Discrete Mathematics: Lecture 20

Part III. Mathematical Logic

predicate logic, quantifiers, WFFs, from NL to WFFs, logic equivalence

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Spring Semester, 2024

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Predicate and Individual

Predicate谓词: describe the property of the subject term (in a sentence)

- A predicate is a function from a domain of individuals to $\{\mathbf{T}, \mathbf{F}\}$
- $\boldsymbol{n}$**-ary predicate**$_{n}$元谓词: a predicate on n individuals
  - I: "is an integer" // unary
  - G: "is greater than" //binary
- **Predicate constant**谓词常项: a concrete predicate // I, G
- **Predicate variable**谓词变项: a symbol that represents any predicate

Individual个体词: the object you are considering (in a sentence)

- "$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}$ is an integer"
- "e^{π} is greater than π^{e}"
  - **Individual Constant**个体常项: $\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}$, e^{π}, π^{e}
  - **Individual Variable**个体变项: x, y, z
  - **Domain**个体域: the set of all individuals in consideration

From Predicates to Propositions

Propositional function命题函数: $P(x_1, \dots, x_n)$, where P is an n-ary predicate

- $P(x, y)$:"x is greater than y"
- $P(x, y)$ gives a proposition when we assign values to x, y
  - $P(e^{\pi}, \pi^{e})$ is a proposition (a true proposition and hence has a truth value)
- $P(x, y)$ is not a proposition

EXAMPLE: p:"Alice's father is a doctor"; q:"Bob's father is a doctor"

- Individuals: Alice's father, Bob's father; Predicate D: "is a doctor"
- $p = D(\text{Alice's father}), q = D(\text{Bob's father})$

Function of Individuals: a map on the domain of individuals

- $f(x) = x$'s father
- $p = D\big(f(\text{Alice})\big); q = D(f(\text{Bob}))$

Universal Quantifier

DEFINITION: Let $P(x)$ be a propositional function. The **universal quantification**全称量化 of $P(x)$ is "$P(x)$ for all x in the domain".

- notation: $\forall x\ P(x)$; read as "for all x $P(x)$" or "for every x $P(x)$"
  - "$\forall$" is called the **universal quantifier**全称量词
  - "$\forall x\ P(x)$" is true iff $P(x)$ is true for every x in the domain
  - "$\forall x\ P(x)$" is false iff there is an x_0 in the domain such that $P(x_0)$ is false
    - **Counterexample**反例: an x_0 such that $P(x_0)$ is false

EXAMPLE: $P(n)$:"$n^2 + n + 41$ is a prime"

- When domain = natural numbers, "$\forall n P(n)$" is "for every natural number n, $n^2 + n + 41$ is a prime"
- When domain is $D = \{0,1, \dots , 39\}$, "$\forall n P(n)$" is "for every $n \in D$, $n^2 + n + 41$ is a prime"

REMARK: If the domain is empty, then "$\forall x\ P(x)$" is true for any P.

Existential Quantifier

DEFINITION: Let $P(x)$ be a propositional function. The **existential quantification**存在量化 of $P(x)$ is "there is an x in the domain such that $P(x)$"

- notation: $\exists x\ P(x)$; read as "for some $x\ P(x)$" or "there is an x s.t. $P(x)$"
  - "$\exists$" is called the **existential quantifier**存在量词
  - "$\exists x\ P(x)$" is true iff there is an x in the domain such that $P(x)$ is true
  - "$\exists x\ P(x)$" is false iff $P(x)$ is false for every x in the domain

EXAMPLE: $P(x)$:"$x^2 - x + 1 = 0$"

- "$\exists x\ P(x)$" is false when $D = \mathbb{R}$ and is true when $D = \mathbb{C}$

REMARK: If the domain is empty, then "$\exists x\ P(x)$" is false for any P.

REMARK: if not stated, the individual can be anything.

