Modal Logic
Solutions to Exercise Set 02

1. Prove that the distribution axioms $\Box(A \to B) \to (\Box A \to \Box B)$ and $\Box(A \wedge B) \to (\Box A \wedge \Box B)$ are valid in every frame, i.e. that they are valid in the minimal modal logic **K**.

   Solution: Let (W, R) be a frame and $(W, R, \Vdash)$ a model based on that frame. Suppose $u \in W$ is a world such that $u \Vdash \Box(A \to B)$. Then for any v such that uRv, $v \Vdash (A \to B)$. Further assume that $u \Vdash \Box A$, so $v \Vdash A$, and hence $v \Vdash B$. Since v was arbitrary, $u \Vdash \Box A$, so discharging our second assumption, $u \Vdash (\Box A \to \Box B)$, and discharging the first, $u \Vdash \Box(A \to B) \to (\Box A \to \Box B)$.

   The argument for $\wedge$ is similar.

2. Construct a model based on a frame with a non-symmetric accessibility relation that makes the B axiom $(A \to \Box\Diamond A)$ valid.

   Solution: There are simpler models with this property, but never mind. Let $W = \mathbb{Z}$, $R = \{(x, y) | y \text{ is the successor of } x\}$, and let $x \Vdash A$ for all atomic propositions A and all worlds $x \in W$.

   $$\cdots \Vdash A \longrightarrow w_{-1} \Vdash A w_0 \Vdash A \longrightarrow w_1 \Vdash A \longrightarrow \cdots$$

3. The following table gives the relationship between modal axioms and frame conditions stated in terms of the accessibility relation.

Logic	**Modal axiom(s)**	**Frame condition(s)**
T	$(\Box A \to A)$ (T)	reflexive
B	T + $(A \to \Box\Diamond A)$ (B)	reflexive, symmetric
K4	$(\Box A \to \Box\Box A)$ (4)	transitive
K5	$(\Diamond A \to \Box\Diamond A)$ (5)	euclidean
S4	T + 4	reflexive, transitive
S5	T + 5	reflexive, transitive, symmetric

   Prove that for every frame F, the modal axiom(s) on the left of a given row are valid in F if and only if the frame condition(s) on the right of that row are true of F (you can omit the implications proved in the lecture).

Solution:

(a) $F \models \mathrm{T} \Leftrightarrow F$ is reflexive: done in lecture.

(b) $F \models \mathrm{T} + \mathrm{B} \Leftrightarrow F$ is reflexive and symmetric.

We show that $F \models \mathrm{B}$ iff F is symmetric. The full equivalence then follows from (a).

($\Leftarrow$) Assume $(W, R, \Vdash)$ is a model based on a symmetric frame. Let $u \in W$ be arbitrary and assume $u \Vdash A$. Let v be any world such that uRv. Then vRu by symmetry, so $v \Vdash \Diamond A$. But v was arbitrary, so $u \Vdash \Box\Diamond A$.

($\Rightarrow$) Let (W, R) be a non-symmetric frame. Then there exist $u, v \in W$ such that uRv but not vRu. Let p be a propositional variable and let $\Vdash = \{(u, p)\}$. Then $u \Vdash p$, but $v \nVdash \Diamond p$, since it's not the case that vRu. Hence because uRv, $u \nVdash \Box\Diamond p$.

(c) $F \models 4 \Leftrightarrow F$ is transitive.

($\Leftarrow$) Assume $(W, R, \Vdash)$ is a model based on a transitive frame. Let $u \in W$ be arbitrary and assume $u \Vdash \Box A$. Let v, w be arbitrary worlds such that uRv and vRw. By transitivity, uRw, so $w \Vdash A$. Because w was arbitrary, $v \Vdash \Box A$, and because v was arbitrary, $u \Vdash \Box\Box A$.

