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STAT150 Markov Chain Overview Topic 1: State type

Formulation: X_n is a discrete time Markov Chain with transition matrix $p(i,j)$: $p(i,j) = P(X_{n+1}=j \mid X_n=i)$

Theorem 1: $P(X_{n+m}=j \mid X_n=i)$ is $p^m(i,j)$

$\Leftrightarrow$ Chapman-Kolmogorov equation : $p^{m+n}(i,j) = \sum_k p^m(i,k)\, p^n(k,j)$

Quick Proof: $P(X_{m+n}=j, X_m=k \mid X_0=i)$ Xm can be anything in $\{X_n\}$ space, hence $\sum_k$

$$= \frac{P(X_{m+n}=j, X_m=k, X_0=i)}{P(X_m=k, X_0=i)} \cdot \frac{P(X_m=k, X_0=i)}{P(X_0=i)}$$

$$= P(X_{m+n} \mid X_m=k) \cdot P(X_m=k \mid X_0=i) = p^m(i,k)\, p^n(k,j)$$

Theorem 2: Strong markov property: if $X_T=y$, $X_0, \dots, X_T$ is irrelavant for predicting future, then X_{T+k} $(k \ge 0)$ behaves like MC with initial y.

Def: $T_y^k = \min\{n > T_y^{k-1} : X_n = y\}$: the time of k-th return to y (subscript)

Also, we let $T_y^1 = T_y$: $T_y = \min\{n \ge 1 : X_n = y\}$ be the first time to return y

☆Def: ρ_{ij} is the probability that X_n returns to j when it starts at i.

Moreover: $\rho_{yy} = P_y(T_y < \infty)$, where we define: $P_x(A) = P(A \mid X_0 = x)$

if $<\infty \Rightarrow$ return to y

starts with y — This means that MC starts with $X_0 = x$

Using theorem 2, we will know: $\rho_{yy}^k = P_y(T_y^k < \infty)$

if $\rho_{yy}^* < 1$, then $\rho_{yy}^k \to 0$ if $k \to \infty$, i.e., as MC proceeds $(k\to\infty)$, X_i probably will never come back to y. In this case, we call y transient

if $\rho_{yy} \overset{*}{=} 1$, then y is recurrent

*: find its way back

Want to know if a state is recurrent or transient? E.g.:

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .7 & .2 & .1 \\ .3 & .5 & .2 \\ .2 & .4 & .4 \end{bmatrix} \end{array}$$

for any X_n, $P(X_{n+1}=3 \mid X_n = ?) \ge 0.1$

$\therefore$ If we wish we start from 3, and come back in following n steps:

$P_3(T_3 > n) \le (0.9)^n$, if $n \to \infty$. $P_3(T_3 > \infty) = \rho_{33} \le \lim_{n\to\infty} (0.9)^n = 0$

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$\because f_{33}=0$, we will return to 3 with probability 1, 3 is recurrent

Lemma: Suppose $P_x(T_y \le k) \ge \alpha > 0$ for all x in $\boxed{\text{state space } S}$ (notation).
Then: $P_x(T_y > nk) \le (1-\alpha)^n$

Def: x communicates with y and write $x \to y$ if:
$\rho_{xy} = P_x(T_y < \infty) > 0$, i.e., x can reach y within finite steps

Lemma: If $x \to y$, $y \to z$, then $x \to z$
Theorem 3: If $\rho_{xy} > 0$, but $\rho_{yx} < 1$, then x is transient:

$\rho_{xx} = P_x(T_x < \infty)$, but $P_x(T_x = \infty)$ seems to be greater than 0 ?

