Modal Logic
Exercise Set 01

To be completed by Thursday 2 May

We will work through these exercises (and possibly some others as well) during the problem class. Exercises marked with a ($\triangle$) are a little more challenging.

1. Show that $(\Box(A \to \Diamond B) \wedge \Diamond(\Box C \vee D))$ is a well-formed modal formula.

2. Use the definition of a modal model to verify the correctness of the possible worlds semantics for the interdefinability of $\Box$ and $\Diamond$: show that for any modal model $M = (W, R, \Vdash)$ and any world $w \in W$,

   (a) $w \Vdash \Box A$ if, and only if, $w \Vdash \neg\Diamond\neg A$;

   (b) $w \Vdash \Diamond B$ if, and only if, $w \Vdash \neg\Box\neg B$.

3. Let $M = (W, R, \Vdash)$ be a model in which every world is accessible to every other, i.e. for all $w, u \in W$, wRu. Show that for all $w \in W$, $w \Vdash (\Diamond P \to \Box\Diamond P)$.

4. Let $M = (W, R, \Vdash)$ be a model and $w \in W$ a world with no other worlds accessible to it, i.e. for no $u \in W$, wRu. Show that at w every formula is necessary, but none are possible, i.e. $w \Vdash \Box A$ for all A, but $w \nVdash \Diamond A$ for any A.

5. ($\triangle$) A formula A is *valid* in a model $(W, R, \Vdash)$ if $u \Vdash A$ for all $u \in W$.

   Construct a (finite) model $M = (W, R, \Vdash)$ with the following properties.

   (a) There are worlds $u, v \in W$ such that $u \neq v$ and uRv;

   (b) There are worlds $w, x \in W$ such that $w \Vdash P$ and $x \nVdash P$;

   (c) $(\Diamond P \to \Box P)$ is valid in M;

   (d) $(\Diamond\neg P \to \Box\neg P)$ is valid in M;

   (e) $(\Diamond P \to \Box P)$ is valid in M^{-1}; and

   (f) $(\Diamond\neg P \to \Box\neg P)$ is valid in M^{-1},

   where $M^{-1} = (X, R^{-1}, \Vdash)$ is the model with the same worlds and forcing relation as M, but with the converse accessibility relation $R^{-1} = \{(y, x) | xRy\}$.