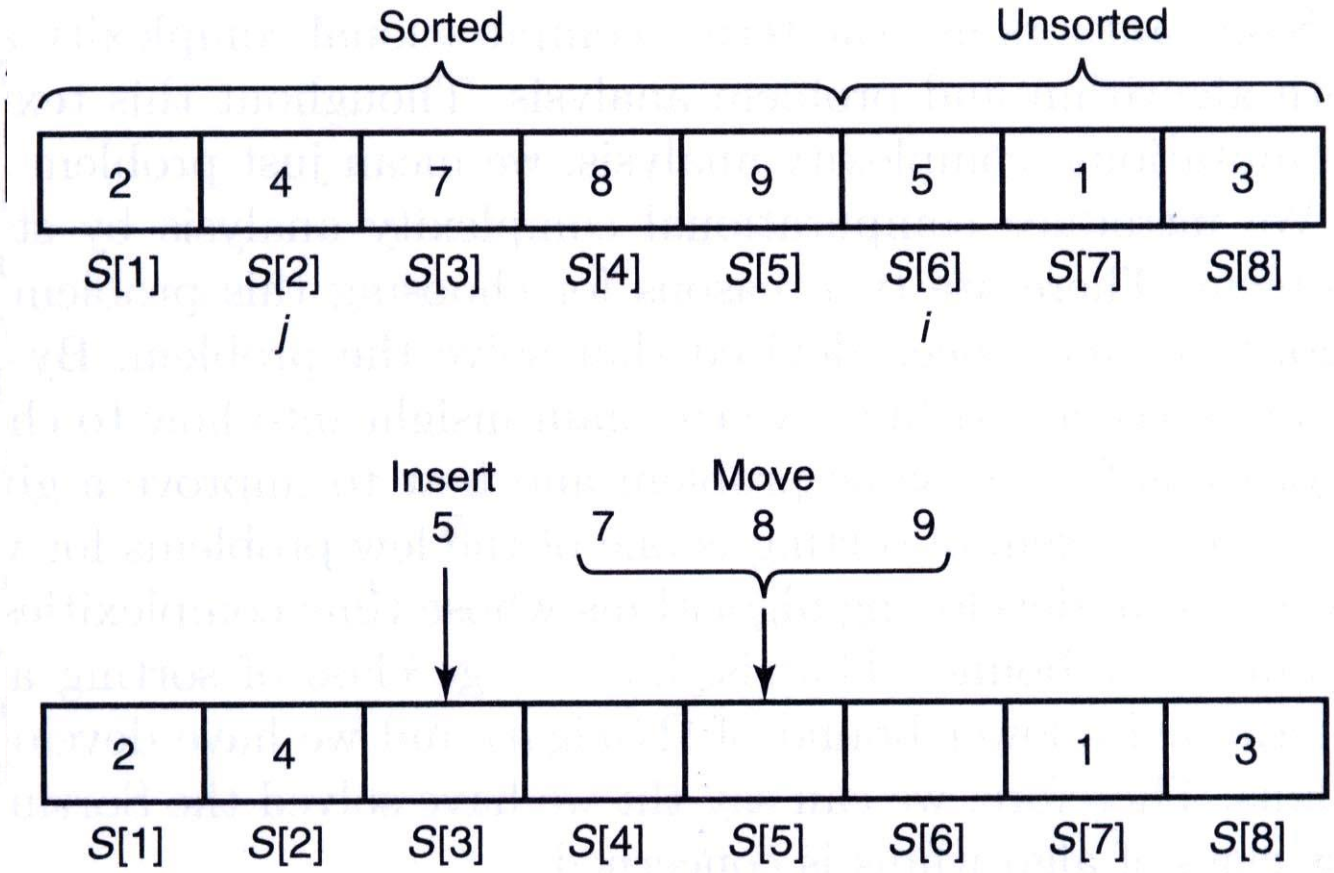
- By inserting records in an existing sorted array.
  - Problem

    Sort n keys in non-decreasing order.
  - Inputs

    Positive integer n; array of keys S indexed from 1 to n.
  - Outputs

    the array S containing the keys in non-decreasing order.

Insertion Sort (2/2)



Inserting Sort Algorithm 7.1

```
void insertionsort(int n, keytype S[]){
     index i,j;
     keytype x;
     for(i=2; i<=n; i++){
       x = S[i];
       j = i - 1;
       while(j>0 && S[j]>x){
         S[j+1] = S[j];
         j--;
       }
       S[j+1] = x;
     }
   }
```

Inserting Sort Algorithm Analysis

- Worst-case Time Complexity Analysis
  - At most (i-1) comparisons for each i.

$$W(n) = \sum_{i=2}^{n} (i-1) = \frac{n(n-1)}{2}$$

- Average-case Time Complexity Analysis

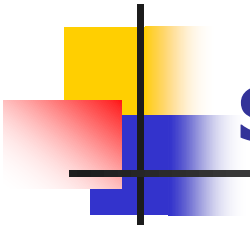
$$\begin{array}{lccccc} Index & i & i-1 & \cdots & 2 & 1 \\ \#Comparison & 1 & 2 & \cdots & i-1 & i-1 \end{array}$$

  - The number of comparisons to insert x

$$1\frac{1}{i} + 2\frac{1}{i} + \cdots + (i-1)\frac{1}{i} + (i-1)\frac{1}{i} = \frac{1}{i}\sum_{k=1}^{i-1} k + \frac{i-1}{i} = \frac{(i-1)i}{2i} + \frac{i-1}{i} = \frac{i+1}{2} - \frac{1}{i}$$

  - Therefore, the average number of comparisons to insert

$$\sum_{i=2}^{n} \left(\frac{i+1}{2} - \frac{1}{i}\right) = \sum_{i=2}^{n} \frac{i+1}{2} - \sum_{i=2}^{n} \frac{1}{i} \approx \frac{(n+4)(n-1)}{4} - \ln n \approx \frac{n^2}{4}$$



Selection Sort

- By selecting records in order and putting them in their proper positions in an array.
  - Problem

    Sort n keys in non-decreasing order.
  - Inputs

    Positive integer n; array of keys S indexed from 1 to n.
  - Outputs

    the array S containing the keys in non-decreasing order.

