Linear Algebra I

Chapter 1. Linear Systems and Matrices

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WELCOME!

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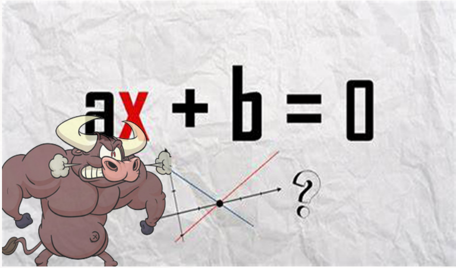
Linear Algebra

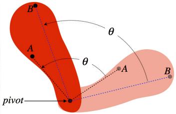
- Problems or examples $\rightleftharpoons$ Abstract concepts $\rightleftharpoons$ Mathematical theory.
- What is LINEAR ALGEBRA?

◇ The study of linear equations and their transformation properties.
◇ The study of a certain algebraic structure called linear space.
◇ The study of linear transformations between such spaces.



Rigid Motion on a Plane

- Core of the Theory of Linear Algebra: To Describe Linear Transforms.



Rigid Rotation

Face Detection/Rotating Images

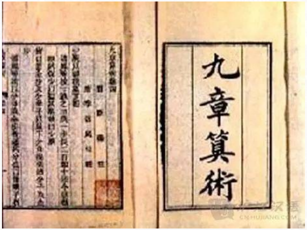
Example. (1) A mass point located at (x, y) is rotated counterclockwise around the origin by θ_0 $(0 \leq \theta_0 < 2\pi)$ radians. What are the new coordinates (u, v)?
(2) Conversely, if the coordinates (u, v) of the new position are known, what are the coordinates (x, y) of the original position?

Overview:

Keywords: linear equation; linear space; point and line; determinant; linear transformation; inverse.

Examples From History

- China (A.D. 263) Cang Zhang & Shoucang Geng:
  "Nine Chapters of the Mathematical Art"

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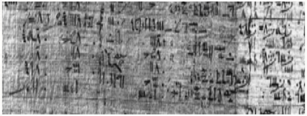
荅曰

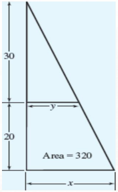
Examples From History

- Egypt (about 1650 B.C.) Problem 40 of the Ahmes Papyrus:
Divide 100 hekats of barley among five men in arithmetic progression so that the sum of the two smallest is one-seventh the sum of the three largest.



Problem 40 of the Ahmes Papyrus

Babylonian clay tablet



- Babylonia (1900-1600 B.C.) A problem on Babylonian clay tablet:
A trapezoid with an area of 320 square units is cut off from a right triangle by a line parallel to one of its sides. The other side has length 50 units, and the height of the trapezoid is 20 units. What are the upper and the lower widths of the trapezoid?

- Greece (third century B.C.) ...
- India (fourth century A.D.) ...

Concepts of Linear System

- Field: $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ (the set of numbers/scalars).
- A general linear system of m equations in n unknowns with $\mathbb{F}$-coefficients:

$$\begin{cases} a_{11}x_1 & +a_{12}x_2 & +\cdots & +a_{1n}x_n & = b_1 \\ a_{21}x_1 & +a_{22}x_2 & +\cdots & +a_{2n}x_n & = b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & +a_{m2}x_2 & +\cdots & +a_{mn}x_n & = b_m \end{cases}$$

Problem: Try to write the above equations with the Σ-notation.

- A solution is a sequence of n numbers for which the substitution

$$x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$$

makes each equation a true statement.
- Classical notation: Ordered n-tuple $(s_1, s_2, \ldots, s_n)$.

- A linear system is said to be

  consistent: at least one solution;

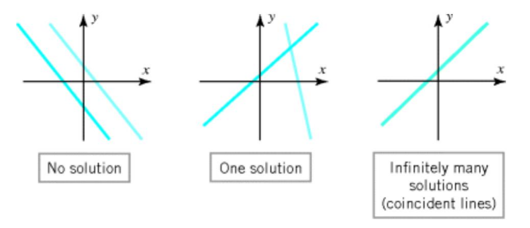
  inconsistent: no solutions.

