Timed Automata

$\langle L, L_0, Act, C, Tr, Inv \rangle$

where

L is a set of locations and $L_0 \subseteq L$ is the set of initial locations

Act is a set of actions and C a set of clocks

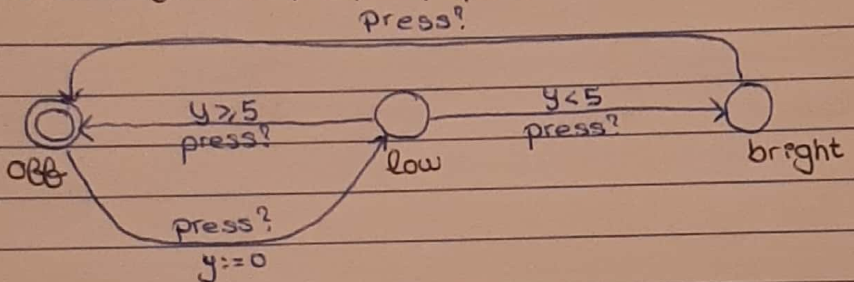
$Tr \subseteq L \times \mathcal{C}(C) \times Act \times \mathcal{P}(C) \times L$ is the transition relation

$$l_1 \xrightarrow{g, a, U} l_2$$

denotes a transition from location l_1 to l_2, labelled by a, enabled if guard g is valid, which, when performed, resets the set U of clocks

$Inv: L \to \mathcal{C}(C)$ is the assignment of invariants to locations where $\mathcal{C}(C)$ denotes the set of clock constraints over a set C of clock variables.

Exercise: Define $\langle L, L_0, Act, C, Tr, Inv \rangle$



$L = \{off, low, bright\}$

$L_0 = \{off\}$

$Act = \{press?\}$

$Tr =$
$\{(off, \{\}, press?, \{y\}, low),$
$(low, \{y \geq 5\}, press?, \{\}, off),$
$(low, \{y < 5\}, press?, \{\}, bright),$
$(bright, \{\}, press?, \{\}, off)\}$

