# Discrete Structures

## Week # 2 Lecture # 2

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# Recap

Negation, AND, OR, Exclusive Or with Examples

Logical Equivalence with Examples

De-Morgan’s Law with Examples

Tautology with Examples

Contradiction with Examples

Laws of Logic (Homework)!

# Today’s Topics

Conditional Statement (Implication)

Examples with Exercise

Implication equivalence

Converse, Inverse, and Contrapositive

Bi-Conditional Statement

# Conditional Statements

- "If you earn an A in Math, then I'll buy a computer."

- If p and q are statement variables, the conditional of q by p is "If p then q" or "p implies q" and is denoted $p \rightarrow q$.

# Conditional Statement or Implication

- The arrow “$\Rightarrow$ “ is the conditional operator

p is called the hypothesis (or antecedent)

q is called the conclusion (or consequent)

# Truth Table for Implication

$$p \rightarrow q$$

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Examples

| STATEMENTS | TRUTH VALUES |
|---|---|
| 1. "If $1 = 1$, then $3 = 3$." | TRUE |
| 2. "If $1 = 1$, then $2 = 3$." | FALSE |
| 3. "If $1 = 0$, then $3 = 3$." | TRUE |
| 4. "If $1 = 2$, then $2 = 3$." | TRUE |
| 5. "If $1 = 1$, then $1 = 2$ and $2 = 3$." | FALSE |
| 6. "If $1 = 3$ or $1 = 2$ then $3 = 3$." | TRUE |

# Alternative Ways of Expressing Implications

- “If p then q”
- “p implies q”
- “if p, q”
- “p only if q”
- “p is sufficient for q”
- “not p unless q”
- “q follows from p”
- “q if p”
- “q whenever p”
- “q is necessary for p”

# Exercise

- Your guarantee is good only if you bought your CD less than 90 days ago.

  If your guarantee is good, then you must have bought your CD player less than 90 days ago.

- To get tenure as a professor, it is sufficient to be world-famous.

  If you are world-famous, then you will get tenure as a professor.

- That you get the job implies that you have the best credentials.

  If you get the job, then you have the best credentials.

- It is necessary to walk 8 miles to get to the top of the peak.

  If you get to the top of the peak, then you must have walked 8 miles.

# Translating English Sentences To Symbols

You do every exercise in this book and You get A on the final, implies, you get an A in the class.

**Solution:** $p \wedge q \rightarrow r$

Getting an A on the final and doing every exercise in this book is sufficient for getting an A in the class.

**Solution:** $p \wedge q \rightarrow r$

# Translating English Sentences To Symbols

**Let p, q, and r be the propositions:**

p = "you have the flu"
q = "you miss the final exam"
r = "you pass the course"

$p \to q$ If you have flu, then you will miss the final exam.

$\sim q \to r$ If you don't miss the final exam, then you will pass the course.

$\sim p \wedge \sim q \to r$ If you neither have flu nor miss the final exam, then you will pass the course.

# Hierarchy of Operations for Logical Connectives

1) $\sim$ (negation)

2) $\wedge$ (conjunction), $\vee$ (disjunction)

3) $\rightarrow$ (conditional)

# Truth Table -- Example 1

| p | q | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

$$p \vee \sim q \to \sim p$$

$p \vee \sim q \to \sim p$ means $(p \vee (\sim q)) \to (\sim p)$

| p | q | $\sim q$ | $\sim p$ | $p \vee \sim q$ | $p \vee \sim q \to \sim p$ |
|---|---|---|---|---|---|
| T | T | F | F | | |
| T | F | T | F | | |
| F | T | F | T | | |
| F | F | T | T | | |

# Truth Table -- Example 1

| p | q | p → q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

$$p \vee \sim q \to \sim p$$

$p \vee \sim q \to \sim p$ means $(p \vee (\sim q)) \to (\sim p)$

| p | q | ~q | ~p | p ∨ ~q | p ∨ ~ q → ~ p |
|---|---|---|---|---|---|
| | | | F | T | F |
| | | | F | T | F |
| | | | T | F | T |
| | | | T | T | T |

| p | q | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Truth Table -- Example 2

$$(p \to q) \wedge (\sim p \to r)$$

| p | q | r | $p \to q$ | $\sim p$ | $\sim p \to r$ | $(p \to q) \wedge (\sim p \to r)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | | | |
| T | T | F | T | | | |
| T | F | T | F | | | |
| T | F | F | F | | | |
| F | T | T | T | | | |
| F | T | F | T | | | |
| F | F | T | T | | | |
| F | F | F | T | | | |