Linear Systems in Two Unknowns

- In the real xy-plane, geometrically study the linear system

$$\begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases}$$

Geometric interpretation: Each solution lines on both lines.



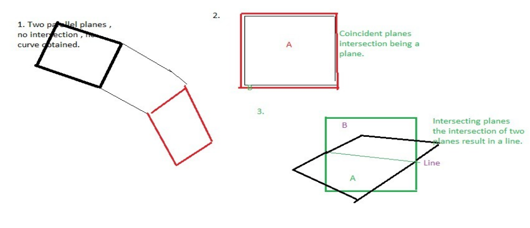
Question: consistent or inconsistent?

Linear Systems in Three Unknowns

- In real xyz-coordinate, geometrically consider the linear system

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$

Geometric interpretation: Each solution lines on both planes.



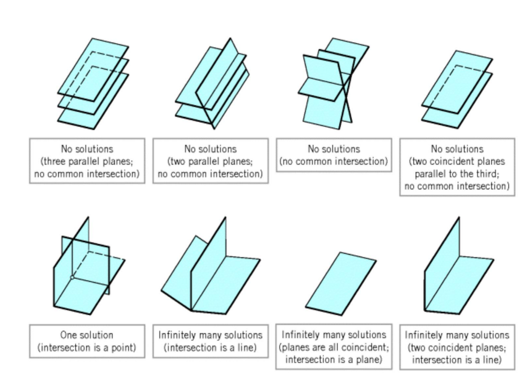
Question: consistent or inconsistent?

Linear Systems in Three Unknowns

- In real *xyz*-coordinate, geometrically consider the linear system

$$\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

Geometric interpretation: Each solution lines on all the planes.



Extracting Key Information from a Linear System

Question: What kind of information is vital in a linear system?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方程 以御錯糅正負
今有上禾三秉中禾二秉下禾一秉實三十九
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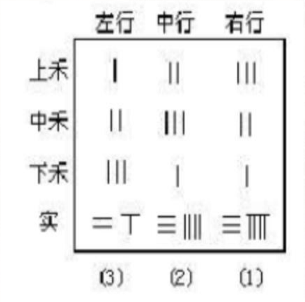


Augmented Matrices

- Aim: Simplifying notations
  $\rightsquigarrow$ Finding key information $\rightsquigarrow$ Establishing further theories.
- Coefficient Matrix and Augmented Matrix

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \rightsquigarrow \left[\begin{array}{cccc:c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right].$$

Example. Try to find out the coefficient matrix and augmented matrix of the following system.

$$\begin{cases} 2x & +4y & +5z & = & 4 \\ 99x & +99y & +198z & = & 198 \\ 3x & -y & +7z & = & 0 \end{cases}$$

Solution:

1

Concepts Related to Matrix

Definition.

◇ Matrix: a rectangular array of numbers.

◇ Entry: a number in the array.

◇ $m \times n$ matrix: the matrix has m rows and n columns.

◇ $M_{m\times n}(\mathbb{F})$: the set of all $m \times n$ matrices with entries in $\mathbb{F}$.

◇ Square Matrix of Order n: an $n \times n$ matrix.

◇ $M_n(\mathbb{F})$: the set of all square matrix of order n with entries in $\mathbb{F}$.

◇ Column vector/matrix: an $m \times 1$ matrix.

◇ Row vector/matrix: a $1 \times n$ matrix.

Example. The following are examples of matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}$$

Arraying Numbers: Tables of Scores

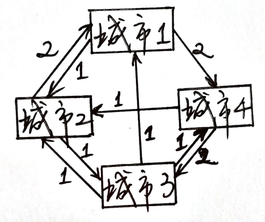
Homework	Algebra	Calculus
Student 1	100	100
Student 2	95	90
Student 3	70	65

Midterm	Algebra	Calculus
Student 1	91	94
Student 2	68	85
Student 3	47	32

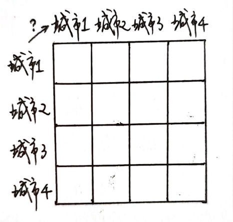
Final	Algebra	Calculus
Student 1	72	68
Student 2	64	70
Student 3	26	63

Arraying Numbers: Graph/Network

- There are 4 cities in the following weighted directed graph. Number of non-stop flights from one city to another is shown beside the corresponding arrow.



