Discrete Mathematics: Lecture 19

Part III. Mathematical Logic

tautological implications, argument, predicate logic, quantifiers

Xuming He
Associate Professor

School of Information Science and Technology
ShanghaiTech University

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Notes by Prof. Liangfeng Zhang and Xuming He

Tautological Implications

DEFINITION: Let A and B be WFFs in propositional variables $p_1, \ldots, p_n$.

- A **tautologically implies** (重言蕴涵) B if every truth assignment that causes A to be true causes B to be true.
  - Notation: $A \Rightarrow B$, called a **tautological implication**
  - $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$; $B^{-1}(\mathbf{F}) \subseteq A^{-1}(\mathbf{F})$

THEOREM: $A \Rightarrow B$ iff $A \rightarrow B$ is a tautology.

- $A \Rightarrow B$ iff $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$ iff $A \rightarrow B$ is a tautology

THEOREM: $A \Rightarrow B$ iff $A \wedge \neg B$ is a contradiction.

- $A \rightarrow B \equiv \neg A \vee B \equiv \neg(A \wedge \neg B)$

Proving $\boldsymbol{A} \Rightarrow \boldsymbol{B}$: (1) $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$; (2) $B^{-1}(\mathbf{F}) \subseteq A^{-1}(\mathbf{F})$; (3) $A \rightarrow B$ is a tautology; (4) $A \wedge \neg B$ is a contradiction

Tautological Implications

Name	Tautological Implication	NO.
Conjunction(合取)	$(P) \wedge (Q) \Rightarrow P \wedge Q$	1
Simplification(化简)	$P \wedge Q \Rightarrow P$	2
Addition(附加)	$P \Rightarrow P \vee Q$	3
Modus ponens(假言推理)	$P \wedge (P \to Q) \Rightarrow Q$	4
Modus tollens(拒取)	$\neg Q \wedge (P \to Q) \Rightarrow \neg P$	5
Disjunctive syllogism(析取三段论)	$\neg P \wedge (P \vee Q) \Rightarrow Q$	6
Hypothetical syllogism(假言三段论)	$(P \to Q) \wedge (Q \to R) \Rightarrow (P \to R)$	7
Resolution (归结)	$(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$	8

Proofs for 5 and 6 **(Method 3)**

EXAMPLE: $\neg Q \wedge (P \to Q) \Rightarrow \neg P$

- $A = \neg Q \wedge (P \to Q), B = \neg P.$
- $A \to B \equiv \neg\big(\neg Q \wedge (P \to Q)\big) \vee \neg P$

$$\equiv \big(Q \vee \neg(P \to Q)\big) \vee \neg P$$

$$\equiv (\neg P \vee Q) \vee \neg(P \to Q)$$

$$\equiv \mathbf{T}$$

EXAMPLE: $\neg P \wedge (P \vee Q) \Rightarrow Q$

- $A = \neg P \wedge (P \vee Q), B = Q.$
- $A \to B \equiv \neg\big(\neg P \wedge (P \vee Q)\big) \vee Q$

$$\equiv \big(P \vee \neg(P \vee Q)\big) \vee Q$$

$$\equiv (\neg(P \vee Q) \vee P) \vee Q$$

$$\equiv \neg(P \vee Q) \vee (P \vee Q)$$

$$\equiv \mathbf{T}$$

Proofs for 7 and 8 **(Method 4)**

EXAMPLE: $(P \to Q) \land (Q \to R) \Rightarrow P \to R$

- $A = (P \to Q) \land (Q \to R);\ B = (P \to R).$
- $A \land \neg B \equiv (\neg P \lor Q) \land (\neg Q \lor R) \land (P \land \neg R)$

$$\equiv \big((\neg P \lor Q) \land P\big) \land \big((\neg Q \lor R) \land \neg R\big)$$

$$\equiv \big((\neg P \land P) \lor (Q \land P)\big) \land \big((\neg Q \land \neg R) \lor (R \land \neg R)\big)$$

$$\equiv (Q \land P) \land (\neg Q \land \neg R)$$

$$\equiv \mathbf{F}$$

EXAMPLE: $(P \lor Q) \land (\neg P \lor R) \Rightarrow Q \lor R$

- $A = (P \lor Q) \land (\neg P \lor R); B = (Q \lor R).$
- $A \land \neg B \equiv (P \lor Q) \land (\neg P \lor R) \land (\neg Q \land \neg R)$

$$\equiv \big((P \lor Q) \land \neg Q\big) \land ((\neg P \lor R) \land \neg R)$$

$$\equiv (P \land \neg Q) \land (\neg P \land \neg R)$$

$$\equiv \mathbf{F}$$

Method 1 & 2

EXAMPLE: $(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \Rightarrow (P \leftrightarrow R)$