Vigorous proof: let $K = \min\{k : p^k(x,y) > 0\}$, then there is a path $\langle x, y_1 \cdots y \rangle$.
$p(x,y_1)\,p(y_1,y_2)\cdots p(y_{K-1},y) > 0$ obviously $y_i \ne x$ (else there's shorter path)
$\therefore P_x(T_x=\infty) \ge \underbrace{\underbrace{p(x,y_1)\,p(y_1,y_2)\cdots p(y_{K-1},y)}_{x \text{ reach out to } y}\,\underbrace{(1-\rho_{yx})}_{y \text{ never goes back to } x}}_{\text{At least one scenario for } x \text{ to never return to its self}} > 0$

Lemma: If x is recurrent and $\rho_{xy} > 0$, then $\rho_{yx} = 1$
Def: Set A is $\boxed{\text{closed}}$ if it is impossible to get out, i.e., if $i \in A$, $j \notin A$, then $p(i,j) = 0$. Set B is $\boxed{\text{irreducible}}$ if $\forall i,j \in B$, $i \to j$

Theorem 4: If set C is a finite closed and irreducible set, then all states in C are recurrent
Theorem 5: If state space S is finite, then S can be written as a disjoint union $T \cup R_1 \cup \cdots \cup R_k$, where T is a set of transient states and the R_i, $1 \le i \le k$, are closed irreducible sets of recurrent states.

Lemma: If x is recurrent and $x \to y$, then y is recurrent
Lemma: In a finite closed set there has to be at least one recurrent state (Intuitively: yes!)

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Let $N(y)$ be the number of visits to y at times $n \geq 1$

Lemma: $E_x N(y) = \frac{\rho_{xy}}{1-\rho_{yy}}$

Proof: $EX = \sum_{k=1}^{\infty} P(X \geq k)$ for X, a integer valued r.v.

$\therefore E_x N(y) = \sum_{k=1}^{\infty} P(N(y) \geq k) \overset{*}{=} \rho_{xy} \sum_{k=1}^{\infty} \rho_{yy}^{k-1} = \frac{\rho_{xy}}{1-\rho_{yy}}$

$*$: $\{N(y) \geq k\} \Leftrightarrow$ k-th return to y occurs: $\{T_y^k < \infty\}$

Lemma $E_x N(y) = \sum_{n=1}^{\infty} p^n(x,y)$.

Proof: $p^i(x,y)$, probability of $X_i = y$ given $x = X_0$.

Since $N(y) = \sum_{n=1}^{\infty} 1\{X_n = y\}$, $E_x N(y) = \sum_{n=1}^{\infty} P(1\{X_n = y\}) = \sum_{n=1}^{\infty} p^n(x,y)$

Theorem 6: y is recurrent if and only if:

$$\sum_{n=1}^{\infty} p^n(y,y) = E_y N(y) = \infty$$

Expected number of visit to y is ∞!

Recap: Generating Functions (PGF, P: Probability)

Def: PGF of X with discrete or continuous space S is:

$$G_X(\xi) = \mathbb{E}(\xi^X) = \sum P(X=i)\xi^i, \quad \xi \in [0,1]$$

$\therefore$ We have: $P_0 = G_X(0)$. $\quad P_n = \frac{1}{n!} \left. \frac{d^n G_X(\xi)}{d\xi^n} \right|_{\xi=1}$

$E(X) = \left. \frac{dG_X(\xi)}{d\xi} \right|_{\xi=1}$ $\quad$ Def $X[n] = X(X-1)\cdots(X-n+1)$

$\therefore Var(X) = E(X^2) - [E(X)]^2 = G_X''(1^-) + G_X'(1^-) - [G_X'(1^-)]^2$

If $X = X_1 + X_2 + \cdots$, then $G_X(\xi) = G_{X_1}(\xi) G_{X_2}(\xi) \cdots$

Topic 2:

Branching Process: Consider a population in which individual in the n-th generation give birth to i.i.d number of children. Def X_n is number of individuals at time n.

Clearly, 0 is absorbing state. let P_k be the prob that a person give offspring

Then $X_{n-1} \cdot \sum_{k=0}^{\infty} k P_k = E(X_n | X_{n-1})$. let $\mu = \sum_{k=0}^{\infty} k P_k$

$\therefore EX_n = \mu^n EX_0$ $\quad$ if $\mu < 1$, will extinct, definitely

what if $\mu \geq 1$??

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We're curious about what's the prob^r of extinction if $\mu \geq 1$

Let ρ be the prob the process die out. $X_0 = 1$. assume $X_1 = k$, if die out, each family line of each child die out.

