Linear Algebra I

Chapter 3. General Vector Spaces

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WELCOME!

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Definition of Vector Space

Definition. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let V be an arbitrary nonempty set of elements on which addition and scalar multiplication are defined, i.e.,

◇ for any $\mathbf{u}, \mathbf{v} \in V$, it satisfies that $\mathbf{u} + \mathbf{v} \in V$;

◇ for any $\mathbf{u} \in V$ and any scalar $k \in \mathbb{F}$, it satisfies that $k\mathbf{u} \in V$.

If the following axioms are satisfied, then V is a called an $\mathbb{F}$-vector space, and elements in V are called vectors.

(A1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad (\forall \mathbf{u}, \mathbf{v} \in V)$.

(A2) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \quad (\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V)$.

(A3) $\exists \mathbf{0} \in V$, s.t. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u} \quad (\forall \mathbf{u} \in V)$.

(A4) $\forall \mathbf{u} \in V$, $\exists \mathbf{v} \in V$, s.t. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} = \mathbf{0}$.

(A5) $1\mathbf{u} = \mathbf{u} \quad (\forall \mathbf{u} \in V)$.

(A6) $(kh)\mathbf{u} = k(h\mathbf{u}) \quad (\forall \mathbf{u} \in V$ and $k, h \in \mathbb{F})$.

(A7) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v} \quad (\forall \mathbf{u}, \mathbf{v} \in V$ and $k \in \mathbb{F})$.

(A8) $(k + h)\mathbf{u} = k\mathbf{u} + h\mathbf{u} \quad (\forall \mathbf{u} \in V$ and $k, h \in \mathbb{F})$.

Remark: The element $\mathbf{0}$ in (3) is called zero vector. The vector $\mathbf{v}$ in (4) is called negative of $\mathbf{u}$ and usually denoted by $-\mathbf{u}$.

Remark: When the scalars are real/complex, we call V a real/complex vector space.

Examples of Vector Spaces

Example. The following are all examples of real vector spaces.

(1) $V = \{\mathbf{0}\}$; $\mathbf{0} + \mathbf{0} = \mathbf{0}; k\mathbf{0} = \mathbf{0}$.

(2) $\mathbb{R}^n$; vector addition and scalar multiplication.

(3) The set of all infinite sequence $\mathbb{R}^{\mathbb{N}} = \{(u_1, u_2, \ldots, u_n, \ldots)\}$;
componentwise addition and scalar multiplication.

(4) The set $\mathbb{R}^{(a,b)}$ of all real-valued functions on (a, b);
function addition and scalar multiplication.

(5) The set P_n of all polynomials with real coefficients and of degree $\leq n$.
polynomial addition and scalar multiplication.

(6) $M_{m\times n}$; matrix addition and scalar multiplication.

(7) The set of all upper triangular square matrices of order n;
matrix addition and scalar multiplication.

Example. (i) $\mathbb{C}^n$ is a $\mathbb{C}$-vector space; (ii) $\mathbb{C}^n$ is a $\mathbb{R}$-vector space.

An Unusual Example

Example. Let V be the set $\mathbb{R}^+$ of all positive real numbers. Define addition and scalar multiplication as

$$u \oplus v = uv, \qquad k \odot u = u^k.$$

Then V forms a real vector space with addition $\oplus$ and scalar multiplication $\odot$.

Basic Properties of Vectors

Theorem. Let V be a $\mathbb{F}$-vector space. Let $\mathbf{u} \in V$ and $k \in \mathbb{F}$. Then the following statements hold.

- $\diamond$ $0\mathbf{u} = \mathbf{0}$;
- $\diamond$ $(-1)\mathbf{u} = -\mathbf{u}$;
- $\diamond$ $k\mathbf{0} = \mathbf{0}$;
- $\diamond$ If $k\mathbf{u} = \mathbf{0}$, then either $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Proof:

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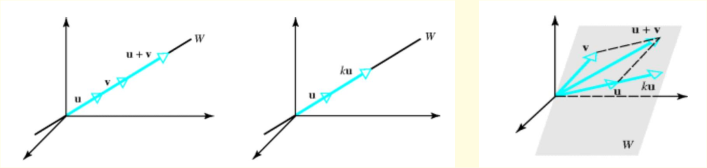
Intuition of a Subspace

Example. In $\mathbb{R}^3$, lines or planes through the origin are also "linear structures" containing $\{\mathbf{0}\}$.



Definition. A non-empty subset W of a vector space V is called a subspace of V, if W is itself a vector space under the same addition and scalar multiplication defined on V.

Definition of Subspace

Question: What kind of properties of a vector space may a subset W not satisfy?

- for any $\mathbf{u}, \mathbf{v} \in W$, it satisfies that $\mathbf{u} + \mathbf{v} \in W$.
- for any $\mathbf{u} \in W$ and any scalar $k \in \mathbb{F}$, it satisfies that $k\mathbf{u} \in W$.

(A1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad (\forall \mathbf{u}, \mathbf{v} \in W)$.

(A2) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w} \quad (\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in W)$.

(A3) $\exists \mathbf{0} \in W$, s.t. $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u} \quad (\forall \mathbf{u} \in W)$.

