Discrete Mathematics

T-step, T-route, number of T-routes, André's reflection principle, Bertrand's ballot problem, Catalan number

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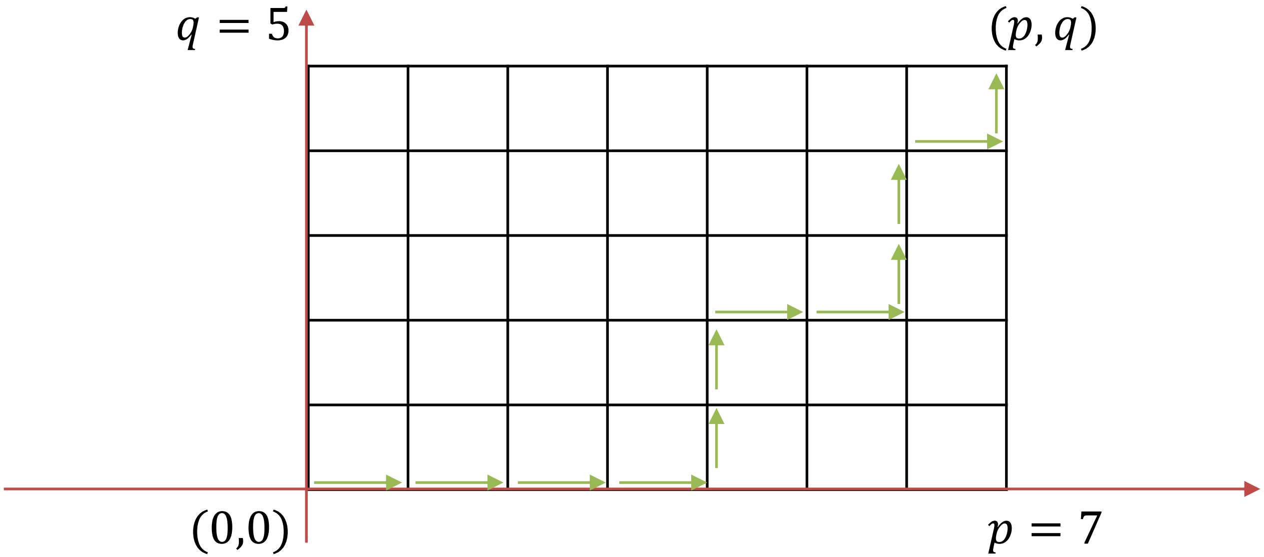
Shortest Path

DEFINITION: A $\boldsymbol{p \times q}$**-grid** is a collection of pq squares of side length 1, organized as a rectangle of side length p and q.



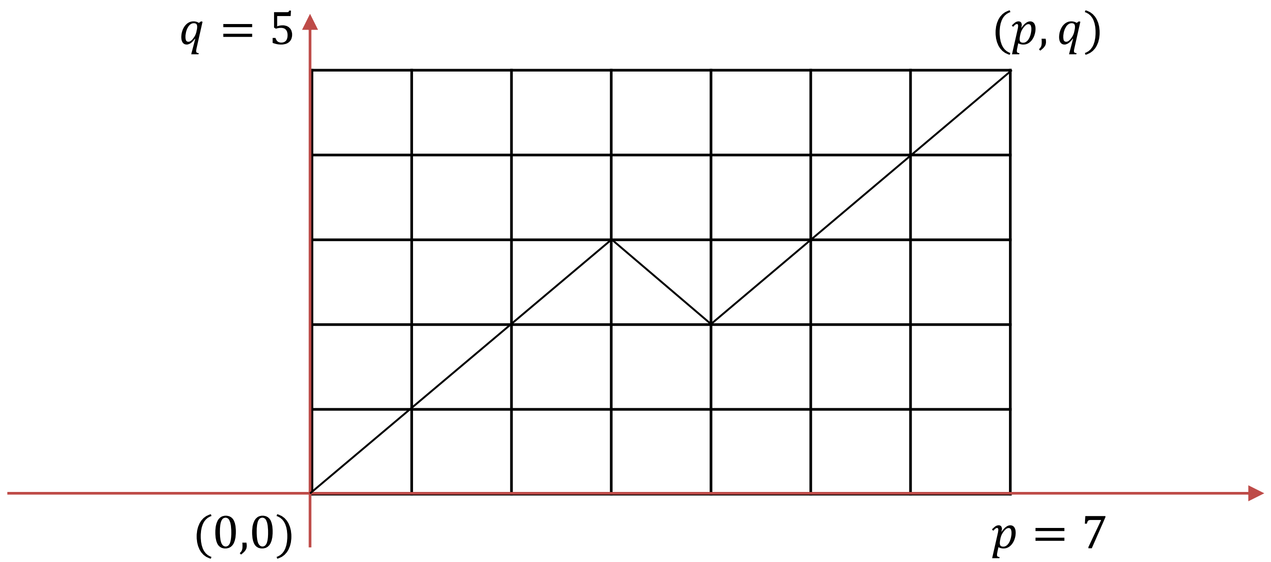
THEOREM: # of shortest paths from (0,0) to (p, q) is $\frac{(p+q)!}{p!q!}$.

