Linear Algebra I

Chapter 5. Inner Product Spaces

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WELCOME!

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Chapter 5. Inner Product Spaces

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A Quick Review

- Fact: Every real n-dimensional vector space V is isomorphic to $\mathbb{R}^n$.

Observation: Vectors in $\mathbb{R}^n$ have norm, distance, angle, orthogonality, etc.
Question: What about vectors in V?

- Dot product in $\mathbb{R}^n$: $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$.
- Norm in $\mathbb{R}^n$: $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$.
- Distance in $\mathbb{R}^n$: $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.
- Angle in $\mathbb{R}^n$: $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$.

General Definition of Inner Product

Definition. Let V be an $\mathbb{F}$-vector space. An inner product on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ such that the following axioms are satisfied.

(i) $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$ $\quad (\forall \mathbf{u}, \mathbf{v} \in V)$.
(ii) $\langle k\mathbf{u} + \ell \mathbf{v}, \mathbf{w} \rangle = k\langle \mathbf{u}, \mathbf{w} \rangle + \ell \langle \mathbf{v}, \mathbf{w} \rangle$ $\quad (\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V, \forall k, l \in \mathbb{R})$.
(iii) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ $\quad (\forall \mathbf{v} \in V)$. And $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$.

In this case, we call $(V, \langle \cdot, \cdot \rangle)$ a inner product space.

Remark: When $\mathbb{F} = \mathbb{R}$, we may ignore the "conjugate".
Remark: When $\mathbb{F} = \mathbb{R}$, we also call it real inner product space;
When $\mathbb{F} = \mathbb{C}$, we also call it unitary space.

Remark: One always has $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{0} \rangle = 0$.
Remark: By (i) and (ii), we have

$$\langle \mathbf{w}, k\mathbf{u} + l\mathbf{v} \rangle = \overline{k}\langle \mathbf{w}, \mathbf{u} \rangle + \overline{l}\langle \mathbf{w}, \mathbf{v} \rangle.$$

Examples

Example. Let $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{R}^n$. For $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$, define

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + \ldots + u_n v_n.$$

Then $(V, \langle \cdot, \cdot \rangle)$ forms a real inner product space.

Example. Let $\mathbb{F} = \mathbb{C}$ and $V = \mathbb{C}^n$. For $\mathbf{z} = (z_1, \ldots, z_n)$ and $\mathbf{v} = (w_1, \ldots, w_n)$, define

$$\langle \mathbf{z}, \mathbf{w} \rangle = z_1 \overline{w_1} + \ldots + z_n \overline{w_n}.$$

Then $(V, \langle \cdot, \cdot \rangle)$ forms a real inner product space.

Examples

Example. Let $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2)$. Prove that $\langle \cdot, \cdot \rangle$ given below is a real inner products on $\mathbb{R}^2$: $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_1 v_2 + u_2 v_1 + 2 u_2 v_2$.

Proof:

①

Example. Let $A \in M_n(\mathbb{R})$ be invertible. Suppose that $\langle \cdot, \cdot \rangle$ is a real inner product on $\mathbb{R}^n$. Prove that $\langle \cdot, \cdot \rangle_A$ defined by (view $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ as column matrices)

$$\langle \mathbf{u}, \mathbf{v} \rangle_A = \langle A\mathbf{u}, A\mathbf{v} \rangle, \quad (\mathbf{u}, \mathbf{v} \in \mathbb{R}^n)$$

is also a real inner product on $\mathbb{R}^n$.

Proof:

②

Norm and Distance

Definition. If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, then the norm of a vector $\mathbf{v} \in V$ is

$$\|\mathbf{v}\| = \sqrt{\langle v, v \rangle}.$$

The distance between two vectors is

$$d(u, v) = \|u - v\|.$$

Vectors with norm 1 are called unit vectors.

Example. In $\mathbb{R}^2$, let $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (0, 1)$. Evaluate $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and $d(\mathbf{u}, \mathbf{v})$ with the following two inner products, respectively.
(a) $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_2 v_2$;
(b) $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + u_1 v_2 + u_2 v_1 + 2 u_2 v_2$.

Solution:

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Basic Properties

Theorem. If $\mathbf{u}, \mathbf{v}$ are vectors in an inner product space, and if k is a scalar, then

- ◇ $\|\mathbf{v}\| \geq 0$ with equality if and only if $\mathbf{v} = 0$.
- ◇ $\|k\mathbf{v}\| = |k| \cdot \|\mathbf{v}\|$.
- ◇ $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$.
- ◇ $d(\mathbf{u}, \mathbf{v}) \geq 0$ with equality if and only if $\mathbf{v} = \mathbf{u}$.

Parallelogram Equation & Polarization identity

Theorem. (Parallelogram Equation) If $\mathbf{u}, \mathbf{v}$ are vectors in an inner product space, then

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right)$$

Theorem. (Polarization identity) Let $\mathbf{u}, \mathbf{v}$ are vectors in an inner product space. When $\mathbb{F} = \mathbb{R}$, we have

$$\langle \mathbf{u}, \mathbf{v} \rangle = \frac{1}{4} \sum_{k=0}^{1} \left\|\mathbf{u} + (-1)^k \mathbf{v}\right\|^2 = \frac{1}{4}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2\right).$$

When $\mathbb{F} = \mathbb{C}$, it satisfies that

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v} \rangle &= \frac{1}{4} \sum_{k=0}^{3} \|\mathbf{u} + \mathrm{i}^k \mathbf{v}\|^2 \\ &= \frac{1}{4}\left(\|\mathbf{u}+\mathbf{v}\|^2 + \mathrm{i}\|\mathbf{u}+\mathrm{i}\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 - \mathrm{i}\|\mathbf{u}-\mathrm{i}\mathbf{v}\|^2\right)\end{aligned}$$

Examples

Example. Show that the following $\langle\cdot,\cdot\rangle$ is an inner products on V, respectively. And compute the inner products, norms and the distances.

(1) $\mathbb{F}=\mathbb{R}$, $V=C[-\pi,\pi]$, $\langle f,g\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)g(t)dt$.

Let $f(t)=t$, $g(t)=t^2$. Evaluate $\|f\|$, $\|g\|$ and $\langle f,g\rangle$.

(2) $\mathbb{F}=\mathbb{R}$, $V=M_n(\mathbb{R})$, $\langle A,B\rangle=\mathrm{tr}(B^TA)$.

Let $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $B=\begin{bmatrix}-1 & 0\\ 3 & 2\end{bmatrix}$. Evaluate $\langle A,B\rangle$, $\|A\|$ and $d(A,B)$.

Solution:

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Problem*: (1)' $\mathbb{F}=\mathbb{C}$, $V=C_{\mathbb{C}}[-\pi,\pi]$ (continuous complex-valued functions),

$$\langle f,g\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\overline{g(t)}dt.$$

Problem*: (2)' $\mathbb{F}=\mathbb{C}$, $V=M_n(\mathbb{C})$, $\langle A,B\rangle=\mathrm{tr}(B^*A)$.

Examples

Example. Show that the following $\langle\cdot,\cdot\rangle$ is an inner products on P_n (with real coefficinets), respectively. And compute the inner products, norms and the distances.

(3) $\mathbb{F}=\mathbb{R}$, $V=P_n$, $\left\langle \sum_{k=0}^n a_k x^k, \sum_{k=0}^n b_k x^k \right\rangle = \sum_{k=0}^n a_k b_k$.

(4) $\mathbb{F}=\mathbb{R}$, $V=P_n$, given distinct $x_0, x_1, \ldots, x_n \in \mathbb{R}$,

$$\langle p, q\rangle = \sum_{j=0}^n p(x_j)q(x_j) \text{ (called evaluation inner product).}$$

In (3) and (4), let $n=2$, $x_0=-2$, $x_1=0$, $x_2=2$. Take $p(x)=x^2$ and $q(x)=1+x$. Evaluate $\langle p,q\rangle$ and $\|p\|$.

