Arquitectura de Computadores Avançada

Design principles for Hadamard codes

António Rui Borges

Computation of the code generating matrix for Hadamard code $[8,4,4]_2$

$$\boldsymbol{x} = \boldsymbol{m} \times \boldsymbol{G} = \| m_1, m_2, \cdots, m_4 \| \times \boldsymbol{G}$$

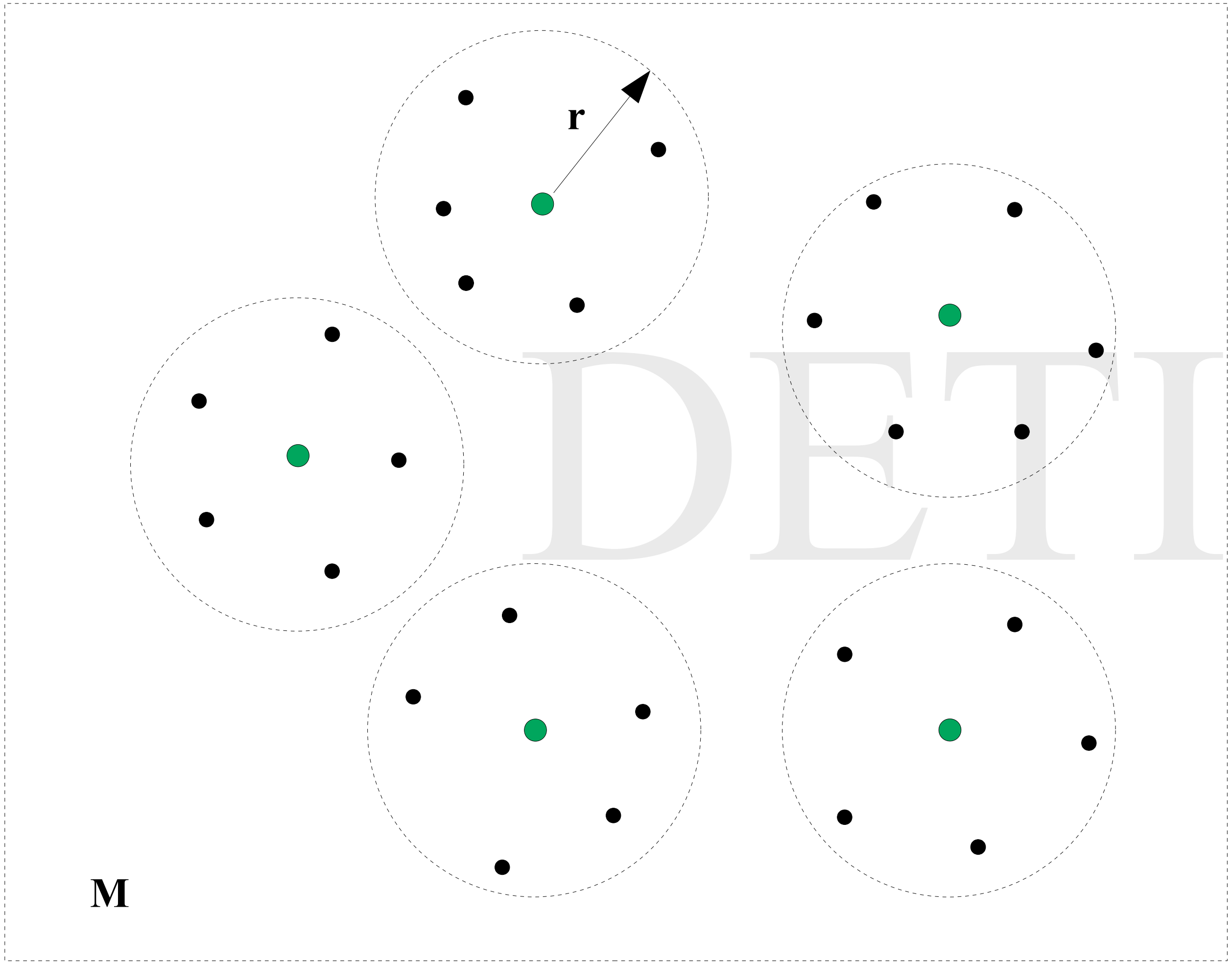
$$\boldsymbol{G} = \begin{Vmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{Vmatrix}$$

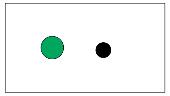
Hadamard code $[8,4,4]_2$

$$\boldsymbol{x} = \| \, x_0 \; x_1 \cdots \, x_7 \, \| = \boldsymbol{m} \times \boldsymbol{G} = \| \, m_1 \; m_2 \cdots \, m_4 \, \| \times \boldsymbol{G}$$

