Class 6
Orbital Mechanics

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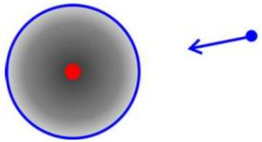
Orbital mechanics: Why do we learn?

- orbit selection
- launch window
- launch site
- ground coverage
- sun angle considerations
- data dump / data storage
- eclipses
- station keeping / orbit
- maintenance
- orbital maneuvers
- interplanetary trajectories
- injection
- mission duration
- propulsion considerations

Motion in Gravitational Field

- Motion in gravitational field is described by Newton's Law of gravitation
- The gravitational force F between two bodies, with masses M_1 and M_2 is

$$\boxed{F = G\frac{M_1 M_2}{R^2}}$$



where $G = 6.6720 \times 10^{-11}\ m^3\ s^{-2}\ kg^{-1}$ is the **universal gravitational constant** and R is the **distance between these two bodies**.

Newton's law of universal gravitation

- Every particle attracts every other particle in the universe with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers.
- That is, every point mass attracts every other point mass by a force acting along the line intersecting the two points. The force is proportional to the product of the two masses, and inversely proportional to the square of the distance between them.
- This is a general physical law derived from empirical observations by what Isaac Newton called inductive reasoning.

Gravitational Field

- Gravitational field can be described by the gravitational potential

$$U_G = -\frac{GM_0}{R} \qquad F = -m\frac{\partial U_G}{\partial R}$$

m: spacecraft mass

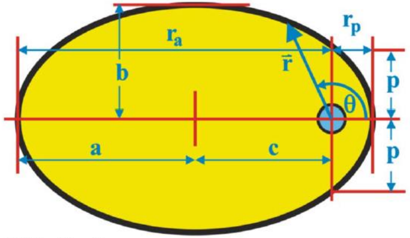
Partial derivative of U_G with respect to R

where M_0 is the mass of the central body.

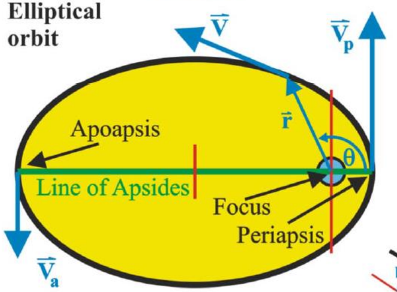
- The universal gravitational constant and central body mass are present as a product in many formulas.
- One can introduce a gravitational parameter, μ, which is a product of the **central body mass** and the **universal gravitational constant**.

$$\mu = GM$$

Ellipse



Elliptical orbit

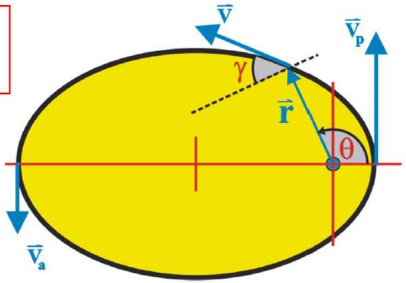


- **a** is the semimajor axis;
- **b** is the semiminor axis;
- $r_{MAX} = r_a$, $r_{MIN} = r_p$ are the maximum and minimum radius-vectors;
- **c** is the distance between the focus and the center of the ellipse;
- **e = c/a** is eccentricity

$$\left(\frac{b}{a}\right)^2 = 1 - e^2$$

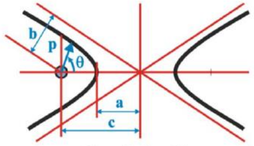
- **2p** is the latus rectum
  (*latus* = *side* and *rectum* = *straight*)
  p — semilatus rectum or semiparameter
- $A = \pi ab$ is the area of the ellipse

