Modal Logic
Exercise Set 02

To be completed by Thursday 9 May

We will work through these exercises (and possibly some others as well) during the problem class. Exercises marked with a ($\triangle$) are a little more challenging.

1. Prove that the distribution axioms $\Box(A \to B) \to (\Box A \to \Box B)$ and $\Box(A \wedge B) \to (\Box A \wedge \Box B)$ are valid in every frame, i.e. that they are valid in the minimal modal logic **K**.

2. Construct a model based on a frame with a non-symmetric accessibility relation that makes the B axiom $(A \to \Box\Diamond A)$ valid.

3. The following table gives the relationship between modal axioms and frame conditions stated in terms of the accessibility relation.

Logic	**Modal axiom(s)**		**Frame condition(s)**
T	$(\Box A \to A)$	(T)	reflexive
B	T + $(A \to \Box\Diamond A)$	(B)	reflexive, symmetric
K4	$(\Box A \to \Box\Box A)$	(4)	transitive
K5	$(\Diamond A \to \Box\Diamond A)$	(5)	euclidean
S4	T + 4		reflexive, transitive
S5	T + 5		reflexive, transitive, symmetric

   Prove that for every frame F, the modal axiom(s) on the left of a given row are valid in F if and only if the frame condition(s) on the right of that row are true of F (you can omit the implications proved in the lecture).

4. Prove that the D axiom, $(\Box A \to \Diamond A)$, is valid in all and only the *serial models*: $(W, R, \Vdash)$ such that for all $u \in W$, there exists $v \in W$ such that uRv.

   (Note that this is a stronger equivalence than the equivalences in the table above between the frame validity of a modal axiom and a frame condition.)

$$
\begin{tikzcd}
 & \mathbf{S5} \arrow[dl, Rightarrow] \arrow[dr, Rightarrow] & \\
\mathbf{B} \arrow[d, Rightarrow] & & \mathbf{S4} \arrow[dll, Rightarrow] \arrow[d, Rightarrow] \\
\mathbf{T} \arrow[d, Rightarrow] & & \mathbf{K4} \arrow[ddl, Rightarrow] \\
\mathbf{D} \arrow[dr, Rightarrow] & & \\
 & \mathbf{K} &
\end{tikzcd}
$$

Figure 1: The relationships between some important propositional modal logics. An arrow $\mathbf{L}_1 \Longrightarrow \mathbf{L}_2$ means that for every frame F, if $F \in \mathbf{L}_1$ then $F \in \mathbf{L}_2$, and that there exists a frame $G \in \mathbf{L}_2$ such that $G \notin \mathbf{L}_1$ (i.e. all implications are *strict*).

5. Figure 1 demonstrates the inclusion relationships between the modal logics in the table above (plus **D**). All of these implications are *strict*, i.e. if $\mathbf{L}_1 \Longrightarrow \mathbf{L}_2$ then there exists a frame $G \in \mathbf{L}_2$ such that $G \notin \mathbf{L}_1$.

   (a) Prove that the B axiom $(A \to \Box\Diamond A)$ is valid in every **S5** frame.

   (b) Construct a four-element frame which is reflexive and symmetric but not transitive.

   (This is sufficient to show that **S5** is strictly stronger than **B**.)

   (c) Prove that **S5** implies **S4**, but not conversely.

6. ($\triangle$) The modal logic **S4.3** is characterised by the frames that are reflexive, transitive, and *linear*: for all $u, v \in W$, either uRv or vRu.

   (a) Show that $\Box(\Box A \to \Box B) \lor \Box(\Box B \to \Box A)$ is valid in all **S4.3** frames.

   (b) Construct an **S4.3** frame in which $(A \to \Box\Diamond A)$ is invalid.

   (c) Show that $(\Diamond\Box(A \to B) \to (\Diamond\Box A \to \Diamond\Box B)$ is valid in all **S4.3** frames, i.e. that $\Diamond\Box$ distributes across logical connectives in **S4.3**.