Types of solutions of straight lines:

i) unique solution ii) infinite solutions

iii) no - solution

⇒ solution is just a point of intersection.

Exp : 1 $\quad x_1 - 2x_2 = -1$

$\underline{-x_1 + 3x_2 = 3}$ $\quad$ ∴ Add

mathematically => $\quad x_2 = 2$

=> $x_1 - 2x_2 = -1$ $\quad$ => $\quad x_1 - 2(2) = -1$

$x_1 = -1 + 4$

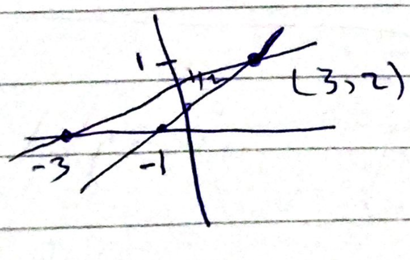
$\boxed{x_1 = 3}$

Point of intersection : $(3, 2)$

∴ unique solution

Geometrically => find intercepts of two lines

line 1 : $x_1 - 2x_2 = -1$: $(0, 1/2)$, $(-1, 0)$
line 2 : $-x_1 + 3x_2 = 3$: $(0, 1)$, $(-3, 0)$



Exp : 2 $\quad x_1 - 2x_2 = -1$

$\underline{-x_1 + 2x_2 = 1}$

mathematically => $\quad 0 = 0$ $\quad$ (correct statement)

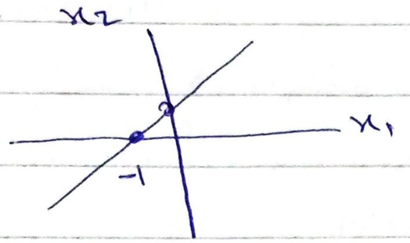
∴ infinite solutions

$\Rightarrow$ Geometrically $\Rightarrow$ line 1 : $x_1 - 2x_2 = -1$

$(0, 1/2), (-1, 0)$

line 2 : $-x_1 + 2x_2 = 1$

$(0, 1/2), (-1, 0)$



$\therefore$ lines are over-lapping and thus infinitly many solutions

$\therefore$ Note : Multiplying any equation with a non-zero factor, it doesn't change the orientation of the line.

In Exp : 2 the equation is multiplied with -1

Exp : 3

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 2x_2 &= 3 \\ \hline 0 &= 2 \end{aligned}$$

mathematically $\Rightarrow$ $0 = 2$

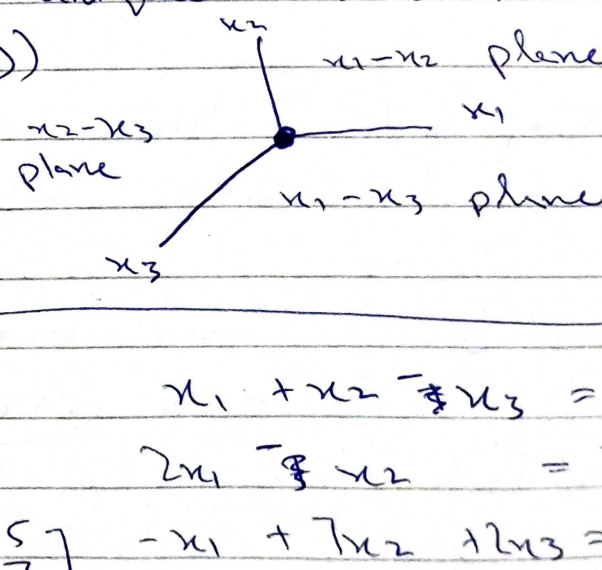
$\therefore$ no - solutions, lines are not intersecting.

Definitions : two systems said to be equivalent if they have same solution.

=> A system of linear equation is said to be "consistent" if it has either unique solution or infinitly many solution.

=> A given system is "inconsistent" if it has no - solution.

∴ Planes have unique solution ~~only~~ at origin. (3D)



Matrix form :

$$x_1 + x_2 - x_3 = 5$$
$$2x_1 - x_2 = 7$$
$$-x_1 + 7x_2 + 2x_3 = 0$$

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ -1 & 7 & 2 \end{bmatrix}}_{\text{coefficient matrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 & -1 & 5 \\ 2 & -1 & 0 & 7 \\ -1 & 7 & 2 & 0 \end{bmatrix} \Rightarrow \text{Augmented matrix}$$

=> Enchelon form : A given matrix is in enchelon form if it satisfies the following 3-properties:

i) All non-zeros rows are above any row of all zeros.

ii) Each leading entry of a row ~~has~~ lies towards the right of the leading entry of the row above it.

iii) All entries below the leading entry must be zero.

=> leading entry : The first non-zero element of the row is called leading entry, and the position of the leading entry in the matrix is called pivot position.

$$3\times4 \quad \begin{bmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & 0 & \boxed{6} & 7 \\ 0 & 0 & 0 & \boxed{4} \end{bmatrix} \qquad \therefore \; \boxed{} = \text{leading entries}$$

=> if a matrix satisfy two additional conditions then it is said to be in "reduced enchelon form".

iv) leading entry must be 1.
v) All elements above the leading entry must be zero.