| p | q | $p \to q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Truth Table -- Example 2

$$(p \to q) \land (\sim p \to r)$$

| p | q | r | $p \to q$ | $\sim p$ | $\sim p \to r$ | $(p \to q) \land (\sim p \to r)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | F | T | T |
| T | T | F | T | F | T | T |
| T | F | T | F | F | T | F |
| T | F | F | F | F | T | F |
| F | T | T | T | T | T | T |
| F | T | F | T | T | F | F |
| F | F | T | T | T | T | T |
| F | F | F | T | T | F | F |

| p | q | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Truth Table -- Example 3

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

| p | q | $p \rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

# Truth Table -- Example 3

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

| p | q | $\sim q$ | $\sim p$ | $p \rightarrow q$ | $\sim q \rightarrow \sim p$ |
|---|---|---|---|---|---|
| T | T | F | F | T | T |
| T | F | T | F | F | F |
| F | T | F | T | T | T |
| F | F | T | T | T | T |

# Implication Law -- Truth Table -- Example 4

$$p \rightarrow q \equiv \sim p \vee q$$

# Implication Law -- Truth Table -- Example 4

$$p \to q \equiv \sim p \vee q$$

| $p$ | $q$ | $p \to q$ | $\sim p$ | $\sim p \vee q$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

# Negation of a Conditional Statement

Since $p \to q \equiv \sim p \vee q$ therefore

$$\sim (p \to q) \equiv \sim (\sim p \vee q)$$

$\equiv \sim (\sim p) \wedge (\sim q)$ De Morgan's law

$\equiv p \wedge \sim q$ Double Negative law

**Example 1 – Negate and find the alternative representation.**

1) If Ali lives in Pakistan then he lives in Lahore.
2) If my car is in the repair shop, then I cannot get to class.
3) If x is prime then $x$ is odd or $x$ is 2.
4) If n is divisible by 6, then n is divisible by 2 and n is divisible by 3.

**Solutions:**

1) Ali lives in Pakistan and he does not live in Lahore.
2) My car is in the repair shop and I can get to class.
3) x is prime but x is not odd and x is not 2.
4) n is divisible by 6 but n is not divisible by 2 or by 3.

# Inverse of a Conditional Statement

The inverse of the conditional statement $p \to q$ is $\sim p \to \sim q$

For instance; for an implication $p \to q$,

Its **inverse** is: $\neg\, p \to \neg\, q$

# Inverse of a Conditional Statement

$p \to q$ is not equivalent to $\sim p \to \sim q$

| $p$ | $q$ | $p \to q$ | $\sim p$ | $\sim q$ | $\sim p \to \sim q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | F | F | F | T | T |
| F | T | T | T | F | F |
| F | F | T | T | T | T |

# Writing Inverse of a Conditional Statement

1. If Today is Friday, Then $2+3 = 5$.
   - If today is not Friday, Then $2 + 3 \neq 5$
2. If it Snows today, I will ski tomorrow
   - If it does not snow today, I will not ski tomorrow.
3. If P is a square, then P is a rectangle.
   - If P is not a square, then P is not a rectangle.
4. If my car is in the repair shop, then I can get to class.
   - If my car is not in the repair shop, then I shall not get to the class.

# Converse of a Conditional Statement

The converse of the conditional statement $p \rightarrow q$ is $q \rightarrow p$

For instance; for an implication $\boldsymbol{p} \rightarrow \boldsymbol{q}$,

Its **converse** is: $\boldsymbol{q} \rightarrow \boldsymbol{p}$

# Converse of a Conditional Statement

| p | q | $p \rightarrow q$ | $q \rightarrow p$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | T | F |
| F | F | T | T |

# Writing Converse of a Conditional Statement

1. If Today is Friday, Then 2+3 = 5.
   - If 2 + 3 = 5, Then today is Friday.

2. If it Snows today, I will ski tomorrow
   - If I will not ski tomorrow then it does snow today.

3. If P is a square, Then P is a rectangle.
   - If P is a rectangle, Then P is a square.

4. If my car is in the repair shop, Then I can get to class.
   - If I go to class, Then my car is in the repair shop.

# Contrapositive of a Conditional Statement

The contrapositive of the conditional statement $p \rightarrow q$ is

$$\sim q \rightarrow \sim p$$

A conditional and its contrapositive are equivalent. Symbolically,

$$p \rightarrow q \equiv \sim q \rightarrow \sim p$$

For instance; for an implication $\boldsymbol{p} \rightarrow \boldsymbol{q}$,

Its **contrapositive** is: $\neg\, \boldsymbol{q} \rightarrow \neg\, \boldsymbol{p}$