- $A = (P \leftrightarrow Q) \wedge (Q \leftrightarrow R); B = (P \leftrightarrow R).$
- $A = \mathbf{T}$ iff $(P \leftrightarrow Q) = \mathbf{T}$ and $(Q \leftrightarrow R) = \mathbf{T}$ iff $P = Q$ and $Q = R$
  - $A^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T}, \mathbf{T}), (\mathbf{F}, \mathbf{F}, \mathbf{F})\}$
- $B = \mathbf{T}$ iff $P = R$
  - $B^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T}, \mathbf{T}), (\mathbf{T}, \mathbf{F}, \mathbf{T}), (\mathbf{F}, \mathbf{T}, \mathbf{F}), (\mathbf{F}, \mathbf{F}, \mathbf{F})\}$
- $A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$

EXAMPLE: $(Q \rightarrow R) \Rightarrow ((P \vee Q) \rightarrow (P \vee R))$

- $A = Q \rightarrow R; B = ((P \vee Q) \rightarrow (P \vee R)).$
- $A = \mathbf{F}$ iff $(Q, R) = (\mathbf{T}, \mathbf{F})$
  - $A^{-1}(\mathbf{F}) = \{(\mathbf{T}, \mathbf{T}, \mathbf{F}), (\mathbf{F}, \mathbf{T}, \mathbf{F})\}$iff $(P, Q) \neq (\mathbf{F}, \mathbf{F})$ and $(P, R) = (\mathbf{F}, \mathbf{F})$
- $B = \mathbf{F}$ iff $(P \vee Q, P \vee R) = (\mathbf{T}, \mathbf{F})$
  - $B^{-1}(\mathbf{F}) = \{(\mathbf{F}, \mathbf{T}, \mathbf{F})\}$
- $A^{-1}(\mathbf{F}) \supseteq B^{-1}(\mathbf{F})$

More Examples

EXAMPLE: $(P \to R) \land (Q \to S) \land (P \lor Q) \Rightarrow R \lor S$

- $A = (P \to R) \land (Q \to S) \land (P \lor Q); B = R \lor S$
- $A \land \neg B \equiv (P \to R) \land (Q \to S) \land (P \lor Q) \land \neg(R \lor S)$

$$
\begin{aligned}
&\equiv (\neg P \lor R) \land (\neg Q \lor S) \land (P \lor Q) \land (\neg R \land \neg S) \\
&\equiv ((\neg P \lor R) \land \neg R)) \land ((\neg Q \lor S) \land \neg S) \land (P \lor Q) \\
&\equiv ((\neg P \land \neg R) \lor (R \land \neg R)) \land ((\neg Q \land \neg S) \lor (S \land \neg S)) \land (P \lor Q) \\
&\equiv ((\neg P \land \neg R) \lor \mathbf{F}) \land ((\neg Q \land \neg S) \lor \mathbf{F}) \land (P \lor Q) \\
&\equiv (\neg P \land \neg R) \land (\neg Q \land \neg S) \land (P \lor Q) \\
&\equiv \neg R \land (\neg Q \land \neg S) \land (\neg P \land (P \lor Q)) \\
&\equiv \neg R \land (\neg Q \land \neg S) \land ((\neg P \land P) \lor (\neg P \land Q)) \\
&\equiv \neg R \land (\neg Q \land \neg S) \land (\mathbf{F} \lor (\neg P \land Q)) \\
&\equiv \neg R \land (\neg Q \land \neg S) \land (\neg P \land Q) \\
&\equiv \neg R \land \neg S \land \neg P \land (\neg Q \land Q) \\
&\equiv \neg R \land \neg S \land \neg P \land \mathbf{F} \\
&\equiv \mathbf{F}
\end{aligned}
$$

Argument

DEFINITION: An **argument** (论证) is a sequence of propositions

- **Conclusion**(结论): the final proposition
- **Premises**(假设)**:** all the other propositions
- **Valid**(有效): the truth of premises implies that of the conclusion
- **Proof** (证明): a valid argument that establishes the truth of a conclusion

EXAMPLE: a valid argument, a proof

- If $\{2^{-n}\}$ is convergent, then $\{2^{-n}\}$ has a convergent subsequence.
- $\{2^{-n}\}$ is convergent.
- $\{2^{-n}\}$ has a convergent subsequence.

Argument Form

DEFINITION: An **argument form**(论证形式) is a sequence of formulas.

- Replacing propositions in an argument with propositional variables
- **Valid**(有效)**:** no matter which propositions are substituted for the propositional variables, the truth of conclusion follows from the truth of premises

EXAMPLE: a valid argument form and an invalid argument form

$p \rightarrow q$

p

q

valid

p: $\{(-1)^n\}$ is convergent.

q: $\{(-1)^n\}$ has a convergent subsequence.

