Discrete Mathematics

recurrence relations, generating functions

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Linear Homogeneous RR

DEFINITION: A **linear homogeneous RR (LHRR) of degree *k* with constant coefficients** is an RR of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$, where $n \geq k$, $\{c_i\}_{i=1}^{k}$ are constant real numbers, and $c_k \neq 0$.

- **degree *k***: every term depends on k terms preceding it
- **constant coefficients:** $c_1, \dots, c_k$ are independent of n
- **linear:** the right-hand side is a linear combination of $a_1, a_2, \dots, a_{n-1}$.
- **homogeneous:** every term is a multiple of some a_j.
  - $f_n = f_{n-1} + f_{n-2}, n \geq 2$ LHRR of degree 2 with constant coefficients
  - $H_n = 2H_{n-1} + 1, n \geq 2$ not homomogenous
- $\{x_n\}_{n\geq 0}$ is a **solution** if $x_n = \sum_{i=1}^{k} c_i x_{n-i}$ for all $n \geq k$

Existence and Uniqueness

THEOREM: For any $a_0, a_1, \ldots, a_{k-1}$, $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ has a unique solution $\{x_n\}_{n\geq 0}$ such that $x_i = a_i$ for every $0 \leq i < k$.

- **Existence**:
  - $x_n = a_n$ for all $0 \leq n < k$
  - $x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$ for all $n \geq k$
- **Uniqueness**:
  a) $x'_n = a_n$ for all $0 \leq n < k$
  b) $x'_n = c_1\, x'_{n-1} + c_2 x'_{n-2} + \cdots + c_k x'_{n-k}$ $(n \geq k)$
  c) $x_n = a_n$ for all $0 \leq n < k$
  d) $x_n = c_1\, x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k}$ $(n \geq k)$
    - a) + c) $\Rightarrow x'_n = x_n$ for all $0 \leq n < k$
    - b) + d) $\Rightarrow x'_n = x_n$ for all $n \geq k$

Characteristic Roots

THEOREM: $\{r^n\}_{n\geq 0}$ is a solution of the LHRR $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k\, r^{n-k}$.

- **characteristic equation**: $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$
- **characteristic roots**: solutions of the characteristic equation.

EXAMPLE: Solve the LHRR $f_n = f_{n-1} + f_{n-2}, n \geq 2$.

- characteristic equation: $r^2 - r - 1 = 0$
- characteristic roots: $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$
  - $\{r_1^n\}_{n\geq 0}$, $\{r_2^n\}_{n\geq 0}$ are solutions

LHRR (no multiple roots)

THEOREM: If $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ has k distinct characteristic roots $r_1, r_2, \dots, r_k$, then $\{x_n\}_{n\geq 0}$ is a solution of the LHRR iff $x_n = \sum_{j=1}^{k} \alpha_j r_j^n$ for some constants $\alpha_1, \dots, \alpha_k$.

EXAMPLE: Solve $f_n = f_{n-1} + f_{n-2}$ with $f_0 = f_1 = 1$.

- Characteristic equation: $r^2 - r - 1 = 0$
- Characteristic roots: $r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}$
- $f_n = \alpha_1 r_1^n + \alpha_2 r_2^n$
  - $f_0 = 1 \Rightarrow \alpha_1 * r_1^0 + \alpha_2 * r_2^0 = 1$
  - $f_1 = 1 \Rightarrow \alpha_1 * r_1^1 + \alpha_2 * r_2^1 = 1$
    - $\alpha_1 = \frac{1}{\sqrt{5}} \cdot \frac{1+\sqrt{5}}{2}, \alpha_2 = -\frac{1}{\sqrt{5}} \cdot \frac{1-\sqrt{5}}{2}$
- $f_n = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \quad (n \geq 0)$

LHRR (multiple roots)

THEOREM: If $a_n = \sum_{i=1}^{k} c_i a_{n-i}$ has distinct characteristic roots $r_1, r_2, \dots, r_t$ with multiplicities $m_1, m_2, \dots, m_t$, then $\{x_n\}_{n\geq 0}$ is a solution of the LHRR iff $x_n = \sum_{j=1}^{t} \left(\sum_{\ell=0}^{m_j-1} \alpha_{j,\ell}\, n^{\ell}\right) r_j^n$ for some constants $\{\alpha_{j,\ell}: j \in [t], 0 \leq \ell < m_j\}$.

