# Modal Logic
# Solutions to Exercise Set 07

A constant domain **K** model is a constant domain first-order modal model in which the accessibility relation has no particular restrictions.

Which of the following sentences are valid in all constant domain **K** models, and which are not valid? In the former case, give a proof that the sentence is valid in all constant domain **K** models; in the latter, either construct an explicit countermodel or derive a contradiction from the assumption that it is valid in all such models.

1. $((\exists x)\Diamond P(x) \wedge \Box(\forall x)(P(x) \to Q(x))) \to (\exists x)\Diamond Q(x)$.

   **Solution:** Let $M$ be a **K** model and let $u$ be a world of $M$ and $v$ a valuation in $M$. Suppose $u \Vdash_v ((\exists x)\Diamond P(x) \wedge \Box(\forall x)(P(x) \to Q(x)))$. There is an $x$-variant $v'$ of $v$ and $z$ such that $uRz$ where $z \Vdash_{v'} P(x)$. $z \Vdash_{v'} P(x) \to Q(x)$ by the definition of $\Vdash$ and our initial assumption, so $z \Vdash_{v'} Q(x)$. $u \Vdash_{v'} \Diamond Q(x)$, and since $v'$ is an $x$-variant of $v$, $u \Vdash_v (\exists x)\Diamond Q(x)$.

2. $(\forall x)\Box P(x) \to \Box(\forall x)P(x)$.

   **Solution:** Let $M$ be a **K** model and let $u$ be a world of $M$ and $v$ a valuation in $M$. Suppose $u \Vdash_v (\forall x)\Box P(x)$, so $z \Vdash_{v'} P(x)$ for any $x$-variant $v'$ of $v$ and any $z$ such that $uRz$. But then $z \Vdash_v (\forall x)P(x)$ by the definition of $\Vdash$, and since $z$ was arbitrary, $u \Vdash_v \Box(\forall x)P(x)$.

3. $\Box(\forall x)P(x) \to (\forall x)\Box P(x)$.

   **Solution:** Let $M$ be a **K** model and let $u$ be a world of $M$ and $v$ a valuation in $M$. Suppose $u \Vdash_v \Box(\forall x)P(x)$. Let $v'$ be an $x$-variant of $v$ and let $z$ be such that $uRz$. $z \Vdash_v (\forall x)P(x)$, so by the definition of $\Vdash$, $z \Vdash_{v'} P(x)$. By the definition of $\Vdash$ again, $u \Vdash_{v'} \Box P(x)$ and thus $u \Vdash_v (\forall x)\Box P(x)$, since $v'$ and $z$ were arbitrary.

4. $(\exists x)\Box P(x) \to \Box(\exists x)P(x)$.

   **Solution:** Let $M$ be a **K** model and let $u$ be a world of $M$ and $v$ a valuation in $M$. Suppose $u \Vdash_v (\exists x)\Box P(x)$. By the definition of $\Vdash$, let $v'$ be an $x$-variant of $v$ such that $u \Vdash_{v'} \Box P(x)$. Let $z$ be an arbitrary world such that $uRz$, so by the definition of $\Vdash$, $z \Vdash_{v'} P(x)$. Since $v'$ is an $x$-variant of $v$, it follows that $z \Vdash_v (\exists x)P(x)$. But since $z$ was arbitrary, $u \Vdash_v \Box(\exists x)P(x)$.

5. $\Box(\exists x)P(x) \to (\exists x)\Box P(x)$.

   **Solution:** We fix the following **K** model $M$: let $W = \{a, b, c\}$ and $R = \{\langle a, b\rangle, \langle a, c\rangle\}$, $D = \{1, 2\}$, and $I(P, a) = \emptyset, I(P, b) = \{1\}, I(P, c) = \{2\}$. Let $v$ be any valuation, and let $v_b$ take the values of $v$ except that $v_b(x) = 1$. Then $b \Vdash_{v_b} P(x)$, and moreover, $b \Vdash_v (\exists x)P(x)$ since $v_b$ is an $x$-variant of $v$ by construction. By a similar argument, $c \Vdash_v (\exists x)P(x)$, and since these are the only worlds accessible from $a$, $a \Vdash_v \Box(\exists x)P(x)$.

   Now, suppose that $a \Vdash_v (\exists x)\Box P(x)$. Then there would be an $x$-variant $v'$ of $v$ such that $b \Vdash_{v'} P(x)$ and $c \Vdash_{v'} P(x)$. By the definition of $\Vdash$, $v'(x) \in I(P, b)$ and $v'(x) \in I(P, c)$. But $I(P, b) \cap I(P, c) = \emptyset$ by construction. Contradiction.

6. $(\exists x)\Diamond(\Box P(x) \to (\forall x)\Box P(x))$.

   **Solution:** This sentence is false in any one-world model of the form $M = \langle W, R, D, I\rangle$ where $W = \{w\}$ and $R = \emptyset$.

7. $(\exists x)\Diamond(P(x) \to (\forall x)\Box P(x))$.

   **Solution:** This sentence is false in any one-world model of the form $M = \langle W, R, D, I\rangle$ where $W = \{w\}$ and $R = \emptyset$.

8. $(\exists x)(\forall y)\Box R(x, y) \to (\forall y)\Box(\exists x)R(x, y)$.

   **Solution:** Let $M$ be a **K** model and let $u$ be a world of $M$ and $v$ a valuation in $M$. Suppose $u \Vdash_v (\exists x)(\forall y)\Box R(x, y)$. Fixing an appropriate $x$-variant $v_1$, we have that for any $y$-variant $v_2$ of $v_1$ and for any world $z$ such that $uRz$, $z \Vdash_{v_2} R(x, y)$, and hence $z \Vdash_{v_2} (\exists x)R(x, y)$. Since $z$ was arbitrary, we have that $u \Vdash_{v_2} \Box(\exists x)R(x, y)$, and since $v_2$ was an arbitrary $y$-variant of $v_1$, $u \Vdash_{v_1} \forall y\Box(\exists x)R(x, y)$. Finally since $\forall y\Box(\exists x)R(x, y)$ is a sentence, we have that $u \Vdash_v (\forall y)\Box(\exists x)R(x, y)$, i.e. $u \Vdash \forall y\Box(\exists x)R(x, y)$.