Selection Sort Algorithm 7.2

```
void selectionsort(int n, keytype S[]){
    index i,j,smallest;
    for(i=1; i<=n-1; i++){
      smallest = i;
      for(j=i+1; j<=n; j++){
        if(S[j]<S[smallest])
          smallest = j;
      exchange S[i] and S[smallest];
    }
  }
```

Selection Sort Algorithm Analysis

- Every-case Time Complexity Analysis
  - The number of comparisons

$$\begin{array}{lcccc} Index & 1 & 2 & \cdots & n-1 \\ \#Comparison & n-1 & n-2 & \cdots & 1 \end{array}$$

  - Therefore, $T(n) = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$
  - The number of assignments
    - 3 assignments for an exchange
    - Therefore, $T(n) = 3(n - 1)$

Analysis summary for Exchange Sort, Insertion Sort, and Selection Sort

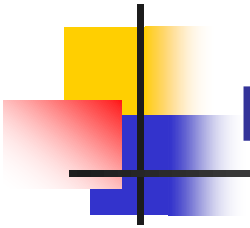
Algorithm	Comparisons of Keys	Assignments of Records	Extra Space Usage
Exchange Sort	$T(n) = \frac{n^2}{2}$	$W(n) = \frac{3n^2}{2}$	In-place
		$A(n) = \frac{3n^2}{4}$	
Insert Sort	$W(n) = \frac{n^2}{2}$	$W(n) = \frac{n^2}{2}$	In-place
	$A(n) = \frac{n^2}{4}$	$A(n) = \frac{n^2}{4}$	
Selection Sort	$T(n) = \frac{n^2}{2}$	$T(n) = 3n$	In-place

Lower Bounds for Algorithms that Remove at Most one inversion per comparison

- [1, 2, 3, … n]
  - n! permutations
  - If $i < j$ and $k_i > k_j$ then it's an inversion.
  - (e.g) How many inversions in [3, 2, 4, 1, 6, 5]?
    - 5 inversions
- Thm 7.1 Any algorithm that sorts n distinct keys only by comparisons of keys and removes one inversion (at most) after each comparison must in the worst case do at least $\frac{n(n-1)}{2}$ comparisons of keys and $\frac{n(n-1)}{4}$ on the average.

Proof for Thm. 7.1

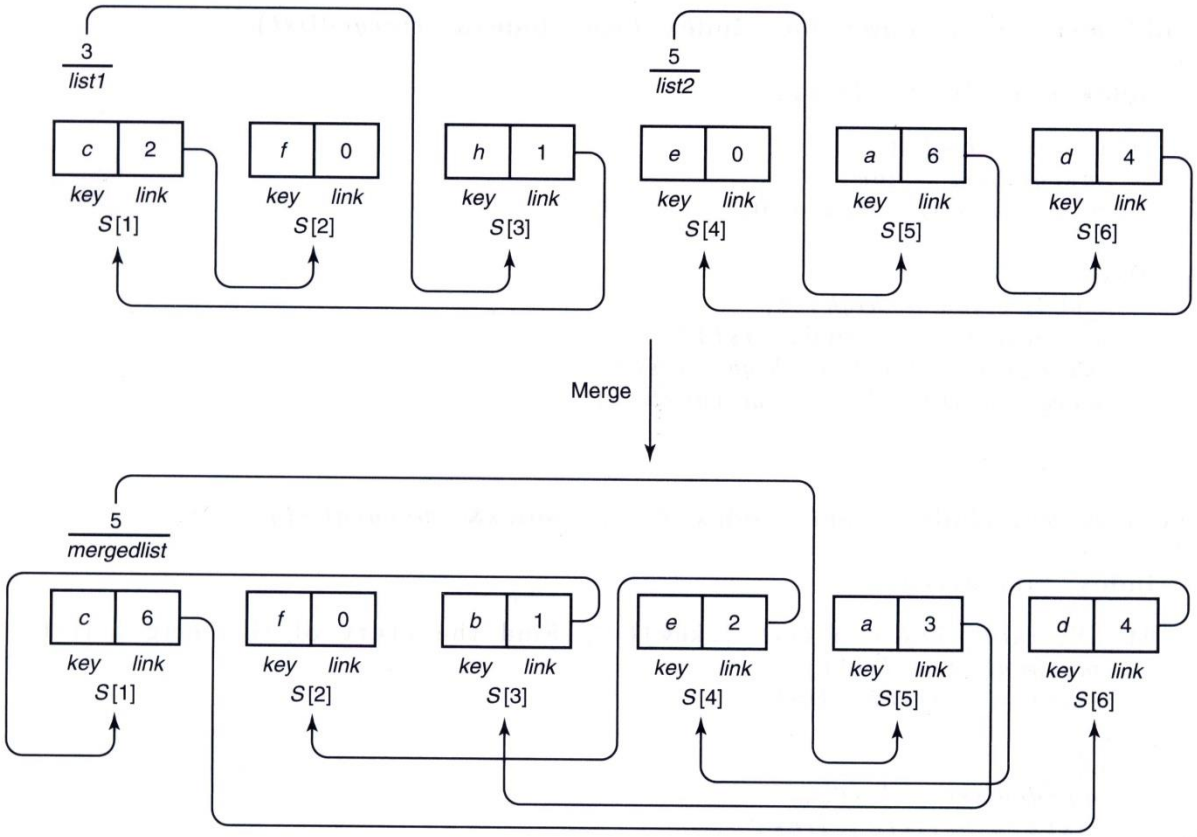
- Worst case
  - The permutation with ½ $n(n\text{-}1)$ inversions, [n, n-1, … 2, 1] is the case.
- Average case
  - $P = [k_1, k_2, \ldots, k_n]$, and its transpose $P^T = [k_n, \ldots, k_2, k_1]$
  - An arbitrary pair (s, r) belongs to P or P^T, always.
  - In other words, I(P) + I(P^T) = ½ $n(n\text{-}1)$.
  - Therefore, on the average, there are ¼ $n(n\text{-}1)$.

