F

Overview of simply typed λ-calculus

Rules

$$\dfrac{A \in \Gamma}{\Gamma \vdash A}\,(ass) \qquad \dfrac{}{\Gamma \vdash 1}\,(trv) \qquad \dfrac{\Gamma \vdash A \times B}{\Gamma \vdash A}\,(\pi_1) \qquad \dfrac{\Gamma \vdash A \times B}{\Gamma \vdash B}\,(\pi_2)$$

$$\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \times B}\,(prd) \qquad \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\,(cry) \qquad \dfrac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}\,(app)$$

Exercise: Prove $A \times B \vdash B \times A$

$$\dfrac{\dfrac{\dfrac{(A \times B) \in \{A \times B\}}{A \times B \vdash A \times B}\,(ass)}{A \times B \vdash B}\,(\pi_2) \quad \dfrac{\dfrac{(A \times B) \in \{A \times B\}}{A \times B \vdash A \times B}\,(ass)}{A \times B \vdash A}\,(\pi_1)}{A \times B \vdash B \times A}\,(prd)$$

Derivable rules

$$\dfrac{\Gamma \vdash A}{\Gamma, B \vdash A} \qquad \dfrac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

Exercise: Prove:

- $A \to B, B \to C \vdash A \to C$

$$\dfrac{\dfrac{\dfrac{\dfrac{B \to C \in \{B \to C\}}{B \to C \vdash B \to C}\,(ass)}{A, A \to B, B \to C \vdash B \to C} \quad \dfrac{\dfrac{\dfrac{A \to B \in \{A \to B\}}{A \to B \vdash A \to B}\,(ass)}{A, A \to B, B \to C \vdash A \to B} \quad \dfrac{\dfrac{A \in \{A\}}{A \vdash A}\,(ass)}{A, A \to B, B \to C \vdash A}}{A, A \to B, B \to C \vdash B}\,(app)}{A \to B, B \to C, A \vdash C}\,(app)}{A \to B, B \to C \vdash A \to C}\,(cry)$$

- $A \to B, A \to C \vdash A \to B \times C$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to B \in \{A \to B\}}{A \to B \vdash A \to B}\,(ass)}{A, A \to B \vdash A \to B} \quad \dfrac{\dfrac{A \in \{A\}}{A \vdash A}\,(ass)}{A, A \to B \vdash A}}{A, A \to B, A \to C \vdash B}\,(app) \quad \dfrac{\dfrac{\dfrac{A \to C \in \{A \to C\}}{A \to C \vdash A \to C}\,(ass)}{A, A \to C \vdash A \to C} \quad \dfrac{\dfrac{A \in \{A\}}{A \vdash A}\,(ass)}{A, A \to C \vdash A}}{A, A \to B, A \to C \vdash C}\,(app)}{A, A \to B, A \to C \vdash B \times C}\,(prd)}{A \to B, A \to C \vdash A \to B \times C}\,(cry)$$

Simply-typed λ-Calculus

$$\frac{x : A \in \Gamma}{\Gamma \vdash x : A}\ (\text{ass}) \qquad \frac{}{\Gamma \vdash * : 1}\ (\text{triv}) \qquad \frac{\Gamma \vdash V : A \times B}{\Gamma \vdash \pi_1 V : A}\ (\pi_1)$$

$$\frac{\Gamma \vdash V : A \quad \Gamma \vdash U : B}{\Gamma \vdash \langle V, U \rangle : A \times B}\ (\text{prd}) \qquad \frac{\Gamma, x : A \vdash V : B}{\Gamma \vdash \lambda x : A . V : A \to B}\ (\text{cry})$$

$$\frac{\Gamma \vdash V : A \to B \quad \Gamma \vdash U : A}{\Gamma \vdash V U : B}\ (\text{app})$$

Examples of λ-terms

$x : A \vdash x : A$ (identity)

$x : A \vdash \langle x, x \rangle : A \times A$ (duplication)

$x : A \times B \vdash \langle \pi_2 x, \pi_1 x \rangle : B \times A$ (swap)

$f : A \to B, g : B \to C \vdash \lambda x : A . g(f x) : A \to C$ (composition)

Exercise: Build a λ-term $f : A \to B, g : A \to C \vdash ? : A \to B \times C$

$f : A \to B, g : A \to C \vdash \lambda x : A . (f x, g x) : A \to B \times C$

Basic facts about functions

Trivial:	$! : X \to \{*\} = 1$	$!(x) = *$
Pair:	$\langle f, g \rangle : X \to A \times B$	$\langle f, g \rangle(x) = (f x, g x)$
Projections:	$\pi_1 : X \times Y \to X$	$\pi_1(x, y) = x$
	$\pi_2 : X \times Y \to Y$	$\pi_2(x, y) = y$
Curry:	$\lambda f : X \to Z^Y$	$\lambda f(x) = (y \mapsto f(x, y))$
App:	$Z^Y \times Y \to Z$	$\text{app}(f, y) = f y$

