Isomorphism theorems, Solvable Groups and the Jordan Holder Theorem

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“Formulae” for roots of polynomials

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubic equations: $ax^3 + bx^2 + cx + d = 0 \Rightarrow$ Cardano’s formula

For quartic equations, there is also a formula.

How about quintic functions? How about polynomials of higher degrees?

“Solvable by radical” theorem

Theorem 0.1 (Solvable by radicals (Dummit Foote Sec 14.7 Theorem 39)): The polynomial $f(x)$ can be solved by radicals (i.e. it’s roots can be written as the combination of $+, -, \times, \div, \sqrt{\cdot}$) and its coefficients, if and only if its Galois group is **solvable**.

I will not explain what are Galois groups today.

Solvable Groups

Definition 0.1 ((Finite) Solvable Groups): A finite group G is **solvable** if there exists a chain of subgroups

$$1 = G_s \lhd G_{s-1} \lhd ... \lhd G_2 \lhd G_1 = G$$

with G_i/G_{i+1} cyclic for $i = 1, ..., s-1$.

But, what are those triangles? What is G_i/G_{i+1}?


就讓我們開始吧

Remark: most of the proofs in this slide are just outlines and incomplete. I hope you can finish the proof, because this is the easiest way to verify that you fully understand how to play with the definitions.

Some definition or tips

- I expect you to know the definition of groups and subgroups. The notation $H \leq G$ means "H is a subgroup of G".
- I expect you to know what a function (mapping) is, what is 1-1 (injective), onto (surjective), and 1-1 onto (bijective).
- Let $f : X \to Y$ be a function, $X' \subseteq X, Y' \subseteq Y$.
  - The set $f(X') := \{f(x) \mid x \in X'\} \subseteq Y$ is called the image.
  - The set $f^{-1}(Y') := \{x \in X \mid f(x) \in Y'\} \subseteq X$ is called the preimage (fiber). Note that f^{-1} is sometimes not "the inverse function".
  - You can make these more clear by drawing a graph.
- Subgroup criterion: Let G be a group and $S \subseteq G$, then S is a subgroup of G if for all $a, b \in S$ we have $ab^{-1} \in S$.
- Generators.
- Cyclic groups / subgroups.
- Groups you are already familiar with: $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Q}^\times, \mathbb{R}^\times, \mathbb{C}^\times$.
- Groups you may not be familiar with: D_{2n}, C_n, Q_8.

- Homomorphism: Let G, G' be groups and $\varphi : G \to G'$ be a mapping. We say φ is a homomorphism if for all $x, y \in G$ we have $\varphi(xy) = \varphi(x)\varphi(y)$. Informally, φ is structure-preserving.
- An isomorphism is a bijective homomorphism.
- Let $\varphi : G \to G'$ be a group homomorphism, $H \leq G$, $H' \leq G'$. Then
  - $\varphi(H) \leq G'$. Especially, $\varphi(G) \leq G'$.
  - $\varphi^{-1}(H') \leq G$. Especially, the kernel $\ker\varphi := \varphi^{-1}(\{e\}) \leq G$.

Normal subgroups

Definition 0.2 (Normal subgroups): A subgroup H of G is normal if $xHx^{-1} = H$ for all $x \in G$. We say "H is normal in G" or write $H \lhd G$.

Remark: some equivalent definitions are

- $xH = Hx$
- $xHx^{-1} \subseteq H$

Example: For any group G, both G and the trivial subgroup $\{e\}$ are normal in G.

Example: Any subgroup of G is normal if G is abelian.

Example: $\langle s \rangle$ is not normal in D_{2n} for $n \geq 3$.

Easy exercise: verify the above statements.

Important exercise: Let $H \leq G$ and $N \lhd G$. Show that $HN \leq G$. Moreover, if $H \lhd G$, then $HN \lhd G$.

Exercise: Let A be an index set (do not assume it's countable), and $\{H_\alpha\}_{\alpha \in A}$ be a collection of normal subgroups in G. Show that $\langle \{H_\alpha\}_{\alpha \in A} \rangle = \left\{ h_1 ... h_n \mid n \in \mathbb{N}, h_i \in \bigcup_{\alpha \in A} H_\alpha \right\}$ is still normal in G.

Cosets and Quotient Groups

Definition 0.3 (Cosets): Let $H \leq G$, then we denote G/H "the set of (left) cosets"

$$G/H := \{xH \mid x \in G\}$$

where $xH := \{xh \mid h \in H\}$.

