Modal Logic
Solutions to Exercise Set 09

1. Give a constant domain tableau proof of a Converse Barcan formula.

   Solution: We construct the constant domain tableau proof as follows.

   $$
   \begin{array}{c}
   1\ \neg(\Box(\forall x)\varphi(x) \to (\forall x)\Box\varphi(x)) \\
   | \\
   1\ \Box(\forall x)\varphi(x) \\
   1\ \neg(\forall x)\Box\varphi(x) \\
   | \\
   1\ \neg\Box\varphi(p) \\
   | \\
   1.1\ \neg\varphi(p) \\
   | \\
   1.1\ (\forall x)\varphi(x) \\
   | \\
   1.1\ \varphi(p) \\
   \times
   \end{array}
   $$

2. Use a failed variable domain proof of a Converse Barcan formula to construct a countermodel. Prove that the Converse Barcan formula in question is not valid in the model.

Solution: A failed, saturated variable domain tableau proof of $\Box(\forall x)P(x) \to (\forall x)\Box P(x)$ is as follows.

$$
\begin{array}{c}
1\ \neg(\Box(\forall x)P(x) \to (\forall x)\Box P(x)) \\
| \\
1\ \Box(\forall x)P(x) \\
1\ \neg(\forall x)\Box P(x) \\
| \\
1\ \neg\Box P(p^1) \\
| \\
1.1\ \neg P(p^1) \\
| \\
1.1\ (\forall x)P(x) \\
| \\
1.1\ P(p^{1.1})
\end{array}
$$

This allows us to construct a variable domain model $M = (W, R, D, I)$ where $W = \{1, 1.1\}$, $R = \{\langle 1, 1.1\rangle\}$, $D(1) = \{p^1\}$, $D(1.1) = \{p^{1.1}\}$, $I(P, 1) = \emptyset$, $I(P, 1.1) = \{p^{1.1}\}$.
We now verify that $M \models \Box(\forall x)P(x)$ but that $M \not\models (\forall x)\Box P(x)$.
Let v be the valuation such that $v(p^\sigma) = p^\sigma$ for any parameter p^σ, and let θ be the assignment such that $\theta(\sigma) = \sigma$ for any prefix σ.
Firstly, note that $M, 1.1 \Vdash_v P(p^{1.1})$. Let w be any x-variant of v at 1.1. Then $w(x) = p^{1.1}$ since $p^{1.1}$ is the sole element of $D(1.1)$. So $M, 1.1 \Vdash_v (\forall x)P(x)$. Finally, $M, 1 \Vdash_v \Box(\forall x)P(x)$ since 1.1 is the unique world such that $1R1.1$.

Now suppose for a contradiction that $M, 1 \Vdash_v (\forall x)\Box P(x)$. It follows that for any x-variant w of v at 1, $M, 1 \Vdash_w \Box P(x)$. $M, 1.1 \Vdash_w P(x)$ since 1.1 is accessible from 1, so $w(x) \in I(P, 1.1)$. $p^{1.1}$ is the unique element of $I(P, 1.1)$, so $w(x) = p^{1.1}$. However, w is an x-variant of v at 1, so by definition, $w(x) \in D(1)$. So $w(x) = p^1$.

3. Prove the universal case of the branch extension lemma for the variable domain tableau system.

References

M. Fitting and R. L. Mendelsohn. *First-Order Modal Logic.* Number 277 in Synthese Library. Springer, 1998.