$p \rightarrow q$: If $\{(-1)^n\}$ is convergent, then $\{(-1)^n\}$ has a convergent subsequence.

$p \rightarrow q$

$\neg p$

$\neg q$

invalid

$\neg p$: $\{(-1)^n\}$ is not convergent.

$\neg q$: $\{(-1)^n\}$ does not have a convergent subsequence.

The truth of $\neg p$ and $p \rightarrow q$ does not imply that of $\neg q$

Rules of inference

- **Rules of inference**(推理规则): relatively simple valid argument forms from tautological implications

Name	Tautological Implication
Conjunction(合取)	$(P) \wedge (Q) \Rightarrow P \wedge Q$
Simplification(化简)	$P \wedge Q \Rightarrow P$
Addition(附加)	$P \Rightarrow P \vee Q$
Modus ponens(假言推理)	$P \wedge (P \to Q) \Rightarrow Q$
Modus tollens(拒取)	$\neg Q \wedge (P \to Q) \Rightarrow \neg P$
Disjunctive syllogism(析取三段论)	$\neg P \wedge (P \vee Q) \Rightarrow Q$
Hypothetical syllogism(假言三段论)	$(P \to Q) \wedge (Q \to R) \Rightarrow (P \to R)$
Resolution (归结)	$(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$

Rule of Inference	*Tautology*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$
p $\therefore p \vee q$	$p \to (p \vee q)$
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$

Building Arguments

QUESTION: Given the premises $P_1, \dots, P_n$, show a conclusion Q, that is, show that $P_1 \wedge \cdots \wedge P_n \Rightarrow Q$.

Name	Operations
Premise	Introduce the given formulas $P_1, \dots, P_n$ in the process of constructing proofs.
Conclusion	Quote the intermediate formula that have been deducted.
Rule of replacement	Replace a formula with a logically equivalent formula.
Rules of Inference	Deduct a new formula with a tautological implication.
Rule of substitution	Deduct a formula from a tautology.

Building Arguments

EXAMPLE: Show that the premises 1, 2, 3, and 4 lead to conclusion 5.

1. “It is not sunny this afternoon and it is colder than yesterday,”
2. “We will go swimming only if it is sunny,”
3. “If we do not go swimming, then we will take a canoe trip,”
4. “If we take a canoe trip, then we will be home by sunset”
5. “We will be home by sunset.”

- **Translating the premises and the conclusion into formulas. Let**
  - p: “It is sunny this afternoon”
  - q: “It is colder than yesterday”
  - r: “We will go swimming”
  - s: “We will take a canoe trip”
  - t: “We will be home by sunset”
    - The premises are $\neg p \land q, r \rightarrow p, \neg r \rightarrow s$, and $s \rightarrow t$.
    - The conclusion is t.
- **Question: ?**$(\neg p \land q) \land (r \rightarrow p) \land (\neg r \rightarrow s) \land (s \rightarrow t) \Rightarrow t$
  - Can be proven with truth table. 32 rows!

Building Arguments

EXAMPLE: Show that the premises 1, 2, 3, and 4 lead to conclusion 5.

1. “It is not sunny this afternoon and it is colder than yesterday,”
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3. “If we do not go swimming, then we will take a canoe trip,”
4. “If we take a canoe trip, then we will be home by sunset”
5. “We will be home by sunset.”

- **Show that** $(\neg p \land q) \land (r \to p) \land (\neg r \to s) \land (s \to t) \Rightarrow t$

(1)	$\neg p \land q$	Premise
(2)	$\neg p$	Simplification using (1)
(3)	$r \to p$	Premise
(4)	$\neg r$	Modus tollens using (2) and (3)
(5)	$\neg r \to s$	Premise
(6)	s	Modus ponens using (4) and (5)
(7)	$s \to t$	Premise
(8)	t	Modus ponens using (6) and (7)

Building Arguments

EXAMPLE: Show that $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S) \Rightarrow S \vee R$

(1)	$P \vee Q$	Premise
(2)	$\neg P \rightarrow Q$	Rule of replacement applied to (1)
(3)	$Q \rightarrow S$	Premise
(4)	$\neg P \rightarrow S$	Hypothetical syllogism applied to (2) and (3)
(5)	$\neg S \rightarrow P$	Rule of replacement applied to (4)
(6)	$P \rightarrow R$	Premise
(7)	$\neg S \rightarrow R$	Hypothetical syllogism applied to (5) and (6)
(8)	$S \vee R$	Rule of replacement applied to (7)

Limitation of Propositional Logic

EXAMPLE: What is the underlying tautological implication in the following proof?