$Inv = \{\}$

$C = \{y\}$

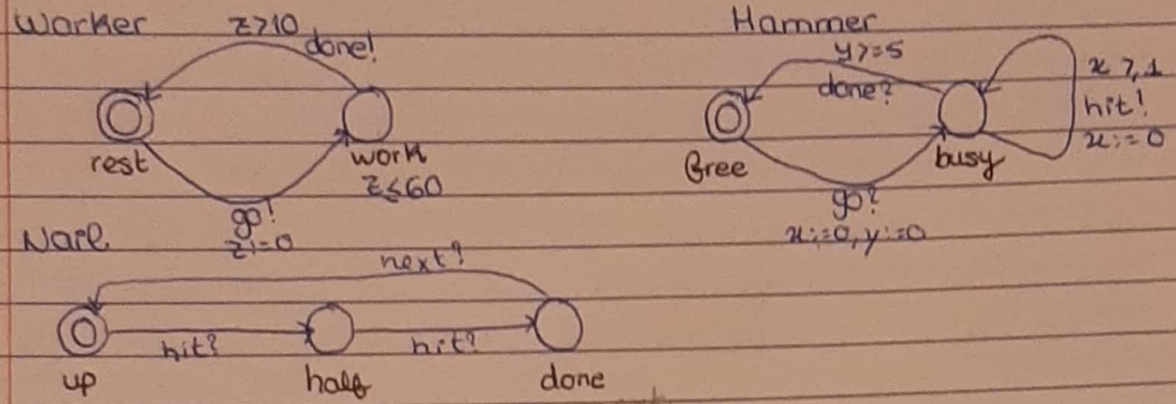
$off \xrightarrow{press?, y} low$

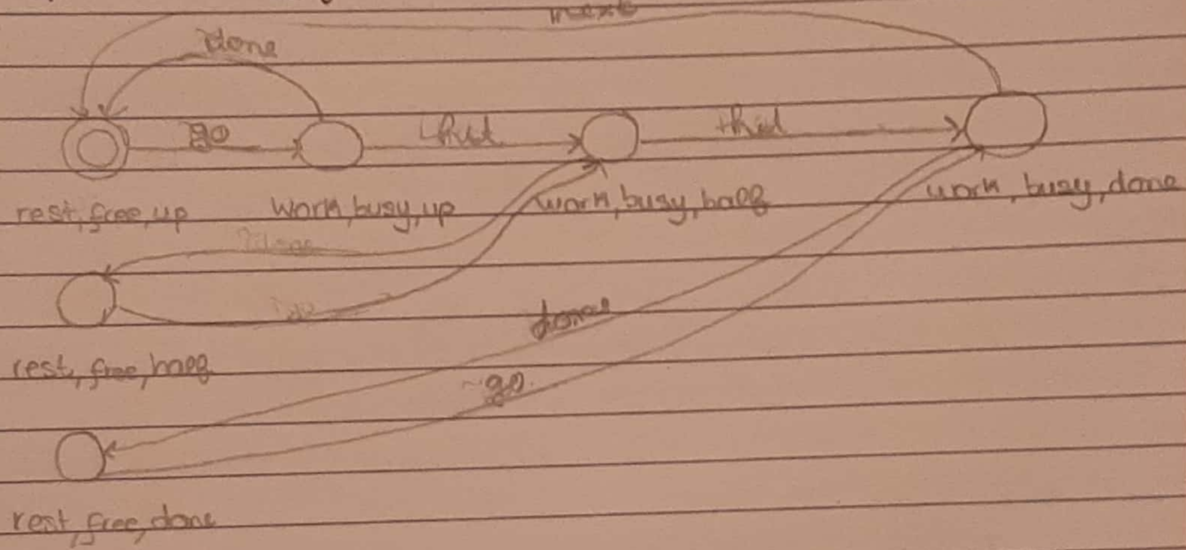
$low \xrightarrow{y \geq 5, press?} off$

$low \xrightarrow{y < 5, press?} bright$

$bright \xrightarrow{press?} off$

Exercise: Define the TA of the composition





Semantics

Timed LTS

Introduce delay transitions to capture the passage of time within a LTS:

$s \xrightarrow{a} s'$ for $a \in Act$, are ordinary transitions due to action ocurrance

$s \xrightarrow{d} s'$ for $d \in \mathbb{R}_0^+$, are delay transitions

subject to a number of constraints, e.g.

- time aditivity
  $(s \xrightarrow{d} s' \wedge 0 \leq d' \leq d) \Rightarrow s \xrightarrow{d'} s'' \xrightarrow{d-d'} s'$ for some state s''
- delay transitions are deterministic
  $(s \xrightarrow{d} s' \wedge s \xrightarrow{d} s'') \Rightarrow s' = s''$

Semantics of TA

Every TA defines a TLTS $T(ta)$ whose states are pairs $\langle$location, clock valuation$\rangle$ with infinitely, even uncountably many states, and infinite branching.

Parallel composition of timed automata

Let $H = Act_1 \cap Act_2$.

$$ta_1 \parallel_H ta_2 := \langle L_1 \times L_2, L_{0,1} \times L_{0,2}, Act_{\parallel H}, C_1 \cup C_2, Tr_{\parallel H}, Inv_{\parallel H} \rangle$$

where

$Act_{\parallel H} = ((Act_1 \cup Act_2) - H) \cup \{\tau\}$

$Inv_{\parallel H} \langle l_1, l_2 \rangle = Inv(l_1) \wedge Inv(l_2)$

$Tr_{\parallel H}$ is given by

$\langle l_1, l_2 \rangle \xrightarrow{g,a,U} \langle l_1', l_2 \rangle$ if $a \notin H \wedge l_1 \xrightarrow{g,a,U} l_1'$

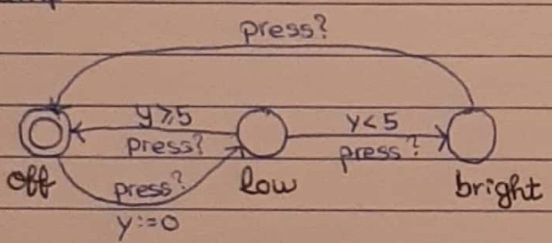
$\langle l_1, l_2 \rangle \xrightarrow{g,a,U} \langle l_1, l_2' \rangle$ if $a \notin H \wedge l_2 \xrightarrow{g,a,U} l_2'$

$\langle l_1, l_2 \rangle \xrightarrow{g,\tau,U} \langle l_1', l_2' \rangle$ if $a \in H \wedge l_1 \xrightarrow{g_1,a,U_1} l_1' \wedge l_2 \xrightarrow{g_2,a,U_2} l_2'$

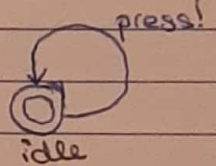
with $g = g_1 \wedge g_2$ and $U = U_1 \cup U_2$

Exercise: Define the TA of the composition.