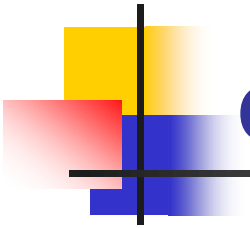


Merge Sort Revisited

- Think [3, 4] and [1, 2].
  - After 3 and 1 are compared, 1 is put in the 1st slot, then two inversions, (3, 1) and (4, 1) are removed.
  - After 3 and 2 are compared, (3, 2) and (4, 2) are removed
- The worst case time complexity
  - $W(h, m) = W(h) + W(m) + h + m - 1$
    $= n \lg n + n - 1$
  - Removing >= 1 inversions after a comparison.
  - We gained significantly!
- 3 improvements
  - Dynamic programming version
  - Linked list version
  - More complex merge algorithm

Figure 7.3 Merging using links. Arrows are used to show how the links work . The Keys are letters to avoid confusion with indices.

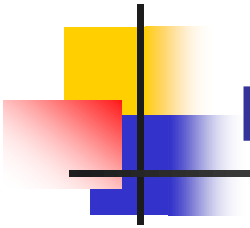




Quick Sort Revisited

- Time complexity

  $W(n) = \Theta(n^2)$ and $A(n) \approx 1.38(n+1)\lg n$
- $A(n)$ is not much worse than Mergesort().
- Technically, QS is not an in-place sorting.
- 5 different ways to reduce the amount of extra space.
  - Check yourself that how much the extra space complexity can be reduced!!
  - The worst-case height of the stack is $\Theta(\lg n)$



Heap Sort

- Unlike MS and QS, HS is truly an in-place sorting algorithm.
  - Worst-case time complexity is $\Theta(n \lg n)$
- Complete binary tree
  - All internal nodes have 2 children.
  - All leaf nodes have a depth of d.
- Essentially CBT
  - It's a complete binary tree up to a depth of $d-1$.
  - The nodes with depth d are as far to the left as possible.
- Heap
  - An ECBT with heap property.

ECBT

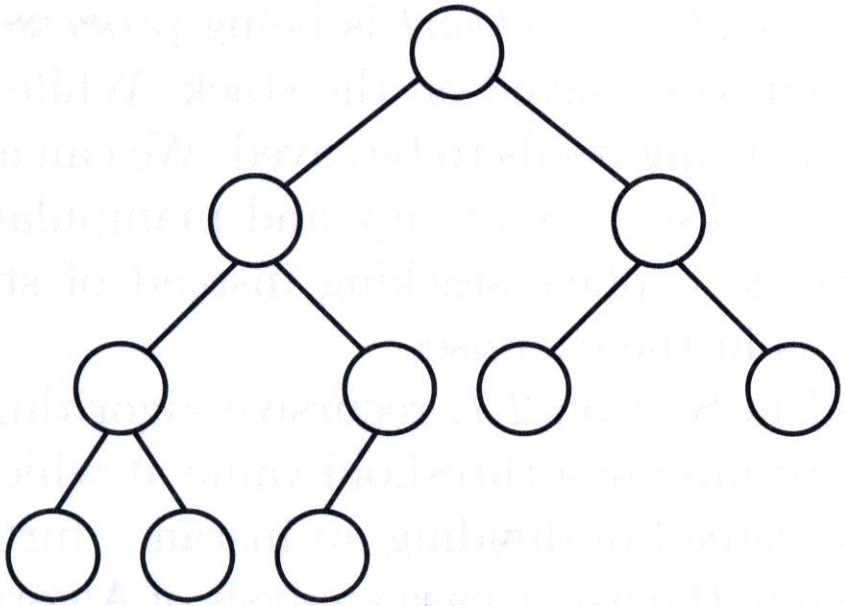


Figure 7.5 A heap.

