Discrete Mathematics
Lecture 15

Liangfeng Zhang
School of Information Science and Technology
ShanghaiTech University

Summary of Lecture 14

LHRR: $a_n = \sum_{i=1}^{k} c_i a_{n-i}$

- $r_1, r_2, \dots, r_t$ are char roots of multiplicities $m_1, m_2, \dots, m_t$
  - Char roots: roots of $r^k - c_1 r^{k-1} - \cdots - c_k$
- $a_n = \sum_{j=1}^{t} \left(\sum_{\ell=0}^{m_j-1} \alpha_{j,\ell}\, n^\ell \right) r_j^n$

LNRR: $a_n = \sum_{i=1}^{k} c_i a_{n-i} + \color{red}{(f_l n^l + \cdots + f_1 n + f_0) s^n}$

- s is a root of $(r^k - c_1 r^{k-1} - \cdots - c_k)$ of multiplicity m
- $a_n = (p_l n^l + \cdots + p_1 n + p_0) s^n n^m$

Summary of Lecture 14

Generating function: $\{a_n\}_{n=0}^{\infty} \leftrightarrow \sum_{n=0}^{\infty} a_n x^n$.

- $A(x) = B(x)$ if $a_n = b_n$ for all $n = 0,1,2,\ldots$
- $A(x) + B(x) = \sum_{n=0}^{\infty}(a_n + b_n)\, x^n$
- $A(x) - B(x) = \sum_{n=0}^{\infty}(a_n - b_n)\, x^n$
- $A(x) \cdot B(x) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_j b_{n-j}\right) x^n$
- $\lambda \cdot A(x) = \sum_{n=0}^{\infty} \lambda a_n x^n$ for any constant $\lambda \in \mathbb{R}$
- $B(x)$ is an **inverse** of $A(x)$ if $A(x)B(x) = 1$

Operations

THEOREM: $A(x)=\sum_{n=0}^{\infty} a_n x^n$ has an inverse iff $a_0 \neq 0$.

EXAMPLE: Let $A(x)=\sum_{n=0}^{\infty} x^n$. Find $A^{-1}(x)$.

- $a_0 = 1 \neq 0$: $A^{-1}(x)$ exists
- Denote $A^{-1}(x)=\sum_{n=0}^{\infty} b_n x^n$; $b_0, b_1, \ldots$ are undetermined coefficients
- $A(x)A^{-1}(x)=1$:
  - $(1+x+x^2+\cdots)(b_0+b_1x+b_2x^2+\cdots)=1+0\cdot x+0\cdot x^2+\cdots$
    - Coefficient of x^0: $b_0=1$
    - Coefficient of x^1: $b_1+b_0=0$
    - Coefficient of x^2: $b_2+b_1+b_0=0$
    - Coefficient of x^n: $b_n+b_{n-1}+\cdots+b_0=0$
      - $b_1=-1, b_2=0, \ldots, b_n=0$
      - $A^{-1}(x)=1-x$

Operations

DEFINITION: $A(x) = \sum_{n=0}^{\infty} a_n x^n$

- $A'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$
  - $A^{(0)}(x) = A(x)$
  - $A^{(k)}(x) = (A^{(k-1)}(x))'$ for all integers $k \geq 1$
- $\int A(x) \mathrm{d}x = \sum_{n=0}^{\infty} \frac{1}{n+1} a_n x^{n+1} + C$, where C is a constant

THEOREM: Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$.

- $\left(\alpha A(x) + \beta B(x)\right)' = \alpha A'(x) + \beta B'(x)$
- $\left(A(x)B(x)\right)' = A'(x)B(x) + A(x)B'(x)$
- $\left(A^k(x)\right)' = kA^{k-1}(x)\, A'(x)$

$(1+\alpha x)^u$

DEFINITION: Let $u \in \mathbb{R}$ and $n \in \mathbb{N}$. The **extended binomial coefficient** $\binom{u}{n} = \begin{cases} u(u-1)\cdots(u-n+1)/n! & n > 0 \\ 1 & n = 0 \end{cases}$

THEOREM: Let x be a real number with $|x| < 1$ and let u be a real number. Then $(1+x)^u = \sum_{n=0}^{\infty} \binom{u}{n} x^n$.