Graph



Adjacent Matrix

- The adjacent matrix collects the number on the arrows.

Three Elementary Operations

Question: How to solve a linear system?

- Aim: ◇ Simplify the systems.
  ◇ Do not alter the solution set.

Recall: Try to simplify the linear system and find the key steps.

$$\begin{cases} 2x & +4y & +5z & = & 4 \\ 99x & +99y & +198z & = & 198 \\ 3x & -y & +7z & = & 0 \end{cases}$$

Remark: Two linear systems are said to be equivalent if they have the same set of solutions.

Three Elementary Operations

- Elementary Operations

on a system:

1. Multiply an equation through by a nonzero constant.
2. Interchange two equations.
3. Add a constant times one equation to another.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad\qquad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$$\downarrow$$

- Elementary (Row) Operations

on a matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

$$\downarrow$$

Question*: Can some operations be fulfilled by the other two?

Problem*: Explore the three elementary column operations of a matrix.

Three Elementary Operations

Example Try to solve the following system with the help of elementary operations on the augmented matrix.

$$\begin{cases} 2x & +4y & +5z & = & 4 \\ 99x & +99y & +198z & = & 198 \\ 3x & -y & +7z & = & 0 \end{cases}$$

Solution:

2

Experience: Finally, each equation has fewer variable than the previous equation does. We obtain an echelon!

Echelon Forms

Definition. A matrix is said to be in (row) echelon form, if the following holds:

(1) Each non-zero row lies above every zero row.
(2) Every leading entry is to the right of the one above it.

Here, the first non-zero element in a non-zero row is called leading entry.

Example. Echelon forms, not reduced:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Echelon Forms

Definition. A matrix in echelon form is said to be in reduced (row) echelon form, if the following holds:

(3) Every leading entry is 1.

(4) The leading entry in each row is the only non-zero entry in its column.

Example. Reduced echelon forms:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Definition. In an echelon form of a matrix,

◇ a pivot position is a location that corresponds to a leading 1;

◇ a leading variable is an unknown corresponds to a pivot column.

Transforming a Matrix to its Echelon Form

Proposition. Any matrix can be transformed into its
(1) echelon forms;
(2) reduced echelon form
by using a series of elementary row operations.

Idea:

③

Problem*: Try to conclude the main steps.

Remark: (1) Forward phase (called Gaussian elimination);
(2) Forward & Backward phases (called Gauss-Jordan elimination).

Remark: (1) All row echelon forms of a matrix have the same number of zero rows, and the same pivot positions;

(2) Every matrix has a unique reduced row echelon form.

Solving by The Elimination Methods

Example. Solve the linear system

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

by (1) Gauss-Jordan elimination;
(2) Gaussian elimination & back-substitution.

Solution:

4

Remark: The second method is usually more efficient on a computer.

- The variables, other than the leading variables, are called free variables.

Definition. If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerial values to the parameters is called a general solution of the system.

Concept of Rank

Question: When an echelon form of an augmented matrix has a row of zero at the bottom, what does it mean?

Definition. Suppose that an echelon form of a matrix A has r non-zero rows. Then we say that A has rank r, and denote $\operatorname{rank}(A) = r$.

Remark: If $A \in M_{m\times n}$, then $0 \leq \operatorname{rank}(A) \leq \min\{m, n\}$.

Remark: If A is the augmented matrix of a linear system, then the number of effective equations in the system is exactly r.