$$\frac{p}{r} = 1 + e\cos\theta$$

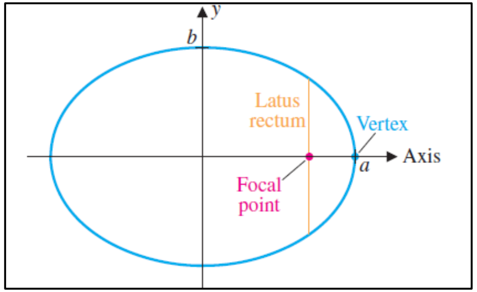


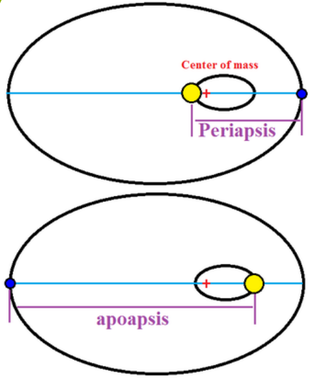
γ – flight-path-angle
θ (ν) – true anomaly (polar angle)

Hyperbolic orbit



Hyperbolic orbit





The two-body system of interacting elliptic orbits: The smaller, satellite body (blue) orbits the primary body (yellow); both are in elliptic orbits around their common center of mass, (red +).

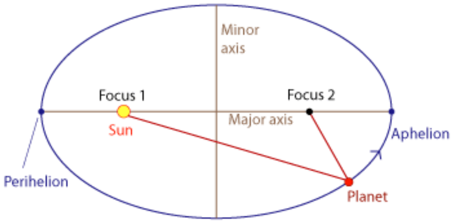
Motion in Gravitational Field

- Three laws by **Johannes Kepler (early 17th century)** are the basis of understanding of planetary and spacecraft motion:
  - ✓ particle (planet, spacecraft, ...) trajectories in the gravitational field of a central body are conic sections (a conic section is an ellipse, parabola or hyperbola; the central body is in the focus)
  - ✓ the line joining the particle (planet, spacecraft, ...) to the central body (radius-vector) sweeps out equal areas in equal times
  - ✓ the square of the period of planetary orbit is proportional to the cube of its semi-major axis

> ✓ **law of gravity**
> ✓ **conservation of energy**
> ✓ **conservation of angular momentum**

Kepler's laws of planetary motion

- The orbit of a planet is an ellipse with the Sun at one of the two foci.
- A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
- The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit.

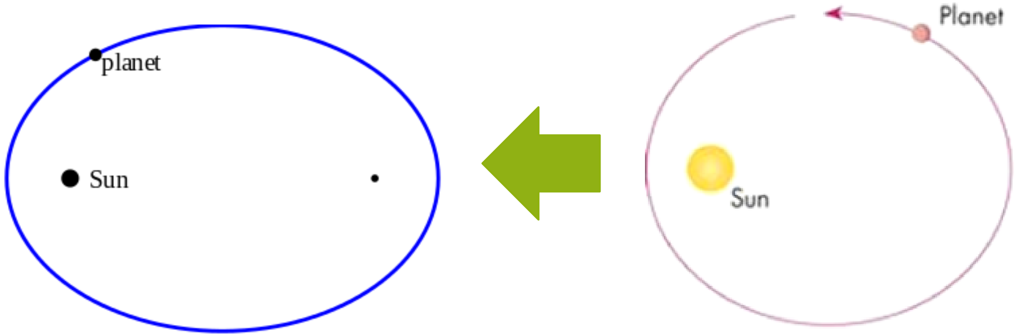


An elliptical orbit of a planet
(greatly exaggerated)

Kepler's laws of planetary motion: 1st Law

- The orbit of a planet is an ellipse with the Sun at one of the two foci.

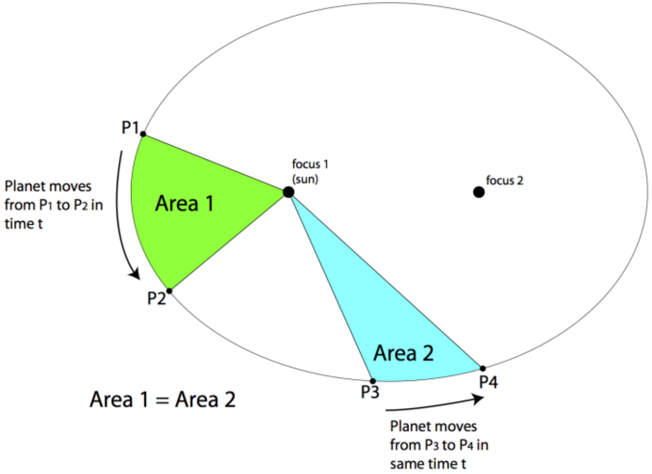
→ The planets orbit the Sun in ellipses with the Sun at one focus (the other focus is empty).

