Probabilistic Changepoint Modeling

An example report for Pittsburgh useR Group

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Summary Our imaginary store receives some number of customers per day. We start an advertising campaign which takes some time to have a full effect on the rate of customers we receive daily, and then has a long-term effect after we have stopped it. Using probabilistic programming languages (PPLs), we can specify a Bayesian model and infer the hidden rates.

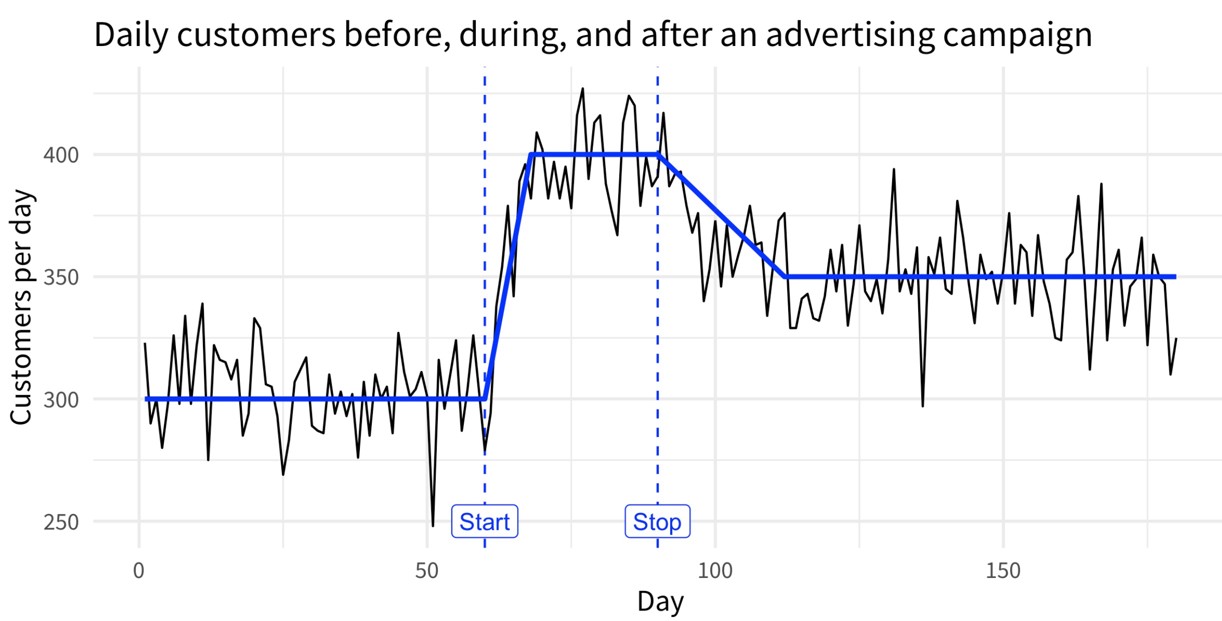
Data simulation

Suppose we are operating a store which receives a random number of customers per day, and specifically it's random according to the Poisson distribution with rate λ_1. Then we start an advertisement campaign – which takes a few days to come to full effect – that changes the rate to λ_2. When we stop the ad, it gradually loses effect but the campaign has had a long-lasting effect (e.g. we've gained new customers who come back regularly), which means our store now receives customers at rate λ_3.

Table 1: Simulation parameters

parameter	value
Rates (unknown)	
λ_1 (old normal)	300
λ_2 (temporary)	400
λ_3 (new normal)	350
Other parameters	
N (total days)	180
T_1 (ad start)	60
T_2 (ad stop)	90
d_1 (time for full effect)	7
d_2 (time to new normal)	21

In the simulation, we have two weights – w_1 and w_2 – which sum to 1 and produce gradual (albeit not smooth) transitions between the different rates. The change is a slope in this toy scenario, but it can also be a more interesting transition like a sigmoid function such as the Gompertz curve.



Modelling

We'll take a look at three similar models – $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ – which model the daily counts of customers y_t as a Poisson-distributed random variable with time-varying rate $\lambda(t)$, up to a maximum of $t = N$ days of data: $y_t \sim \text{Poisson}(\lambda(t)),\ t = 1, \ldots, N$.

