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Semantics for (Hybrid) Programming

The Hybrid While-Language

Fix a stack of variables $X = \{x_1, \dots, x_n\}$. Then, we have,

Linear Terms

$LTerm(X) \ni r \mid r \cdot t \mid x \mid t + s$

r ↓ real number

Atomic Programs

$At(X) \ni x := t \mid x_1' = t_1, \dots, x_n' = t_n \text{ for } t$

↓ run the system of differential equations for t seconds

Hybrid Programs

$Prog(X) \ni a \mid p ; q \mid \text{if } b \text{ then } p \text{ else } q \mid \text{while } b \text{ do } \{p\}$

A language of Linear Terms and its Semantics

$LTerm(X) \ni r \mid r \cdot t \mid x \mid t + s$

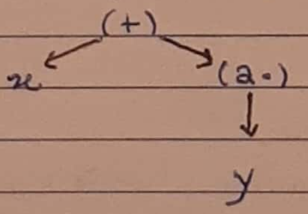
Let $\sigma : X \to \mathbb{R}$ be an environment, i.e a memory on which the program performs computations.

The expression $\langle t, \sigma \rangle \Downarrow r$ tells that the linear expression t outputs r if the current memory is σ.

$$\frac{}{\langle x, \sigma \rangle \Downarrow \sigma(x)}\ (var) \qquad \frac{}{\langle r, \sigma \rangle \Downarrow r}\ (con)$$

$$\frac{\langle t, \sigma \rangle \Downarrow r}{\langle s \cdot t, \sigma \rangle \Downarrow s \cdot r}\ (scl) \qquad \frac{\langle t_1, \sigma \rangle \Downarrow r_1 \quad \langle t_2, \sigma \rangle \Downarrow r_2}{\langle t_1 + t_2, \sigma \rangle \Downarrow r_1 + r_2}\ (add)$$

The linear term $x + 2 \cdot y$ corresponds to the tree



If $\sigma(x) = 3$ and $\sigma(y) = 4$, we can build the following derivation tree:

$$\dfrac{\langle x, \sigma \rangle \Downarrow 3 \qquad \dfrac{\langle y, \sigma \rangle \Downarrow 4}{\langle 2 \cdot y, \sigma \rangle \Downarrow 8}}{\langle x + 2 \cdot y, \sigma \rangle \Downarrow 11}$$

The program $x := x+1;\ x := x+2$ corresponds to the tree

$$\begin{array}{ccc} & (;) & \\ \swarrow & & \searrow \\ x := x+1 & & x := x+2 \end{array}$$

Consider the environment $\sigma = x \mapsto 3$. We build:

$$\dfrac{\dfrac{\langle x+1,\ x\mapsto 3\rangle \Downarrow 4}{\langle x := x+1,\ x\mapsto 3\rangle \Downarrow x\mapsto 4} \qquad \dfrac{\langle x+2,\ x\mapsto 4\rangle \Downarrow 6}{\langle x := x+2,\ x\mapsto 4\rangle \Downarrow x\mapsto 6}}{\langle x := x+1;\ x := x+2,\ x\mapsto 3\rangle \Downarrow x\mapsto 6}$$

Exercise

- $x:=0;\ y:=1;$ while $x \le y$ do $\{x := x+y;\ y := y+1\} \Downarrow ?$

$\sigma = x\mapsto ?, y\mapsto ?$
$\sigma' = x\mapsto 0, y\mapsto ?$
$\sigma'' = x\mapsto 0, y\mapsto 1$
$\sigma''' = x\mapsto 1, y\mapsto 1$
$\sigma'''' = x\mapsto 1, y\mapsto 2$
$\sigma''''' = x\mapsto 3, y\mapsto 2$
$\sigma'''''' = x\mapsto 3, y\mapsto 3$
$\sigma''''''' = x\mapsto 6, y\mapsto 3$
$\sigma'''''''' = x\mapsto 6, y\mapsto 4$

$$\dfrac{\dfrac{\dfrac{}{\langle 0,\sigma\rangle \Downarrow 0}(con)}{\langle x:=0,\sigma\rangle \Downarrow \sigma'}(asg) \quad \dfrac{\dfrac{\dfrac{}{\langle 1,\sigma'\rangle \Downarrow 1}(con)}{\langle y:=1,\sigma'\rangle \Downarrow \sigma''}(asg) \quad \dfrac{\dfrac{*_1}{\langle x\le y,\sigma''\rangle \Downarrow tt} \ \dfrac{*_2}{\langle p,\sigma''\rangle \Downarrow \sigma''''} \ \dfrac{*_3}{\langle \text{while } x\le y \text{ do}\{p\},\sigma''''\rangle \Downarrow \sigma''''''''}}{\langle \text{while } x\le y \text{ do}\{p\},\sigma''\rangle \Downarrow \sigma''''''''}(wh1)}{}}{\langle x:=0;\ y:=1;\ \text{while } x\le y \text{ do}\underbrace{\{x:=x+y;\ y:=y+1\}}_{p},\sigma\rangle \Downarrow \sigma''''''''}(seq)$$

