Modal Logic
Solutions to Exercise Set 08

1. Instances of the Barcan formula are either of the form

(B1) $$(\forall x)\Box\varphi \to \Box(\forall x)\varphi$$

or of the form

(B2) $$\Diamond(\exists x)\varphi \to (\exists x)\Diamond\varphi.$$

Show that every formula of form B1 is equivalent to one of form B2, and conversely.

Solution: Suppose we have a formula of form CB1. By contraposition it is equivalent to $\neg\Box(\forall x)\varphi \to \neg(\forall x)\Box\varphi$. By the equivalence between $\Box\theta$ and $\neg\Diamond\neg\theta$ we have that it is equivalent to $\neg\neg\Diamond\neg(\forall x)\varphi \to \neg(\forall x)\neg\Diamond\neg\varphi$, and by the equivalence between $(\forall x)\theta$ and $\neg(\exists x)\neg\theta$ we have that it is equivalent to $\neg\neg\Diamond\neg\neg(\exists x)\neg\varphi \to \neg\neg(\exists x)\neg\neg\Diamond\neg\varphi$. We then apply double negation elimination and obtain $\Diamond(\exists x)\neg\varphi \to (\exists x)\Diamond\neg\varphi$, which is of the required form B2.

2. Instances of the Converse Barcan formula are either of the form

(CB1) $$\Box(\forall x)\varphi \to (\forall x)\Box\varphi$$

or of the form

(CB2) $$(\exists x)\Diamond\varphi \to \Diamond(\exists x)\varphi.$$

Show that every formula of form CB1 is equivalent to one of form CB2, and conversely.

Solution: Suppose we have a formula of form CB1. By contraposition it is equivalent to $\neg(\forall x)\Box\varphi \to \neg\Box(\forall x)\varphi$. By the equivalence between $\Box\theta$ and $\neg\Diamond\neg\theta$ we have that it is equivalent to $\neg(\forall x)\neg\Diamond\neg\varphi \to \neg\neg\Diamond\neg(\forall x)\varphi$, and by the equivalence between $(\forall x)\theta$ and $\neg(\exists x)\neg\theta$ we have that it is equivalent to $\neg\neg(\exists x)\neg\neg\Diamond\neg\varphi \to \neg\neg\Diamond\neg\neg(\exists x)\neg\varphi$. We then apply double negation elimination and obtain $(\exists x)\Diamond\neg\varphi \to \Diamond(\exists x)\neg\varphi$, which is of the required form CB2.

3. Prove that a variable-domain augmented frame F is *anti-monotonic* if and only if every instance of the *Barcan formula* is valid in every model M based on F.

Solution: For the left-to-right direction, let $F = (W, R, D)$ be a variable domain augmented frame which is anti-monotonic, let $u \in W$ be be any world and v any valuation, and let φ be any formula (so it may, for example, contain any number of free variables). We assume that our instance of the Barcan formula is in the form $\Diamond(\exists x)\varphi \to (\exists x)\Diamond\varphi$. Suppose that $u \Vdash_v \Diamond(\exists x)\varphi$, so there exists $z \in W$ such that uRz and $z \Vdash_v (\exists x)\varphi$, i.e. there is an x-variant w of v at z such that $z \Vdash_w \varphi$ and $w(x) \in D(x)$. Since z is accessible from u, by the $\Diamond$ condition of the definition of truth in a model, $u \Vdash_w \Diamond\varphi$. Moreover, $w(x) \in D(z)$, so by anti-monotonicity, $w(x) \in D(u)$. But this means that w is an x-variant of v at u, so $u \Vdash_v (\exists x)\Diamond\varphi$.
For the right-to-left direction, we reason by contraposition, assuming that anti-monotonicity fails in a frame and constructing a model based on that frame in which an instance of the Barcan formula is invalid. Let $F = (W, R, D)$ be a variable domain augmented frame which is not anti-monotonic, so there exist worlds $y, z \in W$ such that yRz but $D(z) \not\subseteq D(y)$, meaning that there must be some object e such that $e \in D(z)$ but $e \notin D(y)$. Let P be a one-place relation symbol (i.e. a predicate) and let I be defined as

$$I(P, u) = \begin{cases} \{e\} & \text{if } u = z \\ \emptyset & \text{otherwise.} \end{cases}$$

Our intended countermodel is the structure $M = (W, R, D, I)$.
We first show that the antecedent of (an instance of) the Barcan formula $\Diamond(\exists x)P(x) \to (\exists x)\Diamond P(x)$ is true at y. Let v be any valuation, and let w be a valuation such that for all variables x', if $x' = x$ then $w(x') = e$; otherwise $w(x') = v(x)$. Clearly w is an x-variant of v. $z \Vdash_w P(x)$, since $I(P, z) = \{e\}$. As it is also the case that $w(x) \in D(x)$, w is an x-variant of v *at* z, so $z \Vdash_v (\exists x)P(x)$. Finally, since yRz, $y \Vdash_v \Diamond(\exists x)P(x)$.

Now we show that the consequent is false at y. Suppose for a contradiction that it is true, i.e. that $y \Vdash_v (\exists x)\Diamond P(x)$ for all valuations v. Therefore there exists an x-variant w of v at y such that $w(x) \in D(y)$ and $y \Vdash_w \Diamond P(x)$. By the definition of truth in a model, there exists some u such that $u \Vdash_w P(x)$. If $u \neq z$, then $I(P, u) = \emptyset$, so $w(x) \notin I(P, u)$ and hence $u \nVdash_w P(x)$. So $u = z$, and since $I(P, z) = \{e\}$, $w(x) = e$. Since w is an x-variant of v at y, $w(x) = e \in D(y)$. But $e \notin D(y)$ by our initial supposition that (W, R, D) was not anti-monotonic, which is a contradiction. So $y \nVdash (\exists x)\Diamond P(x)$.