∴ If all 5 properties all true then the matrix is in "reduced enchelon form".

Q*: find the solution of the following systems:

$$x_1 - 2x_2 - x_3 = 3$$
$$-2x_1 + 4x_2 + 5x_3 = -5$$
$$3x_1 - 6x_2 - 6x_3 = 8$$

$\Rightarrow \begin{bmatrix} 1 & -2 & -1 & 3 \\ -2 & 4 & 5 & -5 \\ 3 & -6 & -6 & 8 \end{bmatrix} \Rightarrow$ Augmented matrix

=> This matrix is not in enchelon form.

=> formula to make the 2nd rows to the right.

$R_2 \pm \square R_1 \Rightarrow -2 + \boxed{2}(1) = 0$ ✓

$\Rightarrow ∴ R_2 + 2R_1$

1st element ↳ first element

$$\Rightarrow \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 3 & 1 \\ 3 & -6 & -6 & 8 \end{bmatrix}, R_2 + 2R_1$$

Now for 3rd row : $R_3 \pm \square R_1$

$3 - 3(1) = 0$

$$\Rightarrow \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -3 & -1 \end{bmatrix}; R_3 - 3R_1 \Rightarrow \therefore R_3 - 3R_1$$

Now again for 3rd row : $R_3 \pm \square R_2$

$0 + 1(0) = 0$

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, R_3 + R_2 \quad \therefore R_3 + R_2$$

⇒ Here column 1 contains a pivot position.
⇒ So column 1 is pivot column.
So variable corresponding to this column i.e. x_1 is called Basic variable.

⇒ column 2 has no pivot position so column 2 is not a pivot column. So, variable corresponding to this column i.e. x_2 is called free variable.

⇒ Column 3 has pivot position so column 3 is called Pivot column so variable corresponding to this column i.e. is called Basic variable.

∴ A given system has unique solution when all variables are basic variable.

∴ A given system has infinite solutions if there exist at least one free variable.

∴ A given system has no solution when right most column of augmented matrix is pivot.

Exp 1: $x_1 - 2x_2 = -1$
$-x_1 + 3x_2 = 3$ $\Rightarrow \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \boxed{1} & -2 & -1 \\ 0 & \boxed{1} & 2 \end{bmatrix}$; $R_2 + (1)R_1 \Rightarrow$

$x_1 - 2x_2 = -1$
$x_2 = 2$

$-x_2 = 1$

$\therefore$ There x_1 & x_2 are basic variables so given system has unique solution. Thus has 2 pivot columns.

Exp 2: $x_1 - 2x_2 = -1$
$-x_1 + 2x_2 = 1$ $\Rightarrow \begin{bmatrix} 1 & -2 & -1 \\ -1 & 2 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \boxed{1} & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$; $R_2 + (1)R_1$

$\therefore$ Here x_1 is basic variable where x_2 is a free variable, so this system has infinite solutions.

$\Rightarrow x_1 - 2x_2 = -1$ $\therefore$ there x_2 is free
$0 + 0 = 0$ let $x_2 = 0$

$\Rightarrow (-1, 0)$ is a solution.
This is one out of infinite solutions and we can find other solutions by letting x_2 different values.

Exp 3: $x_1 - 2x_2 = -1$

$-x_1 + 2x_2 = 3$

$\Rightarrow \begin{bmatrix} 1 & -2 & -1 \\ -1 & 2 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \boxed{1} & -2 & -1 \\ 0 & 0 & \boxed{2} \end{bmatrix}$, $R_2 + R_1$

∴ As right most column is pivot so given system has no solutions.

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from ①*: $x_1 - 2x_2 - x_3 = 3$

$3x_3 = 1$

$0 = 0$

We know, $x_3 = \frac{1}{3}$, and x_2 is free

let $x_2 = 0$; $x_1 - 0 - \frac{1}{3} = 3$

$\Rightarrow x_1 = \frac{10}{3}$; $\left(\frac{10}{3}, 0, \frac{1}{3}\right)$ is a solution

out of infinite solutions.

Now, let $x_2 = \frac{1}{2}$; $\cancel{\frac{1}{2} - 2x_2 - \frac{1}{3} = 3}$

$\cancel{\frac{1}{2} - 2x_2 = \frac{10}{3}}$ $\quad x_1 - 2\left(\frac{1}{2}\right) = +\frac{1}{3} + 3 \Rightarrow \cancel{x_1 = \frac{1}{3} + 4}$

$\Rightarrow x_1 = +\frac{1}{3} + 4 \Rightarrow \left(\frac{13}{3}, \frac{1}{2}, \frac{1}{3}\right)$

∴ We can perform following operations

1- $R_2 \pm R_1$ 2- interchange any two rows
3- multiply any constant with any row

These operations will make many new systems from just one system.
Those systems will not be equal but they will be equivalent ($\sim$) one solution will satisfy all systems.
These operations just change the orientation but the point of intersection is fixed.