$*_1$
$$\dfrac{\langle x,\sigma''\rangle \Downarrow 0 \quad \langle y,\sigma''\rangle \Downarrow 1 \quad 0 \le 1}{\langle x\le y,\sigma''\rangle \Downarrow tt}(leq)$$

$*_2$
$$\dfrac{\dfrac{\dfrac{\langle x,\sigma''\rangle \Downarrow 0 \quad \langle y,\sigma''\rangle \Downarrow 1}{\langle x+y,\sigma''\rangle \Downarrow 1}}{\langle x:=x+y,\sigma''\rangle \Downarrow \sigma'''} \quad \dfrac{\langle y+1,\sigma'''\rangle \Downarrow 2}{\langle y:=y+1,\sigma'''\rangle \Downarrow \sigma''''}}{\langle p,\sigma''\rangle \Downarrow \sigma''''}$$

$*_3$
$$\dfrac{\langle x\le y,\sigma''''\rangle \Downarrow tt \quad \langle p,\sigma''''\rangle \Downarrow \sigma'''''' \quad \dfrac{*_4}{\langle \text{while}\ldots,\sigma''''''\rangle \Downarrow \sigma''''''''}}{\langle \text{while } x\le y \text{ do } \{p\},\sigma''''\rangle \Downarrow \sigma''''''''}$$

$*_4$
$$\dfrac{\langle x\le y,\sigma''''''\rangle \Downarrow tt \quad \langle p,\sigma''''''\rangle \Downarrow \sigma'''''''' \quad \dfrac{\langle x\le y,\sigma''''''''\rangle \Downarrow ff}{\langle \text{while}\ldots,\sigma''''''''\rangle \Downarrow \sigma''''''''}}{\langle \ldots,\sigma''''''\rangle \Downarrow \sigma''''''''}$$

Exercises

- $\langle 2\cdot x + 2\cdot y, \sigma\rangle \Downarrow\, ?$

$$\sigma(x)=3 \qquad \sigma(y)=4 \qquad \sigma(z)=8$$

$$\dfrac{\dfrac{\dfrac{}{\langle x,\sigma\rangle \Downarrow 3}\,(\text{var})}{\langle 2\cdot x,\sigma\rangle \Downarrow 6}\,(\text{scl}) \qquad \dfrac{\dfrac{}{\langle y,\sigma\rangle \Downarrow 4}\,(\text{var})}{\langle 2\cdot y,\sigma\rangle \Downarrow 8}\,(\text{scl})}{\langle 2\cdot x+2\cdot y,\sigma\rangle \Downarrow 14}\,(\text{add})$$

- $\langle 3\cdot(2\cdot x) + 2\cdot(y+z), \sigma\rangle \Downarrow\, ?$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\langle x,\sigma\rangle \Downarrow 3}\,(\text{var})}{\langle 2\cdot x,\sigma\rangle \Downarrow 6}\,(\text{scl})}{\langle 3\cdot(2\cdot x),\sigma\rangle \Downarrow 18}\,(\text{scl}) \qquad \dfrac{\dfrac{\dfrac{}{\langle y,\sigma\rangle \Downarrow 4}\,(\text{var}) \quad \dfrac{}{\langle z,\sigma\rangle \Downarrow 8}\,(\text{var})}{\langle y+z,\sigma\rangle \Downarrow 12}\,(\text{add})}{\langle 2\cdot(y+z),\sigma\rangle \Downarrow 24}\,(\text{scl})}{\langle 3\cdot(2\cdot x)+2\cdot(y+z),\sigma\rangle \Downarrow 42}\,(\text{add})$$

Equivalence of Linear Terms

$t \sim s$ iff for all environments σ $\langle t,\sigma\rangle \Downarrow r$ iff $\langle s,\sigma\rangle \Downarrow r$