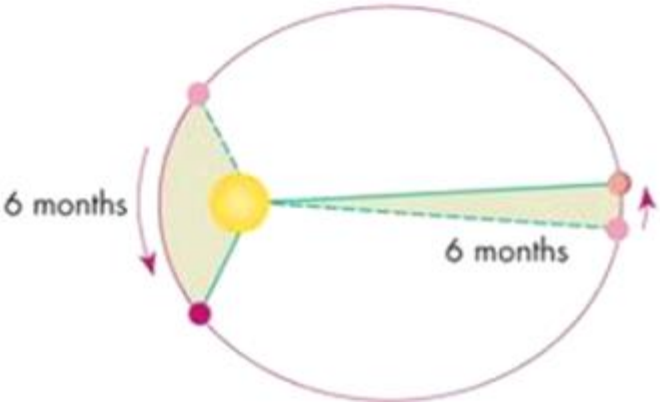


Kepler's first law placing the Sun at the focus of an elliptical orbit

Kepler's laws of planetary motion: 2^{nd} Law

- A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.





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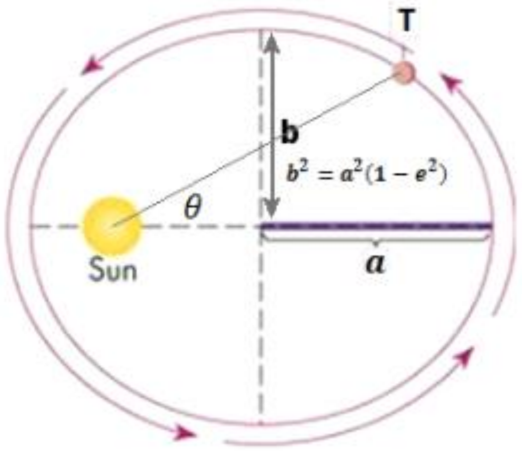
Kepler's laws of planetary motion: 3rd Law

- The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit. → $T^2 \propto a^3$



T: Time to complete orbit
A: semi-major axis

Energy and Angular Momentum

- kinetic energy

$$KE = \frac{mV^2}{2}$$

  where V is the spacecraft velocity
- potential energy

$$PE = -\frac{GM_0 m}{R} = -\frac{\mu m}{R}$$

- total energy is conserved, i.e.,
  E_0 = KE + PE = constant

- **$E_0 < 0$** for an elliptical orbit
- **$E_0 = 0$** for a parabolic orbit
- **$E_0 > 0$** for a hyperbolic orbit

- Angular momentum **H** is the vector product

$$\mathbf{H} = \mathbf{R} \times m\mathbf{V} = \text{constant}$$

- at periapsis and apoapsis, **V** is perpendicular to **R**, and

$$H = R\, m\, V$$

- Therefore

  $H_1 = R_{MIN}\, mV_1$

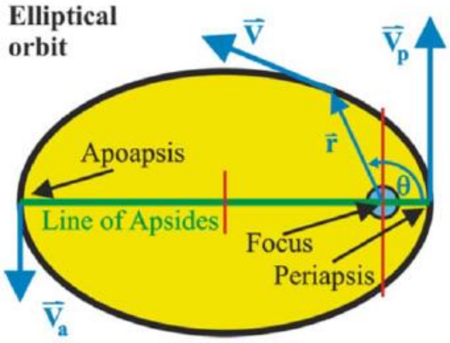
  and

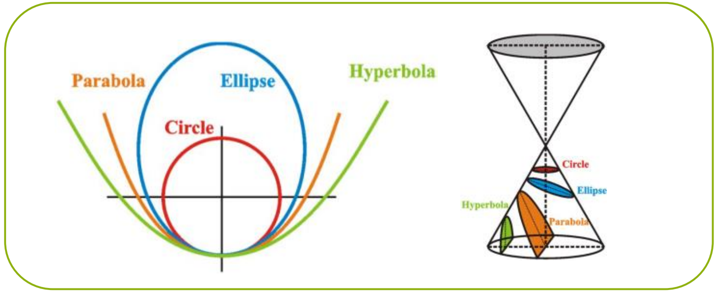
  $H_2 = R_{MAX}\, mV_2$
- Since $H_1 = H_2$

