Modal Logic Exam

LMU Munich

Summer 2019

Instructions:

1. Complete **all questions** from section 1.
2. Complete **1 question** from section 2. The relative difficulty of the questions in section 2 will be taken into account: you do not need to choose a specific question in order to get the highest marks.
3. Submit your answers to me **via email**, as a PDF file, at:

   Benedict.Eastaugh@lrz.uni-muenchen.de.

   I would prefer that your answers be typeset in some reasonable system like LaTeX, but I will accept handwritten answers, as long as I can read them.
4. You must submit your answers by **Monday 23 September**.

Section 1: Answer *all* questions.

1. For each of the following formulas do the following. 1) If it is **K**-valid, provide a tableau proof (please label all modal rules and indicate why the tableau closes). 2) If it is not **K**-valid, then construct a countermodel.

   (a) $\Box(\Box\Box p \vee \neg\Box\Box q)$

   (b) $(\Box p \to \Box q) \to \Box(p \to q)$

   (c) $(\Box p \wedge \Diamond q) \to \Diamond(p \wedge q)$

   (d) $\Box\Box(p \wedge q) \leftrightarrow (\Box\Box p \wedge \Box\Box q)$

2. A frame (W, R) is *directed* if for all $x, y, z \in W$, if xRy and xRz, then there exists $w \in W$ such that yRw and zRw.

   Prove that a frame $F = (W, R)$ is directed if and only if every model $M = (W, R, \Vdash)$ based on F makes all instances of the following scheme valid:

   $$\Diamond\Box\varphi \to \Box\Diamond\varphi.$$

3. For each of the following sentences do the following: i) explain informally the ways in which the following sentences can be read *de re* and *de dicto*; ii) formalise the different readings in the language of first-order modal logic with predicate abstraction.

   (a) Alice believes that someone murdered Humpty Dumpty.

   (b) The number of planets is less than it could have been.

   (c) Whoever wins the sea battle tomorrow, someone will lose it.

   (d) Inspector Clouseau has a suspect in custody, but Commissioner Dreyfus doesn't think that they are the culprit.

4. Let $\mathcal{E}$ be a one-place relation symbol. The *existence relativisation* of a modal first-order formula φ, denoted $(\varphi)^{\mathcal{E}}$, is defined by the following conditions:

   (a) If p is atomic, then $(p)^{\mathcal{E}} = p$;

   (b) $(\neg\psi)^{\mathcal{E}} = \neg(\psi)^{\mathcal{E}}$;

   (c) For a binary connective $\circ$, $(\psi \circ \theta)^{\mathcal{E}} = ((\psi)^{\mathcal{E}} \circ (\theta)^{\mathcal{E}})$;

   (d) $(\Box\psi)^{\mathcal{E}} = \Box(\psi)^{\mathcal{E}}$;

   (e) $(\Diamond\psi)^{\mathcal{E}} = \Diamond(\psi)^{\mathcal{E}}$;

   (f) $((\forall x)(\psi))^{\mathcal{E}} = (\forall x)(\mathcal{E}(x) \to (\psi)^{\mathcal{E}})$;

   (g) $((\exists x)(\psi))^{\mathcal{E}} = (\exists x)(\mathcal{E}(x) \wedge (\psi)^{\mathcal{E}})$.

   Suppose that φ is a first-order modal sentence not containing the predicate $\mathcal{E}$. Prove that if φ is valid in every varying domain model, then $(\varphi)^{\mathcal{E}}$ is valid in every constant domain model.

Section 2: Answer *exactly one* question.

5. Recall the property of *directedness* defined above in section 1, question 2.

   (a) Define two new tableau rules corresponding to the frame condition of directedness. Both should be *necessity rules*: the first should apply to prefixed formulas of the form $\sigma\ \Box\varphi$, and the second should apply to prefixed formulas of the form $\sigma\neg\Diamond\varphi$.

   (b) Prove that if T is an open tableau that is satisfiable by a model $M = (W, R, \Vdash)$ based on a directed frame (W, R) and you apply either of the tableau rules you have just defined, then the result is another tableau T' which is also satisfiable by a directed frame.

   (In other words, prove the cases of the branch extension lemma of the soundness theorem that correspond to your new rules).

6. Prove the completeness of the constant domain **K** rules. That is, prove that if a sentence φ is valid in all constant domain models, then φ has a proof in the constant domain **K** tableau system.

   In answering this question it is sufficient to do the following:

   (a) Provide a systematic construction procedure for constant domain **K** tableaus.

   (b) Give a list of branch conditions that will hold for any open branch $\mathcal{B}$ of a tableau obtained by your systematic construction procedure.