Definition 0.4 (Quotient groups): If $H \lhd G$, then G/H forms a group, where the group operation is defined by

$$xH \cdot yH = xyH$$

The group G/H is called a **quotient group**. Also, let $\pi : G \to G/H$ be defined by $\pi(x) := xH$, then π is called a projection.

Exercise: Show that the group operation on a quotient group is well defined. In other words, the result of the operation is independent of the choice of representatives.

Exercise: Show that π is a homomorphism, and its kernel $\ker \pi = H$.

Use homomorphisms to define normal subgroups

Theorem 0.2: Let $\varphi : G \to G'$ be a group homomorphism, then $\ker \varphi \lhd G$.

Proof: Use the definition of kernel and normal subgroups. $\square$

Theorem 0.3: If N is a normal subgroup of G, then N must be the kernel of some group homomorphism from G.

Proof: Let $\pi : G \to G/N$ defined by $\pi(x) := xN$. Then verify that π is indeed a group homomorphism, and that $N = \ker \pi$. $\square$

Exercise: We already know G and $\{e\}$ are normal in G. Can you find homomorphisms with kernel G and $\{e\}$ respectively?

Exercise: Let H and K be normal subgroups of G. Show that $H \cap K$ is also normal in G by (1) using the definition of normal subgroups directly (2) constructing a group homomorphism.

Exercise: Let A be an index set (do not assume it's countable), and $\{H_\alpha\}_{\alpha \in A}$ be a collection of normal subgroups in G. Show that $\bigcap_{\alpha \in A} H_\alpha$ is still normal in G.

The Isomorphism Theorems

Theorem 0.4 (First Isomorphism Theorem): Let $\varphi : G \to G'$ be a group homomorphism, then

$$G/\ker\varphi \cong f(G)$$

Proof: Denote $N := \ker\varphi$. Let $f : G/N \to \varphi(G)$ with $f(xN) := \varphi(x)$. Show that f is well-defined, bijective and is a homomorphism. $\square$

The First Isomorphism Theorem is very very important.

And yes, there are the second and the third isomorphism theorems, which we will cover later.

Exercise (Dummit Foote 3.3.7): Let M and N be normal subgroups of G such that $MN = G$. Prove that

$$G/(M \cap N) \cong (G/M) \times (G/N)$$

Exercise (Dummit Foote 3.3.8): Let p be a prime and let $P \subseteq \mathbb{C}^\times$ be defined by

$$P := \{z \mid z^{p^n} = 1 \text{ for some } n \in \mathbb{N}\}$$

1. Show that $P \leq \mathbb{C}$. Deduce that $P \lhd \mathbb{C}$ by the fact that $\mathbb{C}^\times$ is abelian.
2. Define $\varphi : P \to P$ by $\varphi(z) = z^p$. Show that φ is a surjective homomorphism. Deduce that P is isomorphic to some of its quotient group.

Now use the First Isomorphism Theorem to prove the second and third.

Theorem 0.5: Let $N \trianglelefteq G$ and $H \leq N_G(N)$, then $HN \leq G$.

Proof: $H \leq N_G(N)$ implies $hNh^{-1} = N$ for all $h \in H$. Let $h_1n_1, h_2n_2 \in HN$ where $h_1, h_2 \in H$ and $n_1, n_2 \in N$. Use the subgroup criterion, i.e. verify that $h_1n_1(h_2n_2)^{-1} \in HN$. $\square$

Theorem 0.6 (Second Isomorphism Theorem): Let $N \trianglelefteq G$ and $H \leq N_G(N)$, then $HN/N \cong H/(H \cap N)$.

Proof: Let $\varphi : H \to HN/N$ be defined by $\varphi(h) = hN$. Verify it is indeed a surjective homomorphism and use the first isomorphism theorem. $\square$

The second Isomorphism theorem is a key step for us to prove Jordan Holder theorem.

Theorem 0.7 (Third Isomorphism Theorem): Let H, K be normal subgroups of G with $K \leq H$, then

$$G/H \cong \frac{G/K}{H/K}$$

Proof: Let $\varphi : G/K \to G/H$ be defined by $\varphi(xK) = xH$. Verify that this map is a well-defined surjective homomorphism and use the first isomorphism theorem. $\square$

This theorem shows that we gain no new structural information from taking quotients of a quotient group.

The second and the third isomorphism theorem can be viewed as important exercises of the application of the first isomorphism theorem.


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大家是不是忘記更重要的事了?

Back to "solvable groups"

Recall the definition: A finite group G is **solvable** if there exists a chain of subgroups

$$1 = G_s \lhd G_{s-1} \lhd ... \lhd G_2 \lhd G_1 = G$$

with G_i/G_{i+1} cyclic for $i = 1, ..., s-1$.