Echelon Form and Consistency

Theorem. Suppose that the augmented matrix of a linear system of equations has an echelon form

$$\left[\begin{array}{ccccccc:c} c_{1j} & & & & & \cdots & c_{1n} & d_1 \\ & c_{2k} & & & & \cdots & c_{2n} & d_2 \\ & & c_{3l} & & & \cdots & c_{3n} & d_3 \\ & & & \ddots & & & \vdots & \vdots \\ & & & & c_{rs} & \cdots & c_{rn} & d_r \\ & & & & & & 0 & d_{r+1} \\ & & & & & & 0 & 0 \\ & & & & & & \vdots & \vdots \\ & & & & & & 0 & 0 \end{array}\right],$$

where, on the pivot columns, $c_{1j}, c_{2k}, c_{3l}, \ldots, c_{rs} \neq 0$. Then

◇ The linear system is consistent if and only if $d_{r+1} = 0$.

◇ The linear system has exactly one solution if and only if

$$d_{r+1} = 0 \text{ and } r = n.$$

◇ The linear system has infinitely many solutions if and only if

$$d_{r+1} = 0 \text{ and } r < n.$$

- For above: m equations; r echelons; r pivot colomns/variables; $n - r$ free variables.

Rank and Consistency

Corollary. Suppose that a linear system has coefficient matrix A and augmented matrix B, where $A \in M_{m\times n}$ and $B \in M_{m\times(n+1)}$.

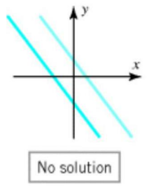
◇ It is inconsistent if and only if $\text{rank}(A) < \text{rank}(B)$.

◇ It is consistent if and only if $\text{rank}(A) = \text{rank}(B)$.

◇ It has exactly one solution if and only if $\text{rank}(A) = \text{rank}(B) = n$.

◇ It has infinitely many solutions if and only if $\text{rank}(A) = \text{rank}(B) < n$.

∘ Example of inconsistent system: two parallel lines on the xy-plane.

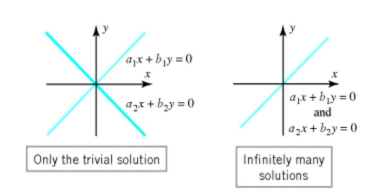


Homogeneous Linear Systems

Definition. A system of linear equations is said to be homogeneous if the constant terms are all zero, i.e.,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{cases}$$

Example. The cases for two unknowns:



- There are exactly two possibilities for a homogeneous linear systems. Why?
  (1) It has only the trivial solution, i.e., $x_1 = x_2 = \ldots = x_n = 0$.
  (2) The system has infinitely many solutions (including the trivial solution).

Recall: Concepts Related to Matrix

Definition.

- ◇ Matrix: a rectangular array of numbers.
- ◇ Entry: a number in the array.
- ◇ $m \times n$ matrix: the matrix has m rows and n columns.
- ◇ $M_{m\times n}$: the set of all $m \times n$ matrices.
- ◇ Square Matrix of Order n: an $n \times n$ matrix.
- ◇ M_n: the set of all square matrix of order n.
- ◇ Column vector/matrix: a $m \times 1$ matrix.
- ◇ Row vector/matrix: a $1 \times n$ matrix.

Example. The following are examples of matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & 0 & -3 \end{bmatrix}, \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 4 \end{bmatrix}$$

Notations Related to Matrix

- When the entries a_{ij} of a matrix A have been specified, we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]_{m\times n} = [a_{ij}]$$

- When the the entries of a matrix A have not been specified, we denote its entry in row i and column j by $(A)_{ij}$.
- We usually write a general row or column vector by

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- For $A \in M_n$, the entries $(A)_{11}, (A)_{22}, \ldots, (A)_{nn}$ are said to be on the main diagonal of A.

Equality of Matrices

Question: Do the following matrices provide us same information?

$$\begin{bmatrix} 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Definition. Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.

More concretely, the equality $[a_{ij}]_{m\times n} = [b_{kl}]_{p\times q}$ holds if and only if

$$m = p, \quad n = q, \quad a_{ij} = b_{ij}\,(1 \le i \le m,\, 1 \le j \le n).$$

Addition and Scalar Multiplication of Matrices

Question: The total score of a student is composed of

30% of homework + 30% of midterm + 40% of final.

