# Discrete Mathematics: Lecture 21

## Part III. Mathematical Logic

### logic equivalence, tautological implication, building arguments

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# Interpretation

**DEFINITION:** an **interpretation**解释 requires one to (remove all uncertainty)

- assign a concrete proposition to every **proposition variable**
- assign a concrete predicate to every **predicate variable**
- restrict the domain of every **bound individual variable**
- assign a concrete individual to every **free individual variable**
- choose a concrete **function**, if there is any

**EXAMPLE**: $\exists x P(x) \to q$

- Domain of $x$={Alice, Bob, Eve}
- $P(x) =$“$x$ gets A+”
- $q =$“I get A+”
- If at least one of Alice, Bob, and Eve gets A+, then I get A+.

# Types of WFFs

**DEFINITION:** A WFF is **logically valid**普遍有效 if it is **T** in every interpretation

- $\forall x\ (P(x) \lor \neg P(x))$ is logically valid

**DEFINITION:** A WFF is **unsatisfiable**不可满足 if it is **F** in every interpretation

- $\exists x\ (P(x) \land \neg P(x))$ is unsatisfiable

**DEFINITION:** A WFF is **satisfiable**可满足 if it is **T** in some interpretation

- $\forall x\ (x^2 > 0)$
  - true when domain= nonzero real numbers

**THEOREM:** Let $A$ be any WFF. $A$ is logically valid iff $\neg A$ is unsatisfiable.

**Rule of Substitution**: Let $A$ be a tautology in propositional logic. If we substitute any propositional variable in $A$ with an arbitrary WFF from predicate logic, then we get a logically valid WFF.

- $p \lor \neg p$ is a tautology; hence, $P(x) \lor \neg P(x)$ is logically valid

# Logical Equivalence

**DEFINITION**: Two WFFs $A$,$B$ are **logically equivalent**等值 if they always have the same truth value in every interpretation.

- notation: $A \equiv B$; example: $\forall x\, P(x) \wedge \forall x\, Q(x) \equiv \forall x\, (P(x) \wedge Q(x))$

**THEOREM**: $A \equiv B$ iff $A \leftrightarrow B$ is logically valid.

- $A \equiv B$
- iff $A, B$ have the same truth value in every interpretation $I$
- iff $A \leftrightarrow B$ is true in every interpretation $I$
- iff $A \leftrightarrow B$ is logically valid

**THEOREM**: $A \equiv B$ iff $A \rightarrow B$ and $B \rightarrow A$ are both logically valid.

- $A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A)$

# Rule of Substitution

**METHOD***:* Applying the rule of substitution to the logical equivalences in propositional logic, we get logical equivalences in predicate logic.

| | |
|---|---|
| $P \vee Q \equiv Q \vee P$ | $A(x) \vee B(y) \equiv B(y) \vee A(x)$ |
| $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ | $\big(A(x) \wedge B(y)\big) \wedge c \equiv A(x) \wedge (B(y) \wedge c)$ |
| $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ | $A(x) \wedge (B(y) \vee c) \equiv \big(A(x) \wedge B(y)\big) \vee (A(x) \wedge c)$ |
| $P \wedge (P \vee Q) \equiv P$ | $A(x) \wedge \big(A(x) \vee B(y)\big) \equiv A(x)$ |
| $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ | $\neg\big(A(x) \wedge B(y)\big) \equiv \neg A(x) \vee \neg B(y)$ |
| $P \to Q \equiv \neg P \vee Q$ | $A(x) \to (\forall y\, B(y)) \equiv \neg A(x) \vee \big(\forall y\, B(y)\big)$ |
| $P \leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$ | $A(x) \leftrightarrow c \equiv (A(x) \to c) \wedge (c \to A(x))$ |

# De Morgan's Laws for Quantifiers

**THEOREM**: $\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$

- <u>Show that $\neg\forall x\ P(x) \rightarrow \exists x\ \neg P(x)$ is logically valid</u>
  - Suppose that $\neg\forall x\ P(x)$ is $\mathbf{T}$ in an interpretation $I$
  - $\forall x\ P(x)$ is $\mathbf{F}$ in $I$
  - There is an $x_0$ such that $P(x_0)$ is $\mathbf{F}$ in $I$
  - There is an $x_0$ such that $\neg P(x_0)$ is $\mathbf{T}$ in $I$
  - $\exists x\ \neg P(x)$ is $\mathbf{T}$ in $I$
- <u>Show that $\exists x\ \neg P(x) \rightarrow \neg\forall x\ P(x)$ is logically valid</u>
  - Suppose that $\exists x \neg P(x)$ is $\mathbf{T}$ in an interpretation $I$
  - There is an $x_0$ such that $\neg P(x_0)$ is $\mathbf{T}$ in $I$
  - There is an $x_0$ such that $P(x_0)$ is $\mathbf{F}$ in $I$
  - $\forall x P(x)$ is $\mathbf{F}$ in $I$
  - $\neg\forall x P(x)$ is $\mathbf{T}$ in $I$

