Modal Logic
Solutions to Exercise Set 10

1. Give variable domain tableau proofs of the following validities:

   (a) $\Diamond(x = y) \to (x = y)$.

   Solution: Construct a tableau as follows:

   $$\begin{array}{c}
   1\ \neg(\Diamond(x = y)) \to (x = y) \\
   | \\
   1\ \Diamond(x = y) \\
   1\ \neg(x = y) \\
   | \\
   1.1\ x = y \\
   | \\
   1\ \neg(x = x) \\
   \times
   \end{array}$$

   (b) $\neg(x = y) \to \Box\neg(x = y)$.

   Solution: Construct a tableau as follows:

   $$\begin{array}{c}
   1\ \neg(\neg(x = y) \to \Box\neg(x = y)) \\
   | \\
   1\ \neg(x = y) \\
   1\ \neg\Box\neg(x = y) \\
   | \\
   1.1\ \neg\neg(x = y) \\
   | \\
   1.1\ (x = y) \\
   | \\
   1\ \neg(x = x) \\
   \times
   \end{array}$$

   (c) $((x = y) \wedge (y = z)) \to (x = z)$.

Solution: Construct a tableau as follows:

$$
\begin{array}{c}
1\ \neg(((x=y)\wedge(y=z))\rightarrow(x=z))\\
|\\
1\ ((x=y)\wedge(y=z))\\
1\ \neg(x=z)\\
|\\
1\ (x=y)\\
1\ (y=z)\\
|\\
1\ (x=z)\\
|\\
1\ \neg(z=z)\\
\times
\end{array}
$$

(d) $(x=y)\rightarrow(\varphi(x)\leftrightarrow\varphi(y))$, where $\varphi(x)$ is a formula in which y does not occur, and $\varphi(y)$ is the result of substituting occurrences of y for free occurrences of x in $\varphi(x)$.

Solution: Construct a tableau as follows:

$$
\begin{array}{c}
1\ \neg((x=y)\rightarrow(\varphi(x)\leftrightarrow\varphi(y)))\\
|\\
1\ (x=y)\\
1\ \neg(\varphi(x)\leftrightarrow\varphi(y))\\
\diagup \qquad \diagdown\\
\begin{array}{cc}
1\ \varphi(x) & 1\ \neg\varphi(x)\\
1\ \neg\varphi(y) & 1\ \varphi(y)\\
| & |\\
1\ \varphi(y) & 1\ \varphi(x)\\
\times & \times
\end{array}
\end{array}
$$

(e) $((\forall x)\varphi(x)\wedge\mathsf{E}(z))\rightarrow\varphi(z)$.

Solution: Construct a tableau as follows:

$$
\begin{array}{c}
1\ \neg(((\forall x)\varphi(x)\wedge\mathsf{E}(z))\rightarrow\varphi(z))\\
|\\
1\ ((\forall x)\varphi(x)\wedge\mathsf{E}(z))\\
1\ \neg\varphi(z)\\
|\\
1\ (\forall x)\varphi(x)\\
1\ \mathsf{E}(z)\\
|\\
1\ p^1=z\\
|\\
1\ \varphi(p^1)\\
|\\
1\ \neg\varphi(p^1)\\
\times
\end{array}
$$

(f) $((\forall x)\Diamond\mathsf{E}(x)\wedge(\exists x)\Box P(x))\rightarrow\Diamond(\exists x)P(x)$.

Solution: Construct a tableau as follows:

$$
\begin{array}{c}
1\ \neg(((\forall x)\Diamond \mathsf{E}(x) \wedge (\exists x)\Box P(x)) \rightarrow \Diamond(\exists x)P(x)) \\
| \\
1\ ((\forall x)\Diamond \mathsf{E}(x) \wedge (\exists x)\Box P(x)) \\
1\ \neg\Diamond(\exists x)P(x)) \\
| \\
1\ (\forall x)\Diamond \mathsf{E}(x) \\
1\ (\exists x)\Box P(x) \\
| \\
1\ \Box P(q^1) \\
| \\
1\ \Diamond \mathsf{E}(q^1) \\
| \\
1.1\ \mathsf{E}(q^1) \\
| \\
1.1\ P(q^1) \\
| \\
1.1\ p^{1.1} = q^1 \\
| \\
1.1\ \neg(\exists x)P(x) \\
| \\
1.1\ \neg P(p^{1.1}) \\
| \\
1.1\ P(p^{1.1}) \\
\times
\end{array}
$$

2. Derive the *Parameter Nonexistence Rule*: A branch of a variable domain tableau containing $\sigma\ \neg\mathsf{E}(p^\sigma)$ closes.

   Solution: Suppose we have a branch containing $\sigma\ \neg\mathsf{E}(p^\sigma)$. We can then carry out the following derivation.

$$
\begin{array}{c}
\sigma\ \neg(\exists y)(y = p^\sigma) \\
| \\
\sigma\ \neg(p^\sigma = p^\sigma) \\
| \\
\sigma\ (p^\sigma = p^\sigma) \\
\times
\end{array}
$$

3. Let $F = (W, R, D)$ be a variable domain frame. Prove that the following conditions are equivalent:

   (a) F is anti-monotonic;

   (b) $\Diamond\mathsf{E}(x) \rightarrow \mathsf{E}(x)$ is valid in every normal model based on F.

Solution: ($\Rightarrow$) Let $F = (W, R, D)$ be a variable domain frame which is anti-monotonic, let $M = (W, R, D, I)$ be a normal model based on F, and let $u \in W$. Suppose v is a valuation in M, and suppose that $M, u \Vdash_v \Diamond\mathsf{E}(x)$. Then there is a world $z \in W$ such that uRz and $z \Vdash_v \mathsf{E}(x)$, so $v(x) \in D(z)$. By anti-monotonicity, $v(x) \in D(u)$. But then $u \Vdash_v (\exists y)(y = x)$ as required.

($\Leftarrow$) Let $F = (W, R, D)$ be a variable domain frame which is not anti-monotonic. We will construct a normal model based on F in which the formula $\Diamond\mathsf{E}(x) \to \mathsf{E}(x)$ is invalid. Let M be the model based on F in which $I(=, u)$ for all worlds u is the equality relation on $D(u)$. Let u, z be worlds such that uRz and there exists $e \in D(z)$ such that $e \notin D(u)$. Let v be a valuation such that $v(x) = v(y) = e$. Then $z \Vdash_v (y = x)$, and moreover, $z \Vdash_v (\exists y)(y = x)$ since $v(x) = e \in D(z)$. So $u \Vdash_v \Diamond\mathsf{E}(x)$. Now suppose for a contradiction that $u \Vdash_v \mathsf{E}(x)$. Then by definition, there exists a y-variant v' of v at u such that $u \Vdash_{v'} (y = x)$. By normality, $v'(y) = v'(x)$. Since v' is an x-variant of v at u, $v'(x) = v(x) = e$ and $e \in D(u)$, a contradiction.