To complete following table of total scores.

HOMEWORK	ALGEBRA	CALCULUS
Student 1	100	100
Student 2	95	90
Student 3	70	65

MIDTERM	ALGEBRA	CALCULUS
Student 1	91	94
Student 2	68	85
Student 3	47	32

FINAL	ALGEBRA	CALCULUS
Student 1	72	68
Student 2	64	70
Student 3	26	63

TOTAL	ALGEBRA	CALCULUS
Student 1	?	?
Student 2	?	?
Student 3	?	?

Addition and Scalar Multiplication of Matrices

Definition. Suppose that $A, B \in M_{m\times n}$ are given by $A = [a_{ij}]$ and $B = [b_{ij}]$. We define the sum $A+B$ to be the $m\times n$ matrix such that

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij}, \quad (1 \le i \le m, 1 \le j \le n)..$$

Attention: Two matrices of different sizes can not add each other.

Definition. Let $c \in \mathbb{F}$. Suppose that $A \in M_{m\times n}$ is given by $A = [a_{ij}]$. We define the scalar multiple cA to be the $m\times n$ matrix such that

$$(cA)_{ij} = c \cdot (A)_{ij} = ca_{ij}, \quad (1 \le i \le m, 1 \le j \le n)..$$

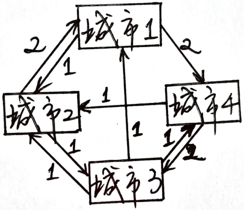
Example. Let $A = \begin{bmatrix} 1 & x \\ 3 & y \end{bmatrix}$, $B = \begin{bmatrix} z & 0 \\ w & 4 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \\ 4 & -2 \end{bmatrix}$. Find x, y, z, w such that $2A - 3B = C$.

Solution:

5

Product of Matrices

Number of non-stop flights:



?→	城市1	城市2	城市3	城市4
城市1	0	1	0	2
城市2	2	0	1	0
城市3	1	1	0	1
城市4	0	1	2	0

?→	城市1	城市2	城市3	城市4
城市1	0	1	0	2
城市2	2	0	1	0
城市3	1	1	0	1
城市4	0	1	2	0

Question: By taking consecutively two non-stop flights, can we start from the ith city and arrive at the jth city? How many different approaches can we take?

Product of Matrices

Definition. Suppose that $A = [a_{ij}] \in M_{m\times r}$, $B = [b_{ij}] \in M_{r\times n}$. The product AB is defined to be the $m \times n$ matrix such that

$$(AB)_{ij} = \sum_{k=1}^{r} a_{ik} b_{kj}, \quad (1 \le i \le m,\ 1 \le j \le n).$$

★ Pay attention to the sizes of the matrices!

Example. Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 5 & 0 \end{bmatrix}$.

Compute AB and BA.

Solution:

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Matrix Form of a Linear System

Example. Show that the system of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

is equivalent to the equation of matrices $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix}$$

Proof:

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Transpose of a Matrix

Definition. Let $A \in M_{m\times n}$. The transpose of A, denoted by A^T, is the $n \times m$ matrix such that

$$(A^T)_{kl} = (A)_{lk}, \quad (1 \le k \le n, 1 \le l \le m).$$

Remark: Rows become columns, and columns become rows.

Example. Find the transposes of the following matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \ \ 3 \ \ 5], \quad D = [4]$$

Solution:

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Example. Let $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. Find AA^T and A^TA.

Solution:

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Trace of a Square Matrix

Definition. The trace of $A \in M_n$, denoted by $\mathrm{tr}(A)$, is defined to be

$$\mathrm{tr}(A) = \sum_{i=1}^{n} (A)_{ii}.$$

Example. Evaluate the trace of the following matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

Solution:

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Example. Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times m}$. Compute $\mathrm{tr}(AB)$ and $\mathrm{tr}(BA)$.
Solution:

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