Neutron balance eigenvalue/eigenvector problem

Computer Project 2 of NPRE 247

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I. INTRODUCTION

Many processes can affect neutron flux in reactor, such as fission, absorption, scattering and so on. From the definition of flux $\phi = nv$ and macroscopic cross section $\sum = n\sigma$, we can see that they all depend on space position $\vec{r}$ and energy E. To achieve neutron balance considering all these processes, we can then get a balance equation with respect to position and energy. This actually turns to an eigenvalue problem, which has the following form

$$B\phi = \lambda\phi, \tag{1.1}$$

from which the neutron flux can be solved.

1.1 Multi-group neutron balance in infinite space

For infinite space (0-D), we will ignore neutron diffusion and treat all involved terms independent on position. The neutron balance equation for group g can be written as,

Loss = Gain

Absorption + Outscattering = Fission + Inscattering

$$\Sigma_{r,g}\phi_g = S_g \tag{1.2}$$

where:

$$\Sigma_{r,g} = \Sigma_{a,g} + \sum_{g' \neq g} \Sigma_{s,g' \leftarrow g}$$

$$S_g = \frac{\chi_g}{k} \sum_{g'} \nu\Sigma_{f,g'}\phi_{g'} + \sum_{g' \neq g} \Sigma_{s,g \leftarrow g'}\phi_{g'}$$

$\Sigma_{a,g}\phi_g$ absorption, $\Sigma_{a,g}$ is absorption macroscopic cross section of group g

$\sum_{g' \neq g} \Sigma_{s,g' \leftarrow g}\phi_g$ outscattering from group g to other group $g' \neq g$

$\frac{\chi_g}{k} \sum_{g'} \nu\Sigma_{f,g'}\phi_{g'}$ fission, Σ_f is fission cross section, ν is number of generated neutrons, χ_g is percentage of group g

$\sum_{g' \neq g} \Sigma_{s,g \leftarrow g'}\phi_{g'}$ inscattering from other group $g' \neq g$ to group g

1.2 Matrix coefficient

Next, we convert equation (1.2) to its matrix form, where we will define multi-group migration, fission matrix and neutron balance eigenvalue/eigenvector matrix. Suppose we have G groups.

For absorption $\Sigma_{a,g}\phi_g$, its matrix form is

$$A = \begin{bmatrix} \Sigma_{a,1} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \Sigma_{a,2} & \cdots & 0 & \cdots & 0 & 0 \\ & & \ddots & & & & \\ 0 & 0 & \cdots & \Sigma_{a,g} & \cdots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & \cdots & 0 & \cdots & \Sigma_{a,G-1} & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \Sigma_{a,G} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_g \\ \vdots \\ \phi_{G-1} \\ \phi_G \end{bmatrix} \tag{1.3}$$

For outscattering $\sum_{g'\neq g}\Sigma_{s,g'\leftarrow g}\phi_g$, its matrix form is

$$S_{out} = \begin{bmatrix} \sum_{g'\neq 1}\Sigma_{s,g'\leftarrow 1} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \sum_{g'\neq 2}\Sigma_{s,g'\leftarrow 2} & \cdots & 0 & \cdots & 0 & 0 \\ & & \ddots & & & & \\ 0 & 0 & \cdots & \sum_{g'\neq g}\Sigma_{s,g'\leftarrow g} & \cdots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & \cdots & 0 & \cdots & \sum_{g'\neq G-1}\Sigma_{s,g'\leftarrow G-1} & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \sum_{g'\neq G}\Sigma_{s,g'\leftarrow G} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_g \\ \vdots \\ \phi_{G-1} \\ \phi_G \end{bmatrix} \tag{1.4}$$