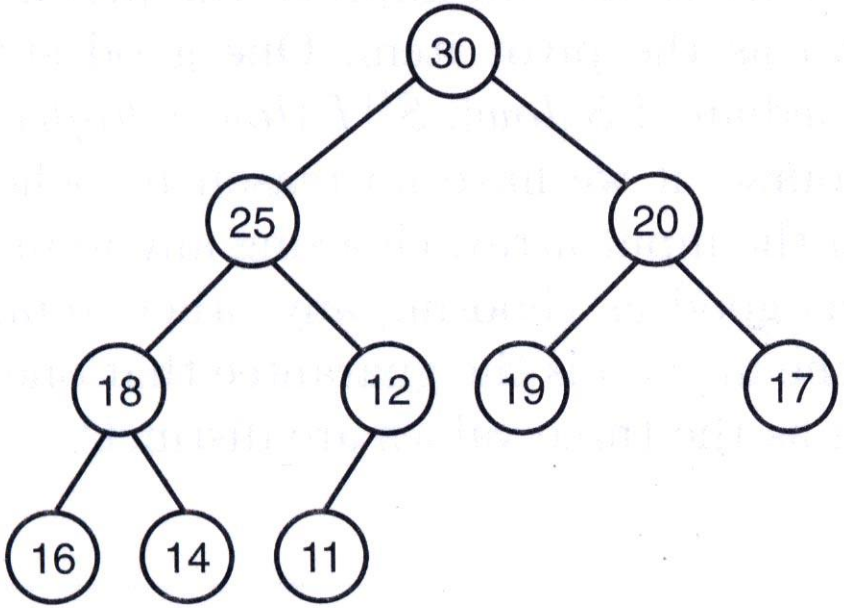


Figure 7.6 Procedure *siftdown* sifts 6 down until the heap property is restored.

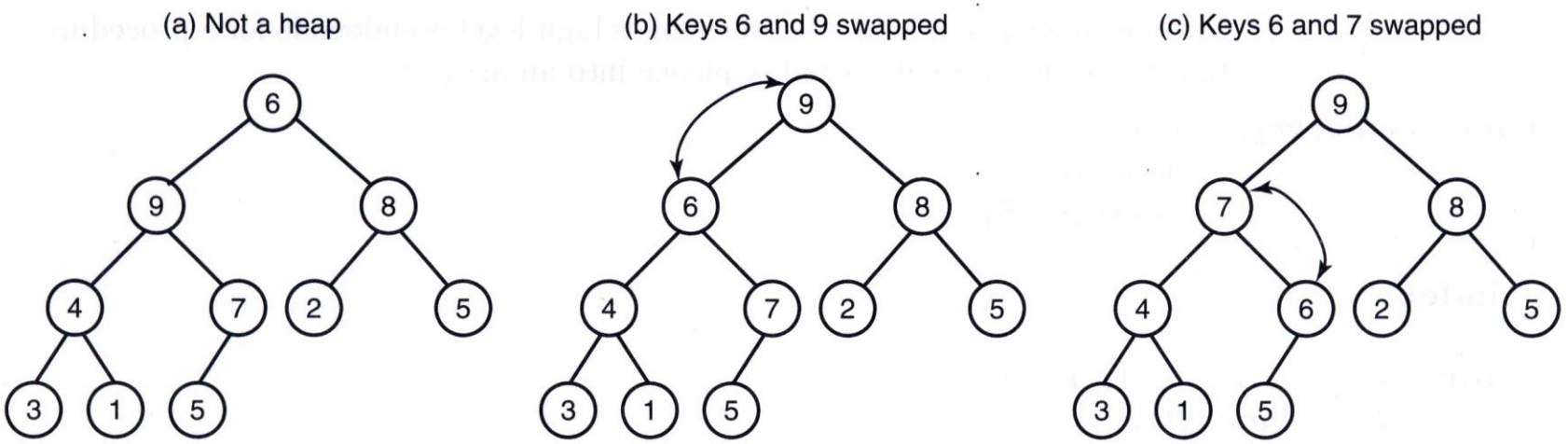
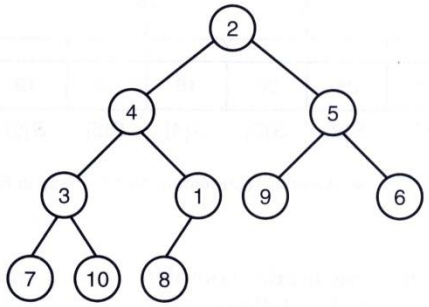
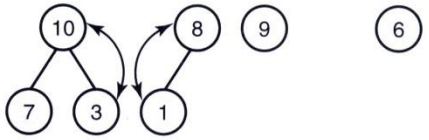


Figure 7.7 Using siftdown to make a heap from an essentially complete binary tree. After steps shown. The right subtree, whose root has depth d – 2, must be made into a heap, and finally the entire tree must be made into a heap.

(a) The initial structure



(b) The subtrees, whose roots have depth $d - 1$, are made into heaps.



(c) The left subtree, whose root has depth $d - 2$, are made into a heap.

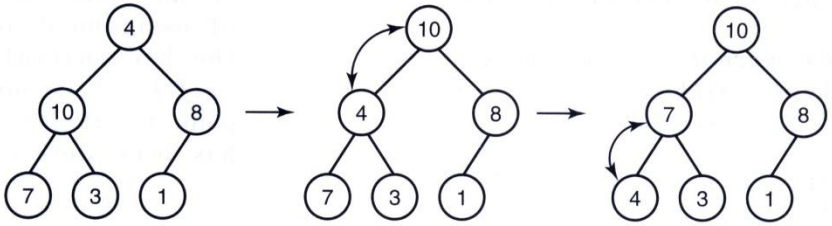


Figure 7.8 The array representation of the heap in Figure 7.5.