  $R_{MIN}\, V_1 = R_{MAX}\, V_2$

  or

  $R_a\, V_a = R_p\, V_p$





Motion in Gravitational Field

- Both spacecraft energy and angular momentum are proportional to the spacecraft mass ***m***.
- It is convenient to use spacecraft energy per unit mass $\boldsymbol{\xi = E_0/m}$ (known as spacecraft specific energy), and spacecraft angular momentum $\boldsymbol{h = H/m}$ (known as spacecraft specific angular momentum).
- From laws of conservation and Newton's Law of gravity, the semimajor axis ***a***, eccentricity ***e***, and period ***P*** are

$$\boxed{a = -\frac{\mu}{2\xi} \qquad e^2 = 1 + \frac{2h^2\xi}{\mu^2} \qquad P = \frac{2\pi a^{3/2}}{\sqrt{\mu}}}$$

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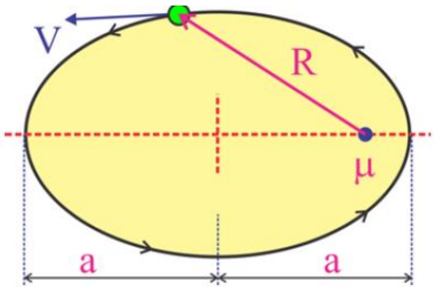
Motion in Gravitational Field

- Combining expressions for total energy and specific energy

$$\boxed{\xi = \frac{E_0}{m} = -\frac{\mu}{2a}} \quad \boxed{\xi = \frac{E_0}{m} = \frac{\mathrm{KE}}{m} + \frac{\mathrm{PE}}{m} = \frac{V^2}{2} - \frac{\mu}{R}} \quad \boxed{\mu = GM}$$

- one obtains the energy integral, which yields the general relation for the **velocity of an orbiting body**:

$$V = \sqrt{\frac{2\mu}{R} - \frac{\mu}{a}}$$



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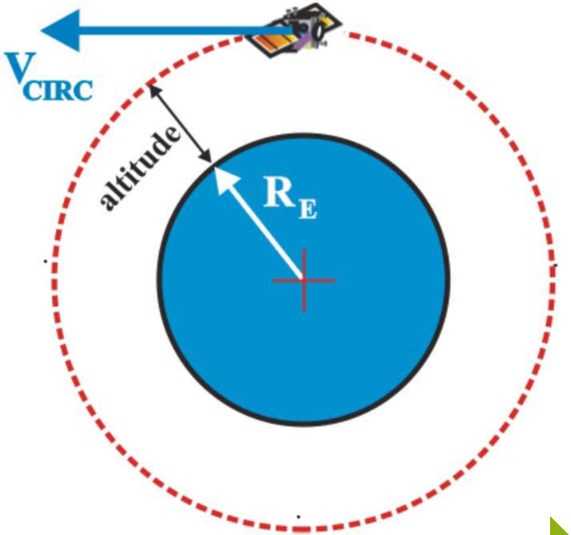
Circular and Escape Velocities



- centrifugal force = gravitational force

$$\frac{mV_{CIRC}^2}{R_{CIRC}} = \frac{GmM_0}{R_{CIRC}^2} = \frac{m\mu}{R_{CIRC}^2}$$

$$V_{CIRC} = \sqrt{\frac{GM_0}{R_{CIRC}}} = \sqrt{\frac{\mu}{R_{CIRC}}}$$ **circular velocity**

