Discrete Mathematics

combination of set, combination of multiset, inverse binomial transform, combinatorial proof

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Combinations of Sets

DEFINITION: Let $A = \{a_1, \dots, a_n\}$ and let $r \in \{0,1,\dots,n\}$.

- ***r*-combination of *A***: an r-subset of A.
  - Notation: $\{a_{i_1}, \dots, a_{i_r}\}$ with $1 \leq i_1 < \cdots < i_r \leq n$
  - $\binom{n}{r}$: the number of r-combinations of an n-element set

THEOREM: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ for all $n \in \mathbb{Z}^+$ and $r \in \{0,1,\dots,n\}$.

DEFINITION: Let $A = \{a_1, \dots, a_n\}$ and let $r \geq 0$.

- ***r*-combination of *A* with repetition**: a multiset $\{x_1 \cdot a_1, \dots, x_n \cdot a_n\}$ of r elements, where $x_1, \dots, x_n \geq 0$ are integers and $x_1 + \cdots + x_n = r$.
  - Notation: $\{a_{i_1}, \dots, a_{i_r}\}$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n$

THEOREM: The number of r-combinations of an n element set with repetition is $\binom{n+r-1}{r}$

Combinations of Sets

- $\mathcal{U}$: the set of all r-combinations of $A = [n]$ with repetition
- $\mathcal{V}$: the set of all r-combinations of $[n+r-1]$ without repetition
  - Let $U = \{u_1, u_2, \dots, u_r\} \in \mathcal{U}$ and $1 \le u_1 \le u_2 \le \cdots \le u_r \le n$.
    - $1 \le u_1 < u_2 + 1 < u_3 + 2 < \cdots < u_r + r - 1 \le n + r - 1$
      - $\{u_1, u_2 + 1, \dots, u_r + r - 1\} \in \mathcal{V}$
      - $f: \mathcal{U} \to \mathcal{V}\ \ \{u_1, u_2, \dots, u_r\} \mapsto \{u_1, u_2 + 1, \dots, u_r + r - 1\}$
    - f is bijective. Hence, $|\mathcal{U}| = |\mathcal{V}| = \binom{n+r-1}{r}$

THEOREM: The number of natural number solutions of the equation $x_1 + x_2 + \cdots + x_n = r$ is $\binom{n+r-1}{r}$.

- $\mathcal{X} = \{(x_1, \dots, x_n): x_1, \dots, x_n \in \mathbb{N}$ and $x_1 + \cdots + x_n = r\}$
- $\mathcal{Y}$: the set of all r-combinations of $[n]$ with repetition
- $f: \mathcal{X} \to \mathcal{Y}\quad (x_1, \dots, x_n) \mapsto \{x_1 \cdot 1, x_2 \cdot 2, \dots, x_n \cdot n\}$
  - f is bijective. Hence, $|\mathcal{X}| = |\mathcal{Y}| = \binom{n+r-1}{r}$.

Application

EXAMPLE: What is the value of k after the program execution?

- $k := 0;$
- for $i_1 := 1$ to n do
  - for $i_2 := 1$ to i_1 do
    - $\vdots$
      - for $i_r := 1$ to i_{r-1} do
        - $k := k + 1;$

Analysis:

- Loop variables: $1 \leq i_r \leq i_{r-1} \leq \cdots \leq i_1 \leq n$
- The number of iterations is equal to the number of r-combinations of the set $[n]$ with repetition
- In every iteration, k increases by 1.
  - After the program execution, $k = \binom{n+r-1}{r}$

Combinations of Multiset

DEFINITION: Let $A = \{n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k\}$ be an n-multiset. Let $r \in \{0,1,\dots,n\}$.

- **r-combination of A**: an r-subset (multiset) of A
  - Notation: $\{x_1 \cdot a_1, x_2 \cdot a_2, \dots, x_k \cdot a_k\}$, where $0 \le x_i \le n_i$ for every $i \in [k]$ and $x_1 + x_2 + \cdots + x_k = r$.

EXAMPLE: $A = \{1 \cdot \mathrm{a}, 2 \cdot \mathrm{b}, 3 \cdot \mathrm{c}\}$

- $\{1 \cdot b, 2 \cdot c\}$ is a 3-combination of A; a 3-subset of A

REMARK:

- For every $r \in \{0,1,\dots,n\}$, an r-combination of $A = \{a_1, a_2, \dots, a_n\}$ without repetition is an r-combination of $\{1 \cdot a_1, 1 \cdot a_2, \dots, 1 \cdot a_n\}$.
- For every $r \ge 0$, an r–combination of $A = \{a_1, a_2, \dots, a_n\}$ with repetition is an r–combination of $\{\infty \cdot a_1, \infty \cdot a_2, \dots, \infty \cdot a_n\}$.

