Discrete Mathematics: Lecture 18

Part III. Mathematical Logic

logically equivalent, rule of replacement, tautological implications

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Truth Table & Types of WFFs (Review)

DEFINITION: Let F be a WFF of $p_1, \ldots, p_n$, n propositional variables

- A **truth assignment** (真值指派) for F is a map $\alpha: \{p_1, \ldots, p_n\} \rightarrow \{\mathbf{T}, \mathbf{F}\}$.
  - There are 2^n different truth assignments.

Tautology(重言式): a WFF whose truth value is **T** for all truth assignment

Contradiction(矛盾式): a WFF whose truth value is **F** for all truth assignment

Contingency(可能式): neither tautology nor contradiction

Satisfiable(可满足的):a WFF is satisfiable if it is true for at least one truth assignment

Rule of Substitution: (代入规则) Let B be a formula obtained from a tautology A by substituting a propositional variable in A with an arbitrary formula. Then B must be a tautology.

Logically Equivalent (Review)

DEFINITION: Let A and B be WFFs in propositional variables $p_1, \dots, p_n$.

- A and B are **logically equivalent** (等值) if they always have the same truth value for every truth assignment (of $p_1, \dots, p_n$)
  - Notation: $A \equiv B$

THEOREM: $A \equiv B$ if and only if $A \leftrightarrow B$ is a tautology.

THEOREM: $A \equiv A$; If $A \equiv B$, then $B \equiv A$; If $A \equiv B, B \equiv C$, then $A \equiv C$

QUESTION: How to prove $A \equiv B$?

Proving $A \equiv B$

Method 1: Show that A, B have the same truth table.

Method 2: Rule of Replacement: (替换规则) Replacing a sub-formula in a formula F with a logically equivalent sub-formula gives a formula logically equivalent o the formula F.

Logical Equivalences

Name	Logical Equivalences	NO.
Double Negation Law 双重否定律	$\neg(\neg P) \equiv P$	1
Identity Laws 同一律	$P \wedge \mathbf{T} \equiv P$	2
	$P \vee \mathbf{F} \equiv P$	3
Idempotent Laws 等幂律	$P \vee P \equiv P$	4
	$P \wedge P \equiv P$	5
Domination Laws 零律	$P \vee \mathbf{T} \equiv \mathbf{T}$	6
	$P \wedge \mathbf{F} \equiv \mathbf{F}$	7
Negation Laws 补余律	$P \vee \neg P \equiv \mathbf{T}$	8
	$P \wedge \neg P \equiv \mathbf{F}$	9

Logical Equivalences

Name	Logical Equivalences	NO.
Commutative Laws 交换律	$P \vee Q \equiv Q \vee P$	10
	$P \wedge Q \equiv Q \wedge P$	11
Associative Laws 结合律	$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$	12
	$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$	13
Distributive Laws 分配律	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	14
	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	15
De Morgan's Laws 摩根律	$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$	16
	$\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$	17
Absorption Laws 吸收律	$P \vee (P \wedge Q) \equiv P$	18
	$P \wedge (P \vee Q) \equiv P$	19

Logical Equivalences

Name	Logical Equivalences	NO.
Laws Involving Implication $\rightarrow$	$P \rightarrow Q \equiv \neg P \vee Q$	20
	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$	21
	$(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \vee Q) \rightarrow R$	22
	$P \rightarrow (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$	23
	$P \rightarrow (Q \rightarrow R) \equiv Q \rightarrow (P \rightarrow R)$	24
Laws Involving Bi-Implication $\leftrightarrow$	$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$	25
	$P \leftrightarrow Q \equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$	26
	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	27
	$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$	28

Proving Types of WFFs

EXAMPLE: Show the following WFF is a tautology or a contradiction:

1. $p \rightarrow (p \vee \neg q \vee r)$

Hint: $\neg p \vee p \vee \neg q \vee r \Leftrightarrow T \vee (\neg q \vee r) \Leftrightarrow T$

2. $\neg((p \vee q) \wedge \neg p \rightarrow q)$

Hint: $\neg\big((p \wedge \neg p) \vee (q \wedge \neg p) \rightarrow q\big)$

$\Leftrightarrow \neg\big((q \wedge \neg p) \rightarrow q\big) \Leftrightarrow \neg(\neg(q \wedge \neg p) \vee q) \Leftrightarrow q \wedge \neg p \wedge \neg q \Leftrightarrow F$