Solution:

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Problem*: (3)' $\mathbb{F}=\mathbb{R}$, $V=P_n$ (with complex coefficients), $\left\langle \sum_{k=0}^n a_k x^k, \sum_{k=0}^n b_k x^k \right\rangle = \sum_{k=0}^n a_k \overline{b_k}$.

Examples

Example.
Show that the following $\langle \cdot, \cdot \rangle$ is an inner products on V.

(5) $V = \mathbb{R}^\infty = \{(x_1, x_2, \ldots, x_n, \ldots) : \sum_{i=1}^\infty x_i^2 < \infty\}$,
$\langle (x_i), (y_i) \rangle = \sum_{i=1}^\infty x_i y_i$.

Proof:

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Self-adjoint Hermitian Matrix

Definition. Let $\mathbf{u}, \mathbf{v} \in M_{n\times 1}(\mathbb{C})$ be column vectors. Then the (complex) inner product of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^* \mathbf{u}.$$

Example. Suppose that $A \in M_{m\times n}(\mathbb{C})$, $u \in M_{n\times 1}(\mathbb{C})$ and $v \in M_{m\times 1}(\mathbb{C})$. Then $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^*\mathbf{v}$.

Definition. A matrix $A \in M_n(\mathbb{C})$ is said to be self-adjoint/Hermitian if $A^* = A$.

Example. The eigenvalues of an Hermitian matrix are real numbers.
Proof:

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Norm and Distance

Now we always assume $(V, \langle\cdot,\cdot\rangle)$ is a real inner product space.

Theorem. For $\mathbf{u},\mathbf{v} \in V$, we have

$$|\langle \mathbf{u},\mathbf{v}\rangle| \leq \|\mathbf{u}\|\,\|\mathbf{v}\|.$$

Proof:

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Remark: $-1 \leq \frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\|\mathbf{v}\|} \leq 1$. So it is reasonable to define the angle θ between vectors u and v as

$$\theta = \arccos\left(\frac{\langle \mathbf{u},\mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right).$$

Corollary. For $\mathbf{u},\mathbf{v} \in V$, the following inequalities hold.

◇ $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$. (Triangular inequality)

◇ $d(\mathbf{u},\mathbf{v}) \leq d(\mathbf{u},\mathbf{w}) + d(\mathbf{w},\mathbf{v})$. (Triangular inequality)

Orthogonality

- $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

Definition. We say that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Example. Prove that the following vectors are pairwise orthogonal in given inner product spaces.

(1) $V = M_2$, $\langle A, B \rangle = \mathrm{tr}(B^T A)$.

Take $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

Proof:

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Example. Prove that the following vectors are pairwise orthogonal in given inner product spaces.

(2) $\mathbb{F}=\mathbb{R}$, $V=C[-\pi,\pi]$, $\langle f,g\rangle=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$.

Take $f_m(t)=\sin mt$, $g_n(t)=\cos nt$ and $h(t)=1/\sqrt{2}$.

(3) $\mathbb{F}=\mathbb{C}$, $V=C_{\mathbb{C}}[-\pi,\pi]$ $\langle f,g\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$.
(the space of all continuous complex-valued functions)

Take $f_n(t)=e^{int}$ $(n\in\mathbb{Z})$.

Proof:

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Gou-Gu Theorem

Theorem. (Gou-Gu Theorem) Suppose $\mathbf{u}$ and $\mathbf{v}$ are orthogonal vectors in an inner product space. Then

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

Example. Let $\mathbb{F}=\mathbb{R}$, $V=P_2$ and $\langle p,q\rangle = \int_{-1}^{1} p(x)q(x)dx$. Verify Gou-Gu Theorem with $p(x)=x$ and $q(x)=x^2$.

Proof:

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Orthogonal Complements

Definition. If W is a subspace of an inner product space V, then the orthogonal complement of W is defined by

$$W^\perp = \{\mathbf{v} : \langle \mathbf{v}, \mathbf{w} \rangle = 0 \, (\forall \mathbf{w} \in W)\}.$$

Theorem. Let W be a subspace of an inner product space V.
(1) $W^\perp$ is a subspace of V.
(2) $W^\perp \cap W = \{\mathbf{0}\}$.
(3) If V is finite-dimensional, then $(W^\perp)^\perp = W$.

Proof:

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Remark: The statement (3) may be incorrect when V is infinite-dimensional. It would be correct provided that W is a "closed" subspace.

Example. Let W be the subspace of $\mathbb{R}^6$ spanned by the vectors

$$\mathbf{w}_1 = (1, 3, -2, 0, 2, 0), \quad \mathbf{w}_2 = (2, 6, -5, -2, 4, -3)$$
$$\mathbf{w}_3 = (0, 0, 5, 10, 0, 15), \quad \mathbf{w}_4 = (2, 6, 0, 8, 4, 18).$$

Find a basis for the orthogonal complement of W under Euclidean inner product.

Solution:

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Examples

Example. Let $V = C[-1,1]$, and

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx, \quad (f, g \in V).$$

Let V_e (V_o) be the space of even (odd, respectively) functions in V. Prove that $V_o^{\perp} = V_e$.

Proof:

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Orthonormal Sets

Philosophy: Basis is designed for a vector space. Now there is a inner product, the basis vectors can have “angles”.

Definition.

◇ A set of two or more vectors in an inner product space is called orthogonal if all pairs of distinct vectors in the set are orthogonal.

◇ An orthogonal set in which each vectors has norm 1 is called orthonormal.

◇ If a basis is orthogonal, then we say it is an orthogonal basis.

◇ If a basis is orthonormal, then we say it is an orthonormal basis.

Example. In $\mathbb{R}^3$ with Euclidean inner product, let $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, where

$$\mathbf{u}_1 = (0, 1, 0), \quad \mathbf{u}_2 = (1, 0, 1), \quad \mathbf{u}_3 = (1, 0, -1).$$

Verify that the set S is orthogonal. Construct an orthonormal basis by normalising these vectors.

Solution:

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Examples

Example. Show that the following sets are orthonormal in given inner product spaces.

(1) $V = M_n$, $\langle A, B\rangle = \mathrm{tr}(B^T A)$, the standard basis.

(2) $V = P_n$, $\left\langle \sum_{k=0}^{n} a_k x^k, \sum_{k=0}^{n} b_k x^k \right\rangle = \sum_{k=0}^{n} a_k b_k$.
$B = \{1, x, x^2, \ldots, x^n\}$.

Proof:

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Examples

Example. Show that the following sets are orthonormal in given inner product spaces.

(3) $V = C[-\pi,\pi]$, $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$.

$$B = \left\{\frac{1}{\sqrt{2}}, \sin x, \sin 2x, \ldots, \sin kx, \cos x, \cos 2x, \ldots, \cos \ell x\right\}.$$

(4) $V = C_{\mathbb{C}}[-\pi,\pi]$, $\langle f,g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$.

$$B = \left\{e^{inx} : |n| \le N\right\}.$$

Proof:

Properties of Orthogonal Basis

Theorem. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of non-zero vectors in an inner product space $(V, \langle\cdot,\cdot\rangle)$, then S is linearly independent.