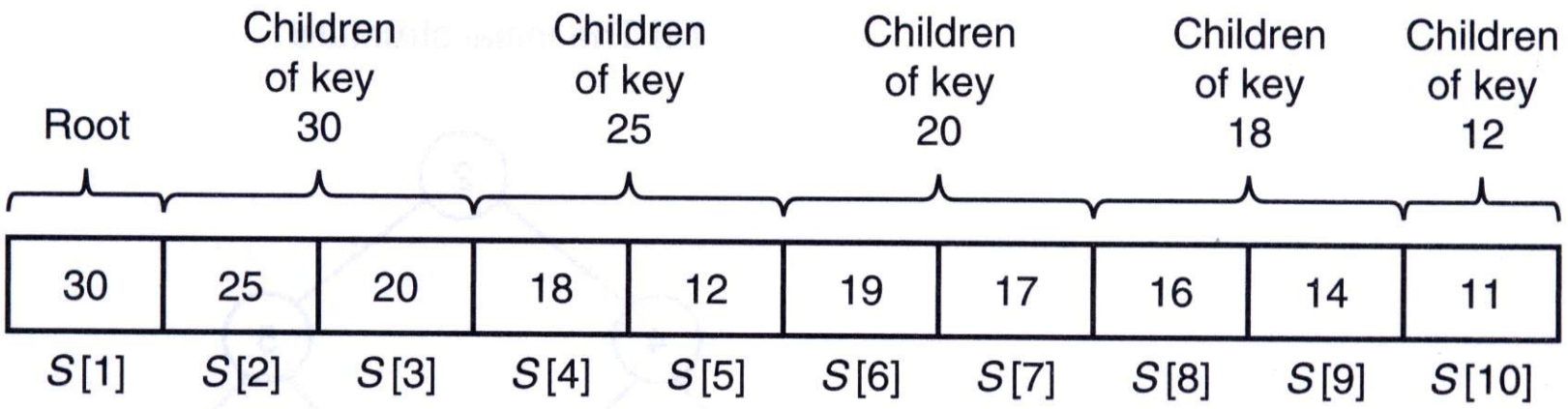


Figure 7.9 An illustration using n = 8, Showing that if an essentially complete binary tree has n node and n is a power of 2, then the depth d of the tree is lg n, there is one node with depth d, and that node has d ancestors. The three ancestors of that node are marked "A".

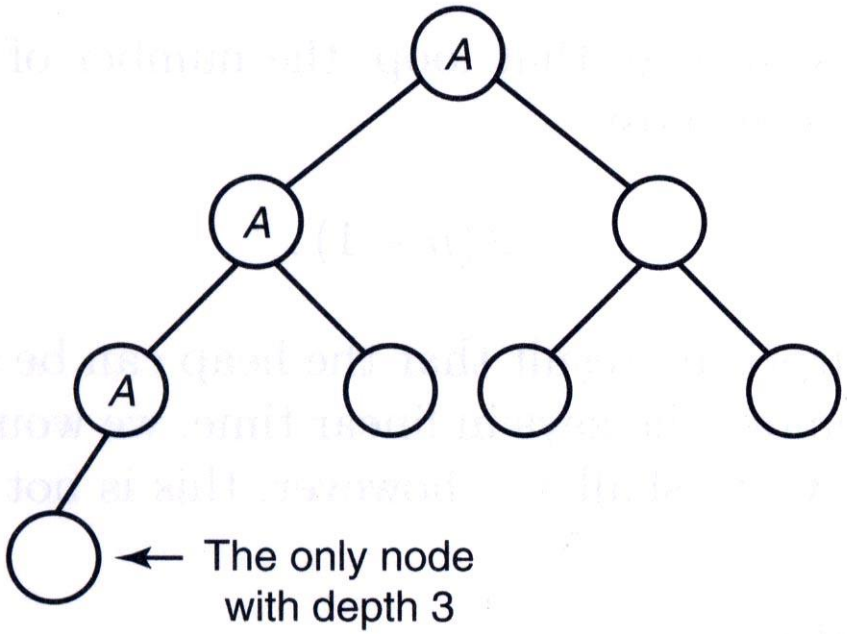
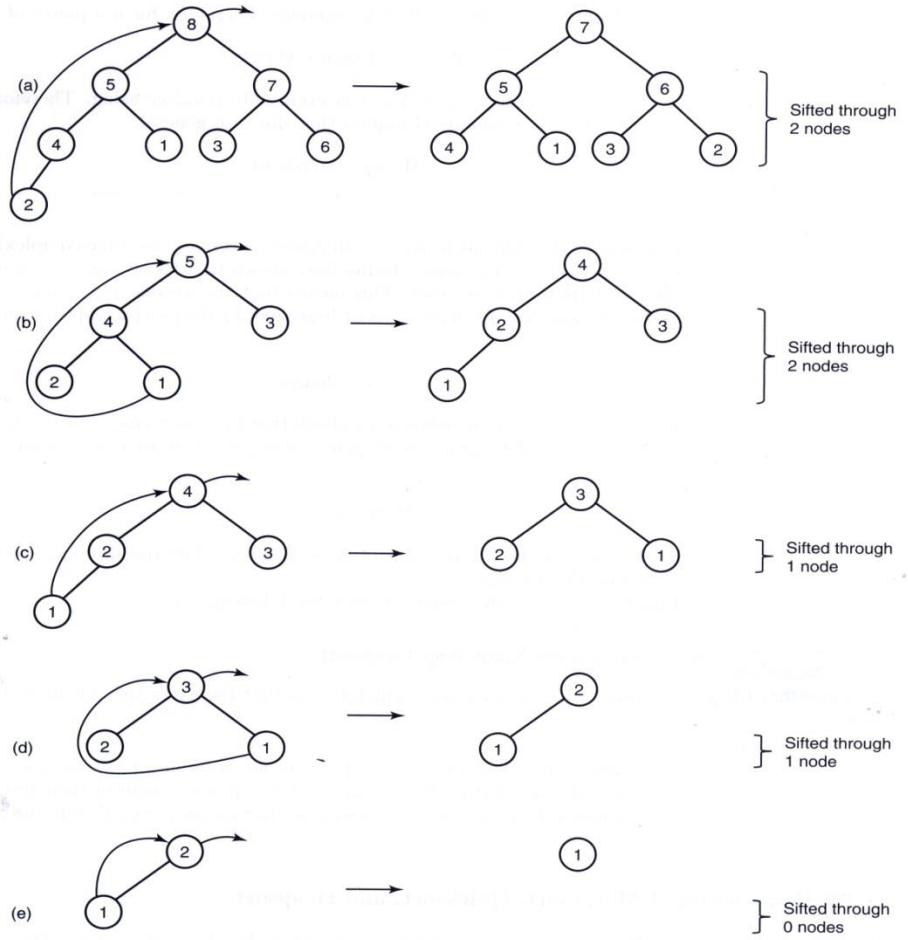


