Discrete Mathematics

Stirling number of the second kind, partitions of integers, recurrence relations

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Distribution Problems

The Problem Statement: distributing n objects into k boxes

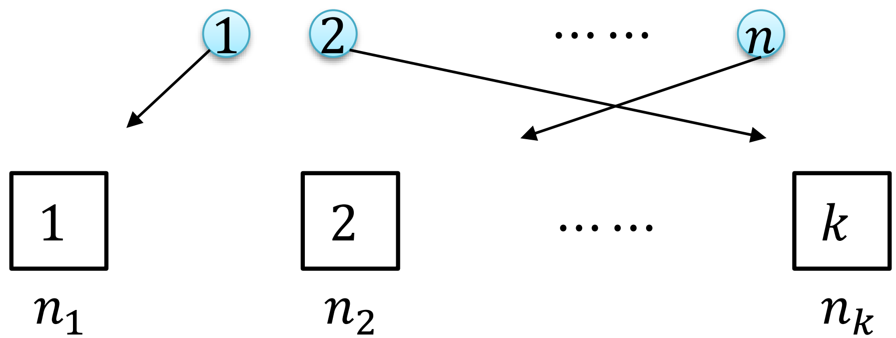
- Objects may be distinguishable (**labeled** with numbers $1,2,\ldots,n$) or indistinguishable (**unlabeled**)
- Boxes may be distinguishable (**labeled** with numbers $1,2,\ldots,k$) or indistinguishable (**unlabeled**)
- ? What is the # of distributing n objects into k?

Problem Type	Objects	Boxes
1	labeled	labeled
2	unlabeled	labeled
3	labeled	unlabeled
4	unlabeled	unlabeled

Problem Classification

Type 1

Problem: distributing n labeled objects into k labled boxes



Classifications

$$n_1 + n_2 + \cdots + n_k = n$$
$$n_1, n_2, \ldots, n_k \in \mathbb{N}$$

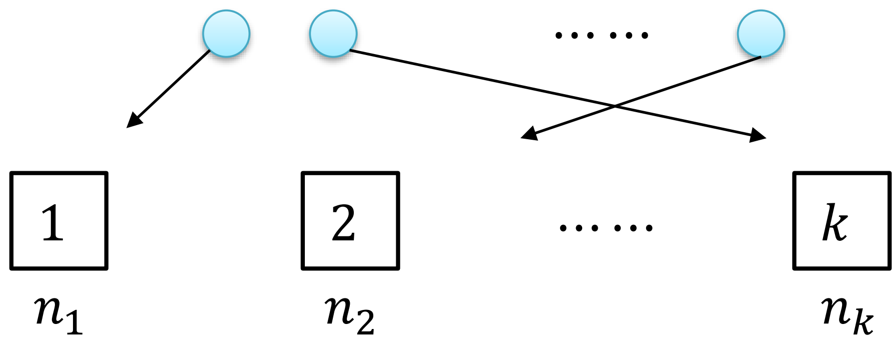
THEOREM: The number of ways of distributing n labeled objects into k labeled boxes such that n_i objects are placed into box i for every $i \in [k]$ is $N_1 = n!/(n_1!\, n_2! \cdots n_k!)$.

- S: the set of the expected distributing schemes
- $|S| = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}$

REMARK: $N_1 =$ # of permutations of $\{n_1 \cdot 1, \ldots, n_k \cdot k\}$.

Type 2

Problem: distributing n unlabeled objects into k labled boxes



Classifications

$$n_1 + n_2 + \cdots + n_k = n$$
$$n_1, n_2, \dots, n_k \in \mathbb{N}$$

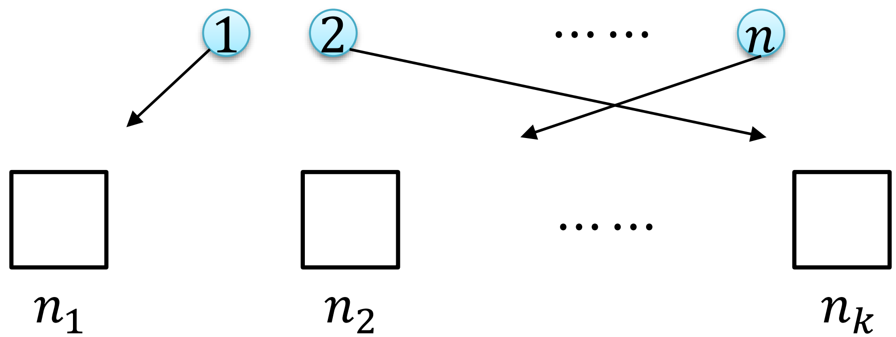
THEOREM: The number of ways of distributing n unlabeled objects into k labeled boxes is $N_2 = \binom{n+k-1}{n}$.