**THEOREM**: $\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$.

# De Morgan's Laws for Quantifiers

**EXAMPLE**: $R(x)$:"$x$ is a real number"; $Q(x)$:"$x$ is a rational number"

- $\neg\forall x\ (R(x) \rightarrow Q(x))$
  - Not all real numbers are rational numbers
- Negation: $\exists x\ \neg\big(R(x) \rightarrow Q(x)\big) \equiv \exists x\ (R(x) \wedge \neg Q(x))$
  - There is a real number which is not rational

**EXAMPLE**: Let the domain be the set of all real numbers. Let $Q(x)$:"$x$ is a rational number" and $I(x)$:"$x$ is an irrational number"

- $\neg\exists x\ (Q(x) \wedge I\ (x))$
  - No real number is both rational and irrational.
- Negation: $\forall x\ \neg\big(Q(x) \wedge I(x)\big) \equiv \forall x\ (\neg Q(x) \vee \neg I(x))$
  - Any real number is either not rational or not irrational.

# Distributive Laws for Quantifiers

**THEOREM**: $\forall x\,\big(P(x) \wedge Q(x)\big) \equiv \forall x\,P(x) \wedge \forall x\,Q(x)$

- Show that $\forall x\,\big(P(x) \wedge Q(x)\big) \rightarrow \forall x\,P(x) \wedge \forall x\,Q(x)$ is logically valid
  - Suppose that $\forall x\,\big(P(x) \wedge Q(x)\big)$ is $\mathbf{T}$ in an interpretation $I$
  - $P(x) \wedge Q(x)$ is $\mathbf{T}$ for every $x$ in $I$
  - $P(x)$ is $\mathbf{T}$ for every $x$ in $I$ and $Q(x)$ is $\mathbf{T}$ for every $x$ in $I$
  - $\forall x\,P(x)$ is $\mathbf{T}$ in $I$ and $\forall x\,Q(x)$ is $\mathbf{T}$ in $I$
  - $\forall x\,P(x) \wedge \forall x\,Q(x)$ is $\mathbf{T}$ in $I$
- Show that $\forall x\,P(x) \wedge \forall x\,Q(x) \rightarrow \forall x\,\big(P(x) \wedge Q(x)\big)$ is logically valid.
  - Suppose that $\forall x\,P(x) \wedge \forall x\,Q(x)$ is $\mathbf{T}$ in an interpretation $I$
  - $\forall x\,P(x)$ is $\mathbf{T}$ in $I$ and $\forall x\,Q(x)$ is $\mathbf{T}$ in $I$
  - $P(x)$ is $\mathbf{T}$ for every $x$ in $I$ and $Q(x)$ is $\mathbf{T}$ for every $x$ in $I$
  - $P(x) \wedge Q(x)$ is $\mathbf{T}$ for every $x$ in $I$
  - $\forall x\,\big(P(x) \wedge Q(x)\big)$ is $\mathbf{T}$ in $I$

**THEOREM**: $\exists x\,\big(P(x) \vee Q(x)\big) \equiv \exists x\,P(x) \vee \exists x\,Q(x)$.

# Tautological Implication

**DEFINITION:** Let $A$ and $B$ be WFFs in predicate logic. $A$ **tautologically implies**(重言蕴涵) $B$ if every interpretation that causes $A$ to be true causes $B$ to be true.

- notation: $A \Rightarrow B$, called a **tautological implication**(重言蕴涵)

**THEOREM:** $A \Rightarrow B$ iff $A \rightarrow B$ is logically valid.

- $A \Rightarrow B$
- iff every interpretation that causes $A$ to be true causes $B$ to be true
- iff there is no interpretation such that $(A, B) = (\mathbf{T}, \mathbf{F})$
- Iff $A \rightarrow B$ is true in every interpretation
- iff $A \rightarrow B$ is logically valid

**THEOREM:** $A \Rightarrow B$ iff $A \land \neg B$ is unsatisfiable.

- $A \rightarrow B \equiv \neg A \lor B \equiv \neg(A \land \neg B)$

# Rule of Substitution

| Name | Tautological Implication | NO. |
|---|---|---|
| Conjunction(合取) | $(P) \wedge (Q) \Rightarrow P \wedge Q$ | 1 |
| Simplification(化简) | $P \wedge Q \Rightarrow P$ | 2 |
| Addition(附加) | $P \Rightarrow P \vee Q$ | 3 |
| Modus ponens(假言推理) | $P \wedge (P \rightarrow Q) \Rightarrow Q$ | 4 |
| Modus tollens(拒取) | $\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$ | 5 |
| Disjunctive syllogism(析取三段论) | $\neg P \wedge (P \vee Q) \Rightarrow Q$ | 6 |
| Hypothetical syllogism(假言三段论) | $(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R)$ | 7 |
| Resolution (归结) | $(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$ | 8 |