∴ In ∞^*, the geometry of the solution R_1 of the system is a plane. Because the solution of the system are infinite (points) and we can join them and make a geometry that will be a plane.

∴ When there is a unique solution in a system, then it is a single point, even in the 3D - space.

∴ When the solutions are multiple of

each other, it means they all lie on a single line, and thus the geometry of the systems will be a line, even in the 3D-space.

Practice: 1.1 = 1 → 4, 17, 18, 19, 20, 21, 22
11, 12, 13, 14
1.2 = 17 → 20 and practice problems.

Q) 02:

$$x_2 - 4x_3 = 8$$
$$2x_1 - 3x_2 + 2x_3 = 1 \quad \Rightarrow \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$
$$5x_1 - 8x_2 + 7x_3 = 1$$

⇒ We need perform one of the operation on this matrix due to the 2nd row.

⇒ Interchanging R_1 and R_2:

$$\Rightarrow \begin{bmatrix} \boxed{2} & -3 & 2 & 1 \\ 0 & \boxed{1} & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$: Now to convert third row, $R_3 \pm \square R_1$

$5 - \frac{5}{2}(2)$

$$\Rightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & \frac{-1}{2} & 2 & \frac{-3}{2} \end{bmatrix}; R_3 - \frac{5}{2} R_1$$ 0

∴ Still it's not in Echelon form

$\Rightarrow R_3 \pm \square R_2 \Rightarrow -\frac{1}{2} \pm \left(\frac{1}{2}\right)(1) \Rightarrow -\frac{1}{2} + \frac{1}{2} = 0$

$\Rightarrow R_3 + \frac{1}{2} R_2 \Rightarrow \begin{bmatrix} \boxed{2} & -3 & 2 & 1 \\ 0 & \boxed{1} & -4 & 8 \\ 0 & 0 & 0 & \boxed{\frac{5}{2}} \end{bmatrix} ; R_3 + \frac{1}{2} R_2$

∴ As right most column is a pivot column, so the given system has no solution.

In 3rd row, we have,

$$0x_1 + 0x_2 + 0x_3 = 5/2$$

$$0 = 5/2$$

not possible

∴ We can solve this using other methods and still get the same answer, like we could multiply any row with non-zero.

$\Rightarrow$ A given system of linear equation $Ax = b \Rightarrow \begin{bmatrix} 0 & 1 & -4 \\ 2 & -3 & 2 \\ 5 & -8 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$ is non-homogeneous if $b \neq 0$ and the given system is homogeneous if $b = 0$. Corresponding to non-homogeneous linear system we have three possible solutions:

-> unique, infinite, no solution; but for homogeneous linear system we have two possible solution

-> unique and infinite solution.

If given system has unique solution, then this solution is called Trivial solution. And mathematically it is given by $\boxed{x = 0}$ ~~x_1, x_2~~ $\rightarrow (0, 0, 0)$
x_1, x_2, x_3

If given homogeneous system has infinite solutions then these solutions are called non-trivial solutions.

Exam question: difference b/w homogeneous and non-homogeneous both mathematically and graphically.

Practice Questions: $2x_1 - x_2 = h$
$-6x_1 + 3x_2 = k$

find the values of h and k, such that given system has;

⇒ unique, infinite, no-solution.

∴ constants are known and doesn't change where as parameters vary based on specific problems.

⇒ h, K are parameters. x_1, x_2 are variables, and $2, -1, -6, 3$ are constants.

⇒ Now the given system can be said; they are systems, because on different values of parameters, it represents a new system.

Now, $\begin{bmatrix} 2 & -1 & h \\ -6 & 3 & K \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & -1 & h \\ 0 & 0 & K+3h \end{bmatrix}; R_2+3R_1$

⇒ this one has no-solution as
$0x_1 + 0x_2 \neq K+3h$

If we let $h=1, K=1$

$\Rightarrow \begin{bmatrix} 2 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow \boxed{\begin{aligned} 2x_1 - x_2 &= 1 \\ -6x_1 + 3x_2 &= 1 \end{aligned}}$

$\Rightarrow$ If we let $h=0, k=0$

$$\begin{bmatrix} \boxed{2} & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ $\Rightarrow$ x_2 is a free variable and thus it has infinite solutions.

$$\Rightarrow \boxed{\begin{aligned} 2x_1 - x_2 &= 0 \\ -6x_1 + 3x_2 &= 0 \end{aligned}}$$

$\Rightarrow$ There is no unique solution.

Q8 Construct 3 different systems whose solution set is $x_1 = 1, x_2 = 2, x_3 = 3$

$$\Rightarrow \begin{aligned} x_1 + 0x_2 + 0x_3 &= 1 \\ 0x_1 + x_2 + 0x_3 &= 2 \\ 0x_1 + 0x_2 + x_3 &= 3 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now for other two systems, multiply any row with any non-zero, they all will be different orientations and equivalent.