Discrete Structures

Lecture # 2

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Today's Topic

The Foundations: Logic, Proposition, Predicate, Definitions

Truth Tables and Negation, OR, AND operators with examples

Exclusive Or with Examples

Logical Equivalence with Examples

De-Morgan's Law with Examples

Tautology with Examples

Contradiction with Examples

Laws of Logic (Homework)!

The Foundations: Logic

- **Mathematical Logic** is a tool for working with compound statements

- **Logic** is the study of correct reasoning

- **Use of logic**

  - **In mathematics:** to prove theorems

  - **In computer science:** to prove that programs do what they are supposed to do

Propositional Logic

- Propositional logic: It deals with **propositions**.

- Predicate logic: It deals with **predicates**.

Definition of a Proposition

<u>Definition</u>: A **proposition** (usually denoted by p, q, r, …) is a declarative statement that is either **True** (T) or **False** (F), but not both or somewhere "in between!".

Propositional Variables

- Variables that represent propositions
- Conventional letters are : p, q, r, s, . . .
- Truth values: T(true), F(false)

Note: Commands and questions are not propositions.

Examples of Propositions

<u>The following are **all** propositions:</u>

- “It is raining” (In a given situation)
- “Amman is the capital of Jordan”
- “1 + 2 = 3” or "2+2 = 3”
- Two plus two is equal to four.
- Toronto is the capital of Canada.
- etc.

Examples of Propositions

But, the following are **NOT** propositions:

- “Who’s there?” (Question)
- “La la la la la.” (Meaningless)
- “Just do it!” (Command)
- “1 + 2” (Expression with a non-true/false value)
- “$1 + 2 = x$” (Expression with unknown value of x)

Operators / Connectives

An **operator** or **connective** combines one or more **operand** expressions into a larger expression. (e.g., "+" in numeric expression.)

- **Unary** operators take 1 operand (e.g. -3);
- **Binary** operators take 2 operands (e.g. 3×4).
- **Propositional** or **Boolean** operators operate on propositions (or their truth values) instead of on numbers.

Compound Statement (Propositions)

- Complicated logical statements build out of simple ones
- Three Symbols
  - ~ (not) --- ~ p (not p)
  - Λ (and) --- pΛq (p and q)
  - V (or) --- pVq (p or q)
- ~ p (Negation), pΛq (Conjunction), pVq (Disjunctions)
- English words to logic
  - “p but q” means “p and q”
  - “neither p nor q” means “~ p and ~ q”

Some Popular Boolean Operators

Formal Name	Nickname	Arity	Symbol
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

The Negation Operator

<u>Definition</u>: Let p be a proposition then $\neg p$ is the **negation** of p (Not p , it is not the case that p).

e.g. If p = "London is a city"

then $\neg p$ = "London is **not** a city" or " it is not the case that London is a city"

The **truth table** for NOT:

p	$\neg p$
F	T
T	F

Operand column

Result column

T :≡ True; F :≡ False
":≡" means "is defined as".

Examples

1. Let p = “Tayyeb’s PC runs Linux”

   - ~ p ?

2. Let H = “It is hot”

   S= “It is Sunny”

   (i). “It is not hot but it is Sunny”

   “_________________”

   (ii). “It is neither hot nor Sunny”

   “_________________”

The Conjunction Operator

<u>Definition</u>: Let p and q be propositions, the proposition "p **AND** q" denoted by ($p \wedge q$) is called the **conjunction** of p and q.

The **conjunction of the statements P** and **Q** is the statement "**P and Q**" and its denoted by **P ∧ Q**. The statement **P ∧ Q** is true only when both P **and** Q are true.

e.g. If p = "I will have salad for lunch" and

q = "I will have steak for dinner", then

$p \wedge q$ = "I will have salad for lunch **and** I will have steak for dinner"

Remember: "$\wedge$" points up like an "A", and it means "AND"

Conjunction Truth Table

- Note that a conjunction $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ of n propositions will have 2^n rows in its truth table.

"And", "But", "In addition to", "Moreover". Ex: The sun is shining but it is raining

Operand columns

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

The Disjunction Operator

<u>**Definition**</u>: Let p and q be propositions, the proposition "p **OR** q" denoted by $(p \vee q)$ is called the **disjunction** of p and q.

The disjunction of the statements **P** and **Q** is the statement "**P or Q**" and its denoted by **P** ∨ **Q**. The statement **P** ∨ **Q** is true only when at least one of P or Q is true.

e.g. p = "My car has a bad engine"

q = "My car has a bad carburetor"

$p \vee q$ = "Either my car has a bad engine **or** my car has a bad carburetor"

Disjunction Truth Table

- Note that $p \lor q$ means that p is true, or q is true, **or both** are true!