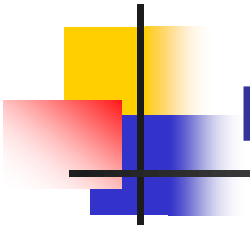
Figure 7.10 Removing the Keys from a heap with eight nodes. The removal of the first Key is depicted in (a): the fourth in (b): the fifth in (c): the sixth in (d): and the seventh in (e). The key moved to the root is sifted through the number of nodes shown on the right.



Heapsort

```
void siftdown(heap& H){
  node parent, largerchild;
  parent = root of H;
  largerchild = parent's child containing larger key;
  while(key at parent is smaller than key at largerchild){
    exchange key at parent and key at largerchild;
    parent = largerchild;
    largerchild = parent's child containing larger key;
  }
}
keytype root(heap& H){
  keytype keyout;
  keyout = key at the root;
  move the key at the bottom node to the root;
  delete the bottom node;
  siftdown(H);
  return keyout;
}
```

```
void removekeys(int n, heap& H, keytype S[]){
  index i;
  for(i=n; i>=1; i--)
    S[i] = root(H);
}
void makeheap(int n, heap& H){
  index i;
  heap Hsub;
  for(i=d-1; i>=0; i--)
    for(all subtree Hsub whose roots have depth i)
      siftdown(Hsub);
}
void heapsort(int n, heap H, keytype S[]){
  makeheap(n,H);
  removekeys(n,H,S);
}
```



Heapsort Analysis: How to?

- S: the greatest number of sift operations.
  - $S1$ = The number of sift operations for a node before d-1
  - $S2$ = The number of sift operations for the last node at d
  - $S = S1 + S2$
- At least 2 comparisons per siftdown
  - Therefore, the total number of worst-case comparison operations is 2 times S.
- Analysis of removekeys()
  - R: the number of comparisons for removekeys()
- Heapsort all together
  - $S+R$ is $\Theta(n \lg n)$

Table 7.2 Analysis summary for Θ (*n lg n*) sorting algorithms*

Algorithm	Comparison of Keys	Assignments of Records	Extra Space Usage
Mergesort	$W(n) = n \lg n$	$T(n) = 2n \lg n$	$\Theta(n)$ records
(Algorithm 2.4)	$A(n) = n \lg n$		
Mergesort	$W(n) = n \lg n$	$T(n) = 0^{\dagger}$	$\Theta(n)$ links
(Algorithm 7.4)	$A(n) = n \lg n$		
Quicksort	$W(n) = n^2/2$		$\Theta(\lg n)$ indices
(with improvements)	$A(n) = 1.38n \lg n$	$A(n) = 0.69n \lg n$	
Heapsort	$W(n) = 2n \lg n$	$W(n) = n \lg n$	In-place
	$A(n) = 2n \lg n$	$A(n) = n \lg n$	