- Let $A = \{p \cdot\rightarrow, q \cdot\uparrow\}$ be a $(p+q)$-multiset.
- # of shortest paths=# of permutations of A.

T-Route

DEFINITION: Let $A = (x, y), B \in \mathbb{Z}^2$. //**integral (lattice) points**

- A **T-Step** at A is a segment from A to $(x + 1, y + 1)$ or $(x + 1, y - 1)$.
- A **T-Route** from A to B is a route where each step is a T-step.



T-Route

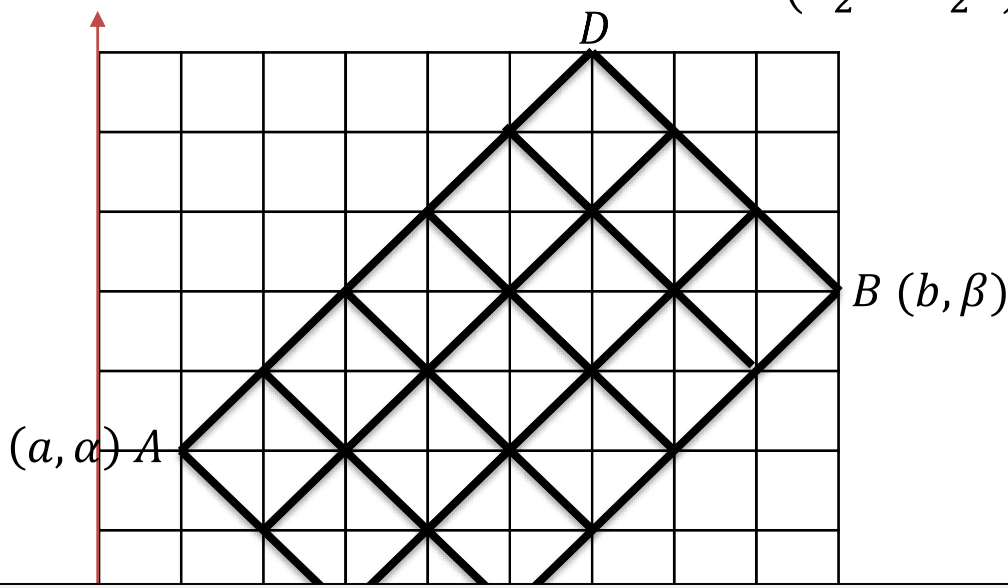
THEOREM: There is a T-route from $A = (a, \alpha)$ to $B = (b, \beta)$ only if (1) $b > a$; (2) $b - a \geq |\beta - \alpha|$; and (3) $2|\,(b + \beta - a - \alpha)$.