($\Rightarrow$) Let (W, R) be a non-transitive frame. Then there exist $u, v, w \in W$ such that uRv and vRw but not uRw. Let p be a propositional variable and let $\Vdash = \{(x, p) | x \in W \wedge uRx\}$. Then $u \Vdash \Box p$, but since uRv and $v \nVdash \Box p$, $u \nVdash \Box\Box p$.

(d) $F \models 5 \Leftrightarrow F$ is euclidean.

($\Leftarrow$) Assume $(W, R, \Vdash)$ is a model based on a euclidean frame. Let $u \in W$ be arbitrary and assume $u \Vdash \Diamond A$, so there exists $x \in W$ such that uRx and $x \Vdash A$. Let $y \in W$ be an arbitrary world such that uRy. By euclideanness, yRx, so $y \Vdash \Diamond A$, and since y was arbitrary, $u \Vdash \Box\Diamond A$.

($\Rightarrow$) Assume (W, R) is a non-euclidean frame, so there are worlds u, v, w such that uRv and uRw, but it is not the case that vRw. Let p be a propositional variable and let $\Vdash = \{(z, p) | z \in W \wedge \neg vRz\}$. $w \Vdash p$ by the construction of $\Vdash$, and because uRw, $u \Vdash \Diamond p$. But by the construction of $\Vdash$ there is no world y such that vRy and $y \Vdash p$, so $v \nVdash \Diamond p$. Consequently, $u \nVdash \Box\Diamond p$.

(e) $F \models \mathrm{T} + 4 \Leftrightarrow F$ is transitive and reflexive.

Combine (a) and (c) above.

(f) $F \models \mathrm{T} + 5 \Leftrightarrow F$ is reflexive, transitive, and symmetric.

We first prove a lemma allowing us to simplify the problem.

Lemma. A reflexive frame is symmetric and transitive iff it's euclidean.

Proof of the lemma: ($\Rightarrow$) Suppose (W, R) is symmetric and transitive, and let $u, v, w \in W$ be such that uRv and uRw. By symmetry, wRu, so by transitivity, wRv. ($\Leftarrow$) Suppose (W, R) is euclidean, and let $u, v, w \in W$ be such that uRv and vRw. By reflexivity, uRu, so by euclideanness, vRu, proving symmetry. But then by euclideanness again (since vRu and vRw), uRw, proving transitivity. QED

But then we can just apply the equivalences (a) and (d).

4. Prove that the D axiom, $(\Box A \to \Diamond A)$, is valid in all and only the *serial models*: $(W, R, \Vdash)$ such that for all $u \in W$, there exists $v \in W$ such that uRv.

(Note that this is a stronger equivalence than the equivalences in the table above between the frame validity of a modal axiom and a frame condition.)

Solution: ($\Rightarrow$) By contraposition: let (W, R) be a non-serial frame, so there exists $u \in W$ such that for no $x \in W$, uRx. Then $u \Vdash \Box A$ and $u \nVdash \Diamond A$, for any A.

($\Leftarrow$) Let (W, R) be a serial frame and let $u \in W$ be arbitrary. Assume $u \Vdash \Box A$. By seriality, there exists a world v such that uRv, and by assumption, $v \Vdash A$. But then $u \Vdash \Diamond A$ by the definition of $\Vdash$.

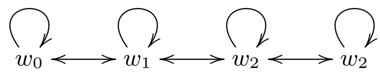
5. Figure 1 demonstrates the inclusion relationships between the modal logics in the table above (plus **D**). All of these implications are *strict*, i.e. if $\mathbf{L}_1 \Longrightarrow \mathbf{L}_2$ then there exists a frame $G \in \mathbf{L}_2$ such that $G \notin \mathbf{L}_1$.

   (a) Prove that the B axiom ($A \to \Box\Diamond A$) is valid in every **S5** frame.

   Solution: Any **S5** frame is symmetric, so it follows from the validity of the B axiom in symmetric frames proved in 3 above.