Proving Types of WFFs

EXAMPLE: Consider the following task: A company plans to relocate John or Mary to its SF office. If John is relocated, then Ben needs to do extra work. If Mary is relocated, then Lisa will be relocated too. It turns out Ben do not need to do extra work. What is the company's arrangement for its employees?

$$A = ((p \vee q) \wedge (p \to r) \wedge (q \to s) \wedge \neg r$$

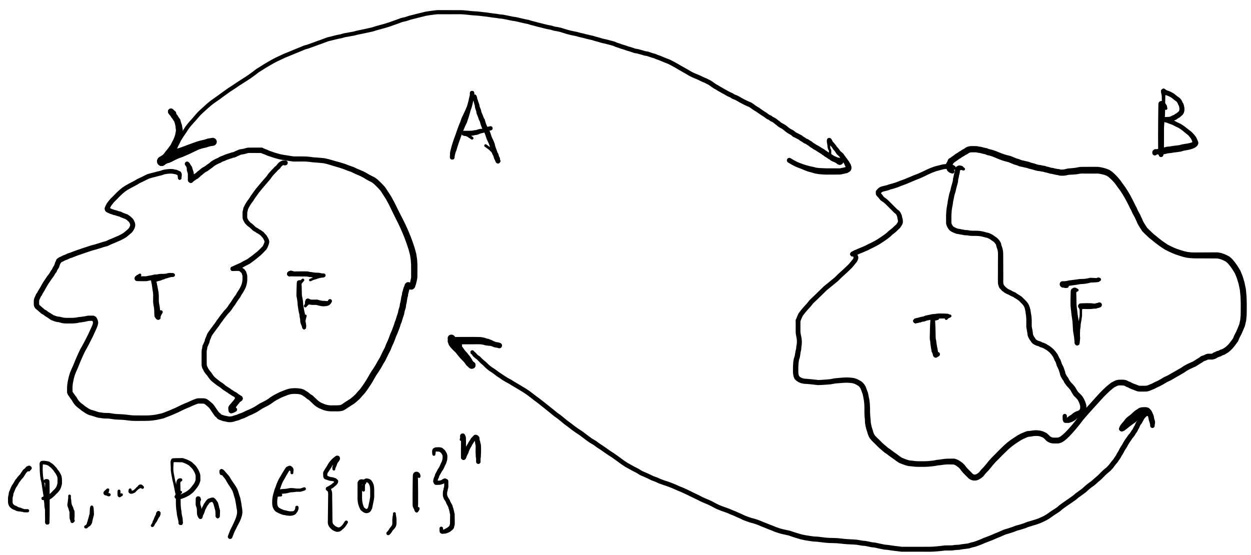
Hint:

$$\begin{aligned}
&\Longleftrightarrow (p \vee q) \wedge (\neg p \vee r) \wedge (\neg q \vee s) \wedge \neg r \\
&\Longleftrightarrow ((p \wedge \neg q) \vee (q \wedge \neg q) \vee (p \wedge s) \vee (q \wedge s)) \wedge ((\neg p \wedge \neg r) \vee (r \wedge \neg r)) \\
&\Longleftrightarrow ((p \wedge \neg q) \vee (p \wedge s) \vee (q \wedge s)) \wedge (\neg p \wedge \neg r) \\
&\Longleftrightarrow (q \wedge s) \wedge (\neg p \wedge \neg r) \Longleftrightarrow \neg p \wedge q \wedge \neg r \wedge s
\end{aligned}$$

Logically Equivalent

THEOREM: Let $A^{-1}(\mathbf{T})$ be the set of truth assignments such that A is true. Then $A \equiv B$ if and only if $A^{-1}(\mathbf{T}) = B^{-1}(\mathbf{T})$.

- $A \equiv B$ if and only if $A^{-1}(\mathbf{F}) = B^{-1}(\mathbf{F})$



Proving $A \equiv B$

EXAMPLE: $P \wedge Q \equiv Q \wedge P$ **//commutative law**

- Idea: Show that $A^{-1}(\mathbf{T}) = B^{-1}(\mathbf{T})$.
- $A = P \wedge Q; B = Q \wedge P$
  - $A = \mathbf{T}$ if and only if $(P, Q) = (\mathbf{T}, \mathbf{T})$
    - $A^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T})\}$
  - $B = \mathbf{T}$ if and only if $(Q, P) = (\mathbf{T}, \mathbf{T})$
    - $B^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T})\}$
- $A^{-1}(\mathbf{T}) = B^{-1}(\mathbf{T})$
- $A \equiv B$

REMARK: $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$ can be shown similarly.

