Specific Instructions for students:

• The time duration for this exam is 100 **minutes**.

• Computers and calculators are prohibited in the exam.

• Answers can be written in **either Chinese or English**.

★ For problems 3-8, please show details of calculations or deductions. A correct answer with no details can not earn points.

Notations and conventions:

• $\mathbb{R}$ is the set of real numbers.

• I denotes an identity matrix of suitable order.

• 0 or $\mathbf{0}$ may denote the number zero, a zero vector, or a zero matrix.

• $\mathbb{M}_{m\times n}$ is the vector space of all $m \times n$ matrices (with real entries).

• For a square matrix $A = [a_{ij}]$, M_{ij} is the minor of entry a_{ij}; C_{ij} is the cofactor of entry a_{ij}; $adj(A)$ is the adjoint matrix of A.

• Give a matrix A, we denote by $null(A)$, $row\,(A)$, $col(A)$ the null space, row space, column space of A respectively. And $nullity\,(A)$ and $r(A)$ denotes the nullity and rank of A.

1. Multiple choice questions.

a). (5 *points*) Which of the following sets are subspaces of the given vector space? ( )

(A) $\{(x_1, x_2, x_3) \subseteq \mathbb{R}^3 : x_1 + 5x_2 + 3x_3 = 0, x_2 - 2x_3 = 0\} \subseteq \mathbb{R}^3$.

(B) $\{(x_1, x_2, x_3) \subseteq \mathbb{R}^3 : x_1 > x_2 > x_3\} \subseteq \mathbb{R}^3$.

(C) $\{(x^2, x, 1) \subseteq \mathbb{R}^3 : x \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

(D) $\{A \in \mathbb{M}_{3\times3} : A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{M}_{3\times3}$, where $\mathbf{x} = [1, 0, 1]^T$.

b). (5 *points*) Let $A \in \mathbb{M}_{n\times n}$ and $B \in \mathbb{M}_{n\times n}$. Determine which of the following statements are true. ( )

(A) If $\det(A - B) = 0$, then $A = B$

(B) If $A^2 = B^2$, then $A = B$ or $A = -B$

(C) If $\det(A - B) = 1$ and there is an $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = B\mathbf{x}$, then $\mathbf{x} = \mathbf{0}$

(D) If $\det(A - B) = 1$, then $dim\,(row\,(A - B)) = n$

c). (5 *points*) Let $U, W \subseteq V$ be 4-dimensional subspaces of a 6-dimensional vector space V, which of the following can not be the possible dimension of $U \cap W$? ( )

(A) 4 (B) 3 (C) 2 (D) 1

2. Fill in the blanks.

a.) (5 *points*) Suppose that $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (0, 1, -2a)$, $\mathbf{v}_3 = (a, 0, 1)$ form a basis for $\mathbb{R}^3$.

If $(-2, -7, -12)$ has coordinates $(3a, -7-3a, a)$ relative to this basis, then a=________.

b). (5 *points*) Suppose that A is a 3×3 matrix with $\det(A) = -3$, then $\det(-2\,adj(A)) =$ ________.

c.) (5 *points*) Let $A = [a_{ij}]$ be a square matrix of size n with all its entries being zero except the $(i, i+1)$-th entries $a_{i,\,i+1}$ which equals 1 for $i = 1, \cdots, n-1$. Then $r(A^{n-1}) =$ ________.

3. (10 *points*) Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -6 & 8 \\ 4 & 5 \\ 2 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}$. Find a matrix X such that $A(X - B) = C$.

4. (10 *points*) Let A be a square matrix of size n with cofactor matrix $C = [C_{ij}]_{1 \le i,j \le n}$. Suppose that the sum of the entries of A in the ith row is equal to i and suppose that the determinant of A is 1. Compute the value of $C_{11} + 2C_{21} + 3C_{31} + \cdots + nC_{n1}$

5. Let W be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

a) (7 *points*) Compute a basis of W and the dimension of W.

b) (8 *points*) Let U be the set of all vectors in $\mathbb{R}^4$ that are orthogonal to all vectors of W (the orthogonal complement of W). Compute a basis and the dimension of U.

6. Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are three linearly independent vectors in $\mathbb{R}^3$. Let $\mathcal{S}$ and $\mathcal{S}'$ be two sets of vectors in $\mathbb{R}^3$ that are given respectively by

$$\mathcal{S} = \{\mathbf{v}_1 + \mathbf{v}_2,\ \mathbf{v}_2 + \mathbf{v}_3,\ \mathbf{v}_1 + \mathbf{v}_3\}; \quad \mathcal{S}' = \{2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3,\ \mathbf{v}_1 + \mathbf{v}_3,\ 2\mathbf{v}_1 + 2\mathbf{v}_2 + 2\mathbf{v}_3\}$$

a) (5 *points*) Verify that both $\mathcal{S}$ and $\mathcal{S}'$ are basis of $\mathbb{R}^3$.

b) (5 *points*) Find the transition matrix from $\mathcal{S}'$ to $\mathcal{S}$.

c) (5 *points*) Suppose that $\mathbf{u} \in \mathbb{R}^3$ has coordinates $[\mathbf{u}]_{\mathcal{S}} = [1, 2, 1]^T$ relative to $\mathcal{S}$, find its coordinates $[\mathbf{u}]_{\mathcal{S}'}$ relative to $\mathcal{S}'$.

7. (10 *points*) Let $\mathbf{v}_1 = (1, 0, 2, 3)$, $\mathbf{v}_2 = (-3, -2, 0, 1)$, $\mathbf{v}_3 = (0, 1, -3, 2)$, $\mathbf{u} = (1, 0, 1, 0)$, $\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$ be vectors in $\mathbb{R}^4$. Suppose that the orthogonal projections of $\mathbf{w}$ along $\mathbf{v}_i$ are the same as that of $\mathbf{u}$ along $\mathbf{v}_i$ for $i = 1, 2, 3$, that is

$$proj_{\mathbf{v_i}}\mathbf{u} = proj_{\mathbf{v}_i}\mathbf{w} \quad i = 1, 2, 3,$$

find the length of $\mathbf{w}$.

8. (10 *points*) Let $\mathbf{u}$ and $\mathbf{v}$ be two linearly independent vectors in $\mathbb{R}^3$. Viewing $\mathbf{u}$, $\mathbf{v}$ as matrices of size 3×1, and compute the rank of A and B that are given by

$$A = \mathbf{u}\mathbf{v}^T \qquad\qquad B = \begin{bmatrix} \mathbf{u}^T\mathbf{u} & \mathbf{u}^T\mathbf{v} \\ \mathbf{v}^T\mathbf{u} & \mathbf{v}^T\mathbf{v} \end{bmatrix}$$