Proof:

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Theorem. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1\rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2\rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_n\rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n.$$

Proof:

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Remark: When S is an orthonormal basis,

$$\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1\rangle\mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2\rangle\mathbf{v}_2 + \cdots \langle \mathbf{u}, \mathbf{v}_n\rangle\mathbf{v}_n.$$

Examples

Example. Let $\mathbf{v}_1 = (0, 2, 0)$, $\mathbf{v}_2 = (3, 0, 3)$, $\mathbf{v}_3 = (-4, 0, 4)$ and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(1) Verify that S form an orthogonal basis for $\mathbb{R}^3$ with Euclidean inner product.
(2) Find $[\mathbf{u}]_B$, where $\mathbf{u} = (1, 2, 4)$.

Solution:

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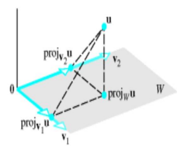
Orthogonal Projections

Theorem. (Projection Theorem) If W is a subspace of a finite-dimensional inner product space V, then every vector $\mathbf{u}$ in V can be expressed in exactly one way as

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2,$$

where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^{\perp}$.



Remark: We denote $\mathbf{w}_1 = \text{proj}_W(\mathbf{u})$ and $\mathbf{w}_2 = \text{proj}_{W^\perp}(\mathbf{u})$.

Theorem. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and W be a subspace of V with an orthogonal basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$. Then for any $\mathbf{u} \in V$,

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_1 + \cdots + \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r$$

Proof:

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Remark: When S is an orthonormal basis for W,

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots, + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r.$$

Examples

Example. Let $\mathbb{F} = \mathbb{R}$, $V = C[-\pi, \pi]$ and

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t)dt, \quad (\forall f, g \in V).$$

Find the orthogonal projection of $h(t) = t$ on the subspace $W = \text{span}\{\cos t, \sin t\}$.

Solution:

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Gram-Schmidt Process

Theorem. Every nonzero finite-dimensional inner product space has an orthonormal basis.

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for V.

Step 1-1. Take $\mathbf{u}_1 = \mathbf{v}_1$.

Step 1-2. Take $\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{v}_2, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = \mathrm{proj}_{\mathrm{span}\{\mathbf{u}_1\}^\perp}(\mathbf{v}_2)$.

Step 1-3. Take $\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 = \mathrm{proj}_{\mathrm{span}\{\mathbf{u}_1, \mathbf{u}_2\}^\perp}(\mathbf{v}_3)$.

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Step 1-n. Take

$$\mathbf{u}_n = \mathbf{v}_n - \frac{\langle \mathbf{v}_n, \mathbf{u}_1 \rangle}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \ldots - \frac{\langle \mathbf{v}_n, \mathbf{u}_{n-1} \rangle}{\|\mathbf{u}_{n-1}\|^2}\mathbf{u}_{n-1} = \mathrm{proj}_{\mathrm{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-1}\}^\perp}(\mathbf{v}_n).$$

Conclusion of step 1:

We obtain an orthogonal basis $S' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for V.

Step 2. Let $\mathbf{w}_i = \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|}$ $(1 \leq i \leq n)$.

Conclusion of step 2:

We obtain an orthonormal basis $S'' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ for V.

Examples

Example. Consider $\mathbb{F} = \mathbb{R}$ and P_2 with the inner product given by

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx, \quad (\forall p, q \in P_2).$$

Apply the Gram-Schmidt process to transform the standard basis $\{1, x, x^2\}$ for P_2 into an orthonormal basis $\{q_1(x), q_2(x), q_3(x)\}$.

Solution:

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Remark: These polynomials are called Legendre polynomials.

Extention of Orthonormal Basis

By the Gram-Schmidt process, orthonormal sets can be extened to orthonormal bases.

Theorem. If W is a finite-dimensional inner product space, then:
(a) Every orthogonal set of nonzero vectors in W can be enlarged to an orthogonal basis for W.
(b) Every orthonormal set in W can be enlarged to an orthonormal basis for W.

QR-Decomposition

Theorem. Suppose that $A \in M_{m\times n}$ and $\operatorname{rank}(A) = n$. Then A can be factored as $A = QR$, where $Q \in M_{m\times n}$ is an matrix with orthonormal column vectors, and R is an invertible upper triangular matrix.

Indeed, let $A = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix}$. Suppose that we obtain orthonormal vectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$ by applying the Gram-Schmidt process to $\mathbf{c}_1, \ldots, \mathbf{c}_n$. Then

$$A = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{bmatrix} \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{q}_1 \rangle & \langle \mathbf{c}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{c}_n, \mathbf{q}_1 \rangle \\ \langle \mathbf{c}_1, \mathbf{q}_2 \rangle & \langle \mathbf{c}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{c}_n, \mathbf{q}_2 \rangle \\ \vdots & \vdots & & \vdots \\ \langle \mathbf{c}_1, \mathbf{q}_n \rangle & \langle \mathbf{c}_2, \mathbf{q}_n \rangle & \cdots & \langle \mathbf{c}_n, \mathbf{q}_n \rangle \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{bmatrix} \begin{bmatrix} \langle \mathbf{c}_1, \mathbf{q}_1 \rangle & \langle \mathbf{c}_2, \mathbf{q}_1 \rangle & \cdots & \langle \mathbf{c}_n, \mathbf{q}_1 \rangle \\ & \langle \mathbf{c}_2, \mathbf{q}_2 \rangle & \cdots & \langle \mathbf{c}_n, \mathbf{q}_1 \rangle \\ & & \ddots & \vdots \\ & & & \langle \mathbf{c}_n, \mathbf{q}_n \rangle \end{bmatrix} = QR.$$

An Example

Example. Find a QR-decomposition of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution:

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Definition of Orthogonal Matrix

Recall: "Orthonormal basis" play an important role in an inner product space.

Question: What is the relationship between two orthonormal bases?

How does the transition matrix look like?

(from an orthonormal basis to standard bases)

Definition. A matrix $A \in M_n(\mathbb{R})$ is said to be orthogonal if one of the following equivalent conditions holds:

(1) $A^T A = AA^T = I_n$;

(2) $A^{-1} = A^T$;

(3) The row vectors of A form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product;

(4) The column vectors of A form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product

Proof of the equivalence:

24

Examples

Example. Show that the matrix

$$\begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}$$

is orthogonal.

Solution:

25

Example. Let $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Whether the matrices R_θ, R_θ^T, R_θ^{-1}, $R_\theta R_{\theta'}$ orthogonal?

Solution:

26

Definition of Unitary Matrix

Definition. A matrix $A \in M_n(\mathbb{C})$ is said to be unitary if one of the following equivalent conditions holds:
(1) $A^*A = AA^* = I_n$;
(2) $A^{-1} = A^*$;
(3) The row vectors of A form an orthonormal set in $\mathbb{C}^n$ with the Euclidean inner product;
(4) The column vectors of A form an orthonormal set in $\mathbb{C}^n$ with the Euclidean inner product

Example. Show that a 1×1 matrix $[z] \in M_1(\mathbb{C})$ is unitary if and only if $|z| = 1$.

Proof:

27

Properties of Orthogonal/Unitary Matrix

Theorem. Let $A, B \in M_n(\mathbb{R})$ be orthogonal matrices. Then
- ◇ A^T is orthogonal;
- ◇ A^{-1} is orthogonal;
- ◇ AB is orthogonal;
- ◇ $\det(A) = \pm 1$. $|\det(A)| = 1$

Proof:

28

Theorem. Let $A, B \in M_n(\mathbb{C})$ be unitary matrices. Then
- ◇ A^* is unitary;
- ◇ A^{-1} is unitary;
- ◇ AB is unitary;
- ◇ $\det(A) = e^{i\theta}$ for some $0 \le \theta < 2\pi$. $|\det(A)| = 1$

Properties of Orthogonal/Unitary Matrix

Theorem. If $A \in M_n(\mathbb{R})$, then the following are equivalent.
(1) A is orthogonal.
(2) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
(3) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
Proof:

29

Remark: As a linear transformation, T_A preserves norm, angle and area. And it preserve (or reverses) the orientation if its determinant is 1 (or -1, respectively).