(A4) $\forall \mathbf{u} \in W$, $\exists \mathbf{v} \in W$, s.t. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} = \mathbf{0}$.

(A5) $1\mathbf{u} = \mathbf{u} \quad (\forall \mathbf{u} \in W)$.

(A6) $(kh)\mathbf{u} = k(h\mathbf{u}) \quad (\forall \mathbf{u} \in W$ and $k, h \in \mathbb{F})$.

(A7) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v} \quad (\forall \mathbf{u}, \mathbf{v} \in W$ and $k \in \mathbb{F})$.

(A8) $(k + h)\mathbf{u} = k\mathbf{u} + h\mathbf{u} \quad (\forall \mathbf{u} \in W$ and $k, h \in \mathbb{F})$.

Definition of Subspace

Question: What kind of properties of a vector space may a subset not satisfy?

Theorem. Let V be a $\mathbb{F}$-vector space and $\emptyset \neq W \subseteq V$. Then W is a subspace if and only if the following conditions hold.
(1) For all $\mathbf{u}, \mathbf{v} \in W$, $\mathbf{u} + \mathbf{v} \in W$.
(2) For all $k \in \mathbb{F}$ and $\mathbf{u} \in W$, $k\mathbf{u} \in W$.

Remark: (1)+(2) $\iff$ For all $\mathbf{u}, \mathbf{v} \in W$ and $k, \ell \in \mathbb{F}$, one has $k\mathbf{u} + \ell\mathbf{v} \in W$.

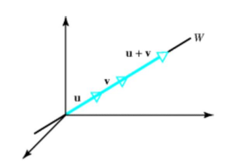
Corollary. Let W be a vector space. If V is a subspace of W, and U is a subspace of V, then U is also a subspace of W.

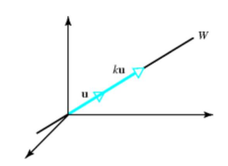
Examples of Subspaces

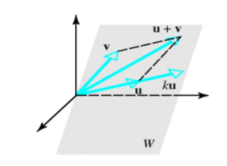
Example. (1) For any vector space V, the subset $W = \{\mathbf{0}\}$ is a subspace of V, which is called zero subspace.
(2) For any vector space V, the subset V is also a subspace of V.

The above two are called the trivial subspaces of V.

Example. In $\mathbb{R}^3$, lines or planes through the origin are also "linear structures" containing $\{\mathbf{0}\}$.







Remark: The line $3x + 4y = 1$ does not form a subspace in $\mathbb{R}^2$, since it does not contain the origin.

Examples of Subspaces

Example. The following are some examples of subspaces.
(1) The set $C(-\infty, +\infty)$ of all continuous functions on $(-\infty, +\infty)$ is a subspace of the vector space $F(-\infty, +\infty)$ of all functions on $(-\infty, +\infty)$.
(2) The set $C^1(-\infty, \infty)$ of all functions on $(-\infty, +\infty)$ with continuous derivative is a subspace of $C(-\infty, +\infty)$.
(3) The set $C^\infty(-\infty, +\infty)$ of all functions on $(-\infty, +\infty)$ which have derivatives of all order is a subspace of $C^1(-\infty, +\infty)$.
(4) The set P_∞ of all polynomials if a subspace of $C(-\infty, +\infty)$.
(5) The set P_n of all polynomials of degree $\leq n$ is a subspace of P_∞.

Question: Is the set of all polynomials of degree n a subspace of P_∞?

Example. The following are some examples of subspaces.
(6) The set U of all symmetric matrix of order n is a subspace of M_n.
(7) The set V of all $n \times n$ upper triangular matrices is a subspace of M_n.
(8) The set of all diagonal matrix of order n is a subspace of either U or V.

Building Subspaces

Theorem. If U and W are subspaces of V, then
(1) $U \cap W$ is a subspace of V;
(2) $U + W$ is a subspace of V.
Here we define

$$U + W \triangleq \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U, \mathbf{w} \in W\}.$$

Proof:

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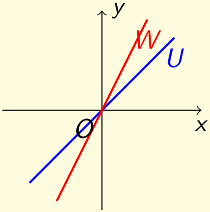
Remark: The set $U \cup W$ may not be a subspace!

Example. Let $V = \mathbb{R}^2$. Let U be the line $y = x$, and W be the line $y = 2x$. Find $U \cap W$, $U + W$ and $U \cup W$.



Linear Combination

Definition. Let V be an $\mathbb{F}$-vector space and let $\mathbf{v} \in V$. A vector $\mathbf{v} \in V$ is said to be a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r \in V$, if $\mathbf{v}$ can be expressed as

$$\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots k_r\mathbf{v}_r,$$

where $k_i \in \mathbb{F}$ (called coefficients).

Example. In the complex vector space $\mathbb{C}^3$, let

$$\mathbf{u} = (\mathrm{i}, 2, -1), \quad \mathbf{v} = (0, -\mathrm{i}, 2).$$

Are the following vectors a linear combination of $\mathbf{u}$ and $\mathbf{v}$?
(1) $\mathbf{w} = (-1, \mathrm{i}, 2-\mathrm{i})$; (2) $\mathbf{w}' = (2, 2+\mathrm{i}, -3)$.