For inscattering $\sum_{g'\neq g}\Sigma_{s,g\leftarrow g'}\phi_{g'}$, its matrix form is

$$S_{in=} \begin{bmatrix} 0 & \Sigma_{s,1\leftarrow 2} & \cdots & \Sigma_{s,1\leftarrow g} & \cdots & \Sigma_{s,1\leftarrow G-1} & \Sigma_{s,1\leftarrow G} \\ \Sigma_{s,2\leftarrow 1} & 0 & \cdots & \Sigma_{s,2\leftarrow g} & \cdots & \Sigma_{s,2\leftarrow G-1} & \Sigma_{s,2\leftarrow G} \\ & & \ddots & & & & \\ \Sigma_{s,g\leftarrow 1} & \Sigma_{s,g\leftarrow 2} & \cdots & 0 & \cdots & \Sigma_{s,g\leftarrow G-1} & \Sigma_{s,g\leftarrow G} \\ & & & & \ddots & & \\ \Sigma_{s,G-1\leftarrow 1} & \Sigma_{s,G-1\leftarrow 2} & \cdots & \Sigma_{s,G-1\leftarrow g} & \cdots & 0 & \Sigma_{s,G-1\leftarrow G} \\ \Sigma_{s,G\leftarrow 1} & \Sigma_{s,G\leftarrow 2} & \cdots & \Sigma_{s,G\leftarrow g} & \cdots & \Sigma_{s,G\leftarrow G-1} & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_g \\ \vdots \\ \phi_{G-1} \\ \phi_G \end{bmatrix} \tag{1.5}$$

For Fission $\frac{\chi_g}{k}\sum_{g'}\nu\Sigma_{f,g'}\phi_{g'}$, its matrix form is

$$F = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_g \\ \vdots \\ \chi_{G-1} \\ \chi_G \end{bmatrix} \begin{bmatrix} \nu\sum_{f1} & \nu\sum_{f2} & \cdots & \nu\sum_{fg} & \cdots & \nu\sum_{fG-1} & \nu\sum_{fG} \end{bmatrix} \tag{1.6}$$

Thus, the neutron balance equation turns to the following matrix form,

$$\left[A + S_{out} - S_{in}\right]\phi = \frac{1}{k}F\phi \tag{1.7}$$

where $A + S_{out} - S_{in}$ is called the migration matrix, and F is called the fission matrix. Equation (1.7) can be rewritten as

$$\left[A + S_{out} - S_{in}\right]^{-1} F\phi = k\phi \tag{1.8}$$

where $\left[A + S_{out} - S_{in}\right]^{-1} F$ is called the neutron balance eigenvalue/eigenvector matrix, k and ϕ is called the eigenvalue and eigenvector of $\left[A + S_{out} - S_{in}\right]^{-1} F$.

1.3 Power iteration

Use the following method, we can get the largest eigenvalue and corresponding eigenvector for the equation

$$B\phi = \lambda\phi \tag{1.9}$$

$k_1 = 1$
$\phi_1 = 1$
do i = 1 to MAXIT
    $\phi_{i+1} = B\,\phi_i \,/\, \|B\,\phi_i\|_2$
    $k_{i+1} = (B\,\phi_{i+1})^T \phi_{i+1} \,/\, (\phi_{i+1}^T \phi_{i+1})$
end

1.4 Data file description

The first column is the group type;

The second column is absorption cross section $\sum_a$;

The third column is $\nu\sum_f$; the forth column is χ.

The other columns are the scattering cross sections.

II. 2G EIGENVALUE/EIGENVECTOR PROBLEM

Table 2.1 Absorption and fission data $\sum_a$, $\nu\sum_f$, χ

Group	$\sum_a$	$\nu\sum_f$	χ
1 2	0.0092 0.0927	0.0046 0.1135	1.0000 0.0000

Table 2.2 Scattering cross section

TO\FROM	1	2
1 2	1.0000 0.0204	0.0000 2.0000

2.1 Migration and fission matrix

According to the matrix defined in Section 1.2, we have

$$A = \begin{bmatrix} 0.0092 & 0 \\ 0 & 0.0927 \end{bmatrix},\ S_{out} = \begin{bmatrix} 0.0204 & 0 \\ 0 & 0 \end{bmatrix},\ S_{in} = \begin{bmatrix} 0 & 0 \\ 0.0204 & 0 \end{bmatrix},\ F = \begin{bmatrix} 0.0046 & 0.1135 \\ 0 & 0 \end{bmatrix}$$