Binding Variables and Scope

DEFINITION: An individual variable x is **bound**约束的 if a quantifier ($\forall, \exists$) is used on x; otherwise, x is said to be **free**自由的**.**

- $\exists x(x + y = 1)$
  - x is bound and y is free
- **scope**辖域 of a quantifier: the part of a formula to which a quantifier is used
  - the scope of $\exists x$ in $\exists x(x + y = 1)$ is $(x + y = 1)$

Predicate Logic谓词逻辑**:** the area of logic that deals with predicates and quantifiers (a.k.a. **predicate calculus**)

- predicate logic is an extension of propositional logic

Well-Formed Formulas

Elements that may appear in Well-Formed Formulas合式公式**:**

- Propositional constants: **T,F**, $p, q, r, \dots$
- Propositional variables: $p, q, r, \dots$
- Logical Connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- Parenthesis: (,)
- Individual constants: $a, b, c, \dots$
- Individual variables: $x, y, z, \dots$
- Predicate constants: $P, Q, R, \dots$
- Predicate variables: $P, Q, R, \dots$
- Quantifiers: $\forall, \exists$
- Functions of individuals: $f, g, \dots$

Well-Formed Formulas

DEFINIITON: well-formed formulas合式公式/formulas

1) propositional constants, propositional variables, and propositional functions without connectives are WFFs
2) If A is a WFF, then $\neg A$ is also a WFF
3) If A, B are WFFs and there is no individual variable x which is bound in one of A, B but free in the other, then $(A \wedge B), (A \vee B), (A \rightarrow B), (A \leftrightarrow B)$ are WFFs.
4) If A is a WFF with a free individual variable x, then $\forall x\ A, \exists x\ A$ are WFFs.
5) WFFs can be constructed with 1)-4).
    - Example: $\forall x\ F(x) \vee G(x), \forall x P(y)$ are not WFFs
    - Example: $\exists x\ (A(x) \rightarrow \forall y\ B(x, y))$ is a WFF

Precedence: $\forall, \exists$ have higher precedence than $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

- $\forall x\ P(x) \rightarrow Q(y)$ means $\big(\forall x P(x)\big) \rightarrow Q(y)$, not $\forall x(P(x) \rightarrow Q(y))$

From Natural Language to WFFs

The Method of Translation:

- Introduce symbols to represent propositional constants, propositional variables, individual constants, individual variables, predicate constants, predicate variables, functions of individuals
- Construct WFFs with 1)-4) such that WFFs reflect the real meaning of the natural language

EXAMPLE: All irrational numbers are real numbers.

- Every irrational number is a real number.
- For every x, if x is an irrational number, then x is a real number.
  - $I(x) =$"x is an irrational number"
  - $R(x) =$"x is a real number"
    - Translation: $\forall x\ (I(x) \to R(x))$

From Natural Language to WFFs

EXAMPLE: Some real numbers are irrational numbers.

- There is a real number which is also an irrational number.
- There is an x such that x is a real number and also an irrational number.
  - $I(x) =$"x is an irrational number"
  - $R(x) =$"x is a real number"
    - Translation: $\exists x\ (R(x) \wedge I(x))$

EXAMPLE: There is a symbol that can not be understood by any person's brain.

- There is a symbol such that any person's brain can not understand it.
- There is an x such that x is a symbol and any person's brain can not understand x.
  - $S(x)$: "x is a symbol"
  - Translation: $\exists x\ (S(x) \wedge (\cdots))$

From Natural Language to WFFs

EXAMPLE: There is a symbol that can not be understood by any person's brain.

- Any person's brain can not understand x.
- For any y, if y is a person, then y's brain cannot understand x.
  - $P(y)$: "y is a person"
  - Translation: $\forall y\ (P(y) \to (\cdots))$
- y's brain cannot understand x
  - $U(z, x)$: "z can understand x"
  - $b(y)$ =the brain of y
    - Translation: $\neg U(b(y), x)$
- Translation: $\exists x\ \Big(S(x) \land \forall y\ \big(P(y) \to \neg U(b(y), x)\big)\Big)$

Interpretation

DEFINITION: an **interpretation**解释 requires one to (remove all uncertainty)

- assign a concrete proposition to every **proposition variable**
- assign a concrete predicate to every **predicate variable**
- restrict the domain of every **bound individual variable**
- assign a concrete individual to every **free individual variable**
- choose a concrete **function**, if there is any

EXAMPLE: $\exists x P(x) \rightarrow q$

- Domain of x={Alice, Bob, Eve}
- $P(x)$ =“x gets A+”
- q =“I get A+”
- If at least one of Alice, Bob, and Eve gets A+, then I get A+.