   (c) Show how to construct a constant domain model $M = (W, R, D, I)$ from the open branch $\mathcal{B}$.

   (d) For your choice of assignment θ of prefixes to worlds, and valuation v assigning variables to elements of D, prove the following two key facts by induction on formula complexity:

   $$\text{If } \sigma\ \varphi \text{ is on } \mathcal{B}, \text{ then } M, \sigma \Vdash_v \varphi$$

   and

   $$\text{If } \sigma\ \neg\varphi \text{ is on } \mathcal{B}, \text{ then } M, \sigma \nVdash_v \varphi.$$

7. A well-known result of Gödel is that the modal logic **S4** interprets intuitionistic propositional logic.

   Specified below is a translation $(\cdot)^\Box$ of formulas in the language of propositional logic to formulas in the language of modal propositional logic, and a tableau proof system for intuitionistic propositional logic.

   Using these, prove the following version of Gödel's result: *If φ has an intuitionistic tableau proof, then $(\varphi)^\Box$ has an **S4** tableau proof.*

   The language of intuitionistic propositional logic is the same as that of classical propositional logic: atomic propositions $p_1, p_2, \ldots$; logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$; brackets. (In all of what follows you may ignore the biconditional $\leftrightarrow$.)

   The translation $(\cdot)^\Box$ is as follows:

   (a) $(p)^\Box = \Box p$ for p an atomic proposition;

   (b) $(\neg\psi)^\Box = \Box\neg(\psi)^\Box$;

   (c) $(\psi \circ \theta)^\Box = \Box((\psi)^\Box \circ (\theta)^\Box)$ where $\circ$ is any one of $\wedge$, $\vee$, $\rightarrow$.

   The tableau rules for intuitionistic propositional logic use *signed prefixes*: in addition to a prefix σ, each formula in the tableau is also annotated by a letter T (denoting that the formula is assertible) or a letter F (denoting that the formula is not assertible).

   A tableau branch is *closed* if it contains both $\sigma\ T\ \varphi$ and $\sigma\ F\ \varphi$ for some prefix σ and formula φ. A *proof of* φ is a tableau starting with a root node $1\ F\ \varphi$ such that every branch is closed.

The tableau rules for intuitionistic propositional logic are as follows:

$$
\begin{array}{c}
\sigma\ T\ (\varphi \wedge \psi) \\
| \\
\sigma\ T\ \varphi \\
\sigma\ T\ \psi
\end{array}
\qquad
\begin{array}{c}
\sigma\ F\ (\varphi \wedge \psi) \\
\diagup \quad \diagdown \\
\sigma\ F\ \varphi \qquad \sigma\ F\ \psi
\end{array}
$$

$$
\begin{array}{c}
\sigma\ T\ (\varphi \vee \psi) \\
\diagup \quad \diagdown \\
\sigma\ T\ \varphi \qquad \sigma\ T\ \psi
\end{array}
\qquad
\begin{array}{c}
\sigma\ F\ (\varphi \vee \psi) \\
| \\
\sigma\ F\ \varphi \\
\sigma\ F\ \psi
\end{array}
$$

$$
\begin{array}{c}
\sigma\ T\ \neg\varphi \\
| \\
\sigma\ F\ \varphi
\end{array}
\qquad
\begin{array}{c}
\sigma\ T\ (\varphi \to \psi) \\
\diagup \quad \diagdown \\
\sigma\ F\ \varphi \qquad \sigma\ T\ \psi
\end{array}
$$

For $\sigma.n$ where σ appears on the branch but $\sigma.n$ is new:

$$
\begin{array}{c}
\sigma\ F\ \neg\varphi \\
| \\
\sigma.n\ T\ \varphi
\end{array}
\qquad
\begin{array}{c}
\sigma\ F\ (\varphi \to \psi) \\
| \\
\sigma.n\ T\ \varphi \\
\sigma.n\ F\ \psi
\end{array}
$$

For $\sigma.n$ that has already appeared on the branch:

$$
\begin{array}{c}
\sigma\ T\ \varphi \\
| \\
\sigma.n\ T\ \varphi
\end{array}
$$

Hint: given an intuitionistic tableau proof T of a sentence φ, show how to construct an **S4** tableau proof T' in which each application of the intuitionistic rules in T corresponds to an application of **S4** rules in T'. Then use induction to prove that T' is a proof of $(\varphi)^{\Box}$. The tableau system for intuitionistic propositional logic given above is defined in section 5 Fitting [2014].

References

M. Fitting. Nested sequents for intuitionistic logics. *Notre Dame Journal of Formal Logic*, 55(1):41–61, 2014. doi:10.1215/00294527-2377869.