Proof:

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Spanning a Subspace

Theorem. Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_r\}$ be a non-empty set of vectors in a vector space V. Then

(1) The set W of all linear combinations of $\mathbf{w}_1, \dots, \mathbf{w}_r$ is a subspace of V.

(2) This set W is the "smallest" subspace of V that contains $\mathbf{w}_1, \dots, \mathbf{w}_r$, i.e., for any subspace U of V satisfying $\mathbf{w}_1, \dots, \mathbf{w}_r \in U$, one has $W \subseteq U$.

Proof:

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Definition. The subspace W in above theorem is called the span of S, and denoted by

$$W \triangleq \operatorname{span}(S) = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_r\}.$$

We also say that the vectors in S span W.

Remark: If we are only interested in information involving $\mathbf{w}_1, \dots, \mathbf{w}_r \in V$, it is sufficient to consider $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_r\}$ instead of the whole V.

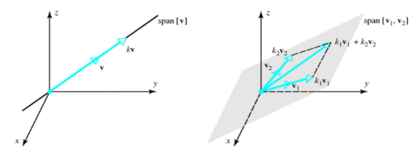
Examples

Example. The standard unit vectors

$\mathbf{e}_1 = \{1, 0, \ldots, 0\}$, $\mathbf{e}_2 = \{0, 1, \ldots, 0\}$, $\ldots$, $\mathbf{e}_n = \{0, \ldots, 0, 1\}$

span $\mathbb{R}^n$.

Example. See span($\{\mathbf{v}\}$) and span$\{\mathbf{v}_1, \mathbf{v}_2\}$ in the figure.



Examples

Example. Show that the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

span the real vector space V composed of all upper triangular square matrices of order 2.

Proof:

Relations between Subspaces

Theorem. Let $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$ and $S_2 = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s\}$ be nonempty sets of vectors in V. Then

$$\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$$

if and only if each vector in S_1 is a linear combination by vectors in S_2.

Example.

$$\mathrm{span}\Big(\{(1,0),(1,2),(-1,1)\}\Big) \subseteq \mathrm{span}\Big(\{(1,1),\ (1,-1)\}\Big).$$

Relations between Subspaces

Theorem. Let $U = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_r\}$ and $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s\}$ be subspaces of V. Then

$$U + W = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_r, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s\}.$$

Proof:

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Question*: How to find $U \cap W$?

Example.

$$\begin{aligned} \text{span}\big(\{(1,0,0), (0,1,0)\}\big) + \text{span}\big((0,0,1), (1,1,0)\big) &= \mathbb{R}^3 \\ &= \text{span}\big(\{(1,0,0), (0,1,0), (0,0,1), (1,1,0)\big). \end{aligned}$$

Solution Spaces of Homogeneous Systems

Theorem. Let $A \in M_{m\times n}(\mathbb{F})$. The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in n unknowns is a subspace of $\mathbb{F}^n$.

Proof:

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Definition. For $A \in M_{m\times n}$, the solution space of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the null space of A. We denote it by $\mathrm{Null}(A)$.

Example. Try to investigate the solutions space of the following system.

$$\begin{cases} x & -2y & +3z & = & 0 \\ 2x & -4y & +6z & = & 0 \\ 3x & -6y & +9z & = & 0 \end{cases}$$

Solution:

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Row and Column Spaces of a Matrix

Definition. Let $A \in M_{m\times n}(\mathbb{F})$. Suppose that

$$A = [a_{ij}]_{m\times n} = \left[\begin{array}{c} \mathbf{r}_1 \\ \hdashline \mathbf{r}_2 \\ \hdashline \vdots \\ \hdashline \mathbf{r}_m \end{array}\right] = \left[\begin{array}{c:c:c:c} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \end{array}\right].$$

The subspace $\text{span}\{\mathbf{r}_1, \dots, \mathbf{r}_m\}$ of $\mathbb{F}^n$ is called the row space of A.
The subspace $\text{span}\{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ of $\mathbb{F}^m$ is called the column space of A.

Notations (only in this class):

$$\mathit{Row}(A) := \text{span}\{\mathbf{r}_1, \dots, \mathbf{r}_m\}, \quad \mathit{Col}(A) := \text{span}\{\mathbf{c}_1, \dots, \mathbf{c}_n\}.$$

Theorem. Let $A \in M_{m\times n}$. A system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b} \in \text{Col}(A)$.

Remark: Let $T_A : \mathbb{F}^n \to \mathbb{F}^m$ be the matrix transformation with standard matrix $A \in M_{m\times n}(\mathbb{F})$. Then $\{T_A(\mathbf{x}) : x \in \mathbb{F}^n\} = \text{Col}(A)$.

Definition of Linear Independence

Definition. Let $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_r\}$ be a non-empty set in a vector space V.

◇ If the equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots k_r\mathbf{v}_r = \mathbf{0}$$

has only the trivial solution $k_i = 0$ $(1 \le i \le r)$, then the set S (or these vectors) is said to be linearly independent.

◇ If the equation has non-trivial solutions, then S (or these vectors) is said to be linearly dependent.