$\therefore \rho = \sum_{k=0}^{\infty} \underbrace{p_k}_{\text{the prob of } k \text{ kids}} \boxed{\rho^k}$, let $\phi(\theta) = \sum_{k=0}^{\infty} p_k \theta^k$, then $\rho = \phi(\rho)$

$\boxed{\rho^k}$: k family line die out !

This has a trivial root $\rho = 1$. The root we want is:

Lemma: The extinction prob ρ is the smallest solution of $\phi(x) = x$ with $0 \leq x \leq 1$

If never die out, $\rho = 0$ has to be the root $\Rightarrow \phi(0) = p_0 = 0$!!

Intuitively reasonable !! No prob to give 0 offspring, how can they die out?

Proof: $X_n = 0 \Rightarrow X_{n+1} = 0$ $\therefore \{X_n = 0\} \subseteq \{X_{n+1} = 0\}$

$\therefore P(X_n = 0) \leq P(X_{n+1} = 0 \cdots) \Rightarrow u_n \leq u_{n+1}$

$\therefore u_n$ is increasing and has a bound $\Rightarrow \lim_{n\to\infty} u_n$ exists

$\therefore u = \lim_{n\to\infty} u_n = \lim_{n\to\infty} \phi(u_n) = \phi(\lim_{n\to\infty} u_n) = \phi(u)$

$\therefore$ The limit, i.e., the extinction prob, is exactly the fix point of ϕ

Corollary: If $p_1 < 1$, the $\lim_{t\to\infty} X_t = \begin{cases} \infty, & p = 1-u \\ 0, & p = u \end{cases}$

Proof: If $P_0 = 0$, then for $k \in \mathbb{Z}^+$, $\rho_{kk} = P_k(T_k < \infty) = p_1^k + p_1^{2k} + \cdots$

$= \frac{p_1^k (1 - p^{kn})}{1 - p^k} = 1 - \frac{1}{1 - p^k} < 1$, so $X_t = k$ are transient states

If $p_0 > 0$, then $\rho_{kk} \leq 1 - (p_0)^k < 1$ so $X_t = k$ are transient states

p_0^k: for k children they all give 0 kids; $1 - p_0^k$, don't extinct, probably may return to k.

$\therefore$ Except for 0 & ∞, all other $k \in \mathbb{Z}^+$ are transient, hence:

$P(\lim_{t\to\infty} X_t = k) = 0 \Rightarrow \lim_{t\to\infty} X_t = \begin{cases} \infty, & p = 1-u \\ 0, & p = u \end{cases}$

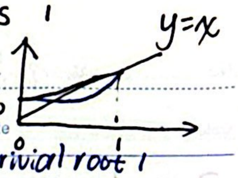
⇒ Corollary: Suppose $p_1<1$, $E(\xi)=\mu\le 1$, $X_0=1$. then prob of $\lim_{n\to\infty} X_n=0$ is 1

Pf. $\frac{d^2\phi(s)}{ds^2} \ge 2p_2 \ge 0$, $0 \le \frac{d\phi(s)}{ds}\Big|_{s=x} \le \frac{d\phi(s)}{ds}\Big|_{s=1} = \mu \le 1$, $p_0=\phi(s)|_{s=0}>0$

$\phi(1)=1$; if intersect, then apparently $\exists s'$ $\frac{d\phi(s')}{ds}\Big|_{s=s'}>1$, contradict

⇒ no root $\in [0,1)$, only trivial root 1

$P\{\lim_{n\to\infty} X_n=0\}=1$



Topic 3 First step Analysis

This methods proceeds by analyzing, or breaking down, the probabilities that can arise at the end of the first transition, and then invoking the law of total probability coupled with Markov Property to establish a relationship among unkown variables.