Figure 7.11 The decision tree corresponding to procedure *sortthree.*

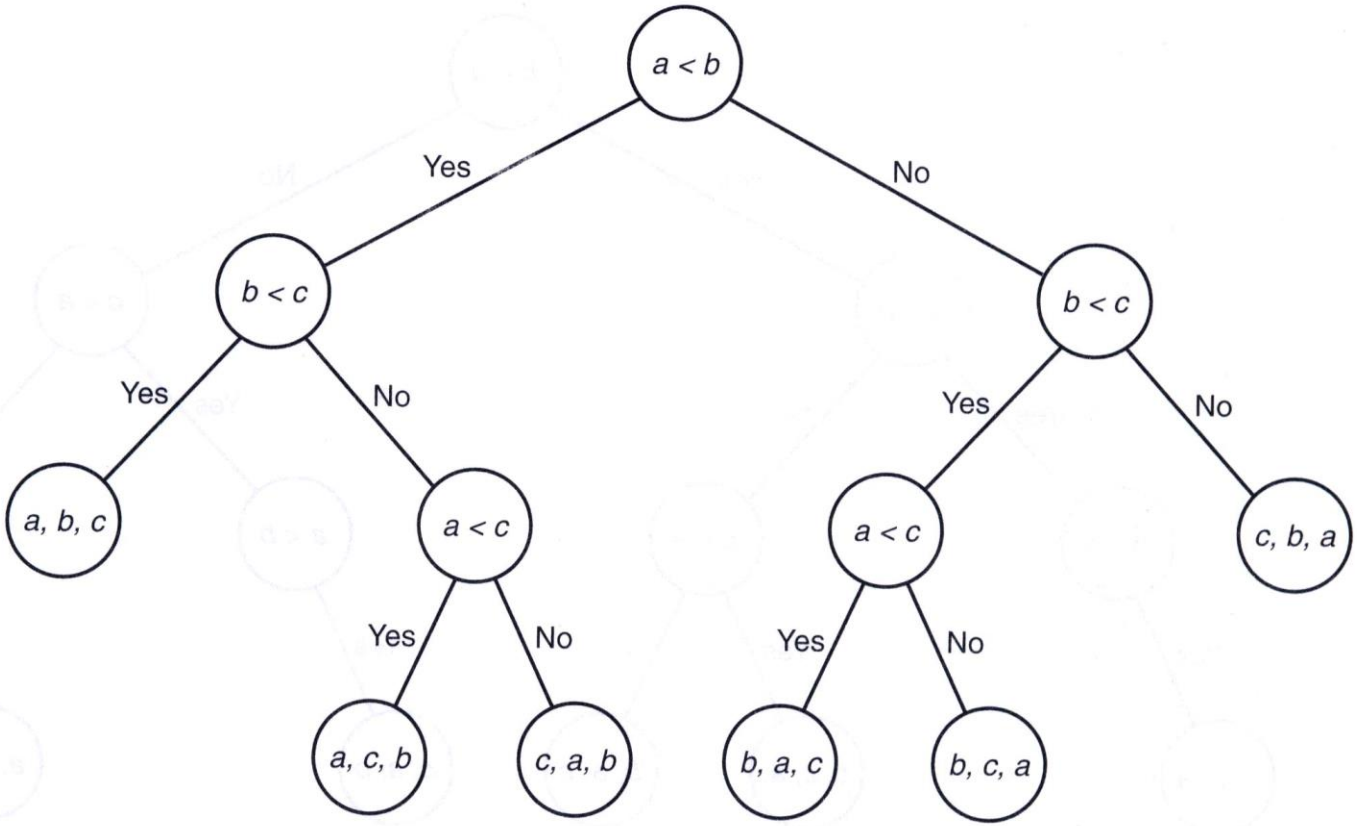
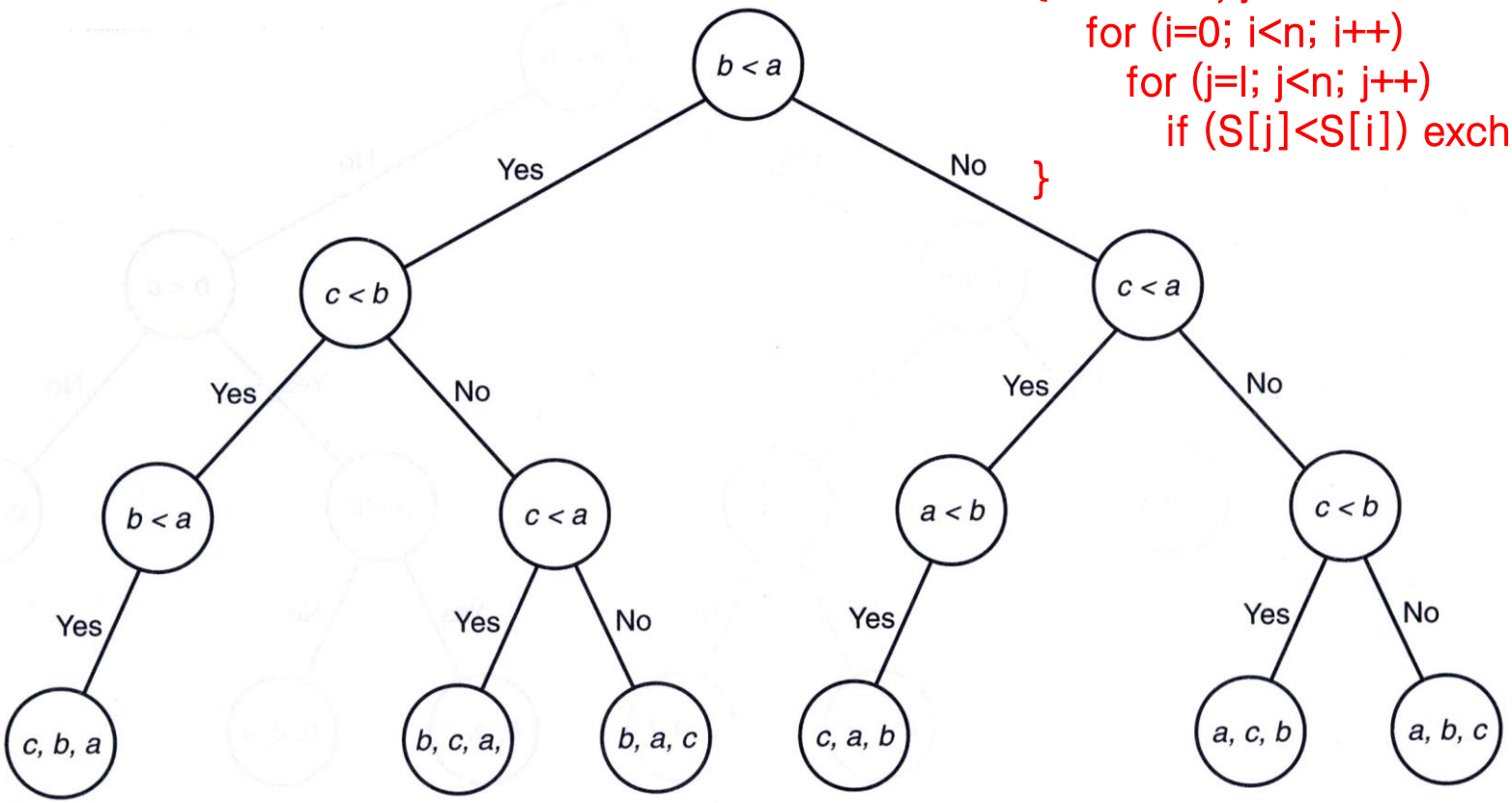
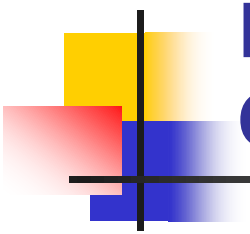