Example. Determine whether the vectors

$$\mathbf{v}_1 = (1, -2, 3), \quad \mathbf{v}_2 = (5, 6, -1), \quad \mathbf{v}_3 = (3, 2, 1)$$

are linearly independent or linearly dependent in the real vector space $\mathbb{R}^3$.

Solution:

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Remark: $(\mathbf{v}_1 \times \mathbf{v}_2) \cdot \mathbf{v}_3 = 0$; they lie in one plane.

Investigation

Example. In P_∞, which of the following sets are linearly independent?

(a) $S = \{0\}$.
(b) $S = \{1 + x\}$.
(c) $S = \{x, x^2\}$.
(d) $S = \{2 - x, 4 - 2x\}$.
(e) $S = \{0, x, 2 - x^2, 4 + x^3, 5 + x^4\}$.

Linear Dependence and Linear Combination

Proposition. (1) $S = \{\mathbf{0}, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is linearly dependent.
(2) $S = \{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.
(3) $S = \{\mathbf{u}, \mathbf{v}\}$ is linearly independent if and only if neither vector is a scalar multiple of the other.

Example.
(1) For $\mathbb{F} = \mathbb{C}$ and $V = \mathbb{C}$, the vectors 1 and i are linearly dependent.
(2) For $\mathbb{F} = \mathbb{R}$ and $V = \mathbb{C}$, the vectors 1 and i are linearly independent.

Theorem. Let S be a set that contains more than 2 vectors. Then S is linearly dependent if and only if at least one vector in S is expressible as a linear combination of other vectors in S.

Proof:

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Examples

Example. Try to express one of the following vectors as the linear combination of the other two.

$$\mathbf{v}_1 = (1, -2, 3), \quad \mathbf{v}_2 = (5, 6, -1), \quad \mathbf{v}_3 = (3, 2, 1).$$

Solution:

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Example. Let

$p_0(x) = 4,\ p_1(x) = 1 + x,\ p_2(x) = 5 + 3x - 2x^2,\ p_3(x) = 1 + 3x - x^2.$

be polynomials in P_2.

(1) Are p_1, p_2, p_3 linearly independent or not?

(2) Are p_0, p_1, p_2, p_3 linearly independent or not?

Solution:

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More Examples

Example. Suppose that the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent. Prove that the vectors

$$\mathbf{x} = \mathbf{u}, \quad \mathbf{y} = \mathbf{u} + \mathbf{v}, \quad \mathbf{z} = \mathbf{u} + \mathbf{v} + \mathbf{w}$$

are also linearly independent.

Proof:

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More Examples

Example. Let $r \in \mathbb{N}$. Prove that the functions

$$\sin x,\ \sin 2x,\ \ldots,\ \sin rx$$

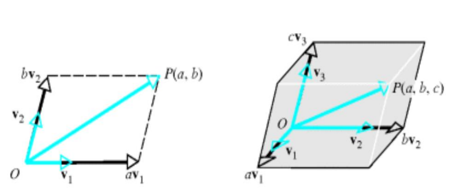
are linearly independent functions in $F(-\infty, +\infty)$.

Proof:

Basis for a Vector Space

Definition. Let V be a vector space and S be a finite set of vectors in V. We call S a basis for V if the following two conditions hold.
(1) S spans V.
(2) S is linearly independent.

Example. The standard basis for $\mathbb{R}^n$ is $\{e_1, e_2, \cdots, e_n\}$.
The standard basis for $\mathbb{R}^3$ is $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.
The following vectors also form bases for $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively.



For example, the vectors $(1, 1, 2)$, $(1, 0, 2)$, and $(2, 1, 3)$ form a basis for $\mathbb{R}^3$.

Example. Show that the following vectors form two bases for P_n, the real vector space of polynomials with real coefficients and of degree $\leq n$.

(A) $p_0(x) = 1,\ p_1(x) = x,\ p_2(x) = x^2,\ \ldots,\ p_n(x) = x^n$,
it is called standard basis for P_n.

(B) $q_0(x) = 1,\ q_1(x) = x + c,\ q_2(x) = (x + c)^2,\ \ldots,$
$q_n(x) = (x + c)^n$, where c is a given non-zero scalar.

Proof:

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Example. The standard basis for $M_{m\times n}$ are the mn different matrices whose entries are zero except for a single entry of 1.

Row Operations and Spaces of a Matrix

Theorem.
(1) Elementary row operations do not change the null space of a matrix.
(2) Elementary row operations do not change the row space of a matrix.
(3) Elementary row operations do not change the dependence relationships among the column vectors.

Explanation:

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- Recall: $\mathrm{span}(S_1) \subseteq \mathrm{span}(S_2)$ if and only if each vector in S_1 is a linear combination by vectors in S_2.

- Remark: Suppose that $\mathbf{c}_1, \ldots, \mathbf{c}_n$ becomes $\mathbf{c}_1', \ldots, \mathbf{c}_n'$ by row operation.

"Do not change dependence relationships" means:

if $k_1\mathbf{c}_1 + \ldots + k_n\mathbf{c}_n = 0$ for some certain scalars $k_1, \ldots, k_n$,

then we also have $k_1\mathbf{c}_1' + \ldots + k_n\mathbf{c}_n' = 0$.

Dependency equations does not change!