**EXAMPLE:** $P \wedge (P \rightarrow Q) \Rightarrow Q$ is a TI in propositional logic.

- $A(x) \wedge \big(A(x) \rightarrow B(y)\big) \Rightarrow B(y)$ must be a TI in predicate logic.
  - Rule of substitution: let $P = A(x)$ and $Q = B(y)$

# Tautological Implications

| | |
|---|---|
| *a.* | $\forall x P(x) \vee \forall x\, Q(x) \Rightarrow \forall x\, (P(x) \vee Q(x))$ |
| *b.* | $\exists x \big(P(x) \wedge Q(x)\big) \Rightarrow \exists x\, P(x) \wedge \exists x Q(x)$ |
| *c.* | $\forall x\, \big(P(x) \rightarrow Q(x)\big) \Rightarrow \forall x P(x) \rightarrow \forall x\, Q(x)$ |
| *d.* | $\forall x\, \big(P(x) \rightarrow Q(x)\big) \Rightarrow \exists x\, P(x) \rightarrow \exists x\, Q(x)$ |
| *e.* | $\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \forall x\, P(x) \leftrightarrow \forall x\, Q(x)$ |
| *f.* | $\forall x\, \big(P(x) \leftrightarrow Q(x)\big) \Rightarrow \exists x\, P(x) \leftrightarrow \exists x\, Q(x)$ |
| *g.* | $\forall x\, \big(P(x) \rightarrow Q(x)\big) \wedge \forall x\, \big(Q(x) \rightarrow R(x)\big) \Rightarrow \forall x\, (P(x) \rightarrow R(x))$ |
| *h.* | $\forall x\, \big(P(x) \rightarrow Q(x)\big) \wedge P(a) \Rightarrow Q(a)$ |

**Additional rule:** Conclusion Premise (附加前提)

- The following two tautological implications are equivalent
  - $P \Rightarrow A \rightarrow B$
  - $P \wedge A \Rightarrow B$

# Examples

**EXAMPLE**: $\underline{\forall x \left(P(x) \to Q(x)\right) \land P(a)} \Rightarrow \underline{Q(a)}$

- Suppose that the left hand side is true in an interpretation $I$ (domain=$D$)
  - $\forall x \left(P(x) \to Q(x)\right)$ is $\mathbf{T}$ and $P(a)$ is $\mathbf{T}$
    - $P(a) \to Q(a)$ is $\mathbf{T}$ and $P(a)$ is $\mathbf{T}$
      - $Q(a)$ is $\mathbf{T}$ in $I$.

**EXAMPLE**: Tautological implication in the following proof?

- All rational numbers are real numbers $\boxed{\forall x\left(P(x) \to Q(x)\right)}$
- 1/3 is a rational number $\boxed{P(1/3)}$
- 1/3 is a real number $\boxed{Q(1/3)}$
  - $P(x)$ =“$x$ is a rational number”
  - $Q(x)$ =“$x$ is a real number”
  - rule of inference: $\forall x \left(P(x) \to Q(x)\right) \land P(1/3) \Rightarrow Q(1/3)$

# Examples

**EXAMPLE**: $\underline{\forall x\,\big(P(x) \to Q(x)\big) \land \forall x\,\big(Q(x) \to R(x)\big)} \Rightarrow \underline{\forall x\,(P(x) \to R(x))}$

- Suppose that the left hand side is $\mathbf{T}$ in an interpretation $I$ (domain=$D$)
  - $\forall x\,\big(P(x) \to Q(x)\big)$ is $\mathbf{T}$ and $\forall x\,\big(Q(x) \to R(x)\big)$ is $\mathbf{T}$
  - $P(x) \to Q(x)$ is $\mathbf{T}$ for all $x \in D$ and $Q(x) \to R(x)$ is $\mathbf{T}$ for all $x \in D$
    - $P(x) \to R(x)$ is $\mathbf{T}$ for all $x \in D$
    - $\forall x\,(P(x) \to R(x))$ is $\mathbf{T}$ in $I$.