Types of WFFs

DEFINITION: A WFF is **logically valid**普遍有效 if it is **T** in every interpretation

- $\forall x\,(P(x) \vee \neg P(x))$ is logically valid

DEFINITION: A WFF is **unsatisfiable**不可满足 if it is **F** in every interpretation

- $\exists x\,(P(x) \wedge \neg P(x))$ is unsatisfiable

DEFINITION: A WFF is **satisfiable**可满足 if it is **T** in some interpretation

- $\forall x\,(x^2 > 0)$
  - true when domain= nonzero real numbers

THEOREM: Let A be any WFF. A is logically valid iff $\neg A$ is unsatisfiable.

Rule of Substitution: Let A be a tautology in propositional logic. If we substitute any propositional variable in A with an arbitrary WFF from predicate logic, then we get a logically valid WFF.

- $p \vee \neg p$ is a tautology; hence, $P(x) \vee \neg P(x)$ is logically valid

Logical Equivalence

DEFINITION: Two WFFs A,B are **logically equivalent**等值 if they always have the same truth value in every interpretation.

- notation: $A \equiv B$; example: $\forall x\, P(x) \land \forall x\, Q(x) \equiv \forall x\, (P(x) \land Q(x))$

THEOREM: $A \equiv B$ iff $A \leftrightarrow B$ is logically valid.

- $A \equiv B$
- iff A, B have the same truth value in every interpretation I
- iff $A \leftrightarrow B$ is true in every interpretation I
- iff $A \leftrightarrow B$ is logically valid

THEOREM: $A \equiv B$ iff $A \rightarrow B$ and $B \rightarrow A$ are both logically valid.

- $A \leftrightarrow B \equiv (A \rightarrow B) \land (B \rightarrow A)$

Rule of Substitution

METHOD*:* Applying the rule of substitution to the logical equivalences in propositional logic, we get logical equivalences in predicate logic.

$P \vee Q \equiv Q \vee P$	$A(x) \vee B(y) \equiv B(y) \vee A(x)$
$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$	$\big(A(x) \wedge B(y)\big) \wedge c \equiv A(x) \wedge (B(y) \wedge c)$
$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	$A(x) \wedge (B(y) \vee c) \equiv \big(A(x) \wedge B(y)\big) \vee (A(x) \wedge c)$
$P \wedge (P \vee Q) \equiv P$	$A(x) \wedge \big(A(x) \vee B(y)\big) \equiv A(x)$
$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$	$\neg\big(A(x) \wedge B(y)\big) \equiv \neg A(x) \vee \neg B(y)$
$P \rightarrow Q \equiv \neg P \vee Q$	$A(x) \rightarrow (\forall y\, B(y)) \equiv \neg A(x) \vee \big(\forall y\, B(y)\big)$
$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$	$A(x) \leftrightarrow c \equiv (A(x) \rightarrow c) \wedge (c \rightarrow A(x))$

De Morgan's Laws for Quantifiers

THEOREM: $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$

- Show that $\neg\forall x\ P(x) \rightarrow \exists x\ \neg P(x)$ is logically valid
  - Suppose that $\neg\forall x\ P(x)$ is $\mathbf{T}$ in an interpretation I
  - $\forall x\ P(x)$ is $\mathbf{F}$ in I
  - There is an x_0 such that $P(x_0)$ is $\mathbf{F}$ in I
  - There is an x_0 such that $\neg P(x_0)$ is $\mathbf{T}$ in I
  - $\exists x\ \neg P(x)$ is $\mathbf{T}$ in I
- Show that $\exists x\ \neg P(x) \rightarrow \neg\forall x\ P(x)$ is logically valid
  - Suppose that $\exists x \neg P(x)$ is $\mathbf{T}$ in an interpretation I
  - There is an x_0 such that $\neg P(x_0)$ is $\mathbf{T}$ in I
  - There is an x_0 such that $P(x_0)$ is $\mathbf{F}$ in I
  - $\forall x P(x)$ is $\mathbf{F}$ in I
  - $\neg\forall x P(x)$ is $\mathbf{T}$ in I

THEOREM: $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$.