Modal Logic
Solutions to Exercise Set 05

We will work through these exercises (and possibly some others as well) during the problem class. Exercises marked with a ($\triangle$) are a little more challenging, and those marked with a ($\blacktriangle$) are more difficult still.

1. Recall that the definitions of an immediate and a strict extension of a tableau T are as follows. A tableau T' is an *immediate extension* of T, in symbols $T \prec_1 T'$, if T' is obtained from T by applying a branch extension rule. A tableau T' *strictly extends* T, in symbols $T \prec T'$, if there is a sequence of k tableaus $\langle T_i \rangle_{i \leq k}$ such that $T_0 = T$, $T_k = T'$, and for all $i < k$, $T_i \prec_1 T_{i+1}$. We call such a sequence a *generating sequence of length* k for T', or that T' is *generated* from T. A tableau T' *extends* T, in symbols $T \preceq T'$, if $T \prec T'$ or $T = T'$.

   Fix a tableau T_0 and consider the restriction of the $\prec$ relation to the set $\{T | T_0 \preceq T\}$. Prove that the following properties hold of the restricted relation.

   (a) $\prec$ is transitive: if $A \prec B$ and $B \prec C$, then $A \prec C$.

   (b) $\prec$ is irreflexive: if $A \prec B$, then $A \neq B$.

   (c) $\preceq$ is antisymmetric: if $A \preceq B$ and $B \preceq A$, then $A = B$.

   (d) $\prec$ (and hence $\preceq$) is wellfounded: there is no function g such that for all $n \in \mathbb{N}$, $g(n+1) \prec g(n)$.

   Solution: For (a), let $A \prec B$ be witnessed by a sequence of tableaus $A = T_0, \ldots, T_k = B$ and let $B \prec C$ be witnessed by $B = T_k, \ldots, T_{k+n} = C$. The sequence $T_0, \ldots, T_{k+n}$ is a generating sequence of length $k+n$ from A to C, so $A \prec C$.
   For (b), if $A \prec B$ then B has at least one more node than A, so $A \neq B$.
   For (c), let $A \preceq B$ and $B \preceq A$ and assume for a contradiction that $A \neq B$. $A \prec B$ and $B \prec A$, so by transitivity, $A \prec A$. But $\prec$ is irreflexive by (b), so $A \neq A$, a contradiction.

   For (d), suppose to the contrary that there is such a g. Let T be any tableau in the range of g, and let n be such that $g(n) = T$. By definition, T is generated by a k-length sequence of tableaus $T_0, T_1, \ldots, T_k = T$. Without loss of generality, let $g(n+k) = T_0$. Then by definition, $g(n+k+1) = T^*$ for some tableau $T^* \prec T_0$. But there can be no such tableau, since T_0 is the minimal element of the ordering.

2. Prove the necessity case of the branch extension lemma. Suppose that T is a tableau with a satisfiable branch B, and that $\sigma\ \Box P$ appears on B. Show that the result of extending B by applying the necessity rule is another satisfiable tableau.

   Solution: Let B be satisfied by $(W, R, \Vdash)$ and assignment f. We apply the necessity rule and extend B with the formula $\sigma.n\ P$, where the prefix $\sigma.n$ already occurred on B.

   $f(\sigma.n)$ is accessible from $f(\sigma)$ by definition, and since $f(\sigma) \Vdash \Box P$, $f(\sigma.n) \Vdash P$. So the existing model and assignment satisfy the extended branch and hence the extended tableau.

3. Prove the other possibility case of the branch extension lemma. Suppose that T is a tableau with a satisfiable branch B, and that $\sigma\ \neg\Box P$ appears on B. Show that the result of extending B by applying the possibility rule is another satisfiable tableau.

   Solution: Let B be satisfied by $(W, R, \Vdash)$ and assignment f. We apply the possibility rule and extend B with the formula $\sigma.n\ \neg P$, where the prefix $\sigma.n$ does not occur on B.
   Since $f(\sigma) \Vdash \neg\Box P$, there must be a world u such that $f(\sigma)Ru$ and $u \Vdash \neg P$. So we define a new assignment

   $$g(\tau) = \begin{cases} u & \text{if } \tau = \sigma.n \\ f(\tau) & \text{otherwise.} \end{cases}$$

   By construction, $g(\sigma.n) \Vdash \neg P$ and $g(\sigma)Rg(\sigma.n)$, and since $g{\restriction}_B = f$, g is an assignment such that all formulas on the extended branch are satisfied by it.

4. ($\triangle$) Re-prove the soundness theorem for **K** by using induction on the length of the generating sequence for the (hypothetical) closed but satisfiable tableau T.

   Hint: Use the trivial tableau T_0 whose only formula is $\sigma\ \neg P$ as the base case, and apply the branch extension lemma to show that T_{n+1} is satisfiable on the assumption that T_n is. The rest of the proof is the same as the one in the lecture.

   Solution: As in the lecture, suppose for a contradiction that P is provable via **K**-tableau, but P is not **K**-valid.
   Let T be a closed tableau starting with $1\ \neg P$, and let T_0 be the initial tableau whose only node is $1\ \neg P$. $T_0 \prec T$, so there is a generating sequence of length k for T from T_0.
   Since P is not **K**-valid, there is a model $(W, R, \Vdash)$ and a world $u \in W$ such that $u \nVdash P$. Let f be such that $f(1) = u$, so T_0 is satisfiable by f and $(W, R, \Vdash)$. Assume that for some $n \leq k$, T_n is satisfiable. By the branch extension lemma, T_{n+1} is satisfiable. So by induction, T_k is satisfiable. But T is also closed, by assumption, and therefore unsatisfiable, giving us our contradiction.