- Let $A = P_0, P_1, \dots, P_k = B$ be a T-route from A to B, where $P_i = (x_i, y_i)$.
  - $x_0 = a, y_0 = \alpha; x_k = b, y_k = \beta;$
  - $x_i - x_{i-1} = 1; y_i - y_{i-1} \in \{\pm 1\}$ for every $i = 1,2, \dots, k$
- $b - a = x_k - x_0 = (x_k - x_{k-1}) + (x_{k-1} - x_{k-2}) + \cdots + (x_1 - x_0) = k > 0$
- $\beta - \alpha = y_k - y_0 = (y_k - y_{k-1}) + (y_{k-1} - y_{k-2}) + \cdots + (y_1 - y_0)$
  - $|\beta - \alpha| \leq |y_k - y_{k-1}| + |y_{k-1} - y_{k-2}| + \cdots + |y_1 - y_0| = k = b - a$
- $b + \beta - a - \alpha = \sum_{i=1}^{k} (y_i - y_{i-1} + x_i - x_{i-1})$
  - $y_i - y_{i-1} + x_i - x_{i-1} \in \{0,2\}$
  - $2|(b + \beta - a - \alpha)$

REMARK: The T-condition (1)+(2)+(3) is also sufficient for the existence of a T-route.

Number of T-Routes

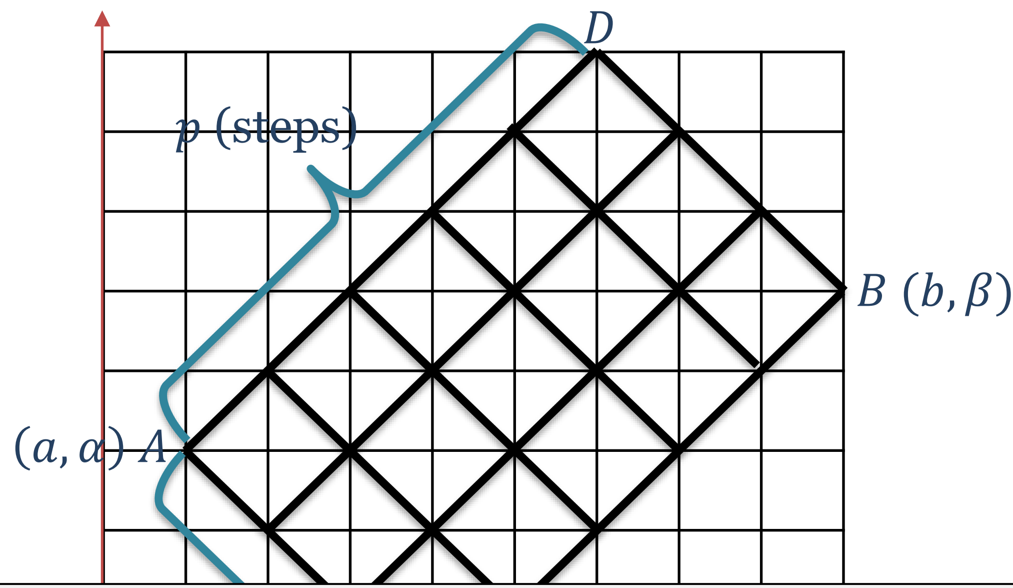
THEOREM: If $A = (a, \alpha), B = (b, \beta)$ satisfy the T-condition. Then the number of T-routes from A to B is $\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta-\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta-\alpha}{2}\right)!}$.



The number of T routes from A to B= the number of shortest paths from A to B on the $p \times q$-grid.

- $AC\colon y-\alpha=-(x-a);\ AD\colon y-\alpha=x-a;$
- $BC\colon y-\beta=x-b;\ BD\colon y-\beta=-(x-b).$
- $p=\frac{1}{2}\cdot(a+b-\alpha+\beta)-a=\frac{1}{2}\cdot(b-a)+\frac{1}{2}\cdot(\beta-\alpha)$
- $q=\frac{1}{2}\cdot(\alpha-\beta+a+b)-a=\frac{1}{2}\cdot(b-a)-\frac{1}{2}\cdot(\beta-\alpha)$