Migration matrix is

$$A + S_{out} - S_{in} = \begin{bmatrix} 0.0296 & 0 \\ -0.0204 & 0.0927 \end{bmatrix}$$

Fission matrix is

$$F = \begin{bmatrix} 0.0046 & 0.1135 \\ 0 & 0 \end{bmatrix}$$

2.2 Analytical calculation of eigenvalues/eigenvectors of the neutron balance

From Section 2.1, we know the migration matrix is

$$A + S_{out} - S_{in} = \begin{bmatrix} 0.0296 & 0 \\ -0.0204 & 0.0927 \end{bmatrix} \tag{2.1}$$

Its inverse matrix is

$$\left[A + S_{out} - S_{in}\right]^{-1} = \begin{bmatrix} 33.7838 & 0 \\ 7.4346 & 10.7875 \end{bmatrix} \tag{2.2}$$

So the eigenvalue/eigenvector matrix is

$$\left[A + S_{out} - S_{in}\right]^{-1} F = \begin{bmatrix} 0.1554 & 3.8345 \\ 0.0342 & 0.8438 \end{bmatrix}$$

Consider the following eigenvalue problem,

$$\left[A + S_{out} - S_{in}\right]^{-1} F\phi = k\phi \tag{2.3}$$

Substitute the values, we have

$$\begin{bmatrix} 0.1554 & 3.8345 \\ 0.0342 & 0.8438 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \varphi_2 \end{bmatrix} = k\begin{bmatrix} \phi_1 \\ \varphi_2 \end{bmatrix} \tag{2.4}$$

We can rewrite it as

$$\begin{bmatrix} 0.1554 - k & 3.8345 \\ 0.0342 & 0.8438 - k \end{bmatrix}\begin{bmatrix} \phi_1 \\ \varphi_2 \end{bmatrix} = 0 \tag{2.5}$$

Its determinant has to be zero, which turns into a quadratic equation

$$0.1554 - k \quad 0.8438 - k \quad - 3.8345 \times 0.0342 = 0 \tag{2.6}$$

Its roots are

$$k_1 = 0, k_2 = 0.9992 \tag{2.7}$$

To get the normalized vector corresponding to the above eigenvalues, we can set $\phi_1 = 1$, solve ϕ_2, then divide them by $\sqrt{1 + \phi_2^2}$.

For eigenvalue $k = 0$, its eigenvector is

$$\phi = \begin{bmatrix} -0.9992 \\ 0.0405 \end{bmatrix} \tag{2.8}$$

For eigenvalue $k = 0.9992$, its eigenvector is

$$\phi = \begin{bmatrix} 0.9766 \\ 0.2149 \end{bmatrix} \tag{2.9}$$

2.3 Numerical calculation of eigenvalues/eigenvectors of the neutron balance

For the following equation,

$$\begin{bmatrix} 0.1554 & 3.8345 \\ 0.0342 & 0.8438 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \varphi_2 \end{bmatrix} = k\begin{bmatrix} \phi_1 \\ \varphi_2 \end{bmatrix} \tag{2.10}$$

Use function *eig* in MATLAB,

[V,D] = eig(Matrix);

which gives the following eigenvalues/eigenvectors,

$$\begin{aligned} k &= 0, \phi = \begin{bmatrix} -0.9992 \\ 0.0405 \end{bmatrix} \\ k &= 0.9992, \phi = \begin{bmatrix} -0.9766 \\ -0.2149 \end{bmatrix} \end{aligned} \tag{2.11}$$

2.4 Calculation of the largest eigenvalue and the corresponding eigenvector using power iteration

After 2 iterations, we get

$k = 0.9992$

$\phi = \begin{bmatrix} 0.9766 \\ 0.2149 \end{bmatrix}$

III. 8G EIGENVALUE/EIGENVECTOR PROBLEM

Table 3.1 Absorption and fission data Σ_a, $\nu\Sigma_f$, χ

Group	Σ_a	$\nu\Sigma_f$	χ
1	0.0056	0.0134	0.3507
2	0.0029	0.0056	0.4105
3	0.0025	0.0011	0.2388
4	0.0133	0.0067	0.0000
5	0.0473	0.0220	0.0000
6	0.0170	0.0222	0.0000
7	0.0538	0.0897	0.0000
8	0.1386	0.2365	0.0000