EXAMPLE: Solve $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1, a_1 = 6$.

- Characteristic equation: $r^2 - 6r + 9 = 0$
- Characteristic roots: $r_1 = 3$
- $a_n = \alpha_{1,0} 3^n + \alpha_{1,1} n\, 3^n$
  - $a_0 = 1 \Rightarrow \alpha_{1,0} * 3^0 + \alpha_{1,1} * 0 * 3^0 = 1$
  - $a_1 = 6 \Rightarrow \alpha_{1,0} * 3^1 + \alpha_{1,1} * 1 * 3^1 = 6$
- $\alpha_{1,0} = 1, \alpha_{1,1} = 1$
- $a_n = 3^n + n3^n = 3^n(n+1)$

Linear Nonhomogeneous RR

DEFINITION: A **linear nonhomogeneous RR (LNRR) of degree *k* with constant coefficients** is an RR of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$, where $c_1, c_2, \ldots, c_k$ are constants, $c_k \neq 0$, and $F(n) \neq 0$.

- **Associated LHRR**: $a_n = \sum_{i=1}^{k} c_i a_{n-i}$
- $\{x_n\}_{n\geq 0}$ is a **solution** if $x_n = \sum_{i=1}^{k} c_i x_{n-i} + F(n)$ for all $n \geq k$.

EXAMPLE: $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$

- $c_1 = 1, c_2 = 1, F(n) = n^2 + n + 1$
- LNRR of degree 2 with constant coefficients
- associated LHRR: $a_n = a_{n-1} + a_{n-2}$

Existence and Uniqueness

THEOREM: For any $a_0, a_1, \ldots, a_{k-1}$, $a_n = \sum_{i=1}^{k} c_i a_{n-i} + F(n)$ has a unique solution $\{x_n\}_{n \geq 0}$ such that $x_n = a_n$ for all $0 \leq n < k$.

- **Existence**:
  - $x_n = a_n$ for all $0 \leq n < k$
  - $x_n = c_1 x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} + F(n)$ for all $n \geq k$
- **Uniqueness**:
  a) $x'_n = a_n$ for all $0 \leq n < k$
  b) $x'_n = c_1\, x'_{n-1} + c_2 x'_{n-2} + \cdots + c_k x'_{n-k} + F(n) \;\; (n \geq k)$
  c) $x_n = a_n$ for all $0 \leq n < k$
  d) $x_n = c_1\, x_{n-1} + c_2 x_{n-2} + \cdots + c_k x_{n-k} + F(n) \;\; (n \geq k)$
  - a) + c) $\Rightarrow x'_n = x_n$ for all $0 \leq n < k$
  - b) + d) $\Rightarrow x'_n = x_n$ for all $n \geq k$

General Solutions

THEOREM: If $\{x_n\}$ is a solution of $a_n = \sum_{i=1}^{k} c_i a_{n-i} + F(n)$, then $\{z_n\}$ is a solution iff $z_n = x_n + y_n$ for some solution $\{y_n\}$ of the associated LHRR $a_n = \sum_{i=1}^{k} c_i a_{n-i}$.