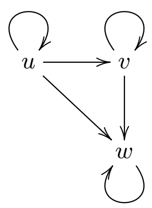
   (b) Construct a four-element frame which is reflexive and symmetric but not transitive.
   (This is sufficient to show that **S5** is strictly stronger than **B**.)

   Solution: The following diagram shows such a frame.



   (c) Prove that **S5** implies **S4**, but not conversely.

   Solution: **S5** frames are reflexive and transitive, so every **S5** frame is an **S4** frame. For the converse, it suffices that reflexivity and transitivity do not imply symmetry.



6. ($\triangle$) The modal logic **S4.3** is characterised by the frames that are reflexive, transitive, and *linear*: for all $u, v \in W$, either uRv or vRu.

   (a) Show that $\Box(\Box A \to \Box B) \vee \Box(\Box B \to \Box A)$ is valid in all **S4.3** frames.

   Solution: Let (W, R) be an **S4.3** frame and let $u \in W$ be arbitrary. Fix arbitrary worlds $x, y \in W$ such that uRx and uRy, and assume that $x \Vdash \Box A$ and $y \Vdash \Box B$. By linearity, either xRy or yRx.
   Suppose the former and let y' be an arbitrary world such that yRy'. By transitivity, xRy' and $y' \Vdash B$, so $x \Vdash \Box B$, and since x was arbitrary, $u \Vdash \Box(\Box A \to \Box B)$.
   Now suppose the latter. By the same argument, $x' \Vdash A$ for all x' such that xRx', so $y \Vdash (\Box B \to \Box A)$ and hence $u \Vdash \Box(\Box B \to \Box A)$.

$$
\begin{tikzcd}
& \mathbf{S5} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\mathbf{B} \arrow[d, Rightarrow] & & \mathbf{S4} \arrow[dll, Rightarrow] \arrow[d, Rightarrow] \\
\mathbf{T} \arrow[d, Rightarrow] & & \mathbf{K4} \arrow[ddl, Rightarrow] \\
\mathbf{D} \arrow[dr, Rightarrow] & & \\
& \mathbf{K} &
\end{tikzcd}
$$

Figure 1: The relationships between some important propositional modal logics. An arrow $\mathbf{L}_1 \Longrightarrow \mathbf{L}_2$ means that for every frame F, if $F \in \mathbf{L}_1$ then $F \in \mathbf{L}_2$, and that there exists a frame $G \in \mathbf{L}_2$ such that $G \notin \mathbf{L}_1$ (i.e. all implications are *strict*).

(b) Construct an **S4.3** frame in which $(A \to \Box\Diamond A)$ is invalid.

Solution: The frame given to solve exercise 5(c) works here too.

(c) Show that $(\Diamond\Box(A \to B) \to (\Diamond\Box A \to \Diamond\Box B))$ is valid in all **S4.3** frames, i.e. that $\Diamond\Box$ distributes across logical connectives in **S4.3**.

Solution: Let (W, R) be an **S4.3** frame and let $u \in W$ be arbitrary. Suppose that $u \Vdash \Diamond\Box(A \to B)$, and that $u \Vdash \Diamond\Box A$. Then there exists a world v such that uRv and $v \Vdash \Box(A \to B)$, and a world w such that uRw and $w \Vdash \Box A$. Let x be any world such that vRx and y any world such that wRy. $x \Vdash (A \to B)$ and $y \Vdash A$. By linearity, either vRw or wRv, so we reason by cases.

Suppose the former holds and let x be any world such that wRx, so $x \Vdash A$. vRx by transitivity, so $x \Vdash (A \to B)$. But then $x \Vdash B$, and since x was arbitrary, $u \Vdash \Diamond\Box B$.

Now suppose the latter holds. Let y be any world such that vRy, so $y \Vdash (A \to B)$. By transitivity, wRy, so $y \Vdash A$, and hence $y \Vdash B$. Since y was arbitrary, $u \Vdash \Diamond\Box B$.