Corollary. All possible eigenvalues of an orthogonal matrix are ± 1.
Proof:

30

Problem*: Suppose that A is an $n \times n$ orthogonal matrix, where n is odd.

Also suppose that $\det(A) = 1$. Prove that 1 is an eigenvalue of A.

Theorem. If $A \in M_n(\mathbb{C})$, then the following are equivalent.
(1) A is unitary.
(2) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in $\mathbb{C}^n$.
(3) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$.

Corollary. Any possible eigenvalue λ of a unitary matrix satisfies that $|\lambda| = 1$.

Transition Matrix

Theorem. Let S be an orthonormal basis for an n-dimensional inner product space V over $\mathbb{F}$. Let $(\mathbf{u})_S = (u_1, u_2, \cdots, u_n)$ and $(\mathbf{v})_S = (v_1, v_2, \cdots, v_n)$. Then

- ◇ $\langle \mathbf{u}, \mathbf{v} \rangle = u_1 \overline{v_1} + \cdots u_n \overline{v_n}$;
- ◇ $\|\mathbf{u}\| = \sqrt{|u_1|^2 + \cdots |u_n|^2}$;
- ◇ $d(\mathbf{u}, \mathbf{v}) = \sqrt{|u_1 - v_1|^2 + \ldots + |u_n - v_n|^2}$.

Proof:

31

Remark: When $\mathbb{F} = \mathbb{R}$, we may ignore the "conjugate".

Theorem. Let V be a finite-dimensional inner product space. If P is the transition matrix from one orthonormal basis to another orthonormal basis for V, then P is an orthogonal/unitary matrix.

Proof:

32

Adjoint Operator*

Theorem. Let $(V, \langle\cdot,\cdot\rangle_V)$ be a finite-dimensional complex inner product spaces. Suppose that $T : V \to V$ is a linear operator. Then there exists a unique linear transformation $T^* : V \to V$ such that

$$\langle T\mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{x}, T^*\mathbf{y}\rangle, \quad (\forall \mathbf{x}, \mathbf{y} \in V).$$

Remark: We do not prove the above theorem here.

Remark: Under a given orthonormal bases, we have $[T^*] = [T]^*$.

The main purpose of this page is:

"(conjugate) transpose" can be defined in an abstract way.

Remark: A linear operator $U : V \to V$ is said to be unitary if

$$UU^* = U^*U = I.$$

§5.1 Inner Products

§5.2 Angle and Orthogonality in Inner Product Spaces

§5.3 Gram-Schmidt Process; QR-Decomposition

§5.4 Orthogonal and Unitary Matrices

§5.5 Orthogonal Diagonalization

§5.6 Singular Value Decomposition

§5.7 Best Approximation; Least Squares

§5.8 Quadratic Forms

§5.9 Polar Decomposition*

Concept of Diagonalisazion

Recall: "Orthonormal basis" play an important role in an inner product space.

Recall: A "diagonalizable" matrix is relatively easy to determine.

Our interest: an operator T on V such that eigenvectors form an orthonormal basis for V.

Example. The projection on the xz-plane in $\mathbb{R}^3$ is given by

$$T(x, y, z) = (x, 0, z), \quad (x, y, z) \in \mathbb{R}^3.$$

Indeed, it has standard matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The vectors $\mathbf{e}_1$ and $\mathbf{e}_3$ are eigenvectors corresponding to the eigenvalue 1; and $\mathbf{e}_2$ is an eigenvector corresponding to 0.

Conditions for Orthogonal Diagonalizability

Definition. Let $A \in M_n(\mathbb{R})$. If

$$P^T AP = P^{-1}AP = D$$

for some orthogonal matrix P and some diagonal matrix D, then we say that A is orthogonally diagonalizable and P orthogonally diagonalizes A.

Remark: In numerical algorithms, the complexity of dealing with P^T is much less than with P^{-1}.

Theorem. Let $A \in M_n(\mathbb{R})$, then the following are equivalent.
(1) A is orthogonally diagonalizable.
(2) A has an orthonormal set of n eigenvectors.
(3) A is symmetric.

Problem*: Prove (1)⇔(2) and (1)⇒(3).

Remark: The proof of (3)⇒(1) is beyond the scope of the course.

Remark: $P = \left[\begin{array}{c:c:c:c} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{array}\right]$, $D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$.

Here $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ $(1 \le i \le n)$, and the set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is orthonormal.

Conditions for Unitary Diagonalizability

Definition. Let $A \in M_n(\mathbb{C})$. If

$$U^*AU = U^{-1}AU = D$$

for some unitary matrix U and diagonal matrix D, then we say that A is unitarily diagonalizable and U unitarily diagonalizes A.

Theorem. Let $A \in M_n(\mathbb{C})$, then the following are equivalent.
(1) A is unitarily diagonalizable.
(2) A has an orthonormal set of n eigenvectors.
(3) A is normal, i.e., $A^*A = AA^*$.

Remark: Hermitian, unitary matrices are all normal.
Remark: The proof of this theorem is beyond the scope of the course.

Orthogonal Diagonalization

Theorem. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix, then eigenvectors from different eigenspaces are orthogonal.

Proof:

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Remark: Same holds for an Hermitian matrix in the complex case.

Example. Suppose that $A \in M_3(\mathbb{R})$ is symmetric and $\text{rank}(A) = 2$. Suppose that

$$A \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & 2 \\ 2 & 1 \end{bmatrix}$$

Find all the eigenvalues of A. For each eigenvalue, find an eigenvector corresponding to it.

Proof:

34

An Example

Fact: Suppose that all the distinct eigenvalues of $A \in M_n$ are $\lambda_1, \ldots, \lambda_k$, and $\{\mathbf{u}_{i,1}, \ldots, \mathbf{u}_{i,i_{n_i}}\}$ are an orthonormal basis for $E_{\lambda_i}(A)$ $(1 \le i \le k)$. Then

$$\{\mathbf{u}_{1,1}, \ldots, \mathbf{u}_{1,n_1}, \mathbf{u}_{2,1}, \ldots, \mathbf{u}_{2,n_2}, \ldots, \mathbf{u}_{k,1}, \ldots, \mathbf{u}_{k,n_k}\}$$

form an orthonormal basis for $\mathbb{R}^n$.

Remark: We only need to apply Gram-Schmidt process on each eigenspaces.

Example. Orthogonal diagonalize the matrix $A = \begin{bmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{bmatrix}$.

Proof:

35

Problem*: Summarize the main steps of orthogonal diagonalization.

An Example

Example. Consider

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{C}).$$

Find a unitary matrix U and unitarily diagonalize A.

Proof:

36

Spectral Decomposition of Symmetric Matrix

Theorem. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. Then there are $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and orthonormal vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \in \mathbb{R}^n$ such that

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$

Proof:

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Remark: The above equality is called a spectral decomposition of A.

Remark: In above theorem, $A\mathbf{u}_i = \lambda_i \mathbf{u}_i$ $(1 \le i \le n)$.
And A is orthogonally diagonalized by $P = \left[\begin{array}{c:c:c:c} \mathbf{u}_1 & \mathbf{u}_2 & \ldots & \mathbf{u}_n \end{array}\right]$.

Problem: Prove that the square matrix $\mathbf{u}_1 \mathbf{u}_1^T$ is the orthogonal projection on $\operatorname{span}\{\mathbf{u}_1\}$.