- $\Leftarrow$: we prove that $z_n = x_n + y_n$ is a solution of the LNRR
  - $x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} + F(n)$
  - $y_n = c_1 y_{n-1} + \cdots + c_k y_{n-k}$
  - $x_n + y_n = c_1(x_{n-1} + y_{n-1}) + \cdots + c_k\,(x_{n-k} + y_{n-k}) + F(n)$
  - $\{x_n + y_n\}$ is a solution of the LNRR
- $\Rightarrow$: we prove that a solution $\{z_n\}$ of the LNRR has the form $z_n = x_n + y_n$
  - $x_n = c_1 x_{n-1} + \cdots + c_k x_{n-k} + F(n)$
  - $z_n = c_1 z_{n-1} + \cdots + c_k z_{n-k} + F(n)$
  - Let $y_n = z_n - x_n$. Then $y_n = c_1 y_{n-1} + \cdots + c_k y_{n-k}$
    - $\{y_n\}$ is a solution of the associated LHRR

Particular Solutions

THEOREM: Let $a_n = \sum_{i=1}^{k} c_i a_{n-i} + F(n)$ be an LNRR with $F(n) = \left(f_l n^l + \cdots + f_1 n + f_0\right) s^n = f(n) s^n$, where $c_i, f_j \in \mathbb{R}$. Suppose that s is a root of $(r^k - c_1 r^{k-1} - \cdots - c_k)$ with multiplicity m, then the LNRR has a particular solution of the form $x_n = \left(p_l n^l + \cdots + p_1 n + p_0\right) s^n n^m$, where $\{p_j\}$ are undetermined coefficients.

EXAMPLE: Particular solution for $a_n = 4a_{n-1} - 4a_{n-2} + n2^n$.

- Characteristic equation of the associated LHRR: $r^2 - 4r + 4 = 0$
- Characteristic roots: $r_1 = 2$ (with multiplicity $m_1 = 2$)
  - Particular solution: $x_n = (p_1 n + p_0) 2^n n^2$

Solving LNRR

EXAMPLE: Solve $a_n = 4a_{n-1} - 4a_{n-2} + n2^n$ with $a_0 = 1, a_1 = 4$.

- Particular solution of the LNRR: $x_n = (p_1 n + p_0)2^n n^2$
- General solution of the associated LHRR: $y_n = \left(\alpha_{1,0} + \alpha_{1,1} n\right)2^n$
- General solution of the LNRR:
  - $z_n = x_n + y_n = \left(\alpha_{1,0} + \alpha_{1,1} n + p_0 n^2 + p_1 n^3\right)2^n$
  - Initial conditions give an equation system:
    - $a_0 = 1$: $\left(\alpha_{1,0} + \alpha_{1,1} \cdot 0 + p_0 \cdot 0^2 + p_1 \cdot 0^3\right)2^0 = 1$
    - $a_1 = 4$: $\left(\alpha_{1,0} + \alpha_{1,1} \cdot 1 + p_0 \cdot 1^2 + p_1 \cdot 1^3\right)2^1 = 4$
    - $a_2 = 20$: $\left(\alpha_{1,0} + \alpha_{1,1} \cdot 2 + p_0 \cdot 2^2 + p_1 \cdot 2^3\right)2^2 = 20$
    - $a_3 = 88$: $\left(\alpha_{1,0} + \alpha_{1,1} \cdot 3 + p_0 \cdot 3^2 + p_1 \cdot 3^3\right)2^3 = 88$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,0} + \alpha_{1,1} + p_0 + p_1 & = 2 \\ \alpha_{1,0} + 2\alpha_{1,1} + 4p_0 + 8p_1 & = 5 \\ \alpha_{1,0} + 3\alpha_{1,1} + 9p_0 + 27p_1 & = 11 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_0 + p_1 & = 1 \\ 2\alpha_{1,1} + 4p_0 + 8p_1 & = 4 \\ 3\alpha_{1,1} + 9p_0 + 27p_1 & = 10 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_0 + p_1 & = 1 \\ 2p_0 + 6p_1 & = 2 \\ 6p_0 + 24p_1 & = 7 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_0 + p_1 & = 1 \\ 2p_0 + 6p_1 & = 2 \\ 6p_1 & = 1 \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_0 + p_1 & = 1 \\ 2p_0 + 6p_1 & = 2 \\ p_1 & = \frac{1}{6} \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_0 & = \frac{5}{6} \\ 2p_0 & = 1 \\ p_1 & = \frac{1}{6} \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} + p_0 & = \frac{5}{6} \\ p_0 & = \frac{1}{2} \\ p_1 & = \frac{1}{6} \end{cases}$$

$$\begin{cases} \alpha_{1,0} & = 1 \\ \alpha_{1,1} & = \frac{1}{3} \\ p_0 & = \frac{1}{2} \\ p_1 & = \frac{1}{6} \end{cases}$$