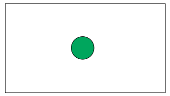
$$\begin{aligned}
\boldsymbol{x}^0 &= \| 0\ 0\ 0\ 0 \| \times G = \| 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \| \\
\boldsymbol{x}^1 &= \| 1\ 0\ 0\ 0 \| \times G = \| 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \| \\
\boldsymbol{x}^2 &= \| 0\ 1\ 0\ 0 \| \times G = \| 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1 \| \\
\boldsymbol{x}^3 &= \| 1\ 1\ 0\ 0 \| \times G = \| 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0 \| \\
\boldsymbol{x}^4 &= \| 0\ 0\ 1\ 0 \| \times G = \| 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1 \| \\
\boldsymbol{x}^5 &= \| 1\ 0\ 1\ 0 \| \times G = \| 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0 \| \\
\boldsymbol{x}^6 &= \| 0\ 1\ 1\ 0 \| \times G = \| 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0 \| \\
\boldsymbol{x}^7 &= \| 1\ 1\ 1\ 0 \| \times G = \| 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1 \| \\
\boldsymbol{x}^8 &= \| 0\ 0\ 0\ 1 \| \times G = \| 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \| \\
\boldsymbol{x}^9 &= \| 1\ 0\ 0\ 1 \| \times G = \| 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \| \\
\boldsymbol{x}^{10} &= \| 0\ 1\ 0\ 1 \| \times G = \| 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0 \| \\
\boldsymbol{x}^{11} &= \| 1\ 1\ 0\ 1 \| \times G = \| 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1 \| \\
\boldsymbol{x}^{12} &= \| 0\ 0\ 1\ 1 \| \times G = \| 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0 \| \\
\boldsymbol{x}^{13} &= \| 1\ 0\ 1\ 1 \| \times G = \| 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1 \| \\
\boldsymbol{x}^{14} &= \| 0\ 1\ 1\ 1 \| \times G = \| 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1 \| \\
\boldsymbol{x}^{15} &= \| 1\ 1\ 1\ 1 \| \times G = \| 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0 \|
\end{aligned}$$

Robustness of Hadamard code $[8,4,4]_2$





metric space **M**
where codewords
are immersed
256 elements

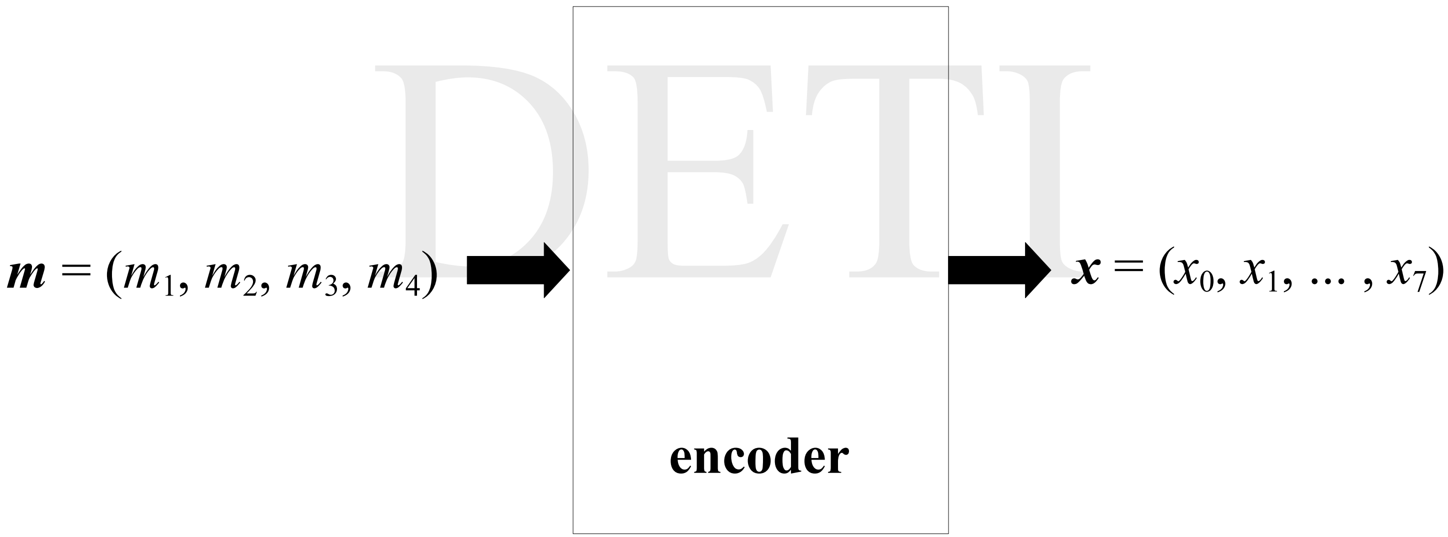


subspace of **M**
consisting solely
of codewords
16 elements

$$r = \frac{d-1}{2} = 1{,}5$$

maximum radius of each codeword
which allows recovery through
nearest-neighbor approximation
error in 1 bit is corrected
error in 2 bits is detected