E.g. Consider MC $\{X_n\}$ whose transition prob matrix is:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ \alpha & \beta & \gamma \\ 0 & 0 & 1 \end{bmatrix} \end{array}, \quad \alpha+\beta+\gamma=1, \text{ then what is } Pr\{X_T=0 \mid X_0=1\}?$$

We can use first step: $\underbrace{Pr\{X_T=0 \mid X_0=1\}}_{u=} = P(X_1=0 \mid X_0=1)\, Pr\{X_T=0 \mid X_1=0\} + \cdots$

$= \alpha + \beta \cdot Pr\{X_T=0 \mid X_0=1\} = \alpha+\beta u \quad \therefore u = \frac{\alpha}{1-\beta}$

Topic 4: Stationary Distribution, Periodicity

Suppose $P=\|P_{ij}\|$ is on a finite number of states, and when raised to some power k, P^k has all of its elements strictly positive.* Then such P and its corresponding MC is called regular.

☆ Fact: Regular MC has a limiting probability distribution $\pi=(\pi_0,\cdots,\pi_N)$ where $\pi_j>0$ for $\forall j\in[0,N]$ and $\sum_j \pi_j=1$

Formula: $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j > 0$; or $\lim_{n\to\infty} Pr\{X_n=j \mid X_0=i\} = \pi_j > 0$

We can tell that: regardless of what initial distribution is!

Theorem: Let P be a regular transition prob matrix, then π is the unique nonnegative solution of equations:

$$\pi_j = \sum_{k=0}^{N} \pi_k P_{kj}, \quad j=0,1,\cdots,N, \quad \sum_{k=0}^{N}\pi_k = 1$$

Pf. $\pi_j = \lim_{n\to\infty} P_{ij}^{(n)} = \lim_{n\to\infty} \sum_{k=0}^{N} P_{ik}^{(n-1)} P_{kj} = \sum_{k=0}^{N} \pi_k P_{kj}$

* $\forall i,j\in S$, $\exists N$, $\forall n\ge N$, $P_{ij}^{(n)}>0$ / $\exists N$, $\forall n\ge N$, $P_{ij}^{(n)}>0$ for $\forall i,j\in S$

*: 记写作 I_i

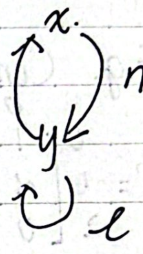
Periodicity: Def: $I_i^{*} = \{n \geq 1,\ P_{ii}^{(n)} > 0\}$ (for n steps in total, there's one way back)

$\Rightarrow$ Period of i: gcd of I_i, $\gcd(I_i)$

Lemma: If $\rho_{xy} > 0$ and $\rho_{yx} > 0$ for $\forall x, y \in S$, than x.y has same period

Proof: Assume: k_1, k_2 be the period of x & y

$\rho_{xy}, \rho_{yx} > 0 \Rightarrow \exists n\ P_{xy}^{(n)} > 0,\ \exists m\ P_{xy}^{(m)} > 0$



$\therefore P_{xy}^{(m+n)} > 0$, $m+n \in I_x$, $k_1 \mid m+n$

Let $l \in I_y$, $P_{yy}^{l} > 0$, then:

$P_{xx}^{(m+n+l)} > P_{xy}^{(n)} \cdot P_{yy}^{(l)} \cdot P_{yx}^{(m)} > 0$, $k_1 \mid m+n+l \Rightarrow k_1 \mid l$

And: $l \in I_y$ $\therefore k_2 \mid l$, $k_1 = cd(I_y)$, $k_2 = \gcd(I_y)$

$\therefore k_1 \leq k_2$, similarly, $k_1 \geq k_2 \Rightarrow k_1 = k_2$

Corollary: If p transition matrix of irreducible MC, than the period of all states in MC is the same. We call it periodicity of MC

Def: An irreducible MC is called aperiodic if its periodicity is 1.