The ad campaign starts on T_1 and stops on T_2. We are interested in inferring λ_1 (the rate before T_1), λ_2 (the rate between T_1 and T_2), and λ_3 (the rate after T_2). We specify $\lambda \sim \mathcal{N}(300, 100)$ and let Stan implicitly assign a default prior to β_1 and β_2.

Model 1

In the simpler model $\mathcal{M}_1$, we ignore the obvious transition periods and model the switch between the rates as immediate:

$$\lambda(t) = \begin{cases} \lambda_1, & \text{if } t \leq T_1, \\ \lambda_2, & \text{if } T_1 < t \leq T_2, \\ \lambda_3, & \text{if } t > T_2. \end{cases}$$

Model 2

In the slightly more complex model $\mathcal{M}_2$, we include the two gradual changes as slopes over d_1 and d_2 days after T_1 and T_2, respectively. We formalize this as follows:

$$\lambda(t) = \begin{cases} \lambda_1, & \text{if } t \leq T_1, \\ \lambda_2, & \text{if } T_1 + d_1 \leq t \leq T_2, \\ \lambda_3, & \text{if } t \geq T_2 + d_2, \\ \lambda_1 + \beta_1(t - T_1), & \text{if } T_1 < t \leq T_1 + d_1, \\ \lambda_2 + \beta_2(t - T_2), & \text{if } T_2 < t \leq T_2 + d_2. \end{cases}$$

Note: in this toy scenario we know d_1 and d_2, but we can also make them hidden parameters to infer from the data. We could formalize our intuition about their values by assigning them priors $\mathcal{N}(7, 2)$ and $\mathcal{N}(14, 4)$, respectively.

Model comparison

Table 2: Posterior probabilities of the three models given data and the estimated rates, calculated using the bridgesampling package.

		Estimate (95% Credible Interval)		
Model	$\Pr(\mathcal{M} \mid \mathcal{D})$	λ_1 (300)	λ_2 (400)	λ_3 (350)
$\mathcal{M}_1$	0.0000	302.9 (298.6–307.1)	387.5 (380.6–394.4)	352.8 (348.9–356.9)
$\mathcal{M}_2$	1.0000	302.8 (298.5–307.1)	395.0 (387.8–402.7)	348.1 (343.8–352.5)

The Bayes factor (BF) can be used to decide between these two competing models – $\mathcal{M}_1$ (instant changes between rates) and $\mathcal{M}_2$ (gradual changes between rates) – by quantifying how much more likely the data $\mathcal{D}$ is under $\mathcal{M}_1$ vs $\mathcal{M}_2$:

$$\text{BF}_{12} = \frac{p(\mathcal{D} \mid \mathcal{M}_1)}{p(\mathcal{D} \mid \mathcal{M}_2)},$$

Using **bridgesampling** (Gronau & Singmann, 2018) allows us to calculate marginal likelihoods – $p(\mathcal{D} \mid \mathcal{M})$ – of Stan models really easily and therefore compute the BF (see `?bf`), which – for $\mathcal{M}_1$ compared to $\mathcal{M}_2$, for example – comes out to be 0, meaning there is no evidence for choosing model 1 over model 2.

Inference results

Table 3: Inference using $\mathcal{M}_2$ (gradual changes between rates).

Parameter	Truth	Point Estimate	95% Credible Interval
Rates			
λ_1	300	302.8	(298.4, 306.9)
λ_2	400	394.9	(388.1, 403.0)
λ_3	350	348.0	(343.9, 352.5)
Differences			
$\lambda_2 - \lambda_1$	100	92.1	(84.1, 100.9)
$\lambda_3 - \lambda_2$	-50	-46.9	(-55.9, -39.1)
$\lambda_3 - \lambda_1$	50	45.2	(38.9, 51.1)

Table 3 shows the inferred differences of rates and we can see that our imaginary advertisement had a positive, statistically significant impact on how many imaginary customers our imaginary store receives per day on average, both during the campaign and long after.

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