EXAMPLE:

- $(1-\alpha x)^{-1} = \sum_{n=0}^{\infty} \alpha^n x^n$
- $(1-\alpha x)^{-u} = \sum_{n=0}^{\infty} \binom{n+u-1}{n} \alpha^n x^n$

Solving RR with GFs

EXAMPLE: Solve the LNRR $a_n = 8a_{n-1} + 10^{n-1}$ with the initial condition $a_0 = 1$ using generating function.

- $A(x) = \sum_{n=0}^{\infty} a_n x^n$

  $= 1 + \sum_{n=1}^{\infty} (8a_{n-1} + 10^{n-1}) x^n$

  $= 1 + 8xA(x) + \frac{x}{1-10x}$

$$A(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

THEOREM: Let $Q(x), P(x) \in \mathbb{R}[x]$ be two polynomials such that $\deg(Q) > \deg(P)$. If $Q(x) = (1 - r_1 x)^{m_1} \cdots (1 - r_t x)^{m_t}$ for distinct non-zero numbers $r_1, \ldots, r_t$ and integers $m_1, \ldots, m_t \geq 1$, then there exist unique coefficients $\{\alpha_{j,u} : j \in [t], u \in [m_j]\}$ s.t.

$$\frac{P(x)}{Q(x)} = \sum_{j=1}^{t} \sum_{u=1}^{m_j} \frac{\alpha_{j,u}}{(1 - r_j x)^u} \; .$$

Solving RR with GFs

EXAMPLE: Solve the LNRR $a_n = 8a_{n-1} + 10^{n-1}$ with the initial condition $a_0 = 1$ using generating function.

- $A(x) = \frac{1-9x}{(1-8x)(1-10x)}$
- $A(x) = \frac{\alpha_{1,1}}{1-8x} + \frac{\alpha_{2,1}}{1-10x}$
  - $\alpha_{1,1} = \alpha_{2,1} = \frac{1}{2}$
- $A(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$

  $= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$
- $a_n = \frac{1}{2}(8^n + 10^n) \quad (n \geq 0)$

GFs of the Catalan numbers $\{C_n\}_{n=0}^{\infty}$:
- $C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0$
- $x\,C(x)^2 = C(x) - 1$

Counting Combinations with GFs

QUESTION: Let $k > 0, N_1, \dots, N_k \subseteq \mathbb{N}$. For every $n \geq 0$, let a_n be the number of n-combinations of $[k]$ with repetition where every $i \in [k]$ appears N_i times. (**Distribution problems: Type 2**)

- $a_n = |\{(n_1, \dots, n_k): n_1 \in N_1, \dots, n_k \in N_k, n_1 + \cdots + n_k = n\}|$
  - This is also the number of ways of distributing n unlabeled objects into k labeled boxes such that N_i objects are sent to box i

THEOREM: $\sum_{n=0}^{\infty} a_n x^n = \prod_{i=1}^{k} \sum_{n_i \in N_i} x^{n_i}$.

- $\prod_{i=1}^{k} \sum_{n_i \in N_i} x^{n_i} = \sum_{n_1 \in N_1} x^{n_1} \cdot \sum_{n_2 \in N_2} x^{n_2} \cdots \sum_{n_k \in N_k} x^{n_k}$

$$= \sum_{n=0}^{\infty} \left(\sum_{n_1 \in N_1, \dots, n_k \in N_k, n_1 + \cdots + n_k = n} 1 \right) x^n$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

Counting Combinations with GFs

EXAMPLE: Let a_n be the number of ways of distributing n identical books to 5 persons such that person 1, 2, 3, and 4 receive $\geq 3, \geq 2, \geq 4, \geq 6$ books, respectively. Calculate a_{20}.