Figure 7.12 The decision tree corresponding to Exchange Sort when sorting three Keys.

```
Void exchangesort()
{   index i, j;
    for (i=0; i<n; i++)
       for (j=I; j<n; j++)
          if (S[j]<S[i]) exchange
}
```





Lower Bounds for Sorting Only by Comparison of Keys

- Terms
  - Decision tree
  - A decision tree is valid if there is a path from the root to a leaf node for each permutation of n keys.
- Lemma 7.1
  - To every deterministic algorithm for sorting n distinct keys, there corresponds a pruned, valid, binary decision tree containing exactly $n!$ leaves.
- Lemma 7.2
  - The worst-case number of comparisons done by a decision tree is equal to its depth.
- Lemma 7.3
  - Suppose m is the number of leaves in a binary tree and d is its depth.
  - Then $d \geq \lceil \lg m \rceil$

Lower Bounds for Sorting Only by Comparison of Keys (Cont'd)

- Theorem 7.2
  - Any deterministic algorithm that sorts n distinct keys only by comparison of keys must in the worst case do at least $\lg(n!)$ comparison of keys.
- Lemma 7.4
  - For any positive integer n,

$$\begin{aligned}
\lg(n!) &= \lg[n(n-1)(n-2)\cdots 2] \\
&= \sum_{i=2}^{n} \lg i \\
&\geq \int_{1}^{n} \lg x \, dx \\
&= \frac{1}{\ln 2}(n \ln n - n + 1) \\
&\geq n \lg n - 1.45n
\end{aligned}$$

Lower Bounds for Sorting Only by Comparison of Keys (Final)

- Theorem 7.3

  Any deterministic algorithm that sorts n distinct keys only by comparison of keys must in the worst case do at least, $\lceil n \lg n - 1.45n \rceil$ comparisons of keys.

Figure 7.14 Sorting by distribution while inspecting the digits from left to right.

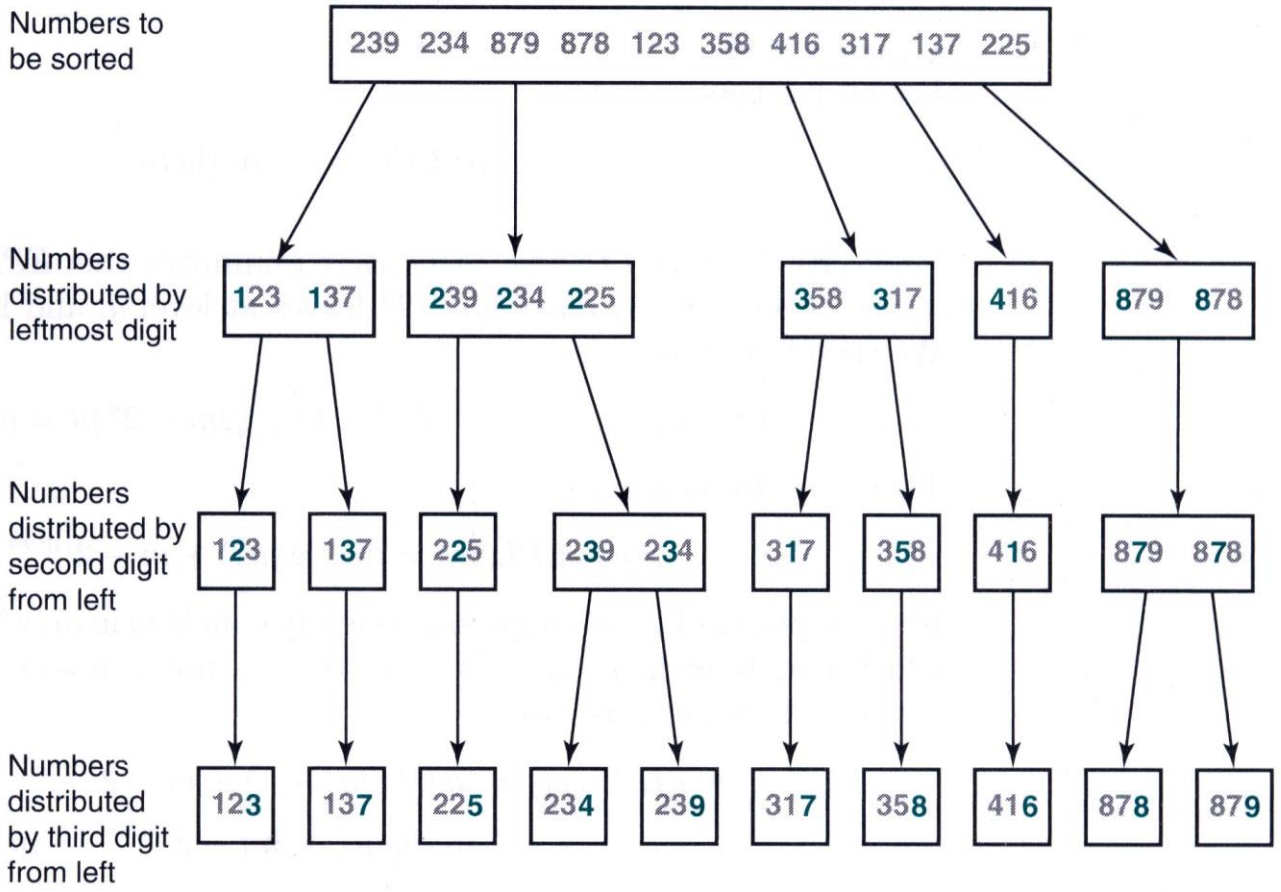


Figure 7.15 Sorting by distribution while inspecting the digits from right to left.