# Writing Inverse of a Conditional Statement

1. If Today is Friday, Then $2+3 = 5$.
   - If $2 + 3 \neq 5$, Then today is not Friday.

2. If it Snows today, I will ski tomorrow
   - I will not ski tomorrow only If it does not snow today.

3. If P is a square, then P is a rectangle.
   - If P is not a rectangle, then P is not a square.

4. If my car is in the repair shop, then I can get to class.
   - If I not get to the class, then my car is not in the repair shop.

# Converse, Inverse, Contrapositive

Some terminology, for an implication $p \rightarrow q$

- Its **converse** is: $q \rightarrow p$
- Its **inverse** is: $\neg p \rightarrow \neg q$
- Its **contrapositive** is: $\neg q \rightarrow \neg p$

| p | q | $\neg$p | $\neg$q | $p \rightarrow q$ | $q \rightarrow p$ (converse) | $\neg p \rightarrow \neg q$ (inverse) | $\neg q \rightarrow \neg p$ (contrapositive) |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T | T |
| T | F | F | T | F | T | T | F |
| F | T | T | F | T | F | F | T |
| F | F | T | T | T | T | T | T |

SAME

# Example of Converse, Inverse, Contrapositive

Write the **converse, inverse and contrapositive** of the statement

**“if $x \neq 0$, then John is a programmer”**

# Example of Converse, Inverse, Contrapositive

**“if $x \neq 0$, then John is a programmer”**

- **Converse**: “if John is a programmer, then $x \neq 0$”
- **Inverse**: “if $x = 0$, then John is not a programmer”
- **Contrapositive**: “if John is not a programmer, then $x = 0$”

# Takeaway

1. An implication is logically equivalent to it's contrapositive not inverse and converse.

2. The inverse and converse of an implication are logically equivalent.

# Bi-Conditional Statement

If **p and q** are statement variables, the **bi-condition** of p and q is "**p if and only if q**" and is denoted as "**$p \leftrightarrow q$**"

The word "**if and only if**" are sometimes abbreviated "iff".

The double headed arrow "$\leftrightarrow$" is bi-conditional operator.

# Truth Table for Bi-Conditional Statement $p \leftrightarrow q$

| p | q | $p \leftrightarrow q$ |
|---|---|---|
| F | F | T |
| F | T | F |
| T | F | F |
| T | T | T |

**Bi-Conditional Statement $p \leftrightarrow q$ -- Example**

1. $1 + 1 = 3$ if and only if earth is flat — True
2. Sky is blue if and only if $1 = 0$ — False
3. Milk is white if and only if birds lay eggs — True
4. 33 is divisible by 4 iff horse has four legs — False
5. $x > 5$ iff $x^2 > 25$ — False

## Bi-Conditional Statement – Logical Equivalence

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

| $p$ | $q$ | $p \leftrightarrow q$ | $p \rightarrow q$ | $q \rightarrow p$ | $(p \rightarrow q) \wedge (q \rightarrow p)$ |
|---|---|---|---|---|---|
| T | T | T | | | |
| T | F | F | | | |
| F | T | F | | | |
| F | F | T | | | |

## Bi-Conditional Statement – Logical Equivalence

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

| $p$ | $q$ | $p \leftrightarrow q$ | $p \rightarrow q$ | $q \rightarrow p$ | $(p \rightarrow q) \wedge (q \rightarrow p)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | F | T | F |
| F | T | F | T | F | F |
| F | F | T | T | T | T |

## Rephrasing Bi-Conditional

$p \leftrightarrow q$ is also expressed as:

"p is necessary and sufficient for q"

"If p then q, and conversely"

"p is equivalent to q"

# Bi-Conditional – Examples

1. **If** it is hot outside you buy an ice cream cone, **and if** you buy an ice cream cone it is hot outside.
    - **Sol:** You buy an ice cream cone **if and only if** it is hot outside.

2. For you to win the contest it is **necessary and sufficient** that you have the only winning ticket.
    - **Sol:** You win the contest **if and only if** you hold the only winning ticket.