Parallel implementation of the encoder for [8,4,4]$_2$ *- 1*



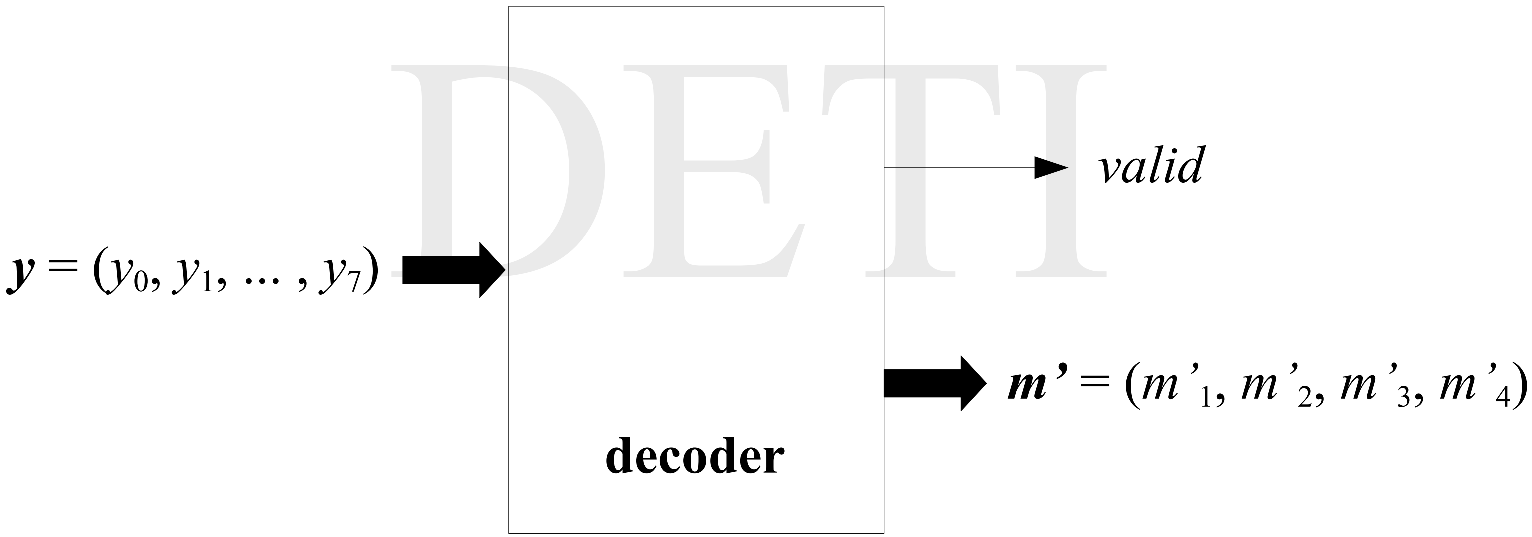
Parallel implementation of the encoder for $[8,4,4]_2$ *- 2*

Straightforward implementation

$$
\begin{aligned}
x_0 &= m_4 \\
x_1 &= m_1 \oplus m_4 \\
x_2 &= m_2 \oplus m_4 \\
x_3 &= m_1 \oplus m_2 \oplus m_4 \\
x_4 &= m_3 \oplus m_4 \\
x_5 &= m_1 \oplus m_3 \oplus m_4 \\
x_6 &= m_2 \oplus m_3 \oplus m_4 \\
x_7 &= m_1 \oplus m_2 \oplus m_3 \oplus m_4
\end{aligned}
$$

- 12 x-ors are needed
- 3 x-or propagation time delays in the worst case.

Parallel implementation of the decoder for $[8,4,4]_2$ *- 1*



Parallel implementation of the decoder for $[8,4,4]_2$ *- 2*

Application of the property of local decodability

$$
\begin{aligned}
c_{11} &= y_0 \oplus y_1 \\
c_{12} &= y_2 \oplus y_3 \\
c_{13} &= y_4 \oplus y_5 \\
c_{14} &= y_6 \oplus y_7 \\
c_{21} &= y_0 \oplus y_2 \\
c_{22} &= y_1 \oplus y_3 \\
c_{23} &= y_4 \oplus y_6 \\
c_{24} &= y_5 \oplus y_7 \\
c_{31} &= y_0 \oplus y_4 \\
c_{32} &= y_1 \oplus y_5 \\
c_{33} &= y_2 \oplus y_6 \\
c_{34} &= y_3 \oplus y_7
\end{aligned}
$$

$c_{\#3}c_{\#2}$ \ $c_{\#1}c_{\#0}$	00	01	11	10
00	0	0	E	0
01	0	E	1	E
11	E	1	1	1
10	0	E	1	E