PH340 Logic 3: Incompleteness and Undecidability
Term 2, 2021–2022

Lecturer

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  Please use your university email and put "PH340" in the subject line.

Website

https://moodle.warwick.ac.uk/course/view.php?id=47743

Readings, announcements, and problem sets will be posted at this address.

Module format for 2021–22

1. **Lectures.** Online asynchronous, $\pm$ 1.5 hours/week, uploaded on Tuesday.
2. **Problem classes** (*attendance expected*)**.** Monday 12:00–13:00 online synchronous on Teams, starting in week 2.
3. **Drop-in problem sessions** (*attendance optional*)**.**
   - Monday 16:00–17:00 by individual appointment, on Teams or in person (please email for a slot).
   - Monday 17:00–18:00 online synchronous on Teams.

Description

Developments in formal logic in the late 19th and early 20th century opened up the prospect for an entirely formalised mathematics, in which all mathematical statements could be expressed by sentences of a formal language, all proofs could be transformed into deductions in a logical system, and all basic mathematical principles could be codified as axioms. This naturally raised a question of completeness: given such a formal language, and an axiomatic theory T expressed in that language, could T either prove or refute every sentence in the formal language, and thus provide a solution (at least in principle) to every mathematical question expressible in that language? Gödel's incompleteness theorems showed that in general the answer is no: for any consistent axiomatic theory T containing a sufficient amount of arithmetic, there will be sentences in the language of T which T can neither prove nor refute (the first incompleteness theorem). Moreover, such a theory T cannot even prove its own consistency (the second incompleteness theorem). This demonstrates the limits of formalisation in mathematics: there can be no universal formal theory capable of answering all mathematical questions, and we can only prove the consistency of our theories by appealing to strictly stronger theories. In this module we will explore the incompleteness theorems: precisely what they say, and how they are proved. Along the way we will develop an understanding of formal theories of arithmetic and elementary computability theory.

Prerequisites

Students are advised to take the module PH210 Logic 2: Metatheory before taking this module. This module can be taken in the same academic year as PH210.

Reading

The main reading for this module will be a customised version of the Open Logic textbook, which is available on the Moodle page. A selection of background and further reading is available below.

Problem class

All students are expected to attend and participate in the problem class. This is particularly important for Philosophy students as attendance contributes to the monitoring point system. This is also your opportunity to clarify anything from the lecture or the readings and to get help with exercises.

Assessment

The module will be assessed (100%) by a two-hour in-person exam in the summer. You will have 2 hours in which to answer 3 questions (from a choice of 6).

Past exam papers for this module are available, and are a good guide to the type and level of difficulty of the questions in this year's exam, although not necessarily to the specific content that will be taught in the module this year. There will be a revision session for the module before the start of the exam period.

How to do well in this module

The lectures will follow our customised version of the Open Logic textbook, as detailed in the schedule below. It will therefore be useful to have read the relevant sections of the textbook before watching the corresponding lectures. Doing the weekly exercises during the term is also essential for building and testing your understanding of the material as we go along. Solutions will be posted on the module website, and discussed in both the problem class and the drop-in problem sessions.

Schedule

The following is an indicative module outline. We may cover a little more or a little less, depending on how things go.

Week	Topics	Textbook
1	First-order logic, the language of arithmetic, the standard model	Ch. 1
2	Primitive recursive functions and relations	Ch. 2
3	Sequences, trees, general recursive functions	Ch. 2
4	Arithmetization of syntax	Ch. 3
5	Representability in **Q** (part 1)	Ch. 4
6	*Reading week (no lecture or problem class)*	
7	Representability in **Q** (part 2)	Ch. 4
8	The fixed-point lemma, the first incompleteness theorem	Ch. 5
9	The derivability conditions, the second incompleteness theorem	Ch. 5
10	Non-standard models of arithmetic	Ch. 6

Resources

Here you can find further reading, including other textbooks that might help by giving a slightly different view on the same material. Links to online versions of some texts, as well as library catalogue details, are available on the module reading list.

1. *Other textbooks*

   (a) *Modern Mathematical Logic*, by Joseph Mileti (Cambridge University Press, 2022).
   `https://webcat.warwick.ac.uk/record=b3861128~S1`
   A contemporary introduction to mathematical logic, covering all the basics of syntax and semantics, completeness and compactness, computation, and incompleteness, together with the basics of set theory and model theory.

   (b) *Computability and Logic*, 3rd ed., by George Boolos, John Burgess, and Richard Jeffrey (Cambridge University Press, 1989).
   `https://encore.lib.warwick.ac.uk/iii/encore/record/C__Rb2786317`
   A good textbook if you'd like a different perspective on the material, or want to read further on topics not covered in the module.

   (c) 'The incompleteness theorems' by Craig Smorynski, in the *Handbook of Mathematical Logic* (pp. 821–865), edited by Jon Barwise (North-Holland, 1977).
   `https://encore.lib.warwick.ac.uk/iii/encore/record/C__Rb3517404`
   An excellent, lucid exposition of Gödel's incompleteness theorems and related results in mathematical logic.

   (d) *Mathematical Logic* by Joseph R. Shoenfield (Addison-Wesley, 1967).
   `https://encore.lib.warwick.ac.uk/iii/encore/record/C__Rb3517405`
   Quite old now, but still a classic textbook. Probably a bit difficult, but will open up a lot more of mathematical logic than this course is able to.

2. *Further reading on incompleteness*

   (a) *Kurt Gödel: Collected Works. I: Publications 1929–1936*, edited by Solomon Feferman, John W. Dawson, Stephen C. Kleene, Gregory H. Moore, Robert M. Solovay, and Jean van Heijenoort (Oxford University Press, 1986).
   `https://encore.lib.warwick.ac.uk/iii/encore/record/C__Rb1661998`
   The first volume of Gödel's collected works contains his famous 1931 paper proving the first incompleteness theorem, 'On formally undecidable statements of Principia Mathematica and related systems'.

   (b) *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931*, edited by Jean van Heijenoort (Harvard University Press, 1967).
   `https://encore.lib.warwick.ac.uk/iii/encore/record/C__Rb3517392`
   This handbook contains many important original papers in mathematical logic from the late 19th century (Frege) up to the discovery of incompleteness in 1930 (Gödel).

3. *Incompleteness and the philosophy of mathematics*

   (a) 'Hilbert's Program' by Richard Zach (Stanford Encyclopedia of Philosophy, 2019).
   An entry in the open-access Stanford Encyclopedia of Philosophy on Hilbert's programme in the foundations of mathematics, and how Gödel's incompleteness theorems impacted it.