Theorem. Let $A \in M_n(\mathbb{C})$ be a normal matrix. Then there are $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and orthonormal vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \in \mathbb{C}^n$ such that

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^* + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^* + \cdots \lambda_n \mathbf{u}_n \mathbf{u}_n^*.$$

Remark: When A is normal, there are matrices $E_1, \ldots, E_k$ with
(i) $E_i^2 = E_i = E^*$ $(1 \leq i \leq k)$;
(ii) $E_i E_j = 0$ $(1 \leq i \neq j \leq k)$;
(iii) $\sum_{i=1}^k E_i = I$,
and distinct numbers $\mu_1, \ldots, \mu_k \in \mathbb{C}$ such that

$$A = \sum_{i=1}^{k} \mu_i E_i.$$

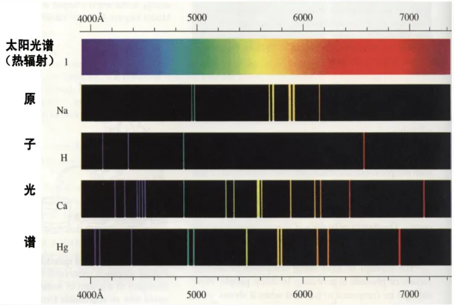
Indeed, the numbers $\mu_1, \ldots, \mu_k$ are all different eigenvalues of A, and E_i is the orthogonal projection on the eigenspace $E_{\mu_i}(A)$.

Spectral Decomposition of Hermitian Matrix*

In quantum physics, each observable can be viewed as an Hermitian (self-adjoint) operator, whose spectral are real. Let

$$A = \sum_{i=1}^{k} \mu_i E_i, \qquad (1)$$

where $\mu_1 < \mu_2 < \ldots < \mu_n$.



Remark: For an operator on infinite-dimensional space, there spectrum may contain "continuous" part. Then the summation '$\sum$' in (1) becomes integration '$\int$'.

Common (Complex) Eigenvectors*

Theorem. Suppose that the two matrices $A, B \in M_n(\mathbb{C})$ commute, i.e., $AB = BA$. Then they have a common eigenvector.

Remark: The eigenvalues of A and B, to which this common eigenvector corresponds, can be different.

Remark: In physics, two commutative observables (i.e., Hermitian operators) have common eigenvectors that forming an orthonormal basis, which means that they can be observed simultaneously.

The Nondiagonalizable Real Matrix*

Theorem. (Schur's Theorem) If the characteristic polynomial of $A \in M_n(\mathbb{R})$ has n roots $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, then there is an orthogonal matrix P such that P^TAP is an upper triangular matrix of the form

$$P^TAP = \begin{bmatrix} \lambda_1 & * & * & \cdots & * \\ 0 & \lambda_2 & * & \cdots & * \\ 0 & 0 & \lambda_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} := S$$

In particular, $A = PSP^T$ is called a Schur decomposition of A.

Remark: In numerical algorithms,

◇ the matrix S is much simpler than A;

◇ the orthogonal matrix does not magnify round-off error, i.e.,

$$\mathbf{x} = \hat{\mathbf{x}} + \mathbf{e} \quad \Longrightarrow \quad \|P\mathbf{x} - P\hat{\mathbf{x}}\| = \|P\mathbf{e}\| = \|\mathbf{e}\|.$$

The Nondiagonalizable Real Matrix*

Theorem. (Hessenberg's Theorem) If $A \in M_n(\mathbb{R})$, then there is an orthogonal matrix P such that

$$P^TAP = \begin{bmatrix} * & * & \dots & * & * & * \\ * & * & \dots & * & * & * \\ 0 & * & \ddots & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & * & * & * \\ 0 & 0 & \dots & 0 & * & * \end{bmatrix} := H.$$

In particular, $A = PHP^T$ is called an upper Hessenberg decomposition of A.

Remark: In numerical algorithms,

◇ the matrix H is much simpler than A;

◇ the orthogonal matrix does not magnify round-off error.

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Property of A^TA

Background: In application, data usually form a matrix A in $M_{m\times n}(\mathbb{R})$, where m and n may be different.

Idea: Find square matrices that contain as much information as A does.
Direction: $A^TA \in M_n(\mathbb{R})$; $AA^T \in M_m(\mathbb{R})$!

Theorem. Let $A \in M_{m\times n}(\mathbb{R})$. Then

- $\diamond$ $\mathrm{Null}(A) = \mathrm{Null}(A^TA)$;
- $\diamond$ $\mathrm{Row}(A) = \mathrm{Row}(A^TA)$;
- $\diamond$ $\mathrm{Col}(A^T) = \mathrm{Col}(A^TA)$;
- $\diamond$ $\mathrm{rank}(A) = \mathrm{rank}(A^TA)$.

Proof:

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Singular Value

Theorem. Let $A \in M_{m\times n}(\mathbb{R})$. Then:
◇ A^TA is orthogonally diagonalizable;
◇ The eigenvalues of A^TA are nonnegative.
Proof:

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Definition. Let $A \in M_{m\times n}(\mathbb{R})$, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A^TA. Then the numbers

$$\sigma_1 = \sqrt{\lambda_1},\ \sigma_2 = \sqrt{\lambda_2},\ \ldots,\ \sigma_n = \sqrt{\lambda_n}.$$

are called the singular values of A.

Remark: We usually assume that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$, and so $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$.

Examples

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find the singular values of A and A^T, respectively.

Solution:

40

Question*: Prove that the singular values of A and A^T are same, except for some zero values.

Singular Value Decomposition (A Quick View)

A quick view on "SVD":

Let $A \in M_{m\times n}(\mathbb{R})$ and $\text{rank}(A) = r$. Then there are orthogonal matrix $U \in M_m$ and $V \in M_n$ such that

$$A = U\Sigma V,$$

where

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \\ & & & \end{bmatrix}_{m\times n}.$$

Notations occur in next several pages: Let $A \in M_{m\times n}(\mathbb{R})$ and $\text{rank}(A) = r$. Eigenvalues of A^TA are

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \ldots = \lambda_n = 0,$$

with corresponding orthonormal eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. The singular values of A are $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > \sigma_{r+1} = \ldots = \sigma_n = 0$.

Theorem. . Let $A \in M_{m\times n}(\mathbb{R})$ and $\operatorname{rank}(A) = r$. There there is

- ⋄ an orthonormal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ for $\mathbb{R}^n$;
- ⋄ an orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ for $\mathbb{R}^m$;
- ⋄ positive scalars $\sigma_1, \ldots, \sigma_r \in \mathbb{R}$, such that

$$A\mathbf{v}_i = \begin{cases} \sigma_i \mathbf{u}_i, & (1 \le i \le r), \\ 0, & (r < i \le n), \end{cases} \quad A^T\mathbf{u}_i = \begin{cases} \sigma_i \mathbf{v}_i, & (1 \le i \le r), \\ 0, & (r < i \le m). \end{cases}$$

Proof:

Take $\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$. Here $\|A\mathbf{v}_i\| = \sigma_i$.

Then extend it to an orthonormal basis.

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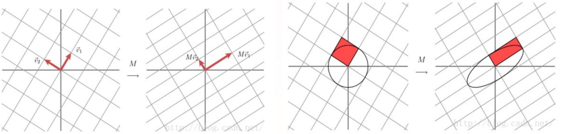
Remark: There exist orthonormal basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ such that $A\mathbf{v}_1, \ldots, A\mathbf{v}_n$ are orthogonal. The quantities σ_i turns $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ into unit vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$.

Remark: In matrix notation, we have $A = U_r \Sigma_r V_r^T$, i.e.,

$$A_{m\times n} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_r \end{bmatrix}_{n\times r} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_r \end{bmatrix}_{m\times r} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}_{r\times r}.$$

Effect of a 2×2 Matrix

By geometric SVD, there must be orthonormal vectors $\mathbf{v}_1, \mathbf{v}_2$ such that $A\mathbf{v}_1, A\mathbf{v}_2$ are orthogonal.