3. If you read the news paper every day, you will be informed and conversely.
    - **Sol:** You will be informed **if and only if** you read the news paper every day.

# Truth Table for $(p \to q) \leftrightarrow (\sim q \to \sim p) \equiv t$

# Truth Table for $(p \to q) \leftrightarrow (\sim q \to \sim p) \equiv t$

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ | $\sim q \to \sim p$ | $(p \to q) \leftrightarrow (\sim q \to \sim p)$ |
|---|---|---|---|---|---|---|
| T | T | | | | | |
| T | F | | | | | |
| F | T | | | | | |
| F | F | | | | | |

# Truth Table for $(p \to q) \leftrightarrow (\sim q \to \sim p) \equiv t$

| $p$ | $q$ | $\sim p$ | $\sim q$ | $p \to q$ | $\sim p \to \sim q$ | $(p \to q) \leftrightarrow (\sim q \to \sim p)$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

# Truth Table for $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

| $p$ | $q$ | $r$ | $p \leftrightarrow q$ | $r \leftrightarrow q$ | $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$ |
|---|---|---|---|---|---|
| T | T | T | | | |
| T | T | F | | | |
| T | F | T | | | |
| T | F | F | | | |
| F | T | T | | | |
| F | T | F | | | |
| F | F | T | | | |
| F | F | F | | | |

# Truth Table for $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$

| **p** | **q** | **r** | $p \leftrightarrow q$ | $r \leftrightarrow q$ | $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow q)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | F | F |
| T | F | T | F | F | T |
| T | F | F | F | T | F |
| F | T | T | F | T | F |
| F | T | F | F | F | T |
| F | F | T | T | F | F |
| F | F | F | T | T | T |

# Truth Table for $p$ A $\sim r \leftrightarrow q \vee r$

| $p$ | $q$ | $r$ | $\sim r$ | $p$ A $\sim r$ | $q \vee r$ | $p$ A $\sim r \leftrightarrow q \vee r$ |
|---|---|---|---|---|---|---|
| T | T | T | | | | |
| T | T | F | | | | |
| T | F | T | | | | |
| T | F | F | | | | |
| F | T | T | | | | |
| F | T | F | | | | |
| F | F | T | | | | |
| F | F | F | | | | |

# Truth Table for $p \wedge \sim r \leftrightarrow q \vee r$

| $p$ | $q$ | $r$ | $\sim r$ | $p \wedge \sim r$ | $q \vee r$ | $p \wedge \sim r \leftrightarrow q \vee r$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | F |
| T | T | F | T | T | T | T |
| T | F | T | F | F | T | F |
| T | F | F | T | T | F | F |
| F | T | T | F | F | T | F |
| F | T | F | T | F | T | F |
| F | F | T | F | F | T | F |
| F | F | F | T | F | F | T |

# Truth Table for $\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \leftrightarrow q$ | $p \leftrightarrow \sim q$ |
|---|---|---|---|---|---|
| T | T | F | F | | |
| T | F | F | T | | |
| F | T | T | F | | |
| F | F | T | T | | |

# Truth Table for $\sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$

| $p$ | $q$ | $\sim p$ | $\sim q$ | $\sim p \leftrightarrow q$ | $p \leftrightarrow \sim q$ |
|---|---|---|---|---|---|
| T | T | F | F | F | F |
| T | F | F | T | T | T |
| F | T | T | F | T | T |
| F | F | T | T | F | F |

# Truth Table for $\sim(p \oplus q) \equiv p \leftrightarrow q$

| $p$ | $q$ | $p \oplus q$ | $\sim(p \oplus q)$ | $p \leftrightarrow q$ |
|---|---|---|---|---|
| T | T | | | |
| T | F | | | |
| F | T | | | |
| F | F | | | |

# Truth Table for $\sim(p \oplus q) \equiv p \leftrightarrow q$

| $p$ | $q$ | $p \oplus q$ | $\sim(p \oplus q)$ | $p \leftrightarrow q$ |
|---|---|---|---|---|
| T | T | F | | |
| T | F | T | | |
| F | T | T | | |
| F | F | F | | |

# Truth Table for $\sim(p \oplus q) \equiv p \leftrightarrow q$

| $p$ | $q$ | $p \oplus q$ | $\sim(p \oplus q)$ | $p \leftrightarrow q$ |
|---|---|---|---|---|
| T | T | F | T | |
| T | F | T | F | |
| F | T | T | F | |
| F | F | F | T | |