- So, this operation is also called **inclusive or**, because it **includes** the possibility that both p and q are true.

p	q	$p \lor q$
F	F	F
F	T	**T**
T	F	**T**
T	T	T

Note the differences from AND

Takeaway

- Rather memorizing, it is easier to remember the rules summarized.

Operator	Symbolic	Summary of Truth Values
Conjunction	$P \wedge Q$	True only when both P and Q are true
Disjunction	$P \vee Q$	False only when both P and Q are false
Negation	(~ or ┐) ~P	Opposite truth value of P

Compound Statements

- Let p, q, r be simple statements. We can form other compound statements, such as
  - $(p \vee q) \wedge r$
  - $p \vee (q \wedge r)$
  - $\neg p \vee \neg q$
  - $(p \vee q) \wedge (\neg r \vee s)$
  - and many others…

Truth Table – Example

- Lets try to build table for

1. **~P ∧ Q**
2. **~P ∧ (Q ∨~P)**
3. **(P ∨ Q) ∧ ~(P ∧ Q)**
4. **~(~P)**

P	**Q**	**~P**	**~P ∧ Q**
T	T		
T	F		
F	T		
F	F		

Truth Table – Example – Cont..

- Lets try to build table for

1. ~P ∧ Q

P	**Q**	**~P**	**~P ∧ Q**
T		F	
T		F	
F		T	
F		T	

Truth Table – Example – Cont..

- Lets try to build table for

1. ~P ∧ Q

P	**Q**	**~P**	**~P ∧ Q**
	T	F	F
	F	F	F
	T	T	T
	F	T	F

Truth Table – Example – Cont..

- Lets try to build table for

1. ~P ∧ Q

P	**Q**	**~P**	**~P ∧ Q**
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

Truth Table – Example – $(p \vee q) \wedge r$

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
F	F	F		
F	F	T		
F	T	F		
F	T	T		
T	F	F		
T	F	T		
T	T	F		
T	T	T		

Truth Table – Example – Cont.. $(p \vee q) \wedge r$

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
F	F	F	F	F
F	F	T	F	F
F	T	F	T	F
F	T	T	T	T
T	F	F	T	F
T	F	T	T	T
T	T	F	T	F
T	T	T	T	T

Truth Table – Example

1. **~P ∧ (Q ∨~R)**

P	Q	R	~R	Q ∨ ~R	~P	~P ∧ (Q ∨~R)
T	T	T				
T	T	F				
T	F	T				
T	F	F				
F	T	T				
F	T	F				
F	F	T				
F	F	F				

Truth Table – Example – Cont..

1. $\sim P \wedge (Q \vee \sim R)$

P	Q	R	$\sim R$	$Q \vee \sim R$	$\sim P$	$\sim P \wedge (Q \vee \sim R)$
		T	F			
		F	T			
		T	F			
		F	T			
		T	F			
		F	T			
		T	F			
		F	T			

Truth Table – Example – Cont..

1. ~P ∧ (Q ∨~R)

P	**Q**	**R**	**~R**	**Q ∨ ~R**	**~P**	**~P ∧ (Q ∨~R)**
	T		F	T		
	T		T	T		
	F		F	F		
	F		T	T		
	T		F	T		
	T		T	T		
	F		F	F		
	F		T	T		

Truth Table – Example – Cont..

1. **~P ∧ (Q ∨~R)**

P	Q	R	~R	Q ∨ ~R	~P	~P ∧ (Q ∨~R)
T					F	
T					F	
T					F	
T					F	
F					T	
F					T	
F					T	
F					T	

Truth Table – Example – Cont..

1. ~P ∧ (Q ∨~R)

P	Q	R	~R	Q ∨ ~R	~P	~P ∧ (Q ∨~R)
				T	F	F
				T	F	F
				F	F	F
				T	F	F
				T	T	T
				T	T	T
				F	T	F
				T	T	T

Truth Table – Example – Cont..

1. **~P ∧ (Q ∨~R)**

P	**Q**	**R**	**~R**	**Q ∨ ~R**	**~P**	**~P ∧ (Q ∨~R)**
T	T	T	F	T	F	F
T	T	F	T	T	F	F
T	F	T	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	F	T	F	F	T	F
F	F	F	T	T	T	T

Truth Table – Example – Cont..