- $a_n = |\{(n_1, \dots, n_5): n_1 \geq 3, n_2 \geq 2, n_3 \geq 4, n_4 \geq 6, n_5 \geq 0, n_1 + \cdots + n_5 = n\}|$
  - $N_1 = \{3,4, \dots\}; N_2 = \{2,3, \dots\}; N_3 = \{4,5, \dots\};$
    $N_4 = \{6,7, \dots\}; N_5 = \{0,1,2 \dots\}$
  - $\sum_{n=0}^{\infty} a_n x^n = \prod_{i=1}^{5} \sum_{n_i \in N_i} x^{n_i}$

$$= (x^3 + \cdots)(x^2 + \cdots)(x^4 + \cdots)(x^6 + \cdots)(1 + x + \cdots)$$

$$= \frac{x^3}{1-x} \frac{x^2}{1-x} \frac{x^4}{1-x} \frac{x^6}{1-x} \frac{1}{1-x} = \frac{x^{15}}{(1-x)^5}$$

$$= x^{15} \sum_{m=0}^{\infty} \binom{-5}{m} (-1)^m x^m$$

- $a_{20} = \binom{-5}{5} (-1)^5 = 126$

GFs for the number of partitions $\{p_k(n)\}$:

- $\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} p_k(n) x^n y^k = \frac{1}{1-yx} \cdot \frac{1}{1-yx^2} \cdot \cdots$

Distribution problems: Type 4

Counting Permutations with GFs

QUESTION: Let $k > 0, N_1, \ldots, N_k \subseteq \mathbb{N}$. For every $n \geq 0$, let a_n be the number of n-permutations of $[k]$ with repetition where every $i \in [k]$ appears N_i times. (**Distribution problems: Type 1**)

- $a_n = \sum_{n_1 \in N_1, n_2 \in N_2, \ldots, n_k \in N_k,\, n_1+n_2+\cdots+n_k=n} \frac{n!}{n_1! n_2! \cdots n_k!}$
  - This is the number of ways of distributing n labeled objects into k labeled boxes such that N_i objects are sent to box i for all $i \in [k]$

THEOREM: $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \prod_{i=1}^{k} \sum_{n_i \in N_i} \frac{x^{n_i}}{n_i!}$.

- $\prod_{i=1}^{k} \sum_{n_i \in N_i} \frac{x^{n_i}}{n_i!} = \sum_{n_1 \in N_1} \frac{x^{n_1}}{n_1!} \cdot \sum_{n_2 \in N_2} \frac{x^{n_2}}{n_2!} \cdots \sum_{n_k \in N_k} \frac{x^{n_k}}{n_k!}$

$$= \sum_{n=0}^{\infty} \left(\sum_{n_1 \in N_1, n_2 \in N_2, \ldots, n_k \in N_k,\, n_1+n_2+\cdots+n_k=r} \frac{n!}{n_1! n_2! \cdots n_k!} \right) \frac{x^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

Counting Permutations with GFs

EXAMPLE: Find $a_n = \{s \in \{1,2,3,4\}^n : s \text{ has an even number of } 1s\}$

- a_n = the number of n-permutations of $\{1,2,3,4\}$ with repetition where 1 appears an even number of times
- $N_1 = \{0,2,4,\ldots\}, N_2 = N_3 = N_4 = \{0,1,2,\ldots\}$
   - $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right)^3$

$$= \frac{e^x + e^{-x}}{2} \cdot e^{3x}$$

$$= \frac{e^{4x} + e^{2x}}{2}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{(4x)^n}{n!} + \frac{(2x)^n}{n!}\right)$$

- $\frac{a_n}{n!} = \frac{1}{2} \cdot \left(\frac{4^n}{n!} + \frac{2^n}{n!}\right)$
   - $a_n = \frac{4^n + 2^n}{2}$

GFs of the Stirling numbers of the 2nd kind $\{S_2(n,j)\}_{n=j}^{\infty}$:

- $\sum_{n=j}^{\infty} \frac{S_2(n,j)}{n!} x^n = \frac{1}{j!}(e^x - 1)^j$

Distribution problems: Type 3

Principle of Inclusion–Exclusion

1718, de Moivre (1667–1754)

Derangements: Let S be the set of permutations of $[n]$. Find $|A|$

for $A = \{x_1x_2 \dots x_n : x_1x_2 \dots x_n \in S;\ x_i \neq i \text{ for all } i \in [n]\}$.

