

# Modal Logic

## Exercise Set 02

To be completed by Thursday 9 May

We will work through these exercises (and possibly some others as well) during the problem class. Exercises marked with a ( $\triangle$ ) are a little more challenging.

1. Prove that the distribution axioms  $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$  and  $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$  are valid in every frame, i.e. that they are valid in the minimal modal logic **K**.
2. Construct a model based on a frame with a non-symmetric accessibility relation that makes the B axiom  $(A \rightarrow \Box \Diamond A)$  valid.
3. The following table gives the relationship between modal axioms and frame conditions stated in terms of the accessibility relation.

Logic	Modal axiom(s)	Frame condition(s)
<b>T</b>	$(\Box A \rightarrow A)$ (T)	reflexive
<b>B</b>	$T + (A \rightarrow \Box \Diamond A)$ (B)	reflexive, symmetric
<b>K4</b>	$(\Box A \rightarrow \Box \Box A)$ (4)	transitive
<b>K5</b>	$(\Diamond A \rightarrow \Box \Diamond A)$ (5)	euclidean
<b>S4</b>	$T + 4$	reflexive, transitive
<b>S5</b>	$T + 5$	reflexive, transitive, symmetric

Prove that for every frame  $F$ , the modal axiom(s) on the left of a given row are valid in  $F$  if and only if the frame condition(s) on the right of that row are true of  $F$  (you can omit the implications proved in the lecture).

4. Prove that the D axiom,  $(\Box A \rightarrow \Diamond A)$ , is valid in all and only the *serial models*:  $(W, R, \Vdash)$  such that for all  $u \in W$ , there exists  $v \in W$  such that  $uRv$ .

(Note that this is a stronger equivalence than the equivalences in the table above between the frame validity of a modal axiom and a frame condition.)

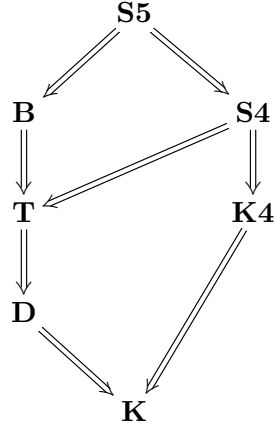


Figure 1: The relationships between some important propositional modal logics. An arrow  $\mathbf{L}_1 \implies \mathbf{L}_2$  means that for every frame  $F$ , if  $F \in \mathbf{L}_1$  then  $F \in \mathbf{L}_2$ , and that there exists a frame  $G \in \mathbf{L}_2$  such that  $G \notin \mathbf{L}_1$  (i.e. all implications are *strict*).

5. Figure 1 demonstrates the inclusion relationships between the modal logics in the table above (plus **D**). All of these implications are *strict*, i.e. if  $\mathbf{L}_1 \implies \mathbf{L}_2$  then there exists a frame  $G \in \mathbf{L}_2$  such that  $G \notin \mathbf{L}_1$ .
  - (a) Prove that the B axiom ( $A \rightarrow \Box \Diamond A$ ) is valid in every **S5** frame.
  - (b) Construct a four-element frame which is reflexive and symmetric but not transitive.  
(This is sufficient to show that **S5** is strictly stronger than **B**.)
  - (c) Prove that **S5** implies **S4**, but not conversely.
6. ( $\Delta$ ) The modal logic **S4.3** is characterised by the frames that are reflexive, transitive, and *linear*: for all  $u, v \in W$ , either  $uRv$  or  $vRu$ .
  - (a) Show that  $\Box(\Box A \rightarrow \Box B) \vee \Box(\Box B \rightarrow \Box A)$  is valid in all **S4.3** frames.
  - (b) Construct an **S4.3** frame in which  $(A \rightarrow \Box \Diamond A)$  is invalid.
  - (c) Show that  $(\Diamond \Box(A \rightarrow B) \rightarrow (\Diamond \Box A \rightarrow \Diamond \Box B))$  is valid in all **S4.3** frames, i.e. that  $\Diamond \Box$  distributes across logical connectives in **S4.3**.