- If 1/3 is a rational number, then 1/3 is a real number.
- 1/3 is a rational number.
- 1/3 is a real number.
  - $q \rightarrow r$:“If 1/3 is a rational number, then 1/3 is a real number.
  - q:“1/3 is a rational number”
  - r:“1/3 is a real number”
    - What is the underlying tautological implication?
      - $(q \rightarrow r) \land q \Rightarrow r$
        - YES. This is a tautological implication.

Limitation of Propositional Logic

EXAMPLE: What is the underlying tautological implication in the following proof?

- All rational numbers are real numbers
- 1/3 is a rational number
- 1/3 is a real number
  - p:"All rational numbers are real numbers"
  - q:"1/3 is a rational number"
  - r:"1/3 is a real number"
    - What is the underlying tautological implication?
      - $p \wedge q \Rightarrow r$?
        - NO. $p \wedge q \rightarrow r$ is not a tautology.
          - Why is this a proof?
            - We need **predicate logic**.

Predicate and Individual

Predicate谓词: describe the property of the subject term (in a sentence)

- A predicate is a function from a domain of individuals to $\{\mathbf{T}, \mathbf{F}\}$
- **$\boldsymbol{n}$-ary predicate**$_{n}$元谓词: a predicate on n individuals
  - I: "is an integer" // unary
  - G: "is greater than" //binary
- **Predicate constant**谓词常项: a concrete predicate // I, G
- **Predicate variable**谓词变项: a symbol that represents any predicate

Individual个体词: the object you are considering (in a sentence)

- "$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}$ is an integer"
- "e^{π} is greater than π^{e}"
  - **Individual Constant**个体常项: $\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}$, e^{π}, π^{e}
  - **Individual Variable**个体变项: x, y, z
  - **Domain**个体域: the set of all individuals in consideration

From Predicates to Propositions

Propositional function命题函数: $P(x_1, \ldots, x_n)$, where P is an n-ary predicate

- $P(x, y)$:“x is greater than y”
- $P(x, y)$ gives a proposition when we assign values to x, y
  - $P(e^{\pi}, \pi^{e})$ is a proposition (a true proposition)
- $P(x, y)$ is not a proposition

EXAMPLE: p:“Alice’s father is a doctor”; q:“Bob’s father is a doctor”

- Individuals: Alice’s father, Bob’s father; Predicate D: “is a doctor”
- $p = D(\text{Alice's father})$, $q = D(\text{Bob's father})$

Function of Individuals: a map on the domain of individuals

- $f(x) = x$’s father
- $p = D\big(f(\text{Alice})\big)$; $q = D(f(\text{Bob}))$

Universal Quantifier

DEFINITION: Let $P(x)$ be a propositional function. The **universal quantification**全称量化 of $P(x)$ is "$P(x)$ for all x in the domain".

- notation: $\forall x\ P(x)$; read as "for all x $P(x)$" or "for every x $P(x)$"
  - "$\forall$" is called the **universal quantifier**全称量词
  - "$\forall x\ P(x)$" is true iff $P(x)$ is true for every x in the domain
  - "$\forall x\ P(x)$" is false iff there is an x_0 in the domain such that $P(x_0)$ is false
    - **Counterexample**反例: an x_0 such that $P(x_0)$ is false

EXAMPLE: $P(n)$:"$n^2 + n + 41$ is a prime"

- When domain = natural numbers, "$\forall n P(n)$" is "for every natural number n, $n^2 + n + 41$ is a prime"
- When domain is $D = \{0,1,\ldots,39\}$, "$\forall n P(n)$" is "for every $n \in D$, $n^2 + n + 41$ is a prime"

REMARK: If the domain is empty, then "$\forall x\ P(x)$" is true for any P.

Existential Quantifier

DEFINITION: Let $P(x)$ be a propositional function. The **existential quantification**存在量化 of $P(x)$ is "there is an x in the domain such that $P(x)$"

- notation: $\exists x\ P(x)$; read as "for some x $P(x)$" or "there is an x $\mathrm{s.t.}\, P(x)$"
  - "$\exists$" is called the **existential quantifier**存在量词
  - "$\exists x\ P(x)$" is true iff there is an x in the domain such that $P(x)$ is true
  - "$\exists x\ P(x)$" is false iff $P(x)$ is false for every x in the domain

EXAMPLE: $P(x)$:"$x^2 - x + 1 = 0$"

- "$\exists x\ P(x)$" is false when $D = \mathbb{R}$ and is true when $D = \mathbb{C}$

REMARK: <u>If the domain is empty, then "$\exists x\ P(x)$" is false for any P.</u>

REMARK: <u>if not stated, the individual can be anything.</u>