Find These Spaces

- Key to find theses spaces of a matrix: Echelon Form.

Example. Finding a basis for the null/row/column space of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}.$$

Solution:

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Problem*: To conclude the main steps of finding a basis for the column space.

Theorem. Suppose that a matrix A has row echelon form R.
◇ The row vectors with the leading $1'$s in R form a basis for $Row(A) = Row(R)$.
◇ The column vectors with the leading $1'$s in R form a basis for $Col(R)$; the corresponding column vectors of A form a basis for $Col(A)$.

More Examples

Example. (1) Find a subset of vectors

$$\mathbf{v}_1 = (1, 2, 0, 2),\ \mathbf{v}_2 = (3, 6, 0, 6),\ \mathbf{v}_3 = (-2, -5, 5, 0),$$

$$\mathbf{v}_4 = (0, -2, 10, 8),\ \mathbf{v}_5 = (2, 4, 0, 4),\ \mathbf{v}_6 = (0, -3, 15, 18)$$

that forms a basis for the space $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_6\}$.

(2) Express each vector as a linear combination of these basis vectors.

Solution:

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Problem*: To summary the main steps of solving the above example.

Unique Expression

Theorem. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for the $\mathbb{F}$-vector space V. Then every $\mathbf{v} \in V$ can be expressed uniquely in the form

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n,$$

where $c_1, c_2, \ldots, c_n \in \mathbb{F}$.

Proof:

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Coordinates Relative to a Basis

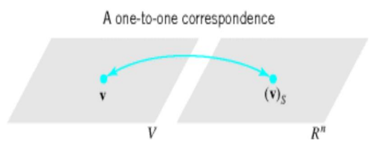
Definition. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be a basis for V and

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n.$$

is the expression of $\mathbf{v} \in V$. Then the coordinate vector of $\mathbf{v}$ relative to S, and the coordinate matrix of $\mathbf{v}$ relative to S, are defined and denoted by

$$(\mathbf{v})_S = (c_1, c_2, \dots, c_n), \qquad [\mathbf{v}]_S = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

respectively.



$$[k\mathbf{v} + \ell\mathbf{w}]_S = k[\mathbf{v}]_S + \ell[\mathbf{w}]_S.$$

$$\{\mathbf{v}_1, \dots, \mathbf{v}_r\} \text{ is independent } \iff \{[\mathbf{v}_1]_S, \dots, [\mathbf{v}_r]_S\} \text{ is independent.}$$

Examples

Example. Find the coordinate vector of $\mathbf{v} = (x, y, z)$ in $\mathbb{R}^3$ relative to the standard basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

Solution:

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Example. A basis for $\mathbb{R}^3$ is $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = (1, 1, 1), \quad \mathbf{v}_2 = (1, 1, 0), \quad \mathbf{v}_3 = (1, 0, 0).$$

(1) Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ relative to S.
(2) Find the vector $\mathbf{w} \in \mathbb{R}^3$ whose coordinates relative to S is $(\mathbf{w})_S = (-1, 3, 2)$.

Solution:

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Examples

Example. (1) Find the coordinate vector of

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

relative to the standard basis in P_n.

(2) Find the coordinate vector of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ relative to the standard basis in $M_{2\times 2}$.

Solution:

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More Examples

Example. Suppose that the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent. Prove that the vectors

$$\mathbf{x} = \mathbf{u}, \quad \mathbf{y} = \mathbf{u} + \mathbf{v}, \quad \mathbf{z} = \mathbf{u} + \mathbf{v} + \mathbf{w}$$

are also linearly independent.

Proof:

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More Examples

Example. (1) Find a subset of vectors

$$p_1(x) = 1 + 2x + 2x^3, \;\; p_2(x) = 3 + 6x + 6x^3,$$
$$p_3(x) = -2 - 5x + 5x^2, \;\; p_4(x) = -2x + 10x^2 + 8x^3,$$
$$p_5(x) = 2 + 4x + 4x^3, \;\; p_6(x) = -3x + 15x^2 + 18x^3$$

that forms a basis for the subspace $\mathrm{Span}\{p_1, \ldots, p_6\}$ in P_3.

(2) Express each vector as a linear combination of these basis vectors.

Solution:

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Definition of Dimension

- A vector space that cannot be spanned by finitely many vectors is said to be infinite-dimensional, whereas those that can are said to be finite-dimensional.
  - Example: The space P_∞ of all polynomials is infinite-dimensional. Why?

Definition. The dimension of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V. In particular, we define $\dim(\{\mathbf{0}\}) = \mathbf{0}$.

Theorem. All bases for a finite-dimensional vector space have the same number of vectors.

Proof:

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Examples

Example. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set in a vector space V, then $\dim(\mathrm{span}(S)) = r$.

Example. (i) $\mathbb{C}$ is a complex vector space of dimension 1.
(ii) $\mathbb{C}$ is a real vector space of dimension 2.