Lamp



User



$L_1 \times L_2 = \{(off, idle), (low, idle), (bright, idle)\}$

$L_{01} \times L_{02} = \{(off, idle)\}$

$Act_{\parallel H} = ((\{press!\} \cup \{press?\}) - \{press\}) \cup \{\tau\} = \{\tau\}$

$C_1 \cup C_2 = \{y\}$

Transitions are

$(off, idle) \xrightarrow[y:=0]{\tau} (low, idle)$

$(low, idle) \xrightarrow{y \geq 5\ \tau} (off, idle)$

$(low, idle) \xrightarrow{y<5\ \tau} (bright, idle)$

$(bright, idle) \xrightarrow{\tau} (low, idle)$

$Tr_{\parallel H} = \{$

$Inv_{\parallel H} =$

Behavioural Equivalence

Traces

A timed trace over a TLTS is a (finite or infinite) sequence $\langle t_1, a_1 \rangle$, $\langle t_2, a_2 \rangle$, ... in $\mathbb{R}_0^+ \times Act$ such that there exists a path

$$\langle l_0, \eta_0 \rangle \xrightarrow{d_1} \langle l_0, \eta_1 \rangle \xrightarrow{a_1} \langle l_1, \eta_2 \rangle \xrightarrow{d_2} \langle l_1, \eta_3 \rangle \xrightarrow{a_2} \ldots$$

such that

$$t_i = t_{i-1} + d_i$$

with $t_0 = 0$ and, for all clock x, $\eta_0 x = 0$.

Exercise: Write 4 possible time traces.
(Lamp - page 2)

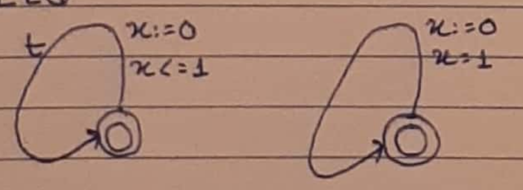
1: $\langle off, \bar{0} \rangle \xrightarrow{3} \langle off, \bar{3} \rangle \xrightarrow{press} \langle low, \bar{0} \rangle \xrightarrow{2} \langle low, \bar{2} \rangle$
2: ...
3: ...
4: ...

Given a timed trace tc, the corresponding untimed trace is $(\pi_2)^\omega$.

Two states s_1 and s_2 of a TLTS are timed-language equivalent if the set of finite timed traces of s_1 and s_2 coincide.

Exercise: Why are they not timed language equivalent?

LEQ



$\langle t, \overline{0.5} \rangle \xrightarrow{t} \langle t, \bar{0} \rangle$ is valid for the first, but not for the second.

Timed bissimulation

A relation R is a timed simulation iff whenever $s_1 R s_2$, for any action a and delay d,

$s_1 \xrightarrow{a} s_1' \Rightarrow$ there is a transition $s_2 \xrightarrow{a} s_2' \wedge s_1' R s_2'$

$s_1 \xrightarrow{d} s_1' \Rightarrow$ there is a transition $s_2 \xrightarrow{d} s_2 \wedge s_1' R s_2'$

And a timed bisimulation if its converse is also a timed simulation.

Clock valuations

A clock valuation η for a set of clocks C is a function

$$\eta : C \longrightarrow \mathbb{R}_0^+$$

assigning to each clock $x \in C$ its current value ηx.

$$\eta \models x \square n \iff \eta x \square n$$
$$\eta \models x - y \square n \iff (\eta x - \eta y) \square n$$
$$\eta \models g_1 \wedge g_2 \iff \eta \models g_1 \wedge \eta \models g_2$$

Operations on clock valuations

Delay

For each $d \in \mathbb{R}_0^+$, valuation $\eta + d$ is given by

$$(\eta + d)x = \eta x + d$$

Reset

For each $R \subseteq C$, valuation $\eta[R]$ is given by

$$\begin{cases} \eta[R]x = \eta x & \Leftarrow x \notin R \\ \eta[R]x = 0 & \Leftarrow x \in R \end{cases}$$

