Modal Logic
Solutions to Exercise Set 06

Branch extension lemma for S4. Suppose that T is a tableau that is satisfiable in a model based on a frame that is reflexive and transitive. Then any tableau T' obtained from T by applying one of the **S4** branch extension rules (i.e. those of **K** plus the special necessity rules (T) and (4)) is also satisfiable by a model based on a frame that is reflexive and transitive.

1. Prove the branch extension lemma for **S4**.

   Solution: Suppose that T is an **S4** tableau that is satisfiable by a model $M = (W, R, \Vdash)$ based on a reflexive and transitive frame (W, R). Let B be a branch through T, and f the assignment that (together with M) satisfies the formulas that appear on B. Let $\sigma\ \Box\varphi$ be any prefixed formula of this form that appears on B.
   For the (4) rules, extend B by $\sigma.n\ \Box\varphi$, where $\sigma.n$ was already on B. By assumption, $f(\sigma)Rf(\sigma.n)$ and $f(\sigma) \Vdash \Box\varphi$. Let u be any world such that $f(\sigma.n)Ru$. By transitivity, $f(\sigma)Ru$, so $u \Vdash \varphi$. But since u was an arbitrary world such that $f(\sigma.n)Ru$, $f(\sigma.n) \Vdash \Box\varphi$. The other case of the (4) rule is similar.

   For the (T) rules, extend B by $\sigma\ \varphi$ (clearly, σ was already on B). By assumption, $f(\sigma) \Vdash \Box\varphi$. By reflexivity, $f(\sigma)Rf(\sigma)$, so $f(\sigma) \Vdash \varphi$. Again, the other case of the (T) rule is similar.

2. Briefly explain (1 paragraph) why proving this modified version of the branch extension lemma for **K** is sufficient to prove the soundness of the **S4** tableau system with respect to the class of reflexive and transitive frames.

   Solution: The other parts of the soundness proof are the lemma that every closed tree is unsatisfiable, and the final proof by induction. The former does not depend on any particular properties of the tableau system, just the fact that on any branch of a closed tableau we have both $\sigma\ \varphi$ and $\sigma\ \neg\varphi$ for some prefix σ and formula φ. The latter's only use of particular properties of the tableau system is via the application of the branch extension lemma: the rest of the proof is just an induction, which depends only on quite general properties of tableaus.