Example. Shears of $\mathbb{R}^2$:

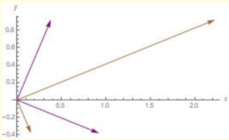
$$M = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \quad M^T = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}, \qquad M^T M = \begin{bmatrix} 1 & k \\ k & 1+k^2 \end{bmatrix}.$$

When $k = 2$, one has

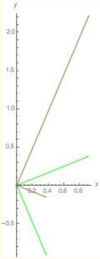
$$\lambda_{1,2} = 3 \pm 2\sqrt{2}, \quad \sigma_{1,2} = \sqrt{2} \pm 1,$$

$$\mathbf{v}_{1,2} = \left(\frac{1}{\sqrt{4 \pm 2\sqrt{2}}}, \frac{1 \pm \sqrt{2}}{\sqrt{4 + 2\sqrt{2}}} \right),$$

$$\mathbf{u}_{1,2} = \left(\frac{\pm 1 + \sqrt{2}}{\sqrt{4 \pm 2\sqrt{2}}}, \frac{\pm 1}{\sqrt{4 \pm 2\sqrt{2}}} \right).$$



$\mathbf{v}_i,\ A\mathbf{v}_i$



$\mathbf{u}_i,\ A^T\mathbf{u}_i$

Singular Value Decomposition (Full Version)

Theorem. Let $A \in M_{m\times n}(\mathbb{R})$ and $\text{rank}(A) = r$. Then there are

- ◇ an orthogonal matrix $U \in M_m(\mathbb{R})$;
- ◇ an orthogonal matrix $V \in M_n(\mathbb{R})$;
- ◇ positive scalars $\sigma_1, \ldots, \sigma_r \in \mathbb{R}$,

such that $A = U\Sigma V^T$, where

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \\ & & & \end{bmatrix}_{m\times n}.$$

Proof:

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Remark: We have

$$U = \left[\begin{array}{c:c:c:c} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m \end{array}\right]_{m\times m}, \qquad V^T = \begin{bmatrix} \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \vdots \\ \hline \mathbf{v}_n^T \end{bmatrix}_{n\times n}.$$

Singular Value Decomposition (Condensed Version)

Theorem. Let $A \in M_{m\times n}(\mathbb{R})$ and $\text{rank}(A) = r$. Then there are

◇ an orthonormal set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ in $\mathbb{R}^n$;

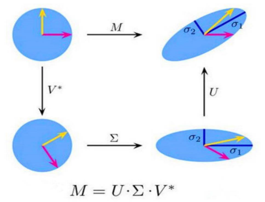
◇ an orthonormal set $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ in $\mathbb{R}^m$;

◇ positive scalars $\sigma_1, \ldots, \sigma_r \in \mathbb{R}$,

such that
$$A = \mathbf{u}_1\sigma_1\mathbf{v}_1^T + \cdots + \mathbf{u}_r\sigma_r\mathbf{v}_r^T,$$

or equivalently, $A = U_r\Sigma_r V_r^T$

$$= \begin{bmatrix} \mathbf{u}_1 & \vdots & \mathbf{u}_2 & \vdots & \cdots & \vdots & \mathbf{u}_r \end{bmatrix}_{m\times r} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix}_{r\times r} \begin{bmatrix} \mathbf{v}_1^T \\ \hline \mathbf{v}_2^T \\ \hline \vdots \\ \hline \mathbf{v}_r^T \end{bmatrix}_{r\times n}.$$



Example. The shear of $\mathbb{R}^2$ with $k = 2$ has SVD:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}+1}{\sqrt{4+2\sqrt{2}}} & \frac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{-1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix} \begin{bmatrix} \sqrt{2}+1 & 0 \\ 0 & \sqrt{2}-1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1+\sqrt{2}}{\sqrt{4+2\sqrt{2}}} \\ \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1-\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \end{bmatrix}.$$

More Properties of SVD*

Fundamental spaces of a matrix:

- ◇ $\mathrm{Col}(A) = \mathrm{Col}(U_r)$.
- ◇ $\mathrm{Null}(A) = \mathrm{Col}(V_{n-r})$.
- ◇ $\mathrm{Row}(A) = \mathrm{Col}(A^T) = \mathrm{Col}(V_r)$.

Approximation of a matrix:

- ◇ When $k \leq r$, the best approximation of matrix is given by

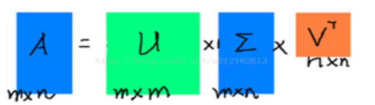
$$\|A - A_k\| = \min_{\mathrm{rank}(B) \leq k} \|A - B\|.$$

Here $\|\cdot\|$ is some certain norm on M_n, and

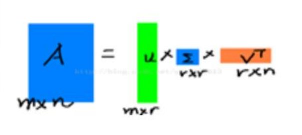
$$A_k = \mathbf{u}_1 \sigma_1 \mathbf{v}_1^T + \ldots + \mathbf{u}_k \sigma_k \mathbf{v}_k^T.$$

The most important information of A gathers at upper left part of SVD!

Applications of SVD



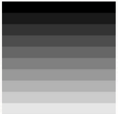
Full Version



Condensed Version

Remark: For applications, the condensed version only remains the most important part, which saves space and simplified computations.

Remark: Recall that $\sigma_1 \geq \sigma_2 \geq \dots$.
One can even discard the smaller singular values: denoise!



$$uv^H = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0.1 & 0.1 & \cdots & 0.1 \\ 0.2 & 0.2 & \cdots & 0.2 \\ \vdots & \vdots & & \vdots \\ 0.9 & 0.9 & \cdots & 0.9 \end{pmatrix}$$

Examples

Example. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Find a singular value decomposition of A .

Solution:

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Chapter 5. Inner Product Spaces

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§5.2 Angle and Orthogonality in Inner Product Spaces

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§5.4 Orthogonal and Unitary Matrices

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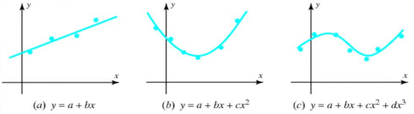
Least Squares Fitting to Data

In this section, we only consider real case, i.e., $\mathbb{F} = \mathbb{R}$.

Problem: To obtain a mathematical relationship $y = f(x)$ by "fitting" a curve to points

$$(x_1, y_1),\ (x_2, y_2),\ \ldots,\ (x_n, y_n).$$



Which coefficients $c_0, c_1, \ldots, c_{n-1}$ of a polynomial of $n-1$ minimizes

$$\sum_{k=1}^{m} \left| y_k - (c_0 + c_1 x_k + \ldots + c_{n-1} x_k^{n-1}) \right|^2?$$

Matrix Form:

Least Squares Solution

Fact: A linear system $A\mathbf{x} = \mathbf{b}$ may be inconsistent.

Aim: Find a vector that is as "close" to a solution as possible.

Problem: Find a vector $\hat{\mathbf{x}}$ that minimizes $\|A\mathbf{x} - \mathbf{b}\|$, i.e.,

$$\sum_{i=1}^{m}(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i)^2$$

We call such a vector a least squares solution of the system

If the least squares solution $\hat{\mathbf{x}}$ exists, then

$$\|A\hat{\mathbf{x}} - \mathbf{b}\| = \min_{\mathbf{w}\in \mathrm{Col}(A)} \|\mathbf{w} - \mathbf{b}\|.$$

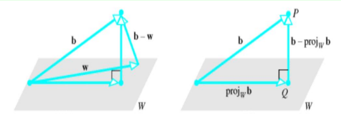
Best Approximation

Theorem. (Best Approximation) If W is a subspace of a finite-dimensional inner product space V, and if $\mathbf{b}$ is a vector in V, then $\text{proj}_W(\mathbf{b})$ is the best approximation to $\mathbf{b}$ from W in the sense that

$$\left\|\mathbf{b} - \text{proj}_W(\mathbf{b})\right\| < \|\mathbf{b} - \mathbf{w}\|$$

for any $\mathbf{w} \in W \setminus \{\text{proj}_W(\mathbf{b})\}$.