Example. For the following real vector spaces, we have

$$\dim(\mathbb{R}^n) = n, \quad \dim(P_n) = n+1, \quad \dim(M_{m\times n}) = mn.$$

Example. Find the dimension of the space of all $n \times n$ real
(1) symmetric matrices; (2)skew-symmetric matrices.
Solution:

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Example. Find a basis for and the dimension of the solution space of the linear system (recalling: $(-3r-4s-2t, r, -2s, s, t, 0)$)

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0 \\ 5x_3 + 10x_4 + 15x_6 = 0 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0 \end{cases}$$

Solution:

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Plus/Minus Theorem

Theorem. (1) Let S be a linearly independent set in V. If $\mathbf{v} \in V$ but $\mathbf{v} \notin \text{span}(S)$, then the set $S \cup \{\mathbf{v}\}$ is still linearly independent.
(2) Suppose that $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S. Then $\text{span}(S) = \text{span}(S \setminus \{v\})$.

Example. Show that

$$p_1(x) = 1 - x^2, \quad p_2(x) = 2 - x^2, \quad p_3(x) = x^3$$

are linearly independent by applying the above theorem.

Proof:

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Some Fundamental Theorems

Theorem. Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S spans V or S is linearly independent.

Theorem. Let S be a finite set of vectors in a finite-dimensional space V.
◇ If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.
◇ If S is a linearly independently set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

Theorem. If W is a subspace of a finite-dimensional vector space V, then
◇ $\dim(W) \leq \dim(V)$.
◇ $W = V$ if and only if $\dim(W) = \dim(V)$.
Proof:

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An Inclusion-exclusion Type Equality

- Inclusion-exclusion principle for two finite sets S and T:

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

Theorem. Let W be a finite-dimensional vector space with U, V two subspaces. Then

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V).$$

Idea:

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Recalling

Definition. For

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} \mathbf{r}_1 \\ \hdashline \mathbf{r}_2 \\ \hdashline \vdots \\ \hdashline \mathbf{r}_m \end{bmatrix} = \left[\begin{array}{c:c:c:c} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n \end{array}\right],$$

define

$$\begin{aligned} \mathrm{Null}(A) &= \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\} \subseteq \mathbb{R}^n, \\ \mathrm{Row}(A) &= \mathrm{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m\} \subseteq \mathbb{R}^n, \\ \mathrm{Col}(A) &= \mathrm{span}\{\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n\} \subseteq \mathbb{R}^m. \end{aligned}$$

Theorem.
(1) Elementary row operations do not change the null space of a matrix.
(2) Elementary row operations do not change the row space of a matrix.
(3) Elementary row operations do not change the dependence relationships among the column vectors.

Recalling

Definition. Suppose that an echelon form of a matrix A has r non-zero rows. Then we say that A has rank r, and denote $\text{rank}(A) = r$.

Theorem. For any matrix $A \in M_{m\times n}$, there is an integer $r \leq \min\{m, n\}$, an invertible matrix $P \in M_m$ and an invertible matrix $Q \in M_n$ such that

$$PAQ = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Rank of a Matrix

Theorem. For any matrix A, we have

$$\mathrm{rank}(A) = \dim(Row(A)) = \dim(Col(A)).$$

Remark: (1) For $A \in M_{m\times n}$, it satisfies that $\mathrm{rank}(A) \leq \min\{m, n\}$.
(2) For any matrix A, one has $\mathrm{rank}(A^T) = \mathrm{rank}(A)$.
(3) Elementary row or column operations do not change the rank of a matrix.

Corollary. Let $A \in M_{m\times n}$. Let $P \in M_m$ and $Q \in M_n$ be invertible matrices. Then $\mathrm{rank}(PA) = \mathrm{rank}(A) = \mathrm{rank}(AQ)$.

Corollary. Partitioned elementary row or column operations do not change the rank of a matrix.

Rank and Minor*

- A minor of A is the determinant of a square submatrix of A.

Theorem. Let $A \in M_{m\times n}$. Then $\operatorname{rank}(A) = r$ if and only if:
- ∘ some $r \times r$ minor of A does not vanish;
- ∘ and every $(r+1) \times (r+1)$ minor of A does vanish.

Some Properties of Rank*

Property. For matrices A, B, C with suitable sizes, we have

- $\diamond$ $\max\{\text{rank}(A), \text{rank}(B)\} \le \text{rank}\left(\begin{bmatrix} A & B \end{bmatrix}\right) \le \text{rank}(A) + \text{rank}(B)$;
- $\diamond$ $\text{rank}\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) \ge \text{rank}\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \text{rank}(A) + \text{rank}(B)$;

Explanation:

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Remark: $Col\left(\begin{bmatrix} A & B \end{bmatrix}\right) = Col(A) + Col(B)$.

Dimension Theorem for Matrices

Definition. The dimension of the null space of A is called the nullity of A denoted by $\text{nullity}(A)$.

Theorem. For $A \in M_{m\times n}$, it satisfies that

$$\text{rank}(A) + \text{nullity}(A) = n.$$

Explanation:

Rank = number of leading variables;
Nullity = number of free variables.

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Corollary. For $A \in M_n$, it is invertible if and only if $\text{rank}(A) = n$, if and only if $\text{nullity}(A) = 0$.

Some Properties of Rank*

Theorem. Let A, B be matrix with suitable sizes. Let λ, μ be scalars. The following inequalities hold.