- The solution $(\alpha_{1,0}, \alpha_{1,1}, p_0, p_1) = \left(1, \frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$ gives

$$a_n = \left(1 + \frac{1}{3}n + \frac{1}{2}n^2 + \frac{1}{6}n^3\right) 2^n$$

Generating Functions

DEFINITION: The **generating function** of a sequence $\{a_r\}_{r=0}^{\infty}$ is defined as $G(x) = \sum_{r=0}^{\infty} a_r x^r$.

- Generating functions are **formal power series**.
- We do not discuss their convergence.

EXAMPLE: generating functions of sequences

- $a_r = 3, G(x) = 3(1 + x + \cdots + x^r + \cdots)$
- $a_r = 2^r, G(x) = 1 + 2x + \cdots + (2x)^r + \cdots$
- $a_r = \binom{n}{r}, G(x) = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n}x^n$

DEFINITION: Let $A(x) = \sum_{r=0}^{\infty} a_r x^r$, $B(x) = \sum_{r=0}^{\infty} b_r x^r$

- $A(x) = B(x)$ if $a_r = b_r$ for all $r = 0,1,2, \ldots$

Operations

DEFINITION: Let $A(x) = \sum_{r=0}^{\infty} a_r x^r$, $B(x) = \sum_{r=0}^{\infty} b_r x^r$

- $A(x) + B(x) = \sum_{r=0}^{\infty}(a_r + b_r)\, x^r$
- $A(x) - B(x) = \sum_{r=0}^{\infty}(a_r - b_r)\, x^r$
- $A(x) \cdot B(x) = \sum_{r=0}^{\infty}(\sum_{j=0}^{r} a_j b_{r-j})\, x^r$
- $\lambda \cdot A(x) = \sum_{r=0}^{\infty} \lambda a_r x^r$ for any constant $\lambda \in \mathbb{R}$
- We say that $B(x)$ is an **inverse** of $A(x)$ if $A(x)B(x) = 1$.
    - The inverse of $A(x)$: $A^{-1}(x)$
    - When $A(x)$ has an inverse, define $\frac{C(x)}{A(x)} = A^{-1}(x) \cdot C(x)$

Operations

THEOREM: $A(x) = \sum_{r=0}^{\infty} a_r x^r$ has an inverse iff $a_0 \neq 0$.

EXAMPLE: Let $A(x) = \sum_{r=0}^{\infty} x^r$. Find $A^{-1}(x)$.

- $a_0 = 1 \neq 0$: $A^{-1}(x)$ exists
- Denote $A^{-1}(x) = \sum_{r=0}^{\infty} b_r x^r$; $b_0, b_1, \ldots$ are undetermined coefficients
- $A(x)A^{-1}(x) = 1$:
  - $(1 + x + x^2 + \cdots)(b_0 + b_1 x + b_2 x^2 + \cdots) = 1 + 0 \cdot x + 0 \cdot x^2 + \cdots$
    - Coefficient of x^0: $b_0 = 1$
    - Coefficient of x^1: $b_1 + b_0 = 0$
    - Coefficient of x^2: $b_2 + b_1 + b_0 = 0$
    - Coefficient of x^r: $b_r + b_{r-1} + \cdots + b_0 = 0$
      - $b_1 = -1, b_2 = 0, \ldots, b_r = 0$
      - $A^{-1}(x) = 1 - x$