Modal Logic
Solutions to Exercise Set 04

1. Use the tableau proof system for **K** to prove the following.

   (a) $(\Box P \wedge \Diamond Q) \rightarrow \Diamond(P \wedge Q)$

   Solution:

   $1\ \neg((\Box P \wedge \Diamond Q) \rightarrow \Diamond(P \wedge Q))$

   $1\ (\Box P \wedge \Diamond Q)$
   $1\ \neg\Diamond(P \wedge Q)$

   $1\ \Box P$
   $1\ \Diamond Q$

   $1.1\ Q$ [Poss.]

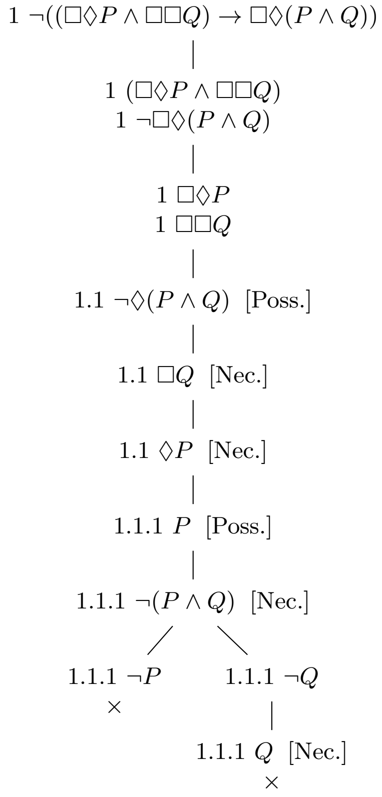
   $1.1\ \neg(Q \wedge P)$ [Nec.]

   $1.1\ \neg Q$ $\qquad$ $1.1\ \neg P$
   $\times$

   $1.1\ P$ [Nec.]
   $\times$

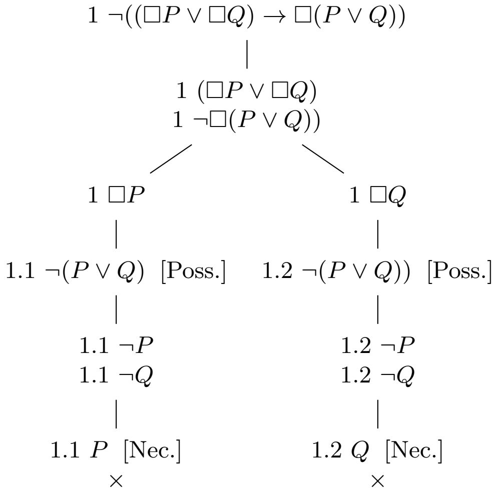
   (b) $(\Box\Diamond P \wedge \Box\Box Q) \rightarrow \Box\Diamond(P \wedge Q)$

Solution:



(c) $(\Box P \vee \Box Q) \to \Box(P \vee Q)$

Solution:



2. ($\triangle$) Show that for any formula A, there is a tableau proof of $\Box A$ if and only if there is a tableau proof of A.

Solution: Any tableau proof of $\Box A$ can be transformed into one of A simply by removing the root node. Conversely, any tableau proof of A can be transformed into one of $\Box A$ by adding a new root above the existing root node, with $1\ \neg\Box A$ at it, and replacing every existing prefix σ with $1.\sigma$.

3. Use the tableau proof system for **T** to prove $\Diamond(P \to \Box P)$.

Solution:

$$\begin{array}{c}
1\ \neg\Diamond(P \to \Box P) \\
| \\
1\ \neg(P \to \Box P)\ \ [\mathrm{T}] \\
| \\
1\ P \\
1\ \neg\Box P \\
| \\
1.1\ \neg P\ \ [\mathrm{Poss.}] \\
| \\
1.1\ \neg(P \to \Box P)\ \ [\mathrm{Nec.}] \\
| \\
1.1\ P \\
1.1\ \neg\Box P \\
\times
\end{array}$$

4. Use the tableau proof system for **K4** to prove $(\Box P \wedge \Box Q) \to \Box(\Box P \wedge \Box Q)$.

Solution:

$$\begin{array}{c}
1\ \neg((\Box P \wedge \Box Q) \to \Box(\Box P \wedge \Box Q)) \\
| \\
1\ (\Box P \wedge \Box Q) \\
1\ \neg\Box(\Box P \wedge \Box Q) \\
| \\
1\ \Box P \\
1\ \Box Q \\
| \\
1.1\ \neg(\Box P \wedge \Box Q)\ \ [\mathrm{Poss.}] \\
\diagup \qquad \diagdown \\
\begin{array}{cc}
1.1\ \neg\Box P & 1.1\ \neg\Box Q \\
| & | \\
1.1\ \Box P\ \ [4] & 1.1\ \Box Q\ \ [4] \\
\times & \times
\end{array}
\end{array}$$

5. Use the tableau proof system for **S4** to prove $(\Box\Diamond P \wedge \Box\Diamond Q) \to \Box\Diamond(\Box\Diamond P \wedge \Box\Diamond Q)$.