- ◇ $\text{rank}(\lambda A + \mu B) \le \text{rank}(A) + \text{rank}(B)$.
- ◇ $\text{rank}(AB) \le \min\{\text{rank}(A), \text{rank}(B)\}$.
- ◇ $\text{rank}(AB) \ge \text{rank}(A) + \text{rank}(B) - k$ for any $A \in M_{m\times k}$, $B \in M_{k\times n}$.

Proof:

Lemma: $\text{Null}(AB) \supseteq \text{Null}(B)$, $Col(AB) \subseteq Col(A)$. 33

Methods: (1) Use null, row and column spaces.
(2) Use (partitioned) elementary operations.

Remark: $AB = 0 \iff Col(B) \subseteq \text{Null}(A)$.

Problem*: Try to prove the following inequalities.
(1) $\text{rank}(AB - CD) \le \text{rank}(A - C) + \text{rank}(B - D)$.
(2) $\text{rank}(ABC) \ge \text{rank}(AB) + \text{rank}(BC) - \text{rank}(B)$.

Orthogonal Complement in $\mathbb{R}^n$

Definition. Let W be a subspace of $\mathbb{R}^n$. The orthogonal complement of W is defined to be

$$W^\perp = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \cdot \mathbf{w} = 0 \, (\forall \mathbf{w} \in W)\}.$$

Theorem. Let W be any subspace of $\mathbb{R}^n$, then

◇ $W^\perp$ is a subspace of $\mathbb{R}^n$;
◇ $W \cap W^\perp = \{\mathbf{0}\}$;
◇ $V = W + W^\perp$;
◇ $(W^\perp)^\perp = W$.

Proof:

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Theorem. Let $A \in M_{m\times n}(\mathbb{R})$. Then we have:

◇ $\mathrm{Null}(A)^\perp = Row(A)$.
◇ $\mathrm{Null}(A^T)^\perp = Col(A)$.

Remark: Here we view both $\mathrm{Null}(A)$ and $Row(A)$ as subspaces of $\mathbb{R}^n$.

Equivalence Theorem (Continued)

Theorem. If $A \in M_n(\mathbb{R})$, then the following statements are equivalent.

(a) A is invertible.
(b) $A\mathbf{x} = 0$ has only the trivial solution.
(c) The reduced row echelon form of A is I_n.
(d) A can be expressed as a product of elementary matrices.
(e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.
(g) $\det(A) \neq 0$.
(h) The column vectors of A are linearly independent.
(i) The row vectors of A are linearly independent.
(j) The column vectors of A span $\mathbb{R}^n$.
(k) The row vectors of A span $\mathbb{R}^n$.
(l) The column vectors of A form a basis for $\mathbb{R}^n$.
(m) The row vectors of A form a basis for $\mathbb{R}^n$.
(n) $\mathrm{rank}(A) = n$.
(o) $\mathrm{nullity}(A) = 0$.
(p) $\mathrm{Null}(A)^{\perp} = \mathbb{R}^n$.
(q) $Row(A)^{\perp} = \{\mathbf{0}\}$.

Recalling

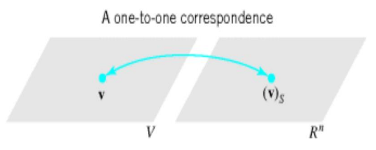
Definition. Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for V and

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_n\mathbf{u}_n.$$

is the expression of $\mathbf{v} \in V$. Then the coordinate vector of $\mathbf{v}$ relative to S, and the coordinate matrix of $\mathbf{v}$ relative to S, are defined and denoted by

$$(\mathbf{v})_S = (c_1, c_2, \ldots, c_n), \qquad [\mathbf{v}]_S = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},$$

respectively.



$$[k\mathbf{v} + \ell\mathbf{w}]_S = k[\mathbf{v}]_S + \ell[\mathbf{w}]_S.$$

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_r\} \text{ is independent } \iff \{[\mathbf{v}_1]_S, \ldots, [\mathbf{v}_r]_S\} \text{ is independent.}$$

Transition Matrix

Physics: Same motion, two different observers. *Motion is relative.*
Mathematics: Same vector, two different bases.

Question: Let $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $B' = \{\mathbf{v}_1', \mathbf{v}_2'\}$ be two bases of $\mathbb{R}^2$. Suppose that

$$\mathbf{v}_1' = a\mathbf{v}_1 + b\mathbf{v}_2,$$
$$\mathbf{v}_2' = c\mathbf{v}_1 + d\mathbf{v}_2.$$

What is the relationship between $[\mathbf{u}]_B$ and $[\mathbf{u}]_{B'}$?

Definition. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}_1', \ldots, \mathbf{v}_n'\}$ be two bases of V. The transition matrix from B' to B is defined as

$$P_{B \leftarrow B'} = \left[\begin{array}{c:c:c:c} [\mathbf{v}_1']_B & [\mathbf{v}_2']_B & \cdots & [\mathbf{v}_n']_B \end{array}\right]$$

Remark: Our textbook uses the notation $P_{B' \to B}$.