A Language of Boolean Terms and its semantics

$BTerm(X) \ni t_1 \leq t_2 \mid b \wedge c \mid \neg b$

$$\dfrac{\langle t_1,\sigma\rangle \Downarrow r_1 \quad \langle t_2,\sigma\rangle \Downarrow r_2 \quad r_1 \leq r_2}{\langle t_1 \leq t_2,\sigma\rangle \Downarrow tt}\,(\text{leq}) \qquad \dfrac{\langle b,\sigma\rangle \Downarrow v}{\langle \neg b,\sigma\rangle \Downarrow \neg v}\,(\text{not})$$

$$\dfrac{\langle t_1,\sigma\rangle \Downarrow r_1 \quad \langle t_2,\sigma\rangle \Downarrow r_2 \quad r_1 \not\leq r_2}{\langle t_1 \leq t_2,\sigma\rangle \Downarrow ff}\,(\text{gtr})$$

$$\dfrac{\langle b_1,\sigma\rangle \Downarrow v_1 \quad \langle b_2,\sigma\rangle \Downarrow v_2}{\langle b_1 \wedge b_2,\sigma\rangle \Downarrow v_1 \wedge v_2}\,(\text{and})$$

A While-language and its Semantics

$Prog(X) \ni x := t \mid p;q \mid \text{if } b \text{ then } p \text{ else } q \mid \text{while } b \text{ do } \{p\}$

$$\dfrac{\langle t,\sigma\rangle \Downarrow r}{\langle x:=t,\sigma\rangle \Downarrow \sigma[r/x]}\,(\text{asg}) \qquad \dfrac{\langle p,\sigma\rangle \Downarrow \sigma' \quad \langle q,\sigma'\rangle \Downarrow \sigma''}{\langle p;q,\sigma'\rangle \Downarrow \sigma''}\,(\text{seq})$$

$$\dfrac{\langle b,\sigma\rangle \Downarrow tt \quad \langle p,\sigma\rangle \Downarrow \sigma'}{\langle \text{if } b \text{ then } p \text{ else } q,\sigma\rangle \Downarrow \sigma'}\,(\text{if1}) \qquad \dfrac{\langle b,\sigma\rangle \Downarrow ff \quad \langle q,\sigma\rangle \Downarrow \sigma'}{\langle \text{if } b \text{ then } p \text{ else } q,\sigma\rangle \Downarrow \sigma'}\,(\text{if2})$$

$$\dfrac{\langle b,\sigma\rangle \Downarrow tt \quad \langle p,\sigma\rangle \Downarrow \sigma' \quad \langle \text{while } b \text{ do } \{p\},\sigma'\rangle \Downarrow \sigma''}{\langle \text{while } b \text{ do } \{p\},\sigma\rangle \Downarrow \sigma''}\,(\text{wh1})$$

$$\dfrac{\langle b,\sigma\rangle \Downarrow ff}{\langle \text{while } b \text{ do } \{p\},\sigma\rangle \Downarrow \sigma}\,(\text{wh2})$$

Examples

$$\dfrac{\dfrac{\langle 1,(x\mapsto 2)\rangle \Downarrow 1 \qquad 1/2<1}{\langle x'=0 \text{ for } 1,(x\mapsto 2),1/2\rangle \Downarrow \text{stop},(x\mapsto 2)}}{\langle (x'=0 \text{ for } 1);(x'=1 \text{ for } 1),(x\mapsto 2),1/2\rangle \Downarrow \text{stop},(x\mapsto 2)}$$

$$\downarrow$$

$$=(x\mapsto 2)[\phi(2,1/2)/x]$$

$$\dfrac{\dfrac{\cdots}{\langle x'=0 \text{ for } 1,(x\mapsto 2),1\rangle \Downarrow \text{skip},(x\mapsto 2)} \qquad \dfrac{\cdots}{\langle x'=1 \text{ for } 1,(x\mapsto 2),1/2\rangle \Downarrow \text{stop},(x\mapsto 2+1/2)}}{\langle (x'=0 \text{ for } 1);(x'=1 \text{ for } 1),(x\mapsto 2),1+1/2\rangle \Downarrow \text{stop},(x\mapsto 2+1/2)}$$

Exercises

- $\langle (x'=1 \text{ for } 1);(x'=-1 \text{ for } 1),(x\mapsto 5),1/2\rangle \Downarrow ?$

$$\dfrac{\dfrac{\langle 1,(x\mapsto 5)\rangle \Downarrow 1 \qquad 1/2<1}{\langle x'=1 \text{ for } 1,(x\mapsto 5),1/2\rangle \Downarrow \text{stop}, x\mapsto 5+1/2}}{\langle (x'=1 \text{ for } 1);(x'=-1 \text{ for } 1),(x\mapsto 5),1/2\rangle \Downarrow \text{stop}, x\mapsto 5+1/2}$$