Solution:

$$\begin{array}{c}
1\ \neg((\Box\Diamond P \wedge \Box\Diamond Q) \to \Box\Diamond(\Box\Diamond P \wedge \Box\Diamond Q)) \\
| \\
1\ (\Box\Diamond P \wedge \Box\Diamond Q) \\
1\ \neg\Box\Diamond(\Box\Diamond P \wedge \Box\Diamond Q) \\
| \\
1\ \Box\Diamond P \\
1\ \Box\Diamond Q \\
| \\
1.1\ \neg\Diamond(\Box\Diamond P \wedge \Box\Diamond Q)\ \ [\mathrm{Poss.}] \\
| \\
1.1\ \neg(\Box\Diamond P \wedge \Box\Diamond Q)\ \ [\mathrm{T}] \\
\diagup \qquad \diagdown \\
\begin{array}{cc}
1.1\ \neg\Box\Diamond P & 1.1\ \neg\Box\Diamond Q \\
| & | \\
1.1\ \Box\Diamond P\ \ [4] & 1.1\ \Box\Diamond Q\ \ [4] \\
\times & \times
\end{array}
\end{array}$$

6. Use the tableau proof system for **S5** to prove $\Box P \vee \Box(\Box P \rightarrow Q)$.

Solution:

$1\ \neg(\Box P \vee \Box(\Box P \rightarrow Q))$

$|$

$1\ \neg\Box P$

$1\ \neg\Box(\Box P \rightarrow Q)$

$|$

$1.1\ \neg(\Box P \rightarrow Q)$ [Poss.]

$|$

$1.1\ \Box P$

$1.1\ \neg Q$

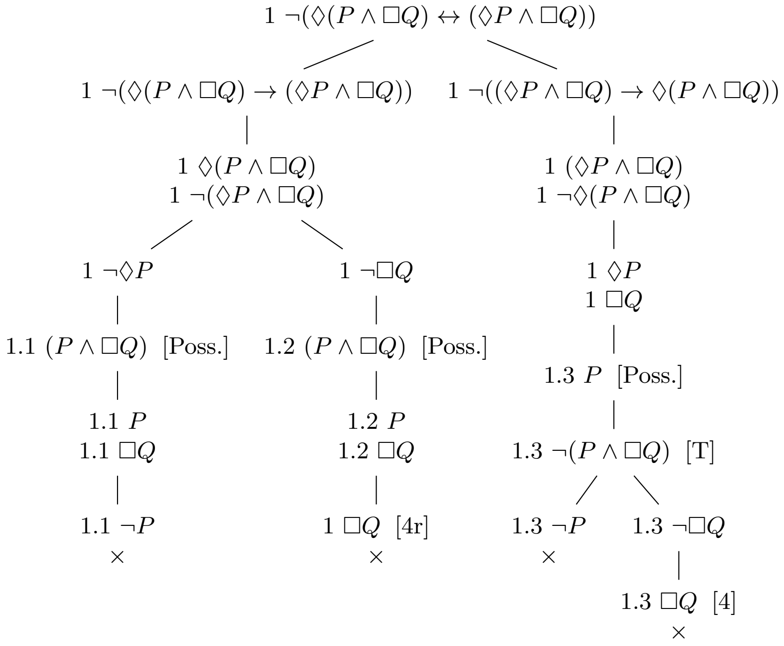
$|$

$1\ \Box P$ [4r]

×

7. Use the tableau proof system for **S5** to prove $\Diamond(P \wedge \Box Q) \leftrightarrow (\Diamond P \wedge \Box Q)$.

Solution:



8. (△) We define a new tableau system as follows. Take prefixes to just be natural numbers n, k. The rules for the truth-functional logical connectives are as usual. The only modal rules are the following. If the prefix k is new to the branch, the possibility rules

$$\frac{n\ \Diamond A}{k\ X} \quad \text{and} \quad \frac{n\ \neg\Box A}{k\ \neg A}$$

may be applied. If the prefix k already appears on the branch, the necessity rules

$$\frac{n\ \Box A}{k\ X} \quad \text{and} \quad \frac{n\ \neg\Diamond A}{k\ \neg A}$$

may be applied.

Prove that the theorems of this tableau system are exactly those of the tableau system **S5** given by the standard modal rules together with the rules T, 4, and 4r.