Modal Logic
Exercise Set 05

To be completed by Thursday 6 June

We will work through these exercises (and possibly some others as well) during the problem class. Exercises marked with a (△) are a little more challenging, and those marked with a (▲) are more difficult still.

1. Recall that the definitions of an immediate and a strict extension of a tableau T are as follows. A tableau T' is an *immediate extension* of T, in symbols $T \prec_1 T'$, if T' is obtained from T by applying a branch extension rule. A tableau T' *strictly extends* T, in symbols $T \prec T'$, if there is a sequence of k tableaus $\langle T_i \rangle_{i \leq k}$ such that $T_0 = T$, $T_k = T'$, and for all $i < k$, $T_i \prec_1 T_{i+1}$. We call such a sequence a *generating sequence of length* k for T', or that T' is *generated* from T. A tableau T' *extends* T, in symbols $T \preceq T'$, if $T \prec T'$ or $T = T'$.

   Fix a tableau T_0 and consider the restriction of the $\prec$ relation to the set $\{T | T_0 \preceq T\}$. Prove that the following properties hold of the restricted relation.

   (a) $\prec$ is transitive: if $A \prec B$ and $B \prec C$, then $A \prec C$.

   (b) $\prec$ is irreflexive: if $A \prec B$, then $A \neq B$.

   (c) $\preceq$ is antisymmetric: if $A \preceq B$ and $B \preceq A$, then $A = B$.

   (d) $\prec$ (and hence $\preceq$) is wellfounded: there is no function g such that for all $n \in \mathbb{N}$, $g(n+1) \prec g(n)$.

2. Prove the necessity case of the branch extension lemma. Suppose that T is a tableau with a satisfiable branch B, and that $\sigma\ \Box P$ appears on B. Show that the result of extending B by applying the necessity rule is another satisfiable tableau.

3. Prove the other possibility case of the branch extension lemma. Suppose that T is a tableau with a satisfiable branch B, and that $\sigma\ \neg\Box P$ appears on B. Show that the result of extending B by applying the possibility rule is another satisfiable tableau.

4. (△) Re-prove the soundness theorem for **K** by using induction on the length of the generating sequence for the (hypothetical) closed but satisfiable tableau T.

   Hint: Use the trivial tableau T_0 whose only formula is $\sigma\ \neg P$ as the base case, and apply the branch extension lemma to show that T_{n+1} is satisfiable on the assumption that T_n is. The rest of the proof is the same as the one in the lecture.