Functional Semantics

Types A are interpreted as sets $[\![A]\!]$

$[\![1]\!] = \{*\}$ $\qquad [\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$ $\qquad [\![A \to B]\!] = [\![B]\!]^{[\![A]\!]}$

A typing context Γ is interpreted as

$$[\![\Gamma]\!] = [\![x_1 : A_1 \times \dots \times x_n : A_n]\!] = [\![A_1]\!] \times \dots \times [\![A_n]\!]$$

A λ-term $\Gamma \vdash V : A$ is interpreted as a function

$$[\![\Gamma \vdash V : A]\!] : [\![\Gamma]\!] \to [\![A]\!]$$

$$\frac{x_i : A \in \Gamma}{[\![\Gamma \vdash x_i : A]\!] = \pi_i} \qquad \frac{}{[\![\Gamma \vdash * : 1]\!] = \,!} \qquad \frac{[\![\Gamma \vdash V : A \times B]\!] = f}{[\![\Gamma \vdash \pi_1 V : A]\!] = \pi_1 \cdot f}$$

$$\frac{[\![\Gamma \vdash V : A]\!] = f \quad [\![\Gamma \vdash U : B]\!] = g}{[\![\Gamma \vdash \langle V, U \rangle : A \times B]\!] = \langle f, g \rangle} \qquad \frac{[\![\Gamma, x : A \vdash V : B]\!] = f}{[\![\Gamma \vdash \lambda x : A.\, V : A \to B]\!] = \lambda f}$$

$$\frac{[\![\Gamma \vdash V : A \to B]\!] = f \quad [\![\Gamma \vdash U : A]\!] = g}{[\![\Gamma \vdash VU : B]\!] = \mathrm{app} \cdot \langle f, g \rangle}$$

Exercises: Prove:

$[\![x : A, y : B \vdash \pi_1 \langle x, y \rangle : A]\!] = [\![x : A, y : B \vdash x : A]\!]$

$[\![x : A, y : B \vdash \pi_1 \langle x, y \rangle : A]\!]$

$=$

$\pi_1 \cdot [\![x : A, y : B \vdash \langle x, y \rangle : A \times B]\!]$

$=$

$\pi_1 \cdot \langle [\![x : A, y : B \vdash x : A]\!], [\![x : A, y : B \vdash y : B]\!] \rangle$

$=$

$[\![x : A, y : B \vdash x : A]\!]$

$[\![\Gamma \vdash V : A \times B]\!] = [\![\Gamma \vdash \langle \pi_1 V, \pi_2 V \rangle : A \times B]\!]$ * está errado nos slides

$[\![\Gamma \vdash \langle \pi_1 V, \pi_2 V \rangle : A \times B]\!]$

$=$

$\langle [\![\Gamma \vdash \pi_1 V : A]\!], [\![\Gamma \vdash \pi_2 V : B]\!] \rangle$

$=$

$\langle \pi_1 \cdot [\![\Gamma \vdash V : A \times B]\!], \pi_2 \cdot [\![\Gamma \vdash V : A \times B]\!] \rangle$

$=$

$\langle \pi_1, \pi_2 \rangle \cdot [\![\Gamma \vdash V : A \times B]\!]$

$=$

$\mathrm{id} \cdot [\![\Gamma \vdash V : A \times B]\!]$

$=$

$[\![\Gamma \vdash V : A \times B]\!]$

$[\![- \vdash (\lambda x. x+1)\ 2 : \mathbb{N}]\!] = [\![- \vdash 3 : \mathbb{N}]\!]$

$[\![- \vdash (\lambda x. x+1)\ 2 : \mathbb{N}]\!]$

$=$

$app \circ \langle [\![- \vdash (\lambda x. x+1) : \mathbb{N} \to \mathbb{N}]\!], [\![- \vdash 2 : \mathbb{N}]\!] \rangle$

$=$

$app \circ \langle \lambda [\![x : \mathbb{N} \vdash x+1 : \mathbb{N}]\!], [\![- \vdash 2 : \mathbb{N}]\!] \rangle$

$=$

$app \circ \langle \lambda succ, 2 \rangle$

$=$

$succ\ 2$

$=$

3

$=$

$[\![- \vdash 3 : \mathbb{N}]\!]$