From ta to $\mathcal{T}(ta)$

Let $ta = \langle L, L_0, Act, C, Tr, Inv \rangle$

$$\mathcal{T}(ta) = \langle S, S_0 \subseteq S, N, T \rangle$$

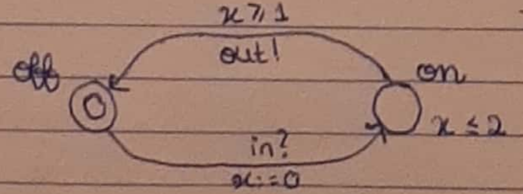
where

- $S = \{ \langle l, \eta \rangle \in L \times \mathbb{R}_0^+ \mid \eta \models Inv(l) \}$
- $S_0 = \{ \langle l_0, \eta \rangle \mid l_0 \in L_0 \wedge \eta x = 0$ for all $x \in C \}$
- $N = Act \cup \mathbb{R}_0^+$
- $T \subseteq S \times N \times S$

$$\langle l, \eta \rangle \xrightarrow{a} \langle l', \eta' \rangle \Leftarrow \exists_{l \xrightarrow{g,a,U} l' \in Tr}\ \eta \models g \wedge \eta' = \eta[U] \wedge \eta' \models Inv(l')$$
$$\langle l, \eta \rangle \xrightarrow{d} \langle l, \eta + d \rangle \Leftarrow \exists_{d \in \mathbb{R}_0^+}\ \eta + d \models Inv(l)$$

Exercise

Define $\mathcal{T}$(SwitchA)



T: $\langle off, \bar{t} \rangle \xrightarrow{d} \langle off, \bar{t} + d \rangle$ $t, d \geq 0$

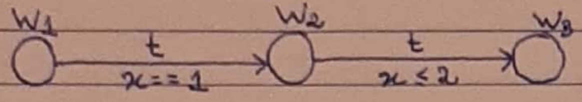
$\langle off, \bar{t} \rangle \xrightarrow{in} \langle on, \bar{0} \rangle$ $t \geq 0$

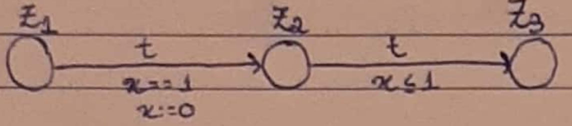
$\langle on, \bar{t} \rangle \xrightarrow{d} \langle on, \bar{t} + d \rangle$ $t, d \geq 0 \wedge t + d \leq 2$

$\langle on, \bar{t} \rangle \xrightarrow{out} \langle off, \bar{t} \rangle$ $1 \leq t \leq 2$

$S = \{ \langle off, \bar{t} \rangle \mid t \in \mathbb{R}_0^+ \} \cup \{ \langle on, \bar{t} \rangle \mid 0 \leq t \leq 2 \}$ where $\bar{t}$ is a shorthand for η such that $\eta x = t$.

Exercise: Are they bisimilar?





$$R = \{\langle\langle w_1, \{x \mapsto d\}\rangle, \langle z_1, \{x \mapsto d\}\rangle\rangle \mid d \in \mathbb{R}_0^+\} \cup$$

$$\{\langle\langle w_2, \{x \mapsto d+1\}\rangle, \langle z_1, \{x \mapsto d\}\rangle\rangle \mid d \in \mathbb{R}_0^+\} \cup$$

$$\{\langle\langle w_3, \{x \mapsto d\}\rangle, \langle z_3, \{x \mapsto e\}\rangle\rangle \mid d, e \in \mathbb{R}_0^+\}$$

Untimed bisimulation

A relation R is an untimed simulation iff whenever $s_1 R s_2$ for any action a and delay t,

$s_1 \xrightarrow{a} s_1' \Rightarrow$ there is a transition $s_2 \xrightarrow{a} s_2' \wedge s_1' R s_2'$

$s_1 \xrightarrow{d} s_1' \Rightarrow$ there is a transition $s_2 \xrightarrow{d'} s_2' \wedge s_1' R s_2'$

And it is an untimed bisimulation iff its converse is also an untimed simulation.