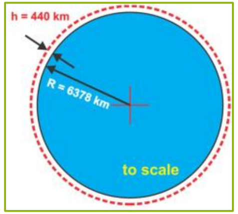
$$P = \frac{2\pi R_{CIRC}}{V_{CIRC}} = 2\pi\sqrt{\frac{R_{CIRC}^3}{\mu}}$$ **period**

Escape velocity

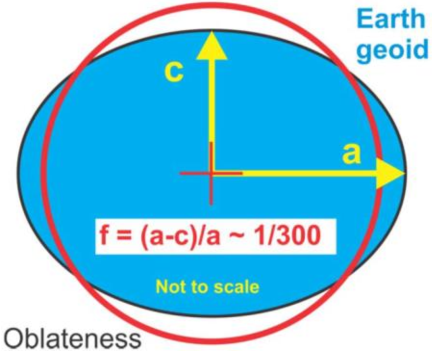
Total energy = kinetic energy + potential energy = 0

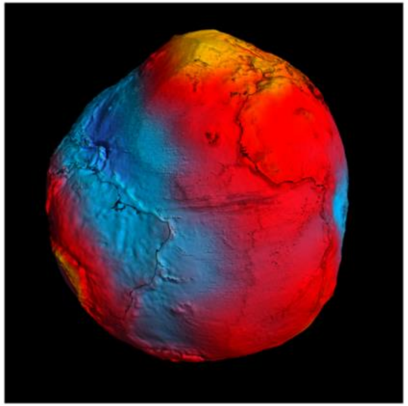
$$\frac{mV_{ESC}^2}{2} = \frac{m\mu}{R} \Rightarrow V_{ESC} = \sqrt{\frac{2GM_0}{R}} = \sqrt{\frac{2\mu}{R}}$$

Earth Oblateness



- **Geoid** = equipotential surface of the gravity field most closely approximating mean sea level
- Accurate geoid model produced by the **Gravity field and steady-state Ocean Circulation Explorer (GOCE) mission**.
- The deviations in height (–100 m to +100 m) from an ellipsoid are exaggerated 10,000 times in the image. The blue colors represent low values and the reds/yellows represent high values.





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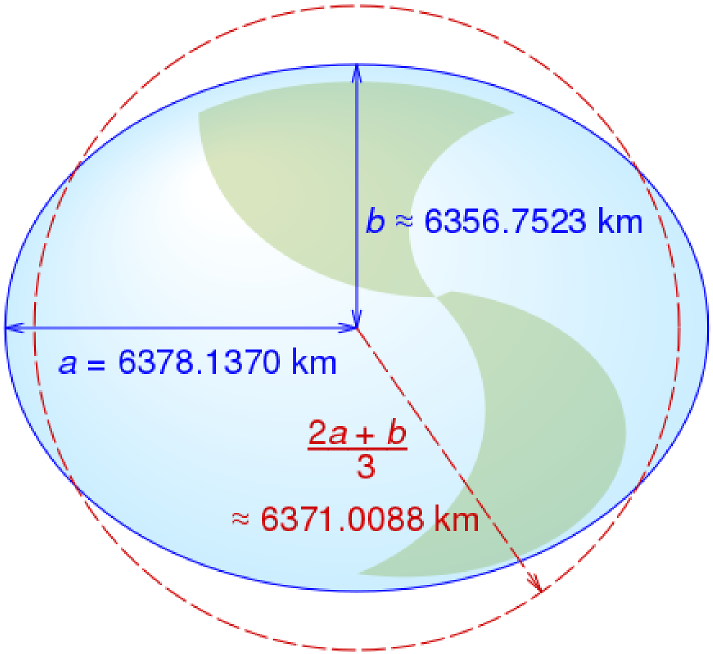
Earth Oblateness



- equatorial radius: $\mathbf{R_e}$ = 6378.14 km
- polar radius: $\mathbf{R_p}$ = 6356.8 km
- volumetric radius: $\mathbf{R_v}$ = 6371.0 km

volumetric mean radius is the radius of a sphere with the same volume

- **orbit altitude** commonly referred to the difference between the orbit radius and the equatorial radius
- ***h* (orbit altitude) = *R* (orbit radius) – R_e (equatorial radius or a=6378 km)**


Thank you!