Modal Logic
Solutions to Exercise Set 11

1. Let $M = (W, R, D, I)$ be a non-rigid variable domain **K** model. Recall that "y is substitutable for x" means that no free occurrence of x in $\varphi(x)$ is within the scope of $(\forall y)$. Now, suppose the variable y is substitutable for x in $\varphi(x)$. Show that the formula $\langle \lambda x.\varphi(x)\rangle(y) \leftrightarrow \varphi(y)$ is valid in M.

   Solution: Fix $u \in W$ and a valuation v in M. We then have that $u \Vdash_v \langle \lambda x.\varphi(x)\rangle(y)$ iff $u \Vdash_w \varphi(x)$, where $w(x) = (v \star I)(x, u) = v(x)$. Since this means $w = v$, we have $u \Vdash_w \varphi(x)$ iff $u \Vdash_v \varphi(x)$, and since y is substitutable for x in $\varphi(x)$, $u \Vdash_v \varphi(x)$ iff $u \Vdash_v \varphi(y)$. We therefore have the equivalence by the definition of truth in a model, and since u and v were arbitrary, the conclusion follows.

2. Show that the formula $\langle \lambda x.(\varphi \rightarrow \psi)\rangle(t) \leftrightarrow \langle \lambda x.\varphi\rangle(t) \rightarrow \langle \lambda x.\psi\rangle(t)$ is valid in all non-rigid variable domain **K** models.

   Solution: Let $M = (W, R, D, I)$ be any non-rigid variable domain **K** model, let u be any world in W, let v be any valuation in M, and let t be a term.

   Suppose $u \Vdash_v \langle \lambda x.(\varphi \rightarrow \psi)\rangle(t)$ and thus that $u \Vdash_w \varphi \rightarrow \psi$, where $w(x) = (v \star I)(t, u)$. Suppose further that $u \Vdash_v \langle \lambda x.\varphi\rangle(t)$, so $u \Vdash_w \varphi$. Then $u \Vdash_w \psi$ from our first assumption, and hence $u \Vdash_v \langle \lambda x.\psi\rangle(t)$ by the definition of truth in a model. By the definition of truth in a model again, $u \Vdash_v \langle \lambda x.\varphi\rangle(t) \rightarrow \langle \lambda x.\psi\rangle(t)$.

3. Construct a non-rigid variable domain **K** model in which $\Diamond\langle \lambda x.P(x)\rangle(c)$ is valid but $\langle \lambda x.\Diamond P(x)\rangle(c)$ is not, where P is a one-place relation symbol and c is a constant.

   Solution: Let $W = \{u, z\}$, $R = \{(u, z), (z, u)\}$, $D(u) = \{c_0\}$, $D(z) = \{c_1\}$, $I(P, u) = \{c_0\}$, $I(P, z) = \{c_1\}$, $I(c, u) = c_0$, and $I(c, z) = c_1$, where $c_0 \neq c_1$.

   We first show that $\Diamond\langle \lambda x.P(x)\rangle(c)$ is valid. Let v be any valuation. $z \Vdash_v \langle \lambda x.P(x)\rangle(c)$ iff $z \Vdash_w P(x)$, where $w(x) = I(c, z) = c_1$. Since $c_1 \in I(P, z)$, $z \Vdash_w P(x)$, so $u \Vdash_v \Diamond\langle \lambda x.P(x)\rangle(c)$. The other case is symmetrical.

   We now show that $\langle \lambda x.\Diamond P(x)\rangle(c)$ is invalid. Let v be any valuation, and suppose for a contradiction that $u \Vdash_v \langle \lambda x.\Diamond P(x)\rangle(c)$. It follows that $u \Vdash_w \Diamond P(x)$, where $w(x) = I(c, u) = c_0$. The only world accessible to u is z, so $z \Vdash_w P(x)$, and hence $c_0 \in I(P, z)$. Since the only element of $I(P, z)$ is c_1, $c_0 = c_1$. But this contradicts our initial assumption.