Table 3.2 Scattering cross section

TO\FROM	1	2	3	4	5	6	7	8
1	0.1179	0	0	0	0	0	0	0
2	0.0530	0.1949	0	0	0	0	0	0
3	0.0301	0.1159	0.5868	0	0	0	0	0
4	0.0001	0.0005	0.0769	0.8234	0	0	0	0
5	0	0	0.0019	0.1961	0.8180	0	0	0
6	0	0	0.0000	0.0050	0.1737	0.6902	0.0023	0
7	0	0	0	0.0007	0.0246	0.2707	0.8626	0.0275
8	0	0	0	0.0001	0.0073	0.0550	0.3589	1.9761

3.1 Migration and fission matrix

The migration matrix $A + S_{out} - S_{in} =$

```
  0.0888         0         0         0         0         0         0         0
 -0.0530    0.1193         0         0         0         0         0         0
 -0.0301   -0.1159    0.0813         0         0         0         0         0
 -0.0001   -0.0005   -0.0769    0.2152         0         0         0         0
       0         0   -0.0019   -0.1961    0.2529         0         0         0
       0         0         0   -0.0050   -0.1737    0.3427   -0.0023         0
       0         0         0   -0.0007   -0.0246   -0.2707    0.4150   -0.0275
       0         0         0   -0.0001   -0.0073   -0.0550   -0.3589    0.1661
```

Fission matrix $F =$

0.0047	0.0020	0.0004	0.0023	0.0077	0.0078	0.0315	0.0829
0.0055	0.0023	0.0005	0.0028	0.0090	0.0091	0.0368	0.0971
0.0032	0.0013	0.0003	0.0016	0.0053	0.0053	0.0214	0.0565
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0

3.2 Numerical calculation of eigenvalues/eigenvectors of the neutron balance

For the following equation,

$$\left[A + S_{out} - S_{in}\right]^{-1} F\phi = k\phi \tag{3.1}$$

where $\left[A + S_{out} - S_{in}\right]^{-1} F =$

0.0529	0.0221	0.0043	0.0265	0.0869	0.0877	0.3543	0.9340
0.0696	0.0291	0.0057	0.0348	0.1143	0.1153	0.4660	1.2287
0.1582	0.0661	0.0130	0.0791	0.2597	0.2621	1.0590	2.7921
0.0567	0.0237	0.0047	0.0284	0.0931	0.0940	0.3797	1.0010
0.0452	0.0189	0.0037	0.0226	0.0742	0.0748	0.3024	0.7972
0.0239	0.0100	0.0020	0.0119	0.0392	0.0395	0.1598	0.4213
0.0222	0.0093	0.0018	0.0111	0.0364	0.0367	0.1484	0.3914
0.0578	0.0242	0.0047	0.0289	0.0950	0.0958	0.3872	1.0208

Use function *eig* in MATLAB,

[V,D] = eig(Matrix);

We can then get the eigenvalues and eigenvectors. Ignore complex values, we have

(1) $k = 1.4063$, $\phi =$

```
-0.2572
-0.3383
-0.7688
-0.2756
-0.2195
-0.1160
-0.1078
-0.2811
```

(2) $k = 0$, there are three eigenvectors, $\phi =$

-0.9894	0.2072	0.0737
0.0509	-0.6138	0.0461
0.1156	-0.1017	-0.8948
0.0415	-0.6514	-0.1128
0.0330	-0.1359	-0.1783
0.0174	-0.2533	-0.3712
0.0162	-0.2099	-0.0598
0.0423	0.1378	0.0762

3.3 Calculation of the largest eigenvalue and the corresponding eigenvector using power iteration

$k = 1.4063$

$\phi =$

0.2572
0.3383
0.7688
0.2756
0.2195
0.1160
0.1078
0.2811