Modal Logic
Solutions to Exercise Set 03

1. Prove that the following frame conditions are equivalent.

   (a) R is an equivalence relation (i.e. it is reflexive, symmetric, and transitive);

   (b) R is reflexive and euclidean;

   (c) R is serial, symmetric, and euclidean;

   (d) R is serial, symmetric, and transitive.

   Solution: We first show (a) $\Rightarrow$ (b). Let R be an equivalence relation. It suffices to prove that R is euclidean. Let u, v, w be such that uRv and uRw. By symmetry wRu so by transitivity, wRv.
   For (b) $\Rightarrow$ (c), reflexivity implies seriality: for any u, u itself witnesses the existence of an x such that uRx. For symmetry, let u, v be such that uRv. By reflexivity, uRu, so by euclideanness, vRu.
   To prove (c) $\Rightarrow$ (d), we only need to prove transitivity. Let u, v, w be such that uRv and vRw. By symmetry vRu, so by euclideanness, uRw.

   Finally we prove (d) $\Rightarrow$ (a). Let u be arbitrary. By seriality there exists an x such that uRx. By symmetry xRu, so by transitivity uRu, proving reflexivity.

2. The class of transitive and converse well-founded frames **GL** is defined by the Löb axiom $\Box(\Box A \to A) \to \Box A$. Prove that every instance of the Löb axiom is valid in every frame $F \in \mathbf{GL}$.

   Solution: Let (W, R) be a transitive, converse well-founded frame. We reason by induction on R^{-1}: let $u \in W$ be arbitrary and assume that for all $x \in W$ such that uRx, $x \Vdash \Box(\Box A \to A) \to \Box A$.
   Assume for a contradiction that $u \Vdash \Box(\Box A \to A)$ and that $u \nVdash \Box A$. Let y be a witness to the latter, i.e. $y \nVdash A$. By the former, $y \Vdash \Box A \to A$, so by modus tollens, $y \nVdash \Box A$.
   By modus tollens on the induction hypothesis, $y \nVdash \Box(\Box A \to A)$, so there exists $z \in W$ such that yRz, $z \Vdash \Box A$, and $z \nVdash A$. By transitivity, uRz, and hence $z \Vdash \Box A \to A$. So by modus ponens, $z \Vdash A$, which is the contradiction we were looking for.

   Consequently, $u \vdash \Box(\Box A \to A) \to \Box A$, so by induction, the Löb axiom holds for every world.

3. Prove that if every instance of the Löb axiom is valid in a frame F then F is transitive and converse well-founded.

Solution: Assume that (W, R) is not transitive, so there are u, v, w such that uRv and vRw but not uRw. Let $x \Vdash p$ iff $x \neq v$ or $x \neq w$. Clearly, $u \nVdash \Box p$. Now let y be any world such that uRy. $y \neq w$ since it's not the case that uRw. If $y = v$ then since $v \nVdash \Box p$, $y \Vdash (\Box p \to p)$. If $y \neq v$ then $y \Vdash p$ by the construction of $\Vdash$, and so $y \Vdash (\Box p \to p)$. Hence, since y was arbitrary, $u \Vdash \Box(\Box p \to p)$.

Now assume that (W, R) is not converse well-founded, so there is an infinite sequence $S = \{x_n | n \in \mathbb{N}\}$ such that for all $n \in \mathbb{N}$, $x_n R x_{n+1}$. Let p be a propositional variable and for all $z \in W$, let $z \Vdash p$ iff $z \notin S$. Let x_k be an arbitrary element of the infinite sequence, so there must be an x_{k+1}. By construction, $x_{k+1} \nVdash p$, so $x_k \nVdash \Box p$.

Let y be any world such that $x_k R y$. If $y \in S$ then $y \nVdash \Box p$ by the same argument as above, and hence $y \Vdash (\Box p \to p)$. If $y \notin S$ then $y \Vdash p$ and so $y \Vdash (\Box p \to p)$. So since y was arbitrary, $x_k \Vdash \Box(\Box p \to p)$, and so the Löb axiom fails at x_k.

4. (▲) A frame (W, R) is *converse well-quasi-ordered* if the converse accessibility relation R^{-1} is a *well-quasi-ordering*: it is well-founded and contains no infinite antichains. Find a modal scheme defining the class of converse well-quasi-ordered frames, or prove that no such scheme exists.