Modal Logic
Solutions to Exercise Set 01

1. *Show that* $(\Box(A \to \Diamond B) \wedge \Diamond(\Box C \vee D))$ *is a well-formed modal formula.*

   To show this it's sufficient to provide a formation sequence by which the formula can be constructed, satisfying the definition of a well-formed formula at each stage.

   (a) A
   (b) B
   (c) C
   (d) D
   (e) $\Diamond B$
   (f) $\Box C$
   (g) $(A \to \Diamond B)$
   (h) $\Box(A \to \Diamond B)$
   (i) $(\Box C \vee D)$
   (j) $\Diamond(\Box C \vee D)$
   (k) $(\Box(A \to \Diamond B) \wedge \Diamond(\Box C \vee D))$

2. *Use the definition of a modal model to verify the correctness of the possible worlds semantics for the interdefinability of* $\Box$ *and* $\Diamond$*: show that for any modal model* $M = (W, R, \Vdash)$ *and any world* $w \in W$*,*

   (a) $w \Vdash \Box A$ *if, and only if,* $w \Vdash \neg\Diamond\neg A$*;*
   (b) $w \Vdash \Diamond B$ *if, and only if,* $w \Vdash \neg\Box\neg B$*.*

   Formally, the proof is by induction on formula complexity. We prove the base case for an atomic proposition, as this is sufficient to give the idea. Let A be any atomic propositional formula. Then

$$
\begin{aligned}
u \Vdash \Box A &\Leftrightarrow \forall w((w \in W \wedge uRw) \Rightarrow w \Vdash A) && \text{[def. of } \Vdash] \\
&\Leftrightarrow \neg\exists w \neg((w \in W \wedge uRw) \Rightarrow w \Vdash A) && \text{[interdefinability of } \forall \text{ and } \exists] \\
&\Leftrightarrow \neg\exists w((w \in W \wedge uRw) \wedge w \nVdash A) && \text{[logic]} \\
&\Leftrightarrow \neg\exists w((w \in W \wedge uRw) \wedge w \Vdash \neg A) && \text{[def. of } \Vdash] \\
&\Leftrightarrow u \nVdash \Diamond\neg A && \text{[def. of } \Vdash] \\
&\Leftrightarrow u \Vdash \neg\Diamond\neg A. && \text{[def. of } \Vdash]
\end{aligned}
$$

3. *Let $M = (W, R, \Vdash)$ be a model in which every world is accessible to every other, i.e. for all $w, u \in W$, wRu. Show that for all $w \in W$, $w \Vdash (\Diamond P \to \Box\Diamond P)$.*

   Let $u \in W$ be an arbitrary world, and assume that $u \Vdash \Diamond P$. Then by the definition of $\Vdash$, there exists $v \in W$ such that $v \Vdash P$. Let $w \in W$ be any world. Then uRw and wRv by our premise concerning the accessibility relation, and hence $w \Vdash \Diamond P$, and (because w was arbitrary) $u \Vdash \Box\Diamond P$.

4. *Let $M = (W, R, \Vdash)$ be a model and $w \in W$ a world with no other worlds accessible to it, i.e. for no $u \in W$, wRu. Show that at w every formula is necessary, but none are possible, i.e. $w \Vdash \Box A$ for all A, but $w \nVdash \Diamond A$ for any A.*

   Let A be any formula. There is no $u \in W$ such that wRu, so by the definition of $\Vdash$, $w \nVdash \Diamond A$. But the clause of the definition of $\Vdash$ for $\Box$ is a conditional whose antecedent is always false, so the conditional is always true, so $w \Vdash \Box A$.

5. *($\triangle$) A formula A is* valid *in a model $(W, R, \Vdash)$ if $u \Vdash A$ for all $u \in W$.*

   Construct a (finite) model $M = (W, R, \Vdash)$ with the following properties.

   (a) There are worlds $u, v \in W$ such that $u \neq v$ and uRv;

   (b) There are worlds $w, x \in W$ such that $w \Vdash P$ and $x \nVdash P$;

   (c) $(\Diamond P \to \Box P)$ is valid in M;

   (d) $(\Diamond\neg P \to \Box\neg P)$ is valid in M;

   (e) $(\Diamond P \to \Box P)$ is valid in M^{-1}; and

   (f) $(\Diamond\neg P \to \Box\neg P)$ is valid in M^{-1},

   where $M^{-1} = (X, R^{-1}, \Vdash)$ is the model with the same worlds and forcing relation as M, but with the converse accessibility relation $R^{-1} = \{(y, x) | xRy\}$.

   There are many simple models with this property, but here's one.

$$u \Vdash P \xrightarrow{R} v \nVdash P$$

   (a) and (b) hold by the construction of the model. For (c), the antecedent is false at every world so the conditional is forced at every world. For (d), the antecedent is only forced at u, but v is the only world accessible to u, so the consequent is forced as well. The satisfaction of the final clauses (e) and (f) follows from symmetrical reasoning to that for (c) and (d).