Proof:



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Remark: $\left\|\mathbf{b} - \text{proj}_W(\mathbf{b})\right\| = \min_{\mathbf{w}\in W} \|\mathbf{b} - \mathbf{w}\|$. The minimum takes uniquely.
Remark: A least square solution $\hat{\mathbf{x}}$ of the system $A\mathbf{x} = \mathbf{b}$ satisfies

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| = \min_{\mathbf{w}\in \text{Col}(A)} \|\mathbf{b} - \mathbf{w}\| \iff A\hat{\mathbf{x}} = \text{proj}_{\text{Col}(A)}(\mathbf{b}).$$

Method of Least Squares

Theorem. (1) Least square solutions of a linear system $A\mathbf{x} = \mathbf{b}$ always exist. For any least squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$, the orthogonal projection of $\mathbf{b}$ on $\mathrm{Col}(A)$ is

$$\mathrm{proj}_{\mathrm{Col}(A)}\mathbf{b} = A\hat{\mathbf{x}}.$$

And it satisfies that $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$.

(2) For every linear system $A\mathbf{x} = \mathbf{b}$, the associated normal system

$$A^T A\mathbf{x} = A^T\mathbf{b}$$

is consistent, and all solutions are least squares solutions of $A\mathbf{x} = \mathbf{b}$.

Proof:

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Remark: $A^T\mathbf{u} = \mathbf{0}$ if and only if $\mathbf{u} \in \mathrm{Col}(A)^{\perp}$

An Example

Example. Find all least squares solutions of the linear system

$$\begin{cases} x_1 + x_2 & = 0 \\ \phantom{x_1 + {}} x_2 & = 1 \\ x_1 & = 2 \end{cases}$$

Solution:

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Remark: The matrix $A^T A$ may be singular.

An Example

Example. Find the orthogonal projection of the vector $\mathbf{b} = (0, 1, 2)$ on the subspace W of $\mathbb{R}^3$ spanned by the vectors

$$\mathbf{w}_1 = (1, 0, 1), \quad \mathbf{w}_2 = (1, 1, 0).$$

Solution:

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Least Squares Fitting to Data

Which coefficients $c_0, c_1, \ldots, c_{n-1}$ minimizes

$$\sum_{k=1}^{m} \left| y_k - (C_0 + c_1 x_1 + \ldots + c_{m-1} x_{m-1}^{n}) \right|^2 ?$$

Method. Consider fitting a polynomial of given degree m, i.e.,

$$y(x) = c_0 + c_1 x + \ldots + c_{n-1} x^{n-1},$$

to m points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$. The coefficients $c_0, c_1, \ldots, c_{n-1}$ can be determined by

$$M^T M \mathbf{c} = M^T \mathbf{y},$$

where the vector $\mathbf{c}$ minimizes $\|\mathbf{y} - M\mathbf{v}\|$ for $\mathbf{v} \in \mathbb{R}^n$. Here

$$M = \begin{bmatrix} 1 & x_1 & \ldots & x_1^{n-1} \\ 1 & x_2 & \ldots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & \cdots & x_m^{n-1} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}.$$

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Definition of Quadratic Form

Definition. A polynomial with terms all of degree 2, i.e.,

$$Q(x_1, x_2, \ldots, x_n) = \sum_{1 \le i \le j \le n} c_{ij} x_i x_j,$$

is called a quadratic form.

Example. The following are quadratic forms.

- ◇ $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^3$.
- ◇ $Q(x, y) = (x/a)^2 + (y/b)^2$.
- ◇ $Q(x, y) = (x/a)^2 - (y/b)^2$.
- ◇ $Q(x, y) = xy$.
- ◇ $Q(x_1, x_2, x_3) = 2x_1^2 + 6x_1x_2 - 5x_2^2 + 3x_2x_3$.

Quadratic Form and Symmetric Matrix

- Since $x_1x_2 = x_2x_1$ as polynomials, we rewrite

$$c_{12}x_1x_2 = \frac{c_{12}}{2}x_1x_2 + \frac{c_{12}}{2}x_2x_1.$$

- Now

$$Q(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j,$$

where $a_{ii} = c_{ii}$ for $1 \leq i \leq n$ and

$$a_{ij} = \frac{1}{2}c_{ij},\ (i < j), \quad a_{ij} = \frac{1}{2}c_{ji},\ (i > j).$$

- The matrices appear as

$$Q(\mathbf{x}) = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \mathbf{x}^T A\,\mathbf{x}.$$

Here A is a symmetric matrix.

Quadratic Form and Symmetric Matrix

- For any matrix $C = [c_{ij}] \in M_n(\mathbb{R})$,
$$Q_C(\mathbf{x}) := \mathbf{x}^T C\,\mathbf{x} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_ix_j$$
defines a quadratic form.

- Fact: One has $C = A + B$, where
$A = \frac{1}{2}(C + C^T)$ is symmetric; $B = \frac{1}{2}(C - C^T)$ is skew-symmetric.

- Fact: One has $Q_B(\mathbf{x}) = 0$, since $B = -B^T$. Why?

- Conclusion: $Q_C(\mathbf{x}) = Q_A(\mathbf{x})$.
The effective part is the symmetric matrix A.

Definition. Let $A \in M_n(\mathbb{R})$ be a symmetric matrix. The quadratic form $Q_A(\mathbf{x})$ associated with A is defined to be
$$Q_A(\mathbf{x}) = \mathbf{x}^T A\,\mathbf{x}, \quad (\forall \mathbf{x} \in \mathbb{R}^n).$$

Remark: If V is a general inner product space, and L is a general operator such that $L^T = L$, then we may also define a quadratic form as $Q_L(\mathbf{v}) = \langle L\mathbf{v}, \mathbf{v}\rangle$ $(\mathbf{v} \in V)$.

Examples

Example. Express the quadratic form in the matrix notation $\mathbf{x}^T A\mathbf{x}$, where A is symmetric.
(1) $2x^2 + 6xy - 5y^2$.
(2) $x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1x_2 - 2x_1x_3 + 8x_2x_3$.

Solution:

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Change of Variable in a Quadratic Form

Definition. ◇ The substitution $\mathbf{x} = P\mathbf{y}$ is called a change of variable if P is invertible.
◇ The substitution $\mathbf{x} = P\mathbf{y}$ is called an orthogonal change of variable if P is orthogonal.

- We have

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T(P^T AP)\mathbf{y}.$$

Remark: When P is orthogonal, the right-hand side equals $\mathbf{y}^T(P^{-1}AP)\mathbf{y}$.

Recall: Any symmetric matrix in $M_n(\mathbb{R})$ is orthogonal diagonalizable.

Theorem. If $A \in M_n(\mathbb{R})$ is a symmetric matrix, then there is an orthogonal change of variable $\mathbf{x} = P\mathbf{y}$ that transforms the quadratic form $Q_A(\mathbf{x})$ into a quadratic form $Q_A(\mathbf{y})$ with no cross product terms, i.e.,

$$\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T(P^T AP)\mathbf{y} = \lambda_1 y_1^2 + \ldots + \lambda_n y_n^2.$$

Here $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A corresponding to the eigenvectors that form the successive columns of P.