Note that G_i need not to be normal in G for $i = 2, ..., s-1$.

Example: $1 \lhd \langle s \rangle \lhd \langle s, r^2 \rangle \lhd D_8$, but $\langle s \rangle$ is not normal in D_8 as we practiced earlier.

We now have to define the term **composition series**.

Subnormal series & Composition series

We just learned how to "decompose" groups by taking quotients. Can we decompose groups just like how we factorize positive integers into prime numbers?

Let me first provide more definitions for simplicity.

Definition 0.5 (Subnormal series & factor groups): A subnormal series of a group G is a chain of subgroups

$$1 = G_s \lhd G_{s-1} \lhd ... \lhd G_2 \lhd G_1 = G$$

The quotient groups G_i/G_{i+1} are called the **factor groups** of the series.

Definition 0.6 (Simple groups): A nontrivial group G is called **simple** if the only normal subgroups are only 1 and G.

Definition 0.7 (Composition series): A **composition series** of G is a subnormal series of **finite length** where all the factor groups are simple.

Example (D_8): All of the followings are composition series of D_8:

- $1 \lhd \langle s \rangle \lhd \langle s, r^2 \rangle \lhd D_8$
- $1 \lhd \langle sr \rangle \lhd \langle sr, r^2 \rangle \lhd D_8$
- $1 \lhd \langle r^2 \rangle \lhd \langle r \rangle \lhd D_8$

All the factor groups are isomorphic to C_2.

But $1 \lhd \langle r \rangle \lhd D_8$ is not a composition series because $\langle r \rangle \cong C_4$ is not simple.

Example (Q_8): All of the followings are composition series of Q_8:

- $1 \lhd \langle -1 \rangle \lhd \langle i \rangle \lhd D_8$
- $1 \lhd \langle -1 \rangle \lhd \langle j \rangle \lhd D_8$
- $1 \lhd \langle -1 \rangle \lhd \langle k \rangle \lhd D_8$

All the factor groups are isomorphic to C_2.

Exercise: Find 2 composition series of C_{12}. What are the factor groups?

Some unsolvable groups

The group S_5 is not solvable. Here is the only composition series of S_5:

$$1 \lhd A_5 \lhd S_5$$

Where A_5 is simple but not cyclic. More generally, S_n is not solvable for $n \geq 5$ because A_n is simple for $n \geq 5$.

Now we have a basic understanding of composition series. Here are some follow up questions:

1. Does composition series of a group always exist?
2. If G has a composition series, is it unique?

For the first question, yes for finite groups, and no for infinite groups.

Exercise: $\mathbb{Z}$ does not have a composition series.

Theorem 0.8 (Jordan Holder (part 1)): Any nontrivial finite group has a composition series.

Proof: Induction on $|G|$. The case $|G| = 2$ is trivial.

Now suppose that the statement is true for groups with order smaller than $|G|$. If G is simple, the proof is trivial. Otherwise, let N be a maximal proper normal subgroup of G. Note that G/N is simple. By the induction hypothesis, N has a composition series $1 \lhd N_s \lhd ... \lhd N_0 = N$. Now we have a composition series of G: $1 \lhd N_s \lhd ... \lhd N_0 = N \lhd G$. □

Uniqueness (?) of composition series of a finite group

For the second question, the answer is obviously no by the previous Q_8 and D_8 examples. However, if we slightly “loosen” the statement, then Jordan Holder theorem says yes for finite groups.

Theorem 0.9 (Jordan Holder (part 2)): Let G be a nontrivial finite group. The factor groups in a composition series of G are unique up to isomorphism and some permutation. More formally, let $1 = N_0 \lhd ... \lhd N_r = G$ and $1 = M_0 \lhd ... \lhd M_s = G$ be two composition series of G, then

- $r = s$
- There is some permutation $\sigma \in S_r$ such that $N_i / N_{i-1} \cong M_{\sigma(i)} / M_{\sigma(i)-1}$ for all $i = 1, ..., r$.

Exercise: Verify the theorem with the groups D_6, D_8, Q_8, C_{12}.

Proof of Jordan Holder Theorem

Proof outline:

- Induction on $|G|$.
  - Trivial for $|G| = 2$
  - Induction on $\min(r, s)$.
    - Trivial for $\min(r, s) = 1$.
    - Case $N_{r-1} = M_{s-1}$.
    - Case $N_{r-1} \neq M_{s-1}$.

This part will be hand-written.