$1/2\cdot(a+b-\alpha+\beta,\alpha+\beta-a+b)$

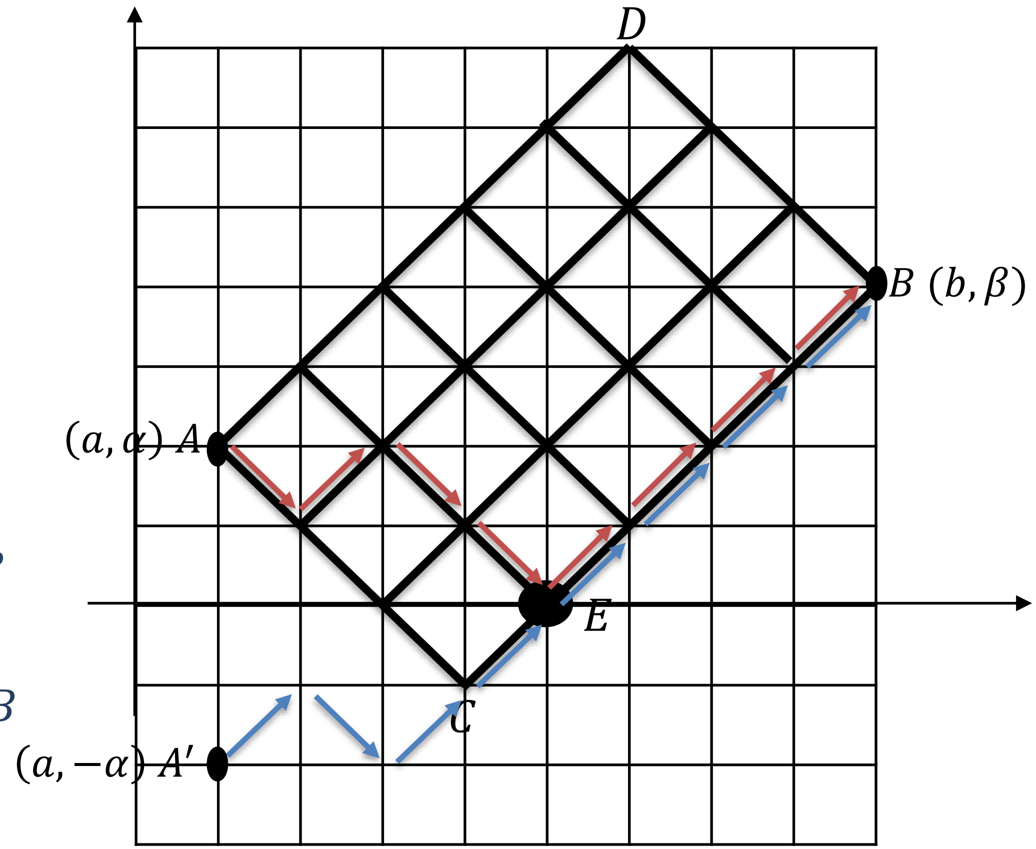


The number of T routes from A to B= the number of shortest paths from A to B on the $p\times q$-grid. This number is $\frac{(p+q)!}{p!q!}=\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta-\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta-\alpha}{2}\right)!}$

Number of T Routes

THEOREM: Let $A=(a,\alpha), B=(b,\beta)$ satisfy the T-condition, where $\alpha,\beta>0$. Then # of T-routes from A to B that intersect the x-axis=# of T routes from $A'(a,-\alpha)$ to B. And this number is $\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta+\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta+\alpha}{2}\right)!}$.

- Ω: the set of T-routes from A to B
- $U=\{\omega\in\Omega: \omega \text{ intersects } y=0\}$
- V: the set of T-routes from A' to B
- $f: U\to V \quad u\mapsto f(u)$
    - u: the brown T route
    - $f(u)$: the blue T route
    - f is a bijection



$$|U|=|V|=\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta+\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta+\alpha}{2}\right)!}$$

André's Reflection Principle-D. André, Solution directe du problème résolu par M. Bertrand, Comptes Rendus Acad. Sci. Paris 105 (1887), 436–437.

Number of T Routes

THEOREM: Let $A = (a, \alpha), B = (b, \beta) \in \mathbb{Z}^2$ satisfy the T-condition, where $\alpha, \beta > 0$. Then # of T routes from A to B that do not intersect the x-axis is

$$\frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta-\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta-\alpha}{2}\right)!} - \frac{(b-a)!}{\left(\frac{b-a}{2}+\frac{\beta+\alpha}{2}\right)!\left(\frac{b-a}{2}-\frac{\beta+\alpha}{2}\right)!}$$

Bertrand's Ballot Problem

History: First published by **W. A. Whitworth** in **1878** but named after **Joseph Louis François Bertrand** who rediscovered it in **1887**.