**EXAMPLE**: Tautological implication in the following proof?

- All integers are rational numbers. $\boxed{\forall x\big(P(x) \to Q(x)\big)}$
- All rational numbers are real numbers. $\boxed{\forall x\big(Q(x) \to R(x)\big)}$
- All integers are real numbers. $\boxed{\forall x\big(P(x) \to R(x)\big)}$
  - $P(x) =$"$x$ is an integer"
  - $Q(x) =$"$x$ is a rational number"
  - $R(x) =$"$x$ is a real number"
  - rule of inference: $\forall x\,\big(P(x) \to Q(x)\big) \land \forall x\,\big(Q(x) \to R(x)\big) \Rightarrow \forall x\,(P(x) \to R(x))$

# Building Arguments

**QUESTION:** Given the premises $P_1, \ldots, P_n$, show a conclusion $Q$, that is, show that $P_1 \wedge \cdots \wedge P_n \Rightarrow Q$.

| Name | Operations |
|---|---|
| Premise | Introduce the given formulas $P_1, \ldots, P_n$ in the process of constructing proofs. |
| Conclusion | Quote the intermediate formula that have been deducted. |
| Rule of replacement | Replace a formula  with a logically equivalent formula. |
| Rules of Inference | Deduct a new formula with a tautological implication. |
| Rule of substitution | Deduct a formula from a tautology. |

# Rules of Inference for ∀, ∃

| Name | Rules of Inference | NO. |
|---|---|---|
| Universal Instantiation<br>全称量词消去 | $\forall\, x\, P(x) \Rightarrow P(a)$<br>$a$ <u>is any</u> individual in the domain of $x$ | 1 |
| Universal Generalization<br>全称量词引入 | $P(a) \Rightarrow \forall x\, P(x)$<br>$a$ <u>takes any</u> individual in the domain of $x$ | 2 |
| Existential Instantiation<br>存在量词消去 | $\exists\, x\, P(x) \Rightarrow P(a)$<br>$a$ is a <u>specific</u> individual in the domain of $x$ | 3 |
| Existential Generalization<br>存在量词引入 | $P(a) \Rightarrow \exists x\, P(x)$<br>$a$ is a <u>specific</u> individual in the domain of $x$ | 4 |

# Building Arguments

**EXAMPLE**: Show that the following premises 1, 2 lead to conclusion 3.

1. "A student in this class has not read the book," $\exists x\big(C(x) \wedge \neg B(x)\big)$
2. "Everyone in this class passed the exam," $\forall x(C(x) \to P(x))$
3. "Someone who passed the exam has not read the book." $\exists x(P(x) \wedge \neg B(x))$

- **Translate the premises and the conclusion into formulas.**
  - $C(x)$: "$x$ is in the class"; $B(x)$: "$x$ has read the book"; $P(x)$: "$x$ passed the exam"
- **?** $\exists x\big(C(x) \wedge \neg B(x)\big) \wedge \forall x(C(x) \to P(x)) \Rightarrow \exists x(P(x) \wedge \neg B(x))$

| | | |
|---|---|---|
| (1) | $\exists x\big(C(x) \wedge \neg B(x)\big)$ | Premise |
| (2) | $C(a) \wedge \neg B(a)$ | Existential instantiation from (1) |
| (3) | $C(a)$ | Simplification from (2) |
| (4) | $\forall x(C(x) \to P(x))$ | Premise |
| (5) | $C(a) \to P(a)$ | Universal instantiation from (4) |
| (6) | $P(a)$ | Modus ponens from (3) and (5) |
| (7) | $\neg B(a)$ | Simplification from (2) |
| (8) | $P(a) \wedge \neg B(a)$ | Conjunction from (6) and (7) |
| (9) | $\exists x(P(x) \wedge \neg B(x))$ | Existential generalization from (8) |

# Building Arguments

**EXAMPLE**: Show that the following premises lead to conclusion.

- $\exists x F(x) \rightarrow \forall y \big(G(y) \rightarrow H(y)\big),\ \exists x M(x) \rightarrow \exists y G(y)$
- $\Rightarrow \exists x (F(x) \wedge M(x)) \rightarrow \exists y H(y)$

| | | |
|---|---|---|
| (1) | $\exists x F(x) \rightarrow \forall y \big(G(y) \rightarrow H(y)\big)$ | Premise |
| (2) | $\exists x M(x) \rightarrow \exists y G(y)$ | Premise |
| (3) | $\exists x (F(x) \wedge M(x))$ | Conclusion Premise |
| (4) | $\exists x F(x) \wedge \exists x M(x)$ | Rule b from (3) |
| (5) | $\exists x F(x)$ | Simplification from (4) |
| (6) | $\forall y \big(G(y) \rightarrow H(y)\big)$ | Modus ponens from (1) and (5) |
| (7) | $\exists x M(x)$ | Simplification from (4) |
| (8) | $\exists y G(y)$ | Modus ponens from (7) and (2) |
| (9) | $G(\mathrm{c})$ | Existential instantiation from (8) |
| (10) | $G(c) \rightarrow H(c)$ | Universal instantiation from (6) |
| (11) | $H(c)$ | Modus ponens from (9) and (10) |
| (12) | $\exists y H(y)$ | Existential generalization from (11) |
| (13) | $\exists x (F(x) \wedge M(x)) \rightarrow \exists y H(y)$ | Conclusion Premise from (3) (12) |