1. $(P \lor Q) \land \sim(P \land Q)$

P	Q	$P \lor Q$	$P \land Q$	$\sim(P \land Q)$	$(P \lor Q) \land \sim(P \land Q)$
T	T				
T	F				
F	T				
F	F				

Truth Table – Example – Cont..

1. $(P \vee Q) \wedge \sim(P \wedge Q)$

P	Q	$P \vee Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$(P \vee Q) \wedge \sim(P \wedge Q)$
T	T	T			
T	F	T			
F	T	T			
F	F	F			

Truth Table – Example – Cont..

1. $(P \vee Q) \wedge \sim(P \wedge Q)$

P	Q	$P \vee Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$(P \vee Q) \wedge \sim(P \wedge Q)$
T	T		T		
T	F		F		
F	T		F		
F	F		F		

Truth Table – Example – Cont..

1. **(P ∨ Q) ∧ ~(P ∧ Q)**

P	Q	P ∨ Q	P ∧ Q	~(P ∧ Q)	(P ∨ Q) ∧ ~(P ∧ Q)
T	T		T	F	
T	F		F	T	
F	T		F	T	
F	F		F	T	

Truth Table – Example – Cont..

1. $(P \vee Q) \wedge \sim(P \wedge Q)$

P	Q	$P \vee Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$(P \vee Q) \wedge \sim(P \wedge Q)$
T	T	T		F	F
T	F	T		T	T
F	T	T		T	T
F	F	F		T	F

Truth Table – Example – Cont..

1. $(P \vee Q) \wedge \sim(P \wedge Q)$

P	Q	$P \vee Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$(P \vee Q) \wedge \sim(P \wedge Q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

A Simple Exercise

Let p = "It rained last night",
q = "The sprinklers came on last night" ,
r = "The grass was wet this morning".

Translate each of the following into English:

$\neg p$ =

$r \wedge \neg p$ =

$\neg r \vee p \vee q$ =

A Simple Exercise

Let p = "It rained last night",
q = "The sprinklers came on last night" ,
r = "The grass was wet this morning".

Translate each of the following into English:

$\neg p$ = **"It didn't rain last night"**

$r \wedge \neg p$ = **"The grass was wet this morning, and it didn't rain last night"**

$\neg r \vee p \vee q$ = **"Either the grass wasn't wet this morning, or it rained last night, or the sprinklers came on last night"**

The Exclusive Or Operator

The binary **exclusive-or** operator "$\oplus$" (XOR) combines two propositions to form their logical "exclusive or" (exjunction?).

e.g. p = "I will earn an A in this course"

q = "I will drop this course"

$p \oplus q$ = "I will either earn an A in this course, **or** I will drop it (but not both!)"

Exclusive-Or Truth Table

- Note that $p \oplus q$ means that p is true, or q is true, but **not both**!
- This operation is called **exclusive or**, because it **excludes** the possibility that both p and q are true.

p	q	$p \oplus q$
F	F	F
F	T	T
T	F	T
T	T	**F**

Note the difference from OR

Natural Language is Ambiguous

Note that <u>English</u> “or” can be <u>ambiguous</u> regarding the “both” case!

“Justin Bieber is a singer or
Justin Bieber is a writer” $\vee$

“John Cena is a man or
John Cena is a woman” $\oplus$

Need context to disambiguate the meaning!

For this class, assume “OR” means <u>inclusive</u>.

Logical Equivalence

- Two statement forms are called logically equivalent if and only if, they have identical truth values for all possible truth values for their statement variables.

- The logical equivalence of statement forms P and Q is denoted by writing $P \equiv Q$.

Truth Table – Example – Cont..

1. $\sim(\sim P) \equiv P$

P	**~P**	**~(~P)**
T	F	T
F	T	F

Same Truth Values in 1st and 3rd row

De Morgan's Laws

- The negation of an **and / or** statement is logically equivalent to the **or / and** statement in which each component is negated.