Special case: there are two candidates A and B in an election. Each receives n votes. What is the probability p_n that A will never trail B during the count of votes?

EXAMPLE. AABABBBAAB is bad, since after seven votes, A receives 3 while B receives 4.

Solution

- Define a variable x_i for $i = 1,2, \ldots, 2n$ ($2n$ votes in total)

$$x_i = \begin{cases} 0 & \text{A receives the } i\text{th vote} \\ 1 & \text{B receives the } i\text{th vote} \end{cases}$$

- The sequence $x_1 x_2 \ldots x_{2n}$ is a ballot sequence such that A never trails B if and only if

$$\begin{cases} x_1 + x_2 + \cdots + x_{2n} = n \\ x_1 + x_2 + \cdots + x_i \leq i/2, i = 1,2, \ldots, 2n-1 \quad (*) \\ x_i \in \{0,1\}, i = 1,2, \ldots, 2n \end{cases}$$

- C_n: The number of solution of the system (*)
- The probability that A never trials B is

$$p_n = C_n / \binom{2n}{n}$$

Catalan Number

1838, Catalan (1814-1894); 1730s, Ming Antu (1692-1763)

THEOREM: C_n is the number of solutions of the equation system

$$\begin{cases} x_1 + x_2 + \cdots + x_{2n} = n \\ x_1 + x_2 + \cdots + x_i \leq i/2, i = 1,2,\ldots,2n-1 \\ x_i \in \{0,1\}, i = 1,2,\ldots,2n \end{cases}$$

In particular, $C_n = \frac{(2n)!}{n!(n+1)!}$

- $\mathcal{C}_n$ is the set of all solutions of the equation system
- $\mathcal{T}_n$: the set of all T-routes from $(1,2)$ to $(2n, 1)$ above the x-axis
- A map $f: \mathcal{C}_n \to \mathcal{T}_n$ Given a solution $(x_1, x_2, \ldots, x_{2n})$ of the equation system
  - Let $P_i = (i, 1 + 1 - 2x_1 + \cdots + 1 - 2x_i)$ for all $i = 1,2, \ldots., 2n$
  - $1 + 1 - 2x_1 + \cdots + 1 - 2x_i > 0$ for $i = 1,2, \ldots 2n$
  - $P_1 = (1, 1 + 1 - 2x_1) = (1,2)$; $P_{2n} = (2n, 1)$
  - $P_1, P_2, \ldots, P_{2n}$ is a T-route above the x-axis

Catalan Number

1838, Catalan (1814-1894); 1730s, Ming Antu (1692-1763)

- A map $g: \mathcal{T}_n \to \mathcal{C}_n$ Let $\{P_i = (u_i, v_i): 1 \le i \le 2n\}$ be the points on a T-Route from $P_1 = (1,2)$ to $P_{2n} = (2n, 1)$, where the T-Route is above the x-axis
  - $x_1 = (2 - v_1)/2 = 0$
  - $x_i = (1 - (v_i - v_{i-1}))/2 \in \{0,1\}, i = 2, \dots, 2n$
  - $x_1 + x_2 + \cdots + x_{2n} = (2n + 1 - v_{2n})/2 = n$
  - $x_1 + x_2 + \cdots + x_i = (i + 1 - v_i)/2 < i/2, i = 1,2, \dots, 2n$
- $A = P_1 = (1,2)$: $a = 1, \alpha = 2; B = P_{2n} = (2n, 1)$: $b = 2n, \beta = 1$
  - $|\mathcal{C}_n| = \frac{(2n-1)!}{(n-1)!n!} - \frac{(2n-1)!}{(n+1)!(n-2)!} = \frac{(2n)!}{n!(n+1)!}$

Parenthesization: Let $a_1, a_2, \dots, a_n, a_{n+1}$ be $n + 1$ numbers. Let $*$ be any binary operator. Let C_n be the number of different ways of parenthesizing $a_1 * a_2 * \cdots * a_n * a_{n+1}$ such that the calculation is not ambiguous. What is C_n?

- Eugène Charles **Catalan** (1838)