# Truth Table for $\sim(p \oplus q) \equiv p \leftrightarrow q$

| $p$ | $q$ | $p \oplus q$ | $\sim(p \oplus q)$ | $p \leftrightarrow q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | T | F | F |
| F | T | T | F | F |
| F | F | F | T | T |

## Laws of Logic

1. ***Commutative Law***: $p \leftrightarrow q \equiv q \leftrightarrow p$

2. ***Implication Law***: $p \rightarrow q \equiv {\sim}p \vee q$

3. ***Exportation Law***: $(p\ \mathsf{A}\ q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

4. ***Equivalence Law***: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

5. ***Reductio ad Absurdum***: $p \rightarrow q \equiv (p\ \mathsf{A}\ {\sim}q) \rightarrow c$

1. ***Commutative Law***: $p \leftrightarrow q \equiv q \leftrightarrow p$
2. ***Implication Law***: $p \to q \equiv {\sim}p \vee q$
3. ***Exportation Law***: $(p\ \mathrm{A}\ q) \to r \equiv p \to (q \to r)$
4. ***Equivalence Law***: $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
5. ***Reductio ad Absurdum***: $p \to q \equiv (p\ \mathrm{A}\ {\sim}q) \to c$

# Applications

***Prove that*** $\boldsymbol{p\ \mathrm{A}\ {\sim}q \to r \equiv {\sim}(p\ \mathrm{A}\ {\sim}q) \vee r}$

***Solution.***

$p\ \mathrm{A}\ {\sim}q \to r \equiv (p\ \mathrm{A}\ {\sim}q) \to r$ *Order of Operation*

$\equiv {\sim}(p\ \mathrm{A}\ {\sim}q) \vee r$ *Implication Law*

1. *Commutative Law:* $p \leftrightarrow q \equiv q \leftrightarrow p$

2. *Implication Law:* $p \rightarrow q \equiv {\sim}p \vee q$

3. *Exportation Law:* $(p \text{ A } q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

4. *Equivalence Law:* $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

5. *Reductio ad Absurdum:* $p \rightarrow q \equiv (p \text{ A } {\sim}q) \rightarrow c$

# Applications

***Prove that*** $(p \rightarrow r) \leftrightarrow (q \rightarrow r) \equiv [{\sim}({\sim}p \vee r) \vee ({\sim}q \vee r)] \text{ A } [{\sim}({\sim}q \vee r) \vee ({\sim}p \vee r)]$

***Solution.***

$(p \rightarrow r) \leftrightarrow (q \rightarrow r) \equiv ({\sim}p \vee r) \leftrightarrow ({\sim}q \vee r)$ *Implication*

$\equiv [({\sim}p \vee r) \rightarrow ({\sim}q \vee r)] \text{ A } [({\sim}q \vee r) \rightarrow ({\sim}p \vee r)]$ *Equivalence of bi-conditional*

$\equiv [{\sim}({\sim}p \vee r) \vee ({\sim}q \vee r)] \text{ A } [{\sim}({\sim}q \vee r) \vee ({\sim}p \vee r)]$ *Implication Law*

## Applications

***Prove that*** $\sim(p \to q) \to p \equiv t$

1. ***Commutative Law***: $p \leftrightarrow q \equiv q \leftrightarrow p$

2. ***Implication Law***: $p \to q \equiv \sim p \vee q$

3. ***Exportation Law***: $(p \wedge q) \to r \equiv p \to (q \to r)$

4. ***Equivalence Law***: $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

# Applications

***Prove that*** $\boldsymbol{\sim(p \to q) \to p \equiv t}$

***Solution.***

| | |
|---|---|
| $\sim(p \to r) \to p \equiv \sim[\sim(p \wedge \sim q)] \to p$ | *Implication Law* |
| $\equiv (p \wedge \sim q) \to p$ | *Double Negation* |
| $\equiv \sim(p \wedge \sim q) \vee p$ | *Implication Law* |
| $\equiv (\sim p \vee q) \vee p$ | *De Morgan's Law* |
| $\equiv (q \vee \sim p) \vee p$ | *Commutative Law of* $\vee$ |
| $\equiv q \vee (\sim p \vee p)$ | *Associative law of* $\vee$ |
| $\equiv q \vee t$ | *Negation Law* |
| $\equiv t$ | *Universal Bound Law* |