- $A_i = \{x_1x_2 \cdots x_n : x_1x_2 \cdots x_n \in S; x_i = i\}, i = 1,2,\dots,n$
  - $A = S - \bigcup_{i=1}^{n} A_i$
  - $|S| = n!$
  - $|\bigcup_{i=1}^{n} A_i| = ?$

Problem: S is a finite set and $A_1, A_2, \dots, A_n \subseteq S$.

- $|\bigcup_{i=1}^{n} A_i| = ?$
- $|\bigcap_{i=1}^{n} A_i| = ?$

Principle of IE (Two Sets)

THEOREM: Let S be a finite set. Let A_1, A_2 be subsets of S. Then

- $|S - A_1| = |S| - |A_1|; |A_1 - A_2| = |A_1| - |A_1 \cap A_2|$
  - $S = A_1 \cup (S - A_1\ ),\ \ A_1 \cap (S - A_1) = \emptyset;$
    - $\{A_1, S - A_1\}$ is a partition of S
      - $|S| = |A_1| + |S - A_1|$
        - $|S - A_1| = |S| - |A_1|$
    - $A_1 - A_2 = A_1 - A_1 \cap A_2$
      - $|A_1 - A_2| = |A_1| - |A_1 \cap A_2|$
- $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
  - $A_1 \cup A_2 = (A_1 - A_2) \cup A_2, (A_1 - A_2) \cap A_2 = \emptyset;$
    - $\{A_1 - A_2, A_2\}$ is a partition of $A_1 \cup A_2$
      - $|A_1 \cup A_2| = |A_1 - A_2| + |A_2| = |A_1| - |A_1 \cap A_2| + |A_2|$
- $|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2|$

Principle of IE (Three Sets)

THEOREM: Let S be a finite set. Let A_1, A_2, A_3 be subsets of S.

Then $\left|\cup_{i=1}^{3} A_i\right| = \sum_{t=1}^{3}(-1)^{t-1} \sum_{1 \le i_1 < \cdots < i_t \le 3} |A_{i_1} \cap \cdots \cap A_{i_t}|$

- $\left|\cup_{i=1}^{3} A_i\right| = |(A_1 \cup A_2) \cup A_3| = |A_1 \cup A_2| + |A_3| - |(A_1 \cup A_2) \cap A_3|$
  - $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$
  - $|(A_1 \cup A_2) \cap A_3| = |(A_1 \cap A_3) \cup (A_2 \cap A_3)|$

    $= |A_1 \cap A_3| + |A_2 \cap A_3| - |(A_1 \cap A_3) \cap (A_2 \cap A_3)|$

    $= |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$
    - $\left|\cup_{i=1}^{3} A_i\right| = |A_1| + |A_2| - |A_1 \cap A_2| + |A_3|$

      $-(|A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|)$
- $\left|\cap_{i=1}^{3} A_i\right| = \sum_{t=1}^{3}(-1)^{t-1} \sum_{1 \le i_1 < \cdots < i_t \le 3} |A_{i_1} \cup \cdots \cup A_{i_t}|$

Principle of IE (n Sets)

THEOREM: Let S be a finite set. Let $A_1, A_2, \dots, A_n \subseteq S$. Then

- $|\bigcup_{i=1}^{n} A_i| = \sum_{t=1}^{n} (-1)^{t-1} \sum_{1 \le i_1 < \cdots < i_t \le n} |A_{i_1} \cap \cdots \cap A_{i_t}|$;
- $|\bigcap_{i=1}^{n} A_i| = \sum_{t=1}^{n} (-1)^{t-1} \sum_{1 \le i_1 < \cdots < i_t \le n} |A_{i_1} \cup \cdots \cup A_{i_t}|$.