Proposition. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}_1', \ldots, \mathbf{v}_n'\}$ be two bases of V. Then for any vector $\mathbf{x} \in V$, it satisfies that

$$[\mathbf{x}]_B = P_{B \leftarrow B'} [\mathbf{x}]_{B'}.$$

Examples of Transition Matrices

Example. Consider bases $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ and $B'' = \{\mathbf{u}''_1, \mathbf{u}''_2\}$ of $\mathbb{R}^2$, where

$$\mathbf{u}'_1 = (1,0), \mathbf{u}'_2 = (1,1), \quad \mathbf{u}''_1 = (1,1), \mathbf{u}''_2 = (2,1).$$

(1) Find the transition matrix $P_{B' \leftarrow B''}$ from B'' to B'.
(2) Find the transition matrix $P_{B'' \leftarrow B'}$ from B' to B''.
(3) If $(x)_{B'} = (3,2)$, find $(x)_{B''}$.

Solution:

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Examples of Transition Matrices

Example. Consider bases $B = \{p_0, p_1, p_2\}$ and $B' = \{q_0, q_1, q_2\}$ for P_2, where

$$p_0(x) = 1,\ p_1(x) = x,\ p_2(x) = x^2,$$
$$q_0(x) = 1,\ q_1(x) = x + c,\ q_2(x) = (x + c)^2.$$

Find the transition matrix $P_{B' \leftarrow B}$.

Solution:

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Theorem. Let B, B', B'' are three bases for V. Then

$$P_{B \leftarrow B''} = P_{B \leftarrow B'} P_{B' \leftarrow B''}$$

Proof:

Corollary. Let B, B' be bases for V. Then $P_{B' \to B}$ is invertible, and

$$P_{B \leftarrow B'}^{-1} = P_{B' \leftarrow B}.$$

Remark: It follows from

$$P_{B\leftarrow B''} = P_{B\leftarrow B'} P_{B'\leftarrow B''}$$

that

$$P_{B'\leftarrow B''} = P^{-1}_{B\leftarrow B'} P_{B\leftarrow B''}$$

When B is the standard basis, one can try

$$\left[\begin{array}{c:c} P_{B\leftarrow B'} & P_{B\leftarrow B''} \end{array}\right] \xrightarrow{\text{row}} \left[\begin{array}{c:c} I & P^{-1}_{B\leftarrow B'} P_{B\leftarrow B''} \end{array}\right]$$

Example. Consider bases $B' = \{\mathbf{u}'_1, \mathbf{u}'_2\}$ and $B'' = \{\mathbf{u}''_1, \mathbf{u}''_2\}$ of $\mathbb{R}^2$, where

$$\mathbf{u}'_1 = (1,0), \mathbf{u}'_2 = (1,1), \quad \mathbf{u}''_1 = (1,1), \mathbf{u}''_2 = (2,1).$$

(1) Find the transition matrix $P_{B'\leftarrow B''}$ from B'' to B'.
(2) Find the transition matrix $P_{B''\leftarrow B'}$ from B' to B''.
(3) If $(x)_{B'} = (3,2)$, find $(x)_{B''}$.

Solution:

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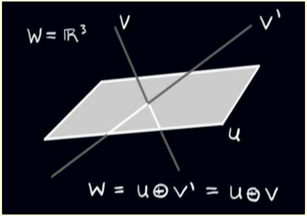
Definition of Direct Sum*

Definition. Let V_1 and V_2 be subspaces of V. Then V is said to be the direct sum of V_1 and V_2, if $V = V_1 + V_2$ and $V_1 \cap V_2 = \{\mathbf{0}\}$. We denote $V = V_1 \oplus V_2$.

Remark: When $V_1 \cap V_2 = \{\mathbf{0}\}$, we say that $V_1 + V_2$ is a direct sum, and denote $V_1 + V_2 = V_1 \oplus V_2$.

Example. The space $\mathbb{R}^3$ is the direct sum of the xOy-plane and the z-axis. More generally, let Π be a plane in $\mathbb{R}^3$ and $\mathbf{w} \in \mathbb{R}^3$ be a vector not in Π. Then $\mathbb{R}^3$ is the direct sum of Π and $\text{span}\{\mathbf{w}\}$.



Equivalent Definitions of Direct Sum*

Theorem. Let V be a finite-dimensional space and V_1, V_2 be its subspaces. Suppose that $V_1 + V_2 = V$. Then the following four statements are equivalent.

- ◇ Any $\mathbf{v} \in V$ can be expressed uniquely in the form $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 \in V_1$ and $\mathbf{v}_2 \in V_2$.
- ◇ The element $\mathbf{0}$ can be expressed uniquely in the form $\mathbf{0} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 \in V_1$ and $\mathbf{v}_2 \in V_2$.
- ◇ $V_1 \cap V_2 = \{\mathbf{0}\}$.
- ◇ $\dim(V) = \dim(V_1) + \dim(V_2)$.

Proof:

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Remark: The first three statements are equivalent even when V is infinite-dimensional.

Examples*

Example. Let $C_e(+\infty, -\infty)/C_o(+\infty, -\infty)$ be the space of all even/odd continuous real-valued functions on $(-\infty, +\infty)$. Show that

$$C(-\infty, +\infty) = C_e(+\infty, -\infty) \oplus C_o(+\infty, -\infty).$$

Example. Show that $M_n(\mathbb{R})$ is the direct sum of the space of all symmetric matrices and the space of all skew-symmetric spaces.

Proof:

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