- S: the set of the expected distributing schemes
- $T = \{(n_1, n_2, \dots, n_k): n_1 + n_2 + \cdots + n_k = n; n_1, n_2, \dots, n_k \in \mathbb{N}\}$
- $f: T \to S \quad (n_1, n_2, \dots, n_k) \mapsto$ a scheme where n_i objects are put into box i
    - f is a bijection. Hence, $|S| = |T| = \binom{n+k-1}{n}$

REMARK: $N_2 = \#$ of n-combinations of $\{\infty \cdot 1, \dots, \infty \cdot k\}$

Type 3

Problem: distributing n labeled objects into k unlabled boxes



Classifications

$$n_1 + n_2 + \cdots + n_k = n$$
$$n_1, n_2, \ldots, n_k \in \mathbb{N}$$
$$n_1 \geq n_2 \geq \cdots \geq n_k$$

EXAMPLE: Assigning 4 employees {a, b, c, d} into 3 unlabeled offices. Each office can contain any number of employees.

- 4 0 0: [abcd − −]
- 3 1 0: [abc d −] [abd c −] [acd b −] [bcd a −]
- 2 2 0: [ab cd −] [ac bd −] [ad bc −]
- 2 1 1: [ab c d][ac b d] [ad b c] [bc a d] [bd a c] [cd a b]

REMARK: The schemes can be classified with $\{n_1, \ldots, n_k\}$

$S_2(n, j)$

1730, Stirling (1692-1770)

DEFINITION: $S_2(n, j)$, the **Stirling number of the second kind**, is defined as the number of different ways of distributing n labeled objects into j unlabeled boxes so that no box is empty.

THEOREM: The number of schemes of distributing n labeled objects into k unlabeled boxes is

$$\sum_{j=1}^{k} S_2(n, j)$$

- $S_2(n, j)$: the number of schemes that use exactly j boxes, $j = 1,2, \dots, k$

$S_2(n,j)$

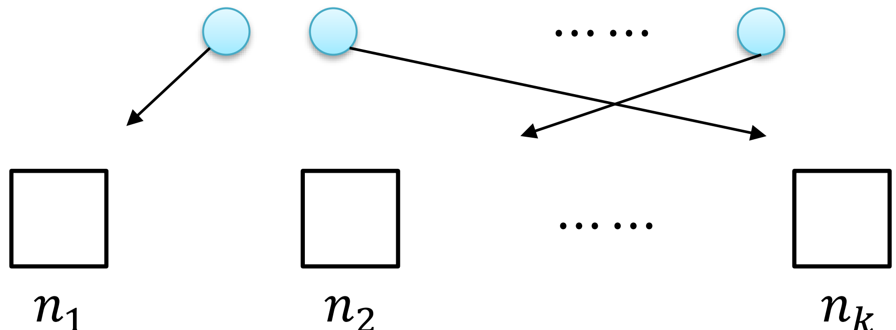
1730, Stirling (1692-1770)

THEOREM: $S_2(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n$ when $n \geq j \geq 1$.

- $T(n,j)$: the number of ways of distributing n labeled objects into j labeled boxes such that no box is empty
  - $T(n,j) = j! \cdot S_2(n,j)$
    - $T(n,j) =?$
- $X:$ the set of ways of distributing n labeled objects into j labeled boxes.
  - By the product rule, $|X| = j^n$
- $X_i \subseteq X$: the set of ways where exactly i boxes are used, $i = 1,2,\dots,j$
  - $\{X_1, X_2, \dots, X_j\}$ is a partition of X and $|X_i| = \binom{j}{i} T(n,i)$
  - $j^n = |X| = \sum_{i=1}^{j}|X_i| = \sum_{i=1}^{j}\binom{j}{i} T(n,i)$
  - $T(n,j) = \sum_{i=1}^{j}(-1)^{j-i}\binom{j}{i} i^n = \sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n$ //inversion
- $S_2(n,j) = \frac{1}{j!} \cdot T(n,j) = \frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i \binom{j}{i}(j-i)^n$

Type 4

Problem: distributing n unlabeled objects into k unlabled boxes



Classifications

$$n_1 + n_2 + \cdots + n_k = n$$
$$n_1, n_2, \ldots, n_k \in \mathbb{N}$$
$$n_1 \geq n_2 \geq \cdots \geq n_k$$

EXAMPLE: # of ways of distributing 4 identical books into 3 identical boxes.