- Symbolically:

$$\sim(P \wedge Q) \equiv \sim P \vee \sim Q \text{ and } \sim(P \vee Q) \equiv \sim P \wedge \sim Q$$

De Morgan's Laws – Example – 1

- Prove $\sim(\boldsymbol{P} \wedge \boldsymbol{Q}) \equiv \sim\boldsymbol{P} \vee \sim\boldsymbol{Q}$

P	Q	~P	~Q	P ∨ Q	~(P ∨ Q)	~P ∧ ~Q
T	T					
T	F					
F	T					
F	F					

De Morgan's Laws – Example – 1

- Prove $\sim(\boldsymbol{P} \wedge \boldsymbol{Q}) \equiv \sim\boldsymbol{P} \vee \sim\boldsymbol{Q}$

P	**Q**	**~P**	**~Q**	**P ∨ Q**	**~(P ∨ Q)**	**~P ∧ ~Q**
T	T	F	F			
T	F	F	T			
F	T	T	F			
F	F	T	T			

De Morgan’s Laws – Example – 1

- Prove $\sim(\boldsymbol{P} \wedge \boldsymbol{Q}) \equiv \sim\boldsymbol{P} \vee \sim\boldsymbol{Q}$

P	Q	~P	~Q	P ∨ Q	~(P ∨ Q)	~P ∧ ~Q
T	T	F	F	T	F	
T	F	F	T	T	F	
F	T	T	F	T	F	
F	F	T	T	F	T	

De Morgan's Laws – Example – 1

- Prove $\sim(\boldsymbol{P} \wedge \boldsymbol{Q}) \equiv \sim\boldsymbol{P} \vee \sim\boldsymbol{Q}$

P	Q	~P	~Q	P ∨ Q	~(P ∨ Q)	~P ∧ ~Q
T	T	F	F			F
T	F	F	T			F
F	T	T	F			F
F	F	T	T			T

De Morgan's Laws – Example – 1

- Prove $\sim(\boldsymbol{P} \wedge \boldsymbol{Q}) \equiv \sim\boldsymbol{P} \vee \sim\boldsymbol{Q}$

P	**Q**	**~P**	**~Q**	**P ∨ Q**	**~(P ∨ Q)**	**~P ∧ ~Q**
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

De Morgan’s Laws – Example – 2

- Prove $\sim(P \wedge Q)$ and $\sim P \wedge \sim Q$ are not equivalent

P	Q	~P	~Q	P ∧ Q	~(P ∧ Q)	~P ∧ ~Q
T	T	F	F			
T	F	F	T			
F	T	T	F			
F	F	T	T			

De Morgan's Laws – Example – 2

- Prove $\sim(\boldsymbol{P} \wedge \boldsymbol{Q})$ ***and*** $\sim\boldsymbol{P} \wedge \sim\boldsymbol{Q}$ are not equivalent

P	Q	~P	~Q	P ∧ Q	~(P ∧ Q)	~P ∧ ~Q
T	T			T		
T	F			F		
F	T			F		
F	F			F		

De Morgan's Laws – Example – 2

- Prove $\sim(P \wedge Q)$ ***and*** $\sim P \wedge \sim Q$ are not equivalent

P	**Q**	**~P**	**~Q**	**P ∧ Q**	**~(P ∧ Q)**	**~P ∧ ~Q**
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

Exercise – 1

- Are the statements $(\boldsymbol{P} \land \boldsymbol{Q}) \land \boldsymbol{R}$ and $\boldsymbol{P} \land (\boldsymbol{Q} \land \boldsymbol{R})$ logically equivalent?

- Are the statements $(\boldsymbol{P} \land \boldsymbol{Q}) \lor \boldsymbol{R}$ and $\boldsymbol{P} \land (\boldsymbol{Q} \lor \boldsymbol{R})$ logically equivalent?

Tautology

- A tautology is a statement from that is always true regardless of the truth values of the statement variables.

- A tautology is represented by the symbol “t”.

Tautology – Example

- The statement $P \vee \sim P$ is Tautology.

$$P \vee \sim P \equiv t$$

P	~P	$P \vee \sim P$
T	F	T
F	T	T

Contradiction

- A contradiction is a statement from that is always false regardless of the truth values of the statement variables.

- A contradiction is represented by the symbol “c”.

Contradiction – Example

- The statement $P \wedge {\sim}P$ is Contradiction.

$P \wedge {\sim}P \equiv c$		
P	~P	P ∧~P
T	F	F
F	T	F

Example – 1

$$(P \wedge Q) \vee (\sim P \vee (P \wedge \sim Q)) \equiv t$$

Example – 2

$$(P \wedge {\sim}Q) \wedge ({\sim}P \vee Q) \equiv c$$

Laws of Logic

Commutative Law: $P \wedge Q \equiv Q \wedge P$ *and* $P \vee Q \equiv Q \vee P$

Associative Law: $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$ *and* $(P \vee Q)$

Distributive Law: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (Q \wedge R)$ *and* $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (Q \vee R)$