Examples

Example. Find a variable change that diagonalizes the quadratic form

$$Q = 2x_1^2 + 2x_2^2 + 2x_3^2 + 6x_1x_2 + 6x_1x_3 + 6x_2x_3,$$

and express Q in terms of the new variables.

Solution:

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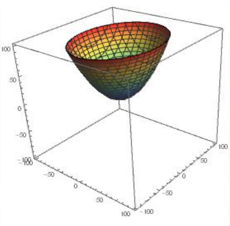
Problem*: Summarize the main steps of the solution.

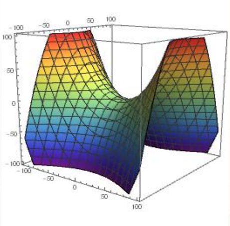
Quadratic Forms in Two Variables

- After orthogonal change of variables, a quadratic form in two variables becomes

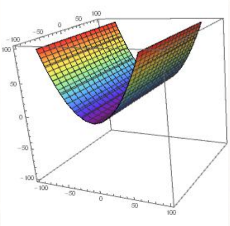
$$Q(x, y) = \lambda_1 x^2 + \lambda_2 y^2.$$



$\lambda_1 > 0,\ \lambda_2 > 0$



$\lambda_1 > 0,\ \lambda_2 < 0$



$\lambda_1 > 0,\ \lambda_2 = 0$

Case 1. $\lambda_1 > 0, \lambda_2 > 0$: $Q(x, y) \geq 0$, but $= 0$ only for $(x, y) = (0, 0)$.
Case 2. $\lambda_1 > 0, \lambda_2 < 0$: $Q(1, 0) > 0$ and $Q(0, 1) < 0$.
Case 3. $\lambda_1 > 0, \lambda_2 = 0$: $Q(x, y) \geq 0$, but $Q(0, 1) = 0$.
For other cases, we reverse the signs of λ_1 or/and λ_2.

Definition. A symmetric matrix $A \in M_n(\mathbb{R})$, or a quadratic form $\mathbf{x}^T A\mathbf{x}$, is said to be

- ◇ positive definite if $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$;
- ◇ negative definite if $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq 0$;
- ◇ semi-positive definite if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x}$;
- ◇ semi-negative definite if $\mathbf{x}^T A\mathbf{x} \leq 0$ for all $\mathbf{x}$;
- ◇ indefinite if it has both positive and negative values.

Theorem. If A is a symmetric matrix, then:
(a) $\mathbf{x}^T A\mathbf{x}$ is positive definite if and only if all eigenvalues of A are positive.
(b) $\mathbf{x}^T A\mathbf{x}$ is semi-positive definite if and only if all eigenvalues of A are non-negative.
(c) $\mathbf{x}^T A\mathbf{x}$ is negative definite if and only if all eigenvalues of A are negative.
(d) $\mathbf{x}^T A\mathbf{x}$ is semi-negative definite if and only if all eigenvalues of A are non-positive.
(e) $\mathbf{x}^T A\mathbf{x}$ is indefinite if and only if A has both positive and negative eigenvalues.

Equivalent Definition of Positive Definite Matrix*

Theorem. Let $A \in M_n(\mathbb{R})$ be symmetric. Then the following statements are equivalent.

(i) A is positive definite, i.e., $x^T Ax > 0$ for any $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$.
(ii) All eigenvalues of A are positive.
(iii) There is some invertible matrix C such that $A = C^T C$.
(iv) There is some positive definite matrix B such that $A = B^2$.
(v...) ...

Theorem. Let $A = [a_{ij}] \in M_n(\mathbb{R})$ be a positive definite matrix. Then
(1) $\det(A) > 0$.
(2) $a_{ii} > 0$ for any $1 \leq i \leq n$.

An Example

Example. Prove that the matrix

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

is positive definite.

Solution:

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Complex Case*

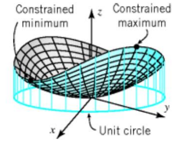
Definition. A matrix $A \in M_n(\mathbb{C})$, or a quadratic form $\mathbf{x}^* A\mathbf{x}$, is said to be

- ◇ positive definite if $\mathbf{x}^* A\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$;
- ◇ negative definite if $\mathbf{x}^* A\mathbf{x} < 0$ for all $\mathbf{x} \neq 0$;
- ◇ semi-positive definite if $\mathbf{x}^* A\mathbf{x} \geq 0$ for all $\mathbf{x}$;
- ◇ semi-negative definite if $\mathbf{x}^* A\mathbf{x} \leq 0$ for all $\mathbf{x}$;
- ◇ indefinite if it has both positive and negative values.

Theorem. If A is a normal matrix, i.e., $AA^* = A^*A$, then:
(a) $\mathbf{x}^* A\mathbf{x}$ is positive definite if and only if all eigenvalues of A are positive.
(b) $\mathbf{x}^* A\mathbf{x}$ is semi-positive definite if and only if all eigenvalues of A are non-negative.

Constrained Extremum Problems

Main Problem:
To find the maximum or minimum values of
a quadratic form $\mathbf{x}^T A\mathbf{x}$ subject of the constraint $\|\mathbf{x}\| = 1$.



Theorem. Let $A(\mathbb{R})$ be a symmetric matrix whose eigenvalues in order of decreasing size are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Then
(1) the quadratic form $\mathbf{x}^T A\mathbf{x}$ attains a maximum value and a minimum value on the set of all unit vectors;
(2) the maximum value attained in part (1) occurs at a unit vector corresponding to the eigenvalue λ_1;
(3) the minimum value attained in part (1) occurs at a unit vector corresponding to the eigenvalue λ_n.

Proof:

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Example. Find the maximum and minimum values of the quadratic form $z = 5x^2 + 5y^2 + 4xy$ subject to the constraint $x^2 + y^2 = 1$.

Solution:

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Polar Decomposition

Observation: Complex number $z = re^{i\theta}$, where $r \geq 0$ and $|e^{i\theta}| = 1$.

Theorem. Let $A \in M_n(\mathbb{R})$. Then there an orthogonal matrices P and two semi-positive definite matrix H, H' such that

$$A = HP = PH'.$$

Proof:

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Remark: If $A = HP$, then $A^T A = P^T H^T HP = P^T H^2 P$. Process to find P and H:

◇ Orthogonally diagonalizion: $A^T A = PDP^T$, where D is a diagonal elements with non-negative diagonal elements $\sigma_1, \ldots, \sigma_n$.

◇ $H = P\sqrt{D}P^T$, where $\sqrt{D}$ is the diagonal elements with non-negative diagonal elements $\sqrt{\sigma_1}, \ldots, \sqrt{\sigma_n}$.

Problem*: Find a process to find P and H' directly.

An Overview for Complex Square Matrix

	$\mathbb{C}$	$M_n(\mathbb{C})$	Eigenvalue
Unitary	$\bar{z}z = z\bar{z} = 1$ $\lvert z\rvert = 1$	$U^*U = UU^* = I_n$	$\lvert\lambda\rvert = 1.$
Hermitian	$\bar{z} = z$ $z \in \mathbb{R}$	$H^* = H$	$\lambda \in \mathbb{R}$
Semi-positive	$z \geq 0$	$\forall x \in \mathbb{C}^n,\ x^*Qx \geq 0$ $\exists B \in M_n(\mathbb{C}),\ Q = B^*B$	$\lambda \geq 0$
Orthogonal Projection	$z = 0$ or 1.	$E^2 = E = E^*$	$\lambda \in \{0, 1\}$
Rectangular Decomposition	$z = x + iy$ $(x, y \in \mathbb{R})$	$A = H + iK$ (H, K Hermitian)	
Polar Decomposition	$z = re^{i\theta}$ $(r \geq 0, 0 \leq \theta < 2\pi)$	$A = QU$ Q semi-positive, U unitary	