- $\langle (x'=1 \text{ for } 1);(x'=-1 \text{ for } 1),(x\mapsto 5),2\rangle \Downarrow$

$$\dfrac{\dfrac{\langle 1,(x\mapsto 5)\rangle \Downarrow 1 \qquad 1=1}{\langle x'=1 \text{ for } 1,(x\mapsto 5),1\rangle \Downarrow \text{skip}, x\mapsto 6} \qquad \dfrac{\langle 1,(x\mapsto 6)\rangle \Downarrow -1 \qquad 1=1}{\langle x'=-1 \text{ for } 1,(x\mapsto 6),1\rangle \Downarrow \text{skip}, x\mapsto 5}}{\langle (x'=1 \text{ for } 1);(x'=-1 \text{ for } 1),(x\mapsto 5),2\rangle \Downarrow \text{skip}, x\mapsto 5}$$

More rules

$$\dfrac{\langle b,\sigma\rangle \Downarrow tt \qquad \langle p,\sigma,t\rangle \Downarrow s,\sigma'}{\langle \text{if } b \text{ then } p \text{ else } q,\sigma,t\rangle \Downarrow s,\sigma'} \qquad \dfrac{\langle b,\sigma\rangle \Downarrow ff \qquad \langle q,\sigma,t\rangle \Downarrow s,\sigma'}{\langle \text{if } b \text{ then } p \text{ else } q,\sigma,t\rangle \Downarrow s,\sigma'}$$

$$\dfrac{\langle b,\sigma\rangle \Downarrow tt \qquad \langle p;\text{while } b \text{ do } \{p\},\sigma,t\rangle \Downarrow s,\sigma'}{\langle \text{while } b \text{ do } \{p\},\sigma,t\rangle \Downarrow s,\sigma'}$$

$$\dfrac{\langle b,\sigma\rangle \Downarrow ff}{\langle \text{while } b \text{ do } \{p\},\sigma,0\rangle \Downarrow \text{skip},\sigma}$$

Equivalence of While-programs

$p \sim q$ if for all environments σ $\langle p, \sigma \rangle \Downarrow \sigma'$ iff $\langle q, \sigma \rangle \Downarrow \sigma'$

Preliminaries about Differential Equations

Consider a stock $X = \{x_1, \dots, x_n\}$ of variables.

Systems of differential equations $x_1' = t_1, \dots, x_n' = t_n$ always have unique solutions.

$$\phi : \mathbb{R}^n \times [0, \infty) \longrightarrow \mathbb{R}^n$$

Example (The Continuous Dynamics of a Vehicle)

$p' = v, \; v' = a$ which admits the solution

$$\phi((x_0, v_0), t) = (x_0 + v_0 t + \tfrac{1}{2} a t^2, \; v_0 + at)$$

We will often abreviate a list $v_1, \dots, v_n$ simply to $\bar{v}$.

$\sigma[\bar{v}/\bar{x}] \Leftrightarrow \forall i \; x_i = v_i$

Example

$$\sigma[v_1, v_2 / x_1, x_2](y) = \begin{cases} v_1 & \text{if } y = x_1 \\ v_2 & \text{if } y = x_2 \\ \sigma(y) & \text{otherwise} \end{cases}$$

We will often treat $\sigma : \{x_1, \dots, x_n\} \to \mathbb{R}$ as $[\sigma(x_1), \dots, \sigma(x_n)]$.

Adding time dependency to Hybrid While-language

$\langle p, \sigma, t \rangle \Downarrow \sigma'$

$$\frac{\langle s, \sigma \rangle \Downarrow r \quad t < r}{\langle \bar{x}' = E \text{ for } s, \sigma, t \rangle \Downarrow stop, \sigma[\phi(\sigma, t)/\bar{x}]}$$

$$\frac{\langle s, \sigma \rangle \Downarrow r \quad t = r}{\langle \bar{x}' = E \text{ for } s, \sigma, t \rangle \Downarrow skip, \sigma[\phi(\sigma, t)/\bar{x}]}$$

$$\frac{\langle t, \sigma \rangle \Downarrow r}{\langle x := t, \sigma, 0 \rangle \Downarrow \sigma[r/x]} \qquad \frac{\langle p, \sigma, t \rangle \Downarrow stop, \sigma'}{\langle p ; q, \sigma, t \rangle \Downarrow stop, \sigma'}$$

$$\frac{\langle p, \sigma, t \rangle \Downarrow skip, \sigma' \qquad \langle q, \sigma', t' \rangle \Downarrow s, \sigma''}{\langle p ; q, \sigma, t + t' \rangle \Downarrow s, \sigma''}$$

Equivalence of While-Programs

$p \sim q$ iff for all environments σ and time instants t,

$\langle p, \sigma, t \rangle \Downarrow s, \sigma'$ iff $\langle q, \sigma, t \rangle \Downarrow s, \sigma'$