Identity Law: $P \wedge t \equiv P$ *and* $P \vee c \equiv P$

Negation Law: $P \vee \sim P \equiv t$ *and* $P \wedge \sim P \equiv c$

Double Negation Law: $\sim(\sim P) \equiv P$

Idempotent Law: $P \wedge P \equiv P$ *and* $P \vee P \equiv P$

DeMorgan's Law: $\sim(P \wedge Q) \equiv \sim P \vee \sim Q$ *and* $\sim(P \vee Q) \equiv \sim P \wedge \sim Q$

Universal Bound Law: $P \vee t \equiv t$ *and* $P \wedge c \equiv c$

Absorption Law: $P \vee (P \wedge Q) \equiv P$ *and* $P \wedge (P \vee Q) \equiv P$

Negations of "t" and "c": $\sim t \equiv c$ *and* $\sim c \equiv t$

Proving Equivalence via Truth Tables

Example: Prove that $p \vee q$ and $\neg(\neg p \wedge \neg q)$ are logically equivalent.

p	**q**	**p∨q**	**~p**	**~q**	**~p∧~q**	**~(~p∧~q)**
F	**F**	**F**	**T**	**T**	**T**	**F**
F	**T**	**T**	**T**	**F**	**F**	**T**
T	**F**	**T**	**F**	**T**	**F**	**T**
T	**T**	**T**	**F**	**F**	**F**	**T**

Proving Equivalence using Logic Laws

Example: Show that $\neg(P \vee (\neg P \wedge Q))$ and $(\neg P \wedge \neg Q)$ are logically equivalent.

$$\neg \; (P \vee (\neg P \wedge Q))$$

$$\equiv \neg P \wedge \neg(\neg P \wedge Q) \quad \text{De Morgan}$$

$$\equiv \neg P \wedge (\neg(\neg P) \vee \neg Q) \quad \text{De Morgan}$$

$$\equiv \neg P \wedge (P \vee \neg Q) \quad \text{Double negation}$$

$$\equiv (\neg P \wedge P) \vee (\neg P \wedge \neg Q) \quad \text{Distributive}$$

$$\equiv \mathbf{F} \vee (\neg P \wedge \neg Q) \quad \text{Negation}$$

$$\equiv (\neg P \wedge \neg Q) \quad \text{Identity}$$

Application

Simplify: $p \vee [\sim(\sim p \wedge q)]$

Solution:

$p \vee [\sim(\sim p \wedge q)]$	
$\equiv p \vee [\sim(\sim p) \vee (\sim q)]$	DeMorgan's Law
$\equiv p \vee [p \vee (\sim q)]$	Double Negative Law
$\equiv [p \vee p] \vee (\sim q)$	Associative Law for $\vee$
$\equiv p \vee (\sim q)$	Indempotent Law

Which is the simplified statement form.

Application

Verify: $\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p$

$\sim(\sim p \wedge q) \wedge (p \vee q)$	
$\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q)$	DeMorgan's Law
$\equiv (p \vee \sim q) \wedge (p \vee q)$	Double Negative Law
$\equiv p \vee (\sim q \wedge q)$	Distributive Law in Reverse
$\equiv p \vee c$	Negation Law
$\equiv p$	Identity Law

Simplifying a Statement

"You will get an A if you are hardworking and the sun shines, or you are hardworking and it rains.:

Solution: Let

p = "You are hardworking'
q = "The sun shines"
r = "It rains"

The condition is then $(p \wedge q) \vee (p \wedge r)$

Proving Equivalence using Logic Laws

Example: Show that $\neg(\neg(P \to Q) \to \neg Q)$ is a contradiction.

$$\neg(\neg(P \to Q) \to \neg Q)$$

$$\equiv \neg(\neg(\neg P \vee Q) \to \neg Q) \text{ Equivalence}$$

$$\equiv \neg((P \wedge \neg Q) \to \neg Q) \text{ De Morgan}$$

$$\equiv \neg(\neg(P \wedge \neg Q) \vee \neg Q) \text{ Equivalence}$$

$$\equiv \neg(\neg P \vee Q \vee \neg Q) \text{ De Morgan}$$

$$\equiv \neg(\neg P \vee \mathbf{T}) \text{ Trivial Tautology}$$

$$\equiv \neg(\mathbf{T}) \text{ Domination}$$

$$\equiv \mathbf{F} \quad \text{Contradiction}$$

Exercise

Use Logical Equivalence to rewrite each of the following sentences more simply.

- It is not true that I am tired and you are smart.
- I forgot my pen or my bag and I forgot my pen or my glasses.
- It is raining and I have forgotten my umbrella, or it is raining and I have forgotten my hat.