- **Associative law**

Proving $A \equiv B$

EXAMPLE: $P \vee Q \equiv Q \vee P$ **//commutative law**

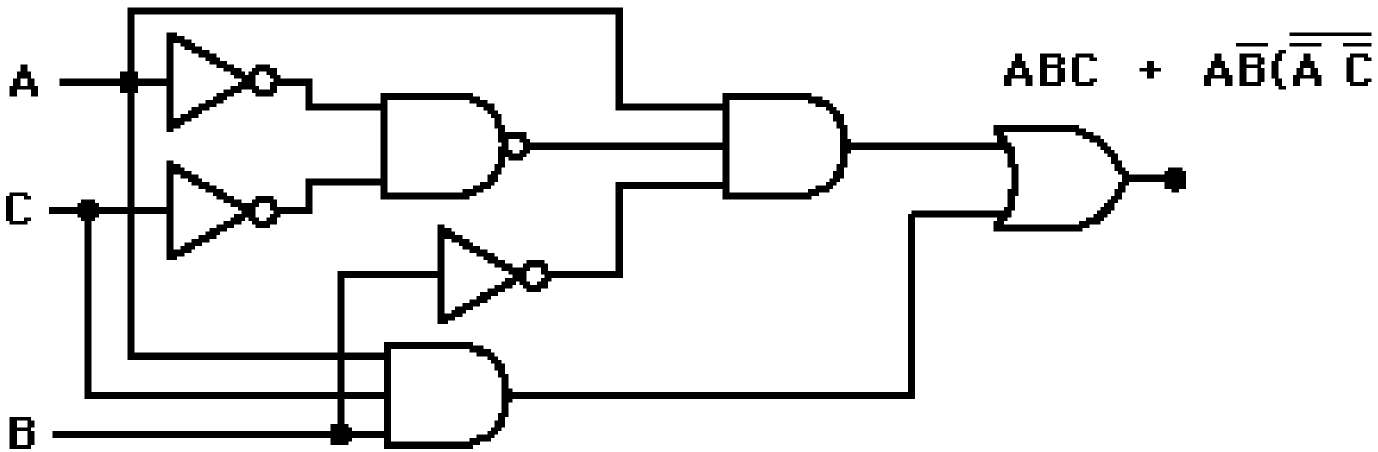
- Idea: Show that $A^{-1}(\mathbf{F}) = B^{-1}(\mathbf{F})$.
- $A = P \vee Q; B = Q \vee P$
  - $A = \mathbf{F}$ if and only if $(P, Q) = (\mathbf{F}, \mathbf{F})$
    - $A^{-1}(\mathbf{F}) = \{(\mathbf{F}, \mathbf{F})\}$
  - $B = \mathbf{F}$ if and only if $(Q, P) = (\mathbf{F}, \mathbf{F})$
    - $B^{-1}(\mathbf{F}) = \{(\mathbf{F}, \mathbf{F})\}$
- $A^{-1}(\mathbf{F}) = B^{-1}(\mathbf{F})$
- $A \equiv B$

REMARK: $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$ can be shown similarly.

- **Associative law**

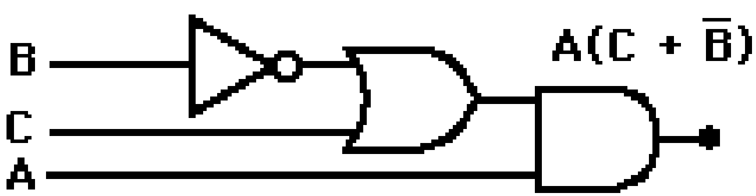
Example: Simplify Logic Circuits

A	B	C	Out
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1



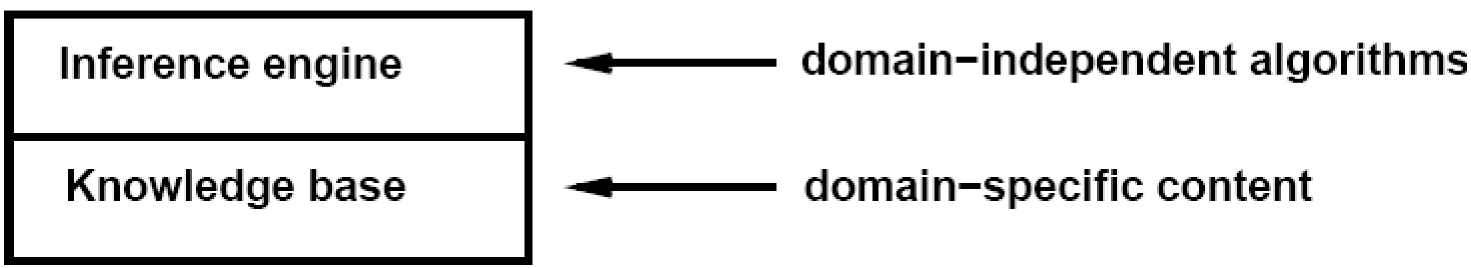
It can be simplified by:

$ABC + A\bar{B}(\bar{\bar{A}} + \bar{\bar{C}})$	DeMorgan's theorem
$ABC + A\bar{B}A + A\bar{B}C$	sum of products form
$ABC + A\bar{B} + A\bar{B}C$	BA=AB and AA = A
$AC(B + \bar{B}) + A\bar{B}$	
$AC + A\bar{B}$	$B + \bar{B} = 1$
$A(C + \bar{B})$	



Logic-based Inference (Review)

- Logic (Knowledge-Based) Inference
  - Knowledge base
    - set of sentences in a formal language to represent knowledge about the “world”
  - Inference engine
    - answers any answerable question following the knowledge base



Tautological Implications

DEFINITION: Let A and B be WFFs in propositional variables $p_1, \dots, p_n$.

- A **tautologically implies** (重言蕴涵) B if every truth assignment that causes A to be true causes B to be true.
  - Notation: $A \Rightarrow B$, called a **tautological implication**
  - $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$; $B^{-1}(\mathbf{F}) \subseteq A^{-1}(\mathbf{F})$

THEOREM: $A \Rightarrow B$ iff $A \rightarrow B$ is a tautology.

- $A \Rightarrow B$ iff $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$ iff $A \rightarrow B$ is a tautology

THEOREM: $A \Rightarrow B$ iff $A \wedge \neg B$ is a contradiction.

- $A \rightarrow B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$

Proving $\boldsymbol{A} \Rightarrow \boldsymbol{B}$: (1) $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$; (2) $B^{-1}(\mathbf{F}) \subseteq A^{-1}(\mathbf{F})$; (3) $A \rightarrow B$ is a tautology; (4) $A \wedge \neg B$ is a contradiction

Proving $A \Rightarrow B$

EXAMPLE: Show the tautological implication "$p \wedge (p \to q) \Rightarrow q$".

- Let $A = p \wedge (p \to q)$; $B = q$. Need to show that "$A \Rightarrow B$"
- $A^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T})\}$; $B^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T}), (\mathbf{F}, \mathbf{T})\}$: $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$.

p	q	$p \to q$	A	B
T	**T**	**T**	**<u>T</u>**	**<u>T</u>**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**F**	**F**

- $$\begin{aligned} A \to B &\equiv \neg\big(p \wedge (p \to q)\big) \vee q \\ &\equiv \big(\neg p \vee \neg(p \to q)\big) \vee q \\ &\equiv (\neg p \vee q) \vee \neg(p \to q) \\ &\equiv (p \to q) \vee \neg(p \to q) \\ &\equiv \mathbf{T} \end{aligned}$$
- $$\begin{aligned} A \wedge \neg B &\equiv \big(p \wedge (p \to q)\big) \wedge \neg q \\ &\equiv (\neg q \wedge p) \wedge (p \to q) \\ &\equiv \neg(p \to q) \wedge (p \to q) \\ &\equiv \mathbf{F} \end{aligned}$$

Tautological Implications

Name	Tautological Implication	NO.
Conjunction(合取)	$(P) \wedge (Q) \Rightarrow P \wedge Q$	1
Simplification(化简)	$P \wedge Q \Rightarrow P$	2
Addition(附加)	$P \Rightarrow P \vee Q$	3
Modus ponens(假言推理)	$P \wedge (P \rightarrow Q) \Rightarrow Q$	4
Modus tollens(拒取)	$\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$	5
Disjunctive syllogism(析取三段论)	$\neg P \wedge (P \vee Q) \Rightarrow Q$	6
Hypothetical syllogism(假言三段论)	$(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R)$	7
Resolution (归结)	$(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$	8

Proofs for 5 and 6

EXAMPLE: $\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$

- $A = \neg Q \wedge (P \rightarrow Q), B = \neg P.$
- $A \rightarrow B \equiv \neg\big(\neg Q \wedge (P \rightarrow Q)\big) \vee \neg P$

$$\equiv \big(Q \vee \neg(P \rightarrow Q)\big) \vee \neg P$$

$$\equiv (\neg P \vee Q) \vee \neg(P \rightarrow Q)$$

$$\equiv \mathbf{T}$$

(To be continued …)