- 4 0 0
- 3 1 0
- 2 2 0
- 2 1 1

REMARK: The schemes are determined by $\{n_1, \ldots, n_k\}$

Partitions of Integers

DEFINITION: $n = a_1 + a_2 + \cdots + a_j$ is called an ***n*-partition** with exactly j parts if $a_1 \geq a_2 \geq \cdots \geq a_j$ are all positive integers.

- $p_j(n) = \left|\left\{\left(a_1, \ldots, a_j\right): a_1 + \cdots + a_j = n, a_1 \geq a_2 \geq \cdots \geq a_j \geq 1 \text{ are integers}\right\}\right|$
  - $p_j(n)$: # of ways of writing n as the sum of j positive integers.
  - $p_1(n) = 1$, $p_n(n) = 1$

EXAMPLE: The integer 4 has four different partitions:

- $4 = 4$
- $4 = 3 + 1$
- $4 = 2 + 2$
- $4 = 2 + 1 + 1$

REMARK: solution to the type 4 problem=$\sum_{j=1}^{k} p_j(n)$

Partitions of Integers

THEOREM: For $n \in \mathbb{Z}^+, j \in [n]$, $p_j(n+j) = \sum_{k=1}^{j} p_k(n)$

- Let $S_k = \{\text{partitions of } n \text{ into } k \text{ positive integers}\}, k \in [j]$
- Let $S = \bigcup_{k=1}^{j} S_k$.
  - $|S| = |S_1| + \cdots + |S_j| = p_1(n) + \cdots + p_j(n)$
- Let $T = \{\text{partitions of } n + j \text{ into } j \text{ positive integers}\}$
  - $|T| = p_j(n+j)$
- $f: S \to T \quad (n_1, \dots, n_k) \mapsto (n_1 + 1, \dots, n_k + 1, \underbrace{1, \dots, 1}_{j-k})$
  - f is bijective
  - $|T| = |S|$

EXAMPLE: determine $p_3(6)$ and $p_4(6)$ with the above theorem

- $p_3(6) = p_3(3+3) = p_1(3) + p_2(3) + p_3(3) = 1 + 1 + 1 = 3$
- $p_4(6) = p_4(2+4) = p_1(2) + p_2(2) + p_3(2) + p_4(2) = 1 + 1 + 0 + 0 = 2$

Computing $p_j(n)$ Recursively

$$p_1(1)$$

$$p_1(2) \quad p_2(2)$$

$$p_1(3) \quad p_2(3) \quad p_3(3)$$

$$p_1(4) \quad p_2(4) \quad p_3(4) \quad p_4(4)$$

$$p_1(5) \quad p_2(5) \quad p_3(5) \quad p_4(5) \quad p_5(5)$$

$$p_1(6) \quad p_2(6) \quad p_3(6) \quad p_4(6) \quad p_5(6) \quad p_6(6)$$

$$p_1(7) \quad p_2(7) \quad p_3(7) \quad p_4(7) \quad p_5(7) \quad p_6(7) \quad p_7(7)$$

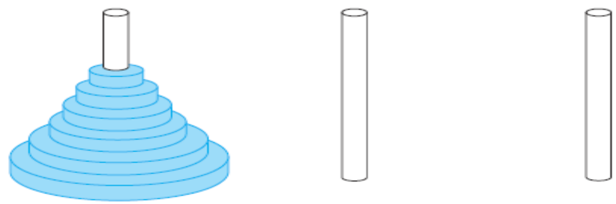
$$p_1(8) \quad p_2(8) \quad p_3(8) \quad p_4(8) \quad p_5(8) \quad p_6(8) \quad p_7(8) \quad p_8(8)$$

$$p_1(9) \quad p_2(9) \quad p_3(9) \quad p_4(9) \quad p_5(9) \quad p_6(9) \quad p_7(9) \quad p_8(9) \quad p_9(9)$$

Recurrence Relation (RR)

Fibonacci Sequence: The solution is a sequence $\{f_n\}_{n\geq 0}$ such that $f_0 = 1, f_1 = 1,\ \ f_n = f_{n-1} + f_{n-2}$ for every $n \geq 2$

The Tower of Hanoi:



- Every time move only 1 disk from one peg to another peg
- Always place a smaller disk on top of a larger disk
- Move all the disks from peg 1 to peg 2.
  - H_n: the smallest number of moves (n disks).
    - $H_1 = 1, H_2 = 3, H_n = 2H_{n-1} + 1$ for $n \geq 2$

QUESTION: $f_n =?\ \ H_n =?$ Find explicit formulas.