EXAMPLE: Let S be the set of permutations of $[n]$. Find $|A|$ for

$$A = \{x_1 x_2 \dots x_n : x_1 x_2 \dots x_n \in S;\ x_i \neq i \text{ for all } i \in [n]\}.$$

- $A_i = \{x_1 x_2 \cdots x_n : x_1 x_2 \cdots x_n \in S; x_i = i\}, i = 1,2, \dots, n$
  - $A = S - \bigcup_{i=1}^{n} A_i$
- $|\bigcup_{i=1}^{n} A_i| = \sum_{t=1}^{n} (-1)^{t-1} \sum_{1 \le i_1 < \cdots < i_t \le n} |A_{i_1} \cap \cdots \cap A_{i_t}|$
  - $\left|A_{i_1} \cap \cdots \cap A_{i_t}\right| = (n-t)!$ for $t = 1,2, \dots, n$
- $|A| = |S| - |\bigcup_{i=1}^{n} A_i|$

$$= n! - \left(\binom{n}{1} * (n-1)! - \binom{n}{2} * (n-2)! + \cdots + (-1)^{n-1} * \binom{n}{n} * 1\right)$$

$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^t \frac{1}{t!} + \cdots + (-1)^n \frac{1}{n!}\right)$$

Cover

DEFINITION: A **cover** of a finite set A is a family $\{A_1, A_2, \dots, A_n\}$ of subsets of A such that $\bigcup_{i=1}^{n} A_i = A$.

LEMMA: Let $\{A_1, A_2, \dots, A_n\}$ be a cover of a finite set A. Then $|A| \leq \sum_{i=1}^{n} |A_i|$.

- $n = 1$: $|A| = |A_1|$
- $n = 2$: $|A| = |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| \leq |A_1| + |A_2|$
- Suppose true when $n \leq k$ ($k \geq 2$).
- When $n = k + 1$, $|A| = \left|\bigcup_{i=1}^{k} A_i \cup A_{k+1}\right|$

$$\leq \left|\bigcup_{i=1}^{k} A_i\right| + |A_{k+1}|$$

$$\leq \sum_{i=1}^{k} |A_i| + |A_{k+1}|$$

$$= \sum_{i=1}^{k+1} |A_i|$$

Pigeonhole Principle

1834, Dirichlet (1805-1859)

THEOREM: (simple form) Let A be a set with $\geq n+1$ elements. Let $\{A_1, A_2, \ldots, A_n\}$ be a cover of A. Then $\exists k \in [n], |A_k| \geq 2$.

- Suppose that $|A_i| \leq 1$ for every $i \in [n]$. Then $n+1 \leq |A| \leq \sum_{i=1}^{n} |A_i| \leq n$.
  - If $\geq n + 1$ objects are distributed into n boxes, then there is at least one box containing ≥ 2 objects.

THEOREM: (general form) Let A be a set with $\geq N$ elements. Let $\{A_1, A_2, \ldots, A_n\}$ be a cover of A. Then $\exists k \in [n], |A_k| \geq \lceil N/n \rceil$.

- If $|A_i| < \lceil N/n \rceil$ for all $i \in [n]$, then $N \leq |A| \leq \sum_{i=1}^{n} |A_i| < n \cdot N/n = N$
  - If we distribute $\geq N$ objects into n boxes, then there is at least one box that contains $\geq \lceil N/n \rceil$ objects.

Pigeonhole Principle

EXAMPLE: Let $n \in \mathbb{Z}^+$. Let $A \subseteq \{1,2,\ldots,2n\}$ have $n+1$ elements.
Then there exist $x, y \in A$ such that $x|y$.

- Let $A = \{a_1, \ldots, a_{n+1}\} \subseteq [2n]$ be any subset of $n+1$ elements.
- $a_j = 2^{u_j} \cdot v_j$, where $u_j \in \mathbb{N}$ and $v_j \in [2n]$ is odd for all $j = 1,2,\ldots,n+1$
  - $\{v_1, v_2, \ldots, v_{n+1}\} \subseteq \{1,3,\ldots,2n-1\}$
- $A_i = \{a_j : v_j = i\}$ for $i = 1,3,\ldots,2n-1$
- $\{A_1, A_3, \ldots, A_{2n-1}\}$ is a partition of A
  - $\exists k \in \{1,3,\ldots,2n-1\}$ such that $|A_k| \geq 2$ //pigeonhole principle
    - $a_s, a_t \in A_k \Rightarrow (a_s = 2^{u_s} \cdot v_s) \wedge (a_t = 2^{u_t} \cdot v_t) \wedge (v_s = v_t = k)$
      - $(x, y) = \begin{cases} (a_s, a_t), & \text{if } u_s \leq u_t \\ (a_t, a_s), & \text{if } u_s > u_t \end{cases}$