Inverse Binomial Transform

DEFINITION: The **binomial transform** of $\{a_n\}_{n\geq s}$ is a sequence $\{b_n\}_{n\geq s}$ such that

$$b_n = \sum_{k=s}^{n} \binom{n}{k} a_k \qquad (1)$$

DEFINITION: The **inverse binomial transform** of $\{b_n\}_{n\geq s}$ is a sequence $\{a_n\}_{n\geq s}$ such that

$$a_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} b_k \quad (2)$$

QUESTION: Given (1), how to find the sequence $\{a_n\}$?

- Answer: $\{a_n\}$ is the inverse binomial transform of $\{b_n\}$
- Application: determine $\{a_n\}$ via $\{b_n\}$
- Proof?

Combinatorial Proofs

DEFINITION: A **combinatorial proof** of an identity $L = R$ is

- **a double counting proof,** which shows that L, R count the same set of objects but in different ways:
  - $L = |X| = R$ and L, R count $|X|$ in different ways.
- **a bijective proof,** which shows a bijection between the sets of objects counted by L and R:
  - $L = |X|, R = |Y|$ and there is a bijection $f: X \to Y$.

EXAMPLE: $\binom{n}{r} = \binom{n}{n-r}$

- $X = \{s \in \{0,1\}^n : s \text{ contains } r \text{ 0s}\} = \{s \in \{0,1\}^n : s \text{ contains } n-r \text{ 1s}\}$
  - $\binom{n}{r} = |X|$
  - $\binom{n}{n-r} = |X|$

Inverse Binomial Transform

LEMMA: $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ for any $n, k, r \in \mathbb{N}$ such that $n \geq k \geq r$.

- Let $U = \{u_1, u_2, \dots, u_n\}$ be a finite set of n elements
- $S = \{(A, B): A \subseteq U, |A| = k, B \subseteq A, |B| = r\}$
  - choose A then choose B: $|S| = \binom{n}{k}\binom{k}{r}$, the left-hand side
  - choose B then choose A: $|S| = \binom{n}{r}\binom{n-r}{k-r}$, the right-hand side

LEMMA: $\sum_{k=r}^{n}(-1)^{n-k}\binom{n}{k}\binom{k}{r} = \begin{cases} 1 & n = r \\ 0 & n > r \end{cases}$.

- $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ as $n \geq k \geq r \geq 0$
- **left** $= \sum_{k=r}^{n}(-1)^{n-k}\binom{n}{r}\binom{n-r}{k-r} = \binom{n}{r}\sum_{k=r}^{n}(-1)^{(n-r)-(k-r)}\binom{n-r}{k-r}$

  $= \binom{n}{r}\sum_{i=0}^{n-r}(-1)^{(n-r)-i}\binom{n-r}{i}$

  $=$ **right**

Inverse Binomial Transform

LEMMA: Let $n, s \in \mathbb{N}, s \le n$. Then $\sum_{k=s}^{n} \underbrace{\sum_{i=s}^{k} a_{k,i}}_{\alpha_k} = \sum_{i=s}^{n} \underbrace{\sum_{k=i}^{n} a_{k,i}}_{\beta_i}$

$k \backslash i$	s	$s+1$	$s+2$	$\cdots$	n	row sum
s	$a_{s,s}$			$\cdots$		α_s
$s+1$	$a_{s+1,s}$	$a_{s+1,s+1}$		$\cdots$		α_{s+1}
$s+2$	$a_{s+2,s}$	$a_{s+2,s+1}$	$a_{s+2,s+2}$	$\cdots$		α_{s+2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
n	$a_{n,s}$	$a_{n,s+1}$	$a_{n,s+2}$	$\cdots$	$a_{n,n}$	α_n
col sum	β_s	β_{s+1}	β_{s+2}	$\cdots$	β_n	$\sum\sum$

THEOREM: Let $\{a_n\}, \{b_n\}$ be two sequences s.t. for all $n \ge s$,

$$a_n = \sum_{k=s}^{n} \binom{n}{k} b_k. \text{ Then } b_n = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k \ \ (n \ge s).$$

- $\sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} a_k = \sum_{k=s}^{n} (-1)^{n-k} \binom{n}{k} \sum_{i=s}^{k} \binom{k}{i} b_i$

$$= \sum_{i=s}^{n} \sum_{k=i}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{i} b_i = b_n$$