## Modal Logic Solutions to Exercise Set 09

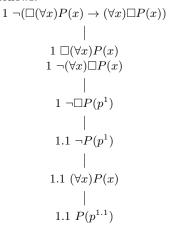
1. Give a constant domain tableau proof of a Converse Barcan formula.

Solution: We construct the constant domain tableau proof as follows.

$$\begin{array}{c|c} 1 \neg (\Box(\forall x)\varphi(x) \rightarrow (\forall x)\Box\varphi(x)) \\ & | \\ 1 \Box(\forall x)\varphi(x) \\ 1 \neg (\forall x)\Box\varphi(x) \\ & | \\ 1 \neg \Box\varphi(p) \\ & | \\ 1.1 \neg\varphi(p) \\ & | \\ 1.1 (\forall x)\varphi(x) \\ & | \\ 1.1 \ \varphi(p) \\ & \times \end{array}$$

2. Use a failed variable domain proof of a Converse Barcan formula to construct a countermodel. Prove that the Converse Barcan formula in question is not valid in the model.

**Solution:** A failed, saturated variable domain tableau proof of  $\Box(\forall x)P(x) \to (\forall x)\Box P(x)$  is as



This allows us to construct a variable domain model M = (W, R, D, I) where  $W = \{1, 1.1\}$ ,  $R = \{\langle 1, 1.1 \rangle\}, \ D(1) = \{p^1\}, \ D(1.1) = \{p^{1.1}\}, \ I(P, 1) = \emptyset, \ I(P, 1.1) = \{p^{1.1}\}.$ 

We now verify that  $M \models \Box(\forall x)P(x)$  but that  $M \not\models (\forall x)\Box P(x)$ .

Let v be the valuation such that  $v(p^{\sigma}) = p^{\sigma}$  for any parameter  $p^{\sigma}$ , and let  $\theta$  be the assignment

such that  $\theta(\sigma) = \sigma$  for any prefix  $\sigma$ . Firstly, note that  $M, 1.1 \Vdash_v P(p^{1.1})$ . Let w be any x-variant of v at 1.1. Then  $w(x) = p^{1.1}$  since  $p^{1.1}$  is the sole element of D(1.1). So  $M, 1.1 \Vdash_v (\forall x) P(x)$ . Finally,  $M, 1 \Vdash_v \Box (\forall x) P(x)$  since 1.1 is the unique world such that 1R1.1.

Now suppose for a contradiction that  $M, 1 \Vdash_v (\forall x) \Box P(x)$ . It follows that for any x-variant w of v at 1,  $M, 1 \Vdash_w \Box P(x)$ .  $M, 1.1 \Vdash_w P(x)$  since 1.1 is accessible from 1, so  $w(x) \in I(P, 1.1)$ .  $p^{1.1}$  is the unique element of I(P, 1.1), so  $w(x) = p^{1.1}$ . However, w is an x-variant of v at 1, so by definition,  $w(x) \in D(1)$ . So  $w(x) = p^1$ .

3. Prove the universal case of the branch extension lemma for the variable domain tableau system.

## References

M. Fitting and R. L. Mendelsohn. First-Order Modal Logic. Number 277 in Synthese Library. Springer, 1998.