Theorem 6: A finite MC is aperiodic and irreducible iff it is regular

Proof: ($\Rightarrow$) $\forall i, \exists N_i$ s.t. $\forall n \geq N_i$, $P_{ii}^{n} > 0$ since a periodic*

*: Frobenius Coin Problem: $\gcd(a,b) = 1$, for $\forall M > ab$, $M = ax + by$, $x.y \in \mathbb{Z}^+$

Also Irreducible $\Rightarrow \forall i, j \in S, \exists m_{ij}$, s.t. $P_{ij}^{(m)} > 0$

$\therefore P_{ij}^{(m+n)} > P_{ii}^{(m)} \cdot P_{ij}^{(n)} > 0$ for $\forall i, j \in S$

$\therefore$ Let $M = \max_{i,j}\{N_i + m_{ij}\}$, $\forall n \geq M$, $\forall ij$, $P_{ij}^{(n)} > 0$

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($\Leftarrow$) P is regular, $\exists N$, $\forall n \geq N$, $P_{ij}^{(n)} > 0$ for $\forall i, j$

$\Rightarrow$ Clearly, MC is irreducible

$\forall i \in S$, $\forall n \geq N$, $n \in T_i$, for instance,

$N \in T_i$, $N+1 \in T_i$. $\Rightarrow$ period $= p \mid N$, $p \mid N+1 \Rightarrow p = 1$

$\Rightarrow$ Aperiodic (Using Corollary).

☆ Regularity $\Leftrightarrow$ aperiodicity & irreducible

Topic 5. Doubly Stochastic Matrices.

Def: A transition probability matrix is called doubly stochastic if: $P = \|P_{ij}\|$ satisfy that:

$$P_{ij} \geq 0, \text{ and } \sum_k P_{ik} = \sum_k P_{kj} = 1 \quad \text{for } \forall i, j$$

Lemma: $S = \{0, 1, \cdots, N-1\}$, if a matrix is doubly stochastic and regular, then $\pi = (1/N, \cdots, 1/N)$ uniform distribution.

Proof: $\pi_j = \sum_k \pi_k P_{kj} = \sum_k P_{kj} = 1 \Rightarrow \pi_k = 1/N$, and ~~this~~ is unique

(Regular matrix has unique solution of equations $\Rightarrow \pi$ (uniqueness)).

π It's called limiting distribution. Interpretation:

Let $R_j(n)$ be the time spent in state j up to time n. (visits)

$$R_j(n) = \sum_{m=0}^{n} \mathbb{1}\{X_m = j\}$$

$$\mathbb{E}\left(\lim_{n\to\infty} \frac{R_j(n)}{n+1}\right) = \mathbb{E}\left(\lim_{n\to\infty} \frac{\sum_{m=0}^{n} \mathbb{1}\{X_m = j\}}{n+1}\right) = \lim_{n\to\infty} \frac{\pi_j \cdot n}{n+1} = \pi_j$$

Topic 6. Classification of State $\Leftrightarrow$ Distribution Existence

Previously, we know the definition of Current state. Now:

Theorem 7: A state i is recurrent iff:

$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$; Equivalently: transient iff $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$

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Recap: Def of Stationary Distribution: $\begin{cases} \sum_j \pi_j = 1 \\ \pi = \pi P \text{, i.e.,} \end{cases}$

$\pi_j \geq 0, \sum_{i=0}^{N/\infty} \pi_i = 1, \quad \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$ for $j = 0, 1, \cdots$

Def of limiting Distribution: $\lim_{n\to\infty} P_{ij}^{(n)} = \pi_j > 0$, or:

$\lim_{n\to\infty} Pr\{X_n = j \mid X_0 = i\} = \pi_j > 0 \quad \Rightarrow \pi = (\pi_1, \pi_2, \cdots)$

If a Stationary Distribution exists, MC doesn't guarantee to converge to that. If MC converge to that, Stationary is equivalent to Limiting.

☆ Conclusion:

① MC is finite, irreducible + aperiodic $\Leftrightarrow$ regular

ergodic thm $\Updownarrow$ limiting

Every state: recurrent $\Leftrightarrow$ Has unique stationary distribution

② MC is infinite, positive recurrent

③ Pipeline: irreducible: $\frac{* \text{ tran}}{\text{rec}}$?

- transient: No stat & lim dis
- recurrent
  - Null: No stat & lim dis
  - Positive: has unique stat dis
    - aperiodic: has limit & lim dis = unique stat dis
    - periodic: no lim dis

* irreducible ⇒ all transient, or all recurrent